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THEOREMS

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Abstract approved

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After a discussion of nets and subnets (both words are here used in broader senses than are customary) two independent topics are treated. One is a generalization of the concept "monotone," as applied to sequences of real numbers, to a related concept for nets in uniform spaces. (Here the nets are as usual functions on directed sets.) A net \( \{S_m, m \in D\} \) in a pseudometric space \( (X, d) \) is "sub-monotone" if whenever \( m_3 \geq m_2 \geq m_1 \), \( d(S_{m_2}, S_{m_3}) \leq d(S_{m_1}, S_{m_3}) \).

A net \( \{S_m, m \in D\} \) in a uniform space \( (X, U) \) is submonotone relative to a base \( V \) for \( U \) if whenever \( m_3 \geq m_2 \geq m_1 \) and \( (S_{m_1}, S_{m_3}) \epsilon V \epsilon V \), also \( (S_{m_2}, S_{m_3}) \epsilon V \). Various sufficient conditions for a net to be submonotone are developed, the manner of convergence of a convergent submonotone net is investigated, conditions for a net to have a submonotone subnet are given, and two theorems are proved in which submonotoneity and a related condition
(respectively) appear as hypotheses. (In particular the familiar fact that a bounded monotone sequence of real numbers has a limit finds a generalization: A submonotone net in a compact space converges.) I hope that it may prove useful to hypothesize submonotoneity in other work.

The second topic is the justification of changing the order in a repeated limit. Here the nets considered are functions on the Cartesian product of two filtered sets; for the principal theorems the range is in a complete uniform space. The repeated limit in one order (and in the particular case of functions on the product of directed sets) is the limit of the subnet defined in the "Theorem on Iterated Limits" on p. 69 of John L. Kelley's General Topology. The repeated limits can exist even though the limits with respect to either argument alone do not. However, when the latter do exist, the repeated limit is identical with the usual iterated limit.

Various necessary and sufficient conditions are given for the two repeated limits to exist and be equal. Some of these are derived from conditions given by Hobson for a real-valued function of two real variables. The remainder are related but original. Finally, the results are applied to the questions of the continuity and differentiability of limit functions. Part of Theorem 3.3.3 is roughly to the following effect: Given a net of functions on a topological space to a complete uniform space which converges pointwise and for
which the discontinuities at a certain point \( x \) of functions sufficiently
late in the net are small, a necessary and sufficient condition that
the limit function be continuous at \( x \) is that for each degree of
closeness there be functions \( f \) arbitrarily late in the net which
closely approximate the limit function in some neighborhood of \( x \)
(which may depend on \( f \)).
NETS IN UNIFORM SPACES: MONOTONEITY, LIMIT THEOREMS

by

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NETS IN UNIFORM SPACES: MONOTONEITY, LIMIT THEOREMS

1. INTRODUCTION

1.1 Outline

In the second and third sections of this chapter nets and subnets are discussed. Both words are used in broader senses than are customary. Thereafter two independent topics are treated. One is a generalization of the concept "monotone," as applied to sequences of real numbers, to a related concept for nets (ordinarily called simply 'nets') in uniform spaces. Various sufficient conditions for a nett to be "submonotone" are developed, the manner of convergence of a convergent submonotone nett is investigated, conditions for a nett to have a submonotone subnett are given, and two theorems are proved in which submonotoneity and a related condition (respectively) appear as hypotheses. (In particular the familiar fact that a bounded monotone sequence of real numbers has a limit finds a generalization: A submonotone nett in a compact space converges.)

I hope that it may prove useful to hypothesize submonotoneity in other work.

The second topic is the justification of changing the order in a repeated limit. There are two theorems (4, p. 409 and 413) in which Hobson gives necessary and sufficient conditions for this change for
the case of a real-valued function defined on a subset of the plane.

Brace (2, p. 158, 1.6) gives a generalization of one of these, changing the setting, of course, and bringing the hypotheses into a compact form. He deals only with the use in which the limits taken on a single variable exist. Here I deal with the case in which they need not, generalize the setting still further, present the conditions in different language, and give besides Hobson's results some similar ones of my own. Finally, the results are applied to the questions of continuity and differentiability of a limit function.

1.2 Nets

I shall use the term 'net' in a broader sense than is customary. I was all but forced to do so by the applications I wished to make in the third chapter. In my treatment I have been influenced by McShane and Botts (6, Ch. 2), Bourbaki (1, Sec. 5-8) and Kelley (5, Ch. 2).

1.2.1 Definition. A filter-base in a set \( D \) is a non-empty collection \( M \) of non-empty subsets of \( D \) such that if \( M_1, M_2 \in M \), there is an \( M_3 \in M \) such that \( M_3 \subseteq M_1 \cap M_2 \). A filter in a set \( D \) is a filter-base \( M \) such that if \( M_1 \in M \) and \( M_2 \supseteq M_1 \), then \( M_2 \in M \). The filter generated by the filter-base \( M \) is the collection of all superset sets in \( M \). A filtered set is a pair \( (D, M) \) such
that $M$ is a filter in $D$. If $(D, M)$ is a filtered set, $D$ is said to be filtered by $M$ and, by an abuse of language, is itself often referred to as a filtered set. A net is a function on a filtered set. The notation $\{S_m, m \in D, M\}$ will be used for the net on the set $D$, filtered by $M$, the value of which at an $m \in D$ is $S_m$. $S_m$ will sometimes be referred to as the $m$th entry. In case reference to the filter $M$ seems unnecessary the notation may be simplified to $\{S_m, m \in D\}$.

1.2.2 Definition. If $\{S_m, m \in D, M\}$ is a net and if there is an $M \in M$ such that for all $m \in M$ $S_m$ has a certain property $P$, then I shall say that $S$ eventually has property $P$ or that all sufficiently late entries $S_m$ have property $P$ whichever seems more natural. Similarly if for all $M \in M$ there is an $m \in M$ such that $S_m$ has $P$, I shall say that $S$ frequently has $P$ or that there are arbitrarily late entries with $P$.

For example I shall as usual say that a net is frequently in a certain set, if it is, but I would rather say in the case of a "double" net $\{S(m, n), (m, n) \in D \times E\}$ (see 1.2.3) that all sufficiently late "rows" $\{S(m, n), n \in E\}$ were convergent than that the net $\{(S(m, n), n \in E), m \in D\}$ of rows was eventually convergent! (In case of ambiguity 'm-late' can be substituted for 'late.')

Now the discussion of nets in topological spaces is much as usual (for example see Kelley (5, Ch. 2)). In particular a net in a
A topological space converges to a point iff it is eventually in each neighborhood of the point. It is necessary (to obtain Kelley's "Theorem on Iterated Limits" (5, p. 69) for example) to define the product of filtered sets as follows:

1.2.3 Definition. If $D = \prod_{a \in A} D_a$ is a Cartesian product of filtered sets, $D_a$ filtered by $M_a$, then the product $\Pi \{M_a : a \in A\}$ = $M$ is the filter generated by the filter-base of all Cartesian products $\prod \{M_a : a \in A\}$ such that $M_a \in M_a$ for all $a \in A$. $D$ filtered by $M$ will be called the product of the filtered sets $D_a$.

Note that this definition is not that of Bourbaki (1, p. 68, 69), and the filter of neighborhoods of a point in a product space is not the product of the filters of neighborhoods of its components.

For the definition of subnet see the next section.

1.2.4 Definition. A partial ordering for a set is a transitive relation on that set. (Note that antisymmetry is not assumed. It is convenient that it not be as will be pointed out. In this I follow Kelley (5, p. 13).) A directed set is a partially ordered set $(D, \geq)$ such that $\geq$ is reflexive and such that if $m_1, m_2 \in D$, there is an $m_3 \in D$ such that $m_3 \geq m_1$ and $m_3 \geq m_2$. A terminal subset of $D$ is a set of the form $\{m : m \in D, m \geq m\} = D_m$, where $m \in D$. The filter-base associated with $\geq$ is the collection of all terminal subsets of $D$.

The filter generated by this filter-base will be called the filter
associated with $\geq$. Any directed set will be regarded as a filtered set (filtered by this filter).

The word 'net' has ordinarily been reserved for functions on directed sets. To distinguish I shall use the word 'nett' for such a function. In view of 1.2.4 every nett is in a natural way a net. Thus any terminology or results for nets apply at once to netts. In particular 'eventually' and 'frequently' have meaning in connection with netts. It is their usual meaning.

My reason for broadening the use of the word 'net' is as follows: One wishes, of course, to consider the limit of a function $f$ on a topological space as the argument approaches some point $a$ in that space. In order to have the discussion of such limits included in the discussion of nets such a function must be considered a net, and so a topological space must be a suitable domain for a net. There seems to be no natural way of directing a space (unless it be a metric space) so as to obtain the desired limit, but there is a natural way of filtering it: the desired filter is simply the filter of all supersets of deleted neighborhoods of $a$.

1.3 Subnets

1.3.1 Definition. Let $S = \{S_m, m \in D, M\}$ be a net in a set $X$. The filter in $X$ associated with $S$ is the collection of all subsets of $X$ which $S$ is eventually in. Ordinarily it is clear what set $X$
is intended, and one can speak simply of the filter associated with $S$ and denote it by $\mathcal{A}(S)$.

It is easy to see that if $K$ is any base for $\mathcal{M}$, $S(K) = \{S(K) : K \in K\}$ is a base for $\mathcal{A}(S)$. In the case of a nett on a directed set $D$ such a base is the collection of all images of terminal subsets of $D$.

Authors differ considerably as to the definition of 'subnet.' The crucial question appears to be what filters appear as filters associated with subnets of a given net, for the filter associated with a net in a topological space determines what its limit and cluster points will be. For example a net converges to a point iff its associated filter does (i.e. iff its associated filter includes the neighborhood system of the point).

With any reasonable definition of 'subnet' if $T$ is a subnet of $S$, $T$ is eventually in any set that $S$ is eventually in, i.e. $\mathcal{A}(T) \supseteq \mathcal{A}(S)$.

1.3.2 Definition. Subnets of a certain type are said to be adequate if given any net $S$ in a set $X$ and any filter $L$ in $X$ such that $L \supseteq \mathcal{A}(S)$, there is a subnet of $S$ of that type with $L$ for associated filter. Similarly, subnets of a certain type are said to be adequate for nets if given any net $S$ and filter $L \supseteq \mathcal{A}(S)$, there is a subnet of $S$ of that type with $L$ for associated filter.
I shall define and demonstrate the adequacy of various types of subnet.

1.3.3 Definition. A subnet of a net S is a net T such that $A(T) \supseteq A(S)$. (Of course, if nets in a set X are under consideration, it is understood that T should be in X.) A subnet of a net S is a nett which is a subnet of S. If $S = \{S_m, m \in I, M\}$ is a net, $M' \supseteq M$, and $S' = \{S'_m, m \in I, M'\}$, then clearly $S'$ is a subnet of S. Such subnets will be referred to as subnets obtained by enlarging the filter on the domain.

