AN ABSTRACT OF THE THESIS OF

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(Name) (Degree) (Major)

Date thesis is presented October 8, 1963

Title INTEGRAL EQUATIONS INVOLVING SPECIAL FUNCTIONS

Abstract approved Redacted for privacy
(Major Professor)

A number of inversion integrals for transformations of the form $\int_x^1 k(t/x)g(t)dt = f(x)$ are found with the aid of Mellin transforms. A typical transformation involves Gauss' hypergeometric function, and the inversion formula is an integral involving Gauss' hypergeometric function and a differential operator. The transformations are contained in a class of transformations whose kernels have the Mellin-Barnes integral representation

$$\bar{\xi}_n(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} z^{-s}\prod_{k=1}^{n}\frac{\Gamma(a_k-s)}{\Gamma(1-c_k-s)}ds \qquad (z>1)$$

where $b_n - n - \sum_{k-1}^{n}(a_k+c_k) > -1$ and $\mathrm{Re}\, a_k > \mathrm{Re}\, s$, $k = 1, \ldots, n$.

Also considered are the dual equations

$$\int_0^\infty y^\alpha G_{pq}^{mn}\left(xy\,\middle|\,{a_1, \ldots, a_p \atop b_1, \ldots, b_q}\right) f(y)dy = e(x) \quad (0<x<1),$$

$$\int_0^\infty G_{pq}^{mn}\left(xy \,\middle|\, \begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}\right) f(y)\,dy = 0 \qquad (x>1)$$

where G_{pq}^{mn} is Meijer's G-function, $0<\alpha<1$, $p<q$, and $p+q = 2m+2n$. A solution, which is given without proof, is a double integral involving a G-function and a function similar to $\bar{\xi}_n$ given above. In the formal proof the equations are transformed similar to Busbridge (1938). The solution is also similar to Busbridge's solution for the special case of Bessel functions.

INTEGRAL EQUATIONS INVOLVING SPECIAL FUNCTIONS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1964

APPROVED:

Redacted for privacy

Associate Professor of Mathematics

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Date thesis is presented October 8, 1963

Typed by Anne Dilworth

ACKNOWLEDGMENT

The author wishes to thank Dr. Buschman for his help and guidance.

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INTEGRAL EQUATIONS INVOLVING SPECIAL FUNCTIONS

CHAPTER I

INTRODUCTION

Inversion integrals have recently been found for some integral transformations of the form

$$\int_x^1 k(t/x)\, g(t)dt = f(x) \quad (0<a\le x\le 1). \tag{1.1}$$

Initially our attention will be directed toward finding more inversion integrals with some special functions as kernels. As a side problem we will also consider the following similar transformations:

$$\int_1^x k(t/x)\, g(t)dt = f(x) \quad (1\le x\le a<\infty), \tag{1.2}$$

$$\int_x^1 k(x/t)\, g(t)dt = f(x) \quad (0<a\le x\le 1), \tag{1.3}$$

$$\int_1^x k(x/t)\, g(t)dt = f(x) \quad (1\le x\le a<\infty). \tag{1.4}$$

Since most of the inversion integrals have been found with the aid of the Mellin transforms, we have, by necessity, investigated Mellin-Barnes integrals. As a result of this investigation, we find that a large number of kernels for which inversion integrals have been found have an integral representation in a class of Mellin-Barnes integrals. In addition we will consider the relation of (1.1) with the transformation

$$\int_0^x k(x-t)\, g(t)dt = f(x) \quad (0 \leq x \leq a < \infty). \tag{1.5}$$

A simple transformation allows us to apply some results concerning (1.5) to (1.1). Finally we examine the dual equations

$$\int_0^\infty G_{pq}^{mn}(xy)y^\alpha f(y)dy = e(x) \quad (0<x<1) \tag{1.6}$$

$$\int_0^\infty G_{pq}^{mn}(xy)f(y)dy = 0 \qquad (x>1) \tag{1.7}$$

(G_{pq}^{mn} is Meyer's G-function, $2m + 2n = p + q$, and $p < q$), which came to our attention during the study of Mellin-Barnes integrals. We leave the discussion of these dual equations until after we have discussed the Mellin-Barnes integrals. First we shall be concerned with the problem of an inversion integral for (1.1).

Ta Li (11) obtained an inversion integral of (1.1) where the kernel k involved a Tchebyshev polynomial, T_n. Using a Legendre polynomial, P_n, as a kernel, R. G. Buschman (3) also found an inversion integral. Later in (4) he extended these results to kernels involving Gegenbauer polynomials, $C_n^{k/2}$, which include both the Legendre and Tchebyshev polynomials. In (5) he also gives inversion integrals for kernels involving Legendre functions, P_ν^μ, and Legendre functions on the cut, $\underline{P}_\nu^\mu$. These results constitute the previous special cases of (1.1).

In order to find the form of the inversion integral of (1.1), we rewrite it, as indicated by R. G. Buschman in (4), in the standard form of a convolution with respect to the Mellin transformation, namely,

$$\int_0^\infty k(t/x)U(t/x - 1)g(t)V(t - 1)dt = f(x)V(x - 1) \quad (1.8)$$

where $U(x) = 1$ for $x > 0$, $U(x) = 0$ for $x < 0$, and $V(x) = 1 - U(x)$. From (9, vol. 1, p. 307-308; 13 and 4) we have

$$M\{k(x)U(x - 1); - s\}M\{g(x)V(x - 1); 1 + s\} = M\{f(x)V(x - 1); s\}.$$

The inverse Mellin transforms are assumed to converge for the individual transforms. Substituting $s - 1$ for s in the last equation and rearranging, we have

$$M\{g(x)V(x - 1); s\} = M\{f(x)V(x - 1); s - 1\}/M\{k(x)U(x - 1); 1 - s\}. \quad (1.9)$$

A formal solution of (1.1) is then

$$g(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{M\{f(x)V(x - 1); s - 1\}}{M\{k(x)U(x - 1); 1 - s\}} x^{-s} ds. \quad (1.10)$$

However, we would prefer to express g in the form of a convolution similar to (1.8). We can not use the general formulas (9, vol. 1, p. 307-308; 13, 14 and 4) directly since the inverse Mellin transform of $[M\{k(x)U(x - 1); 1 - s\}]^{-1}$ will not in general converge.

There are two methods of modifying (1.9) which may allow us to form a convolution. In the first we must find a function h such that

$$M\{h(x)V(x-1);\, s\} = P(s)/M\{k(x)U(x-1);1-s\}$$

where P is a polynomial in s. In the second we insert $\Gamma(s)/\Gamma(s) = (-1+s)(-2+s)\ \ldots\ (-j+s)\Gamma(-j+s)/\Gamma(s)$ in (1.9). We then let $P(s) = (-1+s)(-2+s)\ \ldots\ (-j+s)$ and define

$$h(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}[M\{k(x)U(x-1);1-s\}\Gamma(s)]^{-1}\Gamma(-j+s)x^{-s}ds,$$

provided the integral converges for some positive integer j and

$$M\{h(x)V(x-1);s\} = \Gamma(-j+s)/[\Gamma(s)M\{k(x)U(x-1);1-s\}].$$

In either case (1.9) becomes

$$M\{g(x)V(x-1);s\} = M\{h(x)V(x-1);s\}P(s)M(f(x)V(x-1);s-1\}.$$

With the aid of the general formulas in (9, vol. 1 p. 307-308) we form from P a differential operator which operates on f and which is a polynomial P^* in the operators, d/dy, $y^{-1}d/dy$, $d/dy\, y^2$, and so forth. The coefficients of P^* are rational functions of y. The inversion integral, which is found from (9, vol. 1, p. 308; 14), is then

$$g(t)V(t-1) = \int_0^\infty h(t/y)V(t/y-1)P^*(D)f(y)V(y-1)dy$$

where D is one of the operators, and this is equivalent to

$$g(t) = \int_t^1 h(t/y)P^*(D)f(y)dy. \tag{1.11}$$

This gives us the form of the inversion integral.

It is easier to show that (1.11) is the inversion of (1.1) by substitution rather than by justifying the above steps. Subsequently we will give a detailed proof of an example; however, we will outline the proof here because it gives us a better insight into the form of the differential operator in (1.11). Let I denote the formal integral obtained by substituting (1.11) into the integrand of (1.1), that is,

$$I(x) = \int_x^1 k(t/x)dt \int_t^1 h(t/y)P^*(D)f(y)dy.$$

Using Dirichlet's formula, we have

$$I(x) = \int_x^1 P^*(D)f(y)dy \int_x^y k(t/x)h(t/y)dt. \tag{1.12}$$

If J denotes the inner integral, then J and the differential operator are related. Now we can no longer work with completely general formulas but must give some special results. Hence we list a few values of J and the corresponding differential operator:

$$J(y,x) = (k!)x^{-k}(y - x)^k, \quad P^*(\lambda) = (-d/dy)^{k+1}y^k,$$

$$J(y,x) = (2^k k!)^{-1}(y^2 - x^2)^k y^{n-k} x^{-n-k+1}, \quad P^*(\partial) = y^{-n+k+1}(-y^{-1}d/dy)^{k+1}y^{n+k-1}, \tag{1.13}$$

$$J(y,x) = (k!)^{-1}(1 - x/y)^k y, \quad P^*(\Delta) = y^{-1}(d/dy\, y^2)^{k+1}y^{-k}.$$

Consider, for example, (1.13) and (1.12); the result is

$$I(x) = \int_x^1 (2^k k!)^{-1}(y^2 - x^2)^k x^{-n-k+1} y(-y^{-1}d/dy)^{k+1}[y^{n+k-1}f(y)]dy.$$

If this integral is integrated by parts with the restriction that $f^{(m)}(1) = 0$, $0\leq m\leq k$, then

$$I(x) = -(2^k k!)^{-1}(y^2 - x^2)^k x^{-n-k+1}(-y^{-1}d/dy)^k[y^{n+k-1}f(y)]\Big|_x^1$$
$$-x^{-n-k+1}[(k-1)!]^{-1}2^{-k+1}\int_x^1 (y^2 - x^2)^{k-1}y(-y^{-1}d/dy)^k$$
$$\times[y^{n+k-1}f(y)]dy,$$
$$= -2^{-k+1}[(k - 1)!]^{-1}x^{-n-k+1}\int_x^1 (y^2 - x^2)^{k-1}d\{y^{-1}d/dy)^{k-1}$$
$$\times[y^{n+k-1}f(y)]\}.$$

Repeated integration by parts then gives

$$I(x) = -x^{-n-k+1}\int_x^1 d[y^{n+k-1}f(y)] = f(x).$$

This is the desired result.

At $y = 1$ the condition $f^{(m)}(1) = 0$ cancels half the terms which arise from the integration by parts. The differential operator $P^*(D)$ and the integral J combine at all but the last step in the above integration by parts to give a power of $y^2 - x^2$ which cancels all but one of the remaining terms at $y = x$. The last remaining term is then the desired result. The integral J does not have to be a power of $(y^n - x^n)$. It may be a more general polynomial or a function Q in x and y, provided that $Q(y,x) = 0$ when $y = x$ and that there exists a differential operator. This operator and the function Q must, of course, combine to cancel terms at $y = x$ which arise from the integration by parts. Only a few simple examples have been given.

A similar approach is given by Charles Fox (10) to the transformation

$$\int_0^\infty k(tx)g(t)dt = f(x). \tag{1.14}$$

The substitution $1/y$ for x in (1.8) yields

$$\int_0^\infty k(yt)U(yt-1)g(t)V(t-1)dt = f^*(y)U(y-1) \tag{1.15}$$

where $f^*(y) = f(1/y)$. This brings (1.8) in line with (1.14) so that we may consider (1.1) as a special case of (1.14). However, his method can not be applied to most of the kernels which we consider, at least, not without considerable modifications. When it does apply, the inversion formula requires fewer restrictions on the function f but may be cumbersome, particularly when the function

f must be differentiated more than one or two times. For some simple cases the method is quite useful.

As an example of Fox's method, consider the kernel k whose transform, K, satisfies the functional equation

$$K(s)K(1-s) = [a+bs(1-s)]^{-1} \qquad (1.16)$$

where a and b are constants. If $K(s) = (c+ds)^{-1}$, d is real, and $d < 0$, then k is a special case of both (1.16) and (3.9) $[a = c^2 - cd$ and $b = d^2]$. Now multiply (1.14) by x^{s-1} and integrate with respect to x from 0 to ∞. From (9, vol. 1, p. 308; 13) we find that

$$F(s) = K(s)G(1-s),$$

and this is equivalent to

$$G(s) = \frac{K(s)F(1-s)}{K(s)K(1-s)}. \qquad (1.17)$$

From (1.16) and (1.17) it follows that

$$G(s) = aK(s)F(1-s) + bs(1-s)K(s)F(1-s).$$

We find with the aid of formulas (9, vol. 1, p. 307-308; 3, 9, and 13) that

$$g(t) = a\int_0^\infty k(ut)f(u)du + bd/dt\int_0^\infty tk(ut)d/du\{uf(u)\}du. \qquad (1.18)$$

Under suitable conditions (1.18) can be verified as a solution of (1.14).

