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This paper defines and discusses some of the separation axioms of topological spaces. In the cases considered, a search is made for sets of conditions which would be equivalent in a space satisfying a given separation axiom to the existence of a family of real valued, continuous functions which separates by functional values the points and/or sets corresponding to the given separation axiom.

# PROPERTIES OF REAL VALUED CONTINUOUS FUNCTIONS IN RELATION TO VARIOUS SEPARATION AXIOMS

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William Albert Tryon

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Chairman of the Department of Mathematics

Redacted for Privacy

Dean of Graduate School

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# PROPERTIES OF REAL VALUED CONTINUOUS FUNCTIONS IN RELATION TO VARIOUS SEPARATION AXIOMS

## CHAPTER I

## INTRODUCTION

A topological space $(X, \mathcal{T})$ is said to satisfy the Hausdorff or $T_2$ axiom iff for every two distinct points $a$ and $b$, there are disjoint open sets $U$ and $V$ such that $a \in U$ and $b \in V$. Thus, the points $a$ and $b$ can be "separated" by the open sets $U$ and $V$. The question arises whether two such points $a$ and $b$ can be separated by a function; that is, does there exist a continuous, real valued function $f: X \rightarrow [0,1]$ such that $f(a) = 0$ and $f(b) = 1$ ?

This paper poses the same type of question for topological spaces satisfying other separation axioms, as well as Hausdorff spaces. All of the implications between these separation axioms, holding in arbitrary topological spaces, are also proved in the paper.

# RELATIONSHIP BETWEEN SEPARATION AXIOMS

## CHAPTER II

Since different separation axioms are sometimes referred to by the same terminology, a precise definition of terms is in order.

Definition 2.1 A topological space $(X,\mathcal{T})$ is said to be a $T_0$ space iff for every pair of distinct points, there is an open set which contains one but not the other.

Definition 2.2 A topological space $(X,\mathcal{T})$ is said to be a $T_1$ space iff for every ordered pair of distinct points, there is an open set which contains the first but not the second.

Obviously, a $T_1$ space is a $T_0$ space; however, the following space is frequently cited to demonstrate that not all $T_0$ spaces are $T_1$ spaces.

Example 2.3 $X = \{a,b\}$ $\mathcal{T} = \{\emptyset , \{a\} , X\}$ $(X,\mathcal{T})$ is a $T_0$ space, but fails to be $T_1$ since any open set containing $b$ must also contain $a$ .

Lemma 2.4 A space is $T_1$ iff all singletons are closed.

Proof: Assume $(X,\mathcal{T})$ is $T_1$ and let $a$ be a member of $X$ . For all $b \in X$ such that $b \neq a$ , there is an open set $U_b$ such that $b \in U_b$ and $a \in X - U_b$ .

Therefore $X - \{a\} = \bigcup_{b \neq a} U_b$ is an open set, so $\{a\}$ is closed.

Now assume all singletons are closed, and let $(a,b)$ be an ordered pair of distinct points of $X$. Since $\{b\}$ is closed, $X - \{b\}$ is an open set which contains $a$ but not $b$.

<u>Definition 2.5</u> A topological space $(X, \mathcal{T})$ is said to be a $T_2$ or Hausdorff space iff for every pair of distinct points $a$ and $b$, there are disjoint open sets $U$ and $V$ such that $a \in U$ and $b \in V$.

Again it is obvious that a $T_2$ space is $T_1$, but there are $T_1$ spaces which fail to be $T_2$. Pervin [2] cites the following space as a $T_1$, non-$T_2$ space.

<u>Example 2.6</u> $X = [0,1]$ $\mathcal{T} = \{\emptyset, X$, all complements of countable sets$\}$. $(X, \mathcal{T})$ is a $T_1$ space, since for any pair $(a,b)$, where $a \neq b$, $X - \{b\}$ is an open set containing $a$ but not $b$. However, it is impossible to separate any two points $a$ and $b$ by disjoint open sets $U$ and $V$ such that $a \in U$ and $b \in V$. If $U$ is an open set which contains $a$ but not $b$, then $X - U$ is a countable set. Since $X$ is itself an uncountable set, the only open set contained in $X - U$ is the empty set, which obviously cannot contain $b$.

<u>Definition 2.7</u> A topological space $(X, \mathcal{T})$ is said to be $T_{2\frac{1}{2}}$ or Urysohn space iff for any pair of distinct

points a and b , there are disjoint closed sets F and G such that F is a neighborhood of a and G is a neighborhood of b .

By considering the definition of a neighborhood, we see immediately that a $T_{2\frac{1}{2}}$ space is $T_2$ ; however, there are $T_2$ spaces which fail to satisfy the $T_{2\frac{1}{2}}$ axiom. Urysohn [3] gives the following example of such a space.

Example 2.8 $X = \{(x,y) \mid x \text{ and } y \in N^+\} \cup \{(x,-y) \mid x \text{ and } y \in N^+\} \cup \{(X,0) \mid x \in N^+\} \cup \{(0,1)\} \cup \{(0,-1)\}$ , where $N^+$ denotes the set of all positive integers. A basis for the topology $\mathcal{T}$ is defined in the following manner.

$$\{(x,y)\} \in \mathcal{T}, \quad \text{if} \quad x \neq 0 \neq y$$

$$V^n_{(x,0)} = \{(x,0)\} \cup (\bigcup_{y \geq n} (x,y)) \cup (\bigcup_{y \geq n} (x,-y)) \in \mathcal{T}, \quad \text{if}$$

$x \in N^+$ for $n \in N^+$

$$O^n_{(0,1)} = \{(0,1)\} \cup (\bigcup_{y=1}^{\infty} (\bigcup_{x=n}^{\infty} (x,y))) \in \mathcal{T}, \quad \text{for} \quad n \in N^+$$

$$U^n_{(0,-1)} = (0,-1) \cup (\bigcup_{y=1}^{\infty} (\bigcup_{x=n}^{\infty} (x,-y))) \in \mathcal{T}, \quad \text{for} \quad n \in N^+$$

We can easily verify that this does define a basis for a topology, as claimed, since

$$X = O^1_{(0,1)} \cup U^1_{(0,-1)} \cup (\bigcup_{x \in N^+} V^1_{(x,0)}) ,$$

$$O^n_{(0,1)} \cap U^m_{(0,-1)} = \phi , \text{ for every } m \text{ and } n \in N^+$$

$$O^n_{(0,1)} \cap V^m_{(x,0)} = \begin{cases} \bigcup_{y \geq n+m} \{(x,y)\} , & \text{if } n \leq x \\ \phi , & \text{if } n > x \end{cases}$$

$$U^n_{(0,-1)} \cap V^{m}_{(x,0)} = \begin{cases} \bigcup_{y \geq n+m} \{(x,-y)\} , & \text{if } n \leq x \\ \phi , & \text{if } n > x \end{cases}$$

$$O^n_{(0,1)} \cap \{(x,y)\} = \begin{cases} \{(x,y)\} , \text{ if } n \leq x \text{ and} \\ x > 0 , y > 0 \\ \phi , \text{ if } n > x \text{ or } x > 0 \text{ and} \\ y < 0 \end{cases}$$

$$U^n_{(0,-1)} \cap \{(x,y)\} = \begin{cases} \{(x,y)\} \text{ if } n \leq x \text{ and } x > 0 , \\ y < 0 \\ \phi \text{ if } n > x \text{ or } x > 0 \text{ and} \\ y > 0 . \end{cases}$$

Example 2.8 is a $T_2$ space, but it fails to satisfy Urysohn's axiom. Consider the following chain of relations.

$$(m + n, 0) \in \overline{(O^{m+n}_{(0,1)})} \cap \overline{(U^{m+n}_{(0,-1)})} \subseteq \overline{(O^{n}_{(0,1)})} \cap \overline{(U^{m}_{(0,-1)})},$$

where $(\overline{A})$ denotes the closure of the set $A$. Since the intersection of the closures of any two basic open sets of the points $(0,1)$ and $(0,-1)$ respectively is nonempty, there can be no disjoint closed neighborhoods of the points $(0,1)$ and $(0,-1)$ .

<u>Definition 2.9</u> A topological space $(X,\mathcal{T})$ is said to be a $T_3$ space iff it is $T_1$ and has the property that given any closed set $F$ and $x \in X - F$ , there are disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$ .