1.3.4 Theorem. If $\{S_m, m \in I, M\}$ and $\{T_n, n \in J, N\}$ are nets and F is a net on I to D such that $T = S \circ F$ and such that F is eventually in each $M \in M$, then T is a subnet of S.

Proof. If $K \in A(S)$, then $K \supseteq S(M)$ for some $M \in M$. Since F is eventually in M, $T = S \circ F$ is eventually in $S(M) \subseteq K$ so that $K \in A(T)$.

1.3.5 Definition. If the hypotheses of the preceding theorem hold, F is called a pointer for the subnet T, and T, a subnet of S with pointer.

1.3.6 Theorem. Let $\{S_m, m \in I\}$ and $\{T_n, n \in J\}$ be nets, and let F be a net on I to D such that $T = S \circ F$, F is increasing, and the range of F is cofinal in D (i.e. meets every terminal
subset of D). Then T is a subnett of S, and F is a pointer for T.

Proof. Since F is increasing and meets every terminal subset of D, it is eventually in every terminal subset of D.

1.3.7 Definition. If the hypotheses of the preceding theorem hold, F is called an increasing pointer for T, and T is called a subnett of S with increasing pointer.

1.3.8 Theorem. Given any filter L, there is a net having L for associated filter. Thus subnets are adequate.

Proof. Let \( X = \bigcup \{ L : L \in \mathcal{L} \} \). The net \( \{ x, x \in X, L \} \) is as desired.

Stronger is

1.3.9 Theorem. Given any filter L, there is a nett having L for associated filter. Thus subnetts are adequate.

Proof. Let D be the set of all ordered pairs \( (x, L) \) such that \( x \in L \in \mathcal{L} \). D is directed by the relation \( \geq \) such that \( (x_2, L_2) \geq (x_1, L_1) \) iff \( L_2 \subseteq L_1 \). \( \{ x, (x, L) \in D \} \) is a nett, and L is easily seen to be its associated filter.

Note that the partial ordering used in the proof of 1.3.9 is ordinarily not antisymmetric. The theorem is still true if antisymmetry is required of the partial ordering of a directed set, but the proof is more awkward. The same is true of other theorems.
1.3.10. **Theorem.** Subnets obtained by enlarging the filter on the domain are adequate.

**Proof.** Let \( S = \{ S_m, m \in D, M \} \) be a net in a set \( X \), and let \( L \) be a filter in \( X \) such that \( \underline{L} \supseteq A(S) \). It is easy to see that the collection of all sets of the form \( S^{-1}(L) \cap M \), where \( L \in L \) and \( M \in M \), is a filter-base. Let \( M' \) be the filter generated by this collection.

Then \( S' = \{ S_m, m \in D, M' \} \) is the desired subnet: Certainly \( M' \supseteq M \), and we must only show that \( A(S') = L \). \( A(S') \supseteq L \): If \( L \in L \), then \( S^{-1}(L) \cap D \in M' \), and \( S(S^{-1}(L) \cap D) = S(S^{-1}(L)) \subseteq L \) so that \( L \in A(S') \).

\( A(S') \subseteq L \): If \( K \in A(S') \), then \( K \supseteq S(S^{-1}(L) \cap M) \) for some \( L \in L \) and \( M \in M \). But \( S(S^{-1}(L) \cap M) = L \cap S(M) \) (an identity which is true in general). Since \( L \supseteq A(S) \) and \( S(M) \in A(S) \), \( L \cap S(M) \in L \), so that \( K \in L \).

There are subnetts which have no pointer and subnetts with pointer which have no increasing pointer: \((2,1)\) is a subnett of the sequence \((1, 1, 1, \ldots)\), but (for two reasons) it has no pointer. \((2,1,3)\) is a subnett of \((1,2,3)\) with a unique pointer which is not increasing. Nevertheless, subnetts with pointer are adequate, and subnetts with increasing pointer are adequate for netts:

1.3.11. **Theorem.** If \( S \) is a net and \( L \) is a filter such that \( L \supseteq A(S) \), then \( S \) is frequently in each \( L \in L \).
Proof. Let \( S = \{ S_m, m \in D, M \} \) and let \( L \in \mathcal{L} \). Since \( L \supseteq A(S) \) and \( A(S) \neq \emptyset \), \( L \) meets every set in \( A(S) \), in particular \( S(M) \) for each \( M \epsilon M \), and that is what was to be proved.

1.3.12. **Theorem.** Subnets with pointer are adequate.

**Proof.** Let \( S = \{ S_m, m \in D, M \} \) be a net in a set \( X \) and \( L \) a filter in \( X \) such that \( L \supseteq A(S) \). Let \( E \) be the set of all \( (m, M, L) \in D \times M \times L \) such that \( m \epsilon M \) and \( S_m \epsilon L \). Define a relation \( \geq \) on \( E \) by

\[(m_2, M_2, L_2) \geq (m_1, M_1, L_1) \iff M_2 \subseteq M_1 \text{ and } L_2 \subseteq L_1.\]

Let \( T \) be the function on \( E \) to \( X \) such that \( T(m, M, L) = S_m \). Let \( F \) be the function on \( E \) to \( D \) such that \( F(m, M, L) = m \).

(a) \( E \) is directed by \( \geq \): That \( \geq \) is transitive and reflexive is clear.

Let \( (m_1, M_1, L_1), (m_2, M_2, L_2) \in E \). By 1.3.11 \( S \) is frequently in \( L_1 \cap L_2 \), and so there is an \( m \epsilon M_1 \cap M_2 \) such that \( S_m \epsilon L_1 \cap L_2 \).

\[(m, M_1 \cap M_2, L_1 \cap L_2) \geq (m_1, M_1, L_1) \text{ and } (m, M_1 \cap M_2, L_1 \cap L_2) \geq (m_2, M_2, L_2).\]

(b) \( T \) is a subnet with pointer of \( S \): \( T = S \circ F \), and \( F \) is eventually in each \( M \epsilon M \).

(c) \( A(T) \supseteq L \): Let \( L \in \mathcal{L} \). Since \( S \) is frequently in \( L \), there is an \( (m, M, L) \epsilon E \). The image under \( T \) of the terminal subset \( E_{(m, M, L)} \) is a subset of \( L \), and so \( L \epsilon A(T) \).

(d) \( A(T) \subseteq L \): Let \( K \epsilon A(T) \). Then \( K \supseteq T(E_{(m, M, L)}) \) for some \( (m, M, L) \epsilon E \). \( T(E_{(m, M, L)}) = \{ S_\mu : \mu \epsilon M, S_\mu \epsilon L \} = S(M) \cap L \). But
1.3.13. **Theorem.** Subnets with increasing pointer are adequate for netts.

**Proof.** Let $S = \{S_m, m \in D\}$ be a nett in a set $X$, and let $L$ be a filter in $X$ which includes $\mathcal{A}(S)$. Let $E$ be the set of all $(m, L) \in D \times L$ such that $S_m \in L$. Define a relation $\geq$ on $E$ by

$$(m_2, L_2) \geq (m_1, L_1) \iff m_2 \geq m_1 \text{ and } L_2 \subseteq L_1.$$ 

Let $T$ be the function on $E$ to $X$ such that $T(m, L) = S_m$. Let $F$ be the function on $E$ to $D$ such that $F(m, L) = m$.

The remainder of the proof is very similar to that of 1.3.12.

(a) $E$ is directed by $\geq$: That $\geq$ is transitive and reflexive is clear. Suppose $(m_1, L_1), (m_2, L_2) \in E$. Let $m_3 \in D$ be such that $m_3 \geq m_1$ and $m_3 \geq m_2$. Since by 1.3.11 $S$ is frequently in $L_1 \cap L_2$, there is an $m_4 \geq m_3$ such that $S_{m_4} \in L_1 \cap L_2$. Then $(m_4, L_1 \cap L_2) \geq (m_1, L_1)$ and $(m_4, L_1 \cap L_2) > (m_2, L_2)$.

(b) $T$ is a subnet with increasing pointer of $S$: $T = S \circ F$, $F$ is increasing, and since $S$ is frequently in each $L \in L$, the range of $F$ is cofinal in $D$.

(c) $A(T) \supseteq L$: Let $L \in L$. Since $S$ is frequently in $L$, there is an $(m, L) \in E$. $T(E(m, L)) \subseteq L$, and so $L \in A(T)$.

(d) $A(T) \subseteq L$: Suppose $K \in A(T)$. Then for some $(m, L)$
\[ K \supseteq T(E(m, L)) = \{ S : \mu \geq m \text{ and } S \in L \} = S(D_m) \cap L. \] But
\[ S(D_m) \in L \text{ since } L \supseteq A(S). \] Therefore \( S(D_m) \cap L \in L \), and \( K \in L \).

It is not hard to verify that for each of the various types of subnet which have been introduced, a subnet of a subnet of a net \( S \) is itself a subnet of \( S \) (all three of the given type).

Subnets obtained by enlarging the filter on the domain are essentially the same as McShane and Bott's "subdirected functions" (6, p. 37, 38). Subnets with pointer are Kelley's "subnets" (5, p. 70). Subnets with increasing pointer are especially useful in connection with the "submonotone" nets of the next chapter because a subnet with increasing pointer of a submonotone net is itself submonotone.

No one else seems to be as generous as I with his use of the word 'subnet,' but in view of the results of this section I think there is something to be gained and little to be lost by using the term in the present broad sense. Some theorems (for example those concluding the existence of a subnet) are weakened but usually not in an important way. (The adequacy theorems of this section may restore them to their original form (for example see the proof of 2.7.2), and in any case a stronger theorem can always be stated.) The proofs are if anything simplified. Some theorems (such as those hypothesizing the existence of a subnet) are strengthened without, I believe, much
if any added complication in the proofs. (Because the present
definition is simpler than some, the proofs may actually be simpler.)
The statements of the principle theorems concerning subnets of nets
in topological spaces are unchanged.

1.4. Some Terminology

I wish to use some terminology in connection with uniform
spaces which is not standard but which seems natural and convenient.

1.4.1. Definition. If \((X, U)\) is a uniform space, the members
of \(U\) will be called \textit{closenesses}. If \((x_1, x_2) \in U \subseteq U\), then \(x_1\) will be
said to be \textit{U-close} to \(x_2\). A subset \(W\) of \(X\) will be called \textit{U-small} if any two of its members are \(U\)-close. The \textit{U-neighborhood}
of a point \(x \in X\) is the set of all points \(U\)-close to \(x\). If \(f_1\) and \(f_2\)
are functions with range in \(X\) and \(W\) is a subset of each of their
domains, then \(f_1\) is a \textit{U-approximation} of \(f_2\) on \(W\) if for all
\(w \in W\) \(f_1(w)\) and \(f_2(w)\) are \(U\)-close.