Formula (1.18) applied to (1.15) gives

$$g(t)V(t-1) = a\int_0^\infty k(ut)U(ut-1)f^*(u)U(u-1)du$$

$$+ b\,d/dt\int_0^\infty tk(ut)U(ut-1)d/du[uf^*(u)U(u-1)]du.$$

If the substitution $u = 1/z$ is made, then

$$g(t)V(t-1) = a\int_0^\infty k(t/z)V(t/z-1)f(z)V(z-1)z^{-2}dz$$

$$- b\,d/dt\int_0^\infty tk(t/z)V(t/z-1)d/dz[z^{-1}f(z)V(z-1)]dz,$$

and this is then equivalent to

$$g(t) = a\int_t^1 k(t/z)z^{-2}f(z)dz - b\,d/dt\int_t^1 tk(t/z)d/dz[z^{-1}f(z)]dz. \tag{1.19}$$

The last equation is a solution of (1.1) for only those kernels whose transforms are of the form

$$K(s) = (c-ds)^{-1}.$$

It should also be noted that the kernels in (1.19) and (1.15) are not the same, as might be assumed from the relation of (1.18) to (1.14). In (1.15) the argument of the kernel is greater than one, and in (1.19) it is less than one. The resolvent kernel is the analytic continuation of the transformation kernel. Thus we have found only the form of the solution and must verify in some other way that it is a solution.

We note that (1.2), (1.3), and (1.4) can be written respectively as

$$\int_0^\infty k(t/x)V(t/x - 1)g(t)U(t-1)dt = f(x)U(x-1), \qquad (1.20)$$

$$\int_0^\infty k(x/t)V(x/t - 1)g(t)V(t-1)dt = f(x)V(x-1), \qquad (1.21)$$

$$\int_0^\infty k(x/t)U(x/t - 1)g(t)U(t-1)dt = f(x)U(x-1). \qquad (1.22)$$

The form of the inversion integral can be found in a manner similar to the one already given. Proof that the integral obtained is the inversion integral is similar to the proof that will be given for an example of (1.1).

We next give some examples of inversion integrals with Legendre polynomials as kernels. These inversion integrals are not the only ones possible. The resolvent kernel does not have to be a Legendre polynomial, nor does it have to be the Legendre polynomials used. The transformation

$$\int_x^1 P_n(t/x)g(t)dt = f(x) \qquad (1.23)$$

has inversion integrals of the form

$$g(t) = \int_t^1 P_{n-2}(t/y)y^{2-n}(y^{-1}d/dy)^2[y^n f(y)]dy \qquad (1.24)$$

and

$$g(t) = t^{-2}\int_t^1 P_n(t/y)y\ d^2/dy^2[yf(y)]dy. \qquad (1.25)$$

The transformation

$$\int_x^1 P_n(x/t)g(t)dt = f(x)$$

has inversion integrals of the form

$$g(t) = t^{-2}\int_t^1 P_{n-2}(y/t)y^{2+n}(y^{-1}d/dy)^2[y^{-n+2}f(y)]dy$$

and

$$g(t) = \int_t^1 P_n(y/t)d^2/dy^2 f(y)dy.$$

The transformation

$$\int_1^x P_n(t/x)g(t)dt = f(x) \tag{1.26}$$

has inversion integrals of the form

$$g(t) = \int_1^t P_{n-2}(t/y)y^{2-n}(y^{-1}d/dy)^2[y^n f(y)]dy \tag{1.27}$$

and

$$g(t) = t^{-2}\int_1^t P_n(t/y)y\ d^2/dy^2[yf(y)]dy. \tag{1.28}$$

The inversion integrals (1.27) and (1.28) can be obtained from (1.24) and (1.25) respectively by interchanging the limits of integration without the usual change in sign. This fact is more readily seen from the inner integral in (1.12), that is,

$$J(y,x) = \int_x^y k(t/x)h(t/y)dt \quad (0<a\leq x\leq y\leq 1).$$

If k, h, and J are analytic functions, then by analytic continuation the integral remains valid for $1 \leq y \leq x \leq 1/a$. This relation between the transformations can now be verified by substitutions into the appropriate equations. For example, compare (1.28) and (1.26) with (1.25) and (1.23). The substitution of (1.25) into (1.23) and (1.28) into (1.26) and the inversion in order of integration yield respectively

$$\int_x^1 t^{-2} y\, d^2/dy^2 [y f(y)]\, dy \int_x^y P_n(t/x) P_n(t/y)\, dy = f(x),$$

$$\int_1^x t^{-2} y\, d^2/dy^2 [y f(y)]\, dy \int_y^x P_n(t/x) P_n(t/y)\, dy = f(x).$$

If both pairs of limits of integration are changed in the second equation, the two equations will be the same.

CHAPTER II

GAUSS' HYPERGEOMETRIC FUNCTION

In this chapter we give several inversion integrals of the transformation

$$\int_x^1 k(t/x)g(t)dt = f(x)$$

where the kernel k contains Gauss' hypergeometric function ${}_2F_1$. These inversion integrals include the previously known inversion integrals involving Legendre functions and Gegenbauer polynomials. They also include some special cases not previously considered such as Jacobi polynomials, log functions, and some inverse trigonometric functions.

The hypergeometric equation is

$$z(1-z)d^2u/dz^2 + [c-(a+b+1)z]du/dz - abu = 0$$

where the parameters a, b, and c are arbitrary complex numbers. It has three singularities, 0, 1, and ∞, which are regular singularities. If $c \neq 0, -1, -2, \ldots$, then

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n(b)_n z^n/(c)_n n!$$

is a solution, which is regular at $z = 0$. [By definition $(a)_n = a(a+1) \ldots (a+n-1) = \Gamma(a+n)/\Gamma(a)$]. If a and b are different from 0, -1, -2, ..., the series converges absolutely for all values of $|z| < 1$. It converges absolutely for $|z| = 1$ if $\mathrm{Re}(a+b-c) < 0$. It converges conditionally for $|z| = 1$, $z \neq 1$, if $0 \leq \mathrm{Re}(a+b-c) < 1$ and diverges if $|z| = 1$ and $1 \leq \mathrm{Re}(a+b-c)$. If $1-c$, $b-a$, and $c-b-a$ are not integers, the analytic continuation of ${}_2F_1$ can be obtained from the equation

$$\begin{aligned}{}_2F_1(a,b;c;z) &= \Gamma(c)\Gamma(c-a-b)\{\Gamma(c-a)\Gamma(c-b)\}^{-1} \\ &\quad z^{-a}{}_2F_1(a, a+1-c; a+b+1-c; 1-z^{-1}) \\ &+ \Gamma(c)\Gamma(a+b-c)\{\Gamma(a)\Gamma(b)\}^{-1} z^{a-c}(1-z)^{c-a-b} \\ &\quad {}_2F_1(c-a, 1-a; c+1-a-b; 1-z^{-1}) \qquad |\arg z| < \pi.\end{aligned}$$

For further details see (8, vol. 1, p. 56-119).

If we substitute $x = u - 1$ and $\sigma = 1 - s$ in the formulas (9, vol. 2, p. 400;9) and $x = 1 - u$, $p = s$ in (9, vol. 2, p. 399;4) we obtain respectively

$$\int_1^\infty u^{s-1}(u-1)^{c-1}{}_2F_1(a,b;c;1-u)du = \frac{\Gamma(c)\Gamma(a-c+1-s)\Gamma(b-c+1-s)}{\Gamma(1-s)\Gamma(a+b-c+1-s)}, \tag{2.1}$$

valid for $\mathrm{Re}\, c > 0$, $\mathrm{Re}(a-c+1) > \mathrm{Re}\, s$, and $\mathrm{Re}(b-c+1) > \mathrm{Re}\, s$; and

$$\int_0^1 u^{s-1}(u-1)^{c-1}{}_2F_1(a,b;c;1-u)du = \frac{\Gamma(c)\Gamma(s)\Gamma(c-a-b+s)}{\Gamma(c-a+s)\Gamma(c-b+s)}, \tag{2.2}$$

valid for $\mathrm{Re}\, c > 0$, $\mathrm{Re}\, s > 0$, and $\mathrm{Re}\, s > \mathrm{Re}(a+b-c)$.

The method given in the introduction suggests some possible inversion integrals. We give a few in the following theorem.

Theorem 1. Suppose that $f^{(n+1)}$ is sectionally continuous for $0 < a \leq x \leq 1$ and $f^{(k)}(1) = 0$ for $0 \leq k \leq n$. Let $\lambda = -d/dy$, $\Delta = d/dy\, y^2$, and $\partial = -y^{-1}d/dy$. Now let n be chosen in the equations below such that in

(2.3) $2n+1 > \operatorname{Re}(a+b) > n \geq 0$,

(2.5) $n+1 > \operatorname{Re}(a+b) > 0,\ n \geq 0$,

(2.7) $n+1 > \operatorname{Re} c > 0,\ n \geq 0$,

(2.9) $n \geq 0$,

(2.11) $n+1 > \operatorname{Re} e > 0,\ n \geq 0$,

(2.13) $n+1 > \operatorname{Re} e > 0,\ n \geq 0$.

If

$$[\Gamma(a+b-n)]^{-1}\int_x^1 (t/x-1)^{a+b-n-1}\,{}_2F_1(a,b;a+b-n;1-t/x)\,g(t)\,dt=f(x), \tag{2.3}$$

then

$$g(t) = -[\Gamma(2n-a-b+1)]^{-1}\int_t^1 (1-t/y)^{2n-a-b}$$
$$\times {}_2F_1(n+1-a,n+1-b;2n-a-b+1;1-t/y)\,y^{-1}\Delta^{n+1}[y^{-n}f(y)]\,dy. \tag{2.4}$$

If

$$[\Gamma(a+b)]^{-1}\int_x^1 (t/x-1)^{a+b-1}\,{}_2F_1(a,b;a+b;1-t/x)g(t)dt = f(x), \tag{2.5}$$

then

$$g(t) = [\Gamma(n-a-b+1)]^{-1}\int_t^1 (1-t/y)^{n-a-b}$$
$$\times {}_2F_1(n-a+1,n-b+1,n-a-b+1;1-t/y)\lambda^{n+1}[y^n f(y)]dy. \tag{2.6}$$

If

$$[\Gamma(c)]^{-1}\int_x^1 (t^2/x^2-1)^{c-1}\,{}_2F_1((n+d+1)/2,(2c-n-d-2)/2;c;1-t^2/x^2)$$
$$\times g(t)dt = f(x), \tag{2.7}$$

then

$$g(t) = 2^{-n+1}[\Gamma(n+1-c)]^{-1}\int_t^1 (1-t^2/y^2)^{n-c}y^{n-d}$$
$$\times {}_2F_1((n-d)/2,(n+d-2c+1)/2;n+1-c;1-t^2/y^2)\partial^{n+1}[y^{n+d}f(y)]dy. \tag{2.8}$$

If

$$[\Gamma((n+1)/2)]^{-1}\int_x^1 (t^2/x^2-1)^{(n-1)/2}$$
$$\times {}_2F_1((d-e)/2;(n+e-d)/2;(n+1)/2;1-t^2/x^2)g(t)dt = f(x), \tag{2.9}$$

then

$$g(t) = 2^{-n+1}[\Gamma((n+1)/2)]^{-1}\int_t^1 (1-t^2/y^2)^{(n-1)/2}y^{n+d-e+1}$$
$$\times {}_2F_1((e-d-1)/2,(n-e+d+1)/2;(n+1)/2;1-t^2/y^2)$$

$$\times\, \partial^{n+1}[y^{n-d+e-1} f(y)]dy. \tag{2.10}$$

If

$$[\Gamma(n-e+1)]^{-1} \int_x^1 (t^2/x^2-1)^{n-e}$$

$$\times\, {}_2F_1((n-e+d+1)/2, (n-e-d)/2; n+1-e; 1-t^2/x^2)\, g(t)dt = f(x), \tag{2.11}$$

then

$$g(t) = 2^{-n+1}[\Gamma(e)]^{-1} \int_t^1 (1-t^2/y^2)^{e-1}$$

$$\times\, {}_2F_1((n+e-d)/2, (e+d-n-1)/2; e; 1-t^2/y^2)\, y^{n-d+e} \partial^{n+1}[y^{n+d-e} f(y)dy. \tag{2.12}$$

If

$$[\Gamma(e)]^{-1} \int_x^1 (t^2/x^2-1)^{e-1}\, {}_2F_1((e+d)/2, (e-d-1)/2; e; 1-t^2/x^2)$$

$$\times\, g(t)dt = f(x), \tag{2.13}$$

then

$$g(t) = 2^{-n+1}[\Gamma(n-e+1)]^{-1} \int_t^1 (1-t^2/y^2)^{n-e}$$

$$\times\, {}_2F_1((2n-e-d+1)/2, (d-e)/2; n-e+1; 1-t^2/y^2)\, y^{2n-d-e+1}$$

$$\times\, \partial^{n+1}[y^{d+e-1} f(y)]dy. \tag{2.14}$$

Note 1. Other related forms can be derived from these by use of the transformation formulas for the hypergeometric function (8, vol. 1, p. 110-114).

