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The two dimensional wavemaker problem on a finite domain is derived for nonlinear waves. A numerical method based on the method of lines is developed and applied to two test problems, the nonlinear surface pressure distribution problem and the nonlinear full-flap wavemaker problem. The solutions yield information about the fluid motion and the surface wave at all positions in the domain of the wavetank.

The Nonlinear Wavemaker Problem

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THE NONLINEAR WAVEMAKER PROBLEM

INTRODUCTION

The numerical solution to the two dimensional nonlinear wavemaker problem is needed to determine the waveform on a finite length tank throughout time. No analytical solution exists, due to the nonlinear surface boundary conditions. The method of lines is applied first to a test problem where numerical results are already known, so results from the program used in this paper can be compared with the previously known results to insure accuracy in the program. The nonlinear surface pressure distribution problem is the test problem. The method of lines is then applied to the nonlinear full-flap wavemaker problem.

The method of lines involves discretizing the partial differential equations (PDE's) governing the system, except in one dependent variable, chosen here to be the y variable. To save computer time, the method of lines is applied only to a narrow grid covering the free surface. Successive over-relaxation (SOR) is used to solve Laplace's equation on a coarser grid below and slightly overlapping the surface grid.

The method of lines involves two main parts: the solution of a nonlinear ordinary differential equation (ODE) and the solution of a nonlinear 2×2 system of simultaneous equations. Both have a high arithmetic operation count, and require a large amount of computer memory for data storage. The SOR method has the same drawbacks; however, both of these methods have the advantage that they are easily extended to three dimensions compared to other methods.

In Section 1 the nonlinear equations governing the fluid in an arbitrary wave tank are derived, based on the three assumptions that the fluid is incompressible, irrotational, and inviscid. The classical linear equations are derived as well. The sample problems used are defined in Sections 2 and 3. The numerical method applied is that used by Samuel Ohring in his solution to the first problem; this method is described in Section 4. A discussion of the difficulties encountered in the numerical method follows in Section 5. The results and conclusion follow.

1. THE EQUATIONS

I. Basic Assumptions

To predict a real life phenomena through mathematics, a mathematical model must first be made. The accuracy of the computed results depend on how accurately the assumptions made in the model correspond to what happens in the real situation. In this section the basic model for a generic wavemaker is derived. For the general wave theory used in this paper, the following assumptions are made: the fluid is 1. incompressible, 2. inviscid, and 3. irrotational. The first assumption, that the fluid is incompressible, means the density, ρ, remains constant throughout the fluid.

Draw a closed surface S, fixed in space, and let the fluid flow through it. The total flow over S must vanish, the inflow balanced by outflow. If we let $\vec{q}$ be the fluid velocity and $\vec{n}$ be the unit outward normal to S, then $\vec{q} \bullet \vec{n}$ is the component of velocity normal to S.

Over a small piece of the surface, dS, the outflow of fluid in time dt is

(1) $$\text{(area)(time)(velocity)} = dS\ dt\ (\vec{q} \bullet \vec{n}).$$

For inflow, the velocity changes sign. If we completely cover S by small pieces dS, and then added up the flow through each one, we would get

(2) $$\sum_{i=0}^{n} (\vec{q} \bullet \vec{n})_i\ dt\ dS_i = 0 \quad ,$$

since the total flow must vanish over S. This has to hold for all time, so

(3) $$\sum_{i=0}^{n} (\vec{q} \bullet \vec{n})_i\ dS_i = 0.$$

Letting $dS_i \to 0$ and passing to the limit, we have

(4) $$\int_S \vec{q} \bullet \vec{n}\ dS = 0.$$

Applying the divergence theorem we get $\int_V \operatorname{div}(\vec{q})\ dV = 0$, where V is the volume enclosed by S. This is true for all V, so we get the equation of continuity:

(5) $$\mathrm{div}(\vec{q}) = 0 \ .$$

The second assumption, the fluid is inviscid, indicates that any force acting on the fluid inside the surface S and caused by the fluid outside S is entirely pressure forces acting normal to S. Let $\vec{p}$ be the pressure acting on S by the fluid on the outside of S. Then the force caused by $\vec{p}$ on dS is $-p\vec{n}dS$. We assume the only other force, $\vec{g}\rho dV$, is due to gravity for an element of volume dV, where $\vec{g}$ is the acceleration due to gravity times a unit vector in the vertical downwards direction.

Recall Newton's 2nd Law of Motion : F = ma. The total force (pressure) on S due to fluid outside is $-\int_S p(\vec{l}\bullet\vec{n})\,dS$. The total force on V due to gravity is $\int_V \rho(\vec{l}\bullet\vec{g})\,dV$. In each element dV the mass is ρdV, and the acceleration is Dq/Dt, where D/Dt is the standard notation for total differentiation when following a particular particle. Thus the total force due to mass times acceleration is $\int \rho(\vec{l}\bullet Dq/Dt)\,dV$, where $\vec{l}$ is a unit vector in an arbitrary direction. Putting these into Newton's equation we get:

(6) $$-\int_S p(\vec{l}\bullet\vec{n})\,dS + \int_V \rho(\vec{l}\bullet\vec{g})\,dV = \int_V \rho(\vec{l}\bullet Dq/Dt)\,dV \ .$$

By the divergence theorem, we can rewrite the first integral as

(7) $$-\int_S p(\vec{l}\bullet\vec{n})\,dS = -\int_V \mathrm{div}(p\vec{l}\,)\,dV = \int_V (l\bullet\nabla p)\,dV \ .$$

Rewriting equation (6) we now have:

(8) $$0 = \int_V \vec{l}\bullet\nabla p\;dV - \int_V \rho(\vec{l}\bullet\vec{g})\,dV + \int_V \rho(\vec{l}\bullet\frac{D\vec{q}}{Dt})\;dV \ .$$

Equivalently,

(9) $$\int_V \rho\vec{l}\bullet[\tfrac{1}{\rho}\nabla p - \vec{g} + \frac{D\vec{q}}{Dt}]\;dV = 0.$$

Again, since this holds for any volume V and any direction $\vec{l}$, we get Euler's equation of motion:

$$\frac{1}{\rho}\nabla p - \vec{g} + \frac{D\vec{q}}{Dt} = 0 , \tag{10}$$

$$\text{or} \quad \frac{D\vec{q}}{Dt} = \vec{g} - \frac{1}{\rho}\nabla p . \tag{11}$$

Let (u_1, u_2, v) be the velocities corresponding to the space directions (x_1, x_2, y). Then the component form of (11) is:

$$\rho\frac{Du_1}{Dt} + \frac{\partial p}{\partial x_1} = 0 \tag{12a}$$

$$\rho\frac{Du_2}{Dt} + \frac{\partial p}{\partial x_2} = 0 \tag{12b}$$

$$\rho\frac{Dv}{Dt} + \frac{\partial p}{\partial y} + \rho g = 0, \tag{12c}$$

where $\vec{g} = (0, 0, -g)$, and g is the usual gravitational constant. Expanding the total derivatives D/Dt, we get:

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + v\frac{\partial u_1}{\partial y}\right) + \frac{\partial p}{\partial x_1} = 0 \tag{13a}$$

$$\rho\left(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + v\frac{\partial u_2}{\partial y}\right) + \frac{\partial p}{\partial x_2} = 0 \tag{13b}$$

$$\rho\left(\frac{\partial v}{\partial t} + u_1\frac{\partial v}{\partial x_1} + u_2\frac{\partial v}{\partial x_2} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} + \rho g = 0 \tag{13c}$$

Rewriting (5) using the (u_1, u_2, v) notation and multiplying through by ρ yields:

$$\rho\frac{\partial u_1}{\partial x_1} + \rho\frac{\partial u_2}{\partial x_2} + \rho\frac{\partial v}{\partial y} = 0 \tag{14}$$

Now multiply (14) by u_1 and add it to (13a) to get:

$$\rho\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x_1}(\rho u_1^2 + p) + \frac{\partial}{\partial x_2}(\rho u_1 u_2) + \frac{\partial}{\partial y}(\rho u_1 v) = 0 \tag{15a}$$

Similarly, multiplying (14) by u_2 and v, and then adding to (13b) and (13c) respectively, we get:

$$\rho\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x_1}(\rho u_1 u_2) + \frac{\partial}{\partial x_2}(\rho u_2^2 + p) + \frac{\partial}{\partial y}(\rho u_2 v) = 0 \tag{15b}$$

$$\rho\frac{\partial v}{\partial t} + \frac{\partial}{\partial x_1}(\rho v u_1) + \frac{\partial}{\partial x_2}(\rho v u_2) + \frac{\partial}{\partial y}(\rho v^2 + p + gy) = 0 \tag{15c}$$

These are the components for Euler's equation in a more symmetrical form .

The third assumption, the fluid is irrotational, means the individual particles of the fluid don't rotate. Mathematically, this means

$$(16) \qquad \text{curl } \vec{q} = \vec{0}$$

Expanding eq. (16),

$$(17) \qquad \left(\frac{\partial q_3}{\partial x_2} - \frac{\partial q_2}{\partial x_3} \, , \, \frac{\partial q_1}{\partial x_3} - \frac{\partial q_3}{\partial x_1} \, , \, \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} \right) = \vec{0}$$

For (17) to hold, we must have $\vec{q} = \nabla\phi$, for some scalar ϕ . From (5) we have $\text{div}(\nabla\phi) = 0$, and so we get Laplace's equation: $\Delta\phi = 0$.

The system of coordinates is set up with y in the vertical upwards direction and x_1, x_2 horizontal in mutually perpendicular directions. With these axes, Laplace's equation becomes:

$$(18) \qquad \frac{\partial^2\phi}{\partial {x_1}^2} + \frac{\partial^2\phi}{\partial {x_2}^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \ .$$

Bernoulli's equation can be derived using the equations above. Recall that for any function $f = f(x_1, x_2, y, t)$, the chain rule for partial differentiation is:

$$(19) \qquad \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{Dx_1}{Dt} + \frac{\partial f}{\partial x_2}\frac{Dx_2}{Dt} + \frac{\partial f}{\partial y}\frac{Dy}{Dt} \ .$$

Note that $\vec{q} = (\frac{Dx_1}{Dt}, \frac{Dx_2}{Dt}, \frac{Dy}{Dt})$; so we get $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{q} \bullet \nabla f)$.

