A DEVICE FOR FOCUSING ELECTRONS IN AN ELECTROSTATIC FIELD

by

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TABLE OF CONTENTS

TABLE OF ILLUSTRATIONS

A DEVICE FOR FOCUSING ELECTRONS IN AN ELECTROSTATIC FIELD

INTRODUCTION

In view of the growing importance of electron optics, especially electron focusing in such apparatus as cathode ray tubes, television tubes, and electron microscopes, it has become desirable to have an apparatus that will demonstrate electron focusing in the laboratory. The following is a description of an apparatus that will demonstrate a number of principles of electron focusing.

Electron optics deals with the trajectories of electrons in electric and magnetic fields considered from the point of view of geometric optics.

The index of refraction of light, which is the ratio of the velocity of light in a vacuum to the velocity of light of a particular wave length in any substance, is analogous to electron velocity in electron optics. An electron moving at an angle other than zero to the lines of force in an electrostatic field follows a curved path. According to electron optics the path is curved because the electron passes through a medium in which the index of refraction is a function of position.

The system to be discussed illustrates a case of an electrostatic field having axial symmetry. Such a system will be found to be similar to a glass lens for light.

Electrostatic lenses having axial symmetry are formed by applying various potentials to electrodes, such as co-axial cylinders having axial symmetry. The focusing action of such an electron lens is uniquely and readily determined for paraxial electrons. Hence, given a source of electrons and an electron lens, the position and size of the electron image of the source may be easily determined (as with light) from a knowledge of the size of the source and the distance between the lens and the source.

The two principal problems of electron optics are (1) lens analysis and (2) lens synthesis. Lens analysis includes the determination of the focal and principal points, and the determination of lens aberrations. Lens constants and aberrations may be determined from electron trajectories through the electrostatic field; therefore, the major factor of lens analysis is electron-trajectory determination. Lens synthesis involves the determination of electric fields or the electrode arrangement necessary to produce a lens of given constants and aberration. Lens synthesis is by far the more difficult. The aberrations will not be considered.

It was known to the earliest investigators that an object in the path of a cathode-ray beam would cast a sharp shadow on a fluorescent screen. In this respect cathode rays resembled light rays.

Over one-hundred years ago, Sir William Hamilton (2, p.1 and 67) showed that there is an analogy between the path of a ray of light through refracting media and the path of a particle through conservative fields of force.

The laws formulated by Hamilton were verified soon after the discovery of the electron. Shortly after the discovery of the electron such workers as Classen, Busch and later Knoll and Ruska (3, p.1388) began the calculation of electron paths and thus laid the groundwork for the science of electron optics.

FUNDAMENTAL CONCEPTS

In the present discussion the electrons may be thought of as small electrically charged spheres with a diameter in the order of 10^{-12} cm, a charge of 1.6×10^{-19} coulomb, and a mass of 9×10^{-28} gram. According to the Bohr theory, every nucleus is thought of as being surrounded by an atmosphere of these minute charged particles coursing about it in planetary orbits, the individual electrons being prevented from escaping by the strong attracting electric field of the positively charged particles. This attractive force is least for the electrons traveling in the outer fringe; where, in a large part, they are balanced by the repellent forces exerted by the remaining electrons. The outer electrons, or valence electrons, are more readily detached from the atoms than the rest.

Under normal conditions free electrons are not free to leave the metal since the attractive forces exerted by the positively charged atom cores and ions near the surface prevents their escape. One way to get the electrons out of the metal into free space is by heating the metal to a high temperature. The ions are set into strong vibrations causing collisions between free electrons and ions, and even between free electrons, giving some of the electrons sufficient energy to escape. If the potential energy of an electron inside a metal is less than outside, the difference may be considered as the work (or kinetic energy converted into potential) that an electron has to do against the electrical surface forces in order to escape. This idea is commonly expressed by saying that the metal is surrounded by a "potential barrier." For example, if nickel has a potential barrier of 16 volts, an electron at rest within the metal can escape from the surface only after obtaining in some manner velocity whose component normal to the surface is equal to that which might have been obtained by being subject to a potential difference of 16 volts. The higher the temperature of a piece of metal such as a lamp filament, the more electrons will have the energy necessary to escape. The number of electrons that escape from the metal per unit time per unit area at any temperature T°K is

$$i = AT^2 \exp -\left(\frac{Wa - Wi}{kT}\right)^{1} . \qquad (1)$$

A is a constant, Wa is the height of the potential energy barrier, Wi is the maximum kinetic energy of the free electrons inside a metal, and k is Boltzman's constant. The quantity $\theta = \frac{300(Wa - Wi)}{e}$ is the work function and is the additional energy which must be supplied to an electron

1. For derivation see Zworykin Television, p. 11-22.

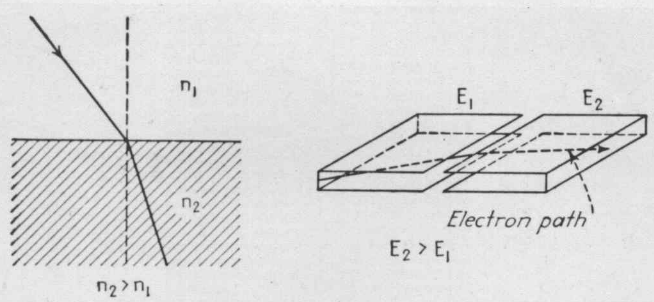
inside the metal in order to enable it to escape from the metal. $(W_a - W_i)$ is in ergs and θ is in volts. θ is ap- aproximately 5 volts for nickel. If an oxide coated cathode is used θ can be reduced to about 1 volt. The cathode material is nickel and is coated with a mixture of the carbonates of barium and strontium.

ANALOGY BETWEEN ELECTRON AND LIGHT OPTICS

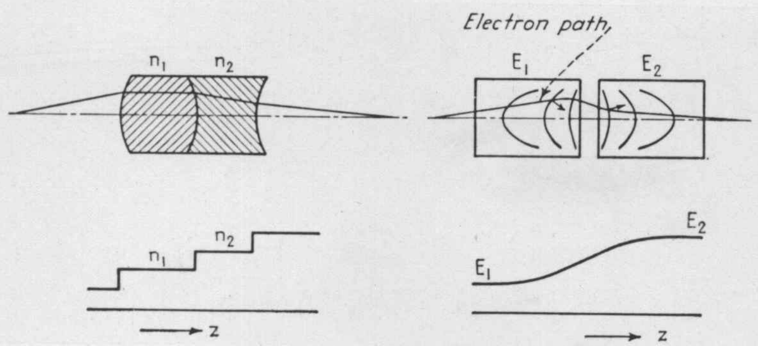
A knowledge of the electron paths through fields with radial symmetry is necessary for an understanding of this problem. An electrostatic field can be shown to cause the electron paths to bend in a manner similar to the way in which light is bent when going through a lens. It is therefore possible to replace a radially symmetric electronic system with an analogous optical system.