Note 2. In the proof of (2.8), (2.10), (2.12), and (2.14)

$$M\{\int_0^\infty f_1(x^2y^2)f_2(y^2)dy;s\} = \tfrac{1}{4}M\{f_1(x);s/2\}M\{f_2(x);1/2-s/2\}$$

is used instead of (9, vol. 1, p. 308;13).

Note 3. We can weaken the restriction on $f^{(n+1)}$ to be just integrable instead of sectionally continuous. In order that the inversion integral be convergent, it is needed in addition to being integrable that the resolvent kernel be bounded almost everywhere (14, p. 113). The key steps in the proof are the change in order of integration and the integration by parts. For the change in order of integration we require that the kernels be integrable on appropriate intervals in order to apply Fubini's Theorem (14, p. 121). For the integration by parts it is sufficient that the (n-1)-th derivative of J [as defined in the proof below] with respect to y be absolutely continuous (14, p. 153).

Proof. We prove that (2.4) is an inversion integral of (2.3). The proofs of the other inversion integrals are similar.

Consider the integral, I, obtained by substituting (2.4) into the integrand of (2.3), that is,

$$I(x) = -[\Gamma(a+b-n)]^{-1}\int_x^1 (t/x-1)^{a+b-n-1}\,{}_2F_1(a,b;a+b-n;1-t/x)$$

$$\times dt[\Gamma(2n-a-b+1)]^{-1}\int_t^1 (1-t/y)^{2n-a-b}$$

$$\times {}_2F_1(n+1-a,n+1-b;2n-a-b+1;1-t/y)y^{-1}\Delta^{n+1}[y^{-n}f(y)]dy.$$

Using Dirichlet's formula, we obtain

$$I(x) = -\int_x^1 \Delta^{n+1}[y^{-n}f(y)]dy\int_x^y [\Gamma(a+b-n)]^{-1}(t/x-1)^{a+b-n-1}$$

$$\times {}_2F_1(a,b;a+b-n;1-t/x)[\Gamma(2n-a-b+1)]^{-1}(1-t/y)^{2n-a-b}$$

$$\times {}_2F_1(n+1-a,n+1-b;2n-a-b+1;1-t/y)y^{-1}dt.$$

If the substitution $z = t/y$ is made in the inner integral, then

$$I(x) = -\int_x^1 \Delta^{n+1}[y^{-n}f(y)]dy\int_{x/y}^1 [\Gamma(a+b-n)]^{-1}(yz/x-1)^{a+b-n-1}$$

$$\times {}_2F_1(a,b;a+b-n;1-yz/x)[\Gamma(2n-a-b+1)]^{-1}(1-z)^{2n-a-b}$$

$$\times {}_2F_1(n+1-a,n+1-b;2n-a-b+1;1-z)dz.$$

Next we let $u = y/x$, denote the inner integral by J, and write it in the standard form of a convolution, that is,

$$J(u)U(u-1) = \int_0^\infty [\Gamma(a+b-n)]^{-1}(uz-1)^{a+b-n-1}$$

$$\times {}_2F_1(a,b;a+b-n;1-uz)U(uz-1)[\Gamma(2n-a-b+1)]^{-1}(1-z)^{2n-a-b}$$

$$\times {}_2F_1(n+1-a,n+1-b;2n-a-b+1;1-z)V(z-1)dz$$

where U and V are as defined for (1.8). The Mellin transform of both sides is, from (9, vol. 1, p. 308;13),

$$M\{J(u)U(u-1);s\} = M\{[\Gamma(a+b-n)]^{-1}(u-1)^{a+b-n-1}{}_2F_1(a,b;a+b-n;1-u)$$

$$\times U(u-1);s\} M\{[\Gamma(2n-a-b+1)]^{-1}(1-z)^{2n-a-b}$$

$$\times {}_2F_1(n+1-a,n+1-b;2n-a-b+1\,;\,1-z)V(z-1);1-s\}.$$

From (2.1) and (2.2) we have

$$M\{J(u)U(u-1);s\} = \frac{\Gamma(n-b-1-s)\Gamma(n-a+1-s)}{\Gamma(1-s)\Gamma(1+n-s)}\,\frac{\Gamma(1-s)\Gamma(-s)}{\Gamma(n-b-s+1)\Gamma(n-a-s+1)}$$

$$= \Gamma(-s)/\Gamma(n-s).$$

But from (9, vol. 1, p. 311;32, and p. 307;3) we find that $(u-1)^n u^{-n}(n!)^{-1}U(u-1)$ has the same Mellin transform. Hence

$$J(u) = (1 - 1/u)^n/n!, \quad \text{for} \quad u > 1,$$

and we have

$$I(x) = -(n!)^{-1}\int_x^1 (1-x/y)^n \Delta^{n+1}[y^{-n}f(y)]dy$$

$$= -(n!)^{-1}(1-x/y)^n y^2 \Delta^n[y^{-n}f(y)]\Big|_x^1$$

$$-((n-1)!)^{-1}\int_x^1 (1-x/y)^{n-1} x\Delta^n[y^{-n}f(y)]dy.$$

Repeated integration by parts and application of the condition $f^{(k)}(1) = 0$ then yields

$$I(x) = -\int_x^1 x^n d[y^{-n}f(y)] = f(x).$$

The proof is complete.

If we specialize the parameters in (2.9) and (2.13), we can get the transformations of the form (1.1) which were considered in (11), (3), (4), and (5). For from (8, vol. 1, p. 176;21) with n even, we have

$$C_n^{k/2}(x) = (k)_n(n!)^{-1}{}_2F_1(-n/2, n/2+k/2, k/2+1/2; 1-x^2).$$

If in (2.9) we let $d = 0$, $e = n$, and $n = k$, we get, except for a constant, the transformation considered in (4). The inversion integral (2.10) is, except for a constant, the inversion integral given in (4). If we compare Mellin transforms of the kernels, we find the same result holds for odd n. Likewise comparing (2.1) and (9, vol. 1, p. 320;3) we find

$$(1-x^2)^{-\mu/2}\underline{P}_\nu^\mu(x) = 2^\mu(1-x^2)^{-\mu}{}_2F_1((1-\mu+\nu)/2, (-\mu-\nu)/2; 1-\mu; 1-x^2),$$

and from (8, vol. 1, p. 126;20) we find

$$(x^2-1)^{-\mu/2}P_\nu^\mu(x) = 2^{-\mu+1}(x^2-1)^{-\mu}{}_2F_1((1-\mu+\nu)/2, (-\mu-\nu)/2; 1-\mu; 1-x^2).$$

The substitutions $f(x) = x^{1-\mu}h(x)$, $e = \mu$, and $d = \nu$ in (2.13) and (2.14) give one of the transformations in (5) and its inversion. The other transformation considered in (5) has the form (1.2). Since the transformations in (3) and (11) are special cases of the one considered in (4), all previous transformations of the form (1.1) are contained in (2.9) and (2.13).

We can also obtain other special cases which have not been considered previously. If suitable substitutions are made in the equations (8, vol. 1, p. 102;13, 14, and 15), then

$$\sin^{-1}((1-z)^{1/2}) = (1-z)^{1/2}{}_2F_1(1/2,1/2;3/2;1-z),$$

$$\tan^{-1}((z-1)^{1/2}) = (z-1)^{1/2}{}_2F_1(1/2,1;3/2;1-z),$$

and

$$\ln z = (z-1){}_2F_1(1,1;2;1-z).$$

If $n = 2$, $d = 2$, and $e = 1$ in (2.9) and (2.10), we find

$$i[\Gamma(3/2)]^{-1}\int_x^1 \sin^{-1}((1 - t^2/x^2)^{1/2})g(t)dt = f(x)$$

and

$$g(t) = [6\Gamma(3/2)]^{-1}\int_t^1 (1-t^2/y^2)^{1/2}(4t^2/y^2-1)y^4\partial^3 f(y)dy,$$

where ∂ denotes $-y^{-1}d/dy$, and since the hypergeometric series in (2.10) terminates for this particular case. If $n = 1$, $a = 1/2$, and $b = 1$ in (2.5) and (2.6), it follows with the aid of (8, vol. 1, p. 101;5) that

$$[\Gamma(3/2)]^{-1}\int_x^1 \tan^{-1}((t/x -1)^{1/2})g(t)dt = f(x)$$

and

$$g(t) = [\Gamma(1/2)]^{-1}\int_t^1 (1-t/y)^{-1/2}$$

$$\times[1/2(1+(1-t/y)^{-2})+1/2(1-(1-t/y)^{-2})]\lambda^3[y^2 f(y)]dy,$$

where λ denotes $-d/dy$. If $n = 2$, $a = 1$, and $b = 1$ in (2.5) and (2.6) we find with the aid of (8, vol. 1, p. 101;7) that

$$[\Gamma(2)]^{-1}\int_x^1 \ln(t/x)g(t)dt = f(x)$$

and

$$g(t) = \int_t^1 (t/y)^{-3}(2-t/y)\lambda^4[y^3 f(y)]dy.$$

From (8, vol. 2, p. 170;16)

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}{}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; 1/2-1/2x),$$

so that from Theorem 1 we can find a large number of inversion integrals to transformations involving Jacobi polynomials. In most cases the resolvent kernel will not be a Jacobi polynomial. However, if we substitute $e = \alpha + 1$ and $d = -2n - \alpha - 1$ in (2.13) and make adjustments for the 1/2 in the function ${}_2F_1$, we find that

$$\left[\Gamma(\alpha+1)\binom{n+\alpha}{n}\right]^{-1}\int_x^1 (t^2/x^2-1)^{\alpha} P_n^{(\alpha,o)}(t^2/x^2)g(t)dt = f(x)$$

has the inversion integral

$$g(t) = (1/2)^{1/2}2^{-k+1}\left[\Gamma(k-\alpha)\binom{n+k}{n+\alpha+1}\right]^{-1}\int_t^1 (1-t^2/y^2)^{k-\alpha-1}$$

$$\times P_{n+\alpha+1}^{(k-\alpha-1,\alpha+1/2)}(t^2/y^2)y^{2k+2n+1}\partial^{k+1}[y^{-2n-1}f(y)]dy,$$

provided α is an integer, $\alpha \geq 0$, and $k - \alpha > 0$. The last inversion integral is the only obvious one in which the resolvent kernel is a Jacobi polynomial.

CHAPTER III

MELLIN-BARNES INTEGRALS

We generalize the results of Chapter 1 to some functions which have a Mellin-Barnes integral representation. These functions are connected with the generalized hypergeometric function ${}_nF_{n-1}$.

The hypergeometric differential equation of order n,

$$z^n(1-z)d^ny/dz^n + \sum_{k=1}^{n-1}(\alpha_k - \beta_k z)z^k d^k y/dz^k = 0,$$

has three singularities, 0, 1, and ∞. N. E. Nørlund (13) rewrites the equation in the form

$$(\delta-c_1)\ \dots\ (\delta-c_n)y - z(\delta+a_1)\ \dots\ (\delta+a_n)y = 0$$

where $\delta = zd/dz$. He lets ξ_n denote the solution which is valid in the vicinity of the singularity at 1. He also obtains a solution $\overline{\xi}_n$ from ξ_n by substituting $1/z$ for z and interchanging the a_i's and c_i's. The solution $\overline{\xi}_n$ is also valid in the vicinity of the singularity at 1. The details are too involved to be given here, and so we refer merely to N. E. Nørlund's paper. We give only those results related to the Mellin transforms of ξ_n and $\overline{\xi}_n$.

The solution ξ_n and $\overline{\xi}_n$ have the following integral representations:

$$\xi_n(z) = \frac{1}{2\pi i}\Gamma(b_n+1)\int_{\gamma-i\infty}^{\gamma+i\infty} z^{-s}\prod_{k=1}^{n}\frac{\Gamma(s+c_k)}{\Gamma(s-a_k+1)}ds \quad (0<z<1) \tag{3.1}$$

where $\mathrm{Re}(s+c_k) > 0$, $k = 1, \ldots, n$, $\mathrm{Re}\, b_n > -1$, and

$$b_n = n-1-\sum_{k=1}^{n}(a_k+c_k),$$

$$\bar{\xi}_n(z) = \frac{1}{2\pi i}\Gamma(b_n+1)\int_{\gamma-i\infty}^{\gamma+i\infty} z^{-s}\prod_{k=1}^{n}\frac{\Gamma(a_k-s)}{\Gamma(1-c_k-s)}ds \quad (z>1) \tag{3.2}$$

where $\mathrm{Re}\, a_k > \mathrm{Re}\, s$, $k = 1, \ldots, n$, $\mathrm{Re}\, b_n > -1$, and b_n is the same in both cases. The integral (3.1) is vanishing for $z > 1$, and (3.2), for $0 < z < 1$. We also note that

$$\int_0^1 z^{s-1}\xi_n(z)dz = \Gamma(b_n+1)\prod_{k=1}^{n}\frac{\Gamma(s+c_k)}{\Gamma(s-a_k+1)} \tag{3.3}$$

and

$$\int_1^\infty z^{s-1}\bar{\xi}_n(z)dz = \Gamma(b_n+1)\prod_{k=1}^{n}\frac{\Gamma(a_k-s)}{\Gamma(1-c_k-s)}. \tag{3.4}$$

All of the above integrals are absolutely convergent.