Replacing f by $\vec{q} = \nabla\phi$, we get :

$$(20) \qquad \frac{D\vec{q}}{Dt} = \frac{\partial}{\partial t}(\nabla\phi) + (\vec{q} \bullet \nabla)\vec{q}$$

After some work, we can rewrite eq. (20) as:

$$(21) \qquad \frac{D\vec{q}}{Dt} = \frac{\partial}{\partial t}(\nabla\phi) + \nabla(\tfrac{1}{2} \, |\vec{q}|^2) - (\vec{q} \text{ x curl } \vec{q}) \ .$$

where $| \vec{q} |^2$ = speed. Since we assumed $\text{curl } \vec{q} = 0$, and assuming we can interchange the order in which we take the partial derivative and the gradient of ϕ, we have:

$$\frac{D\vec{q}}{Dt} = \nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{q}|^2 \right) . \tag{22}$$

Substituting Euler's equation in for $D\vec{q}/Dt$ yields:

$$-\frac{1}{\rho}\nabla p + \vec{g} = \nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{q}|^2 \right) . \tag{23}$$

Since $\vec{g}$ is only in the negative y direction , $\vec{g} = -g(0, 0, 1) = -g\nabla y$, eq. (23) becomes:

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{q}|^2 + \frac{p}{\rho} + gy \right) = 0 . \tag{24}$$

Integrating (24) , we get Bernoulli's equation:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\vec{q}|^2 + \frac{p}{\rho} + gy = f(t), \qquad \nabla f(t) = 0 . \tag{25}$$

II. Boundary Conditions

In the previous section, the domain of the fluid was considered arbitrary, but in a wavemaker problem, at least part of the boundary is held fixed and certain boundary conditions are imposed there. There are two kinds of boundary conditions: kinematic and dynamic. Kinematic describes the motion of the fluid, and dynamic relates the motion to the forces associated with it and to the properties of the moving objects.

The kinematic boundary condition comes from the basic idea of a continuum. Suppose a surface S is drawn in the fluid, and let S move with the fluid. If each particle on the surface S is followed, the same particles always make up S, and the particles originally inside S always remain inside S. Let the equation $S(x_1, x_2, y, t) = 0$ describe the surface. Then as long as x_1, x_2, y and t satisfy $S = 0$, the particle (x_1, x_2, y) remains on the surface for time t. Equivalently, for any surface S,

$$\frac{DS}{Dt} = 0 \tag{26}$$

Expanding the total derivative yields:

(27) $$\frac{\partial S}{\partial t} + u_1\frac{\partial S}{\partial x_1} + u_2\frac{\partial S}{\partial x_2} + v\frac{\partial S}{\partial y} = 0 .$$

Suppose a part of S is chosen to be part of the "free surface" between the water and the air. Any motion of the air affecting the water is neglected. Define this free surface by $y = \eta(x_1, x_2, t)$. Then for S the following holds

(28) $$S = \eta(x_1, x_2, t) - y = 0.$$

Note that η doesn't depend on y, so in applying (27) we get:

(29) $$\frac{\partial \eta}{\partial t} + u_1 \frac{\partial \eta}{\partial x_1} + u_2 \frac{\partial \eta}{\partial x_2} - v = 0 \qquad \text{on } y = \eta(x,t).$$

This is called the kinematic surface boundary condition (KSBC). Similarly, if S is part of the bottom of the tank or river bed, then:

(30) $$\frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x_1} + u_2\frac{\partial h}{\partial x_2} + v = 0 \qquad \text{on } y = -h(x,t).$$

If it is a rigid bed, so that $y = -h(x)$, then:

(31) $$u_1 \frac{\partial h}{\partial x_1} + u_2\frac{\partial h}{\partial x_2} + v = 0 \qquad \text{on } y = -h(x).$$

If it is a rigid, flat bed, then h is constant and (31) simplifies to:

(32a) $$v = 0$$

as the boundary condition.

Similarly, the boundary condition at a vertical wall would be

(32b) $$u = 0 ,$$

so there would be no flow through the wall. Equations (29) and (31) can be rewritten in terms of the velocity potential, where U_1 and U_2 represent uniform horizontal currents in the x_1 and x_2 directions respectively:

(33) $$(u_1, u_2, v) = \left(U_1 + \frac{\partial \phi}{\partial x_1} , U_2 + \frac{\partial \phi}{\partial x_2} , \frac{\partial \phi}{\partial y}\right) .$$

(34) $$\frac{\partial \eta}{\partial t} + \left[U_1 + \frac{\partial \phi}{\partial x_1}\right] \frac{\partial \eta}{\partial x_1} + \left[U_2 + \frac{\partial \phi}{\partial x_2}\right] \frac{\partial \eta}{\partial x_2} - \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = \eta$$

(35) $$\left[U_1 + \frac{\partial \phi}{\partial x_1}\right] \frac{\partial h}{\partial x_1} + \left[U_2 + \frac{\partial \phi}{\partial x_2}\right] \frac{\partial h}{\partial x_2} + \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = -h$$

The dynamic boundary condition applies only on the free surface. Since we assume that with the absence of any motion in the air, the pressure there is constant. We will take this constant to zero. Then the only pressure on the water surface is surface tension. If we draw a line on the fluid surface, the fluid on the left will exert a tension τ per unit length of line on the fluid to the right. τ is the surface tension coefficient; it is different for various fluids and depends on temperature.

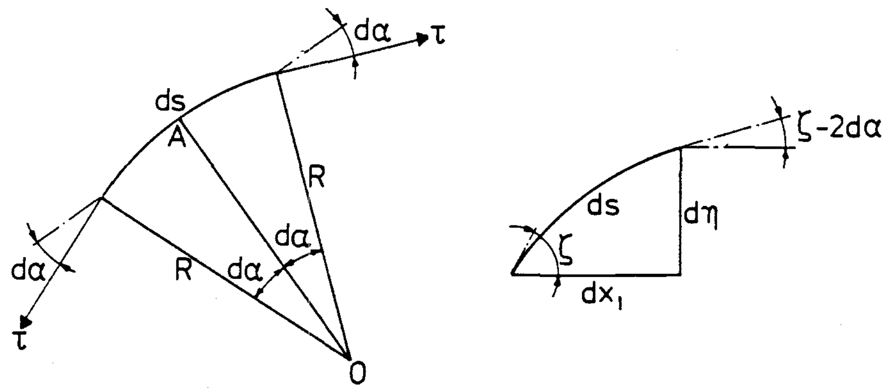

Figure 1. Surface Pressure Components on a Free Surface

For simplicity, consider a two dimensional case at a fixed time. The surface can then be represented as $y = \eta(x_1)$. Imagine a small piece of the free surface , length ds in the x_1 direction and unit length in the x_2 direction. Referring to Figure 1. , draw the segment AO perpendicular to the tangent to the surface at the midpoint of ds. As we let ds$\rightarrow$0 , it approaches a piece of an arc of the circle centered at 0 with radius AO. The component of the tension τ which is perpendicular to AO gives a net force along AO of $\tau\sin(d\alpha)$. τ occurs on both the left and the right, so the total force will be

$$2\tau\sin(d\alpha) \approx 2\tau d\alpha \tag{36}$$

when $d\alpha$ is very small.

This force must be balanced by an increased pressure inside the fluid, since the pressure of the air was taken to be zero, so that

$$2\tau d\alpha = p ds \tag{37}$$

Let R = AO be the radius of curvature of the arc ds. Then from the arclength formula,

$$ds = 2 d\alpha R \tag{38}$$

Substituting ds in (38) into (37) the surface pressure is:

$$p = \frac{\tau}{R} = \tau\kappa \tag{39}$$

where $\kappa = 1/R$ is the curvature of the surface, and is taken to be positive if the surface is concave down. By substituting the surface pressure p from (39) into Bernoulli's equation (25), the result is the dynamic surface boundary condition:

$$\text{(40) DSBC:} \qquad \frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{q}|^2 + \frac{\tau\kappa}{\rho} + g\eta = f(t) \qquad \text{on } y = \eta,$$

where f(t) is a function of time only, and satifies $\nabla f = 0$.

Equations (34), (35) and (40), along with Laplace's equation are the governing equations for a generic wavemaker problem for nonlinear waves. The nonlinear KSBC and DSBC must be evaluated at the unknown free surface η. For this reason, no analytic solutions exist at this time. It is also these nonlinear boundary conditions that make the nonlinear problem so difficult to solve.

III. Approximations

The problem up to this point consists of solving Laplace's equation, which is linear, along with several nonlinear boundary conditions. Generally, to get linear solutions, we have to make some approximations. Here the assumption is made that the waves are small compared to the water depth.

