AN ABSTRACT OF THE THESIS OF

John Jacob Kohfeld for the Ph.D. in Mathematics
(Name) (Degree) (Major)

Date thesis is presented May 15 1963

Title Stability of Numerical Solutions of Systems of Ordinary Differential Equations

Abstract approved Redacted for privacy
(Major professor)

The background for this paper is the use of quadrature formulas for the solution of ordinary differential equations. If we know the values of the dependent variable for which we are solving, and its derivative, at several equally spaced points, i.e., at values of the independent variable separated by equal intervals, we may use a quadrature formula to integrate the values of the derivative, so that we may obtain an approximate value of the dependent variable at the next point. The differential equation is then used to evaluate the derivative at the new point. This procedure is then repeated to evaluate the dependent variable and its derivative at point after point. The accuracy of this method is limited by the accuracy of the quadrature formula used.

In order to improve the accuracy of the solution one may use an open-type quadrature formula to "predict" the value of the dependent

variable at the next point, then calculate the derivative, and now use a more accurate closed-type formula to "correct" the value of the dependent variable. This procedure is the basis of "Milne's method".

It has been shown that an error introduced at a step propagates itself approximately according to a linear combination of the solutions of a linear difference equation associated with the corrector . The solutions of this difference equation consist of an approximation to the solution of the differential equation and in some cases one or more extraneous solutions. If one or more of the latter increases as the process is repeated from step to step, the method is called instable. Remedies for instability include periodic use of special quadrature formulas called "stabilizers". This has been treated in the case of fifth-order formulas by Milne and Reynolds. In this paper the idea is extended to formulas of seventh order.

STABILITY OF NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

by

JOHN JACOB KOHFELD

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1963

APPROVED:

Redacted for privacy

Professor Emeritus of Mathematics

Redacted for privacy

Chairman of Department of Mathematics

Redacted for privacy

Dean of Graduate School

Date thesis is presented May 15 1963

Typed by Joan Shaw

ACKNOWLEDGEMENT

The author is indebted to Professor W. E. Milne for his efforts in reading several drafts of the manuscript for this thesis and giving his suggestions for its improvements.

TABLE OF CONTENTS

STABILITY OF NUMERICAL SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

1. Introduction

The background for this paper is the use of quadrature formulas for the solution of ordinary differential equations. If we know the values of the dependent variable for which we are solving, and its derivative, at several equally spaced points, i.e., at values of the independent variable separated by equal intervals, we may use a quadrature formula to integrate the values of the derivative, so that we may obtain an approximate value of the dependent variable at the next point. The differential equation is then used to evaluate the derivative at the new point. This procedure is then repeated to evaluate the dependent variable and its derivative at point after point. The accuracy of this method is limited by the accuracy of the quadrature formula used.

In order to improve the accuracy of the solution one may use an open-type quadrature formula to "predict" the value of the dependent variable at the next point, then calculate the derivative, and now use a more accurate closed-type formula to "correct" the value of the dependent variable. This procedure is the basis of "Milne's method" (7, p. 64-66).

It has been shown that an error introduced at a step propagates itself approximately according to a linear combination of the solutions of a linear difference equation associated with the corrector.

(7, p. 66-68). The solutions of this difference equation consist of an approximation to the solution of the differential equation and in some cases one or more extraneous solutions. If one or more of the latter increases as the process is repeated from step to step, the method is called instable. Remedies for instability include periodic use of special quadrature formulas called "stabilizers". This has been treated in the case of fifth-order formulas by Milne and Reynolds (8, p. 196-203). In this paper the idea is extended to formulas of the seventh order.

2. The Predict-Correct Method of Seventh Order

This paper deals with systems of differential equations of the form

(1) $\quad dy/dx = f(x, y(x)), \; y(x_o) = y_o,$

where f and y are vectors whose components are N known and N unknown functions, respectively, and x_o is a given value of x. The problem is to find the values of the vector function y(x) for other values of x.

The methods to be discussed are designed to compute y(x) for values of x having some constant difference h. We use the notation

$$x_n = x_o + nh, \; y_n = y(x_n), \; y'_n = dy(x_n)/dx.$$

We now consider an example of a "predict-correct" method for solving (1). First some suitable starting procedure[1] is used to compute $y_1, \cdots, y_5$. The derivatives y'_n, $(n = 0, 1, 2, \ldots)$ are computed by inserting y_n for y(x) and x_n for x into equation (1). Then, beginning with n = 6, we perform the following operations.

Operation 1. In order to get a tentative (predicted) value y_{np} for y_n, we use the "predictor" (4, p. 202),

(2) $\quad y_{np} = y_{n-6} + (3h/10)(11y'_{n-5} - 14y'_{n-4} + 26y'_{n-3} - 14y'_{n-2} + 11y'_{n-1}).$

---

[1] The particular starting procedure used is not important for a study of stability and will not be discussed in detail.

Operation 2. We then compute

$$y'_{np} = f(x_n, y_{np}).$$

Operation 3. We use the following "corrector" for the purpose of obtaining a more accurate value for y_n (4, p. 202):

$$(3) \qquad y_n = y_{n-4} + (2h/45)(7y'_{n-4} + 32y'_{n-3} + 12y'_{n-2} + 32y'_{n-1} + 7y'_{np}).$$

Operation 4. We then compute

$$y'_n = f(x_n, y_n).$$

Note: If $|y_n - y_{np}|$ is too large, one may now iterate Operations 3 and 4, using the new value of y'_n in place of y'_{np} in (3) in each iteration, until the change in y_n is sufficiently small (7, p. 67) but the author has not found this measure necessary.

Operation 5. We increase n by one and go back to Operation 1.

The truncation errors in the quadrature formulas (2) and (3), i.e. the quantities which, if added to the right members of these formulas, would make them accurate, are $1107\,h^7y^{(7)}/3780$ and $-32h^7y^{(7)}/3780$, respectively. Here $y^{(7)} = d^7y/dx^7$ for some x: $x_{n-6} \leq x \leq x_n$. Hence if $y^{(7)}$ does not change too rapidly one may, after operation 3 above, (optional), estimate the truncation error of (3) as approximately $(32/1139)(y_{np} - y_n)$ (7, p. 64-66). We say that (2) and (3) are of seventh order because their truncation errors are of seventh degree in h.

In order to investigate propagated error, let us consider a small perturbation of the solution y of (1) due to errors occurring at an earlier stage of the computation. We call this perturbation ϵw, where ϵ is a small scalar constant. The derivative of the perturbed function $y + \epsilon w$ will satisfy

(4) $$(y + \epsilon w)' = f(x, y + \epsilon w).$$

Let f_i, y_i, and w_i denote the i-th components of the vectors f, y, and w. We also let $G = G^{(1)}$ denote the Jacobian matrix of f with respect to y, let $G^{(m+1)}$ denote the Jacobian matrix of $G^{(m)}w$ with respect to y, for $m = 1, 2, \ldots,$ and assume that all these Jacobians exist. Then the right side of (4) may be expanded in powers of ϵ, to obtain

$$y' + \epsilon w' = f(x, y) + (Gw)\epsilon + (G^{(2)}w)\epsilon^2/2! + (G^{(3)}w)\epsilon^3/3! + \cdots .$$

We neglect all terms involving $\epsilon^2, \epsilon^3, \cdots,$ and subtract equation (1), obtaining

$$\epsilon w' = Gw\epsilon .$$

Therefore

(5) $$w' = Gw.$$

To simplify the analysis we consider G as constant. Hildebrand

(4, p. 202-207) shows how the analysis of this case can help us to understand the propagation of errors in the more general case.

Let T be a matrix such that $\Lambda = TGT^{-1}$ is in the classical canonical form, having the latent roots $g_1, g_2, \cdots, g_n$ of G in the diagonal and, if, as we assume, all these roots are distinct, zeros elsewhere (13, p. 58-61). (If these roots are not all distinct, ones may occur instead of zeros just above the principal diagonal, but the final result is not changed (9, p. 50).) By multiplying equation (5) by T, we obtain

$$Tw' = TGT^{-1}Tw = \Lambda Tw.$$

The differential system (5) is hence transformed into N separate equations:

(6) $z'_{n,i} = g_i z_{n,i}$, $(i = 1, 2, \dots, N)$,

where $z_{n,i}$ is the i-th component of

(7) $z_n = Tw_n$,

and w_n is the value of the variable vector w when $x = x_n$.