<u>Lemma 2.10</u> In a $T_3$ space $(X,\mathcal{T})$, closed neighborhoods form a basis for the topology $\mathcal{T}$ .

<u>Proof:</u> We need only show that given any open set $O$ and $x \in O$ , there is a closed neighborhood $G$ of $x$ such that $G \subseteq O$ . Since our space is $T_3$ , $\{x\}$ and $X - O$ are disjoint closed sets, which are contained in the disjoint open sets $U$ and $V$ . This means that $U \subseteq X - V \subseteq O$ . Since $X - V$ is closed, $(\overline{U})$ is a closed neighborhood of $x$ which is contained in $O$ .

A $T_3$ space is also $T_{2\frac{1}{2}}$ , but the converse implication fails. If $(X,\mathcal{T})$ is a $T_3$ space and $a$ and $b$ are distinct points of $X$ , we may consider $a$ as a point and $\{b\}$ as a singleton closed set not containing $a$ . Using our $T_3$ axiom , we may choose disjoint open sets $U$ and $V$ which contain $a$ and $\{b\}$ respectively. By Lemma 2.10, we may find closed neighborhoods $F$ and $G$ of $a$ and $b$ respectively such that $a \in F \subseteq U$ and $b \in G \subseteq V$ . Since $U$ and $V$ are disjoint, $F$ and $G$

will be disjoint, closed neighborhoods of a and b respectively, as we desired.

<u>Example 2.11</u> Let $A = \{(x,y) \mid x,y$ are real and $y > 0\}$ and $B = \{(x,0) \mid x$ is real$\}$. Our space shall be $X = A \cup B$ . We shall define a basis for the topology in the following manner.

$$O^n_{(x,y)} = B((x,y), \tfrac{1}{n}) \cap X \text{ , if } (x,y) \in A \text{ (and } n \in N^+)$$

$$V^n_{(x,0)} = (B((x,0), \tfrac{1}{n}) \cap A) \cup \{(x,0)\}, \text{ if } (x,0) \in B \text{ , (and } n \in N^+) \text{ where}$$

$$B((x_0,y_0), \tfrac{1}{n}) = \{(x,y) \in R^2 \mid d((x_0,y_0) , (x,y)) < \tfrac{1}{n}\}$$

and d is the usual Euclidean distance in $R^2$ . Checking the conditions needed for the existance of a basis, we see immediately that we do have a topology.

Now let $(x_1,y_1)$ and $(x_2,y_2)$ be distinct points in our space; therefore $d((x_1,y_1) , (x_2,y_2)) \geq \frac{1}{m_0}$ for some $m_0 > 0$ .

$$\overline{\left(O^{4m_0}_{(x_1,y_1)}\right)} \cap \overline{\left(O^{4m_0}_{(x_2,y_2)}\right)} = \phi \text{ , if } y_1 \neq 0 \neq y_2$$

$$\overline{\left(V^{4m_0}_{(x_1,0)}\right)} \cap \overline{\left(O^{4m_0}_{(x_2,y_2)}\right)} = \phi \text{ , if } y_1 = 0 \neq y_2$$

$$\overline{\left(V^{4m_0}_{(x_1,0)}\right)} \cap \overline{\left(V^{4m_0}_{(x_2,0)}\right)} = \phi \text{ , if } y_1 = y_2 = 0 .$$

Therefore, Example 2.11 is $T_2$ space.

This space, which is cited by Pervin, is not a $T_3$ space, however. The point $(0,0)$ is not in the closed set $F = \{(x,0) \quad x \neq 0\}$. If $U$ is any open set containing $(0,0)$, there is an $n \in N^+$ such that $V^n_{(0,0)} \subseteq U$. The point $(\frac{1}{2n},0)$ is a member of $F$, yet no basic open set $V^m_{(\frac{1}{2n},0)}$ of $(\frac{1}{2n},0)$ has a nonempty intersection with $V^n_{(0,0)}$, and therefore with $U$. Since any open set $V$ containing $F$ must contain a basic open set $V^m_{(\frac{1}{2n},0)}$ of $(\frac{1}{2n},0)$, there can be no open set containing $V$ which fails to meet $U$.

Definition 2.12 A topological space $(X,\mathcal{U})$ is a $T_{3\frac{1}{2}}$ space iff it is a $T_1$ space and has the property that given any closed set $F$ and $x \in X - F$, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

Since the inverse images under $f$ of the open sets $]-\infty,\frac{1}{2}[$ and $]\frac{1}{2},\infty[$ are open, disjoint sets which contain a and $F$ respectively, it is clear that a $T_{3\frac{1}{2}}$ space is $T_3$. There are $T_3$ spaces which fail to be $T_{3\frac{1}{2}}$, however. Before we give an example of such a space, we shall first prove the following lemma.

Lemma 2.13 The product of two $T_3$ spaces is also a $T_3$ space (with the product topology.)

Proof: Let $X$ and $Y$ be two $T_3$ spaces, and let $(x_1,y_1)$ and $(x_2,y_2)$ be distinct points in $X \times Y$.

Therefore, either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$, then there is an open set $U \subseteq X$ such that $x_1 \in U$ and $x_2 \in X - U$. Therefore $U \times Y$ is an open set in $X \times Y$ which contains $(x_1,y_1)$ but not $(x_2,y_2)$. The argument is quite similar if $y_1 \neq y_2$. Since for any ordered pair of distinct points in $X \times Y$, we can find an open set $O \subseteq X \times Y$ which contains the first point but not the second, $X \times Y$ is a $T_1$ space.

Now, let $F$ be a closed set in $X \times Y$ and $(x,y) \in X \times Y - F$. Since $F$ is closed, there is a basic open set $U \times V \subseteq X \times Y - F$ such that $(x,y) \in U \times V$. Since $X$ is $T_3$ and $x$ is not a member of the closed set $X - U$, there are disjoint open sets $A$ and $B$ such that $x \in A$ and $(X - U) \subseteq B$. This means that $A \subseteq X - B \subseteq U$. Since $X - B$ is closed, $(\overline{A})^X \subseteq X - B \subseteq U$, where $(\overline{A})^X$ denotes the closure of $A$ in the space $X$. Similarly, we can find an open set $O \subseteq Y$ such that $y \in O \subseteq (\overline{O})^Y \subseteq V$, where $(\overline{O})^Y$ denotes the closure of $O$ in the space $Y$. Therefore, $(x,y) \in A \times O$ and $F \subseteq X \times Y - (\overline{A}^X \times \overline{O}^Y) = X \times Y - (\overline{A \times O})^{X \times Y}$, which means that $X \times Y$ is a $T_3$ space.

Example 2.14 Let $X$ denote the set of all ordinals less than or equal to the first uncountable ordinal $\Omega$, and let $Y$ denote the set of all ordinals less than or equal to the first infinite ordinal $\omega$. We define the bases for the topologies of $X$ and $Y$ in the following manner. In $X$, we shall let each set consisting of a single countable ordinal be open. Neighborhoods of $\Omega$ will consist of all sets which contain $\Omega$ and have countable complements. Similarly, we shall let each set consisting of a single finite ordinal be open in $Y$, as well as all sets containing $\omega$ that have finite complements.

It is obvious that both spaces are $T_1$, since all singletons are closed. Now let $F$ be a closed set in $X$, and $x \in X - F$. If $\Omega \notin F$, then $F$ is open, so $F$ and $X - F$ will be disjoint open sets containing $F$ and $x$ respectively. If $\Omega \in F$, then $\{x\}$ is open, so $\{x\}$ and $X - \{x\}$ will be disjoint open sets containing $x$ and $F$ respectively. Therefore, $X$ is a $T_3$ space.

A similar exercise will show that $Y$ is also a $T_3$ space.

By Lemma 2.13, we know that $X \times Y$ is also a $T_3$ space. A quick check of the definition of the relative topology will verify that the subspace $Z = X \times Y - \{(\Omega\ \omega)\}$ must also be $T_3$.

<u>Lemma 2.15</u> If $f\colon Z \to R^1$ is a continuous function with the property that there is a sequence $\{i_n\}_{n \in N^+}$ of distinct ordinals $< \omega$ such that $f(\Omega, i_n) \geq r$ for all $n \in N^+$, then there is a $\delta_0 < \Omega$ such that $f(\delta, \omega) \geq r$

for all $\delta > \delta_0$. The lemma also holds if both the inequalities are reversed.