Let $\epsilon > 0$ be a small parameter, and suppose there exist $\eta_i = \eta_i(x_1, x_2, t)$ and $\phi_i = \phi_i(x_1, x_2, y, t)$ such that

(41) $$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots$$

(42) $$\phi = U_1x_1 + U_2x_2 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$$

Laplace's equation is then

(43) $$\epsilon\Delta\phi_1 + \epsilon^2\Delta\phi_2 + \cdots = 0$$

which should hold for any ϵ, so each ϕ_i must be harmonic. The kinematic surface boundary condition (KSBC) becomes

(44) $$\epsilon\left[\frac{\partial\eta_1}{\partial t} + \frac{\partial\eta_1}{\partial x_1}U_1 + \frac{\partial\eta_1}{\partial x_2}U_2 - \frac{\partial\phi_1}{\partial y}\right] +$$

$$\epsilon^2\left[\frac{\partial\eta_2}{\partial t} + \frac{\partial\eta_2}{\partial x_1}U_1 + \frac{\partial\eta_1}{\partial x_1}\frac{\partial\phi_1}{\partial x_1} + \frac{\partial\eta_2}{\partial x_2}U_2 + \frac{\partial\eta_1}{\partial x_2}\frac{\partial\phi_1}{\partial x_2} - \frac{\partial\phi_1}{\partial y}\right] + \ldots = 0 .$$

For Eqn. (44) to hold for any ϵ, the sum of all the order ϵ terms must be zero, and similarly for all the order ϵ^2 terms, and so on. Similarly, the dynamic surface boundary condition (DSBC) becomes

(45) $$\epsilon\left[\frac{\partial\phi_1}{\partial t} + U_1\frac{\partial\phi_1}{\partial x_1} + U_2\frac{\partial\phi_1}{\partial x_2} + g\eta_1\right] + \frac{\tau\kappa}{\rho} + \frac{1}{2}(U_1{}^2 + U_2{}^2) +$$

$$\epsilon^2\left[\frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\phi_1}{\partial x_2}\right)^2 + \left(\frac{\partial\phi_1}{\partial y}\right)^2\right] + U_1\frac{\partial\phi_2}{\partial x_1} + U_2\frac{\partial\phi_2}{\partial x_2}\right] + \ldots = f(t).$$

Both equations are evaluated at $y = \eta$. These equations are still nonlinear, and still have the difficulty that they must be evaluated at the unknown position $y = \eta$.

One way to get around the difficulty is to satisfy the boundary conditions at a mean value of η, say at $y = 0$, using Taylor's theorem. For any function $F = F(x_1, x_2, y, t)$ we have

(46) $$F(x_1, x_2, \eta, t) = F(x_1, x_2, 0, t) + \eta\frac{\partial F}{\partial y}(x_1, x_2, 0, t) + \cdots$$

Since the waves are assumed to be small, all the terms in Eqn. (46) with η's can be neglected. Apply this to each of the ϕ_i in the asymptotic expansion (42) of ϕ. Substituting (41) and (42) into the KSBC (34), and linearizing the equation by keeping only terms of order ϵ , we have:

(47) $$\epsilon \left[\frac{\partial \eta_1}{\partial t} + U_1 \frac{\partial \eta_1}{\partial x_1} + U_2 \frac{\partial \eta_1}{\partial x_2} - \frac{\partial \phi_1}{\partial y} \right] = 0 \quad \text{on } y = 0.$$

Similarly, for the DSBC, evaluated at $y = 0$:

(48) $$\epsilon \left[\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x_1} + U_2 \frac{\partial \phi_1}{\partial x_2} + g\eta_1 \right] + \frac{\tau\kappa}{\rho} + \frac{1}{2}(U_1{}^2 + U_2{}^2) = f(t)$$

Often $f(t)$ is chosen so as to cancel with the constant on the left, which is $\frac{1}{2}(U_1{}^2 + U_2{}^2)$ here. Since it was assumed all the ϵ^2 terms are negligible, $\epsilon\eta_1$ can be replaced by η, and $\epsilon\phi_1$ by ϕ. With this change of notation, the linear surface boundary conditions are, along with Laplace's equation:

(49) KSBC: $$\frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x_1} + U_2 \frac{\partial \eta}{\partial x_2} = \frac{\partial \phi}{\partial y} \quad \text{on } y = 0,$$

(50) DSBC: $$\frac{\partial \phi}{\partial t} + U_1 \frac{\partial \phi}{\partial x_1} + U_2 \frac{\partial \phi}{\partial x_2} + g\eta_1 - \frac{\tau\kappa}{\rho} = 0 \quad \text{on } y = 0,$$

(51) $$U_1 \frac{\partial h}{\partial x_1} + U_2 \frac{\partial h}{\partial x_2} - \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = h',$$

where h' is some mean value of h. In the case of no current, all the U_1 and U_2 terms are zero.

2. THE NONLINEAR SURFACE PRESSURE DISTRIBUTION PROBLEM

The two dimensional surface pressure distribution problem is defined on a rectangular wavetank of finite length and depth. The waves are generated by the instantaneous acceleration, from rest to a constant speed U, of a pressure distribution on the surface of initially calm water.

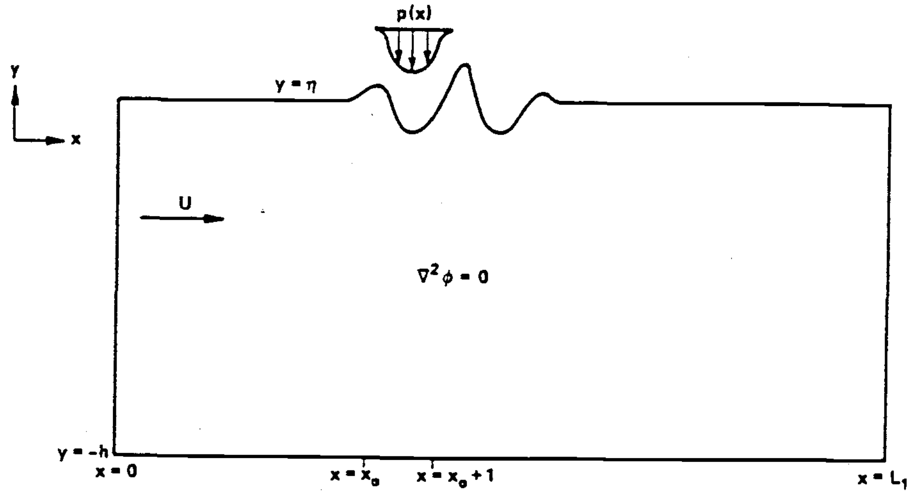

Figure 2. Domain for the Surface Pressure Distribution Problem.

The same assumptions that the water is incompressible, irrotational, and inviscid are made, so the previous equations from Section 1 apply, although only two dimensions are used here. In particular, eq. (18), (34), (35) and (40) are used. It is assumed that initially everything in the system is at rest, and then accelerated impulsively to speed U. The pressure distribution can be thought of as modelling the effects of the air cushion under a "hovercraft" style boat.

Before solving the equations derived in Section 1, they must be non-dimensionalized. The following change of variables is made:

$$\eta' = L\eta$$

(52)
$$t' = \frac{L}{U}t$$
$$\phi' = LU\phi$$
$$p' = Pp$$

where L is the length of the pressure distribution, (which is 1 in this problem), U is the constant speed of the current, and P is the max pressure in the given pressure distribution. Primes indicate the old dimensional variables. Note that in the new notation:

(53)
$$\frac{\partial \phi'}{\partial t'} = \frac{\partial \phi'}{\partial \phi}\frac{\partial \phi}{\partial t}\frac{\partial t}{\partial t'} = U\phi_t U = U^2\phi_t \ ,$$

(54) and
$$\frac{\partial \phi'}{\partial x} = \frac{\partial \phi'}{\partial \phi}\frac{\partial \phi}{\partial x} = U\phi_x$$

Applying the change of variables to eq. (18), (34), (35) and (40) yields:

(34′)
$$U\eta_t + U\eta_x + U\eta_x\phi_x - U\phi_y = 0 \quad \text{on } y = \eta$$

Dividing through by U ,

(55) KSBC:
$$\eta_t + \eta_x(1 + \phi_x) - \phi_y = 0 \quad \text{on } y = \eta$$

Similarly, on $y = \eta$,

(40′)
$$\phi_t + \phi_x + (g\eta/U^2) + (pP/\rho U^2) + \tfrac{1}{2}({\phi_x}^2 + {\phi_y}^2) = 0$$

Define $\delta = P/\rho gL$ and $Fr^2 = U^2/gL$. Fr represents the Froude number based on L, and δ is the hydrostatic surface displacement caused by the surface pressure p, divided by L. Both are dimensionless. Substituting into (40′):

(56) DSBC: $\phi_t + \phi_x + (\eta/Fr^2) + (\delta p/Fr^2) + \tfrac{1}{2}({\phi_x}^2 + {\phi_y}^2) = 0$ on $y=\eta$

The boundary condition on the bottom and sides become $U\phi_y = 0$ and $U\phi_x = 0$; Laplace's equation becomes $U^2\phi_{xx} + U^2\phi_{yy} = 0$. Rewriting:

(57)
$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for } x \ \varepsilon \ [0,L_1] \ , \ y \ \varepsilon \ [-h,\eta]$$

(58)
$$\phi_y = 0 \quad \text{on } y = -h$$

(59)
$$\phi_x = 0 \quad \text{on } x = 0, L_1,$$

where L_1 is the length of the wavetank.

The surface pressure distribution problem can be represented by the initial boundary value problem consisting of eq. (55) – (59), where

(60) $$p = \sin^2[\,\pi(x-x_0)] \qquad \text{for } x_0 \le x \le x_0+1$$

$$\text{and} \quad p = 0 \qquad \text{otherwise.}$$

At time = 0,

(61) $$\phi = 0 \quad \text{everywhere, and}$$

(62) $$\eta = -\delta p.$$

The origin of the coordinate system is in the undisturbed free surface.

3. THE NONLINEAR FULL- FLAP WAVEMAKER PROBLEM

The two dimensional full-flap wavemaker problem is defined on the same wavetank as the surface pressure distribution problem, except the left end of the tank is taken to be a full-flap wavemaker. (Fig. 3) The flap can be thought of as being hinged to the bottom of the tank and allowed to oscillate back and forth in the horizontal direction. The waves are thus generated by the instantaneous acceleration, from rest at time zero, of the full-flap wavemaker.