Now we consider the integral transformation

$$\int_x^1 \bar{\xi}_n(t/x)g(t)dt = f(x). \tag{3.5}$$

The methods given in the introduction suggest that both ξ_n and ξ_{n+1} can be used for inversion kernels. Thus if $\bar{\xi}_n$ is given by (3.2), we find that (3.5) has the following inversion integrals:

$$g(t) = -\int_t^1 \Gamma(b_n+1)[\Gamma(j-b_n+1)]^{-1} \xi_{n+1}(t/y) y^{-1}$$

$$\times \Delta^j [y^{-j+1} f(y)] dy \qquad (3.6)$$

where $\Delta = d/dy\, y^2$, $\mathrm{Re}(j - b_n + 1) > 0$, and

$$\xi_{n+1}(z) = \frac{1}{2\pi i} \Gamma(j-b_n+1) \int_{\gamma-i\infty}^{\gamma+i\infty} z^{-s} \prod_{k=1}^{n} \frac{\Gamma(-c_k+s)}{\Gamma(a_k-1+s)} \frac{\Gamma(-1+s)}{\Gamma(-1+j+s)} ds$$

and

$$g(t) = \int_t^1 \Gamma(b_n+1)[\Gamma(j-b_n+1)]^{-1} \xi^*_{n+1}(t/y) y^{-1}$$

$$\times (\lambda)^k [y^{k-1} f(y)] dy \qquad (3.7)$$

where $\lambda = -d/dy$, $\mathrm{Re}(j - b_n + 1) > 0$, and

$$\xi^*_{n+1}(z) = \frac{1}{2\pi i} \Gamma(j-b_n+1) \int_{\gamma-i\infty}^{\gamma+i\infty} z^{-s} \prod_{k=1}^{n} \frac{\Gamma(-c_k+s)}{\Gamma(a_k-1+s)} \frac{\Gamma(-j+s)}{\Gamma(s)} ds.$$

If we want the inversion kernel to be in the class ξ_n, we must restrict the parameters in (3.2). The restriction can be made in many ways, but one of the simplest is to restrict one of the a_i's in (3.2) to be zero, say a_n. The inversion integral is then

$$g(t) = -\int_t^1 \Gamma(b_n+1)[\Gamma(j-b_n+1)]^{-1} \xi^*_n(t/y) y^{-1} \Delta^j [y^{-j+1} f(y)] dy \qquad (3.8)$$

where $\mathrm{Re}(j-b_n+1) > 0$ and

$$\xi_n^*(z) = \frac{1}{2\pi i}\Gamma(j-b_n+1)\int_{\gamma-i\infty}^{\gamma+i\infty} z^{-s}\Pi_{k=1}^{n-1}\frac{\Gamma(-c_k+s)}{\Gamma(a_k-1+s)}\,\frac{\Gamma(c_n+s)}{\Gamma(-1+j+s)}ds.$$

Other inversion integrals can be found. The proof that these are inversion integrals is very similar to the one given in Theorem 1 and hence is not repeated. It involves the substitution of (3.6), (3.7), or (3.8) into (3.5), the interchange in the order of integration, simplification by the use of a special convolution integral of the type

$$\int_x^y \overline{\xi}_n(t/x)\xi_{n+1}(t/y)dt,$$

and a j-fold integration by parts.

The relation of the functions ξ_2 and $\overline{\xi}_2$ to Gauss' hypergeometric function is found by comparison of equations (2.1) and (3.4) and equations (2.2) and (3.3). For example, if $x > 1$, then

$$\Gamma(c)(x-1)^{c-1}{}_2F_1(a,b;c;1-x) = \overline{\xi}_2(x)$$

$$= \frac{1}{2\pi i}\Gamma(c)\int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s}\,\frac{\Gamma(a-c+1-s)\Gamma(b-c+1-s)}{\Gamma(1-s)\Gamma(a+b-c+1-s)}ds.$$

Thus we see that (3.6) and (3.7) give us further inversion integrals for transformations involving ${}_2F_1$. These are inversion integrals of a more general transformation than given in Theorem 1 since the parameters do not have to be restricted in the transformation kernel itself.

The Mellin-Barnes integral (3.2) leads us to consider the slightly more general integral

$$R(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(b_1-B_1 s)\,\ldots\,\Gamma(b_n-B_n s)}{\Gamma(d_1-D_1 s)\,\ldots\,\Gamma(dq-Dqs)}(pz)^{-s}ds \quad (3.9)$$

where the B_i's and D_i's are real,

$$\alpha = \sum_{j=1}^{n} B_j - \sum_{j=1}^{q} D_j = 0$$

$$\Lambda = \mathrm{Re}(\sum_{j=1}^{n} b_j - \sum_{j=1}^{q} d_j + 1/2(q-n) < 0.$$

$$\rho = \Pi_{j=1}^{n} B_j^{-Bj}\Pi_{j=1}^{q} D_j^{Dj}.$$

For $\Lambda < -1$ the integral is absolutely convergent for all positive z [see (6) or (8, vol. 1, p. 49-50)]. For $-1 < \Lambda < 0$ the integral converges but not absolutely. If B_i's and D_i's are rational. we can use the multiplication formula of Gauss and Legendre (8, vol. 1, p. 4; 11),

$$\Pi_{r=0}^{m-1}\Gamma(z+r/m) = (2\Pi)^{1/2(m-1)}m^{1/2-mz}\Gamma(mz) \quad (m=2,3, \ldots),$$

to transform R into $\overline{\xi}_n$, that is, $R(z) = C'\overline{\xi}_n(az^P)$. We follow a similar example given by J. Boersma (1). We let δ be the least common denominator of $B_1, \ldots, B_n$ and $D_1, \ldots, D_q$, and define

$$\nu_j = \delta B_j, \ j=1, \ \ldots, \ n,$$

$$\mu_j = \delta D_j, \ j=1, \ \ldots, \ q,$$

$$s = \delta w.$$

Hence ν_j and μ_j are integers, and we have

$$R(z) = \frac{\delta}{2\pi i}\int_{\gamma/\delta - i\infty}^{\gamma/\delta + i\infty} \frac{\Pi_{j=1}^{n}\Gamma(\nu_j(-w+b_j/\nu_j))}{\Pi_{j=1}^{q}\Gamma(\mu_j(-w+d_j/\mu_j))}(\rho z)^{-\delta w} dw$$

$$= \frac{\delta}{2\pi i}\,\frac{\Pi_{j=1}^{n}(2\Pi)^{(\mu_j-1)/2}\mu_j^{1/2-d_j}}{\Pi_{j=1}^{q}(2\Pi)^{(\nu_j-1)/2}\nu_j^{1/2-b_j}}$$

$$\times\int_{\gamma/\delta - i\infty}^{\gamma/\delta + i\infty} \frac{\Pi_{j=1}^{n}\Pi_{m_j=0}^{\nu_j-1}\Gamma(-w+(b_j+m_j)/\nu_j)}{\Pi_{j=1}^{q}\Pi_{\ell_j=0}^{\mu_j-1}\Gamma(-w+(d_j+\ell_j)/\mu_j)}(\rho^{\delta}z^{\delta})^{-w}dw.$$

Thus we conclude that if $\Lambda < -1$ and the parameters are rational, R is vanishing for $0 < z < 1$ and

$$\int_1^{\infty} z^{s-1}R(s)dz = \frac{\Gamma(b_1-B_1 s)\ \ldots\ \Gamma(b_n-B_n s)}{\Gamma(d_1-D_1 s)\ \ldots\ \Gamma(d_q-D_q s)}\rho^{-s}. \qquad (3.10)$$

It now follows by the Lebesgue convergence theorem (14, p. 104) or (17, p. 337) that R is vanishing for $0 < z < 1$ and that (3.10) holds if the parameters are irrational.

We again consider (1.1) with R as the kernal, that is,

$$\int_x^1 R(t/x)g(t)dt = f(x).$$

An inversion integral is

$$g(t) = \int_t^1 T(t/y)y^{-1}(-D)^j[y^{j-1}f(y)]dy \qquad (3.11)$$

where $D = d/dy$, $-j - \Lambda < -1$ and

$$T(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(d_1-D_1+D_1 s)\ \ldots\ \Gamma(d_q-D_q+D_q s)\Gamma(-j+s)}{\Gamma(b_1-B_1+B_1 s)\ \ldots\ \Gamma(b_n-B_n+B_n s)\Gamma(s)}(\rho z)^{-s}\frac{ds}{\rho}.$$

(3.12)

The proof that (3.11) is an inversion integral is similar to the proof given in Theorem 1 and outlined for the inversion integral (3.6), (3.7), and (3.8).

CHAPTER IV

RELATED INTEGRAL EQUATIONS

We consider in this chapter a relation between the integral equation

$$\int_0^x r(x-z)h(z)dz = \ell(x) \tag{4.1}$$

and the integral equation

$$\int_x^1 k(t/x)g(t)dt = f(x). \tag{4.2}$$

If we substitute $x = e^{-y}$ and $t = e^{-z}$ in (4.2), we obtain

$$\int_0^y k(e^{y-z})g(e^{-z})e^{-z}dz = f(e^{-y}), \tag{4.3}$$

and this is the same form as (4.1). E. C. Titchmarsh (16) [Also G. Doetsch (7, p. 151-163) and other places] gives a few theorems which concern (4.1), and we convert the two most important theorems so that they apply directly to (4.2).

<u>Theorem 2</u>. Let

$$\int_1^\infty |k(x)x^{-k-1}|dx < \infty$$

and

$$\int_0^1 |f(x)x^{k-1/2}|^2 dx < \infty.$$

Then in order that there should be a solution g of (4.2) such that

$$\int_0^1 |g(x)x^{k+1/2}|^2 dx < \infty,$$

it is necessary and sufficient that

$$\int_{-\infty}^{\infty} \left|\frac{M\{f(x)V(x-1);u-iv\}}{M\{k(x)U(x-1);-u+iv\}}\right|^2 dv \leq M$$

where $U(x) = 1$ for $x > 0$, $U(x) = 0$ for $x < 0$, $V(x) = 1 - U(x)$, and M is a constant independent of u, for all $u \geq k$.

Proof. We apply Theorem 149 (16, p. 322) to (4.3) and revert to (4.2) by suitable substitutions.

Theorem 2 gives conditions for (1.10) to be a solution of (1.1).

Theorem 3. If g is integrable over any interval $(x,1)$, $0 < x < 1$; k is integrable over $(1, 1/x)$; k is not null; and

$$\int_x^1 k(t/x)g(t)dt = 0 \quad (0<x<1),$$

then g is null in $(x,1)$.

Proof. We apply Theorem 152 (16, p. 325) to (4.3) with $f(e^{-y}) \equiv 0$ and revert to (4.2).

Theorem 4. If

$$g(t) = \int_t^1 h(t/y)P(D)f(y)dy$$

and

$$g*(t) = \int_t^1 h*(t/y)P*(D*)f(y)dy$$

are inversion integrals of

$$\int_x^1 k(t/x)g(t)dt = f(x),$$

then $g(t) = g*(t)$ almost everywhere.

Proof. We have

$$\int_x^1 k(t/x)[g(t)-g*(t)]dt = f(x) - f(x) = 0.$$

By Theorem 3, $g(t) - g*(t) = 0$ almost everywhere. Hence $g(t) = g*(t)$ almost everywhere.

For example, when the inversion integrals (1.24) and (1.25) are applied to the same function f, they will give the same function g almost everywhere. Any other inversion integral of (1.23) will also give the same function g almost everywhere, provided, of course, it can be applied to the function f. In the application of (1.24) and (1.25) we require that $f(1) = f'(1) = 0$. Other inversion integrals may require a greater or lesser number of derivatives to be zero at 1. Thus if any two inversion integrals can be applied to the same function, the resulting functions will be equal almost everywhere.