More standard is

1.4.2. Definition. If \((X, U)\) is a uniform space, \(U \subseteq U\), and
\(V \subseteq U\) is such that \(V \circ V \subseteq U\), \(V\) is called a \textit{half} of \(U\). 'Third,'
'fourth,' etc., are defined similarly.
2. MONOTONEITY

2.1. Submonotone Nets

2.1.1. Definition. Let $S = \{S_m, m \in D\}$ be a nett in a uniform space $(X, U)$. Let $V$ be a subset of $U$. $S$ will be called $V$-submonotone if whenever $m_3 \geq m_2 \geq m_1$ ($m_1, m_2, m_3 \in D$) and $(S_{m_1}, S_{m_3}) \in V$, then also $(S_{m_2}, S_{m_3}) \in V$. If a nett $S$ is $V$-submonotone where $V$ is some subbase for the uniformity $U$, then if $V$ is clear from the context, $S$ may be called simply submonotone.

2.1.2. Definition. If $X$ is a set and $d$ is a family of pseudometrics on $X$, then the symbol $V_d$ will denote the family of all sets $V_{d, r} = \{(x, y) : x, y \in X, d(x, y) < r\}$ for $d \in d$, $r > 0$. If \{X, d\} is a pseudometric space, 'submonotone' will mean $V_{\{d\}}$ - submonotone.

A submonotone nett in a pseudometric space has, of course, a simple description in terms of the pseudometric. \{S_m, m \in D\} in $(X, d)$ is submonotone iff whenever $m_3 \geq m_2 \geq m_1$, $d(S_{m_2}, S_{m_3}) \leq d(S_{m_1}, S_{m_3})$. It is worth drawing a picture to see what this requirement looks like in the plane for a single triple $(m_1, m_2, m_3)$.

One might be tempted to take for $V$ simply the entire uniformity $U$ and have a standard notion of submonotoneity for an arbitrary uniform space. However, $U$-submonotoneity is not apt to
be a very reasonable or interesting property. For example, if \( R \) is the set of real numbers with its usual metric \( d \) and uniformity \( U \), then no sequence in \( R \) which assumes three values is \( U \)-submonotone. (Proof: If \( a, b, c \) are distinct, \( m_3 \geq m_2 \geq m_1 \) and \( S_{n_1} = a, S_{n_2} = b, S_{n_3} = c \), then if \( U = V_d |b - c| \cup \{(a, c)\} \), \((S_{m_1}, S_{m_3}) \not\in U \) but \((S_{m_2}, S_{m_3}) \| U \).)

The following is easy to verify.

2.1.3. Theorem. (i) If \( V \) is a subbase for a uniformity and \( V' \) is the base consisting of all finite intersections of members of \( V \), then a nett is \( V' \)-submonotone iff it is \( V \)-submonotone.

(ii) A nett \( S \) is \( U \{V_a : a \in A \} \) - submonotone iff \( S \) is \( V_a \)-submonotone for each \( a \in A \).

2.2 Relationship to Ordinary Monotoneity

A monotone (i.e. non-decreasing or non-increasing) real nett is easily seen to be submonotone. A submonotone sequence of real numbers, however, need not be monotone, as will follow from

2.2.1. Theorem. If \( S = \{S_m : m \in D \} \) is a nett in a pseudo-metric space \((X, d)\) and if there is an \( x \in X \) such that if \( m_2 \geq m_1 \), \( d(S_{m_2}, x) \leq \frac{1}{2} d(S_{m_1}, x) \), then \( S \) is submonotone.

Proof. Suppose \( m_3 \geq m_2 \geq m_1 \) so that \( d(S_{m_3}, x) \leq \frac{1}{2} d(S_{m_2}, x) \) and \( d(S_{m_2}, x) \leq \frac{1}{2} d(S_{m_1}, x) \). Then
\[ d(S_{m_1}, S_{m_2}) \geq d(S_{m_1}, x) - d(S_{m_3}, x) \]

\[ \geq 2d(S_{m_2}, x) - \frac{1}{2} d(S_{m_2}, x) \]

\[ = \frac{3}{2} d(S_{m_2}, x) \]

but

\[ d(S_{m_2}, S_{m_3}) \leq d(S_{m_2}, x) + d(S_{m_3}, x) \]

\[ \leq d(S_{m_2}, x) + \frac{1}{2} d(S_{m_2}, x) \]

\[ \leq \frac{3}{2} d(S_{m_2}, x). \]

Wherever used, 'ω' will stand for the set of non-negative integers with its natural order. It follows from 2.2.1 that a sequence \( S \) for which there is an \( x \) such that \( d(S_{m+1}, x) \leq \frac{1}{2} d(S_m, x) \), \( m \in \omega \), is submonotone. In particular \( \{(-\frac{1}{2})^m, m \in \omega\} \) = \( \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} \) is submonotone whereas it certainly is not monotone.

One can obtain monotoneity as a conclusion by adding hypotheses. For example it is not hard to see that a real submonotone nett which converges to a point and assumes only values to the \( \left\{ \text{left} \right\} \) of that point is non- \( \{\text{decreasing}\} \). Also:

2.2.2 Definition. A nett \( S \) in a uniform space \((X, U)\) is \( V \)-monotone where \( V \subseteq U \) if whenever \( m_3 \geq m_2 \geq m_1 \), and
(S_{m_1}, S_{m_3}) \in V \ then \ both \ (S_{m_2}, S_{m_3}) \ and \ (S_{m_1}, S_{m_2}) \ are \ in \ V.

Again it may be worthwhile to draw a picture.

2.2.3 Theorem. A monotone nett of real numbers on a linearly ordered set (a sequence, for example) is monotone. If the directed set is not linearly ordered, the conclusion need not hold.

Proof. Let S be a nett as described.

(a) If m_3 > m_2 > m_1, then (since |S_{m_2} - S_{m_3}| \leq |S_{m_1} - S_{m_3}|
and |S_{m_1} - S_{m_2}| \leq |S_{m_1} - S_{m_3}|) S_{m_1}, S_{m_2}, S_{m_3}
appear either in order or in reverse order on the line.

(b) If m_4 > m_3 > m_2 > m_1 then S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}
appear either in order or in reverse order: Apply (a) to m_3, m_2, m_1 and to m_4, m_3, m_2 or, in case S_{m_2} = S_{m_3}, to m_4, m_3, m_1.

(c) If S is constant, S is certainly monotone.

(d) Suppose S is not constant and, say, there exist m_1, m_2 such that m_2 > m_1, S_{m_1} < S_{m_2}. Suppose m_4 > m_3. Arrange m_1, m_2, m_3, m_4, in order. An application of (b) shows that S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4} appear in the same order, in particular S_{m_3} < S_{m_4}.

(e) Similarly if there exist m_1, m_2 such that m_2 > m_1 and S_{m_1} > S_{m_2}, then whenever m_4 > m_3, S_{m_3} > S_{m_4}.

(f) Finally, the diagrams
are of \( V \)-monotone nets which are neither non-decreasing nor non-increasing. (The vertices of a diagram are the elements of the directed set. One vertex follows another if it can be reached by traveling along a broken line always moving up. The value of the nett at a vertex is written next to it.)

2.3 \textbf{Sufficient Conditions for Submonotoneity of Netts and Sequences in Pseudometric Spaces}

2.2.1 provides the first sufficient condition for submonotoneity. If a nett in a pseudometric space \((X, d)\) converges fast enough to a limit, it is \( V \)-monotone.

The condition which comes immediately from the definition is that for \( m_3 > m_2 > m_1 \)

\[
d(S_{m_2}^{m_3}, S_{m_3}) \leq d(S_{m_1}^{m_3}, S_{m_3}).
\]

Since \( d(S_{m_1}^{m_3}, S_{m_3}) \geq d(S_{m_1}^{m_2}, S_{m_2}) - d(S_{m_2}^{m_2}, S_{m_3}) \), it is enough to show that for \( m_3 > m_2 > m_1 \)

\[
d(S_{m_2}^{m_3}, S_{m_3}) \leq \frac{1}{2} d(S_{m_1}^{m_2}, S_{m_2}). \quad (*)
\]

The condition (*) requires for a sequence \( \{S_m, m \in \omega\} \) that in
particular \( d(S_{m+1}, S_{m+2}) \leq \frac{1}{2} d(S_m, S_{m+1}) \), for all \( m \in \omega \). However, this is not sufficient for submonotoneity as will appear below. If we try the condition

\[
d(S_{m+1}, S_{m+2}) \leq f d(S_m, S_{m+1}),
\]

we obtain

2.3.1. **Theorem.** A sequence \( \{S_m, m \in \omega\} \) in a pseudometric space \((X, d)\) is submonotone if for all \( m \in \omega \)

\[
d(S_{m+1}, S_{m+2}) \leq \frac{1}{3} d(S_m, S_{m+1}).
\]

**Proof.** Let \( d(S_m, S_{m+1}) = d_m \). Suppose that for all \( m \in \omega \)

\[
d_{m+1} \leq fd_m.
\]

Then if \( m_3 > m_2 > m_1 \),

\[
d(S_{m_1}, S_{m_3}) \geq d_{m_1} - (d_{m_1+1} + d_{m_1+2} + \ldots + d_{m_3-1})
\]

\[
\geq d_{m_1} \left(1 - \sum_{\mu=1}^{m_3-m_1-1} f^\mu\right)
\]

\[
> d_{m_1} \left(1 - \frac{f}{1 - f}\right)
\]

\[
= d_{m_1} \left(\frac{1 - 2f}{1 - f}\right)
\]

and
\[ d(S_{m_2}, S_{m_3}) \leq d_{m_2} + d_{m_2+1} + \ldots + d_{m_3-1} \]

\[ \leq d_{m_2} \sum_{\mu=0}^{m_2-m_2-1} f^\mu \]

\[ \leq \frac{d_{m_1} \cdot f^{m_2-m_1}}{1-f} \leq d_{m_1} \left( \frac{f}{1-f} \right) \]

Thus it is enough to have \( f \leq 1 - 2f \) or \( f \leq \frac{1}{3} \).

An \( f \) larger than \( \frac{1}{3} \) definitely does not guarantee submonotonicity as is shown by the sequence of partial sums of the series

\[ 1 + (-1) + f + f^2 + \ldots + f^m + \ldots \]

The sequence is \( 1, 0, f, f + f^2, \ldots, \ldots, \sum_{1}^{m} f^\mu, \ldots \).