The equations used in the surface pressure distribution problem will also be used in this problem, except it is assumed that there is no current in the water, and the surface pressure is negligible compared to the forces caused by the wavemaker. With these changes the governing equations become:

(63) KSBC: $\quad \eta_t = -\eta_x \phi_x + \phi_y \quad$ on $y = \eta$,

(64) DSBC: $\quad \phi_t = -g\eta - \frac{1}{2}({\phi_x}^2 + {\phi_y}^2) \quad$ on $y = \eta$,

(65) Laplace: $\quad \phi_{xx} + \phi_{yy} = 0 \quad$ for $x \varepsilon [0,L_1]$, $y \varepsilon [-h,\eta]$,

(66) $\quad \phi_x = [\,(y + \eta + h)\cos(t)]\,/\,5 \quad$ at $x = 0$,

$$\phi_x = 0 \quad \text{at } x = L_1,$$

$$\phi_y = 0 \quad \text{at } y = -h,$$

$$\phi = 0\,,\quad \eta = 0 \quad \text{everywhere at } t = 0.$$

Equation (66) represents the wavemaker function. In this paper it was chosen to coincide with the wavemaker function used in Cooper's solution of the same problem using the linear equations.

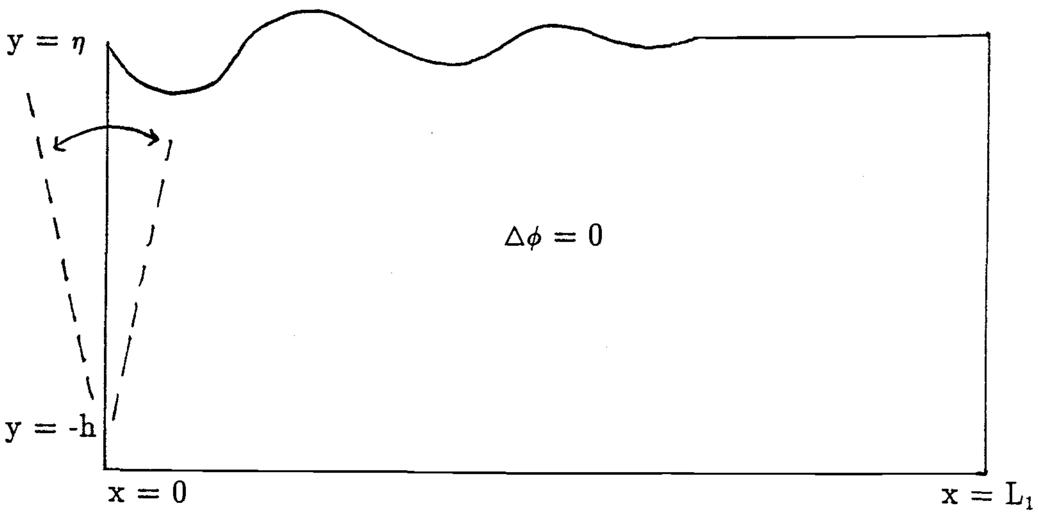

Figure 3. Domain for the Full-Flap Wavemaker Problem.

4. THE NUMERICAL METHOD: THE METHOD OF LINES

The method of lines is applied as used by Samuel Ohring. The wavetank domain is covered by two overlapping grid systems, as shown in Figure 4.

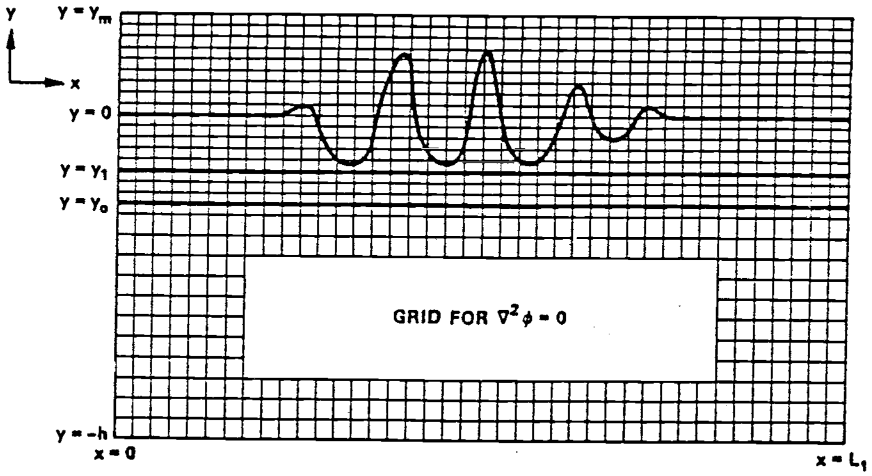

Figure 4. Grid System Used by the Method of Lines.

The upper grid, for $0 \le x \le L_1$, $y_0 \le y \le y_m$ is that used by the method of lines. The bottom grid, for $0 \le x \le L_1$, $-h \le y \le y_1$ is that used by the SOR method. The equations are discretized in all variables but y in the upper grid, and in all variables in the lower grid. Finite differences replace derivatives in the discretized directions. The points on the grid where lines intersect are called nodes. In the upper grid, solutions will be calculated continuously in the y direction for each vertical line $x=x_i$, but data will only be stored for values on the nodes. Linear interpolation is used between node values when necessary.

When computing the solution at time step t^{n+1} , it is assumed that a solution is known for the entire wavetank for time step t^n . Sweeping through the upper grid on the kth iteration sweep from the upstream end (x=0) to the downstream end ($x=L_1$), eq. (55)-(57) and (59) are solved for each line x_i, subject to a Dirichlet boundary condition at $y = y_0$. When solving these equations up each line $x=x_i$, assuming the k^{th} iterate solution is known at line x_{i-1} , and the k-1^{st} iterate solution at x_{i+1}. After this pass, Laplace's equation is solved on the bottom grid subject to the boundary conditions on the sides and bottom, plus a Dirichlet boundary condition at $y=y_1$. The convergence criteria (67) are then checked for each i. If they are met, the k^{th} iterate is taken to be the solution for time step t^{n+1}. If the criteria are not met, a k+1^{st} sweep is made (and additional sweeps if necessary) until the convergence criteria:

$$(67) \qquad \begin{aligned} [\eta^{(k)}(x_i, t^{n+1}) - \eta^{(k-1)}(x_i, t^{n+1})] &< \epsilon \\ [\phi^{(k)}(x_i, y_0, t^{n+1}) - \phi^{(k-1)}(x_i, y_0, t^{n+1})] &< \epsilon \end{aligned}$$

are met.

The following discussion applies only to the upper grid for the kth iteration sweep of the time step t^{n+1} at the line $x=x_i$. Euler's modified method , an implicit time differencing method, is applied to equations (55) and (56). One gets, at the line $x=x_i$,

$$(68) \qquad -\eta^{(k)}(x_i, t^{n+1}) + \eta(x_i, t^n) + (\tfrac{\Delta t}{2})[F_{n+1}{}^{(k)} + F_n] = 0,$$

$$(69) \qquad -\phi^{(k)}(x_i, \eta^{(k)}, t^{n+1}) + \phi(x_i, \eta, t^n) + (\tfrac{\Delta t}{2})[G_{n+1}{}^{(k)} + G_n] = 0,$$

where $\eta^{(k)} = \eta^{(k)}(x_i, t^{n+1})$, $\eta = \eta(x_i, t^n)$, and

$$(70) \qquad \begin{aligned} F_{n+1}{}^{(k)} = {} & \phi_y{}^{(k)}(x_i, \eta^{(k)}, t^{n+1}) \\ & -\frac{1}{2\Delta x}[\eta^{(k-1)}(x_{i+1}, t^{n+1}) - \eta^{(k)}(x_{i-1}, t^{n+1})]\,[1 + \phi_x{}^{(k)}(x_i, \eta^{(k)}, t^{n+1})], \end{aligned}$$

$$(71) \qquad \begin{aligned} G_{n+1}{}^{(k)} = {} & -\eta^{(k)}(x_i, t^{n+1})/Fr^2 - \delta p/Fr^2 - \phi_x{}^{(k)}(x_i, \eta^{(k)}, t^{n+1}) \\ & -\tfrac{1}{2}(\,[\phi_y{}^{(k)}(x_i, \eta^{(k)}, t^{n+1})]^2 + [\phi_x{}^{(k)}(x_i, \eta^{(k)}, t^{n+1})]^2\,) \\ & + \phi_y{}^{(k)}(x_i, \eta^{(k)}, t^{n+1})\, F_{n+1}{}^{(k)}, \end{aligned}$$

$$(72) \quad \phi_x^{(k)}(x_i, \eta^{(k)}, t^{n+1})$$
$$= [\phi^{(k-1)}(x_{i+1}, \eta^{(k)}, t^{n+1}) - \phi^{(k)}(x_{i-1}, \eta^{(k)}, t^{n+1})]/2\Delta x \, .$$

F_n and G_n are also given by (70) and (71) by replacing t^{n+1} by t^n and removing all superscripts (k) and (k-1). All dependent variables on lines other then x_i are considered known, and the solution for time step t^n is known for all lines. Recall that the values of the dependent variables are saved only at node points. Generally, the free surface will not cross at a node point, so the terms $\phi^{(k-1)}(x_{i+1}, \eta^{(k)}, t^{n+1})$ and $\phi^{(k)}(x_{i-1}, \eta^{(k)}, t^{n+1})$ in eq. (72) are expressed as linear functions of the unknown $\eta^{(k)}(x_i, t^{n+1})$ in terms of the known values of $\phi^{(k-1)}(x_{i+1}, y_j, t^{n+1})$ and $\phi^{(k)}(x_{i-1}, y_j, t^{n+1})$ nearest the known free surface $\eta^{(k-1)}(x_{i+1}, t^{n+1})$ and $\eta^{(k)}(x_{i-1}, t^{n+1})$ respectively. Using central differencing for Laplace's equation about the node (x_i, y_j) gives

$$(73) \qquad \phi_{yy}^{(k)}(x_i, y, t^{n+1}) = \frac{2\phi^{(k)}(x_i, y, t^{n+1})}{(\Delta x)^2} - \left[\frac{\phi^{(k-1)}(x_{i+1}, y, t^{n+1}) + \phi^{(k)}(x_{i-1}, y, t^{n+1})}{(\Delta x)^2}\right] .$$

At the free surface $y = \eta^{(k)}$, the unknowns in equations (68),(69), and (73) are $\eta^{(k)}(x_i, t^{n+1})$, $\phi^{(k)}(x_i, \eta^{(k)}, t^{n+1})$, and $\phi_y^{(k)}(x_i, \eta^{(k)}, t^{n+1})$. This gives three equations and three unknowns, or a 3x3 system of simultaneous equations.