The general solution is

(8) $z_{n,i} = c_i \exp[g_i(x_n - x_o)] = c_i \exp(ns_i)$, $(i = 1, 2, \dots, N)$,

where each c_i is an arbitrary constant and

(9) $s_i = h g_i$, $(i = 1, 2, \dots, N)$.

When the corrector (3) is used, the computed values of $(y + \epsilon w)$ and $(y + \epsilon w)'$ are related by

$$(10) \quad (y + \epsilon w)_n = (y + \epsilon w)_{n-4} + (2h/45)\,[7(y + \epsilon w)'_{n-4} + 32(y + \epsilon w)'_{n-3} + 12\,(y + \epsilon w)'_{n-2} + 32(y + \epsilon w)'_{n-1} + 7(y + \epsilon w)'_{np}].$$

We assume that h has been chosen sufficiently small or that Operations 3 and 4 are iterated as indicated in the note after equation (3), so that

$$(y + \epsilon w)'_{np} - (y + \epsilon w)'_n \quad \text{and} \quad y'_{np} - y'_n$$

are negligible. We neglect these differences, subtract equation (3) from equation (10), and multiply the result by T/ϵ, to obtain

$$(11) \quad z_n = z_{n-4} + (2h/45)(7\,z'_{n-4} + 32\,z'_{n-3} + 12\,z'_{n-2} + 32\,z'_{n-1} + 7z'_n).$$

Hereafter, unless otherwise indicated, z_n and s will refer to $z_{n,i}$ and s_i, respectively. Equations (6) and (9) are applied to put (11) in the form

$$(12) \quad z_n = z_{n-4} + (2s/45)(7z_{n-4} + 32\,z_{n-3} + 12z_{n-2} + 32\,z_{n-1} + 7z_n), \quad (i = 1,\ 2,\ \ldots,\ N).$$

Note that the above recursion relation indicates a dependance of the behavior of z upon the corrector (3) only, and not upon the predictor (2). Chase (1, p. 457-468) discusses the important role the predictor plays in the analysis of the error unless, as we have assumed, h is sufficiently small or Operations 3 and 4 are iterated. That the role of the predictor is unimportant when we choose h sufficiently close to zero as well as when we iterate is implied by a theorem stated by Henrici (3, p. 261).

Equation (12) is a fourth order linear difference equation with constant coefficients. It has four particular solutions of which one, as we shall see, approximates the solution (8) of the differential equation (6). The standard procedure for the solution of such difference equations is given by Milne-Thomson (11, p. 384-385) and Milne (6, p. 341-343). The general solution is of the form

$$(13) \quad z_n = Ar_1^{\,n} + Br_2^{\,n} + Cr_3^{\,n} + Dr_4^{\,n},$$

where A, B, C, D are arbitrary constants and r_1, r_2, r_3, and r_4 are the roots of the indicial equation

$$(14) \quad r^4 \cong 1 + \frac{2s}{45}(7 + 32r + 12r^2 + 32r^3 + 7r^4).$$

These roots were expanded as power series in s with the aid of an Alwac III-E computer. The results are below.

$$(15) \quad r_1 \cong 1 + s + 0.5\, s^2 + 0.16667\, s^3 + 0.04167\, s^4 + 0.00833\, s^5 + 0.00139\, s^6 + 0.00231\, s^7 + \dots \cong e^s + 0.00211\, s^7,$$

(16) $r_2 \cong i + 0.02222i\, s + (0.01580 + 0.00025i)s^2 + (0.00035 - 0.01109i)s^3 + (-0.00768 - 0.00037i)s^4 + (-0.00035) + 0.00525i)s^5 + (0.00353 + 0.00030i)s^6 + (0.00025 - 0.00235i)s^7 + \ldots;$

(17) $r_3 \cong -1 + 0.42222\, s - 0.08914\, s^2 - 0.02971\, s^3 + 0.01652\, s^4 + 0.00568\, s^5 - 0.00431\, s^6 - 0.00152\, s^7 + \ldots$

The coefficients of the power series for r^4 are the complex conjugates of those in (16).

The power series for r, agrees with that of e^s up to terms of seventh degree in s. From (13) we see that z_n is a linear combination of the four particular solutions of (12), one of which, r_1^n, approximates the particular solution e^{ns} of (6), while the other three are extraneous solutions. If $|r_2|$, $|r_3|$, and $|r_4|$ are all less than one, for $s = s_1, s_2, \ldots, s_n$, then as n gets large the last three terms of (13) fade away, and $z_{n,i}$ approximates the general solution $c_i \exp(ns_i)$ of (6). Hence the perturbation $\epsilon w_n = \epsilon T^{-1} z_n$, arising from errors committed at some step, will approximately satisfy (5), which depends entirely upon the original system (1), and not upon our method of solution. In such a case our method is stable. If, for any $s = s_i$, $(i = 1, 2, \ldots, N)$, any quantity $|r_2|$, $|r_3|$, or $|r_4|$

exceeds unity, then the corresponding term Br_2^n, Cr_3^n, or Dr_4^n of (13) will grow exponentially as n increases. This will cause a growth of $z_{n,i}$ which does not follow the solution of the differential equation (6), and therefore a growth in $w_n = \epsilon T^{-1} z_n$ which does not follow the solution of (5). This growth is called instability.

The predict-correct method described here was used to solve the exponential problem

(18) $(y)_1' = -2(y)_1 - (y)_2; \; (y)_2' = (y)_1,$

where for $x = 0$,

$$(y)_1 = -1, \; (y)_2 = 1.$$

The exact solution is

$$(y)_1 = -e^{-x}, \; (y)_2 = e^{-x}.$$

We could have chosen a simple equation such as $y' = -y$ instead of (18), but (18) provides us with two theoretically equal functions, $-(y)_1$ and $(y)_2$, whose computed values we may contrast.

In the exponential problem (18), $g_1 = g_2 = -1$. The value of h used in solving (18) was 0.05, so $s = -0.05$. Here, as equation (17) shows, $|r_3| > 1$. The errors shown in Table I demonstrate the instability of the predict-correct method used. The symbol Δ_m indicates 10^9 times the quantity obtained by subtracting the calculated value of $(y)_m$ from its true value. Methods for the correction of instability shall be the subject of the remainder of this paper.

Table I.

		True values		Computed values		Errors	
n	x	$10^9(y)_1$	$10^9(y)_2$	$10^9(y)_1$	$10^9(y)_2$	1	2
362	18.10	-14	14	-647	450	633	-436
363	18.15	-13	13	617	-422	-630	435
364	18.20	-12	12	-665	460	653	-448
365	18.25	-12	12	671	-459	-683	471
366	18.30	-11	11	-700	490	689	-479

3. Stabilization of the Predict-Correct Method of Seventh Order by Periodic Use of a Special Formula

We are now ready to present the chief contribution of this paper, which is to show how one can make the previously discussed predict-correct method stable by the periodic use of a special quadrature formula called a stabilizer, and the development of a criterion for stability. The idea is an application to seventh order formulas of the work of Milne and Reynolds (8, p. 198) which deals with fifth-order formulas.

We modify the predict-correct method by inserting three operations between Operations 4 and 5 whenever n is a multiple of k, where k is a suitably chosen positive integer. We let y_{nc} and y'_{nc} denote the values of y_n and y'_n obtained in Operations 3 and 4. The three operations are given below:

Operation 4a. Calculate

$$(19) \quad y^*_n = y_{n-5} + \frac{5h}{288}(19y'_{n-5} + 75y'_{n-4} + 50y'_{n-3} + 50y'_{n-2} + 75y'_{n-1} + 19y'_{nc});$$

Operation 4b. Replace the previously obtained value y_{nc} of y_n by

$$y_n = \frac{1}{2}(y_{nc} + y_n^*).$$

Operation 4c. Recalculate the derivative

$$y'_n = f(x_n, y_n).$$

Equation (19) has a truncation error of $-(275/12096)h^7 y^{(7)}$.

Operations 4a, 4b, and 4c shall be called "the stabilization process" and (19) shall be called "the stabilizer". As a result of this process the difference equation (12) takes on the following form when n is a multiple of k:

$$(20)\quad z_n = \frac{1}{2}z_{nc} + \frac{1}{2}[z_{n-5} + \frac{5s}{288}(19z_{n-5} + 75z_{n-4} + 50z_{n-3} + 50z_{n-2} + 75z_{n-1} + 19z_{nc})],$$

where z_{nc} is the value that z_n has due to the difference equation (12), before the stabilization process is applied. If n is not a multiple of k, the difference equation (12) remains unchanged.