Electron optics have nothing to do, except by analogy, with optics in the ordinary sense of the word, but is the subject dealing with the paths of electrons in an electrostatic or magnetic field. The analogy is between the behavior of an electron in an electrostatic field and the behavior of a light ray in a medium of variable index of refraction. Electrons can be reflected, refracted and focused as in the case of light rays. (9, p.4-13). Corresponding to Fermat's principle of least time for physical optics, there is the Hamiltonian principle of least action that applies to electron optics. (See Appendix 1)

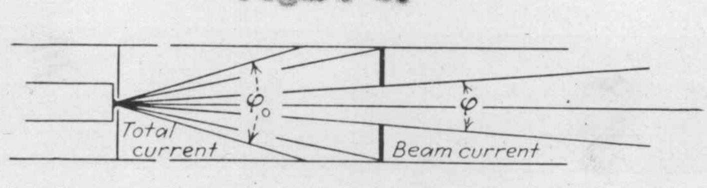
In light optics a ray is refracted upon passing a boundary between a medium of one index of refraction and a medium of another. A ray passing from a region of refraction n_1 (Figure 1) to a region of higher index n_2 is bent toward the normal. Similarly in the corresponding electron-optic case, an electron passing from a region in



Refraction of an electron and of a light beam
Figure 1.



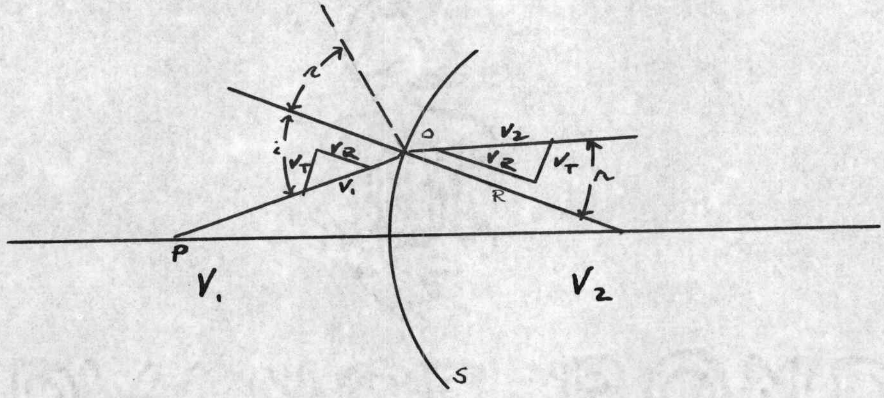
Simple electron-optical lens
and physical equivalent
Figure 2.



Solid angles subtended at cathode
by total and beam currents
Figure 3.

which it has a velocity v_1, to a region in which it has a velocity v_2, greater than v_1, is bent toward the line normal to the boundary between the two regions. The potential changes gradually rather than abruptly, and any changing potential field is capable of causing an electron to follow a path characterized by a curve.

To illustrate the focusing action of the electrostatic field consider the artificial case of a spherical surface separating two media of different indices of refraction. Let S of Figure 4 be such a surface, and let R be its radius; further let the electrostatic potential to the left of S be V_1 and to the right V_2. Now consider an electron moving in the direction PO with the velocity v_1. As it arrives at the surface a force normal to the surface in the direction of R will act on it, and when it has passed through the surface its velocity will have changed to, say, v_2. The force being normal to the surface, only the component of initial velocity v_r normal to the surface will change; the tangential component of velocity v_r will be the same on both sides of the surface.



Electron refraction and reflection at spherical surface
Figure 3a.

It is noted that

$$v_t = v_2 \sin r = v_1 \sin i, \qquad (2)$$

where i and r are the angles of incidence and refraction, respectively. Therefore,

$$\frac{\sin i}{\sin r} = \frac{v_2}{v_1} = u \text{ (a constant)} \qquad (3)$$

The work done by the field on the electron when it goes from the first to the second medium is $e(V_2 - V_1)$. From the law of conservation of energy it follows that

$$\tfrac{1}{2}mv_2{}^2 = \tfrac{1}{2}mv_1{}^2 + e(V_2 - V_1) \qquad (4)$$

and so by dividing 4 by $\frac{1}{2}mv_1^2$, it becomes

$$u = \frac{v_2}{v_1} = \sqrt{1 + \frac{(V_2 - V_1)e}{\frac{1}{2}mv_1{}^2}} \qquad (5)$$

The electron velocity v_1 corresponds to the voltage

V_1, then $\frac{1}{2}mv_1^2 = eV_1$, and (1, p.1100)

$$u = \frac{v_2}{v_1} = \sqrt{1 + \frac{(V_2 - V_1)}{V_1}} = \sqrt{\frac{V_2}{V_1}} \tag{6}$$

If V_2 V_1 so that $V_2 - V_1$ is negative, and if in absolute magnitude it is larger than $\frac{1}{2}m(v_1 \cos i)^2$ - the part of the kinetic energy of the electron corresponding to the normal component of its velocity - then the electron will be reflected back from the surface with its normal velocity component reversed.

Any electrostatic field that has a rotational symmetry about some straight-line axis has the properties of a lens. The symmetrical electrostatic field is produced by any set of electrodes that have rotational symmetry about the axis. The simplest type electron lens is a hole in a circular plate. It consists of a circular aperture placed between planes where there is a difference in the potential gradient on the two sides. There is a penetration of the field and the lens action is divergent.

A system equivalent to that used in this particular case is approximated by a lens consisting of the field between two coaxial cylinders placed end to end and maintained at different potentials.