Proof: Since $f$ is continuous, for every $m$ and $n$ $N^+$ there is an ordinal $\delta_{(m,n)} < \Omega$ such that $f(\delta, i_n) > r - \frac{1}{m}$ for all $\delta > \delta_{(m,n)}$. Let $\delta_o = \sup \{\delta_{(m,n)} \mid m,n \in N^+\}$. Since $\{\delta_{(m,n)} \mid m,n \in N^+\}$ is a countable set contained in $X - \{\Omega\}$, $\delta_0 < \Omega$. If $\delta > \delta_0$, $f(\delta, i_n) \geq \sup \{r - \frac{1}{m} \mid m \in N^+\} = r$. Since $f$ is continuous, $\lim_{n\to\infty} f(\delta, i_n) = f(\delta, \omega) \geq r$ for all $\delta > \delta_0$. The argument is quite similar if the inequalities are reversed.

Now let $I$ denote the space of all integers — positive, negative and zero — with the discrete topology. Forming the product $I \times Z$, we have a $T_3$ space in which points may conveniently be represented in the form $(n, \delta, i)$ where $n \in I$, $\delta \in X$ and $i \in Y$. All points of the form $(n, \Omega, \omega)$ are omitted.

We now form a new space $K$ by taking the quotient space formed by the following equivalence relation $r$.

$$r = \Delta_{(I\times Z) \times (I\times Z)} \cup \{(n,\delta,\omega),(n+1,\delta,\omega)) \mid n \text{ is even}\}$$

$$\cup \{((n+1,\delta,\omega),(n,\delta,\omega)) \mid n \text{ is even}\}$$

$$\cup \{((n,\Omega,i),(n+1,\Omega,i)) \mid n \text{ is odd}\}$$

$$\cup \{((n+1,\Omega,i),(n,\Omega,i)) \mid n \text{ is odd}\}$$, where $\Delta_{(I\times Z)\times(I\times Z)}$

is the diagonal in the space $(I \times Z) \times (I \times Z)$ . Since $(I \times Z)/_r = K$ does form a quotient space, a set $G \subseteq K$ will be open (or closed) iff $f^{-1}(G)$ is open (or closed) in $I \times Z$ . We want to show that $K$ is a $T_3$ space. Let $[z]$ be an equivalence class in $K$ . By the manner in which we defined $r$ , $[z]$ will consist of either one or two points of $I \times Z$ . Therefore $f^{-1}([z])$ will be a set in $I \times Z$ consisting of either one or two points. In either case, $f^{-1}([z])$ is closed, since $I \times Z$ is a $T_3$ space. Therefore $K$ is a $T_1$ space, since singleton sets are closed. Now let $F$ be a closed set in $K$ , and let $[x] \in K - F$ . If $[x]$ consists of only one point of $I \times Z$ , then $x$ is of the form $(n,\delta,i)$ where $\delta < \Omega$ and $i < \omega$ . Since all such points are open sets $\{[x]\}$ is open as well as closed. Therefore $\{[x]\}$ and $K - \{[x]\}$ are disjoint open sets containing $[x]$ and $F$ respectively. If $[x]$ consists of two points of $I \times Z$ , we may represent $[x]$ by $[(n_0,\delta_0,\omega)]$ where $n_0$ is even or by $[(n_0,\Omega,i_0)]$ where $n_0$ is odd. If $[x] = [(n_0,\delta_0,\omega)]$ , the $f^{-1}([x]) = \{(n_0,\delta_0,\omega)\} \cup \{(n_0+1,\delta_0,\omega)\}$ . Since $[x] \notin F$ , there is an open set $U' \subseteq K - F$ such that $[x] \in U'$ . Therefore $f^{-1}(U')$ is an open set in $I \times Z$ which contains $f^{-1}([x])$ . There is an $i_0 < \omega$ such that $U = \{(m,\delta_0,i) \mid m = n_0 \text{ or } n_0+1 , i > i_0\} \subseteq f^{-1}(U')$ . Examining the equivalence relation $r$ , we see that $f^{-1}(f(U)) = U$ . Since $f(U) \subseteq U' \subseteq K - F$ , and $I \times Z - U$ is the union of basic open sets in $I \times Z$ , $f(U)$ and $K - f(U)$ are disjoint open sets which contain $[x]$ and $F$ respectively. The argument is in the other case.

Now let us adjoin two new points $a_+$ and $a_-$ to $K$ . Neighborhoods $U_n(a_+)$ of $a_+$ consist of all triples $(j,\delta,i) \in K$ such that $j > n > 0$ along with $a_+$ itself. Similarly, neighborhoods $U_n(a_-)$ of $a_-$ consist of all triples $(-j,\delta,i) \in K$ such that $j > n > 0$ along with $a_-$ itself. The families $\{U_n(a_+) \mid n \in N^+\}$ and $\{U_n(a_-) \mid n \in N^+\}$ are a bases for the neighborhood systems of the points $a_+$ and $a_-$ respectively.

Let us denote the space $K \cup \{a_+\} \cup \{a_-\} = A$ , with the topology $\mathcal{T}$ generated by the union of the topology for $K$ and the open neighborhoods for $a_+$ and $a_-$ . This space is the one to which Example 2.14 refers. It is clear that $(A,T)$ is a $T_1$ space, since every singleton is closed. Let $F$ be a closed set in $A$ , and $x \in A - F$ . If $x = [(n,\delta,i)]$ where $\delta < \Omega$ and $i < \omega$, then $\{x\}$ and $A - \{x\}$ will be disjoint open sets. If $x = [(n,\Omega,i)]$ , there is a $\delta_0 < \Omega$ such that $O = \{[(n,\delta,i)] \mid \delta > \delta_0]\}$ fails to meet $F$ . $A - O$ may be written as the union of basic open sets, however, so $A - O$ and $O$ will be disjoint open sets containing $F$ an $x$ respectively.
Similarly, if $x = [(n , \delta , \omega)]$ , $a_+$ or $a_-$ ,

we can find an open set $O$ such that $x \in O$ , $O \cap F = \phi$ and $A - O$ can be written as the union of basic open sets. Since the various cases considered are the only possiblilities, $(A, \mathcal{T})$ is a $T_3$ space.

We will now show that no continuous real valued function can separate $a_+$ and $a_-$ . Since $\{a_-\}$ is a closed set, this is sufficient to demonstrate that $(A, \mathcal{T})$ is not a $T_{3\frac{1}{2}}$ space. It will suffice to show that if $f : A \to R^1$ is a continuous function such that $f(a_+) = 0$ , then $f(a_-) = 0$ .

Let us assume that $f : A \to R^1$ is a continuous function and $f(a_+) = 0$ , and let $\varepsilon > 0$ be arbitrary. Since $f$ is continuous, and $f(a_+) = 0$ , there is an odd integer $n$ such that $f([(n,\Omega,i)]) \leq \varepsilon/2$ for all $i < \omega$ . By Lemma 2.15, we know that there is a $\delta_1 < \Omega$ such that $f([(n,\delta,\omega)]) = f([(n-1,\delta,\omega)]) \leq \varepsilon/2$ for all $\delta < \delta_1$ . From this we can conclude that $f([(n-2,\Omega,i)]) = f([(n-1,\Omega,i)]) < \frac{2\varepsilon}{3}$ for all but a finite number of $i < \omega$. If not, we have $f([(n-1,\delta,\omega)]) = f([(n,\delta,\omega)]) \geq \frac{2\varepsilon}{3}$ for all $\delta > \delta_2$ , where $\delta_2 < \Omega$ . Since this would mean that $f([(n,\delta_1 + \delta_2 + 1 ,\omega)]) \geq \frac{2\varepsilon}{3}$ and $f([(n, \delta_1 + \delta_2 + 1,\omega)]) \leq \varepsilon/2$ , an obvious contradiction, our conclusion must hold. We can use a similar argument to show that $f([(n-4,\Omega,i)]) < \frac{4\varepsilon}{5}$ for all but a finite number of ordinals $< \omega$ , since $f([(n-2,\Omega,i)]) < \frac{2\varepsilon}{3}$ for all but a finite number of ordinals $< \omega$ . Continuing by induction, we can show that for every even integer, $p > 0$ , there is a finite ordinal $i_p < \omega$ such that $f([(n-p,\Omega,i)]) < \frac{p\cdot\varepsilon}{p+1}$ for all ordinals $i > i_p$ . Since $f$ is continuous, $f(a_-) \leq \varepsilon$ . Since we can also show that $f(a_-) \geq -\varepsilon$ , $f(a_-) = 0$ .