If \( f > \frac{1}{3} \), then

\[ \sum_{1}^{\infty} f^\mu > \frac{1}{2}, \]

and so for sufficiently large \( m \) the entry

\[ \sum_{1}^{m} f^\mu \]

of the sequence is greater than \( \frac{1}{2} \), i.e. closer to the first entry than to the second.
2.4 **Series, Power Series**

One can obtain criteria for submonotoneity of the sequence of partial sums of a complex series by translating the various conditions mentioned in 2.3.

**2.4.1 Theorem.** Let \( \sum_{\nu=0}^{\infty} u_\nu \) be a series of complex numbers. Let

\[
S_n = \sum_{\nu=0}^{n} u_\nu \quad \text{and} \quad d_{n_1, n_2} = S_{n_2} - S_{n_1} = \sum_{n_1+1}^{n_2} u_\nu.
\]

Then \( (S_n, n \in \omega) \) is submonotone if any of the following hold:

(a) Whenever \( n_1 < n_2 < n_3 \), \( |d_{n_2, n_3}| \leq |d_{n_1, n_3}| \).

(b) Whenever \( n_1 < n_2 < n_3 \), \( |d_{n_2, n_3}| \leq \frac{1}{2} |d_{n_1, n_2}| \)

(c) \( \sum_{\nu=0}^{\infty} u_\nu \) converges and for each \( n \) \( |\sum_{n+1}^{\infty} u_\nu| \leq \frac{1}{2} |\sum_{n}^{\infty} u_\nu| \)

(d) \( |u_{n+1}| \leq \frac{1}{3} |u_n| \)

**Proof.** (a) follows from the definition, (b) from (*) of 2.3, (c) from 2.2.1, and (d) from 2.3.1.

Although it does not seem easy to find necessary and sufficient conditions on the \( u_\nu \) for 2.4.1(a) or (b) to hold, 2.4.1(d) yields a
necessary and sufficient condition for the sequence $S$ of partial sums of a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

to be submonotone in a neighborhood of 0:

2.4.2. **Theorem.** The sequence of partial sums of a complex power series

$$\sum_{n=0}^{\infty} a_n z^n$$

is submonotone in a neighborhood of 0 if and only if there is an $r > 0$ such that the quantities

$$\left| \frac{a_{n+m}}{a_n} \right| r^m,$$

where $a_n, a_{n+m}$ are consecutive nonzero coefficients, are bounded. (In particular if all the quantities

$$\left| \frac{a_{n+1}}{a_n} \right|$$

are bounded, the sequence of partial sums is submonotone.) (In the neighborhood which is constructed in case the condition above holds, 2.4.1(d) holds so that the series converges.)

**Proof I.** Suppose that
is a series for which the condition on the coefficients holds. By 2.4.1(d) the sequence of partial sums will be submonotone if whenever \(a_n, a_{n+m}\) are consecutive nonzero coefficients,

\[
\frac{a_{n+m}}{a_n} \cdot |z|^m \leq \frac{1}{3}.
\]

Choose \(r, N > 0\) such that \(\frac{a_{n+m}}{a_n} \cdot r^m < N\) (for all pairs \(a_n, a_{n+m}\)) so that

\[
\frac{a_{n+m}}{a_n} \cdot |z|^m < N \left(\frac{|z|}{r}\right)^m.
\]

Since \(m > 1\), if we choose \(z\) so that \(|z| \leq \text{Min}\left(\frac{r}{3N}, r\right)\), we will have \(\frac{|z|}{r} < 1\) and \(N \left(\frac{|z|}{r}\right)^m \leq N \frac{|z|}{r} \leq \frac{1}{3}\).

II. Suppose that

\[
\sum_{v=0}^{\infty} a_v z^v
\]

is a series for which the condition on the coefficients fails. We wish to show that the sequence of partial sums is not submonotone in any neighborhood of 0. Select a neighborhood containing, say, \(\{z : |z| < r\}\), where \(r > 0\). If we can find a \(z\) in this neighborhood
and consecutive nonzero \( a_n, a_{n+m} \) such that

\[
\left| a_{n+m} z^{n+m} \right| > \frac{1}{2} \left| a_n z^n \right| \quad \text{(or} \quad \frac{a_{n+m}}{a_n} \left| z^m \right| > \frac{1}{2} \} \right)
\]

and

\[
\frac{a_{n+m}}{a_n} z^n z^{n+m} \quad \text{is negative (or} \quad \arg\left(\frac{a_{n+m}}{a_n} z^m \right) = \pi, \right)
\]

\( S_{n+m} \) is closer to \( S_{n-1} \) than \( S_n \), and we are through.

Find \( a_n, a_{n+m} \) such that \( \frac{|a_{n+m}|}{|a_n|} r^m > \frac{1}{2} \).

If we choose \( z \) so that \( |z| = r \) and \( \arg z = \frac{1}{m} \left( \pi - \arg \frac{a_{n+m}}{a_n} \right) \),

then \( z \) is in the chosen neighborhood and satisfies the condition (*)

While the condition of 2.4.2 is met by Taylor’s Series of most functions that come to mind, it is not hard to construct a power series which fails to meet it. The MacLaurin series

\[
1 + z + \frac{1}{2} z^2 + \frac{1}{4} z^3 + \frac{1}{2} z^4 + \frac{1}{5} z^5 + \ldots + z^{2n} + \frac{1}{2^n} z^{2n+1} + \ldots
\]

for the rational function

\[
\frac{1}{1 - z^2} + \frac{z}{1 - \frac{1}{2} z^2}
\]

clearly fails to meet it. The series converges in \( \{z : \ |z| < 1\} \)

but the partial sums form a submonotone sequence in no neighborhood of 0.
2.5 Submonotone Nets with Cauchy Subnets

A monotone sequence of real numbers with a Cauchy subsequence is easily seen to be Cauchy. This fact generalizes nicely to

2.5.1. Theorem. Let \((X, U)\) be a uniform space, \(V\) a sub-base for \(U\). A \(V\)-submonotone nett which has a Cauchy subnett is itself Cauchy.

Proof. By 2.1.3 we may suppose that \(V\) is a base for \(U\). Let \(\{S_m, m \in D\}\) be a \(V\)-submonotone nett, and let \(\{T_n, n \in E\}\) be a Cauchy subnett. Given \(U \in U\), we want to show that there is an \(M \in D\) such that if \(m_1, m_2 \geq M\), then \((S_{m_1}, S_{m_2}) \in U\). Choose a symmetric half \(U'\) of \(U\). Choose \(V \in V\), \(V \subseteq U'\). Choose \(N_1\) so that if \(n_1, n_2 \geq N_1\), then \((T_{n_1}, T_{n_2}) \in V\). Choose \(N_2 > N_1\) such that \(T_{N_2} = S_M\) for some \(M \in D\).

If \(m_1, m_2 \geq M\), then \((S_{m_1}, S_{m_2}) \in U\).

For choose \(m_3 \geq m_1, m_2\) and then \(N_3 \geq N_1\) such that \(T_{N_3} = S_{m_4}\) where \(m_4 \geq m_3\). (See diagram. \(m_1\) and \(m_2\) might be related in the ordering. All that the diagram is to imply is that the indicated relations hold.) Since \(N_2, N_3 \geq N_1\), \((T_{N_2}, T_{N_3}) \in V\), i.e.
Since $S$ is $V$-submonotone, $(S, S_{m_4}) \in V$ and $(S, S_{m_4}) \in V$. Therefore, since $V \subseteq U'$ and $U'$ is symmetric, $(S, S_{m_2}) \in U' \circ U' \subseteq U$.

2.5.2 Corollary. Let $X$, $U$, and $V$ be as in 2.5.1. A $V$-submonotone nett in a compact subspace always converges.

Proof. If $S$ is a $V$-submonotone nett in a compact subspace, then $S$ has a convergent subnett $T$. $T$ is Cauchy, so $S$ is also Cauchy and converges to any limit of $T$.

This corollary generalizes the familiar fact that a bounded non-decreasing or non-increasing nett of real numbers converges.

2.6 Submonotone Convergence.

A monotone nett \{$(S, m) \in D$\} of real numbers which converges to a point $x$ has the property that if $m_2 > m_1$, then $|S_{m_2} - x| \leq |S_{m_1} - x|$. It turns out that $V$-submonotone netts have a similar property provided that $V$ is a subbase with a certain property, a property possessed for example by the subbases $V_d$, $d$ a generating family of pseudometrics (see 2.1.2).

2.6.1 Definition. Let $(X, U)$ be a uniform space and let $V$ be a subbase for $U$. A nett \{$(S, m) \in D$\} will be said to converge $V$-submonotonically to a point $x \in X$ if it converges to $x$ and if whenever $m_2 > m_1$ and $(S, x) \in V \in V$, then also $(S, x) \in V$. This may be shortened to 'submonotonically' if the intended
subbase is clear.

Again it is easy to verify that if \( S \) converges \( V \)-submonotonically to \( x \) and \( V' \) is the base of finite intersections of elements of \( V \), then \( S \) also converges \( V' \)-submonotonically to \( x \).

2.6.2 Definition. Let \( X \) be a topological space. A collection \( M \) of subsets of \( X \) will be called regular if whenever \( x \in M \cap M \), there is an \( M' \in M \) such that \( x \in M' \) and \( M' \subseteq M \). The terminology is suggested by the concept "regular topological space."

A space is regular if for each \( x \), every neighborhood of \( x \) includes a closed neighborhood of \( x \) or, equivalently, the closure of a neighborhood of \( x \). A collection \( M \) of open sets is regular if for each \( x \), every "\( M \)-neighborhood" of \( x \) includes the closure of an \( M \)-neighborhood of \( x \).

2.6.3 Theorem. If \( V \) is a regular open subbase for a uniformity \( \mathcal{U} \) and if \( \{S_m, m \in D\} \) is \( V \)-submonotone and converges to a point \( x \), then \( S \) converges \( V \)-submonotonically to \( x \).

Proof. Suppose \( (S_{m_1}, x) \in V \) and \( m_2 > m_1 \). Choose \( V' \) such that \( (S_{m_1}, x) \in V \) and \( V' \subseteq V \). Since \( V' \) is open, there is a \( U \in \mathcal{U} \) such that \( U[S_{m_1}] \times U[x] \subseteq V' \), and so since \( \lim_{m} S_m = x \), \( \{ (S_{m_1}, S_m), m \in D \} \) is eventually in \( V' \). Therefore since \( S \) is \( V \)-submonotone, \( \{ (S_{m_2}, S_m), m \in D \} \) is eventually in \( V' \).

\[ \lim_{m} (S_{m_2}, S_m) = (S_{m_2}, x) \] and so \( (S_{m_2}, x) \in \bar{V}' \subseteq V \).
The conclusion need not hold if the assumption of regularity is dropped. For example, let \( X = \mathbb{R} \), the set of all real numbers, let \( d \) be the usual metric for \( \mathbb{R} \), and let \( V \) be the base for the metric uniformity consisting of all \( V_d, 2^{-s}, s \in \omega \). Then the sequence \( \left( \frac{3}{4}, 1, 1, \frac{1}{8}, \frac{1}{16}, \ldots \right) \) converges to 0 and is \( V \)-submonotone, but does not converge \( V \)-submonotonically to 0 (\( (\frac{3}{4}, 0) \in V_{d, 1} \), but \( (1, 0) \notin V_{d, 1} \)).