The equivalent physical lens consists of a combination of convex and concave surfaces between regions of successively higher indices of refraction. In the equal-diameter cylinder lens, as this is called, the equipotentials in the first cylinder are concave in direction of increasing potential, while those in the second cylinder are convex. An electron entering from the low potential side at a small angle with the axis will experience a force toward the axis and then experience a force away from the axis. The first part of the lens has a convergent action while the second part of the lens has a divergent action. The convergent action is stronger because the electron velocity is lower in the convergent portion of the lens and hence the electron here is deflected more for the same force.

It is noted in Figure (2) that the front surface of the physical lens components are curved in the same way as are the equipotentials in the electron lenses. This gives an indication of how an electron lens will behave merely by inspection. There is a correlation between the nature of the equipotential surfaces and the curvature of the curve of potential along the axis against distance in the axial direction. It is noted that when the curvature of the potential distribution along the axis is upward, then the equipotentials are concave in the direction of increasing potential and the lens has a convergent action.

The potential distribution for a configuration can be obtained by solving the reduced Laplace equation. (See

Appendix 5)

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0 \quad [1] \tag{7}$$

subject to the boundary condition that V assume the given values of potential on the electrodes. In fields of rotational symmetry, the potential at any point in the field can be determined if the variation of potential with distance along the axis is known. For most boundary conditions the solution of the differential equation is difficult and no analytic solution is possible.

The potential distribution in space is uniquely determined if the distribution of potential along the axis together with its even derivatives are known. This is shown as follows: $V(r,z)$ due to the axial symmetry, can be expanded in an infinite series containing only even powers of r

$$V(r,z) = V_o(z) + r^2 V_2(z) + r^4 V_4(z) + \ldots + r^{2n} V_{2n}(z) + \ldots \tag{8}$$

where $V(r,z)$ = potential at a radial distance r from the axis and at an axial distance z from some reference point on the axis.

$V_o(z)$ = potential on the axis at the point $(0,z)$

[1] The equation is similar to the equation of continuity in that the same number of electrostatic lines must leave as enter.

Substituting (8) in (7) and equating the coefficients of equal powers of r to zero, there results:

$$V(r,z) = V_o(z) - \frac{r^2}{2^2}\, V_o''(z) + \frac{r^4}{2^2 4^2}\, V_o''''(z)$$

$$+ \dots + \frac{(-1)^n\, r^{2n}}{2^2 4^2 \dots (2n)^2}\, V_o^{2n}(z) + \dots \qquad (9)$$

where $V_o''(z)$ = second derivative of axial potential with respect to axial distance z, and

$V_o^{(2n)}(z)$ = 2 nth derivative of axial potential with respect to axial distance z.

By setting $r = 0$ it is noted that $V(0,z) = V_o(z)$. $V_o(z)$ represents the distribution of potential along the axis. Hence, if the function $V_o(z)$ together with its even derivatives are known, then the potential distribution along the axis can be found. There are graphic methods for plotting electron paths on an equipotential map which are fairly easy, rapid, and sufficiently accurate for most practical problems. (8, p.79)

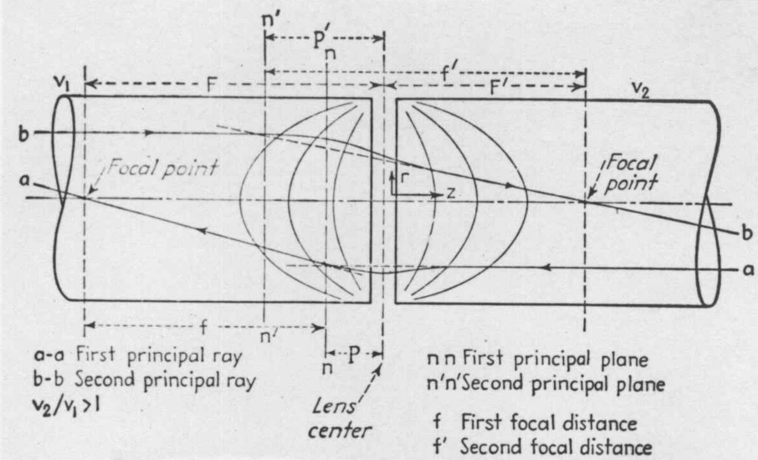
The differential equation of an electron path through a field of rotational symmetry is (See Appendix 3)

$$\frac{d^2r}{dz^2} - \frac{V_o'(z)\, dr}{2V_o(z)\, dz} + V_o''\, \frac{r}{2} = 0 \qquad (10)$$

This equation is valid only for "paraxial" electrons, i.e., those electrons that are within a few percent of the electrode radius from the axis, and that make an angle of a few degrees or less with the axis. The above equation is incapable of exact solution, but good numerical approximations can be made by various methods.

Certain deductions can be made from the differential equation (10), regarding the nature of the electron paths. Since it is of the second order, its complete solution may be expressed as the sum of two linear independent solutions, which means that any electron path through the lens may be described as the sum of two different and distinct paths. (See Appendix 4). The pair of independent paths used represents (1) an electron entering the lens parallel to the axis and (2) an electron leaving the lens parallel to the axis. These paths are the <u>principal rays</u> of the lens and are indicated in Figure (4). The ray that leaves the lens parallel to the axis is the <u>first principal ray</u>. The ray that enters the lens parallel to the axis (from the left) is known as the <u>second principal ray</u>. The focal points of the lens are those points at which the principal rays cross the axis.

The intersection of the initial and final straight-line portions of a principal ray lies in a principal plane. Each lens will in general have two principal planes. The distance from a focal point to its corresponding principal plane is the focal distance or focal length of the lens. From Figure (4) it is noted that the electrons are affected by the field only through a limited



Terminology of electron lens

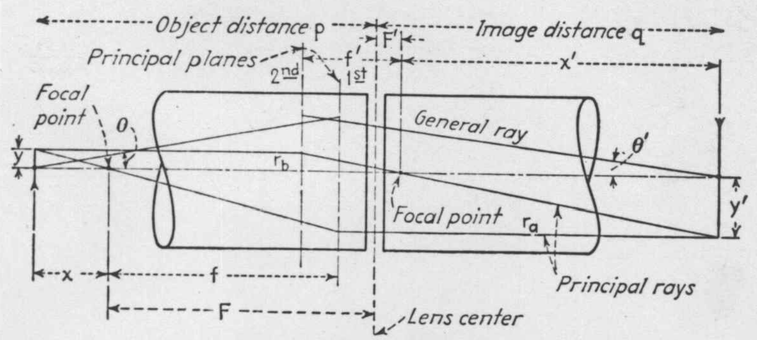


Image formation in electron lens

Figure 4.