Definition 2.16 A topological space $(X,\mathcal{T})$ is said to be a $T_4$ space iff $(X,\mathcal{T})$ is $T_1$ and has the property that given any pair of disjoint closed sets $F$ and $G$ , there are disjoint open sets $U$ and $V$ such that $F \subseteq U$ and $G \subseteq V$ .

Since singletons are closed in a $T_1$ space, the following lemma and theorem will demonstrate that a $T_4$ space is also a $T_{3\frac{1}{2}}$ space.

Lemma 2.17 In a $T_4$ space $(X,\mathcal{T})$ , if $F$ is a closed set contained in the open set $U$ , then there is an open set $V$ such that $F \subseteq V \subseteq \overline{V} \subseteq U$ .

Proof: Let $F$ and $U$ be sets as described in the Lemma. Since $F \subseteq U$ , $F$ and $X - U$ are disjoint closed sets. This means that there are disjoint open sets $O$ and $V$ such that $F \subseteq V$ and $(X - U) \subseteq 0$ . Since $V \subseteq (X - 0) \subseteq U$ and $(X - 0)$ is closed, $F \subseteq V \subseteq \overline{V} \subseteq (X - O) \subseteq U$ .

Theorem 2.18 (Urysohn's Lemma) If $(X,\mathcal{T})$ is a $T_4$ space, and $F$ and $G$ are disjoint closed sets, then there is a continuous function $f\colon X \to [0,1]$ such that $f(F) = \{0\}$ and $f(G) = \{1\}$ .

Proof: Let $F$ and $G$ be disjoint closed sets in the $T_4$ space $(X, \mathcal{T})$ . By Lemma 2.17, there is an open set $U_0$ such that $F \subseteq U_0 \subseteq \overline{U}_0 \subseteq X - G$ . Let $X - G = U_1$ . Ordering the rationals in $]0,1[$ as follows: $\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{5},\ldots\} = \{r_1, r_2, r_3, \ldots\}$ and invoking Lemma 2.17 again, we define $U_{\frac{1}{2}}$ as an open set such that $U_0 \subseteq U_{\frac{1}{2}} \subseteq \overline{U}_{\frac{1}{2}} \subseteq U_1$ . We continue our induction by taking the smallest $r_j$ and largest $r_k$ such that $j$ and $k$ are less than $m$ and $r_k < r_m < r_j$ and defining the open set $U_{r_m}$ such that $\overline{U}_{r_k} \subseteq U_{r_m} \subseteq \overline{U}_{r_m} \subseteq U_{r_j}$ . Again, Lemma 2.17 guarantees us that such a set exists. We now define a relation $g(x)$ in the following manner.

$$g(x) = \begin{cases} \inf\limits_{n \in N^+} \{r_n \mid x \in U_{r_n}\} , & \text{if } x \in U_1 \\ 1 , & \text{if } x \notin X - U_1 . \end{cases}$$

The relation $g$ is obviously a well-defined function, and $g(X) \subseteq [0,1]$ . Also $g(F) = \{0\}$ and $g(G) = \{1\}$ , since $F \subseteq U_0$ and $G \subseteq X - U_1$ . Continuity is a consequence of the following equations.

$$g^{-1}([0,\beta[) = \bigcup_{r_n<\beta} U_{r_n} , \quad \text{for } \beta > 0$$

$$g^{-1}(]\alpha,1]) = \bigcup_{r_n>\alpha} (X - \overline{U}_{r_n}) , \quad \text{for } \alpha < 1 .$$

Since the inverse images of sub-basic open sets are open, $g$ is continuous.

However, there are $T_{3\frac{1}{2}}$ spaces which fail to satisfy the $T_4$ axiom. The following space is often used as an example of such a space.

Example 2.19 Our space $X$ shall consist of the same points as in Example 2.11. We shall also denote the sets $A$ and $B$ in the same manner. We shall define a basis for our topology as follows.

$$0^n_{(x,y)} = B((x,y), \tfrac{1}{n}) \cap X \ , \quad \text{if} \quad (x,y) \in A$$

$$V^m_{(x,0)} = B((x, m), \tfrac{1}{2m}) \cup \{(x,0)\} \ , \quad \text{if} \quad (x,0) \in B \ ,$$

where $B((x,y), \alpha)$ is the normal open Euclidean ball of radius $\alpha > 0$ and center $(x,y)$, with $m$ and $n \in N^+$ .

Now let $F$ be a closed set and $(x_0,y_0) \in X - F$ . Since $F$ is closed, there must be an $n_0 \in N^+$ such that $0^{n_0}_{(x_0,y_0)}$ (or $V^{n_0}_{(x_0,0)}$) is contained in $X - F$ . Let us assume $(x_0,y_0) \in A$ , so our basic neighborhood will be $0^{n_0}_{(x_0,y_0)}$.

In order to construct our function, we must extend our notation slightly.

$$O^{\alpha}_{(x,y)} = B((x,y), \tfrac{1}{\alpha}) \cap X , \quad \text{for} \quad \alpha > 0 , \text{ if } \quad (x,y) \in A$$

$$V^{\alpha}_{(x,0)} = B((x,\tfrac{1}{2\alpha})\tfrac{1}{2\alpha}) \cap \{(x,0)\} , \quad \text{for} \quad \alpha > 0 , \text{ if}$$

$(x,0) \in B$ .

We shall now define the function which gives us the desired separation.

$$f(x,y) = \begin{cases} 1, \text{ if } \quad (x,y) \in X - O^{n_0}_{(x_0,y_0)} \\ n_0 \quad \inf \{\frac{1}{\alpha} \mid (x,y) \in O^{\alpha}_{(x_0,y_0)} , \text{ if} \\ (x,y) \in O^{n_0}_{(x_0,y_0)} . \end{cases}$$

It is clear that $f$ is a well defined function and that $f(x_0,y_0) = 0$ and $f(F) = \{1\}$ . Continuity follows from the following equations.

$$f^{-1}(]0,\beta[) = O^{\beta\cdot n_0}_{(x_0,y_0)} \quad \text{for} \quad 0 < \beta \leq 1 .$$

$$f^{-1}(]\alpha,1]) = X \ \overline{(O^{\alpha\cdot n_0}_{(x_0,y_0)})} , \quad \text{for} \quad 0 \leq \alpha < 1 .$$

Since the inverse images of sub-basic open sets are open, $f$ is continuous.

The proof is quite similar for the case when $(x_0,y_0) \in B$ .

It remains to show that $(X, \mathcal{I})$ is not a $T_4$ spce. Let $F = \{(x,o) \mid x$ is rational$\}$ and $G = \{(x,0) \mid x$ is irrational$\}$ . It is easy to show that F and G are disjoint closed sets.

Now let V be any open set which contains G . We shall define a sequence of sets $\{G_n\}_{n \in N^+}$ in the following manner.

$$G_n = \{(x,0) \in G \mid V^n_{(x,0)} \subseteq V\} \text{ , for } n \in N^+ .$$

Clearly $\bigcup_{n \in N^+} G_n = G$ .

A famous theorem known as the Baire Category Theorem states that the complement of a set of first category is dense. Since G is of the second category $R^1 \times \{0\}$ with the usual relative topology from the metric space $R^2$ , there must be an $n_0 \in N^+$ such that the interior of $(\overline{G_{n_0}})$ is nonempty in the usual metric topology. This means that there is a rational $x_0$ such that $(x_0,0) \in (\overline{G_{n_0}})$ , where we are again talking about the usual metric topology for $R^1 \times \{0\}$ A short argument now shows that for any $m \in N^+$ , there is a point $(y_m,0) \in G_{n_0}$ such that $V^m_{(x_0,0)} \cap V^{n_0}_{(y_m,0)} \neq \phi$ . But any open set U which contains the closed set F must contain a basic open set of the form $V^m_{(x_0,0)}$ for some

$m \in N^+$ . Therefore $V \cap U \neq \phi$ , for any such open set $U$ ; ie, the space of Example 2.19 fails to be a $T_4$ space.