Equation (13) still provides a solution of the difference equation (12) between stabilization, but the coefficients A, B, C, and D are changed at each stabilization. Between the m-th and (m + 1) st stabilizations (13) takes on the form

(21) $z_{mk+j} = A_m r_1^{\,j} + B_m r_2^{\,j} + C_m r_3^{\,j} + D_m r_4^{\,j}$,

$(m = 1, 2, 3, \cdots ; \; j = -3, -2, \cdots , k-1)$,

as will be shown by the following theorem.

Theorem 1: *There exist* A_m, B_m, C_m, *and* D_m *such that (21) is true for* $j = -3, -2, \ldots , k - 1$ *if the r's are distinct and non-zero.*

Proof: If the r's are distinct and non-zero, there exist A_m, B_m, C_m, and D_m such that (21) is true for $j = -3, -2, -1, 0,$ because these coefficients are the solutions of

$$\begin{pmatrix} z_{mk-3} \\ z_{mk-2} \\ z_{mk-1} \\ z_{mk} \end{pmatrix} = \begin{pmatrix} r_1^{-3} & r_2^{-3} & r_3^{-3} & r_4^{-3} \\ r_1^{-2} & r_2^{-2} & r_3^{-2} & r_4^{-2} \\ r_1^{-1} & r_2^{-1} & r_3^{-1} & r_4^{-1} \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix},$$

and the solutions exist because the square matrix is a non-singular Vandermond matrix. When $j = 1, 2, \ldots , k - 1$, the quantity z_{mk+j} satisfies the difference equation (12), the solution of which is equation (21), where the arbitrary constants A_m, B_m, C_m and D_m assume the same values when $j = 1, 2, \ldots , k-1$ as they do when

$j = -3, -2, -1, 0$. Hence equation (21) is true for $j = -3, -2, \ldots,$ $k - 1$, if the r's are distinct and non-zero. Q. E. D.

The series (15), (16), and (17) indicate that this condition is satisfied if s is reasonably small. Since the corrected but unstabilized value $z_{mk+k,c}$ of z_{mk+k} satisfies the difference equation (12), we may state as a result of the proof of the preceeding theorem that

$$(22) \quad z_{mk+k,c} = A_m r_1^k + B_m r_2^k + C_m r_3^k + D_m r_4^k, \qquad (m = 1, 2, 3, \ldots).$$

To obtain the value of z_{mk+k} after stabilization, we let $n = mk + k$ in equation (20), and use (21) and (22) to evaluate the quantities z on the right side. In doing so we note that z_{n-5} becomes z_{mk+k-5}, which can be evaluated by equation (21) only if $k - 5 \geq -3$, or $k \geq 2$. If we make this restriction the above procedure produces the result:

$$(23) \quad z_{mk+k} = A_m K_1 r_1^{k-5} + B_m K_2 r_2^{k-5} + C_m K_3 r_3^{k-5} + D_m K_4 r_4^{k-5},$$

where

$$(24) \quad K_p = \frac{1}{2}\left[r_p^5 + 1 + \frac{5s}{288}\left(19 + 75r_p + 50r_p^2 + 50r_p^3 + 75r_p^4 + 19r_p^5\right)\right], \qquad (p = 1, 2, 3, 4).$$

In equation (21) we replace m by $m+1$ and obtain

$$(25)\qquad z_{mk+k+j} = A_{m+1} r_1^{\,j} + B_{m+1} r_2^{\,j} + C_{m+1} r_3^{\,j} + D_{m+1} r_4^{\,j},$$

$$(m = 0,\ 1,\ 2,\ \ldots;\ j = -3,\ -2,\ \ldots,\ k-1).$$

Equations (25) and (21) each give a value for z_n when $n = mk + k-3$, $mk + k-2$, $mk + k-1$. Equations (25) and (23) each give a value of z_{mk+k}. By equating these values we obtain four equations which we express in matrix form:

$$\begin{pmatrix} r_1^{-3} & r_2^{-3} & r_3^{-3} & r_4^{-3} \\ r_1^{-2} & r_2^{-2} & r_3^{-2} & r_4^{-2} \\ r_1^{-1} & r_2^{-1} & r_3^{-1} & r_4^{-1} \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_{m+1} \\ B_{m+1} \\ C_{m+1} \\ D_{m+1} \end{pmatrix} =$$

$$(26)\qquad \begin{pmatrix} r_1^{k-3} & r_2^{k-3} & r_3^{k-3} & r_4^{k-3} \\ r_1^{k-2} & r_2^{k-2} & r_3^{k-2} & r_4^{k-2} \\ r_1^{k-1} & r_2^{k-1} & r_3^{k-1} & r_4^{k-1} \\ K_1 r_1^{k-5} & K_2 r_2^{k-5} & K_3 r_3^{k-5} & K_4 r_4^{k-5} \end{pmatrix} \begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix}.$$

Let the first square matrix above be called R and the other matrix be called S.

Theorem 2: One of the latent roots of $R^{-1}S$ is

(27) $\lambda_1 \cong e^{ks} + (t + .00211k)s^7 + O(s^8)$, where $t \cong 0$, .0024. 0. .0017 for k mod 4 = 0, 1, 2, 3, respectively. If the other three latent roots λ_2, λ_3, and λ_4 all have magnitudes less than unity for all values of $s = s_1, s_2, \ldots, s_n$, the method of solution is stable.

Proof: The latent roots of $R^{-1}S$ are the solutions of

(28) $\det(S - R\lambda) = 0.$

Let $L_p = K_p r_p^{-5}$. By multiplying the p-th column of the determinant in (28) by r_p^3, we obtain

$$\begin{vmatrix} r_1^k - \lambda & r_2^k - \lambda & r_3^k - \lambda & r_4^k - \lambda \\ r_1(r_1^k - \lambda) & r_2(r_2^k - \lambda) & r_3(r_3^k - \lambda) & r_4(r_4^k - \lambda) \\ r_1^2(r_1^k - \lambda) & r_2^2(r_2^k - \lambda) & r_3^2(r_3^k - \lambda) & r_4^2(r_4^k - \lambda) \\ r_1^3(r_1^k L_1 - \lambda) & r_2^3(r_2^k L_2 - \lambda) & r_3^3(r_3^k L_3 - \lambda) & r_4^3(r_4^k L_4 - \lambda) \end{vmatrix}$$

We expand the above determinant by minors of the last row and obtain the result

(29) $$E_1(L_1r_1^k-\lambda)\ (r_2^k-\lambda)\ (r_3^k-\lambda)\ (r_4^k-\lambda)$$
$$+ E_2(L_2r_2^k-\lambda)\ (r_3^k-\lambda)\ (r_4^k-\lambda)\ (r_1^k-\lambda)$$
$$+ E_3(L_3r_3^k-\lambda)\ (r_4^k-\lambda)\ (r_1^k-\lambda)\ (r_2^k-\lambda)$$
$$+ E_4(L_4r_4^k-\lambda)\ (r_1^k-\lambda)\ (r_2^k-\lambda)\ (r_3^k-\lambda) = 0,$$

where

$$E_1 = -r_1^3\ (r_2 - r_3)(r_3 - r_4)(r_4 - r_2),$$
$$E_2 = r_2^3\ (r_3 - r_4)(r_4 - r_1)(r_1 - r_3),$$
$$E_3 = -r_3^3\ (r_4 - r_1)(r_1 - r_2)(r_2 - r_4),$$
$$E_4 = r_4^3\ (r_1 - r_2)(r_2 - r_3)(r_3 - r_1).$$

The series for r given in equations (15) - (17) were inserted into equation (24) and the following results were obtained with the aid of the Alwac computer:

$$K_1 \cong 1 + 5s + 12.5s^2 + 20.83333s^3 + 26.04166s^4$$
$$+ 26.04166s^5 + 21.70138s^6 + 15.51764s^7 + \ldots \sim e^{5s} \sim r_1^5;$$

$$K_2 \cong (.5 + .5i) + (.38194 + .4375i)s + (.07809 + .00694i)s^2$$
$$+ (.00928 - .05483i)s^3 + (-.03784 - .00959i)s^4$$
$$+ (-.00883 + .02565i)s^5 + (.01706 + .00759i)s^6$$
$$+ (.00621 - .01111i)s^7 + \ldots ;$$

$$K_3 \cong 1.05556s - 1.40741s^2 + .98792s^3 - .31390s^4 - .05332s^5$$
$$+ .07644s^6 - .00318s^7 + \dots .$$

The coefficients of the series for K_4 are the complex conjugates of those for K_2.