distance. When this distance is length comparable to the focal length, the lens is known as a thick lens. For any lens there exists the relation: (8, p.325)

$$\frac{f}{p + P} + \frac{f'}{q + P'} = 1 \qquad (11)$$

where f = first focal length

f' = second focal length

p = distance between object and lens center

q = distance between image and lens center

P = distance between first principal plane and lens center

P' = distance between second principal plane and lens center

Equation (11) indicates the position of the image for any position of the object in terms of four fundamental quantities of lens, f, f', P and P'. The object and image distances as measured from their corresponding principal planes are related by Newton's formula. (See Appendix 5)

$$pq = ff' \qquad (12)$$

The ratio of the two focal distances is related to the ratio of the electrode potentials by (See Appendix 5)

$$\frac{f'}{f} = \left[\frac{V_2}{V_1}\right]^{\frac{1}{2}} \qquad (13)$$

where V_1 = initial potential, that of first electrode

V_2 = final potential, that of second electrode

The relationship, known as Lagrange's law (See Appendix 5), between the lateral magnification, the angular magnification, and the electrode voltage ratio, is

$$m_1 m_a \left[\frac{V_2}{V_1}\right]^{\frac{1}{2}} = 1 \quad (9, p.91) \qquad (14)$$

where $m_1 = y'/y$ in Figure (4), lateral magnification

$m_a = \theta'/\theta$ in Figure (4), angular magnification

V_2 = second electrode potential

V_1 = first electrode potential

It is noted that (1) the principal planes are crossed, i.e., the object and image spaces overlap; (2) the focal length of the image space f' is greater than the focal length of the object space f; (3) the principal planes are located inside the first anode for ($V_1 < V_2$).

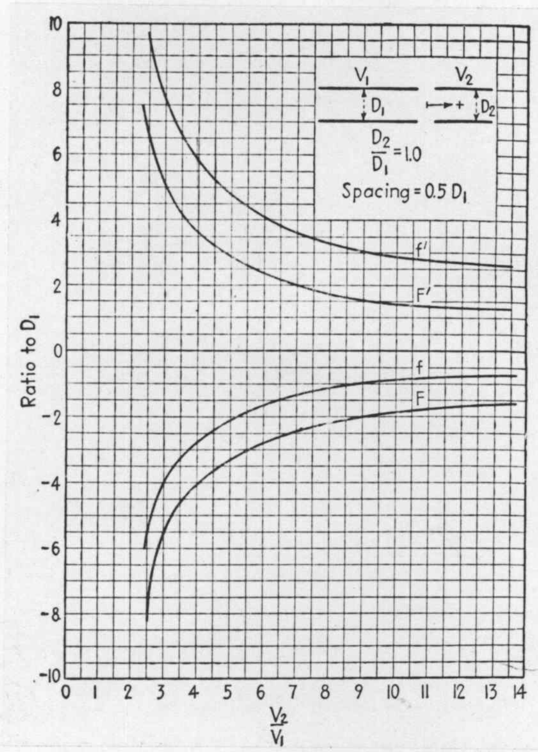
Cardinal points (f', F', f, and F) shown in Figure (4) may be determined by calculating the fundamental trajectories. (6, p.325) (3, p.1398)

FOCUSING CHARACTERISTICS OF LENSES

As already shown, the lenses of electron optics are completely described by four quantities, the two focal lengths and the two principal planes. All four of these quantities depend upon the voltage ratio, so that it became necessary to draw curves showing the dependence of the four quantities as functions of the voltage ratio. (See Figure 5)

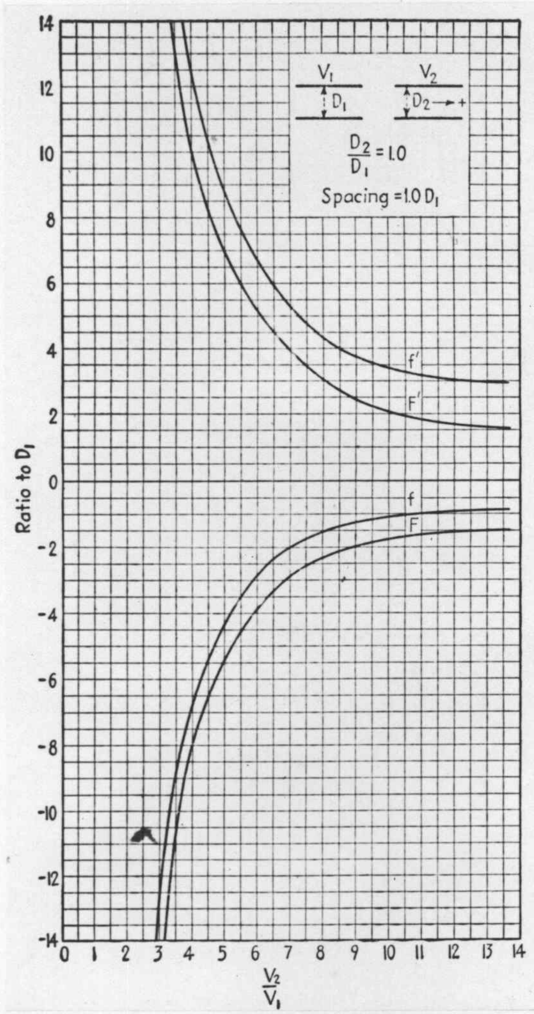
Figure (5) enables one to determine the location and size of the image for any voltage ratio and location of object. It is desirable to present the relationship between object distance, image distance, and magnification in a form that does not involve the focal lengths. Figure (6) gives such curves, the "object-image-distance curves." These curves show that the magnification of all useful lenses will be approximately 0.8 (6, p.332) of the ratio of the image to the object distance. They also show that there is always some particular electrode configuration that makes it possible to achieve any magnification, provided that the proper voltage ratio is chosen.