Definition 2.20 A topological space $(X, \mathcal{T})$ is said to be a $T_5$ space iff it is a $T_1$ space and has the property that for every pair of separated subsets $A$ and $B$ , there are disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$ . (Two sets $A$ and $B$ are separated iff $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \phi$).

Since any two disjoint closed sets clearly satisfy the above relation, a $T_5$ space is also $T_4$ . Before we give an example of a $T_4$ space which fails to be $T_5$ , we will first prove the following two lemmas.

Lemma 2.21 A compact Hausdorff space $(X, \mathcal{T})$ is $T_4$.

Proof: Let $F$ and $G$ be disjoint closed sets in the compact $T_2$ space $(X, \mathcal{T})$ . Fix $x \in F$ . For every $y \in G$ , there are disjoint open sets $U_y$ and $V_y$ such that $x \in U_y$ and $y \in V_y$ . Since $G$ is a closed subset of a compact space, $G$ is also compact. Since $G \subseteq \bigcup_{y \in G} V_y$ , there is a finite subset $\{y_1, y_2, \ldots, y_n\} \subseteq G$ such that $G \subseteq \bigcup_{i-1}^{n} V_{y_i}$ . $\bigcap_{i=1}^{n} U_{y_i}$ is therefore an open set containing $x$ which fails to meet $\bigcup_{i=1}^{n} V_{y_i}$ . Therefore, $(X, \mathcal{T})$ is a $T_3$ space.

Now, for every $x \in F$ , there are disjoint open sets

$U_x$ and $V_x$ such that $x \in U_x$ and $G \subseteq V_x$ . Since F is compact and $F \subseteq \bigcup_{x \in F} U_x$ , there is a finite subset $\{x_1, x_2, \ldots, x_m\} \subseteq G$ such that $G \subseteq \bigcup_{i=1}^{m} U_{x_i}$ . Therefore $\bigcup_{i=1}^{m} U_{x_i}$ and $\bigcap_{i=1}^{m} V_{x_i}$ are disjoint open sets which contain F and G respectively. Since X is $T_1$ , it is therefore a $T_4$ space.

Lemma 2.22 A topological space $(X, \mathcal{T})$ is a $T_5$ space iff every subspace $(X^*, \mathcal{T}^*)$ is $T_4$ which the relative topology.

Proof: Suppose $(X, \mathcal{T})$ is a $T_5$ space and let F and G be disjoint (relatively) closed subsets of the subspace $(X^*, \mathcal{T}^*)$ . Denoting the relative closure of a set $A \subseteq X^*$ by $\overline{A}^* = \overline{A} \cap X^*$ , we have $F \cap \overline{G} = (F \cap X^*) \cap \overline{G} = F \cap \overline{G}^* = F \cap G = \phi$ . Similarly, $\overline{F} \cap G = \phi$. Therefore, there are disjoint open sets (in X ) U and V such that $F \subseteq U$ and $G \subseteq V$ . Then $U^* = U \cap X^*$ and $V^* = V \cap X^*$ are disjoint open sets (in $X^*$) such that $F \subseteq U^*$ and $G \subseteq V^*$ . Since $T_1$ is a hereditary property, $(X^*, \mathcal{T}^*)$ is a $T_4$ space.

Now assume that every subspace is $T_4$ , and let A and B be separated subsets in X . $X^* = X - (\overline{A} \cap \overline{B})$ is an open subset of X . Let us consider $X^*$ as a subspace with the relative topology. $X^* \cap \overline{A}$ and $X^* \cap \overline{B}$

will be disjoint closed subsets of $X^*$ . Since $X^*$ is a $T_4$ space by hypothesis, there must be open sets $U$ and $V$ in $X$ such that $U^* = U \cap X^*$ and $V^* = V \cap X^*$ are disjoint (open) sets in $X^*$ that contain $A$ and $B$ respectively. Since $X^*$ is open in $X$ , however, $U^*$ and $V^*$ are disjoint open sets in $X$ . Since $(X, \mathcal{T})$ is $T_1$ , $(X, \mathcal{T})$ is a $T_5$ space

Example 2.23 (Tichonov) Let $X^* = X \cup \{\alpha\}$ be the one point compactification of an uncountable discrete space $X$ . Similarly, let $Y^* = Y \cup \{\beta\}$ be the one point compactification of an infinite discrete space $Y$ . Since both $X^*$ and $Y^*$ are compact Hausdorff spaces, $X^* \times Y^*$ is a compact Hausdorff space. By Lemma 2.21, $X^* \times Y^*$ is a $T_4$ space. In order to show that $X^* \times Y^*$ is not a $T_5$ space, we only need exhibit a subspace $Z^*$ which is not $T_4$ . Let $Z^* = X^* \times Y^* - \{(\alpha,\beta)\}$ , and define $F$ and $G$ in the following manner.

$$F = \{(\alpha,y) \in Z^*\}$$

$$G = \{(x,\beta) \in Z^*\}$$

Clearly $F$ and $G$ are disjoint sets. If $(x_0,y_0) \notin F$ , then $\{x_0\} \times Y^*$ is an open set containing $(x_0,y_0)$ that fails to meet $F$ . A similar exercise will prove that $G$ is closed as well.

Now suppose that $U$ and $V$ are disjoint open sets that contain $F$ and $G$ respectively. Pick any infinite sequence $\{y_n\}_{n \in N^+}$ of distinct points in $Y$ . For each $n \in N^+$ , the set $O_n = \{(x,y_n) \mid x \in X\}$ must be contained in $U$ for all but a finite number of terms. Now, the sets $V_x = \{(x,y) \mid y \in Y^*\}$ can be entirely contained in $V$ for only finitely many $x \in X$ . If not, $O_1$ would have more than a finite number of terms in the complement of $U$ . Next, we see that the sets $V_x$ that have all points but 1 belonging to $V$ must also be finite in number. If not, $U$ would not contain all but a finite number of points of $O_1 \cup O_2$ . Continuing by induction, we see that for any $n$, the number of $x \in X$ such that $V_x$ contains all but exactly $n$ points in $V$ must be finite. If not, $O_1 \cup O_2 \cup O_3, \ldots, \cup O_n$ would not have all but a finite number of terms contained in $U$ , which is a contradiction. Therefore, only a countable number of points in $X$ have the property that $V_x \cap (Z - V)$ is finite. If this is true, then $V$ cannot be an open set containing $G$ , since $V_x \cap (Z - V)$ must be finite for all of the points in the uncountable set $X$ . This contradiction leads us to conclude that there cannot be disjoint open sets that contain $F$ and $G$ respectively, so $Z^*$ is not $T_4$ . Therefore $X^* \times Y^*$ fails to be a $T_5$ space

We have proven the followng implications

$$T_5 \Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_{2\frac{1}{2}} \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0 .$$

We also showed that the above implications are, in general, not reversible. Therefore, the above chain of implications gives the only valid interrlation of the separation axioms in arbitrary topological spaces. It should be noted, however, that a separation axiom along with an additional property will sometimes guarantee a stronger axiom.

The topological spaces satisfying the various separation axioms above are the ones with which we shall concern ourselves in this thesis. It should be noted there are other separation axioms, but we shall not examine them in any detail.

# CHAPTER III

## PROPERTIES OF REAL-VALUED, CONTINUOUS FUNCTIONS IN RELATION TO VARIOUS SEPARATION AXIOMS

We know that if a topological space $(X, \mathcal{T})$ is $T_2$, then for any pair of distinct points a and b, there are disjoint open sets U and V such that $a \in U$ and $b \in V$. Since topologists are often concerned with continuous maps from arbitrary topological spaces into the real line $R^1$, it is natural to consider the possibility of a continuous function $f\colon X \to R^1$ such that $f(a) \neq f(b)$.

If there is such a function f, then there is a continuous function $g\colon X \to [0,1]$ such that $g(a) = 0$ and $g(b) = 1$. This is a consequence of the following compositions.

$$g(x) = \min\{\max\{\frac{f(x) - f(a)}{f(b) - f(a)} ; 0\} ; 1\}$$

If f is continuous, then g must also be continuous, and g has the desired bounds and values at a and b.