A special case of (4.1) is Abel's integral equation,

$$\int_0^x (x-t)^{\alpha} g(t)dt = f(x) \quad (0<\alpha<1). \qquad (4.4)$$

This equation can be transformed into (1.1); however, it has a better appearance in the form (1.4). If $x = \ln y$ and $t = \ln v$ are substituted into (4.4), it becomes

$$\int_1^y (\ln y/v)^{-\alpha} g(\ln v)\, v^{-1} dv = f(\ln y). \qquad (4.5)$$

The solution (16, p. 331-332) of (4.4),

$$g(t) = \Pi^{-1} \sin\Pi\alpha\; d/dt \int_0^t (t-y)^{\alpha-1} f(y)dy,$$

becomes

$$g(\ln v) = \Pi^{-1} \sin\Pi\alpha\; v d/dv \int_1^v (\ln v/x)^{\alpha-1} f(\ln x)\, x^{-1} dx. \qquad (4.6)$$

We rewrite (4.5) in the form

$$\int_1^x (\ln x/t)^{\beta}\, h(t)dt = k(x) \qquad (4.7)$$

and apply the method given in the introduction. The method suggests the solution

$$h(t) = \int_1^t [\Gamma(\beta+1)\Gamma(j-\beta)]^{-1} (y/t)(\ln t/y)^{j-\beta-1}$$

$$\times y^{-1} (d/dy\; y)^{j+1} [y^{-1} k(y)] dy, \qquad (4.8)$$

where $j - \beta > 0$. This solution is not as general as (4.6) since in general we must restrict some of the derivatives of k to be zero at 1. However, since $-1 < \beta < 0$ implies j may be selected so that $j = 0$, the solution (4.8) may be written in a form close to (4.6),

$$h(t) = -(\Pi t)^{-1} \sin \Pi\beta \int_1^t (\ln t/y)^{-\beta-1} d(k(y)),$$

where only the restriction $k(0) = 0$ is used.

CHAPTER V

DUAL EQUATIONS

We consider in this chapter the dual equations

$$\int_0^\infty f(g)y^\alpha G_{pq}^{mn}\left(xy\left|\begin{matrix}a_1, & \dots, & a_p\\ b_1, & \dots, & b_q\end{matrix}\right.\right)dy = e(x) \quad (0<x<1), \qquad (5.1)$$

$$\int_0^\infty f(y)G_{pq}^{mn}\left(xy\left|\begin{matrix}a_1, & \dots, & a_p\\ b_1, & \dots, & b_q\end{matrix}\right.\right)dy = 0 \qquad (x>1) \qquad (5.2)$$

where m, n, p, and q are integers with $1\leq m\leq q$, $0\leq n\leq p$, $p<q$, and $2(m+n) = p+q$; and G_{pq}^{mn} is Meijer's G-function,

$$G_{pq}^{mn}\left(x\left|\begin{matrix}a_1, & \dots, & a_p\\ b_1, & \dots, & b_q\end{matrix}\right.\right) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Pi_{j=1}^{m}\Gamma(b_j-s)\Pi_{j=1}^{n}\Gamma(1-a_j+s)}{\Pi_{j=m+1}^{q}\Gamma(1-b_j+s)\Pi_{j=n+1}^{p}\Gamma(a_j+s)}$$

$$\times x^s ds,$$

where $\mathrm{Re}\, b_j > \gamma > -1+\mathrm{Re}\, a_k$, $j=1, \dots, m$; $k=1, \dots, n$. Our solution has the same form as I. W. Busbridge's solution (2) of the dual equations

$$\int_0^\infty f(y)y^\alpha J_\nu(xy)dy = e(x) \quad (0<x<1), \qquad (5.3)$$

$$\int_0^\infty f(y)J_\nu(xy)dy = 0 \qquad (x>1). \qquad (5.4)$$

The solution can be reduced to Busbridge's solution for the particular case of Bessel functions. Unfortunately, the solution contains only a few other obvious special cases.

These are J_ν^2, G. N. Watson's function $\omega_{\mu\nu}$ (18), and a linear combination of exponential, sine, and cosine functions; these are new. The solution's usefulness is to show that there exist solutions to dual equations with kernels other than Bessel functions of the first kind, J_ν. Except for some trigonometric kernels (15), which can actually be considered as a special case of Bessel functions, the only dual equations considered in the literature involve the Bessel function J_ν.

As we remarked in the introduction, we were led to consider these dual equations during the study of Mellin-Barnes integrals, in particular the integral

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Pi_{k=1}^{n} \frac{\Gamma(s+a_k)}{\Gamma(s+b_k)}\, x^{-s} ds. \qquad (5.5)$$

We have noted previously that the integral is vanishing for $x > 1$. If s is changed to $-s$ in the gamma functions, the integral is vanishing for $0<x<1$. It was with these facts in mind that we happened to look at E. C. Titchmarsh's formal solution (16, p. 337-339) of equations (5.3) and (5.4). To solve these equations, he applied Parseval's formula (16, p. 95, Theorem 72) for the Mellin transform to the left-hand sides and obtained

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\, 2^{-s+\alpha} x^{s-1-\alpha} ds = e(x) \quad (0<x<1),$$

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}2^{1-s}x^{s-2}ds = 0 \quad (x>1)$$

where F denotes the Mellin Transform of f. Putting

$$F(s) = 2^{s-\alpha}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\alpha-\frac{1}{2}s)}Y(s),$$

he obtained

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}Y(s)x^{s-1-\alpha}ds = e(x) \quad (0<x<1), \tag{5.6}$$

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}Y(s)x^{s-1}ds = 0 \quad (x>1). \tag{5.7}$$

We noticed a similarity between (5.5) and (5.6), especially in view of the use which we made of the multiplication formula of Gauss and Legendre in Chapter 3. Thus we contemplated obtaining products of gamma functions in equations similar to (5.6) and (5.7). And considering Meijer's G-function, we felt that such equations were obtainable. Therefore, the same procedure was applied to equations (5.1) and (5.2), and the result then led to further research.

We apply the formal manipulations given by E. C. Titchmarsh to (5.1) and (5.2). These manipulations are fairly short, and they give an outline of the proof to be given for Theorem 5. This proof is rather long and involved so that the outline may prove helpful to the reader. Since we obtain only a formal solution first, we will not worry about restrictions on the parameters of the G-function at this point. Further we let

$$\left.\begin{aligned} \Pi_{j=1}^{m}\Gamma(1+b_j-s) &= \Gamma_m(1+b-s), \\ \Pi_{j=m+1}^{q}\Gamma(-b_j+s) &= \Gamma_q(-b+s), \\ \Pi_{j=1}^{n}\Gamma(-a_j+s) &= \Gamma_n(-a+s), \\ \Pi_{j=n+1}^{p}\Gamma(1+a_j-s) &= \Gamma_p(1+a-s), \\ \Pi_{j=m+2}^{q}\Gamma(-b_j+s) &= \Gamma_q^*(-b+s), \end{aligned}\right\} \qquad (5.8)$$

in order to simplify the writing.

<u>Outline of Proof for Theorem 5</u>. If Parseval's formula is applied to the left-hand sides of equations (5.1) and (5.2), the result is

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\frac{\Gamma_m(1+\alpha+b-s)\Gamma_n(-\alpha-a+s)}{\Gamma_q(-\alpha-b+s)\Gamma_p(1+\alpha+a-s)}x^{s-\alpha-1}ds = e(x) \quad (0<x<1), \qquad (5.9)$$

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\frac{\Gamma_m(1+b-s)\Gamma_n(-a+s)}{\Gamma_q(-b+s)\Gamma_p(1+a-s)}x^{s-1}ds = 0 \qquad (x>1). \qquad (5.10)$$

The substitution

$$F(s) = \frac{\Gamma_q(-b+s)\Gamma_p(1+\alpha+a-s)}{\Gamma_m(1+\alpha+b-s)\Gamma_n(-a+s)}Y(s) \qquad (5.11)$$

into the last two equations gives

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} Y(s)\frac{\Gamma_q(-b+s)\Gamma_n(-\alpha-a+s)}{\Gamma_q(-\alpha-b+s)\Gamma_n(-a+s)}x^{s-\alpha-1}ds = e(x) \quad (0<x<1), \qquad (5.12)$$

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} Y(s)\frac{\Gamma_m(1+b-s)\Gamma_p(1+\alpha+a-s)}{\Gamma_m(1+\alpha+b-s)\Gamma_p(1+a-s)}x^{s-1}ds = 0 \quad (x>1). \qquad (5.13)$$

Multiplying (5.12) by $x^{\alpha-w}$, where $\mathrm{Re}(s-w) > 0$, and integrating over $(0,1)$, we obtain

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma_q(-b+s)\Gamma_n(-\alpha-a+s)}{\Gamma_q(-\alpha-b+s)\Gamma_n(-a+s)}\,\frac{Y(s)}{s-w}ds = \int_0^1 e(x)x^{\alpha-w}dx = E(\alpha-w+1)$$

$$(\mathrm{Re}\, w < k).$$

Moving the line of integration from $\mathrm{Re}\, s = k$ to $\mathrm{Re}\, s = k' < \mathrm{Re}\, w$ and assuming $\mathrm{Re}\, b_j < k'$, $j = m+1, \ldots, q$ and $\mathrm{Re}(a_j+\alpha) < k'$, $j=1, \ldots, n$, we obtain

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} \frac{\Gamma_q(-b+s)\Gamma_n(-\alpha-a+s)}{\Gamma_q(-\alpha-b+s)\Gamma_n(-a+s)}\,\frac{Y(s)}{s-w}ds = E(1+\alpha-w)$$

$$- \frac{\Gamma_q(-b+w)\Gamma_n(-\alpha-a+w)}{\Gamma_q(-\alpha-b+w)\Gamma_n(-a+w)}Y(w).$$

Now the integral occurring on the left-hand side of this equation is a regular function of w for $\operatorname{Re} w > k'$. Therefore so is the function on the right-hand side. Hence so also is

$$Y(w) - \frac{\Gamma_q(-\alpha-b+w)\Gamma_n(-a+w)}{\Gamma_q(-b+w)\Gamma_n(-\alpha-a+w)}E(1+\alpha-w).$$

If we assume suitable conditions at infinity, we have

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\left\{Y(s) - \frac{\Gamma_q(-\alpha-b+s)\Gamma_n(-a+s)}{\Gamma_q(-b+s)\Gamma_n(-\alpha-a+s)}E(1+\alpha-s)\right\}\frac{ds}{s-w} = 0 \quad (\operatorname{Re} w < k). \tag{5.14}$$

Similarly multiplying (5.13) by ρ^{-w}, $\operatorname{Re}(s-w) < 0$, and integrating over $(1,\infty)$, we obtain

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma_m(1+b-s)\Gamma_p(1+\alpha+a-s)}{\Gamma_m(1+\alpha+b-s)\Gamma_p(1+a-s)}\,\frac{Y(s)}{s-w}ds = 0 \quad (\operatorname{Re} w > k').$$

We conclude as before that

$$\frac{\Gamma_m(1+b-s)\Gamma_p(1+\alpha+a-s)}{\Gamma_m(1+\alpha+b-s)\Gamma_p(1+a-s)}Y(s),$$

and so Y, is regular for $\operatorname{Re} s < k$. Hence

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} \frac{Y(s)}{s-w}ds = 0 \quad (\operatorname{Re} w > k').$$

Moving the line of integration from $\operatorname{Re} s = k'$ to

Re s = k, we have

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k+i\infty} \frac{Y(s)}{s-w}ds = Y(w) \quad (\text{Re}\, w < k). \qquad (5.15)$$

It follows from (5.14) and (5.15) that

$$Y(s) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma_q(-\alpha-b+s)\Gamma_n(-a+s)}{\Gamma_q(-b+s)\Gamma_n(-\alpha-a+s)} \frac{E(1+\alpha-s)}{s-w}ds. \qquad (5.16)$$

If Mellin's inversion formula is applied to (5.11), then

$$f(y) = \frac{1}{2\pi i}\int_{k+i\infty}^{k+i\infty} \frac{\Gamma_q(-b+s)\Gamma_p(1+\alpha+a-s)}{\Gamma_m(1+\alpha+b-s)\Gamma_n(-a+s)} Y(s)y^{-s}ds. \qquad (5.17)$$

Equations (5.16) and (5.17) give a solution to (5.1) and (5.2).

Equations (5.16) and (5.17) can not in general be simplified. We may attempt a simplification by defining

$$\xi_{q-m+n}(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma_q(-\alpha-b+s)\Gamma_n(-a+s)}{\Gamma_q(-b+s)\Gamma_n(-\alpha-a+s)} x^{-s}ds$$

and

$$G_{p\ q}^{q-m\ p-n}\left(z\,\middle|\,\begin{matrix}-\alpha-a_{n+1}, \ldots, -\alpha-a_p, -a_1, \ldots, -a_n\\ -b_{m+1}, \ldots, -b_q, -\alpha-b_1, \ldots, -\alpha-b_m\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma_q(-b+s)\Gamma_p(1+\alpha+a-s)}{\Gamma_m(1+\alpha+b-s)\Gamma_n(-a+s)} z^{-s}ds;$$

however, the integral defining ξ_{q-m+n} will not be absolutely convergent in general. Thus when

$$E(1+\alpha-s) = \int_0^1 e(u)u^{\alpha-s}du$$

and

$$(s-w)^{-1} = \int_0^1 v^{s-w-1}dv$$

are substituted into (5.16), it may not be possible to justify the change in the order of integration. Nevertheless, if we do change the order of integration, we obtain

$$Y(w) = \int_0^1 e(u)u^{\alpha}du\int_0^1 v^{-w-1}\xi_{q-m+n}(u/v)dv.$$

From Chapter 3 we note that $\xi_{q-m+n}(u/v) = 0$ for $v < u$; thus we can write

$$Y(w) = \int_0^1 e(u)u^{\alpha}du\int_u^1 v^{-w-1}\xi_{q-m+n}(u/v)dv$$

$$= \int_0^1 v^{-w-1}dv\int_0^v e(u)u^{\alpha}\xi_{q-m+n}(u/v)du.$$

If this equation is substituted into (5.17) and the order of integration is changed, the result is

$$f(w) = \int_0^1 v^{-1}G_{p\ q}^{q-m\ p-n}(vy)dv\int_0^v e(u)u^{\alpha}\xi_{q-m+n}(u/v)du. \quad (5.18)$$

This would also be a solution if we could justify the changes in the order of integration.