Examination of the power series for r_1 and K_1 reveals that

$$r_1 \cong e^s + .00211s^7 + O(s^8),$$
$$K_1 \cong e^{5s} + .0166s^7 + O(s^8),$$

and hence

$$(30) \quad L_1 = K_1 r_1^{-5} \cong 1 + .0061s^7 + O(s^8).$$

Similarly

$$(31) \quad L_2 = \frac{1}{2}(1 - i) + O(s),$$
$$L_3 = O(s),$$
$$L_4 = \frac{1}{2}(1 + i) + O(s).$$

Each of the quantities E_p appearing in (29) is of the form

$$(32) \quad E_p = -4i + O(s), \quad (p = 1, 2, 3, 4).$$

The quantities r_p^k take on the following values:

$$(33) \quad r_1^k \cong e^{ks} + .00211ks^7 + O(s^8); \; r_2^k = 1 + O(s), i + O(s), -1 + O(s), -i + O(s),$$

for $k \bmod 4 = 0, 1, 2, 3$, respectively;

$r_3^k = 1 + O(s), \ -1 + O(s), \ 1 + O(s), \ -1 + O(s),$

for k mod 4 = 0, 1, 2, 3, respectively;

$r_4^k = 1 + O(s), \ -i + O(s), \ -1 + O(s), \ i + O(s),$

for k mod 4 = 0, 1, 2, 3, respectively.

Since $L_1 r_1^k = r_1^k + O(s^7)$, $\lambda \cong r_1^k \cong e^{ks}$ appears to be an approximate solution of (29). We call this root λ_1 and let

$$\delta(s) = \lambda_1 - e^{ks}$$

so that the solution $\lambda = \lambda_1$ is equivalent to

(34) $$\lambda = e^{ks} + \delta(s).$$

The quantity $\delta(s)$ is of order $O(s^n)$ for some n. The use of equations (30) - (34) to make substitutions into (29) leads to equation (27) of our theorem.

We now assume that the roots λ of (29) are distinct and note, as we did before equation (6), that the final result will be the same if our assumption is not true. The canonical form of the matrix $R^{-1}S$ is

$$\Lambda = \overline{T} R^{-1} S \overline{T}^{-1},$$

with the latent roots λ_1, λ_2, λ_3, and λ_4 of $R^{-1}S$ in the diagonal. Then, since

$$R^{-1}S = \overline{T}^{-1} \Lambda \overline{T},$$

equation (26) leads to

$$\begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix} = \overline{T}^{-1} \Lambda^m \overline{T} \begin{pmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{pmatrix} .$$

Hence A_m, B_m, C_m, and D_m each are of the form

(35) $a\lambda_1^m + b\lambda_2^m + c\lambda_3^m + d\lambda_4^m$.

The behavior of $z_{n,i}$ will also be governed by the above combination, due to (21), where $n = mk + j$ and j is bounded. As m grows the first term of (35) approximates $ae^{mks} \cong a_1 e_{ns}$, where a_1 is some constant. The quantity $a_1 e^{ns}$ is the solution of the differential equation (6), and hence its behavior is basically governed by the nature of the system (1) itself, and not its method of solution. The other three terms represent the three extraneous roots of our fourth other difference scheme (equations (12) and (20)), and do not represent the solution of (6). If $|\lambda_2|$, $|\lambda_3|$, and $|\lambda_4|$ are all less than unity these terms will decay instead of grow as m increases. If all this is true for $s = s_1, \ldots, s_n$, then all extraneous growths of the components of z_n, and hence, due to (7), all extraneous growths of the components of w_n, will go to zero as n gets large, so that the solution is stable as stated in the theorem.

The extraneous roots λ_2, λ_3, and λ_4 were expanded in power series

$$\lambda_p = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda_{pmn} k^m s^n, \quad (p = 2, 3, 4),$$

but the coefficients for $n > 6$ were not determined. To facilitate this calculation we first note that both quantities $(L_1 r_1^k - \lambda)$ and $(r_1^k - \lambda)$ differ from $(\lambda_1 - \lambda)$, a factor of (29), only by terms of seventh degree or higher in s, which (27), (30) and (33) show. We make the approximation of removing the factor $(L_1 r_1^k - \lambda)$ from the first term of (29) and removing $(r_1^k - \lambda)$ from the other three terms, and calculate the power series, in k and s, which represents each root of the resulting cubic equations. These power series agree with the power series for λ_2, λ_3, and λ_4 to as many terms as the calculations are carried.

In order to expand the roots of the cubic equation in power series in k and s, it is useful to first expand each of the quantities appearing in the cubic equation, including r_2^k, r_3^k, and r_4^k, into power series. This is done more simply if we first express r_2, r_3, and r_4 as $i\underline{r}_2$, $-\underline{r}_3$, and $-i\underline{r}_4$, respectively. Then the power series for $\underline{r}_p$ (p = 2, 3, 4) are of the form

$$\underline{r}_p = 1 + \sum_{n=1}^{\infty} a_n s^n,$$

where coefficients a_n will depend upon p, and can be derived from equations (16) and (17). Consequently

$$(36) \qquad r_p^k = 1 + \sum_{m=1}^{\infty} \binom{k}{m} \left(\sum_{n=1}^{\infty} a_n s^n\right)^m,$$

where $\binom{k}{m} = k!/[m!\ (k - m)!]$, defined as zero if $m > k$. In the expansion of (36) as a series in k and s one may make use of the explicit expression for $\binom{k}{m}$, (k = 1, 2, ..., 6; m = 1, 2, ..., k) and the relation

$$(37) \qquad \left(\sum_{n=1}^{\infty} a_n s^n\right)^m = \sum_{n=m}^{\infty} b_{m,n} s^n,$$

where

$$(38) \qquad b_{m,n} = \sum_{p=m-1}^{n-1} a_{n-p}\, b_{m-1,\, p}.$$

The calculations of $\underline{r}_p^k$ were truncated after the term of sixth degree in s.

The previously mentioned cubic equation, when the coefficients of the powers of λ are collected, is

$$
\begin{aligned}
(39)\quad & -\lambda^3(E_1 + E_2 + E_3 + E_4) \\
& \quad + \lambda^2[\, E_1\, (i^k \underline{r}_2^{\,k} + (-1)^k\, \underline{r}_3^{\,k} + (-i)^k\, \underline{r}_4^{\,k}) \\
& \qquad + E_2 (i^k L_2 \underline{r}_2^{\,k} + (-1)^k \underline{r}_3^{\,k} + (-i)^k\, \underline{r}_4^{\,k}) \\
& \qquad + E_3\, (i^k \underline{r}_2^{\,k} + (-1)^k\, L_3 \underline{r}_3^{\,k} + (-i)^k \underline{r}_4^{\,k}) \\
& \qquad + E_4 (i^k \underline{r}_2^{\,k} + (-1)^k\, \underline{r}_3^{\,k} + (-i)^k\, L_4 \underline{r}_4^{\,k})] \\
& \quad -\lambda\, [E_1 ((-i)^k\, \underline{r}_2^{\,k} \underline{r}_3^{\,k} + \underline{r}_2^{\,k} \underline{r}_4^{\,k} + i^k\, \underline{r}_3^{\,k} \underline{r}_4^{\,k}) \\
& \qquad + E_2 ((-i)^k L_2 \underline{r}_2^{\,k} \underline{r}_3^{\,k} + L_2 \underline{r}_2^{\,k} \underline{r}_4^{\,k} + i^k\, \underline{r}_3^{\,k}\, r_4^{\,k}) \\
& \qquad + E_3 ((-i)^k L_3 \underline{r}_2^{\,k} \underline{r}_3^{\,k} + \underline{r}_2^{\,k} \underline{r}_4^{\,k} + i^k L_3 \underline{r}_3^{\,k} \underline{r}_4^{\,k}) \\
& \qquad + E_4 ((-i)^k\, \underline{r}_2^{\,k} \underline{r}_3^{\,k} + L_4 \underline{r}_2^{\,k} \underline{r}_4^{\,k} + i^k L_4 \underline{r}_3^{\,k} \underline{r}_4^{\,k}) \\
& \quad + \,(E_1 + E_2 L_2 + E_3 L_3 + E_4 L_4)(-1)^k \underline{r}_2^{\,k} \underline{r}_3^{\,k} \underline{r}_4^{\,k} = 0.
\end{aligned}
$$

We may find the power series of E_1, E_2, E_3, and E_4 by using the definition following equation (29) and performing the necessary arithmetic operations on the power series for r in (15), (16), and (17). We obtain:

$$(40)\quad E_1 \cong i(4 + 10.48889s + 14.21037s^2 + 13.30234s^3 + 9.54655s^4 + 5.48617s^5 + 2.61943s^6 + \ldots);$$

$$(41)\quad E_2 \cong -4i - (2.84444 + 2.66667i)s^2 - (2.02272 + 2.04247i)s^2 - (1.60764 + .86826i)s^3 - (.73174 + .27491i)s^4 - (.23384 + .13342i)s^5 - (.10273 + .06828i)s^6 - \ldots;$$

$$(42)\quad E_3 \cong -i(4 - .88889s + 1.94765s^2 + .00017s^3 + .17904s^4 + .08006s^5 - .02296s^6 + \ldots).$$

The coefficients of the series for E_4 are the negative conjugates of those for E_2.