From the data of Figures (5) and (6) it is noted that the following focal length properties are common to all lenses. "(1) Focal lengths are always uniformly decreasing functions of voltage ratio. (2) Principal planes lie



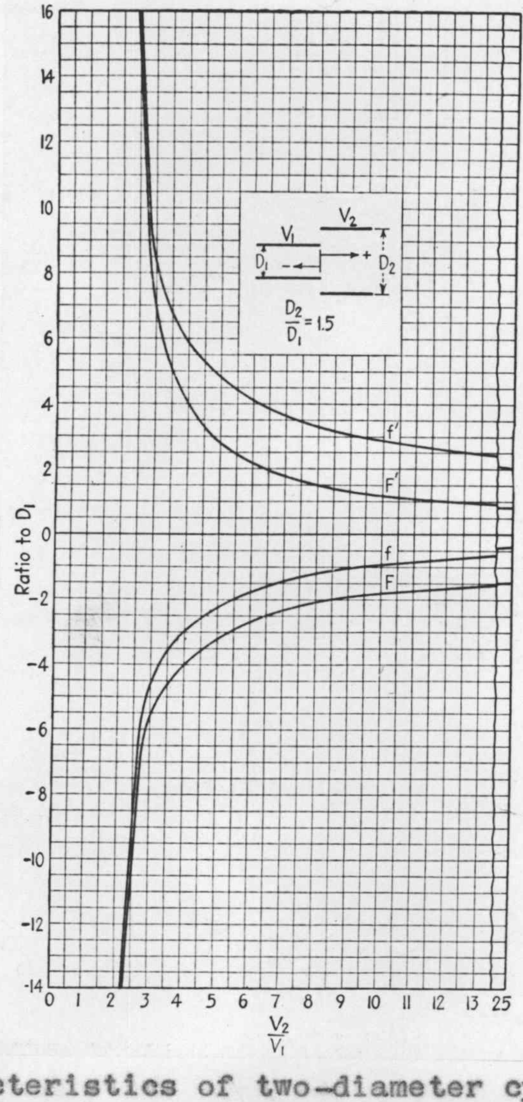
Focal characteristics of two-diameter cylinder lenses as a function of electrode voltage ratio.

Figure 5.



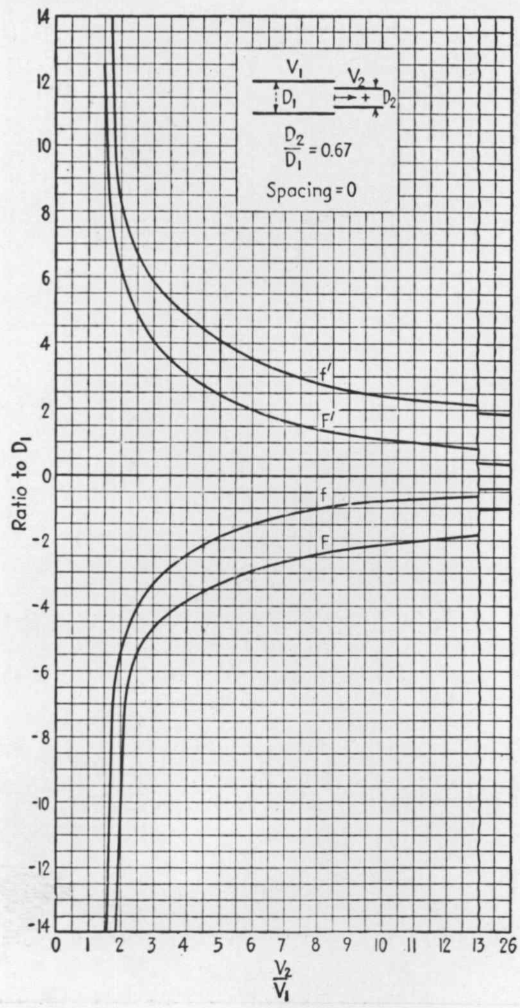
Focal characteristics of two-diameter cylinder lenses as a function of electrode voltage ratio.

Figure 5a.



Focal characteristics of two-diameter cylinder lenses as a function of electrode voltage ratio.

Figure 5b.



Focal characteristics of two-diameter cylinder lenses as a function of electrode voltage ratio.

Figure 5c.

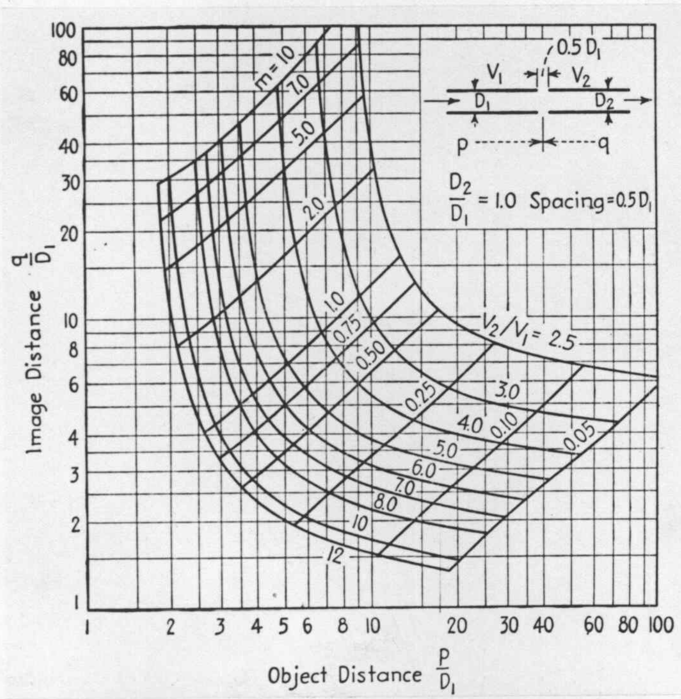


Image-object distance curves
of two-diameter cylinder lenses.

Figure 6.