Note that in (73) the last term is known, so really it is an ODE for $\phi^{(k)}$ at x_i for y in successive intervals $[y_j, y_{j+1}]$. The values of $\phi^{(k-1)}(x_i, y_0, t^{n+1})$ and $\phi_y^{(k-1)}(x_i, y_0, t^{n+1})$ are used as the initial conditions. Thus (73) has an analytic solution over each interval $[y_j, y_{j+1}]$, and we could solve (73) by starting at y_0 and working up. To solve (73) this way would give a solution which was independent of the free surface conditions in (68) and (69). Solving

(73) downwards form the unknown free surface is also impossible, so we need a method which will 1. uncouple the free surface conditions from the integration of the ODE representing Laplace's equation along the line $x=x_i$; and 2. yield a solution which satisfies both Laplace's equation and the free surface conditions.

Following Meyer's method, a Ricatti transformation that relates ϕ and ϕ_y through the auxiliary variables R(x, y) and W(x, y) is used:

$$(74) \qquad \phi^{(k)}(x_i, y, t^{n+1}) = R(x_i, y)\,\phi_y^{(k)}(x_i, y, t^{n+1}) + W(x_i, y)$$

Note that the grid ordinate y_k has been replaced by the continuous variable y. This will apply to eq. (73) as well in the following discussion. Substituting (74) in for $\phi^{(k)}(x_i, y, t^{n+1})$ in (73) yields:

$$(75) \quad \phi_{yy}^{(k)}(x_i, y, t^{n+1}) = (2/(\Delta x)^2)\ R(x_i, y)\,\phi_y^{(k)}(x_i, y, t^{n+1})$$
$$-\Big[\,\phi^{(k-1)}(x_{i+1}, y, t^{n+1}) + \phi^{(k)}(x_{i-1}, y, t^{n+1}) - 2W(x_i, y)\Big]/(\Delta x)^2.$$

This is a first order ODE for ϕ_y once R and W are known.

To obtain R and W, differentiate (74) with respect to y to get:

$$(76) \qquad \phi_y^{(k)} = R_y\phi_y^{(k)} + R\phi_{yy}^{(k)} + W_y\ .$$

Substitute (75) in for ϕ_{yy} :

$$(77) \quad \phi_y^{(k)} = R\Big[2R\phi_y^{(k)} - \phi^{(k-1)}(x_{i+1}) - \phi^{(k)}(x_{i-1}) + 2W\Big]/(\Delta x)^2$$
$$+\ R_y\phi_y^{(k)} + W_y\ \text{, or}$$

$$(78) \qquad 0 = \phi_y^{(k)}[R_y + (2/(\Delta x)^2)R^2 - 1] + (2/(\Delta x)^2)RW + W_y$$
$$+\ (R/(\Delta x)^2)\,[-\phi^{(k-1)}(x_{i+1}) - \phi^{(k)}(x_{i-1})].$$

To solve (78), assume the first term on the right is zero; then R must satisfy:

$$(79) \qquad R_y\,(x_i, y) = 1 - (2/(\Delta x)^2)R^2(x_i, y)\ , \quad R(x_i, y_0) = 0.$$

An analytical solution for R in (79) is known:

$$(80) \qquad R(y) = \frac{\Delta x}{\sqrt{2}} \left[\frac{\exp[\frac{2^{1.5}(y-y_0)}{\Delta x}] - 1}{\exp[\frac{2^{1.5}(y-y_0)}{\Delta x}] + 1}\right]$$

The equation for W is then:

$$(81) \qquad Wy(x_i, y) = (1/(\Delta x)^2)\, R(x_i, y)\, [\phi^{(k-1)}(x_{i+1}) + \phi^{(k)}(x_{i-1})]$$
$$- (2/(\Delta x)^2)\, R(x_i, y)\, W(x_i, y) \ , \quad W(x_i, y_0) = \phi^{(k-1)}(x_i, y_0, t^{n+1})$$

The initial conditions implied on R and W arise from the known initial condition $\phi^{(k-1)}(x_i, y_0, t^{n+1})$ obtained from solving the bottom grid.

The numerical solution for W is obtained on each successive interval $[y_j, y_{j+1}]$, beginning at y_0 and moving up to the first y_k beyond the last known iterate value of η, using a second order Runge-Kutta method. The term $[\phi^{(k-1)}(x_{i+1}) + \phi^{(k)}(x_{i-1})]$ in eq. (81) is known only at each node (the endpoints of each interval), and is expressed as a linear function over the interval $[y_j, y_{j+1}]$ based on its values at the nodes.

At the unknown free surface $y = \eta^{(k)}(x_i, t^{n+1})$, eq. (74) is:

$$(82) \quad \phi^{(k)}(x_i, \eta^{(k)}, t^{n+1}) = R(x_i, \eta^{(k)})\phi_y^{(k)}(x_i, \eta^{(k)}, t^{n+1}) + W(x_i, \eta^{(k)})$$

Similarly, we can express R and W in terms of $\eta^{(k)}(x_i, t^{n+1})$ at the free surface. (Linear interpolation is used for W).

Eq. (68),(69) and (82) now form a nonlinear 3x3 system of simultaneous equations with the unknowns $\eta^{(k)}(x_i, t^{n+1})$, $\phi_y^{(k)}(x_i, \eta^{(k)}, t^{n+1})$, and $\phi^{(k)}(x_i, \eta^{(k)}, t^{n+1})$. $\phi_y^{(k)}$ can be expressed in terms of the other two unknowns using eq. (68), and then substituted into eq. (69) and (82) to reduce the system to a 2x2 system of simultaneous equations. Newton's method for systems was used. Once $\eta^{(k)}(x_i, t^{n+1})$ and $\phi^{(k)}(x_i, \eta^{(k)}, t^{n+1})$ are known, they can be substituted back into eq. (68) to get $\phi_y^{(k)}(x_i, \eta^{(k)}, t^{n+1})$.

$\phi_y^{(k)}(x_i, \eta^{(k)}, t^{n+1})$ is used as the initial condition for numerically solving the ODE in eq. (75) for $\phi_y^{(k)}$ along the line $x = x_i$. The numerical solution starts at the free surface and advances downward to $y = y_0$ along the node points on x_i . Note that the first interval will be of a variable length depending on the value of $\eta^{(k)}(x_i, t^{n+1})$. A second order Runge-Kutta method is used on each successive interval, with the final solution at the end of one interval serving as the initial condition for the next interval. Linear interpolation is used on any terms known only on the node points, such as $W(x_i, y)$, $\phi^{(k-1)}(x_{i+1}, y, t^{n+1})$ and $\phi^{(k)}(x_{i-1}, y, t^{n+1})$.

The velocity potential $\phi^{(k)}(x_i, y, t^{n+1})$ is then obtained at the node points on x_i from the free surface down to $y=y_0$ using eq. (75). This completes the k^{th} iterate solution for time step t^{n+1} at the line x_i in the upper grid.

In the bottom grid, eq. (57)-(59) are solved for ϕ subject to a Dirichlet boundary condition at $y=y_1$, where $\phi^{(k)}(x_i, y_1, t^{n+1})$ is known from the solution on the upper grid. The bottom grid is solved using successive over-relaxation (SOR) in the following manner. Each of the node points in the lower grid is numbered, starting at the lower left hand corner and working upward row by row to end at the upper right hand corner. (See Fig. 5). The actual corner points are not included because of the singularities at those points.

Because finite differences were used, $\phi(x_i, y_j)$ can be represented as a linear combination of the values of ϕ at neighboring nodes. Thus, if there are N nodes in the lower grid, N equations in N unknowns can be written, and the resulting NxN system of simultaneous equations solved. Let A represent the NxN matrix corresponding to the system of equations. In this problem, the value of ϕ at each node depends on no more than the values of ϕ at four other nodes, so A is a large, sparse matrix. Let $\vec{u} = (u_1, u_2, \ldots, u_N)$ represent the variable ϕ at each of the numbered nodes. Then solving the lower grid amounts to solving the system

$$A\vec{u} = \vec{v}, \tag{83}$$

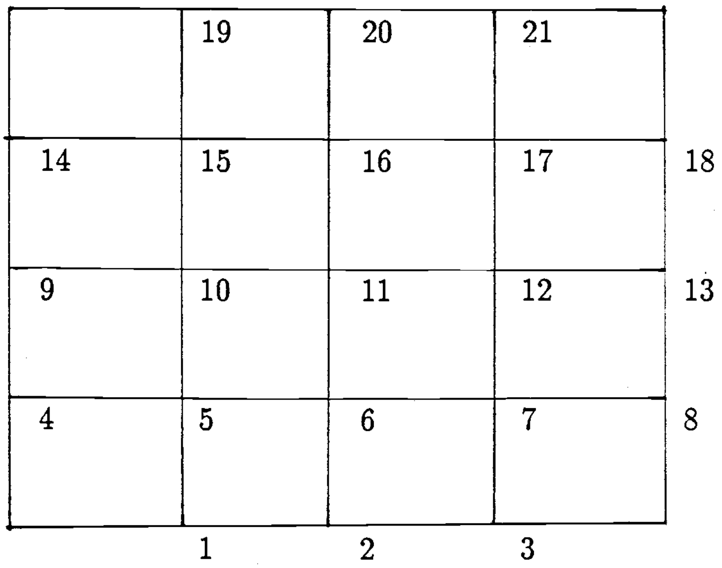

Figure 5. SOR Node Numbering System.