Similarly, if $g\colon X \to [0,1]$ is a continuous function such that $g(a) = 0$ and $g(b) = 1$, then for any pair of real numbers $\alpha$ and $\beta$ such that $\alpha < \beta$, there is a continuous function $h\colon X \to [\alpha,\beta]$ which has the property that $h(a) = \alpha$ and $h(b) = \beta$ . This also follows from a composition of several continuous functions.

$h(x) = \min\{\max\{((\alpha)(1 - g(x)) + \beta \cdot g(x)); \alpha\}; \beta\}$. If $g$ is continuous, $h$ must also be continuous, and $h$ certainly has the values at $a$ and $b$ and the bounds that we desired.

Therefore, it suffices to just consider the existence of a continuous function $g\colon X \to [0,1]$ such that $g(a) = 0$ and $g(b) = 1$ . Throughout this chapter, we shall restrict ourselves to functions corresponding to the function $g$ above, as they will exist iff the more arbitrary continuous, real-valued functions of the type discussed above exist.

We have examined several separation axioms, and now wish to discuss them further in the manner mentioned at the beginning of this chapter. The next theorem will show that we need only consider spaces which satisfy the $T_{2\frac{1}{2}}$ separation axiom.

<u>Theorem 3.1</u> Let $(X, \mathcal{T})$ be a topological space. If, for every pair of distinct points $a$ and $b$ in $X$ , there is a continuous function $f\colon X \to [0,1]$ such that $f(a) = 0$ and $f(b) = 1$ , the space is $T_{2\frac{1}{2}}$ .

<u>Proof:</u> Let $a$ and $b$ be distinct points in a topological space $(X, \mathcal{T})$ which possesses a family of continuous, real-valued functions which separates points.

Let f be a member of this family such that $f(a) = 0$ and $f(b) = 0$. Therefore $f^{-1}([0,\frac{1}{4}])$ and $f^{-1}([\frac{3}{4},1])$ are disjoint closed sets which are neighborhoods of the points a and b respectively.

This leads one to wonder whether there are any necessary and sufficient conditions in order to guarantee that an arbitrary $T_{2\frac{1}{2}}$ space $(X,\mathcal{T})$ has the property that for any pair of distinct points a and b there is a continuous, real-valued function $f: X \to [0,1]$ such that $f(a) = 0$ and $f(b)$.

We know by Lemma 2.21 that if a $T_{2\frac{1}{2}}$ space is compact, it is a $T_4$ space. Since singletons are closed in a $T_{2\frac{1}{2}}$ space, Theorem 2.18 assures us that every compact, $T_{2\frac{1}{2}}$ space has a family of continuous, real-valued functions which separate points. However, the condition of compactness is by no means a necessary one, since $R^1$ is a non-compact $T_4$ space.

The following example by Hewitt demonstrates that several other powerful properties are not enough to guarantee the existence of our family of functions.

<u>Example 3.2</u> Our space shall be defined by the following sets.

$X = \{(x,y) \mid x,y$ are rational, $x,y \in ]0,1[\} \cup$
$\{(0,0)\} \cup \{(1,0)\} \cup \{(\frac{1}{4},y) \mid y = r\sqrt{2}$ , r is

rational and $r\sqrt{2} \in ]0,1[\} \cup \{(\frac{1}{2},y) \mid y = r\sqrt{3}$ , $r$ is rational and $r\sqrt{3} \in ]0,1[\} \cup$ $\{(\frac{3}{4},y) \mid y = r\sqrt{5}$ , $r$ is rational and $r\sqrt{5} \in [0,1]\}$ .

A basis for the topology is defined in the following manner.

$$O^n_{(0,0)} = \{(x,y) \in X \mid 0 \leq x < \tfrac{1}{4} \text{ , } 0 < y < \tfrac{1}{n}\} \text{ , for } n \in N^+ \text{ ,}$$

$$V^n_{(1,0)} = \{(x,y) \in X \mid \tfrac{3}{4} < x \leq 1 \text{ , } 0 < y < \tfrac{1}{n}\} \text{ , for } n \in N^+$$

$$U^n_{(\frac{1}{2},r\sqrt{3})} = \{(x,y) \in X \mid \tfrac{1}{4} < x < \tfrac{3}{4} \text{ , } r\sqrt{3} - \tfrac{1}{n} < y < r\sqrt{3} + \tfrac{1}{n}\} \text{ , for } n \in N^+ \text{ .}$$

All remaining points will have the usual Euclidean balls of radii $\frac{1}{n}$ , for $n \in N^+$ , with center at the particular point in question as a neighborhood basis.

A quick check shows that this does define a topology, and that the resulting topological space is $T_{2\frac{1}{2}}$ , however, no continuous, real-valued function separates the points $(0,0)$ and $(1,0)$ . Let us assume that there is a continuous function $f: X \to [0,1]$ such that $f((0,0)) = 0$ and $f((1,0)) = 1$ . Since $f$ is continuous, $f^{-1}([0,\frac{1}{4}]) = A$ and $f^{-1}([\frac{3}{4},1]) = B$ are disjoint closed sets which are neighborhoods of $(0,0)$ and $(1,0)$ respectively. This implies that there is an $m_0$ $N^+$ such that $\overline{(O^{m_0}_{(0,0)})} \subseteq A$ and $\overline{(V^{m_0}_{(1,0)})} \subseteq B$. Picking a rational $r_0$ such that $0 < r_0\sqrt{3} < \frac{1}{m_0}$, we denote $f((\frac{1}{2},r_0\sqrt{3}))$ by $\alpha$. Therefore, $f^{-1}([\alpha - \frac{1}{8}, \alpha + \frac{1}{8}]) = C$ is a closed neighborhood of $(\frac{1}{2},r_0\sqrt{3})$. Therefore there is a $k \in N^+$ such that $\overline{U^k_{(\frac{1}{2},r_0\sqrt{3})}} \subseteq C$. Since $r_0$ is pick- in such a manner that $\overline{U^k_{(\frac{1}{2},r_0\sqrt{3})}}$ meets $\overline{O^{m_0}_{(0,0)}}$ for any $K$ $N^+$, the sets $A$ and $C$ must also meet. This means that $\alpha \leq \frac{3}{8}$. We can similarly show that $B \cap C \neq \phi$, so $\alpha \geq \frac{5}{8}$. Since this is an obvious contradiction, there can be no continuous, real-valued function which separates the points $(0,0)$ and $(1,0)$.

Example 3.2 is a space which is countable, and therefore separable. It is a $C_I$ space, (i.e., has a countable neighborhood base at each point) and is therefore a $C_{II}$ space (i.e., has a countable base for the topology). Since Example 3.2 was a $T_{2\frac{1}{2}}$ space which failed to possess the family of functions that we desired, none of the above properties will be sufficient to guarantee the existence of such a family.

Many other elementary properties fail to be necessary for the existence of such a family. Discreteness is certainly sufficient, since any function from a discrete space into any other space is continuous, but it is by no means necessary. Connectedness and the Lindeloff property are neither necessary nor sufficient, since Example 3.2 is a connected, Lindeloff space, and an uncountable discrete space is neither connected nor Lindeloff. Metrizability is sufficient, since every metric space is $T_4$ , but Example 2.19 is a non-metrizable $T_3$ space.

We have not shown that there exists no set of conditions which is equivalent in a $T_{2\frac{1}{2}}$ space to the existence of a family of real-valued, continuous functions which separates points. In fact, it would be very impractical to attempt such a feat, since we would have to demonstrate that every conceivable set of properties fails to be equivalent in a $T_{2\frac{1}{2}}$ space to the existence of a family of functions of the type we have discussed. We have, however, demonstrated that all the conditions considered, which constitute many of the basic properties of topology, fail to have the equivalence that we desired.

Turning to the case of $T_3$ spaces, we ask ourselves if there is any set of conditions which is necessary and sufficient in a $T_3$ space to the existence of a family of continuous, real-valued functions which separates points from disjoint closed sets. (We shall use the terminology above to mean that for any closed set $F$ and $x \in X - F$ , there is a function $f\colon X \to [0,1]$ such that $f(x) = 0$

and $f(F) = \{1\}$ .) If we examine our proposal more closely we see that we are seeking a set of conditions that is satisfied in every $T_{3\frac{1}{2}}$ space, as well as making every $T_3$ space that satisfies that set of conditions also satisfy the $T_{3\frac{1}{2}}$ axiom.