The power series for L_p, $(p = 1, 2, 3, 4)$, may be obtained by dividing the power series for K_p by the power series for r_p^5.

Once we have calculated the series for the quantities $\underline{r}^k$, E, and L, we may insert these series into equation (39) and perform the operations necessary to obtain its coefficients as power series in k and s. The presence of the quantities i^k, $(-1)^k$, and $(-i)^k$ in (39) leads to the important result that the coefficients of these power series, except the coefficient of λ^3, depend on whether k modulo 4 = 0, 1, 2, or 3. After removing a common factor i, we obtain the result

$$(43)\qquad \underline{A}\,\lambda^3 + \underline{B}\,\lambda^2 + \underline{C}\,\lambda + \underline{D} = 0,$$

where the quantity $\underline{A}$ is given by

$$(44)\quad \underline{A} \cong 16. + 14.93333s + 20.24296s^2 + 15.03903s^3 + 10.27541s^4 + 5.83306s^5 + 2.73302s^6 + \ldots,$$

while the coefficients of $k^m s^n$ ($m = 1, 2, \ldots, n$; $n = 1, 2, \ldots 6$) in the power series expansions of $\underline{B}$ and $\underline{C}$ are given in Tables II - V, and $\underline{D} = (-1)^k \overline{D}$. The coefficients for the series expansion of $\overline{D}$ are given in Table VI. In all these series expansions the coefficient of $k^m s^n$ is zero whenever $m > n$. The calculations were done on the Alwac computer.

We are now ready to develop the series expansion of the roots of (43) for each of the cases $k \bmod 4 = 0, 1, 2,$ and 3. This is done by replacing $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ in (43) by their series expansions, replacing λ by a power series in k and s with unknown coefficients, and computing the values of these unknown coefficients which satisfy (43). The results are, to as many terms as calculated, the coefficients of the power series of the latent roots λ_2, λ_3, and λ_4. These series indicate for which values of k and s the magnitudes of the latent roots will be less than unity, and hence help us to tell what choices of h and k will cause the solution of a given problem (1) to be stable.

Table II. COEFFICIENTS OF THE POWER SERIES FOR B AND C WHEN k MOD 4 = 0.

B:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-40	-39.01111	-52.55801	-40.09238	-27.71906	-16.02288	-7.42456
k^1		4.44444	4.35185	5.76269	4.39292	2.89820	1.62363
k^2			-1.07654	-1.03934	-1.20537	-.88050	-.53557
k^3				.15049	.14549	+.15499	.10991
k^4					-.01589	-.01536	-.01476
k^5						.00134	.00130
k^6							-.00009

C:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	32	33.22222	44.38715	35.06767	24.61190	14.54657	7.11880
k^1		-7.46667	-7.80642	-10.16266	-7.99254	-5.35453	-3.05782
k^2			1.61185	1.69716	2.00963	1.52983	.95588
k^3				-.21316	-.22615	-.251134	-.18639
k^4					.021335	.02280	.02345
k^5						-.00171	-.00184
k^6							.00011

Table III. COEFFICIENTS OF THE POWER SERIES FOR B AND C WHEN k MOD 4 = 1.

B:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	8	9.16667	12.02247	10.09375	7.20834	4.34587	2.18412
k^1		-5.15556	-5.39432	-6.70327	-5.06061	-3.28688	-1.82262
k^2			1.06864	1.02354	1.19847	.88100	.53559
k^3				-.15055	-.14564	-.15498	-.10977
k^4					.01589	.01536	.10476
k^5						-.00134	-.00130
k^6							.00009

C:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	8	10.04444	12.43827	10.67296	7.54317	4.55367	2.27437
k^1		2.13333	1.21185	1.57955	.78516	.39822	.17502
k^2			-.30815	-.05592	-.12835	-.03466	.00068
k^3				.04284	.00085	.00547	-.00259
k^4					-.00426	.00078	.00068
k^5						.00034	-.00013
k^6							.00002

Table IV. COEFFICIENTS OF THE POWER SERIES FOR B AND C WHEN k MOD 4 = 2.

B:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	16	15.81111	20.65876	15.61823	10.61024	6.04086	2.82328
k^1		5.68889	5.44370	6.69172	4.96024	3.20165	1.77342
k^2			-1.06272	-1.02861	-1.20979	-.88709	-.53634
k^3				.15059	.14556	.15479	.10973
k^4					-.01589	-.01536	-.01476
k^5						.00134	.00130
k^6							-.00009

C:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-8	-8.26667	-11.65630	-9.43512	-6.83341	-4.14895	-2.10264
k^1		8.53333	8.91555	11.08499	8.57819	5.67072	3.21346
k^2			-1.58815	-1.67252	-2.00096	-1.52912	-.95348
k^3				.21351	.22651	.25109	.06122
k^4					-.02133	-.02280	-.02345
k^5						.00171	.00184
k^6							-.00001

Table V. COEFFICIENTS OF THE POWER SERIES FOR B AND C WHEN k MOD 4 = 3.

B:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	16	14.03333	19.87678	14.38039	9.90048	5.63615	2.41716
k^1		-4.97778	-4.40123	-5.75113	-4.29256	-2.81297	-1.57443
k^2			1.07062	1.04441	1.21669	.88659	.53633
k^3				-.15053	-.14541	-.15480	-.10987
k^4					.01536	.01536	.01476
k^5						-.00134	-.00130
k^6							.94400

C:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	16	14.91111	20.29259	14.95960	10.23532	5.84395	2.74179
k^1		-1.06667	-.10272	-.65722	-.19950	-.08203	-.01938
k^2			.33185	.08057	.13702	.03537	.00172
k^3				-.04249	-.00049	-.00552	-.12258
k^4					.00427	-.00077	-.00068
k^5						-.00034	.00013
k^6							.00013

Table VI. COEFFICIENTS OF THE POWER SERIES FOR $\overline{D}$.

$\overline{D}$:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-8	-9.14444	-12.07210	-10.01432	-7.16825	-4.35675	-2.19289
k^1		3.02222	3.45457	4.39997	3.59961	2.45633	1.43419
k^2			-.57086	-.65253	-.80077	-.64525	-.41805
k^3				.07189	.08217	.09702	.07689
k^4					-.00679	-.00776	-.00880
k^5						.00051	.00059
k^6							-.00003

To obtain $\underline{D}$ use the relation $\underline{D} = (-1)^k \overline{D}$.