2.6.4 Corollary. If \( d \) is a family of pseudometrics which generates \( U \), then if a nett is \( V_d \)-submonotone and converges to a point, it converges \( V_d \)-submonotonically to that point. In particular if \((X, d)\) is a pseudometric space, a submonotone nett which converges does so submonotonically.

Proof. \( V_d \) is easily seen to be open and regular.

On the other hand, the sequence \( (1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots) \) converges submonotonically to 0 but is not submonotone.

2.6.5 Theorem. Let \( f \) and \( f_m \) for each \( m \) in a directed set \( D \) be defined on a topological space \( X \) and be real-valued and continuous. If for each \( x \in X \), \( f_m(x) \uparrow f(x) \) (i.e., \( \{f_m(x), m \in D \} \) is a non-decreasing nett and \( f_m(x) \to f(x) \)), then \( \{f_m, m \in D \} \) converges to \( f \) uniformly on compacta.

It was the observation that this theorem, due to Dini, could be generalized to functions with range in a uniform space that led me to the investigations of this chapter:
2.6.6. **Theorem.** Let \( f \) and \( f' \) be continuous functions on a topological space \( X \) to a uniform space \((X, U)\). If there is a subbase \( V \) for \( U \) such that for each \( x \in X \) \( \{f_m(x), x \in D\} \) converges \( V \)-submonotonically to \( f(x) \), then \( \{f_m, m \in D\} \) converges to \( f \) uniformly on compacta.

**Proof.** We must show that given any \( V \in \mathcal{V} \) and compact \( C \subseteq X \) there is an \( M \in D \) such that if \( m > M \), \( (f_m(x), f(x)) \in V \) for all \( x \in C \).

Let \( V' \) be a symmetric third of \( V \). For each \( y \in C \) find an \( M \in D \) such that if \( m > M \), \( (f_m(y), f(y)) \in V' \). Since \( f_m, f \) are continuous, we can find for each \( y \) an open neighborhood \( U_y \) of \( y \) such that if \( x \in U_y \), then \( (f(y), f(x)) \in V' \) and \( (f_m(y), f_m(x)) \in V' \).

Since by choice of \( M \), we have also \( (f_m(y), f(y)) \in V' \), \( (f_m(x), f(x)) \in V' \), \( (f_m(x), f(x)) \in V' \cap V' = V, \) for all \( x \in U_y \). Since \( C \) is compact and \( C \subseteq \bigcup \{U_y : y \in C\} \), there exist \( y_1, \ldots, y_r \) such that

\[
C \subseteq \bigcup_{\nu=1}^{r} U_{y_\nu}
\]

Let \( M > M_{y_1}, \ldots, M_{y_r} \). If \( m > M \), then \( (f_m(x), f(x)) \in V \) for all \( x \in C \). If \( x \in U_{y_i} \), then \( (f_m(y_i), f(x)) \in V \). Since \( f_m(x) \rightarrow f(x) \) \( V \)-submonotonically and \( m > M_{y_i} \), \( (f_m(x), f(x)) \in V \).

2.7. **The Existence of Submonotone Subnets**

In general a net in a uniform space need not have \( V \)-submonotone subnets even if it is convergent and \( V \) is a very
reasonable sort of subbase. We can have \((X, d)\) a metric space, \(e\) a countably infinite family of metrics which generates the uniformity associated with \(d\), and \(S\) a convergent sequence on \(X\), yet \(S\) need not have a \(\frac{V}{e}\)-submonotone subnett:

Let \(X\) be the set of all bounded real sequences. Let pseudo-metrics \(d_n, n \in \omega\), be defined by

\[
    d_n(x, y) = d_n((\eta_1, \eta_2)) = |\eta_1 - \eta_2|,
\]

and let

\[
    d = \sum_{\nu=0}^{\infty} 2^{-\nu} d_\nu
\]

so that \(d\) is a metric on \(X\). Let \(e_n = d + d_n, e = \{e: e = e_n\text{ for some } n \in \omega\}\). It is not hard to see that \(e_n < (2^n + 1) d\) and \(d < e_n\), so that each \(\frac{V}{e}\) and certainly \(\frac{V}{e_n}\) are subbases (even bases) for the uniformity associated with \(d\).

However, the sequence

\((1, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, 0, \ldots), \ldots\)

though it converges \(\frac{V}{d}\)-submonotonically even) to \((0, 0, 0, \ldots)\), has no \(\frac{V}{e}\)-submonotone subnett: For let \(\{T_n, n \in \mathbb{E}\}\) be a subnett. Choose \(n_1, n_2, n_3, \in \mathbb{E}\) such that \(n_3 > n_2 > n_1\) and \(T_{n_1} = S_{m_1}\) for some \(m_1\), \(T_{n_2} = S_{m_2}\) for some \(m_2 > m_1\), and \(T_{n_3} = S_{m_3}\) for some \(m_3 > m_2\). Since
\[ e_m^2 \left( S_{m_2}, S_{m_3} \right) = \frac{1}{2^{m_2}} + \frac{1}{2^{m_3}} + 1 > \frac{1}{2^{m_1}} + \frac{1}{2^{m_3}} = e_m^2 \left( S_{m_1}, S_{m_3} \right), \]

\( T \) is not \( V_e \)-submonotone.

However,

2.7.1. Theorem. If \((X, U)\) is a uniform space and \( V = V_d \), where \( d \) is a finite set of pseudometrics, then any nett with a cluster point has a \( V \)-submonotone subnett with increasing pointer.

It is convenient first to prove two lemmas:

2.7.2. Theorem. If \((X, d)\) is a pseudometric space, then any nett with a cluster point has a submonotone subnett with increasing pointer.

Proof. If \( T^{(2)} \) is a subnett of \( T^{(1)} \) with increasing pointer \( F_2 \) and \( T^{(3)} \) is a subnett of \( T^{(2)} \) with increasing pointer \( F_3 \), then \( T^{(3)} \) is a subnett of \( T^{(1)} \) with increasing pointer \( F_2 \circ F_3 \). Also if \( x \) is a cluster point of a nett, then there is a subnett which converges to \( x \), and so by the adequacy for netts of subnetts with increasing pointer, there is a subnett with increasing pointer which converges to \( x \).

Therefore without loss of generality we may restrict our attention to
Let \( \{S_m, m \in D\} \) be a nett in \( X \) which converges to \( x \). If \( S \) is frequently in \( \{x\} \), then if \( D' = \{m : m \in D, S_m \in \{x\}\} \), the restriction of \( S \) to \( D' \) is a submonotone subnett, and the obvious pointer is increasing.

Otherwise let \( E \) be the set of all \( m \in D \) such that \( S_m \notin \{x\} \).

Define a relation \( \geq' \) on \( E \) by

\[
m_2 \geq' m_1 \text{ iff } (m_2 = m_1 \text{ or } m_2 >' m_1) \]

where

\[
m_2 >' m_1 \text{ iff } (m_2 \geq m_1, \text{ and } d(S_{m_2}, x) < \frac{1}{2} d(S_{m_1}, x)) \]

Let \( T \) be the restriction of \( S \) to \( E \). Then

(a) \( E \) is directed by \( \geq' \): \( \geq' \) is easily seen to be transitive. If \( m_1, m_2 \in E \), then \( S_{m_1} \notin x, S_{m_2} \notin x \), and so \( c = \text{Min}(d(S_{m_1}, x), d(S_{m_2}, x)) \neq 0 \). Choose \( m_3 > m_1, m_2 \) such that \( 0 \neq d(S_{m_3}, x) \leq \frac{c}{2} \). Then \( m_3 >' m_1, m_2 \).

(b) \( T \) with the injection map \( i \) on \( E \) to \( D \) is a subnett of \( S \) with increasing pointer:

\[
T = S \circ i. \text{ The range } E \text{ of } i \text{ is cofinal in } D \text{ since } S \text{ is not eventually (not even frequently) in } \{x\}. \text{ } i \text{ is increasing by definition of } >' .
\]

(c) \( T \) is submonotone: This is immediate by 2.2.1.
2.7.3 **Theorem.** If \( V \) is a subset of \( U \) and if a nett \( S \) is \( V \)-submonotone, then any subnett of \( S \) with increasing pointer is also \( V \)-submonotone.

**Proof.** If \( M \) is an increasing pointer for \( \{T_n, n \in E\} \) as a subnett of \( \{S_m, m \in D\} \) and if \( n_3 \geq n_2 \geq n_1 \), then \( T_{n_1} = S_{M(n_1)} \) and \( M(n_3) \geq M(n_2) \geq M(n_1) \). The desired relationship among the \( T_{n_i} \) follow since \( S \) is \( V \)-submonotone.

Now the proof of 2.7.1 is easy:

**Proof.** Let \( d = \{d_1, \ldots, d_r\} \), and let \( S \) be a nett on \( X \) which converges to \( x \). By 2.7.2 \( S \) has a \( V_{\{d_1\}} \)-submonotone subnett \( T^{(1)} \) with increasing pointer, which certainly also converges to \( x \). Again by 2.7.2 \( T^{(1)} \) has a \( V_{\{d_2\}} \)-submonotone subnett \( T^{(2)} \) with increasing pointer. By 2.7.3 and 2.1.3(ii) \( T^{(2)} \) is a \( V_{\{d_1, d_2\}} \)-submonotone subnett of \( S \) with increasing pointer. Increasing in this way, we eventually obtain the desired subnett.

A non-convergent sequence in a metric space need not have a submonotone subnett: Let \( (X, d) \) be the space introduced at the beginning of this section, and let \( \{S_m, m \in \omega\} \) be defined by

- \( S_0 = (0, 0, 0, \ldots, \ldots) \),
- \( S_1 = (4, 0, 0, \ldots, \ldots) \),
- \( S_2 = (0, 4^2, 0, \ldots, \ldots) \),
- \( S_3 = (0, 0, 4^3, 0, \ldots, \ldots) \),
- etc.
Then $S$ has no submonotone subnett: As before if $T$ is a subnett choose $n_1, n_2, n_3$ such that $n_3 \geq n_2 \geq n_1$, $T_{n_i} = S_{m_i}$, $i = 1, 2, 3$, where $m_3 > m_2 > m_1$.

$$d(T_{n_2}, T_{n_3}) = d(S_{m_2}, S_{m_3}) = 2^{m_2} + 2^{m_3} \geq 2^{m_1} + 2^{m_3} = d(T_{n_1}, T_{n_3}).$$

Finally, however

2.7.4 **Theorem.** If $(X, \| \cdot \|)$ is a (real or complex) finite-dimensional Banach space, then any nett in $X$ has a submonotone subnett with increasing pointer.