<u>Theorem 3.3</u> A topological space $(X, \mathcal{T})$ is a $T_3$ space iff it is homeomorphic to a subset of a compact Hausdorff space $(Y, \quad)$ .

<u>Proof</u>: Let us assume that $(X, \mathcal{T})$ is homeomorphic to a subset of the compact $T_2$ space $(Y, \eta)$ . By Lemma 2.21, we know that $(Y, \eta)$ is a $T_4$ space, and therefore $T_{3\frac{1}{2}}$ . Since every subspace of a $T_{3\frac{1}{2}}$ space is $T_{3\frac{1}{2}}$ , $(X, \mathcal{T})$ is homeomorphic to a $T_{3\frac{1}{2}}$ space. Let us denote that space by $(Y_1, \ _1)$ and let $f$ be the bicontinuous bijection from $X$ to $Y_1$ . Therefore, if $F$ is a closed set in $X$ and $x \in X - F$ , we know that $f(F) = F^1$ and $f(x) = x^1$ are a closed set and disjoint point in $Y_1$ . Since $Y_1$ is $T_3$ , there is a continuous function $g\colon Y_1 \to [0,1]$ such that $g(x^1) = 0$ and $g(F^1) = \{1\}$ . Therefore the composition function $g \circ f$ will be a continuous function taking $X$ into $[0,1]$ and maps $x$ into $0$ and $F$ into $\{1\}$ .

Now, let us assume that $(X, \mathcal{T})$ is a $T_{3\frac{1}{2}}$ space. We shall let $\{f_\lambda\}_{\lambda \in A}$ be the collection of all real-valued, bounded, continuous functions defined on $X$ , and let

$\{I_\lambda\}_{\lambda \in A}$ be a collection of closed and bounded intervals in $R^1$ such that $f_\lambda(X) \subseteq I_\lambda$. Since the product of compact spaces is compact, and the product of $T_2$ spaces is also $T_2$, $\prod_{\lambda \in A} I_\lambda$ is a compact, Hausdorff space. Let us define a mapping $g\colon X \to \prod_{\lambda \in A} I_\lambda$ by setting $g(x) = f_\lambda(X)$ for all $x \in X$, and denote $Y = g(X) \quad \prod_{\lambda \in A} I_\lambda$. To show that $g$ is a homeomorphism from $X$ to $Y$, we only need show that $g$ is a continuous, open injection. $g$ is continuous, since the $\lambda$th projection $\Pi_\lambda \circ g = f_\lambda$ is continuous for all $\lambda \in A$. If $x$ and $y$ are distinct points in $X$, there is an $\lambda_0 \in A$ such that $f_{\lambda_0}(x) = 0$ and $f_{\lambda_0}(y) = 1$. Obviously, $g(x) \neq g(y)$ since the two points of $Y$ differ in the $\lambda_0$th place. Now let $G$ be an open set in $X$, and let $x$ be in $G$. Since $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$, there must be an $\alpha \in A$ such that $f_\alpha(x) = 0$ and $f_\alpha(X - G) = \{1\}$. Clearly $Y \cap \Pi_a^{-1}(]-\infty,1[) = U$ is an open set in $Y$. If $z \in U$, then $g^{-1}(z) \in G$, and $z \in g(G)$. Therefore $g(x) \in g(U) \subseteq g(G)$, and $g(G)$ is open in $Y$.

Clearly, if a topological space $(X, \mathcal{T}_1)$ is homeomorphic to a subspace of a compact Hausdorff space $(Y, \mathcal{T}_2)$ then $(X, \mathcal{T}_1)$ can be embedded in its closure in $(Y, \mathcal{T}_2)$, which is a compact Hausdorff space. In light of Theorem 3.3 and the above remark, the following theorem is immediate.

<u>Theorem 3.4</u> A $T_3$ space $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ iff $(X, \mathcal{T})$ is homeomorphic to a subspace of a compact Hausdorff space.

In the case of $T_4$ spaces, we find that there is even less to prove. We know by Urysohn's Lemma (Theorem 2.18) that every $T_4$ space has a family of real-valued continuous functions that separates disjoint closed sets. Conversely, if a $T_1$ space possesses such a family, and F and G are disjoint closed sets, then there is a function f in the family such that $f(F) = \{0\}$ and $f(G) = \{1\}$. Obviously $f^{-1}(]-\infty, \frac{1}{2}[)$ and $f^{-1}(]\frac{1}{2}, \infty[)$ are disjoint open sets which contain F and G respectively. The following theorem is an immediate consequence of the preceding remarks.

<u>Theorem 3.5</u> A topological space $(X, \mathcal{T})$ is a $T_4$ space iff it is $T_1$ and has a family of continuous, real-valued functions which separates disjoint closed sets.

In the case of $T_5$ spaces, we are seeking a set of conditions which is equivalent in a $T_5$ space to the existence of a family of continuous, real-valued functions which separates separated subsets; (i.e., for any nonempty sets A and B such that $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$, there is a member of the family f such that $f(A) = \{0\}$ and $f(B) = \{1\}$.)

<u>Theorem 3.6</u> A $T_5$ space $(X, \mathcal{T})$ has a family of

continuous real-valued functions which separates separated subsets iff every pair of separated subsets have disjoint closures.

<u>Proof</u>: Assume every pair of separated subsets have disjoint closures, and let A and B be separates subsets. Therefore $\overline{A} \cap \overline{B} = \phi$ . Since $(X, \mathcal{T})$ is $T_5$, it is $T_4$ , so there is a continuous function $f\colon X \to [0,1]$ such that $f(\overline{A}) = \{0\}$ and $f(\overline{B}) = \{1\}$ .

Now assume there is a family of continuous, real-valued functions which separates separated subsets, and let A and B be nonempty separated subsets of X . By hypothesis, there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$ . Since $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are disjoint closed sets which contain A and B respectively, A and B will have disjoint closures.

Even though we have found a set of conditions which gives the desired equivalence in a $T_5$ space, we do not really have a good intuitive picture of what these conditions mean in a $T_5$ space. Before we dwell more on this, let us first prove the following lemma and theorem.

<u>Lemma 3.7</u> Let $(X, \mathcal{T})$ be a $T_2$ space. If x is a point of X such that there is a sequence of distinct points $\{x_n\}_{n \in N^+} \subseteq X - \{x\}$ such that $\{x_n\}_{n \in N^+} \to x$, then the sets $\{x_{2n} \quad n \in N^+\}$ and $\{x_{2n-1} \quad n \in N^+\}$ are

separated subsets of $X$ .

<u>Proof</u>: Let $(X, \mathcal{T})$ be a $T_2$ space and let $x$ and $\{x_n\}_{n \in N^+}$ satisfy the hypothesis. It will suffice to show that $\overline{\{x_{2n} \mid n \in N^+\}} = \{x_{2n} \mid n \in N^+\} \cup \{x\}$ and $\overline{\{x_{2n-1} \mid n \in N^+\}} = \{x_{2n-1} \mid n \in N^+\} \cup \{x\}$ , since the two sets are disjoint and $x$ belongs to neither set.

Let $y$ be a point in $X - (\{x_{2n} \mid n \in N^+\} \cup \{x\})$ . Then there are disjoint open sets $U$ and $V$ such that $y \in U$ and $x \in V$ . Since $\{x_{2n}\}_{n \in N^+} \to x$ , there is an $m$ $N^+$ such that $x_{2k} \in V$ if $k > m$ . For each $k \leq m$ , there are disjoint open sets $U_k$ and $V_k$ such that $y \in U_k$ and $x_k \in V_k$ . Therefore $U \cap (\bigcap_{k=1}^{m} U_k)$ is an open set containing $y$ which fails to meet $\{x_{2n} \mid n \in N^+\} \cup \{x\}$ . Since $x \in \overline{\{x_{2n} \mid n \in N^+\}}$ , $\overline{\{x_{2n} \mid n \in N^+\}} = \{x_{2n} \mid n \in N^+\} \cup \{x\}$ . Similarly, we can show that $\overline{\{x_{2n-1} \mid n \in N^+\}} = \{x_{2n-1} \mid n \in N^+\} \cup \{x\}$ .