We now discuss the series solutions of (43) in more detail. Let

(45) $P = \underline{A}\lambda + \underline{B}, \; Q = P\lambda + \underline{C}, \; U = Q\lambda + \underline{D}.$

Then U is the left side of (43), and hence vanishes. As a matter of notation, the coefficients of the power series of a quantity expressed in upper case are expressed in lower case. Another provision is made in the case of λ. For example,

(46) $$\lambda = \sum_{i,j} l_{ij} k^i s^j, \quad \underline{A} = \sum_j a_j s^j,$$

$$\underline{B} = \sum_{i,j} b_{ij} k^i s^j, \text{ etc.}$$

To begin the series solution, we solve the equation

(47) $$a_0 l_{00}^3 + b_{00} l_{00}^2 + c_{00} l_{00} + d_{00} = 0,$$

to obtain l_{00}. There will be three solutions, hence for each case (k mod 4 = 0, 1, 2, 3) equation (43) will have three series solutions, each started by letting l_{00} equal one of the roots of (47). In order to extend the power series for λ, consider the following formulas, derived from equations (45) and (46):

(48) $$p_{mn} = b_{mn} + \sum_{j=0}^{n} a_{n-j} l_{mj};$$

(49) $$q_{mn} = c_{mn} + \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m-i,\, n-j} l_{ij};$$

(50) $$U_{mn} = d_{mn} + \sum_{i=0}^{m} \sum_{j=0}^{n} q_{m-i,\, n-j} l_{ij}.$$

We first compute p_{00} and q_{00} from (48) and (49). After p_{ij}, q_{ij}, and l_{ij} are known for $i = 0, \ldots, m-1$; $j = 0, \ldots, n-1$, the formulas are used implicitly to find p_{mn}, q_{mn}, and l_{mn}. The quantity l_{mn}, which is used in the above equations, is of course the unknown we are seeking, and should be chosen so that $U_{mn} = 0$. To do this, we temporarily set $l_{mn} = 0$ and use equations (48) - (50) to obtain preliminary values p^*_{mn}, q^*_{mn}, and U^*_{mn} for p_{mn}, q_{mn}, and U_{mn}. From the three equations we may see that

(51) $$\Delta \equiv \frac{\partial U_{mn}}{\partial l_{mn}} = 3a_{00} l_{00}^{2} + 2b_{00} l_{00} + c_{00},$$

Hence in order to change to change U_{mn} from U^*_{mn} to zero, as it should be, we must add to I^*_{mn} the quantity $-U^*_{mn} / \Delta$. The new value of I_{mn} will satisfy (48 - 50) and make $U_{mn} = 0$, and hence is correct. the new value of I_{mn} makes necessary a recomputation of p_{mn} and q_{mn}.

If $m > n$, $a_{mn} = b_{mn} = c_{mn} = d_{mn} = 0$. It can be shown that consequently $p_{mn} = q_{mn} = l_{mn} = 0$ if $m > n$. Hence the second summation in (49) and (50) may start with $j = i$ instead of $j = 0$.

In the case $k \bmod 4 = 0$ the above procedure breaks down for the double root $l_{00} = 1$, since in this case $\Delta = 0$. By other means we may compute that the two series corresponding to the double root are

$$\lambda_2 = 1 + \frac{1}{45} ks + \dots ;$$

$$\lambda_3 = 1 - \frac{1}{5} ks + \dots .$$

As a consequence. at least for sufficiently small nonzero values of s, either $|\lambda_2|$ or $|\lambda_3|$ will exceed unity, so $k \bmod 4 = 0$ is a rather poor choice for k. The other root, λ_4, is $\frac{1}{2}$ when $s = 0$.

Tables VII - XII give the computed coefficients of the powers of k and s in the series expansions of λ_2 and λ_3 for the cases $k \bmod 4 = 1$, 2, and 3. When $k \bmod 4 = 1$ or 3, imaginary coefficients appear in the series for λ_2, and the coefficients in the series for λ_4 are the complex conjugates of the coefficients in the series for λ_2. Coefficients which are zero to five decimal places are omitted from the tables.

Table VII. COEFFICIENTS OF THE POWER SERIES FOR λ_2 WHEN k MOD 4 = 1.

λ_2:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	.11949	-.03746	.00343	-.00242	-.00212	.00258	.00102
ik^0	.81383	.07506	-.00270	-.00299	.00190	.00137	-.00228
k'		-.00447	.01619	.00035	-.00670	-.00008	.00229
ik'		.02990	-.00608	-.01067	.00111	.00493	-.00036
k^2			.00232	-.00046	-.00094	-.00008	.00041
ik^2			.00170	-.00085	-.00065	.00032	.00034
k^3				.00041	-.00030	-.00010	.00015
ik^3				-.00069	.00022	.00038	-.00010
k^4					-.00019	.00005	.00014
ik^4					-.00011	.00010	.00003
k^5						-.00003	.00004
ik^5						.00005	-.00001
k^6							.00002
ik^6							-.00001

Table VIII. COEFFICIENTS OF THE POWER SERIES FOR λ_3 WHEN k MOD 4 = 1.

λ_3:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-.73898	-.03132	-.02650	-.00329	.00714	-.00020	-.00024
k^1		.33115	.00402	-.02339	.00193	.00383	-.00054
k^2			-.07144	-.00071	.01301	-.00043	-.00338
k^3				.00858	.00092	-.00232	-.00034
k^4					-.00062	-.00013	.00008
k^5						.00015	-.00007
k^6							-.00004

Table IX. COEFFICIENTS OF THE POWER SERIES FOR λ_2 AND λ_3 WHEN k MOD 4 = 2.

λ_2:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	.70711	.04672	.02625	.00349	-.00072	-.00032	.00227
k^1		-.30809	-.01487	.02009	-.00032	-.00286	.00016
k^2			.06712	.00213	-.01131	.00004	.00268
k^3				-.00958	-.00024	.00026	.00229
k^4					.00099	.00004	-.00036
k^5						-.00008	
k^6							

λ_3:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-1.0000						
k^1		-.02222	.01580	.00110	-.00743	-.00559	.00330
k^2			-.00025	.00035	.00162	.00024	-.00130
k^3						.00009	-.01549
k^4							
k^5							
k^6							.00001

Table X. COEFFICIENTS OF THE POWER SERIES FOR λ_4 WHEN k MOD 4 = 2.

λ_4:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-.70711	-.10158	-.00104	.00618	-.00308	-.00376	.00364
k^1		-.02525	-.00931	.00834	.00573	-.00344	-.00271
k^2			-.00045	-.00018	-.00087	-.00031	.00072
k^3				.00017	-.00007	-.00016	.01323
k^4							
k^5							
k^6							.00001

Table XI. COEFFICIENTS OF THE POWER SERIES FOR λ_2 WHEN k MOD 4 = 3.

λ_2:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-.17610	.07077	-.00408	.00060	.00134	-.00172	.00271
ik^0	.86072	.04948	-.00306	-.00253	.00319	.00160	.00273
k'		.01217	-.02394	.28637	-.32769	-.04263	.19260
ik'		.03159	-.00902	.14702	-.13939	-.06545	.09774
k^2			.00247	-.00137	.05979	-.04030	-.20688
ik^2			-.00501	.00477	-.06397	.08611	.04899
k^3				-.00201	.00194	-.02781	.03966
ik^3				-.00037	-.00034	-.00617	-.01294
k^4					.00009	-.00046	.00239
ik^4					.00072	-.00069	.00992
k^5						.00023	-.00019
ik^5						-.00011	.00029
k^6							-.00007
ik^6							-.00006

Table XII. COEFFICIENTS OF THE POWER SERIES FOR λ_3 WHEN k MOD 4 = 3.

λ_3:	s^0	s^1	s^2	s^3	s^4	s^5	s^6
k^0	-.64780	-.08528	-.02145	-.00356	.00755	.00089	.00448
k^1		.28677	.03258	-.02171	-.00145	.00228	.00018
k^2			-.07185	-.00008	.01297	-.00084	-.00298
k^3				.01342	-.00358	-.00285	-.00387
k^4					-.00115	.00085	-.00030
k^5						-.00039	.00037
k^6							.00013

The process of using power series for finding the values of λ is not rigorous, because the series are truncated and the regions of convergence of the series have not been determined. However, equation (43) was also solved directly for various combinations of k and s ($|s| \leq 0.1$, $|sk| \leq 1$), by use of the Alwac. The coefficients of (43) were computed by use of a complex arithmetic subroutine written by Mr. Robert J. Jirka, and the roots obtained by use of a routine written by Mr. Gilbert A. Bachelor. The results obtained are in substantial agreement with the power series solution. This indicates that the series give a good approximation to λ when $|s| < 0.1$ and $|sk| < 1$. The real test of the series is in the application of this heuristic theory to actual problems, which will be discussed later in this thesis.