We now make some preparation for the proof of Theorem 5, which is based on I.W. Busbridge's proof (2). Busbridge makes a transformation in order to simplify the analysis, and we will make a similar transformation. The large number of parameters in the G-function complicates the proof some, but for the main part the proof is similar.

We let

$$h(x) = x^{b_{m+1}} \int_0^x u^{-b_{m+1}} e(u)du$$

and transform (5.1) so that

$$h(x) = x^{b_{m+1}} \int_0^x u^{-b_{m+1}} du \int_0^\infty y^\alpha f(y) G_{pq}^{mn}(uy)dy.$$

If the G-function is expressed as a Mellin-Barnes integral, the order of integration is changed, and use is made of the recurrence relation for the gamma function, then

$$h(x) = \int_0^\infty y^{\alpha-1} f(y)dy \frac{1}{2\pi i} \int_{\gamma-1-i\infty}^{\gamma-1+i\infty} \frac{\Gamma_m(1+b-s)\Gamma_n(-a+s) \ (yx)^s ds}{\Gamma_q^*(-b+s)\Gamma_p(1+a-s)\Gamma(1-b_{m+1}s)}$$

where Γ_q^* is defined in (5.8). We again transform the last equation so that

$$x^{-b_{m+1}} h(x) = d/dx\{\int_0^x u^{-b_{m+1}} h(u)du\}$$

$$= d/dx\left\{x^{-b_{m+1}}\int_0^\infty y^{\alpha-2} f(y)\,dy\right.$$

$$\left.\times\frac{1}{2\pi i}\int_{\gamma-2-i\infty}^{\gamma-2+i\infty}\frac{\Gamma_m(2+b-s)\Gamma_n(-1-a+s)}{\Gamma_q^*(-1-b+s)\Gamma_p(2+a-s)}\frac{(yx)^s ds}{\Gamma(1-b_{m+1}+s)}\right\}. \tag{5.19}$$

Similarly for (5.2) we have

$$0 = d/dx\left\{\int_0^x u^{-b_{m+1}} du \int_0^\infty f(y) G_{pq}^{mn}(uy)\,dy\right\}$$

$$= d/dx\left\{x^{-b_{m+1}}\int_0^\infty y^{-1} f(y)\,dy\right.$$

$$\left.\times\frac{1}{2\pi i}\int_{\gamma-1-i\infty}^{\gamma-1-i\infty}\frac{\Gamma_m(1+b-s)\Gamma_n(-a+s)}{\Gamma_q^*(-b+s)\Gamma_p(1+a-s)}\frac{(yx)^s ds}{\Gamma(1-b_{m+1}+s)}\right\}. \tag{5.20}$$

If h is an integral over $(0,1)$, then its Mellin transform must be regular for $\operatorname{Re} s > 0$, and, on integrating by parts, we see that

$$|H(s)| = \left|\int_0^1 h(x)x^{s-1}dx\right|$$

$$= |s|^{-1}\left|h(1) - h(0) - \int_0^1 x^s h'(x)dx\right|$$

$$= |s|^{-1}\left|h(1) - E(s+1) - b_{m+1}(s+b_{m+1})^{-1}[E(1-b_{m+1}) - E(s+1)]\right|$$

$$= O(|I_m s|^{-1})$$

uniformly for $\operatorname{Re} s \geq \eta > 0$. The simple behavior of H at infinity and the recurrence relation for the Bessel function are apparently the reasons why Busbridge transformed (5.3) and (5.4).

We need the following lemma, and then we are ready to state and prove Theorem 5.

Lemma. Let Φ be an analytic function of $w = u+iv$, which is regular in a strip containing the line $u = c$; let

$$|\Phi(c+iv)| = O(|v|^{-\beta})$$

where $0 < \beta < 1$. Let

$$\Psi(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Phi(w)\frac{dw}{w-s} \quad (\mathrm{Re}\, s \neq c).$$

Then Ψ is regular for $\mathrm{Re}\, s \neq c$ and

$$|\Psi(s)| \leq A|\sigma-c|^{-\delta}|t|^{-\beta+\delta}$$

where $0 < \delta < \beta$ and A is a constant depending on δ. Also, for every $\epsilon > 0$

$$|\Psi(s)| = O(|t|^{-\beta+\epsilon})$$

uniformly for $|\mathrm{Re}\, s - c| \geq n > 0$.

Proof. See (2, p. 117).

Theorem 5. If h is an integral over $(0,1)$ and

$$H(s) = \int_0^1 x^{s-1}h(x)dx,$$

then the equations

$$d/dx\{x^{-b_{m+1}}\int_0^\infty y^{\alpha-2}f(y)$$

$$\times G_{pq}^{mn}\left(xy\middle|\begin{matrix}a_1+2, & \dots, & a_p+2 \\ b_1+2, & \dots, & b_m+2, b_{m+1}, b_{m+2}+2, & \dots, & b_q+2\end{matrix}\right)$$

$$\times dy\} = x^{-b_{m+1}}h(x) \quad (0<x<1), \qquad (5.21)$$

$$d/dx\{x^{-b_{m+1}}\int_0^\infty y^{-1}f(y)$$

$$\times G_{pq}^{mn}\left(xy\middle|\begin{matrix}a_1+1, & \dots, & a_p+1 \\ b_1+1, & \dots, & b_m+1, b_{m+1}, b_{m+2}+1, & \dots, & b_q+1\end{matrix}\right)dy\} = 0$$

$$(x>1) \qquad (5.22)$$

have one and only one solution, f, in the class of functions whose Mellin transforms are regular for $\max(a_1, b_{m+1}) < \sigma < d(q-p)^{-1} + \alpha + 1/2$ and $O(|t|^{(q-p)(\sigma-\alpha-\frac{1}{2})-d+\epsilon})$ for every $\epsilon > 0$, in any interior strip. It is given by

$$f(x) = \frac{x^{\frac{1}{2}-k}}{2\pi i} \underset{T\to\infty}{\mathrm{l.i.m.}} \int_{k-iT}^{k+iT} Y(s) \frac{\Gamma_q(-b+s)\Gamma_p(1+a+\alpha-s)}{\Gamma_n(-a+s)\Gamma_m(1+b+\alpha-s)} x^{k-\frac{1}{2}-s} ds, \qquad (5.23)$$

$$Y(s) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)} \frac{H(\alpha-w)}{w-s} dw \qquad (5.24)$$

where $\max(a_1, b_{m+1}) < k < (d-1/2)(q-p)^{-1} + \alpha + 1/2$,

$k < c' < d(q-p)^{-1} + \alpha + 1/2$, $1 \leq m \leq q$, $0 \leq n \leq p$, $2(m+n) = p+q$,

$0 < q-p$, $1 > \alpha > 0$, $1 > \frac{1}{2}(q-p)\alpha > 0$, $s = \sigma + it$,

$$\mathrm{Re}\, b_j \geq b_1 \geq 0,\ j=1, \ldots, m,$$

$$\mathrm{Re}\, b_j \leq b_{m+1} - \alpha,\ j=m+2, \ldots, q,\ b_{m+1} \leq 0,$$

$$\mathrm{Re}\, a_j \leq a_1,\ j=1, \ldots, n, \qquad a_1 \leq 0,$$

$$\mathrm{Re}\, a_j \geq b_1,\ j=n+1, \ldots, p,$$

$$d = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right),$$

where $-1/2 \geq d(q-p)^{-1} > -1/2 - \alpha$, if $\alpha > 1/2$ and $d(q-p)^{-1} = -1/2$ if $1/2 \geq \alpha > 0$.

Note 4. In the proof we make frequent use of asymptotic expansions for products of gamma functions without referring to them directly. These expansions can be found from (6) or (8, vol. 1, p. 49-50). For example, we find that

$$\left|\frac{\Gamma_m(a+s)}{\Gamma_m(b+s)}\right| \quad \text{or} \quad \left|\frac{\Gamma_m(a-s)}{\Gamma_m(b-s)}\right| = O(|t|^{d'}) \quad (|t| \to \infty)$$

where $d' = \sum_{j=1}^{m} (a_j - b_j)$. Likewise the product of gamma functions

$$\frac{\Gamma_m(b+s)\Gamma_n(1-a-s)}{\Gamma_q(1-b-s)\Gamma_p(a+s)},$$

which is the Mellin transform of the G-function, is

$$O(e^{-\frac{1}{2}\pi[2(m+n)-(q+p)]|t|}|t|^{(q-p)\gamma+d+\frac{1}{2}(p-q)}) \qquad (|t| \to \infty)$$

where d is defined in Theorem 5 and γ is real and related to the path of integration in the G-function. The restriction that $2(m+n) = p+q$ eliminates the exponential factor in the last order term. This elimination is needed in order to get convergence of both (5.23) and the G-function.

Proof. We follow Busbridge (2) closely in the proof.

The Necessity of the Form (5.23) for the Solution. We assume that equations (5.21) and (5.22) have a solution, f, whose Mellin transform, F, is regular for $\max(a_1, b_{m+1}) < \sigma < d(q-p)^{-1} + \alpha + 1/2$, and that

$$|F(s)| = O(|t|^{(q-p)(\sigma-\alpha-\frac{1}{2})-d+\epsilon})$$

uniformly in any interior strip. It follows from this, by a suitable choice of ϵ, that, if $\max(a_1, b_{m+1}) < k < (d-1/2)(q-p)^{-1} + \alpha + 1/2$, then $F(k+iT)$ belongs to $L^2(-\infty,\infty)$ and therefore that $x^{k-\frac{1}{2}}f(x)$ belongs to

$L^2(0,\infty)$. Now let

$$\max(a_1, b_{m+1}, (d-1/2)(q-p)^{-1}+\alpha) < k < (d-1/2)(q-p)^{-1}+\alpha+1/2;$$

then both $y^{k-\frac{1}{2}}f(y)$ and

$$y^{-k+\frac{1}{2}}y^{\alpha-2}G_{pq}^{mn}\left(xy\,\middle|\,\begin{matrix}a_1+2, \ldots, a_p+2 \\ b_1+2, \ldots, b_m+2, b_{m+1}, b_{m+2}+2, \ldots, b_q+2\end{matrix}\right)$$

belong to $L^2(0,\infty)$. We may therefore apply Parseval's theorem (16, Theorem 72) for functions of L^2 to the integral on the left-hand side of (5.21), and we have

$$\int_0^\infty G_{pq}^{mn}\left(xy\,\middle|\,\begin{matrix}a_1+2, \ldots, a_p+2 \\ b_1+2, \ldots, b_m+2, b_{m+1}, b_{m+2}+2, \ldots, b_q+2\end{matrix}\right) y^{\alpha-2}f(y)dy$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\frac{\Gamma_m(1+b+\alpha-s)\Gamma_n(-a-\alpha+s)}{\Gamma_q^*(-b-\alpha+s)\Gamma_p(1+a+\alpha-s)}\,\frac{x^{s-\alpha+1}}{\Gamma(2-b_{m+1}-\alpha+s)}ds, \tag{5.25}$$

and therefore

$$x^{-b_{m+1}}h(x) = d/dx\left\{\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\right.$$

$$\left.\times F(s)\frac{\Gamma_m(1+b+\alpha-s)\Gamma_n(-a-\alpha+s)}{\Gamma_q^*(-b-\alpha+s)\Gamma_p(1+a+\alpha-s)}\,\frac{x^{s-\alpha-b_{m+1}+1}}{\Gamma(2-b_{m+1}-\alpha+s)}ds\right\}. \tag{5.26}$$

The integrand is regular for $\max(a_1, b_{m+1}) < \sigma < d(q-p)^{-1}$

$+\alpha+1/2$, and it is $O(|t|^{-2+\epsilon})$, so we may take $k = (d-1/2)(q-p)^{-1}+\alpha+1/2$. Let $\Phi(x)$ denote the expression in brackets so that $\Phi'(x) = x^{-b_{m+1}}h(x)$ $(0<x<1)$. Let $\mathrm{Re}\, w < (d-1/2)(q-p)^{-1}+\alpha+1/2 \leq \alpha$. Then

$$H(\alpha-w) = \int_0^1 h(x)x^{\alpha-w-1}dx$$

$$= \int_0^1 \Phi'(x)x^{\alpha-w+b_{m+1}-1}dx$$

$$= \Phi(1) + (w-\alpha-b_{m+1}+1)\int_0^1 \Phi(x)x^{\alpha-w+b_{m+1}-2}dx,$$

since $\lim_{x\to 0} x^{\alpha+b_{m+1}-\mathrm{Re}\, w-1}\Phi(x) = 0$. Hence substituting for $\Phi(x)$, inverting the order of integration, integrating and collecting the expressions together, we obtain

$$H(\alpha-w) = \frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} F(s)\frac{\Gamma_m(1+b+\alpha-s)\Gamma_n(-a-\alpha+s)}{\Gamma_q^*(-b-\alpha+s)\Gamma_p(1+a+\alpha-s)}$$

$$\times[\Gamma(1-b_{m+1}-\alpha+s)]^{-1}(s-w)^{-1}ds \quad (\mathrm{Re}\, w < k') \qquad (5.27)$$

where the change in order of integration is justified by absolute convergence.