where $\vec{v}$ is a constant vector. The matrix A can be split into two parts: the diagonal elements, D, and the off-diagonal elements, B. Eq. (83) can be rewritten as:

$$\vec{u} = D^{-1}(\vec{v} - B\vec{u}). \tag{84}$$

This suggests using a recursive formula, given an initial guess $\vec{u_0}$:

$$\vec{u_k} = D^{-1}(\vec{v} - B\vec{u_{k-1}}) \quad \text{for } k = 1, 2, \ldots \tag{85}$$

The i^{th} component of $\vec{u_k}$ is then

$$u_k(i) = [v(i) - \sum_{j=1}^{N} B(i,j)\, u_{k-1}(j)] \,/\, D(i,i). \tag{86}$$

or equivalently, with γ=1,

$$u_k(i) = u_{k-1}(i) + \gamma\left[\frac{v(i) - \sum_{j=1}^{N} B(i,j)\, u_{k-1}(j) \;-\; D(i,i)\, u_{k-1}(i)}{D(i,i)}\right]. \tag{87}$$

The SOR method involves "over-correcting" the k-1^{st} guess, so γ is chosen such that $\gamma \varepsilon [1,2)$. The recursive formula (87) is used until the maximum difference between two successive guesses, $u_k(i)$ and $u_{k-1}(i)$, is smaller than a fixed tolerance. The values of ϕ from the last iteration of the lower grid are used as the initial guess.

If a grid point (x_i, y_j) is on one of the boundaries of the wave tank, the equations must be modified. When x_i=0 or L_1, the centered difference method cannot be used on the derivatives, so eq. (68), (69), and (73)-(81) must be modified, and then solved in the same manner as before. Eq. (59) applies on these boundaries; a forward difference method is used for x_i= 0 and a backward difference method for x_i= L_1. 3-point formulas were used in both the forward and backward difference methods; for example:

$$\eta_x^{(k)}(x_0, t^{n+1}) = \frac{-3\eta^{(k-1)}(x_0, t^{n+1}) + 4\eta^{(k-1)}(x_1, t^{n+1}) - \eta^{(k-1)}(x_2, t^{n+1})}{2\Delta x} \tag{88}$$

With these changes, the following equations apply when x_i= 0:

(73′) $$\phi_{yy}{}^{(k)}(x_0, y_k, t^{n+1}) = \frac{-\phi^{(k)}(x_0, y_k, t^{n+1}) + 2\phi^{(k-1)}(x_1, y_k, t^{n+1}) - \phi^{(k-1)}(x_2, y_k, t^{n+1})}{(\Delta x)^2}$$

(75′) $$\phi_{yy}{}^{(k)}(x_0, y_k, t^{n+1}) = \frac{1}{(\Delta x)^2}\Big[R(x_0, y)\phi_y{}^{(k)}(x_0, y, t^{n+1})\Big] + $$
$$+ \frac{1}{(\Delta x)^2}\Big[-W(x_0, y) + 2\phi^{(k-1)}(x_1, y, t^{n+1}) - \phi^{(k-1)}(x_2, y, t^{n+1})\Big].$$

Again, differentiate (74) with respect to y and substitute in for ϕ_{yy} in (75′) :

(78′) $$W_y(x_0, y) = \phi_y{}^{(k)}(x_0, y)\Big[1 - R_y(x_0, y) + (1/(\Delta x)^2)R^2(x_0, y)\Big]$$
$$- \frac{R(x_0, y)}{(\Delta x)^2}\Big[-W(x_0, y) + 2\phi^{(k-1)}(x_1, y, t^{n+1}) - \phi^{(k-1)}(x_2, y, t^{n+1})\Big].$$

Equation (78′) can be solved if R is required to satisfy:

(79′) $$R_y(x_0, y) = 1 + (1/(\Delta x)^2)R^2(x_0, y)$$

This has the analytic solution:

(80′) $$R(x_0, y) = \Delta x \tan[(y - y_0)/\Delta x] , \qquad R(x_0, y_0) = 0.$$

The first order ODE for W is then:

(81′) $$W_y(x_0, y) =$$
$$\frac{R(x_0, y)}{(\Delta x)^2}\Big[W(x_0, y) - 2\phi^{(k-1)}(x_1, y, t^{n+1}) + \phi^{(k-1)}(x_2, y, t^{n+1})\Big] ,$$
$$W(x_0, y_0) = \phi^{(k-1)}(x_0, y_0, t^{n+1}).$$

These primed equations are used the same way as the unprimed equations. When $x_i = L_1$, the equations for $x_i = 0$ can be used after making the change of variable:

(89) $$\begin{aligned} 0 &\rightarrow L_1 \\ x_1 &\rightarrow x_{L-1} \\ x_2 &\rightarrow x_{L-2} . \end{aligned}$$

The Cartesian grid used in this paper is defined as follows: beginning at x = 0, which is the line $x = x_0$,

(90) $$x_i = i\,\Delta x, \ \text{ for } i= 1, \dots , 100;$$

and beginning at $y = -h$ and progressing up in the positive y-direction,

$$y_j = -h + j\Delta y, \quad \text{for } j = 1, \ldots, 76 . \tag{91}$$

The values $\Delta x = 0.1$, $\Delta y = 0.02$, and $\Delta t = 0.03$ were used. With these values, $L_1 = 10.0$, $h = 1.36$, and $y_m = 0.16$. Note that the notation y_0 and y_1 are special in the preceding discussion; in terms of (91) , $y_0 = y_{63} = -0.1$ and $y_1 = y_{65} = -0.06$. Similarly, the x_0 used in eq. (67) and (68) is defined by $x_0 = x_{27} = 2.7$; so that the surface pressure distribution is applied from x_i = 27 through $x_i = 36$. The values $Fr = 0.35$ and $\delta = 0.0125$ were also used.

5. DIFFICULTIES WITH THE NUMERICAL METHOD

Many difficulties were encountered in the programming of the numerical method. Ohring's method was followed as closely as possible so the results of my program could be compared for accuracy before applying the numerical method to the wavemaker problem. However, several errors found in Ohring's paper had to be corrected.

Ohring's discretization of the wavetank domain involved using variable step sizes in both the x and y directions. His reasoning was to use a finer grid size in the critical areas, which are the areas immediately surrounding the surface and directly below the pressure distribution. A coarser grid was used in the less critical areas where the potential ϕ does not change very rapidly, as on the bottom of the wavetank. However, Ohring uses a centered difference method to approximate derivatives at the node points, and this method requires the step sizes to be equal. Using variable step size increases the order of the error in the method from $O(h^2)$ to $O(h)$ or possibly larger.

Additionally, Ohring states that the average number of SOR iterations used within each cycle was 1, with up to 4 or 5 required near time zero. It is more common for SOR, when applied to a time dependent problem such as these, to take 100 or more iterations before it converges within a given tolerance. This implies that Ohring's tolerance may have been comparitively large. The computer program written for this paper used an average of 25-35 SOR iterations within each cycle, with more than 150 iterations used in some cycles. The error in each iteration was computed by substituting the n^{th} iterate solution vector, $\vec{u}^n$, back into the original equation it was supposed to solve:

$$A\vec{u} = \vec{v}. \tag{92}$$

The error is then taken to be:

(93) $$\text{error} = \text{abs}\left[\text{maximum}\ \{\ [A\vec{u}^n]_i - v_i\ \}\right] \text{ over all i.}$$

The error tolerance is taken to be 0.005. Usually the error it taken to be the maximum difference between two consecutive iterates:

(94) $$\text{error} = \text{abs}\left[\text{maximum}\ (u^{n+1}_i - u_i^n)\right].$$

However, if the SOR method is converging very slowly, this error could converge to zero while the error in eq. (93) is still quite large. It is likely that Ohring used the error bound described in eq. (94).

Ohring's convergence criteria for completing each time step is that the difference between two consecutive cycles of η is less than 0.0002 . Considering the maximum values of η near time zero are only 0.01, this is probably a poor choice for the error bound. It would be better to choose an error bound dependent on the values of η, such as

(95) $$\left| \frac{\eta^{(k+1)}(x_j,\ t^{n+1}) - \eta^{(k)}(x_j,\ t^{n+1})}{\eta^{(k)}(x_j,\ t^{n+1})} \right| < \epsilon.$$

The most limiting factor in the numerical method was the lack of sufficient computer memory storage. At each node point on the grid, ϕ and ϕ_y had to be stored in two large matrix structures in the computer memory. Although full use was made of the available storage, the step sizes in the x and y directions were more than twice the step sizes used by Ohring. These differences in step size make comparisons between the two programs difficult, especially since my computations break down before t = 0.6 seconds, which is the first time step Ohring gives data for.

6. RESULTS

The results were not as accurate as hoped for when the project was first started. Computer speed and memory size limitations were the main cause for the lack of accuracy. Figures 6 -9 show the calculated waveforms for various times for each of the problems solved. Figure 10 shows one of Ohring's results. Note the instability that occurs at x = 2.8 in Ohring's graph.

Ohring applies a filtering function, due to Longuet-Higgins, on the upstream end of the domain (from i=0 to i=28) to smooth instabilities which develop at the upstream end of the pressure distribution on the surface. These instabilities are inherent in the problem because of the current running into the pressure distribution. This smoothing function is justified in a physical sense, but it has no meaning mathematically, and it would be better to use a finer grid and smaller time steps to smooth results. Even after applying the filtering function Ohring's results still show instabilities in this region. (See Figure 10.) The results from my program have instabilities at precisely the same places as Ohrings', although they are more pronounced due to a coarser grid size than that used by Ohring.

Ohring briefly mentions in the back of his paper that for some cases the calculations break down at the first downstream crest before the second crest has fully developed, but he essentially only emphasizes his successes. He only computes two cases out to 6.3 seconds; the other five cases break down earlier.