**Proof.** It is known that the spheres and balls $\{x: \| x \| = p\}$ and $\{x: \| x \| \leq p\}$ in a Banach space are compact iff the space is finite dimensional. Let $A$ be the unit sphere, and find a finite covering of $A$ by sets $A_1, \ldots, A_r$ of diameter less than or equal to $\frac{1}{3}$.

Let $\{S_m, m \in D\}$ be a nett on $X$. If $S$ is frequently in some compact set, then it has a cluster point, and the result follows from 2.7.2. So we may suppose that $S$ is eventually outside of each compact subset of $X$.

Let $C_\nu = \{x: \frac{x}{\| x \|} \in A_\nu\}$. Then since $X = C_1 \cup \ldots \cup C_r$, there is a $j$ such that $S$ is frequently in $C_j = C$. Thus $S$ has a subnett with increasing pointer in $C$, and without loss of generality we may suppose $S$ to be in $C$.

Define a new ordering $\geq'$ on $D$ by
\[ m_2 \geq' m_1 \text{ iff } (m_2 = m_1 \text{ or } m_2 >' m_1) \]

where
\[ m_2 >' m_1 \text{ iff } (m_2 \geq m_1 \text{ and } \| S_{m_2} \| \geq 2 \| S_{m_1} \|) \]

Then

(a) D with \( \geq' \) is a directed set: Transitivity and reflexivity are easy. If \( m_1, m_2 \in D \), then there is an \( m_3 \in D \) such that \( m_3 \geq m_1, m_2 \). Since \( S \) is eventually outside each ball, there is an \( m_4 \geq m_3 \) such that \( \| S_{m_4} \| \geq 2 \max (\| S_{m_1} \|, \| S_{m_2} \|) \).

\[ m_4 \geq' m_1, m_2. \]

(b) \( \{S_m, m \in D, \geq'\} \) is a subnet of \( \{S_m, m \in D, \geq\} \), for which the identity map is an increasing pointer: This is clear since if
\[ n_2 \geq' n_1, n_2 \geq n_1. \]

(c) \( \{S_m, m \in D, \geq'\} \) is monotone: Suppose \( m_3 \geq' m_2 \geq' m_1 \).

Then \( \| S_{m_2} \| \geq 2 \| S_{m_1} \| \) and \( \| S_{m_3} \| \geq 2 \| S_{m_2} \| \).

Let
\[ \frac{S_{m}}{\| S \|_{m}} = x_i, \text{ i.e. } \{1, 2, 3\}, \text{ so that } x_i \in C. \]

\[ \| S_{m_3} - S_{m_2} \| = \| S_{m_3} \| x_3 - \| S_{m_2} \| x_2 \| \]

\[ = \| (\| S_{m_3} \| - \| S_{m_2} \| x_3 + \| S_{m_2} \| (x_3 - x_2) \| \]

\[ \leq (\| S_{m_3} \| - \| S_{m_2} \|) \| x_3 \| + \| S_{m_2} \| \| x_3 - x_2 \|. \]
\[ \| S_{m_3} \| - \| S_{m_2} \| + \| S_{m_2} \| \cdot \frac{1}{3} \leq \| S_{m_3} \| - \frac{2}{3} \| S_{m_2} \|. \]

and

\[ \| S_{m_3} - S_{m_1} \| = \| (\| S_{m_3} \| - \| S_{m_1} \|) x_3 + \| S_{m_1} \| (x_3 - x_1) \| \]

\[ \geq \| S_{m_3} \| - \| S_{m_1} \| - \frac{1}{3} \| S_{m_1} \| \]

\[ = \| S_{m_3} \| - \frac{4}{3} \| S_{n_1} \| \]

Thus we will have \( \| S_{m_3} - S_{m_2} \| \leq \| S_{m_3} - S_{m_1} \| \) if

\[ \| S_{m_3} \| - \frac{2}{3} \| S_{m_2} \| \leq \| S_{m_3} \| - \frac{4}{3} \| S_{m_1} \| , \] which follows from

\[ \| S_{m_2} \| \geq 2 \| S_{m_1} \| . \]

2.8. Remarks

One question which the last section does not answer is whether or not a nett (for which convergence might be assumed) in a uniform space necessarily has a subnet which is \( V \)-submonotone for some subbase \( V \). The only theorems in this thesis which deduce results from assumptions of submonotoneity (the theorem guaranteeing Cauchyness of submonotone nets with Cauchy subnets and the generalization of Dini's theorem) assume only \( V \)-submonotoneity for some subbase \( V \).

Since filters are also used in discussing convergence, one
might wonder whether the discussion could be carried over to them.

Apparently the answer is no. At least it is easy to see that two real
netts, one submonotone, one not, can have the same associated filter.

(The two sequences \((1, 2, 3, 4, 5, \ldots)\) and \((1, 3, 2, 5, 4, \ldots)\)
furnish an example.)
3. REPEATED, ITERATED, AND DOUBLE LIMITS

3.1 Definitions

In this chapter I shall use \( S(m) \) instead of \( S_m \) for the value of a net \( S \) at an element \( m \) of its domain.

3.1.1 Definition. If \( D \) and \( E \) are filtered sets and \( S \) is a net on the filtered set \( D \times E \), then \( S \) will be called a double net. Since, when \( S \) is in a topological space, various limits other than the ordinary limit of \( S \) will be discussed, the term 'double limit' will often be used for the ordinary limit. The nets \( \{S(m, n), n \in D\} \) will be called the rows of \( S \), and the nets \( \{S(m, n), m \in D\} \), the columns. The row limit \( \lim_n S(m, n) \) will be denoted by \( S(m, \cdot) \), the column limit \( \lim_n (Sm, n) \) by \( S(\cdot, n) \), and the two iterated limits

\[ \lim_n (S(m, \cdot)) = \lim_m (\lim_n S(m, n)) \]  
\[ \lim_n (S(\cdot, n)) = \lim_m (\lim_n S(m, n)) \]

by \( S(:, \cdot) \) and \( S(\cdot, :) \), respectively.

3.1.2 Definition. If a double net \( \{S(m, n), (m, n) \in D \times E, M \cdot N\} \) is being discussed, then \( T \) will denote the net on \( D \times E^D \) such that \( T(m, f) = S(m, f(m)) \). The symbol \( T(S) \) can be used to avoid confusion in a discussion involving more than one double net. (Note that in accordance with the definition of the product of filters, the intended filter on \( D \times E^D \) is that generated by the filter-base of all \( M \times \bigwedge \{N_m : m \in D\} \) such that \( M \in M \) and \( N_m \in N \) for each \( m \in D \).)
Similarly, \( T : \) will denote the net on \( D^E \times E \) such that \( T : (f,n) = S(f(n),n) \). Whenever \( S \) is in a topological space and one of the nets \( T : \) and \( T : \) has a limit, it will be called a repeated limit of \( S \) and denoted by \( S(: \cdot) \) or \( S(: \cdot) \), respectively.

Alternatively \( \lim \lim S(m,n) \) and \( \lim \lim S(m,n) \) may be used.

3.1.3 Theorem. Let \( S \) be a double net. \( A(T :.) \) is the collection of all sets \( A \) such that all sufficiently late rows of \( S \) are eventually in \( A \). Consequently if \( S \) is in a topological space, \( S(: \cdot) \) is defined and \( S(: \cdot) = x \) if and only if for each neighborhood \( U \) of \( x \) all sufficiently late rows of \( S \) are eventually in \( U \). Also if \( S \) is in a uniform space, \( T : \) is Cauchy if and only if for each closeness \( U \) there is a \( U \)-small set \( X \) such that all sufficiently late rows of \( S \) are eventually in \( X \).

The statements for \( T : \), etc., are entirely similar.

Proof. \( [T : \text{ is eventually in } A] \) iff [there is an element \( M \times \bigcap \{N_m : m \in D\} \) of the filter on \( D \times E^D \) such that if \( (m,f) \in M \times \bigcap \{N_m : m \in D\}, \quad T : (m,f) = S(m,f(m)) \epsilon A \) iff [there is an \( M \epsilon M \) such that for each \( m \epsilon M \) there is an \( N_m \) such that if \( n \epsilon N_m \), \( S(m,n) \epsilon A \) iff [all sufficiently late rows of \( S \) are eventually in \( A \)].

Note that there are only minor differences between the notations for the repeated limits and those for the iterated limits. \( S(:, \cdot) = \)

\[
\lim \lim S(m,n) \quad \text{and} \quad S(:, \cdot) = \lim \lim S(m,n) \quad \text{are intimately related.}
\]
3.1.4 Theorem. Let $S$ be a double net in a topological space. If $S(:, .)$ is defined, then so is $S(\cdot \cdot)$ and $S(\cdot \cdot) = S(:, \cdot \cdot)$. Conversely, if the space is regular, $S(m, \cdot \cdot)$ is defined for all $m$, and $S(\cdot \cdot)$ is defined, then $S(:, \cdot \cdot)$ is defined, and $S(:, \cdot \cdot) = S(\cdot \cdot)$.

Proof. If $S(\cdot \cdot) = \lim_{m} S(m, \cdot \cdot) = x$, then for any open neighborhood $U$ of $x$ all sufficiently late $S(m, \cdot \cdot)$ are in $U$. But since $U$ is open and $S(m, \cdot \cdot) = \lim_{n} S(m, n)$, if $S(m, \cdot \cdot) \in U$, the $m^{th}$ row is eventually in $U$. Thus all sufficiently late rows are eventually in $U$, and by 3.1.3 $S(\cdot \cdot) = x$.

For the (partial) converse let $U$ be a closed neighborhood of $S(\cdot \cdot)$. All sufficiently late rows are eventually in $U$. But since $U$ is closed, if the $m^{th}$ row is eventually in $U$, so is its limit $S(m, \cdot \cdot)$. Thus all sufficiently late $S(m, \cdot \cdot)$ are in $U$. Since the space is regular, the closed neighborhoods form a base, and so $\lim_{m} S(m, \cdot \cdot) = S(\cdot \cdot) = S(\cdot \cdot)$.

On the other hand it is easy to construct examples of nets for which the repeated limits exist but the iterated limits do not. The diagram below of a net on $\omega \times \omega$ ($S(m, n)$ appearing in the $m^{th}$ row at the $n^{th}$ column) furnishes such an example. (In this case the double limit fails to exist, but replacing the '1's on the main diagonal by '0's would change that.) (To understand the net read to the right and down from entries on the main diagonal.)
The first half of 3.1.4 is derived from the "Theorem on Iterated Limits" in Kelley (5, p. 69). The second half is the content of a paper of Varsavsky's (7).