Equation (5.22) may be reduced in the same way to a form similar to (5.21). We find that Parseval's theorem for function of L^2 may be used if

$\max[a_1, b_{m+1}, (d-(\alpha+1)/2)(q-p)^{-1}+\alpha+1/2]<k<(d-1/2)(q-p)^{-1}+\alpha+\frac{1}{2},$

and we have

$$0 = d/dx\left\{\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\frac{\Gamma_m(1+b-s)\Gamma_n(-a+s)}{\Gamma_q^*(-b+s)\Gamma_p(1+a-s)}\,\frac{x^{s-b_{m+1}}\,ds}{\Gamma(1-b_{m+1}+s)}\right\}.$$

The integrand is regular for $\max(a_1, b_{m+1}) < \sigma <$ $\min(b_1+1), d(q-p)^{-1}+\alpha+1/2)$, and it is $O(|t|^{(p-q)\alpha-1+\epsilon})$, so that we may again take $k = k' = (d-1/2)(q-p)^{-1}+\alpha+1/2$. Multiplying by $x^{-w+b_{m+1}}$ where $\mathrm{Re}\, w > k'$ and integrating over $(1,\infty)$, we obtain as before

$$0 = \frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} F(s)\frac{\Gamma_m(1+b-s)\Gamma_n(-a+s)}{\Gamma_q(-b+s)\Gamma_p(1+a-s)}\,\frac{ds}{s-w}\ (\mathrm{Re}\, w>k'). \tag{5.28}$$

We write, as in (5.11),

$$Y(s) = F(s)\frac{\Gamma_n(-a+s)\Gamma_m(1+b+\alpha-s)}{\Gamma_q(-b+s)\Gamma_p(1+a+\alpha-s)} \tag{5.29}$$

so that Y is regular for $\max(a_1, b_{m+1})<\sigma<d(q-p)^{-1}+\alpha+1/2$; moreover, it is $O(|t|^{\frac{1}{2}(p-q)\alpha+\epsilon})$ uniformly in any interior strip. Then equations (5.27) and (5.28) become

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} Y(s)\frac{\Gamma_q(-b+s)\Gamma_n(-a-\alpha+s)}{\Gamma_q^*(-b-\alpha+s)\Gamma_n(-a+s)\Gamma(1-b_{m+1}-\alpha+s)}\frac{ds}{s-w} = H(\alpha-w)$$

$$(\mathrm{Re}\, w < k'), \quad (5.30)$$

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} Y(s)\frac{\Gamma_m(1+b-s)\Gamma_p(1+a+\alpha-s)}{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-s)}\frac{ds}{s-w} = 0 \quad (\mathrm{Re}\, w > k').$$

$$(5.31)$$

Now let

$$\max(a_1, b_{m+1}) < c < k' < c' < \min(b_1+1, d(q-p)^{-1}+\alpha+1/2),$$

and let $c < \mathrm{Re}\, w < k'$ in (5.30). Since the integrand is regular in the strip $\max(a_1, b_{m+1}) < \sigma < d(q-p)^{-1}+\alpha+1/2$, except for a simple pole at $s = w$, and since it is $O(|t|^{-2+\epsilon})$, we may move the line of integration to $\sigma = c$; this move gives (by Cauchy's theorem)

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(s)\frac{\Gamma_q(-b+s)\Gamma_n(-a-\alpha+s)}{\Gamma_q^*(-b-\alpha+s)\Gamma_n(-a+s)}\,\frac{}{\Gamma(1-b_{m+1}-\alpha+w)}\,\frac{ds}{s-w}$$

$$= H(\alpha-w) - Y(w)\frac{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a-w)\Gamma(1-b_{m+1}-\alpha+w)}.$$

$$(5.32)$$

Since the left-hand side is regular for $\mathrm{Re}\, w > c$, so is the right-hand side, and since

$$\frac{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}$$

is regular for $\operatorname{Re} w > \max(a_1, b_{m+1})$, and therefore for $\operatorname{Re} w > 0$, it follows, by integrating around a large rectangle to the right of $\operatorname{Re} w = c'$, that, if $\operatorname{Re} z < c'$, (compare to (5.14))

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\left\{\frac{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}H(\alpha-w)-Y(w)\right\}$$

$$\times ds/w-z = 0, \tag{5.33}$$

provided the integrand is of a suitable order at infinity. To prove this fact, we use the lemma with

$$\Phi(w) = Y(w)\frac{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha+w)}.$$

This is $O(|v|^{-1+\epsilon})$ (where $w = u+iv$) when $\operatorname{Re} w = c$; hence, it follows from the lemma that the right-hand side of (5.32) is less than

$$|A|\,|u-c|^{-\delta}|v|^{-1+\epsilon+\delta} \quad (0<\delta<1-\epsilon)$$

where A depends on δ. Since

$$\left|\frac{\Gamma_q^*(-b-\alpha+w)\ \Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha-w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}\right| = O(|w|^{1+\frac{1}{2}(p-q)\alpha})$$

uniformly for $|\arg w| \leq \frac{1}{2}\pi$, the integrand in (5.33) is less than

$$A|u-c|^{-\delta}|v|^{-1+\epsilon+\delta}(u^2+v^2)^{\frac{1}{4}(p-q)\alpha}$$

where A may now depend on z and δ. If we integrate the function around the rectangle whose corners are the points $c'-iT$, $T-iT$, $T+iT$, and $c'+iT$ where $T > 1$, we have

$$\left|\int_{T-iT}^{T+iT}\right| \leq \frac{2A}{(T-c)^{\delta}} T^{\frac{1}{2}(p-q)\alpha} \int_0^T \frac{dv}{v^{1-\epsilon-\delta}} = O(T^{\frac{1}{2}(p-q)\alpha+\epsilon}),$$

$$\left|\int_{c'+iT}^{T+iT}\right| \leq AT^{-1+\frac{1}{2}(p-q)\alpha+\delta-1} \int_{c'}^{T} \frac{du}{(u-c)^{\delta}} = O(T^{\frac{1}{2}(p-q)\alpha+\epsilon}).$$

Similarly

$$\left|\int_{c'-iT}^{T+iT}\right| = O(T^{\frac{1}{2}(p-q)\alpha+\epsilon}).$$

Since $1/2(p-q)\alpha < 0$, ϵ may always be chosen so that all these integrals tend to zero as $T \to \infty$, and (5.33) follows.

Now consider (5.31) when $k' < \operatorname{Re} w < c'$. We transform the equation in a manner similar to that given for (5.30). The integrand is regular in the strip

$\max(a_1, b_{m+1}) < \sigma < \min(b_1+1, d(q-p)^{-1} + \alpha + 1/2)$, except for a simple pole at $s = w$, and it is $O(|t|^{-1+(p-q)\alpha+\epsilon})$. Hence moving the line of integration to $\operatorname{Re} s = c'$, we have

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} Y(s)\frac{\Gamma_m(1+b-s)\Gamma_p(1+a+\alpha-s)}{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-s)}\frac{ds}{s-w} = Y(w)\frac{\Gamma_m(1+b-w)\Gamma_p(1+a+\alpha-w)}{\Gamma_m(1+b+\alpha-w)\Gamma_p(1+a-w)}. \tag{5.34}$$

The left-hand side of this equation is regular for $\operatorname{Re} w < c'$; hence, so is the right-hand, and since

$$\frac{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-w)}{\Gamma_m(1+b-w)\Gamma_p(1+a+\alpha-w)}$$

is regular for $\operatorname{Re} w < 1$, it follows that Y must be regular for $\operatorname{Re} w < c'$ and, therefore, for $\operatorname{Re} w < d(q-p)^{-1}+\alpha+1/2$. Now

$$\left|Y(s)\frac{\Gamma_m(1+b-s)\Gamma_p(1+a+\alpha-s)}{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-s)}\right| = \begin{cases} O(|t|^{(p-q)\alpha+\epsilon}) & (q-p)\alpha<1 \\ O(|t|^{-1+\epsilon}) & (q-p)\alpha\geq 1 \end{cases}$$

on the line $\operatorname{Re} s = c'$. Hence it follows from the lemma with $\beta = \gamma-\epsilon$, where $\gamma = \min((q-p)\alpha, 1)$, that the right-hand side of (5.34) is less than

$$A|u-c'|^{-\delta}|v|^{-\gamma+\epsilon+\delta} \quad (0<\delta<\gamma-\epsilon).$$

Since

$$\left|\frac{\Gamma_m(1+b+\alpha-w)\Gamma_p(1+a-w)}{\Gamma_m(1+b-w)\Gamma_p(1+a+\alpha-w)}\right| = O(|w|^{-\frac{1}{2}(p-q)\alpha})$$

uniformly for $\frac{1}{2}\pi \leq \arg w \leq \frac{3}{2}\pi$, we have

$$\left|\frac{Y(w)}{w-z}\right| \leq A|u-c'|^{-\delta}|v|^{-\gamma+\epsilon+\delta}(u^2+v^2)^{-\frac{1}{2}-\frac{1}{4}(p-q)\alpha}$$

when $u = \operatorname{Re} w < \operatorname{Re} z$ (A depending on z and δ). Let $\operatorname{Re} z > c$; then integrating $Y(w)/w-z$ round the rectangle whose corners are the points $c-iT$, $c+iT$, $-T-iT$, $-T-iT$ ($T>|\operatorname{Im} z|$), we find that

$$\left|\int_{c+iT}^{-T+iT}\right|, \left|\int_{-T+iT}^{-T-iT}\right|, \left|\int_{-T-iT}^{c-iT}\right|$$

are all $O(T^{-(\gamma+\frac{1}{2}(p-q)\alpha-\epsilon)})$, and therefore, since $\gamma > \frac{1}{2}(p-q)\alpha$,

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} Y(w)\frac{dw}{w-z} = 0 \quad (\operatorname{Re} z > c). \tag{5.35}$$

Finally let $c < \operatorname{Re} z < c'$, and let us move the line of integration in (5.35) to $\operatorname{Re} w = c'$. Then we have, as in (5.15),

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} Y(w)\frac{dw}{w-z} = Y(z); \tag{5.36}$$

the move is justified by the fact that $|Y(w)|$ is

$O(|v|^{(p-q)\alpha+\epsilon})$ for $\text{Re}\, w < d(q-p)^{-1} + \alpha + 1/2$. Substituting from (5.36) in (5.33), we have the form similar to (5.16)

$$Y(z) = \int_{c'-\infty}^{c'+i\infty} \frac{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)} \frac{H(\alpha-w)}{w-z} dw,$$

which is equation (5.24). This equation will be true for $\text{Re}\, z < c'$. Equation (5.23) now follows from (5.29) by the theory of Mellin transform for functions of L^2.

<u>The Sufficiency of the Condition</u>. We first show that, if f is defined by (5.23) and (5.24), then its Mellin transform, F, satisfies the stated condition.

Consider (5.24) when $\text{Re}\, w = c'$; $H(\alpha-w)$ is $O(|v|^{-1})$ since $c' < d(q-p)^{-1} + \alpha + 1/2 \leq \alpha$, and

$$\left|\frac{\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)\Gamma(1-b_{m+1}-\alpha+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}\right| = O(|v|^{1+\frac{1}{2}(p-q)\alpha}).$$

It therefore follows from the lemma with $\beta = -1/2(p-q)\alpha$ that Y is regular for $\text{Re}\, s < c'$ and that

$$|Y(s)| = O(|t|^{\frac{1}{2}(p-q)\alpha+\epsilon})$$

uniformly for $\text{Re}\, s < c'-\eta$. Since c' may be taken as near to $d(q-p)^{-1} + \alpha + 1/2$ as we please, it follows that Y is regular for $\text{Re}\, s < d(q-p)^{-1} + \alpha + 1/2$. Hence

$$F(s) = Y(s)\frac{\Gamma_q(-b+s)\Gamma_p(1+\alpha+a-s)}{\Gamma_n(-a+s)\Gamma_m(1+b+\alpha-s)} \tag{5.37}$$

is regular for $\max(a_1, b_{m+1}) < \operatorname{Re} s < d(q-p)^{-1} + \alpha + 1/2$, and, using the asymptotic expansion for the gamma function (see Note 4), we see that

$$|F(s)| = O(|t|^{(q-p)(\sigma-\alpha-\frac{1}{2})-d+\epsilon})$$

uniformly in any interior strip.