The instabilities at the upstream end of the pressure distribution indirectly cause my program to fail. The resulting steep slopes of η cause the first few guesses of $\eta(i)$, calculated by Newton's method, to be 2-3 times larger than the actual value $\eta(i)$. Although the value $\eta(i)$ quickly converges, an error occurs if one of the first guesses is higher than the top of the grid, where y is 0.24 . Outside of the grid, none of the variables used in approximating $\eta(i)$ are known, and the computer arbitrary assigns values. The grid needs to be computed further up in the y direction, but the grid used in my program was already the largest allowed by the computer. This would not be a problem on a larger computer.

The grid size also causes poor resolution of the waveform in the areas of instability. This accounts for the ragged appearance of the waves. Note that the instabilities in the pressure distribution problem were inherent in the problem. Problems using flap or piston type wavemakers do not have these instabilities. The full-flap wavemaker problem solved in this paper worked correctly until the calculated waves were large enough to cause the problem with Newton's method described above. For this method to be useful in the future for predicting information about nonlinear waves, a finer grid should be used, and perhaps even a smaller time step. The method should be checked to see if it converges to a solution as the step sizes get smaller, and the results should be analyzed to see how closely the model represents real waves.

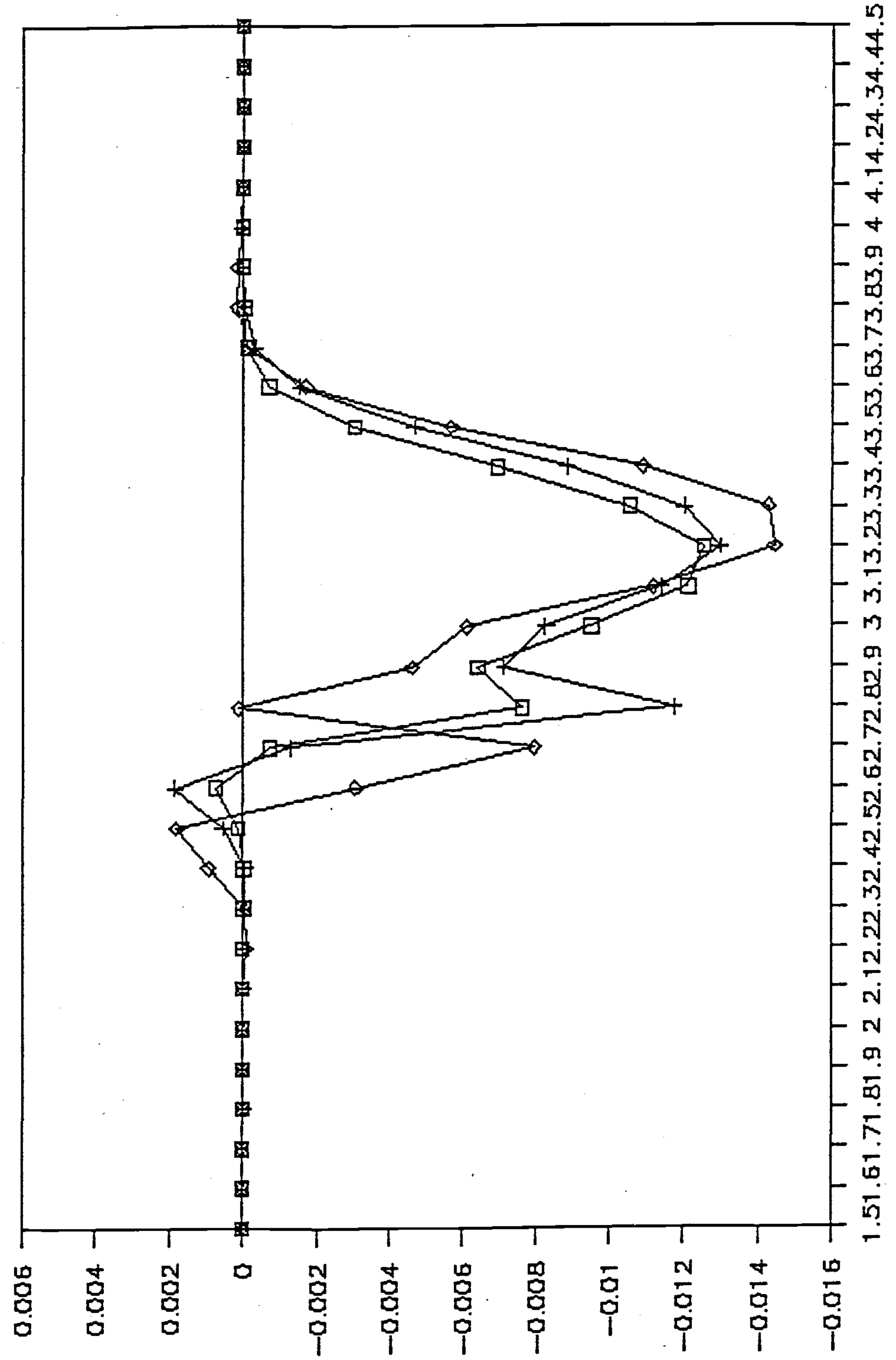

Figure 6. Surface Elevation for the Surface Pressure Distribution Problem.

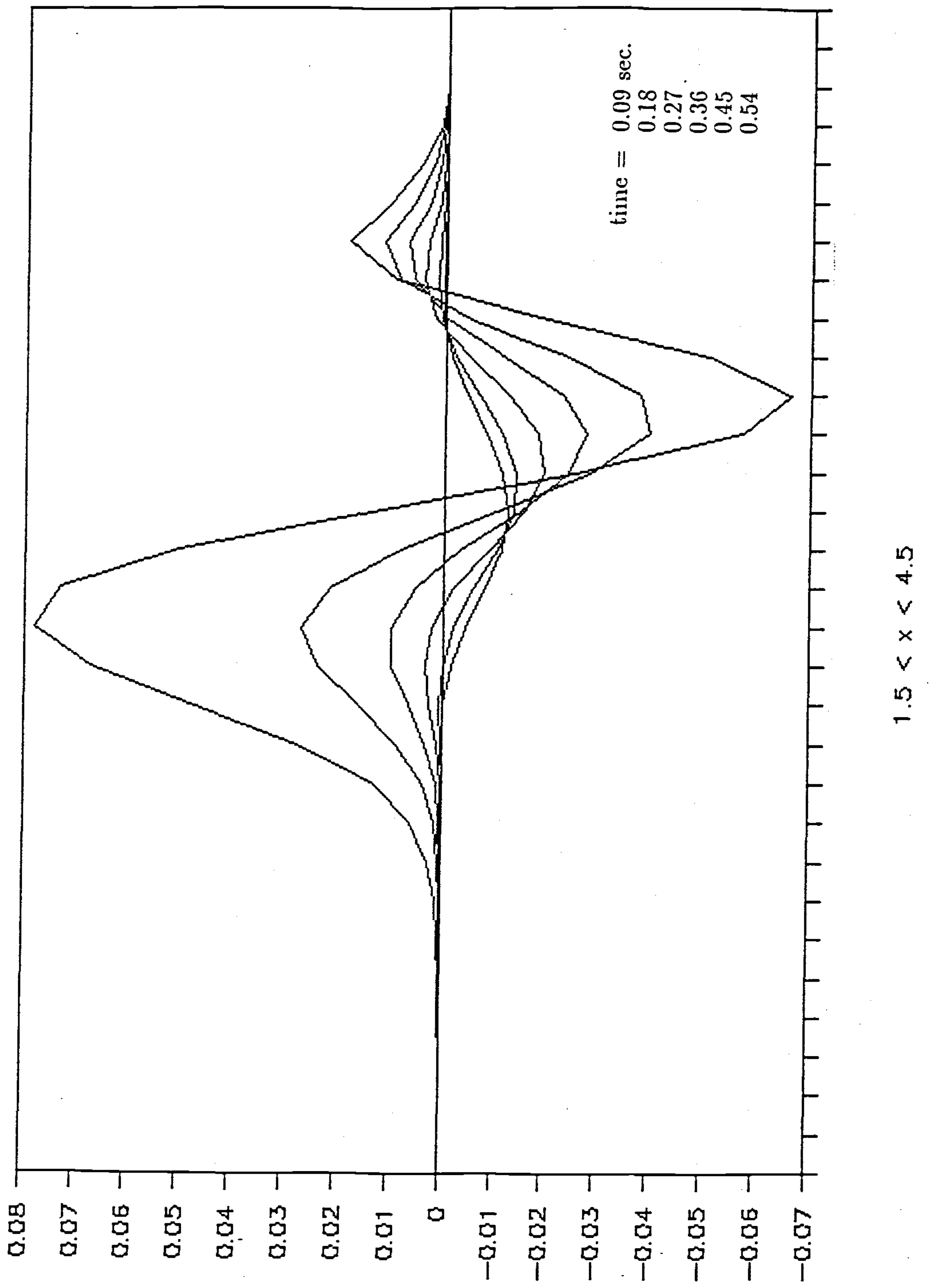

Figure 7. Surface Elevation for the Surface Pressure Distribution Problem Without Application of the Filtering Function.