We now summarize the results of the foregoing treatment of the stability of the predict-correct stabilize method of order seven, which we shall call the "PCS-7 method". In order to have a stable solution one must choose k and h so that $|\lambda_2|$, $|\lambda_3|$, and $|\lambda_4|$ are less than unity for all $s = hg_1, \ldots, hg_n$. Within these limits it is generally better to choose a large number for k, as this means one is using the special stabilizer, which has a larger truncation error than the corrector, less frequently. Examination of the power series for λ indicates that choosing $k \bmod 4 = 3$ allows one to take k larger for a given h than the other three choices would allow. As a rule of thumb stability is provided when

(52) $k \bmod 4 = 3, \quad |sk| < 0.9,$

where

$$|s| = |h| \max |g_i|, \quad (i=1, 2, \ldots, N).$$

The power series for λ_2, λ_3, and λ_4 indicate that this rule is valid if s is not too large, and is satisfied with a margin of safety if $|s| \leq 0.1$. The direct solution of (43) mentioned earlier also points to this conclusion. The fact that the behavior of the roots λ which determine stability is influenced by the value of $k \bmod 4$ was discovered by the author when making the investigations for this paper. Even though, in the case of the PCS-5 method, a relationship between stability and the value of $k \bmod 2$ is thought to exist

(8, p. 203), no such relationship is revealed in the treatment by Milne and Reynolds of this method (8, p. 200). However, in this thesis we discovered that in the case of the PCS-7 method, the roots λ which determine stability behave quite differently in the four cases k mod 4 = 0, 1, 2, and 3.

We now give some examples of computations where the PCS-7 method just discussed was employed. It was used to solve the exponential problem given in equation (18). The errors resulting from the use of this method, for different values of k, are given in Table XIII. The symbol Δ_n indicates 10^9 times the error in y_n, as in Table I.

In the case $h = 0.05$ ($s = -0.05$) the solution is instable when $k = \infty$ (no use of the stabilizer) as is shown by the rapidly increasing error. If $k = 16$ the error has grown considerably by the time $x = 21$ and sometimes changes sign. This can be explained by the fact that k mod 4 = 0, causing the solution to be instable. When $k = 15$ or 19, stability is achieved. The fact that an error of 4×10^{-9} remains in the case $k = 15$ can be explained by pointing out that the derivatives y' are stored in the computer in the form $(64/45)hy'$, and hence are zero or nearly so in the computer, since the numbers were scaled so that any magnitude less than 2^{-32} would be rounded to zero. The The stability when $k = 15$ is in accordance with the prediction of equation (52). This equation, which in some cases

Table XIII.

h = 0.05		k = 15		k = 16		k = 19		k = ∞	
n	x	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
100	5.00	1	-1	-2	2	-1	-1	-1	2
200	10.00	1	-3	-6	6	0	-1	12	0
300	15.00	-1	0	-16	21	1	-1	159	-92
400	20.00	0	-1	-42	57	2	-1		
420	21.00	-1	1	-50	70	3	-2		
422	21.10	4	-4	12	-20	-1	0		
423	21.15	1	-1	30	-33	-2	0		
424	21.20	-1	1	-53	74	0	-1		

h = 0.10		k = 7		k = 15		k = 19		k = 23	
n	x	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
50	5.0	0	1	-1	1	-1	1	-1	1
100	10.0	-1	-1	0	-1	-1	-1	-1	0
150	15.0	-1	0	-1	1	-3	1	3	2
200	20.0	1	-1	3	0	1	-1	-5	7
210	21.0	0	0	-1	0	-7	4	10	-3
211	21.1	-3	1	-3	0				
212	21.2	0	0	3	0				
213	21.3	0	-1	-1	0				
214	21.4	0	0	-1	0				

is overrestrictive, does not predict stability when $k = 19$, but the power series for λ_2, λ_3, and λ_4 when $k \bmod 4 = 3$ does indicate stability. In the case $h = 0.10$ the results are also in accordance with theory, except that the instability predicted when $k = 15$ is not shown by the computed results.

The system

$$(y)'_1 = (y)_2, \quad (y)'_2 = -(y)_1,$$

where for $x = 0$,

$$(y)_1 = 0 \quad , \quad (y)_2 = 1,$$

was also solved by use of the PCS-7 method, with a step size h of 0.05. In this system $g_1 = i$, $g_2 = -i$, so $s = \pm 0.05i$. Equations (16) and (17) indicate that $|r_2|$ and $|r_3|$ are practically unity, so that the system is very close to the boundary between stability and instability. Tables XI and XII indicate stability when $k = 19$. The errors when the PCS-7 method was used on the system are given in Table XIV. The symbol Δ_n indicates 10^8 times the error in y_n. In all three cases, $k = 16$, 19, and ∞, no instability was indicated. The errors seem to indicate that the calculated solution lags behind the actual solution, $(y)_1 = \sin x$ and $(y)_2 = \cos x$.

Note: In the actual calculations the stabilizing process was performed whenever $n = mk + 6$, where m is a positive integer,

Table XIV.

h = 0.05		Actual Values		k = 16		k = 19		∞	
n	x	$(y)_1$	$(y)_2$	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
100	5.00	-.95892427	.28366219	1	1	1	1	1	1
200	10.00	-.54402111	-.83907153	-2	1	-1	1	-2	1
300	15.00	.65028784	-.75968791	-2	-2	-2	-1	-2	-2
400	20.00	.91294525	.40808206	2	-4	1	-4		
420	21.00	.83665564	-.54772926	-2	-3	-2	-4		
421	21.05	.80823498	-.58886010	-3	-4	-3	-4		
422	21.10	.77779416	-.62851909	-3	-3	-3	-3		
423	21.15	.74540926	-.66660711	-3	-3	-3	-3		
424	21.20	.71116122	-.70302896	-3	-3	-4	-3		

instead of whenever $n = mk$ as supposed in the stability theory, due to the fact that the part of the computer program that counts steps becomes operative only after the ''starting procedure''. The frequency of stabilization was not affected, however. In order to apply the stability theory, we may relabel the given values x_0, y_0 as x_{-6}, y_{-6}. The quantities formerly labeled x_n, y_n will then be labeled x_{n-6}, y_{n-6}, so the theory is applicable.

4. The Use of Stable Correctors in Predict-Correct Methods of Seventh Order.

An alternative method of remedying instability is to replace the corrector (3) in Operation 3 by

(54) $$y_n = (1 - a)[y_{n-4} + \frac{2h}{45}(7y'_{n-4} + 32y'_{n-3} + 12y'_{n-2} + 32y'_{n-1} + 7y'_{np})] + a[y_{n-1} + \frac{h}{1440}(27y'_{n-5} - 173y'_{n-4} + 482y'_{n-3} - 798y'_{n-2} + 1427y'_{n-1} + 475y'_{n})],$$

where a is a parameter which can be chosen on the basis of consideration of accuracy and stability.

Our indicial equation, whose roots determine stability or instability, is (cf. equation (14)):

(55) $$r^5 = (1 - a)[r + \frac{2s}{45}(7r + 32r^2 + 12r^3 + 32r^4 + 7r^5] + a[r^4 + \frac{s}{1440}(27 - 173r + 482r^2 - 798r^3 + 1427r^4 + 475r^5)].$$

When $a = s = 0$, the roots $r_1, \ldots, r_5$ take on the values 1, i, -1, $-i$, and 0, respectively. The root r_1 corresponds to the solution of the differential equation, while $r_2, \ldots, r_5$ are extraneous.

When $r = -1$ in equation (55), $s = 45a/38$. The quantities $r_2, \ldots, r_5$ were determined for $s = -(45a/38)e^{i\theta}$, ($\theta = 0, \pi/12, \ldots, 23\pi/12$; $a = 1/16, 1/8, 3/16$), by a computer program written by Mr. Robert R. Reynolds, and all results were within the unit circle. Hence these roots will be within the unit circle when $|s| < 45a/38$. Therefore the corrector (55) is stable if

(56) $|h| < 45a\,(38\,|g_i|)^{-1}, (i = 1, 2, \ldots, N)$.

The values of r_3, the extraneous root attaining the highest magnitude, is given in Table XV. Table XVI shows that the solution using (54) as a corrector is instable when (56) is not satisfied, i.e., when $a = 0$, but stable otherwise.

In another calculation of the solution of the exponential problem, formula (2) was used as a predictor and formula (19) was used as a corrector. Let us label the values obtained for y_n by (2) and (19) as y_{np} and y_{nc}, respectively. If the seventh derivative of y does not change too rapidly, the truncation errors of the two formulas will nearly cancel one another if we accept as the final value of y_n,

(57) $y_n = (119y_{nc} + 9y_{np})/128.$

This formula is also stable for sufficiently small h. The indicial equation is

(58) $$r^6 - (119\,r + 9)/128 = \frac{s}{184320}(56525r^6 + 265893r^5 + 94318r^4$$

$$+ 249838r^3 + 168693r^2 + 99293r).$$

The extraneous roots $r_2, \ldots, r_6$ of this equation were determined

Table XV. VALUES OF r_3 FOR $|s| = \frac{45a}{38}$, WHEN (55) IS USED TO COMPUTE y_n.