We now show that the function f satisfies (5.21) and (5.22). The left-hand side of (5.21), when the value of f from (5.23) is substituted, may be reduced to the right-hand side of (5.26) where $F(s)$ has the value (5.37), that is,

$$d/dx\{x^{-b_{m+1}}\int_0^\infty y^{\alpha-2} f(y)$$

$$\times G_{pq}^{mn}\left(xy\middle|\begin{matrix} a_1+2, & \ldots, & a_p+2 \\ b_1+2, & \ldots, & b_m+2, b_{m+1}\, b_{m+2}+2, & \ldots, & b_q+2\end{matrix}\right)dy\}$$

$$= d/dx\{\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} Y(s)\frac{\Gamma_q(-b+s)\Gamma_p(-a-\alpha+s)}{\Gamma_q^*(-b-\alpha+s)\Gamma_p(-a+s)}\,\frac{x^{s-\alpha-b_{m+1}+1}}{\Gamma(1-b_{m+1}-\alpha+s)}ds\}$$

If we substitute for Y from (5.24), the right-hand side of this last equation becomes

$$d/dx\{\frac{-1}{4\pi^2}\int_{k'-i\infty}^{k'+i\infty}\frac{\Gamma_q(-b+s)\Gamma_n(-a-\alpha+s)\ x^{s-\alpha-b_{m+1}+1}}{\Gamma_q^*(-b-\alpha+s)\Gamma_p(-a+s)\ \Gamma(2-b_{m+1}-\alpha+s)}ds$$

$$\times\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1-b_{m+1}-\alpha+w)\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}\ \frac{H(\alpha-w)}{w-s}dw\}$$

$$= d/dx\{\frac{-1}{4\pi^2}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1-b_{m+1}-\alpha+w)\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}H(\alpha-w)dw$$

$$\times\int_{k'-i\infty}^{k'+i\infty}\frac{\Gamma_q(-b+s)\Gamma_n(-a-\alpha+s)\ x^{s-\alpha-b_{m+1}+1}}{\Gamma_q^*(-b-\alpha+s)\Gamma_p(-a+s)\Gamma(2-b_{m+1}-\alpha+s)}\ \frac{ds}{w-s}\ , \qquad (5.38)$$

provided the inversion of the order of integration is justified. Since $c' > k'$, the value of the inner integral, when $0<x<1$, is

$$-2\pi i\ \frac{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}{\Gamma_q^*(-b-\alpha+w)\Gamma_p(-a+w)}\ \frac{x^{s-\alpha-b_{m+1}+1}}{\Gamma(2-b_{m+1}-\alpha+s)},$$

and we have

$$d/dx\{x^{-b_{m+1}}\int_0^\infty y^{\alpha-2}f(y)G_{pq}^{mn}(xy)dy\}$$

$$= d/dx\{\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{H(\alpha-w)}{w-\alpha-b_{m+1}+1}\ x^{s-\alpha-b_{m+1}+1}dw\} = x^{-b_{m+1}}h(x)$$

when $0<x<1$. The last step follows from the theory of Mellin transforms for functions of L^2, since H belongs to $L^2(-\infty,\infty)$ when $\operatorname{Re} s > 0$. The inversion of the order of integration in (5.37) is justified by absolute convergence. Since by an extension of the theorem of the arithmetic and geometric means, given δ such that $0<\delta<1$, a constant k depending on δ exists such that

$$[(\sigma-c)^2+(v-t)^2]^{1/2} \geq k|\sigma-c|^{\delta}|v-t|^{1-\delta}$$

for all σ, v, and t, the modulus of the integrand is less than a constant multiple

$$|v|^{\frac{1}{2}(p-q)\alpha}|t|^{\frac{1}{2}(q-p)\alpha-2}(c'-k')^{-\delta}|v-t|^{-1+\delta}x^{k'-\alpha-b_{m+1}+1}$$

when v and t are large. The substitution $t = v\xi$ shows that the double integral is absolutely convergent. The proof that f satisfies (5.21) is complete.

The proof that f satisfies (5.22) is similar. By Parseval's theorem we have

$$d/dx\{x^{-b_{m+1}}\int_0^\infty y^{-1}f(y)$$

$$\times G_{pq}^{mn}\left(xy\middle|\begin{matrix}a_1+1, & \ldots, & a_p+1 \\ b_1+1, & \ldots, & b_m+1, b_{m+1}, b_{m+2}+1, & \ldots, & b_q+1\end{matrix}\right)dy\}$$

$$= d/dx\{\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty} Y(s)\frac{\Gamma_m(1+b-s)\Gamma_p(1+a+\alpha-s)}{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-s)}x^{s-b_{m+1}}\frac{ds}{s-b_{m+1}}\}$$

$$= d/dx\Big\{-\frac{1}{4\pi^2}\int_{k'-i\infty}^{k'+i\infty}\frac{\Gamma_m(1+b-s)\Gamma_p(1+a+\alpha-s)}{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-s)}x^{s-b_{m+1}}\frac{ds}{s-b_{m+1}}$$

$$\times\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1-b_{m+1}-\alpha+w)\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}\frac{H(\alpha-w)}{w-s}dw\Big\}$$

$$= d/dx\Big\{-\frac{1}{4\pi^2}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1-b_{m+1}-\alpha+w)\Gamma_q^*(-b-\alpha+w)\Gamma_n(-a+w)}{\Gamma_q(-b+w)\Gamma_n(-a-\alpha+w)}H(\alpha-w)dw$$

$$\times\int_{k'-i\infty}^{k'+i\infty}\frac{\Gamma_m(1+b-s)\Gamma_p(1+a+\alpha-s)}{\Gamma_m(1+b+\alpha-s)\Gamma_p(1+a-s)\,s-b_{m+1}}x^{s-b_{m+1}}\frac{ds}{w-s}\Big\}. \tag{5.39}$$

The inversion of the order of integration is again justified by absolute convergence. The value of the inner integral when $x > 1$ is

$$2\pi i\frac{\Gamma_m(1+b-b_{m+1})\Gamma_p(1+a+\alpha-b_{m+1})}{\Gamma_m(1+b+\alpha-b_{m+1})\Gamma_p(1+a-b_{m+1})}\;\frac{1}{w-b_{m+1}}$$

from the residue at $s = b_{m+1}$. Since this is independent of x, the last expression in (5.39) is zero when $x > 1$, and f satisfies (5.22). The proof is complete.

We now specialize the parameters to obtain some special cases. From (8, vol. 1, p. 216;3) we have

$$G^{10}_{02}(x|a,b) = x^{\frac{1}{2}(a+b)} J_{a-b}(2x^{\frac{1}{2}}).$$

We set $b = -1-a$ and let $a > 0$ in order to satisfy the conditions of Theorem 5. Thus equations (5.1), (5.2), (5.21), (5.22), (5.23) and (5.24) become respectively

$$\int_0^\infty f(y)y^\alpha (xy)^{-\frac{1}{2}} J_{2a+1}(2(xy)^{\frac{1}{2}})dy = e(x) \qquad (0<x<1),$$

$$\int_0^\infty f(y)(xy)^{-\frac{1}{2}} J_{2a+1}(2(xy)^{\frac{1}{2}})dy = 0 \qquad (x>1),$$

$$d/dx\{x^{1+a}\int_0^\infty y^{\alpha-2} f(y)(xy)^{\frac{1}{2}} J_{2a+3}(2(xy)^{\frac{1}{2}})dy\} = x^{1+a}h(x) \qquad (0<x<1), \quad (5.40)$$

$$d/dx\{x^{1+a}\int_0^\infty y^{-1} f(y) J_{2a+2}(2(xy)^{\frac{1}{2}})dy\} = 0 \quad (x>1), \qquad (5.41)$$

$$f(x) = \frac{x^{\frac{1}{2}-k}}{2\pi i} \underset{T\to\infty}{\mathrm{l.i.m.}} \int_{k-iT}^{k+iT} Y(s)\frac{\Gamma(1+a+s)}{\Gamma(1+a+\alpha-s)} x^{k-\frac{1}{2}-s} ds, \qquad (5.42)$$

$$Y(s) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma(2+a-\alpha+w)}{\Gamma(1+a+w)} \frac{H(\alpha-w)}{w-s} dw . \qquad (5.43)$$

If $a = \nu/2 - 1$, $x = u^2$, and $y = (v/2)^2$ are substituted in (5.40) and (5.41), they become respectively

$$d/du\{u^{\nu+1}\int_0^\infty v^{2\alpha-2} 2^{-2\alpha+1} f((v/2)^2) J_{\nu+1}(uv)dv\} = 2u^{\nu+1}h(u^2),$$

$$d/du\{u^{\nu}\int_0^{\infty} v^{-1} 2f((v/2)^2) J_{\nu}(uv)dv\} = 0.$$

If $2^{-2\alpha+2} f((v/2)^2) = \overline{f}(v)$, $2h(u^2) = \overline{h}(u)$, and $2\alpha = \alpha'$, we get the equations considered by Busbridge (2), that is,

$$d/du\{u^{\nu+1}\int_0^{\infty} J_{\nu+1}(uv)v^{\alpha'-2}\overline{f}(v)dv\} = u^{\nu+1}\overline{h}(u) \quad (0<u<1),$$

$$d/du\{u^{\nu}\int_0^{\infty} J_{\nu}(uv)v^{-1}\overline{f}(v)dv\} = 0 \qquad (u>1).$$

If $a = \nu/2-1$, $x = (v/2)^2$, $s = s'/2$, $2\alpha = \alpha'$, $2k = k'$, and $Y(s'/2) = X(s')$ are substituted into (5.42), and the result is multiplied by $2^{-2\alpha+2}$ and simplified, then

$$\overline{f}(v) = 2^{-2\alpha+2} f((v/2)^2) = v^{\frac{1}{2}-k} \underset{T\to\infty}{\text{l.i.m.}} \int_{k'-i\infty}^{k'+i\infty} X(s')$$

$$\times 2^{s'-\alpha+1} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s')}{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\alpha'-\frac{1}{2}s')} x^{k'-\frac{1}{2}-s'} ds',$$

which is part of the solution given by Busbridge. Finally if $a = \nu/2-1$, $s = s'/2$, $w = w'/2$, $Y(s'/2) = X(s')$, $2\alpha = \alpha'$, and $H(\alpha-w) = \overline{H}(\alpha'-w')$ are substituted in (5.43), then

$$X(s') = \frac{1}{2\pi i}\int_{2c'-i\infty}^{2c'+i\infty} \frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w')}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w')} \overline{H}(\alpha'-w') \frac{dw'}{w'-s'} \quad (k'<2c'<\alpha'),$$

which is the remaining part of Busbridge's solution.

A second example is the function $\tilde{\omega}_{\mu\nu}$ which G. N. Watson (18) defines. From (12) we have

$$\frac{1}{2}x^{1/2}G^{20}_{04}(x^2/16|\mu/2,\nu/2,-\mu/2,-\nu/2) = \tilde{\omega}_{\mu\nu}(x)$$

and in conjunction with (8, vol. 1, p. 209;8)

$$G^{20}_{04}(x|\mu/2+\sigma,\nu/2+\sigma,-\mu/2+\sigma,-\nu/2+\sigma) = 4x^{1/4+\sigma}\tilde{\omega}_{\mu\nu}(4x^{1/2}).$$

If $\sigma = -1/2$, $\mu > 0$, $\nu > 0$, and $\nu \geq \mu + \alpha$ or $\mu \geq \nu + \alpha$ (since $\tilde{\omega}_{\mu\nu}$ is a symmetric function in μ and ν), then $\tilde{\omega}_{\mu\nu}$ is a special case of Theorem 5.

A third example can be formed from (see (12))

$$\frac{1}{4}x^{3/2}G^{20}_{04}((x/4)^4|1/8, 3/8, -1/8, -3/8) = \Pi^{-1/2}[e^{-x} - \cos x + \sin x].$$

Again from (8, vol. 1, p. 209;8) we have

$$G^{20}_{04}(x|1/8+\sigma,3/8+\sigma,-1/8+\sigma,-3/8+\sigma)$$

$$= 4\pi^{-1/2}x^{\sigma}(4x)^{-3/8}[e^{-(4x)^{1/4}} - \cos(4x)^{1/4} + \sin(4x)^{1/4}].$$

If $\sigma = -1/2$ and $0<\alpha<1/4$, then the above function is a special case of Theorem 5.

A fourth example can be found from (8, vol. 1, p. 217;20 and p. 209;8), namely,

$$G^{1\,1}_{1\,3}\left(x\,\middle|\,\begin{matrix}\tfrac{1}{2}+\sigma \\ a+\sigma,0+\sigma,-a+\sigma\end{matrix}\right) = \pi^{\frac{1}{2}}x^{\sigma}J^{2}_{a}(x^{\frac{1}{2}}).$$

The conditions for application of Theorem 5 are $\sigma = -1/2$, $1/2<\alpha<1$, and $a>1/2+\alpha$.

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