If we define for a net S of real numbers

\[ \lim_{m} S(m) = \{ x : \lim_{m} S(m) < x \leq \lim_{m} S(m) \}, \]

then we have as a corollary to 3.1.3.

3.1.5 Theorem. Let S be a double net of real numbers. Then 'S(: .)' is defined and \( S(: .) = x \) iff given any neighborhood U of x, for all sufficiently late rows

\[ \lim_{n} S(m, n) \subseteq U. \]

This indicates that the present definition of repeated limit generalizes that used by Hobson (4, p. 407).
It is easy to establish a relationship between the repeated limits and the double limit:

3.1.6 **Theorem.** \( T_\cdot \) and \( T_\cdot \) are subnets of \( S \). Consequently if \( S \) is in a topological space and has a double limit, then \( S \) has repeated limits in both orders, and they are equal to the double limit.

**Proof.** Let \( S = \{ S(m,n), (m,n) \in D \times E, M \cdot N \} \) and let \( A \in A(S) \).

Then \( A \supseteq S(M \times N) \) for some \( M \in M \) and \( N \in N \). Thus all sufficiently late rows of \( S \) are eventually in \( A \), and by 3.1.3 \( A \subseteq A(T_\cdot) \).

Similarly \( A(S) \subseteq A(T_\cdot) \).

3.2 **Conditions for Existence and Equality of the Repeated Limits**

In the following theorem, parts (b), (b'), (d), (d'), and (e), a row or column is selected in some way and then some statement is made about all "sufficiently late" entries in that row or column.

The understanding is that how late is "sufficiently late" may depend on the row or column selected.

3.2.1 **Theorem.** Let \( S \) be a double net in a complete uniform space. The following are equivalent:

(a) \( 'S(\cdot.)' \) and \( 'S(\cdot:) \) are defined, and \( S(\cdot.) = S(\cdot:) \).

(b) For each closeness \( U \), if we choose any sufficiently late column
and then any sufficiently late entry in that column, the row and column in which it stands are eventually in its U-neighborhood.

(c) \( S(.:) \) is defined, and for each closeness U, if we choose any sufficiently late column, there are arbitrarily late entries \( y \) in that column such that the row and column in which \( y \) stands are eventually in its U-neighborhood.

(d) \( S(:.) \) is defined, and for each closeness U, there are arbitrarily late rows \( R \) such that for all sufficiently late entries \( y \) in \( R \) the column in which \( y \) stands is eventually in its U-neighborhood.

(b') For each closeness U, if we choose any sufficiently late row and then any sufficiently late entry in that row, the row and column in which it stands are eventually in its U-neighborhood.

(c') \( S(:.) \) is defined, and for each closeness U, if we choose any sufficiently late row, there are arbitrarily late entries \( y \) in that row such that the row and column in which \( y \) stands are eventually in its U-neighborhood.

(d') \( S(:.) \) is defined, and for each closeness U, there are arbitrarily late columns \( C \) such that for all sufficiently late entries \( y \) in \( C \) the row in which \( y \) stands is eventually in its U-neighborhood.

(e) For each closeness U, there are arbitrarily late rows \( R \) such that for all sufficiently late entries \( y \) in \( R \) the column in which \( y \)
stands is eventually in its $U$-neighborhood, and there are arbitrarily late columns $C$ such that for all sufficiently late entries $y$ in $C$ the row in which $y$ stands is eventually in its $U$-neighborhood.

Proof. Let $S = \{S(m, n), (m, n) \in D \times E, M \cdot N\}$.

If (a), then (b): Let $S(\_ \_ \_) = S(\_ \_) = x$. Let $U$ be a closeness, and $V$, a symmetric half of $U$. By 3.1.3 all sufficiently late rows and all sufficiently late columns are eventually in $V[x]$. Let $N \in \mathbb{N}$ be such that for all $n \in \mathbb{N}$ the $n^{th}$ column is eventually in $V[x]$. It is enough now to show that for any $n \in \mathbb{N}$ the $n^{th}$ column has the property required by (b).

To this end fix $n \in \mathbb{N}$, and let $M \in \mathbb{M}$ be such that (i) all entries $S(m, n)$ with $m \in M$ are in $V[x]$ and (ii) the $m^{th}$ row for any $m \in M$ is eventually in $V[x]$. If $m \in M$, then $S(m, n) \in V[x]$, and the row and column in which $S(m, n)$ stands are eventually in $V[x]$. Consequently the row and column in which $S(m, n)$ stands are eventually in $U[S(m, n)]$.

If (b), then (c): We have only to show that $T$ is Cauchy, i.e. (see 3.1.3) that for each closeness $U$ there is a $U$-small set $X$ such that all sufficiently late rows are eventually in $X$. Select a closeness $U$, and let $V$ be a symmetric fourth of $U$. By (b) we can find an $S(m, n)$ and an $M \in \mathbb{M}$ such that for all $\mu \in M$ $S(\mu, n)$ is $V$-close to $S(m, n)$ and has its row eventually in its $V$-neighborhood.
Then for any $\mu \in M$ the $\mu^{th}$ row is eventually in $V \circ V[S(m, n)]$, a U-small set.

If (c), then (a): Let $U$ be a closeness, and let $V$ be a symmetric third of $U$. Choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the $n^{th}$ column contains arbitrarily late entries $y$ such that the row and column in which $y$ stands are eventually in its $V$-neighborhood. It is enough to show that for all $n \in \mathbb{N}$ the $n^{th}$ column is eventually in $U[S(\cdot : \cdot)]$. Fix $n \in \mathbb{N}$, and choose $M \in \mathbb{M}$ so that for all $m \in M$ the $m^{th}$ row is eventually in $V[S(\cdot : \cdot)]$. By the way in which $N$ was chosen we can find an $m \in M$ such that the $m^{th}$ row and the $n^{th}$ column are eventually in $V[S(m, n)]$. But by the way $M$ was chosen the $m^{th}$ row is also eventually in $V[S(\cdot : \cdot)]$. Thus the $n^{th}$ column is eventually in $V \circ V \circ V[S(\cdot : \cdot)]$.

If (a), then (e): Let $S(\cdot : \cdot) = S(\cdot : \cdot) = x$. Let $U$ be a closeness, and let $V$ be a symmetric half of $U$. All sufficiently late rows and all sufficiently late columns are eventually in $V[x]$. Because of this and because of the symmetry of (a) and (e) it is enough to show that any row which is eventually in $V[x]$ has the property of the first part of (e). So let $m \in D$ be such that the $m^{th}$ row is eventually in $V[x]$. Find $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$, (i) $S(m, n) \in V[x]$ and (ii) the $n^{th}$ column is eventually in $V[x]$. Then if $n \in \mathbb{N}$, the $n^{th}$ column is eventually in $V \circ V[S(m, n)]$.

If (e), then (d): We must show that if $U$ is any closeness,
there is a $U$-small set $X$ such that all sufficiently late rows are eventually in $X$. So let $U$ be a closeness, let $V$ be a symmetric fourth of $U$, and let $W$ be a symmetric half of $V$.

By (e) we can find a row $R$ such that for all sufficiently late entries $y$ in $R$ the column in which $y$ stands is eventually in $W[y]$, a $V$-small set. Thus all sufficiently late columns are eventually in $V$-small sets.

Thus by (e) we can find an $n$ such that the $n^{th}$ column is eventually in a $V$-small set and has the property of the second part of (e). It follows that there is an $M \in \mathbb{M}$ such that $\{S(m, n) : m \in M\}$ is $V$-small and such that for all $m \in M$ the $m^{th}$ row is eventually in $V[S(m, n)]$. But then if we choose any $m_0 \in M$, for all $m \in M$ the $m^{th}$ row is eventually in $V \ast V[S(m_0, n)]$, a $U$-small set.

If (d), then (a): Let $U$ be a closeness, and $V$ a symmetric half of $U$. We want to show that all sufficiently late columns are eventually in $U[S(\cdot, \cdot)]$. All sufficiently late rows are eventually in $V[S(\cdot, \cdot)]$, and so we can find an $m \in \mathbb{D}$ such that the $m^{th}$ row is eventually in $V[S(\cdot, \cdot)]$ and has the property of the second part of (d). It follows that there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ $S(m, n) \in V[S(\cdot, \cdot)]$ and the $n^{th}$ column is eventually in $V[S(m, n)]$. But then for all $n \in \mathbb{N}$ the $n^{th}$ column is eventually in $V \ast V[S(\cdot, \cdot)]$.

That (b') (c'), and (d') are equivalent to (a) follows from the fact that they are obtained from (b), (c), and (d), respectively, by
reversing the order of the arguments of $S$ and the fact that (a) is unaffected by that change.

In the above theorem there are several places at which because of the possibility of concise wording (a) or (b) of the following theorem was assumed for some rows or columns. One should be aware that either could be replaced by (c).

3.2.2 Theorem. Let $T$ be a net. The following are equivalent:

(a) Whatever closeness $U$ we select, for all sufficiently late entries $y$ $T$ is eventually in $U[y]$.

(b) For all closenesses $U$ there are arbitrarily late entries $y$ such that $T$ is eventually in $U[y]$.

(c) For all closenesses $U$, $T$ is eventually in a $U$-small set.

The proof is easy and is omitted.

The equivalence of part (a) of 3.2.1 with parts (b), (b'), (d), and (d') are derived from results of Hobson's (4, p. 409, 413). Brace (2, p. 158, 1.6) states a result in a different setting which is again essentially the equivalence of (a) and (b).

As a corollary to 3.2.1 we have

3.2.3 Theorem. Let $S$ be as in 3.2.1.

The following are equivalent:
(a) Either 'S(:.)' and 'S(.:)' are both undefined, or else they are both defined and \( S(:.) = S(.:) \).

(b) For each closeness \( U \) if we choose any sufficiently late column, there are arbitrarily late entries \( y \) in that column such that the row and column in which \( y \) stands are eventually in its \( U \)-neighborhood, and if we choose any sufficiently late row, there are arbitrarily late entries \( y \) in that row such that the row and column in which \( y \) stands are eventually in its \( U \)-neighborhood.

It is a simple matter to apply the last two theorems to the case in which the rows and columns have limits:

3.2.4 Theorem. Let \( S \) be a double net in a complete uniform space for which \( 'S(m,.)' \) and \( 'S(.,n)' \) are defined for all \( m \in D, n \in E \). Then the following are equivalent:

(a) \( 'S(:,.)' \) and \( 'S(.,:)' \) are defined and \( S(:,.) = S(.,:) \).

(b) For each closeness \( U \) if we choose any sufficiently late column, any sufficiently late entry in that column is \( U \)-close to the limit of the row in which it stands.