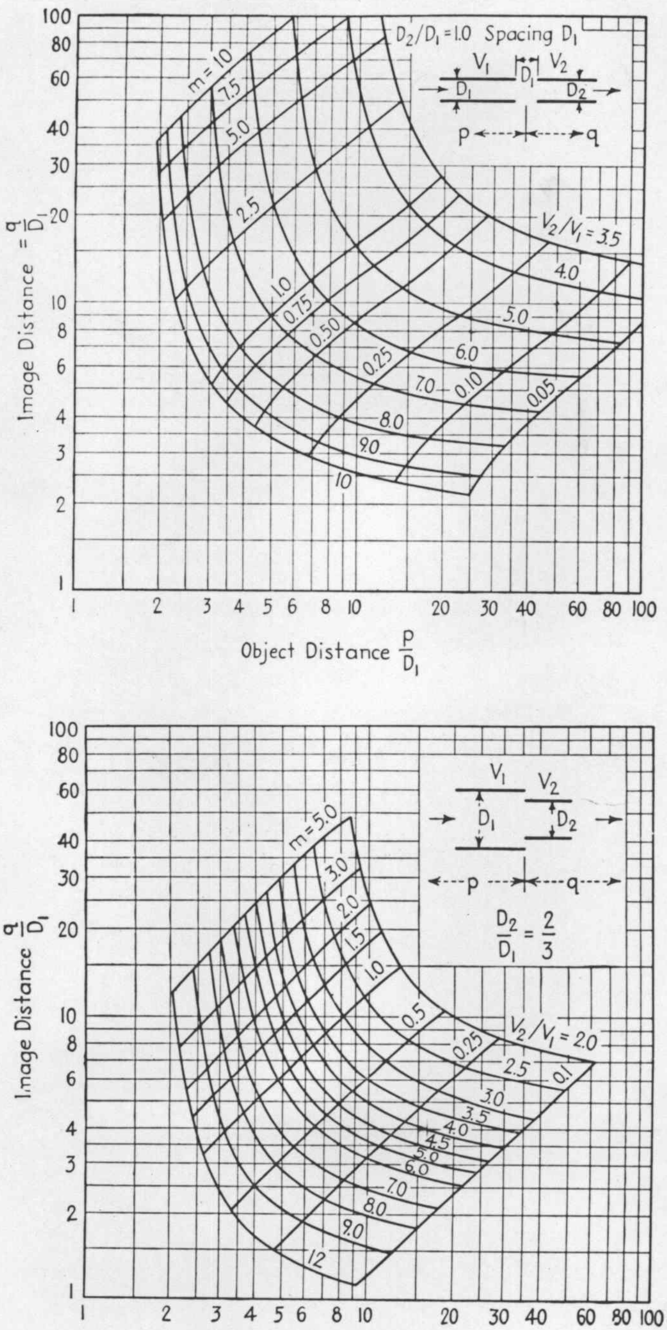


Image-object distance curves
of two-diameter cylinder lenses.

Figure 6a.

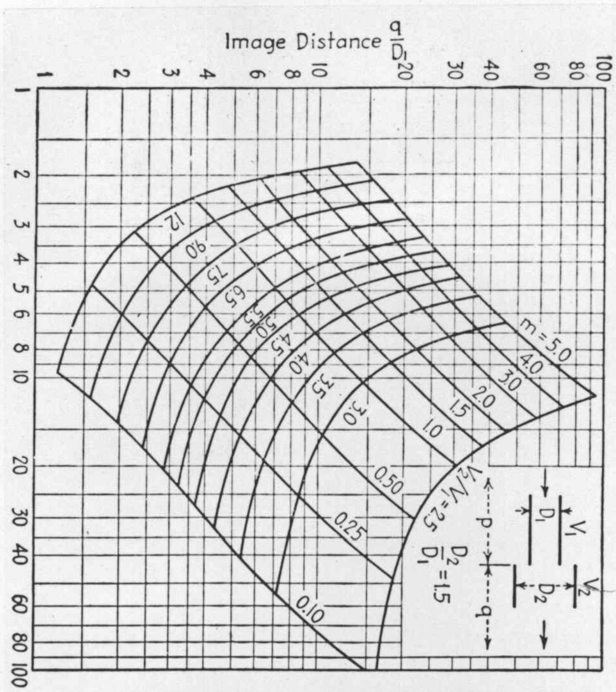


Image-object distance curves of two-diameter cylinder lenses.

Figure 6b.

on the low voltage side of the lens center. (3) Principal planes are crossed, with the exception of large-diameter aperture lenses, i.e., the first principal plane lies between the second principal plane and the lens center on the low-voltage side of the lens. (4) Focal length in the direction of increasing potential is always greater than the focal length in the other direction. (5) Position of principal planes does not change much with voltage ratio."

Properties of the focal lengths for the particular cases indicate "(1) The focal length of two-diameter cylinder lenses increases (the lens grows weaker) for all but the highest voltage ratios, as the ratio of second to first cylinder diameter increases. (2) The focal length of equal-diameter cylinder lenses increases, that is, the lens grows weaker, as the axial spacing of the cylinders increases. The change is small for small spacings but increases rapidly as the spacing is increased. (3) The focal length of aperture lenses (a single short ring or a hole in a plate) increases (the lens grows weaker) as the aperture diameter increases. The change is small for small diameters but increases rapidly as the diameter increases. (4) Aperture lenses, have for the most part shorter focal lengths than cylinder lenses (when the unit of length is the diameter of the first cylinder) if the aperture spacing be taken equal to first cylinder diameter. (5) Cylinder-aperture lenses with axial spacing of one diameter have the longest focal length of all the lenses."

The magnification of the lenses may be summarized as follows: "(1) Contours of constant magnification in the object-image distance curves are approximately straight lines with a slope of one. (2) An approximate universal magnification formula that fits all lenses tested is:

$$M = \frac{kq}{p} \tag{15}$$

where p and q are object and image distances, respectively. The value of the constant k is usually within 1 per cent of 0.8. (3) Magnification in two diameter lenses decreases as the diameter ratio increases. The change is appreciable except between spacings of 0.5 and 1.0. (4) Magnification of aperture lenses with small apertures decreases as aperture diameter increases."

Considerations important in design that are common to all lenses are: "(1) As object distance is increased at a given voltage ratio the corresponding image distance decreases, as does also the magnification. (2) For a given object distance the image distance and magnification decrease as the voltage ratio is increased. (3) In any lens there is a minimum object distance that can be used at any given voltage ratio. The minimum object distance is determined by the vertical asymptote to the contour of constant voltage ratio."

"The effect of object distance and voltage ratio on the relative strength of the lenses can be summarized as follows: (1) Two-diameter cylinder lenses become stronger as diameter ratio decreases for voltage ratios between one and four. This holds for all object and image distances. (2) For larger voltage ratios the lens strength is nearly independent of diameter ratio. (3) Equal diameter lenses become stronger as the actual separation of the cylinders increases for all voltage ratios and normal values of object and image distance. (4) Aperture lenses become slightly stronger as the aperture diameter is enlarged from a very small size to a practical value, and then become greatly weaker."

DESIGN PROCEDURE

Guided by the above summary and the foregoing Figures (5) and (6), optical systems have been designed for four different electron guns. From the focal characteristic curves, Figure (5), the values for f', F', f, and F were determined. From the image-object distance curves (Figure 6) p and q were found. With these six characteristics determined, diagrams were constructed similar to "Image Formation in Thick Lens," Figure (4).