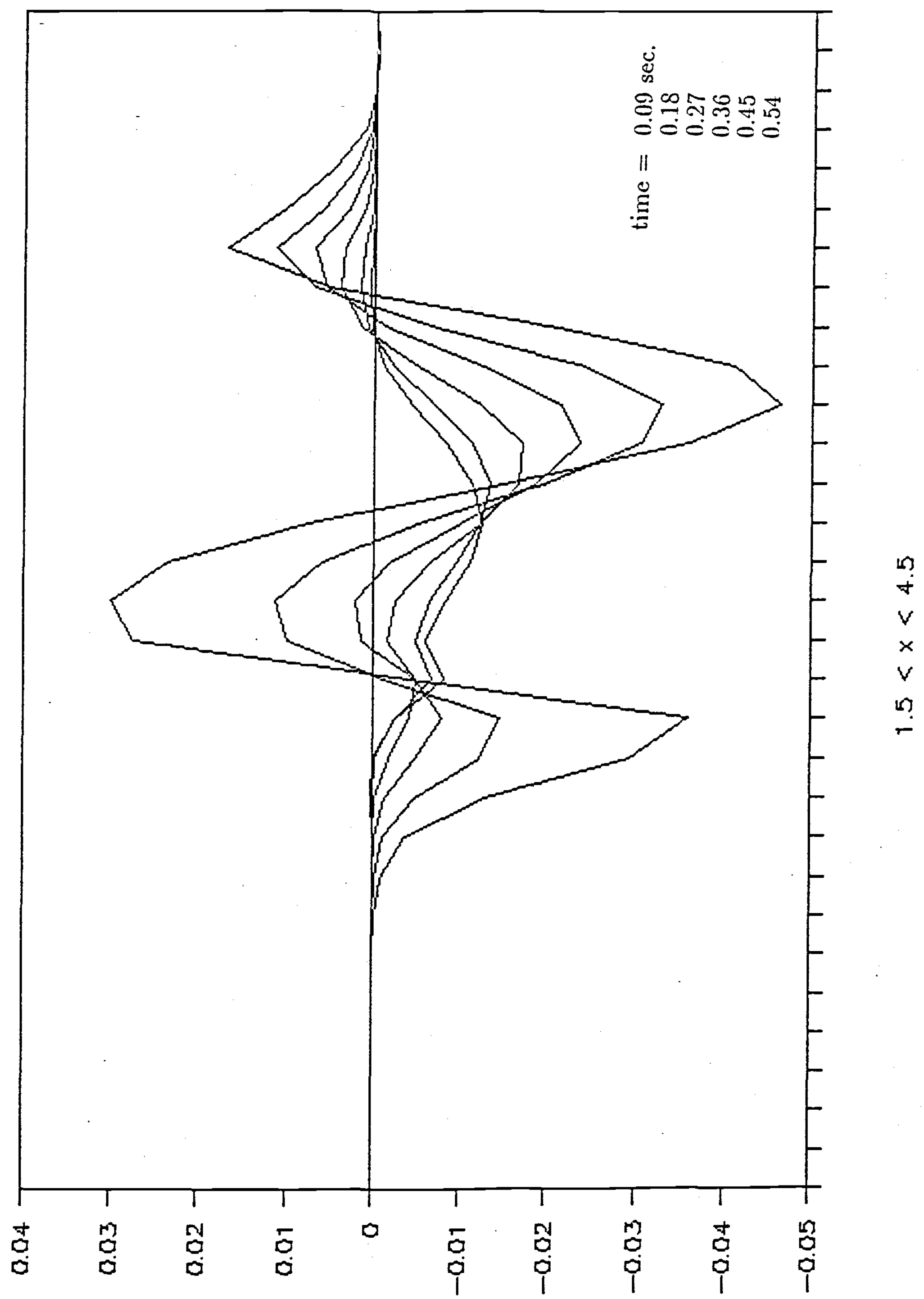

Figure 8. Surface Elevation for the Surface Pressure Distribution Problem With Application of the Filtering Function.

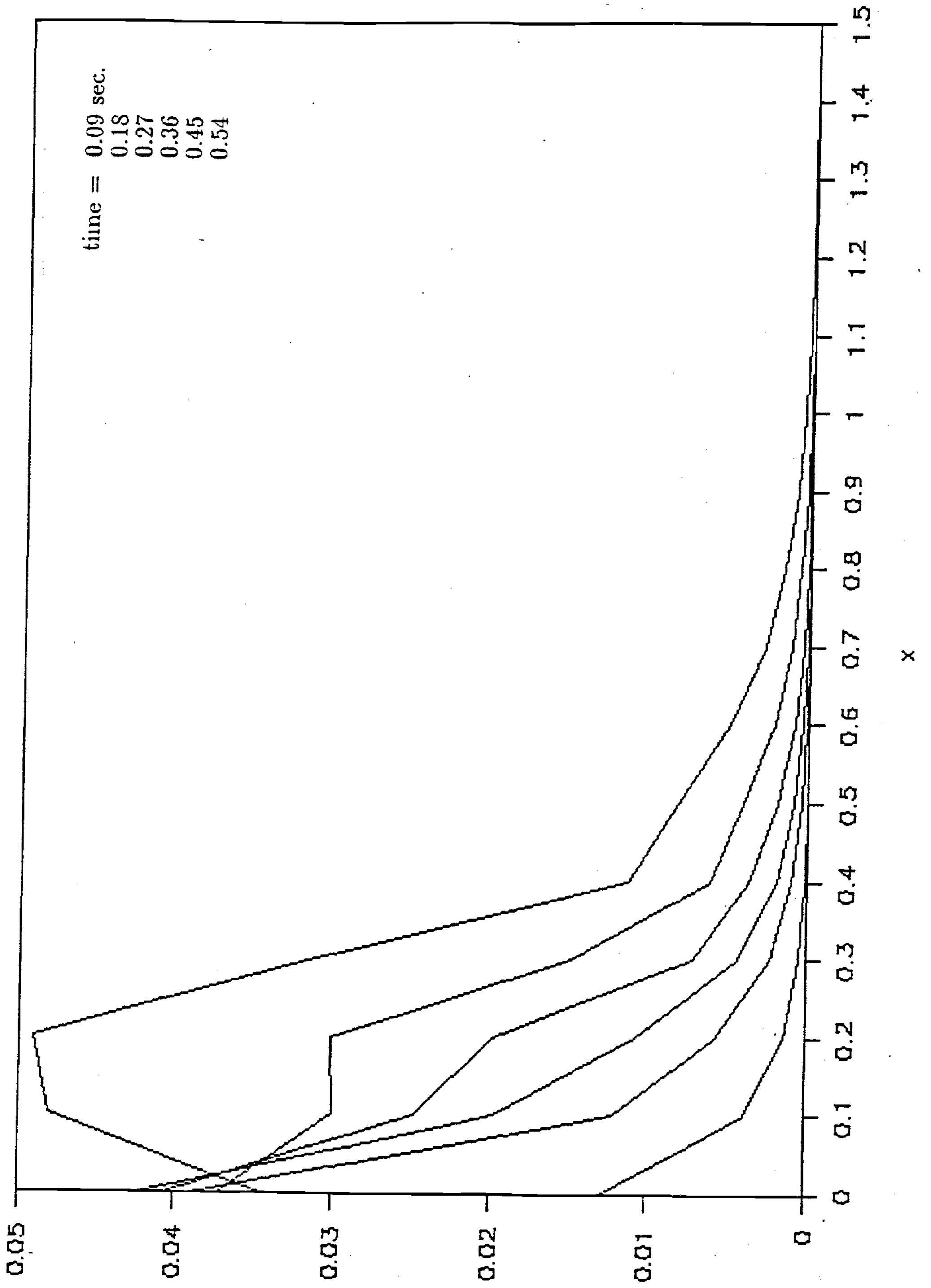

Figure 9. Surface Elevation for the Wavemaker Problem.

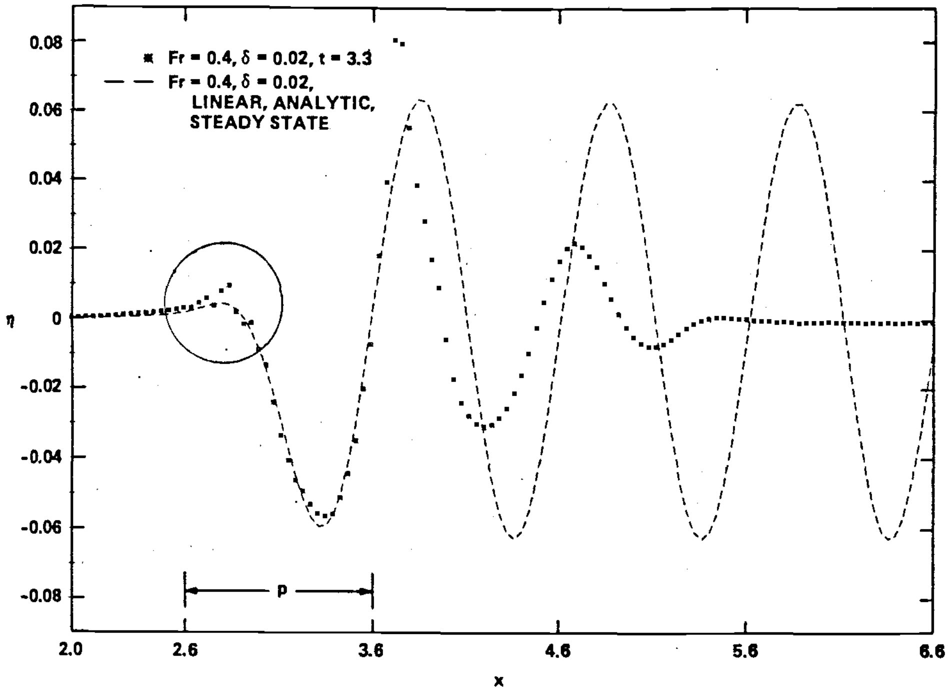

Figure 10. Ohring's Results for the Surface Pressure Distribution Problem.

7. CONCLUSION

The governing equations for a generic wavemaker model for nonlinear waves are derived and then applied to two cases: the surface pressure distribution problem and the full-flap wavemaker problem. These problems are then solved numerically using the method of lines.

The method of lines is shown to simulate the problem given, but accuracy is limited by the speed and memory storage of the computer used. The method should be checked to see if it converges as Δx, Δy, and Δt decrease to zero. Since this method does not rely on the existence of analytic solutions, once convergence has been shown the method of lines can be applied to the wavemaker problem with almost any wavemaker configuration. The method of lines has the additional advantage that it is easily extended to three dimensions.

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