<u>Theorem 3.8</u> Let $(X, \mathcal{T})$ be a topological space satisfying the $C_I$ and Hausdorff axioms. Then every pair of separated subsets have disjoint closures iff $\mathcal{T}$ is the discrete topology.

<u>Proof</u>: If $\mathcal{T}$ is the discrete topology, then $(\bar{A} \cap B) \cap (A \cap \bar{B}) = \phi$ implies that $\bar{A} \cap \bar{B} = \phi$ , since $A = \bar{A}$ and $B = \bar{B}$ .

Conversely, let us assume that $\mathcal{T}$ is not the discrete topology. Therefore, there is an $x \in X$ such that $x \in \overline{(X - \{x\})}$ . This means that there is a sequence of distinct points $\{x_n\}_{n \in N^+} \subseteq X - \{x\}$ such that $\{x_n\}_{n \in N^+} \to x$ . By Lemma 3.7, we have that $\{x_{2n} \mid n \in N^+\}$ and $\{x_{2n-1} \mid n \in N^+\}$ are separated subsets of $X$ . However, these two separated sets do not have disjoint closures, since $x$ is a member of the closures of both sets. Invoking the contrapositive, we have the result we desired.

The preceding theorem shows us that a $C_I$ , $T_5$ space has a family of continuous, real-valued functions that separates separated subsets iff the space is a discrete space. We are tempted to try to "extend" this result to arbitrary $T_5$ spaces by using a method similar to the one used to prove Theorem 3.8. In a non - $C_I$ space, however, we must use a type of convergence more general than sequences. Instead, we must use nets or filters, which are not necessarily indexed by either a countable or linearly ordered set. The method of proof used in proving Lemma 3.7 relied quite heavily on both these properties, which often escape us when using nets or filters.

In fact, there are non- $C_I$ , non-discrete $T_5$ spaces which have the property that any pair of separated subsets have disjoint closures.

<u>Example 3.9</u>[1] Let $X$ be the real line, and let $\mathfrak{F}^*$ be the filter generated by the filter basis $\{]0,\frac{1}{n}[ \mid n \in N^+\}$ . One of the properties of filters (which we will not discuss in any detail here) is that every filter can be extended to an ultrafilter. Accordingly, let us extend $\mathcal{F}^*$ to an ultrafilter . We shall define our topology $\mathcal{T}$ in the following manner.

$$\mathcal{T} = \{A \subseteq X \mid 0 \notin A\} \cup \{F \cup \{0\} \mid F \in \mathfrak{F}\}$$

To show that this is a topology as claimed, we must demonstrate that arbitrary unions of members of $\mathcal{T}$ and finite intersections of members of $\mathcal{T}$ are also members of $\mathcal{T}$ .

Let $\{O_\alpha \mid \alpha \in A\}$ be a collection of members of $\mathcal{T}$ . If $0 \notin O_\alpha$ for all $\alpha \in A$ , then $0 \notin \bigcup_{\alpha\in A} O_\alpha$ , so $\bigcup_{\alpha\in A} O_\alpha \in \mathcal{T}$. If there is an $\alpha_0 \in A$ such that $0 \in O_{\alpha_0}$ , $O_{\alpha_0} = F \cup \{0\}$ for some $F \in \mathfrak{F}$ . Since $F \subseteq O_{\alpha_0} \subseteq \bigcup_{\alpha\in A} O_\alpha$, $\bigcup_{\alpha\in A} O_\alpha \in \mathfrak{F}$ . Therefore $\{0\} \cup (\bigcup_{\alpha\in A} O_\alpha) = \bigcup_{\alpha\in A} O_\alpha \in \mathfrak{F}$.

Now let $\{O_k \mid 1 \leq k \leq n\}$ be a finite collection of members of $\mathcal{T}$ . If there is an $m$ such that $0 \leq m \leq n$ and $0 \notin O_m$ , then $0 \notin \bigcup_{k=1}^{n} O_k$ ; i.e., $\bigcap_{k=1}^{n} O_k \in \mathcal{T}$. If $0 \in O_k$ for all $k$ such that $1 \leq k \leq n$ , then

[1] This space was suggested to me by Darrell Kent, Professor of Mathematics, Washington State University.

$O_k = F_k \cup \{0\}$ for all $k$, where $F_k \in \mathfrak{F}$. Since $\mathfrak{F}$ is a filter, $\bigcap_{k=1}^{n} F_k \in \mathfrak{F}$, so $\{0\} \cup (\bigcap_{k=1}^{n} F_k) = \bigcap_{k=1}^{n} O_k \in \mathcal{T}$.

We now wish to show that $(X, \mathcal{T})$ is a $T_1$ space. Let $(\alpha, \beta)$ be an ordered pair of distinct points. If $\alpha \neq 0$, then $\{\alpha\}$ is an open set which fails to contain $\beta$. If $\alpha = 0$, then there is an $n$ $N^+$ such that $\beta$ $]0, \frac{1}{n}[$. Therefore, $[0, \frac{1}{n}[$ is an open set containg 0 but not $\beta$.

In order to show that our space is $T_5$, it will suffice by Lemma 2.22 to show that every subspace is $T_4$. Therefore, let $(X^*, \mathcal{T}^*)$ be a subspace of $(X, \mathcal{T})$, and let $A$ and $B$ be disjoint, closed subsets in $X^*$. If $0 \notin A$ and $0 \notin B$, then $A$ and $B$ will be open in $X$, so they naturally will be open in $X^*$. If $0 \in A$, then $0 \notin B$. Again $B$ will be open in $X$, so will also be open in $X^*$. Since $B$ is also closed, $B$ and $X^* - B$ will be disjoint open sets in $X^*$ which contain $B$ and $A$ respectively. Since the $T_1$ property is a hereditary property, $(X^*, \mathcal{T}^*)$ is a $T_4$ space.

Now, let $A$ and $B$ be separated subsets of $X$, (i.e., $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \phi$.) We want to show that $\overline{A} \cap \overline{B} = \phi$.

Case I: $0 \in A$. If $0 \in A$, then $A = \overline{A}$, so

$$\overline{A} \cap \overline{B} = A \cap \overline{B} = \phi \quad .$$

Case II: $0 \in \overline{A} - A$ . This means that $F \cap A \neq \phi$ for all $F \in \mathfrak{F}$. Since $\mathfrak{F}$ is an ultrafilter, $A \in \mathfrak{F}$. Clearly, $0 \notin B$ . If $0 \in \overline{B}$ , then $B \in \mathfrak{F}$ , so $A \cap B \neq \phi$ , an obvious contradiction. Since $0 \notin \overline{B}$ , $\overline{B}$ is open. Therefore $A$ is contained in the closed set $X - \overline{B}$ , and $\overline{A} \subseteq X - \overline{B}$ .

Case III: $0 \notin \overline{A}$ . If $x \in X - (A \cup \{0\})$ , then $\{x\}$ is an open set which fails to meet $A$ . Therefore $A = \overline{A}$ , and $\overline{A} \cap \overline{B} = A \cap \overline{B} = \phi$ .

Example 3.9 is a $T_5$ space in which every pair of separated subsets have disjoint closures. However, the space is not discrete. If $\{0\}$ is open, then $\{0\}$ must be a member of the ultrafilter $\mathfrak{F}$ . This means that $\{0\}$ and $]0,\frac{1}{2}[$ must have a non-empty intersection, since $]0,\frac{1}{2}[ \in \mathfrak{F}$ . Clearly, this is a contradiction, so $\{0\} \notin \mathcal{T}$ . By Theorem 3.8, this space cannot be a $C_I$ space.

The attempt to generalize Theorem 3.8 to the case of arbitrary $T_5$ spaces must fail, since Example 3.9 is a counter-example to such a theorem. However, it does restrict the number of $T_5$ spaces which will have the equivalence that we desired in the case of $T_5$ spaces.

Of the cases considered, the results of the $T_3$ and $T_4$ cases were already known. Although no positive results were found in the case of $T_{2\frac{1}{2}}$ spaces, the properties that we considered, which constitute many of the basic properties of topological spaces, all either failed to be necessary or sufficient. In the case of $T_5$ spaces, an equivalence of the type we sought was found. We showed that this equivalence would exist in a $C_I$, $T_5$ space iff the space was discrete. However, there are non $C_I$, $T_5$ spaces which have the equivalence that we desired, but are not discrete.

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