Figure (7) is a schematic drawing of the four designs worked out from Figures (5) and (6). Cases I, II, III and IV are indicated on the drawing, Figure (7). Values were chosen for p and q that would be convenient to construct. The last three cases were chosen so that their p and q values would be the same as Case I. The lens center was assumed different for Cases I and II, while for III and IV it was assumed constant. Diagrams for the above determinations are convenient when drawing fixed object and image positions. In all cases p plus q was maintained at approximately 8.37 inches.

Case I $\frac{f'}{D_1} = 5.2$; $f' = 2.6"$

$\frac{F'}{D_1} = 3.2$; $F' = 1.6$

$\frac{f}{D_1} = 2.4$; $f = 1.2$

$\frac{F}{D_1} = 3.6$; $F = 1.8$

$\frac{p}{D_1} = 5$; $p = 2.5"$

$\frac{q}{D_1} = 12$; $q = 6.0"$

$D_1 = 0.5"$

$D_2 = 0.5"$

Spacing 0.5 of $D_1 = 0.25"$

$\frac{V_2}{V_1} = 4.5$; $\frac{D_2}{D_1} = 1$

Case II $\frac{f'}{D_1} = 6.0$; $f' = 3.0$

$\frac{F'}{D_1} = 4.5$; $F' = 2.25$

$\frac{f}{D_1} = 2.3$; $f = 1.15$

$\frac{F}{D_1} = 3.4$; $F = 1.7$

Case II (Cont.) $\frac{p}{D_1} = 5.3"$; $p = 2.65"$

$\frac{q}{D_1} = 12"$; $q = 6.0"$

Spacing $1.0\ D_1 = 1.0 \times 0.5 = 0.5"$

$\frac{V_2}{V_1} = 6.5$; $\frac{D_2}{D_1} = 1$

Case III $\frac{f'}{D_1} = 5.0$; $f' = 2.5$

$\frac{F'}{D_1} = 3.0$; $F' = 1.5$

$\frac{f}{D_1} = 2.4$; $f = 1.2$

$\frac{F}{D_1} = 3.4$; $F = 1.7$

$\frac{p}{D_1} = 4.74$; $p = 2.37"$

$\frac{q}{D_1} = 12$; $q = 6$

Spacing .01 $\frac{V_2}{V_1} = 4.75$; $\frac{D_2}{D_1} = 1.5$

Case IV $\frac{f'}{D_1} = 4.4$; $f' = 2.2"$

$\frac{F'}{D_1} = 2.7$; $F' = 1.35"$

$\frac{f}{D_1} = 2.0$; $f = 1.0"$

Case IV
(Cont.)

$\frac{F}{D_1} = 3.5$; $F = 1.7"$

$\frac{p}{D_1} = 4.75$; $p = 2.37"$

$\frac{q}{D_1} = 12$; $q = 6"$

Spacing .01 D: $\frac{V_2}{V_1} = 4.5$; $\frac{D_2}{D_1} = 0.67$

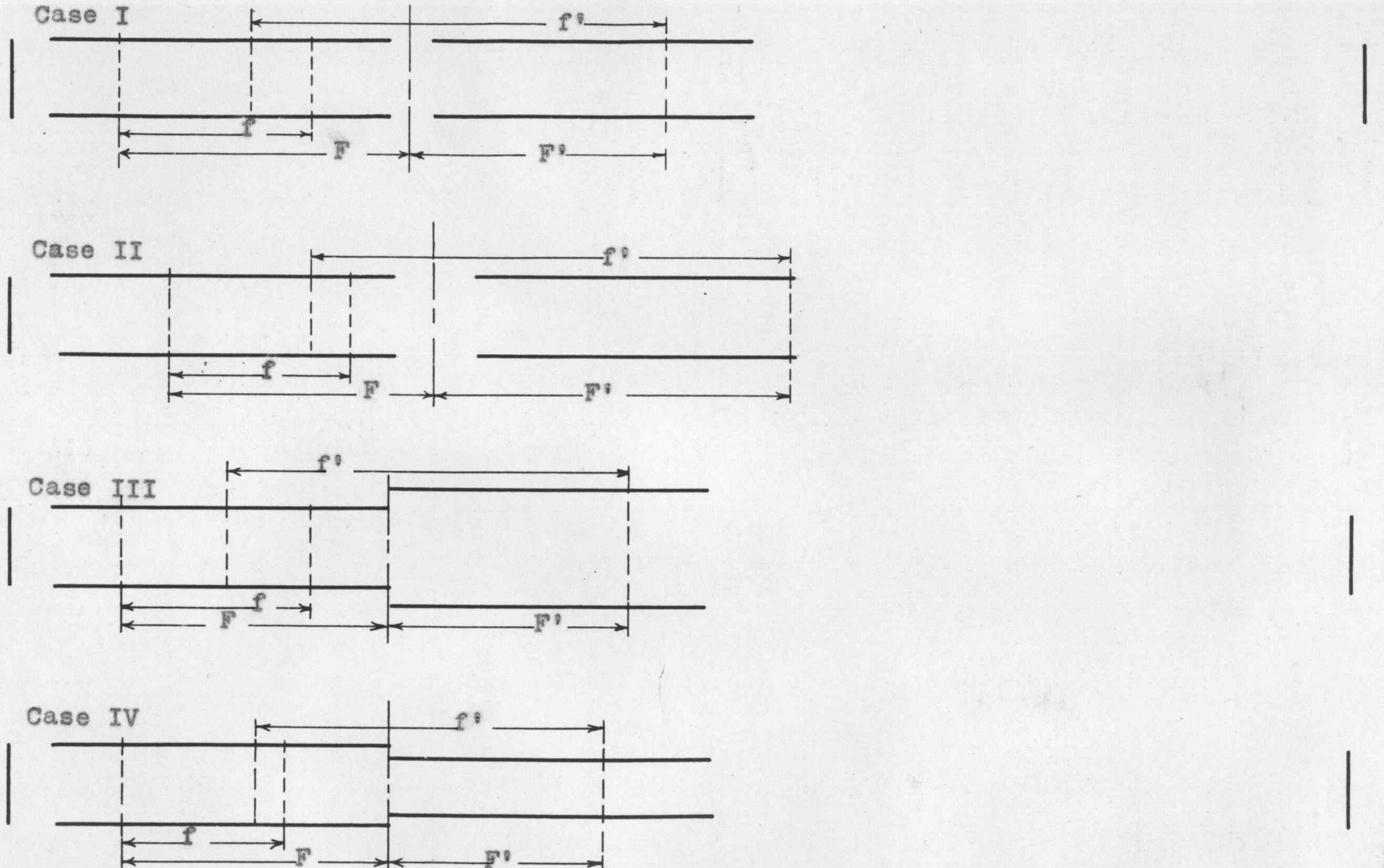


Figure 7. Design Diagrams.