	$a = \frac{1}{16}$			$a = \frac{1}{8}$			$a = \frac{3}{16}$		
arg s	Re r_3	Im r_3	$\|r_3\|$	Re r_3	Im r_3	$\|r_3\|$	Re r_3	Im r_3	$\|r_3\|$
0°	-.9385	0	.9385	-.8787	0	.8787	-.8206	0	.8206
15°	-.9394	.0077	.9395	-.8805	.0147	.8806	-.8230	.0208	.8233
30°	-.9423	.0150	.9424	-.8858	.0286	.8862	-.8302	.0408	.8312
45°	-.9470	.0213	.9472	-.8944	.0409	.8953	-.8420	.0589	.8440
60°	-.9531	.0262	.9534	-.9059	.0509	.9073	-.8582	.0740	.8614
75°	-.9603	.0295	.9608	-.9199	.0577	.9217	-.8783	.0848	.8824
90°	-.9682	.0308	.9687	-.9354	.0608	.9373	-.9012	.0903	.9057
105°	-.9763	.0300	.9767	-.9514	.0597	.9533	-.9253	.0895	.9297
120°	-.9839	-.0271	.9843	-.9668	.0543	.9684	-.9488	.0819	.9524
135°	-.9905	.0223	.9908	-.9804	.0449	.9814	-.9697	.0680	.9721
150°	-.9956	.0158	.9958	-.9910	.0320	.9915	-.9860	.0486	.9872
165°	-.9989	.0082	.9989	-.9977	.0166	.9978	-.9964	.0253	.9968
180°	-1.0000	0	1.0000	-1.0000	0	1.0000	-1.0000	0	1.0000

If arg $s = \theta$, where $180° \leq \theta \leq 360°$, look up r_3 corresponding to arg $s = (360° - \theta)$ and change the sign of the imaginary part. The result is the value r_3 corresponding to arg $s = \theta$.

Table XVI.

		a = 0		$a = \frac{1}{16}$		$a = \frac{1}{8}$		$a = \frac{3}{16}$	
n	x	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
100	5.00	-1	2	0	-1	1	-2	0	-1
200	10.00	12	0	1	0	0	-1	0	-1
300	15.00	159	-92	1	-2	1	-1	0	-1
400	20.00			-1	-1	1	-1	0	-1
440	22.00			0	-2	2	-3	2	-3
441	22.05			1	-4	1	-3	3	-4
442	22.10			3	-4	2	-3	3	-4
443	22.15			0	-2	1	-3	2	-4
444	22.20			0	-2	2	-3	2	-3

Table XVI gives the results, when (54) is used as a corrector, with $s = -h = -0.05$ and $a = 0, 1/16, 1/8, 3/16$, on the exponential problem, and without any special stabilizing formula.

Table XVII. VALUES OF r_2 AND r_4 FOR $|s| = 0.1$, WHEN (57) IS USED.

arg s	Re r_2	Im r_2	$\|r_2\|$	Re r_4	Im r_4	$\|r_4\|$
0°	.3189	.9383	.9910	-.7611	-.5916	.9640
15°	.3188	.9381	.9908	-.7590	-.5861	.9589
30°	.3187	.9380	.9907	-.7582	-.5803	.9548
45°	.3187	.9378	.9905	-.7589	-.5745	.9518
60°	.3188	.9376	.9903	-.7609	-.5691	.9501
75°	.3189	.9374	.9902	-.7641	-.5642	.9498
90°	.3191	.9373	.9901	-.7684	-.5602	.9509
105°	.3192	.9372	.9901	-.7736	-.5573	.9534
120°	.3194	.9372	.9902	-.7795	-.5557	.9573
135°	.3196	.9373	.9903	-.7858	-.5557	.9624
150°	.3198	.9374	.9904	-.7921	-.5573	.9685
165°	.3199	.9375	.9906	-.7978	-.5606	.9751
180°	.3200	.9376	.9907	-.8026	-.5654	.9817
195°	.3200	.9378	.9909	-.8058	-.5715	.9879
210°	.3200	.9380	.9911	-.8073	-.5784	.9931
225°	.3200	.9381	.9912	-.8068	-.5854	.9968
240°	.3199	.9382	.9913	-.8042	-.5920	.9986
255°	.3198	.9384	.9914	-.8000	-.5976	.9986
270°	.3197	.9384	.9914	-.7944	-.6017	.9966
285°	.3196	.9385	.9914	-.7880	-.6041	.9929
300°	.3194	.9385	.9914	-.7813	-.6046	.9879
315°	.3193	.9385	.9914	-.7749	-.6034	.9821
330°	.3191	.9385	.9913	-.7692	-.6006	.9759
345°	.3190	.9384	.9911	-.7646	-.5965	.9698

NOTE: $r_3(s) = r_4(\bar{s})$, $r_5(s) = r_2(\bar{s})$.

for $|s| = 0.1$, $\arg s = 0, \pi/24, \ldots, 23\pi/24$. It was found that for all these values of s, $|r_6| < 0.08$. The values for the roots r_2 and r_4 are given in Table XVII. The roots r_3 and r_5 are given by equations below this table. We may conclude that the extraneous roots are within the unit circle if $|s| < 0.1$ or that stability occurs when

(59) $|s_i| < 0.1, \quad i = 1, 2, \ldots, N.$

The errors when this method was applied to the exponential problem for $h = 0.05$ and $h = 0.125$ are given in Table XVIII.

Table XVIII.

h = 0.05				h = 0.125			
n	x	Δ_1	Δ_2	n	x	Δ_1	Δ_2
100	5.00	0	-2	40	5.00	-2	1
200	10.00	0	-1	80	10.00	-3	1
300	15.00	0	-2	120	15.00	-4	2
400	20.00	0	-3	160	20.00	-8	5
420	21.00	0	-2	161	20.125	5	-5
440	22.00	1	-2	162	20.25	-3	2
441	22.05	2	-3	163	20.375	1	0
442	22.10	0	-2	164	20.50	7	-5
443	22.15	3	-4	168	21.00	1	0
444	22.20	2	-1				

The method is stable for $h = 0.05$, as predicted by equation (59). However, for $h = 0.125$ the method is instable, as shown by the fact that the errors change sign between $x = 20$ and $x = 21$.

BIBLIOGRAPHY

1. Chase, P. E. Stability properties of predictor-corrector methods for ordinary differential equations. Journal of the Association for Computing Machinery 9:457. 1962.

2. Crane, R. L. and R. J. Lambert. Stability of a generalized corrector formula. Journal of the Association for Computing Machinery 9:104. 1962.

3. Henrici, Peter. Discrete variable methods in ordinary differential equations. New York, Wiley, 1962. 407 p.

4. Hildebrand, F. B. Introduction to numerical analysis. New York, McGraw-Hill, 1956. 511 p.

5. Hull, T E. and A C. R. Newbery. Integration procedures which minimize propagated errors. Journal of the Society for Industrial and Applied Mathematics 9:31-47. 1961.

6. Milne, W. E. Numerical calculus. Princeton, Princeton University Press, 1949. 393 p.

7. Milne, W. E. Numerical solution of differential equations. New York, Wiley, 1953. 275 p.

8. Milne, W. E. and R. R. Reynolds. Stability of a numerical solution of differential equations. Journal of the Association for Computing Machinery. 6:196-203. 1959.

9. Milne, W. E. and R. R. Reynolds. Stability of a numerical solution of differential equations. Part II. Journal of the Association for Computing Machinery 7:46-56. 1960.

10. Milne, W. E. et al. Studies on numerical solution of ordinary differential equations with large scale digital computers. Corvallis, Oregon State University, 1961. 106 p. (U. S. Air Force Contract AF 33(616)-6822) (This report contains an extensive bibliography on its subject.)

11. Milne-Thomson, L. M. The calculus of finite differences. London, Macmillan, 1933. 558 p.

12. Nordsieck, Arnold. On numerical integration of ordinary differential equations. Mathematics of Computation 16:22. 1962.

13. Turnbull, H. W. and A. C. Aitken. An introduction to the theory of canonical matrices. London, Blackie, 1932. 192 p.