(c) \( 'S(:,.)' \) is defined, and for each closeness \( U \) if we choose any sufficiently late column, there are arbitrarily late entries \( y \) in that column such that \( y \) is \( U \)-close to the limit of the row in which it stands.
(d) 'S(:, .)' is defined, and for each closeness U there are arbitrarily late rows R such that any sufficiently late entry y in R, is U-close to the limits of the column in which it stands.

(b') For each closeness U if we choose any sufficiently late row, any sufficiently late entry in that row is U-close to the limit of the column in which it stands.

(c') 'S(., :)' is defined, and for each closeness U if we choose any sufficiently late row, there are arbitrarily late entries y in that row such that y is U-close to the limit of the column in which it stands.

(d') 'S(., :)' is defined, and for each closeness U there are arbitrarily late columns C such that any sufficiently late entry in C is U-close to the limit of the row in which it stands.

(e) For each closeness U there are arbitrarily late rows R such that any sufficiently late entry in R is U-close to the limit of the column in which it stands, and there are arbitrarily late columns C such that any sufficiently late entry in C is U-close to the limit of the row in which it stands.

3.2.2 becomes in this setting

3.2.5 Theorem. Let S be as in 3.2.4. The following are equivalent:

(a) Either 'S(:, .)' and 'S(., :)' are both undefined, or else they
are both defined and \( S(:, \cdot) = S(\cdot, \cdot) \).

(b) For each closeness \( U \) if we choose any sufficiently late column, there are arbitrarily late entries \( y \) in that column such that \( y \) is \( U \)-close to the limit of the row in which it stands, and if we choose any sufficiently late row, there are arbitrarily late entries \( y \) in that row such that \( y \) is \( U \)-close to the limit of the column in which it stands.

3.3 Some Examples Related to the Theorems of 3.2

One might hope that if (c) of 3.2.4 held except that \( 'S(:, \cdot)' \) was not defined, then \( S(\cdot, :) \)' would also be undefined. This is not the case as is shown by the example

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & No \ limit & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ddots & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{array}
\]

A similar hope for (d) is shattered by the same example.

If (b) of 3.2.5 holds, the case "\( 'S(:, \cdot)' \) and \( 'S(\cdot, :)' \) are both undefined" of (a) can occur as is shown by the example
3.4 Continuity of the Limit of Not Necessarily Continuous Functions

3.4.1 Definition. Let \( f \) be a function on a topological space \( X \) to a uniform space \((Y, V)\). A closeness \( V \in V \) will be called a bound on the discontinuity of \( f \) at a point \( x \in X \) if there is a neighborhood \( U \) of \( x \) such that if \( u \in U \), \((f(u), f(x)) \in V\). \( V \) will be called a bound on the discontinuity of \( f \) if it is a bound on the discontinuity of \( f \) at each \( x \in X \).

3.4.2 Theorem. Suppose that for each \( m \) in a filtered set \( D \), \( f_m \) is a function on a topological space \( X \) to a uniform space \((Y, V)\). Suppose that \( \lim_{m \to \infty} f_m(\xi) \) is defined for all \( \xi \in X \), and let \( f \), on \( X \) to \( Y \) be the pointwise limit of the net of functions \( f_m \). Let \( x \in X \), and suppose that each \( V \in V \) is a bound on the discontinuity of all sufficiently late \( f_m \) at \( x \). Then
\[ f(.) = \lim_{m \to x} \lim_{m \to x} f(m) = \lim_{m \to x} f(m) = f_s(x). \]

**Proof.** Given any closeness \( V \in V \), let \( W \) be a symmetric half of \( V \). \( W \) is a bound on the discontinuity of all sufficiently late \( f_m \) at \( x \). That is, for all sufficiently late \( f_m \), there is a neighborhood of \( x \) on which \( f_m \) is \( W \)-close to \( f_m(x) \). But by the definition of \( f_s \) for all sufficiently late \( f_m \), \( f_m(x) \) is \( W \)-close to \( f_s(x) \). Thus for all sufficiently late \( f_m \), there is a neighborhood of \( x \) on which \( f_m \) is \( V \)-close to \( f_s(x) \). That this hold for arbitrary \( V \in V \) is precisely the condition for \( f_s(.) \) to be \( f_s(x) \).

In the following theorem 3.2.1 is applied, and all the results retained for the sake of seeing what they all look like in a particular case. I have, however, used brackets around the letters designating some that seem to me less interesting than others.

**3.4.3 Theorem.** Let \( f_m, m \in D \), and \( f_s \) be as in 3.4.2. Let \( Y \) be complete. The following are equivalent:

(a) \( f_s \) is continuous at \( x \).

[a'] For each closeness \( V \) if we choose any \( \xi \) sufficiently close to \( x \), all sufficiently late \( f_m(\xi) \) are \( V \)-close to \( f_s(x) \).

(a'') For each closeness \( V \) if we choose any \( \xi \) sufficiently close to \( x \), there are arbitrarily late \( f_m(\xi) \) \( V \)-close to \( f_s(x) \).
[b] For each closeness $V$ if we choose any $\zeta$ sufficiently close to $x$, all sufficiently late $(f_m(\zeta), f_m(x))$ are in $V$.

(c) For each closeness $V$ if we choose any $\zeta$ sufficiently close to $x$, there are arbitrarily late $(f_m(\zeta), f_m(x))$ in $V$.

(d) For each closeness $V$ there are arbitrarily late $f_m$ which $V$-approximate $f_*$ in some neighborhood of $x$.

[b'] For each closeness $V$ all sufficiently late $f_m$ $V$-approximate $f_*$ in some neighborhood of $x$.

(c') $\lim_{\zeta \to x} f_*(\zeta)$ is defined, and for each closeness $V$ if we choose any sufficiently late $f_m$, there are points in any neighborhood of $x$ at which $f_m$ and $f_*$ are $V$-close.

[d'] $\lim_{\zeta \to x} f_*(\zeta)$ is defined, and for each closeness $V$ there are points $\zeta$ in any neighborhood of $x$ such that all sufficiently late $(f_m(\zeta), f_m(x))$ are in $V$.

[e] For each closeness $V$ there are points $\zeta$ in any neighborhood of $x$ such that all sufficiently late $(f_m(\zeta), f_m(x))$ are in $V$, and there are arbitrarily late $f_m$ which $V$-approximate $f_*$ in some neighborhood of $x$.

Proof. Since by 3.4.2 $f.(.) = f.(x)$ and since (a) is equivalent to the equality of $f_*(.)$ and $f_*(x)$, we are in a position to apply 3.2.1.

Parts [b] - [e] of the present theorem are obtained rather
mechanically from the corresponding parts of 3.2.1 but with some changes occasioned by the fact that \( f'(\xi) \) and \( f(\xi) \), for \( \xi \in X \), are defined and by the fact that each closeness \( V \) is a bound on the discontinuity at \( x \) of all sufficiently late \( f_m(a') \) is simply the condition of 3.1.3 that \( f_x(\cdot) \) be \( f_x(x) \). (a') and (a'') are equivalent in view of the fact that \( f'(\xi)' \) is defined.

3.4.3 is, of course, a catalog, not anything one could have in detail at his fingertips. But if one wished to demonstrate continuity in a particular case, it would certainly provide a variety of avenues that could be followed. Part (d) seems the most useful. It certainly makes it obvious that if \( f_m \rightarrow f \) uniformly in some neighborhood of each point and either each \( f_m \) is continuous or else for each closeness \( V, V \) is a bound on the discontinuity of all sufficiently late \( f_m \), then \( f \) is continuous. The equivalence of (d) and (a) strengthens a result of Brace's (3,1.3).

3.5 Differentiability of the Limit of Differentiable Functions

3.5.1 Theorem. Suppose that for each \( m \) in a filtered set \( D \), \( f_m \) is a real-valued function differentiable at the real number \( x \). Suppose that for all \( \xi \) in some neighborhood of \( x \), \( f_m(\xi) \rightarrow f(\xi) \). Then the following are equivalent:

(a) \( f_m \) is differentiable at \( x \), \( \lim_{m \to \infty} f_m'(x) \) is defined, and
\[ f_m'(x) = \lim_{m \to \infty} f_m'(x). \]

(b) For each \( \epsilon > 0 \) if we choose any \( \xi \) sufficiently close to \( x \) and then any sufficiently late \( f_m' \),

\[
\frac{f_m(\xi) - f_m(x)}{\xi - x}
\]

is within \( \epsilon \) of \( f_m'(x) \).

Proof. Let \( S(m, \xi) = \frac{f_m(\xi) - f_m(x)}{\xi - x} \). Then \( S(m, \cdot) = f_m'(x) \), and

\[
S(\cdot, \xi) = \frac{f(\xi) - f(x)}{\xi - x}
\]

for all \( \xi \) in some neighborhood of \( x \). The equivalence of parts (a) and (b) of 3.2.4 is the desired result.

3.5.2 Definition. Let \( f \) be a function on a pseudometric space \((X, d)\) to a pseudometric space \((X, e)\). Let us say that \( \delta \) \( \epsilon \)-restrains \( f \) at \( x \) if whenever \( d(x, \xi) < \delta \), \( e(f(x), f(\xi)) < \epsilon \).

3.5.3 Theorem. Let \( f_m, m \in D \), and \( f \) be as in the preceding theorem, and let there be a neighborhood of \( x \) in which all \( f_m \) are differentiable. If for any \( \epsilon > 0 \) there is a \( \delta \) which \( \epsilon \)-restrains all sufficiently late \( f_m' \) at \( x \), then \( f \) is differentiable at \( x \),

\[
\lim_{m \to \infty} f_m'(x) \quad \text{is defined, and} \quad f'(x) = \lim_{m \to \infty} f_m'(x).
\]

Proof. Choose \( \epsilon > 0 \), and let \( \delta \) be so small that (i) for all \( m \)

\[
f_m \to f \quad \text{and} \quad f_m \text{ is differentiable in the } \delta\text{-neighborhood } U \text{ of } x
\]

and (ii) \( \delta \) \( \epsilon \)-restrains all sufficiently late \( f_m' \) at \( x \). Using the
mean value theorem, for each \( m \) and for \( \zeta \in \mathcal{U} \) we can write

\[
\frac{f^m(\zeta) - f^m(x)}{\zeta - x} = f^m'(\eta)
\]

for some \( \eta \in \mathcal{U} \). By (ii), (b) of 3.5.1 holds, and therefore we can conclude (a) of 3.5.1 as desired.

Of course, we can replace the assumption "for any \( \epsilon > 0 \) there is a \( \delta \) which \( \epsilon \)-restrains all sufficiently late \( f^m' \)" by the stronger "the \( f^m' \) are equicontinuous at \( x \)."