THE ELECTRON GUN

The electron gun is a device for concentrating, controlling and focusing the electron beam. It has an axially symmetric structure and comprises a cathode, grid, a first anode and a second anode. (4, p.371; 7, p.182). A tubular cathode of sheet nickel with a flat oxide coated emitting surface at one end is indirectly heated (Figure 8). A control grid of sheet nickel in the form of an enveloping cylinder with an aperture surrounds the cathode. The electrostatic field created by the potential difference between the cathode and the first anode penetrates the grid and causes the emitted electrons to move nearly parallel to the axis of its cylinder, forming the beam. Apertures within the cylinder serve to remove from the beam strongly divergent electrons that would produce a fuzzy spot on the fluorescent screen. The first anode is frequently referred to as the accelerating anode. The second anode is the focusing electrode. The principal focusing action occurs in the electrostatic field between the first anode and the second anode. In the second anode the electron is given a radial velocity component directed towards the axis of symmetry. The beam is focused by changing the second anode potential, while the beam intensity is controlled by changing the control grid. These two control-actions are almost independent.

In the design of an electron gun, the following components, together with the value used in the present design, are considered as independent variables. (See Figure 8). These values were taken from experimental curves by Epstein and Maloff.

δ	Cathode-grid spacing	0.012"
a_g	Diameter of grid aperture	0.035"
d_g	Diameter of grid skirt	0.5"
l_g	Length of grid skirt	2.0mm
δ'	Grid-first anode spacing	0.1593"
l_f	First-anode length	2.125"
a_s	Diameter of first anode stopping aperture	0.052"
l_a	Distance between stopping aperture and cathode	1.406"
d_1	Diameter of first anode	0.5"
E_g	Voltage on Grid	
E_{p_1}	Voltage on first anode	Variable within power supply limits
E_{p_2}	Voltage on second anode	

The following components are plotted as a function of the cutoff voltage: Grid aperture, cathode grid spacing, grid skirt length; backing aperture voltage on first anode.

The following components are plotted as a function of the total current: Grid skirt length, voltage on first anode. All other components mentioned above are given for each curve. From these curves it was possible to arrive at suitable values for each component.

The useful current is that portion of the total current which constitutes the beam that strikes the screen to form the luminescent spot. Only that portion of the total current which passes through the stopping aperture is the useful current or beam current. If it is assumed that the current density along any cross section of the total current is constant; then to a good approximation the beam current I_{p_2} is given by the relation (Figure 3)

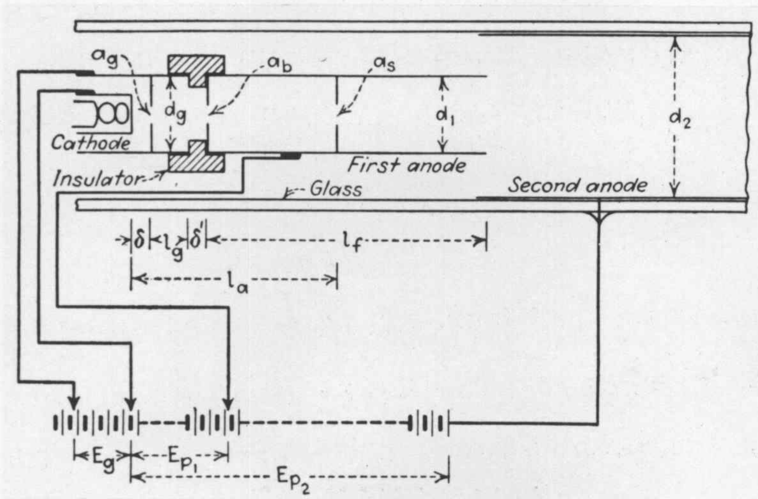
$$I_{p_2} = I_t \, \phi/\phi_o \; (\phi < \phi_o) \quad (2, \text{p.77}) \qquad (16)$$

where ϕ is the solid angle subtended at the point of intersection between the cathode and axis by the stopping aperture, and ϕ_o is the solid angle subtended at the same point by the total current I_t. For the small solid angle ϕ in radians^2 is given by

$$\phi = \pi (a_s/l_a)^2 \qquad (17)$$

where a_s is the diameter of first-anode stopping aperture and l_a = distance between stopping aperture and cathode.

From the experimental curves (2, p.172-175) the largest valid ϕ (the solid angle subtended at the cathode by the stopping aperture) is 10^{-5} radians^2. Substituting



The electron gun.

Figure 8.

$\phi = 10^{-5}$ radians^2 in equation (17) allows one to choose a_s and l_a.

The screen is coated with a fluorescent material which emits light upon bombardment by electrons in a vacuum. Common fluorescent materials are phosphors of zinc, cadmium, and calcium whose characteristics are given by Terman. (6, p.343). Coatings are made by settling out the material from a water suspension. Screens can also be made by spraying, using a volatile organic liquid such as acetone with a small amount of binder.

The anode lengths and also the grid length were chosen so as to fit conveniently between the cardinal points. (See Figure 7).

These dimensions are as follows:

Grid	
length	0.75"
diameter	0.5" I.D.
aperture	0.035
skirt length	2.0 mm
First Anode	
length	2.125"
diameter	0.5" I.D.
aperture	0.053"
aperture distance from cathode	1.406"

Second Anode

No. 1

length	2.25"
diameter	0.5" I.D.
distance from first anode	0.25"

No. 2

length	2.25"
diameter	0.5" I.D.
distance from first anode	0.5"

No. 3

length	2.0"
diameter	0.75" I.D.
distance from first anode	0.01"

No. 4

length	2.00"
diameter	0.375" I.D.
distance from first anode	0.01"

Other details of design are indicated in Figure (9).

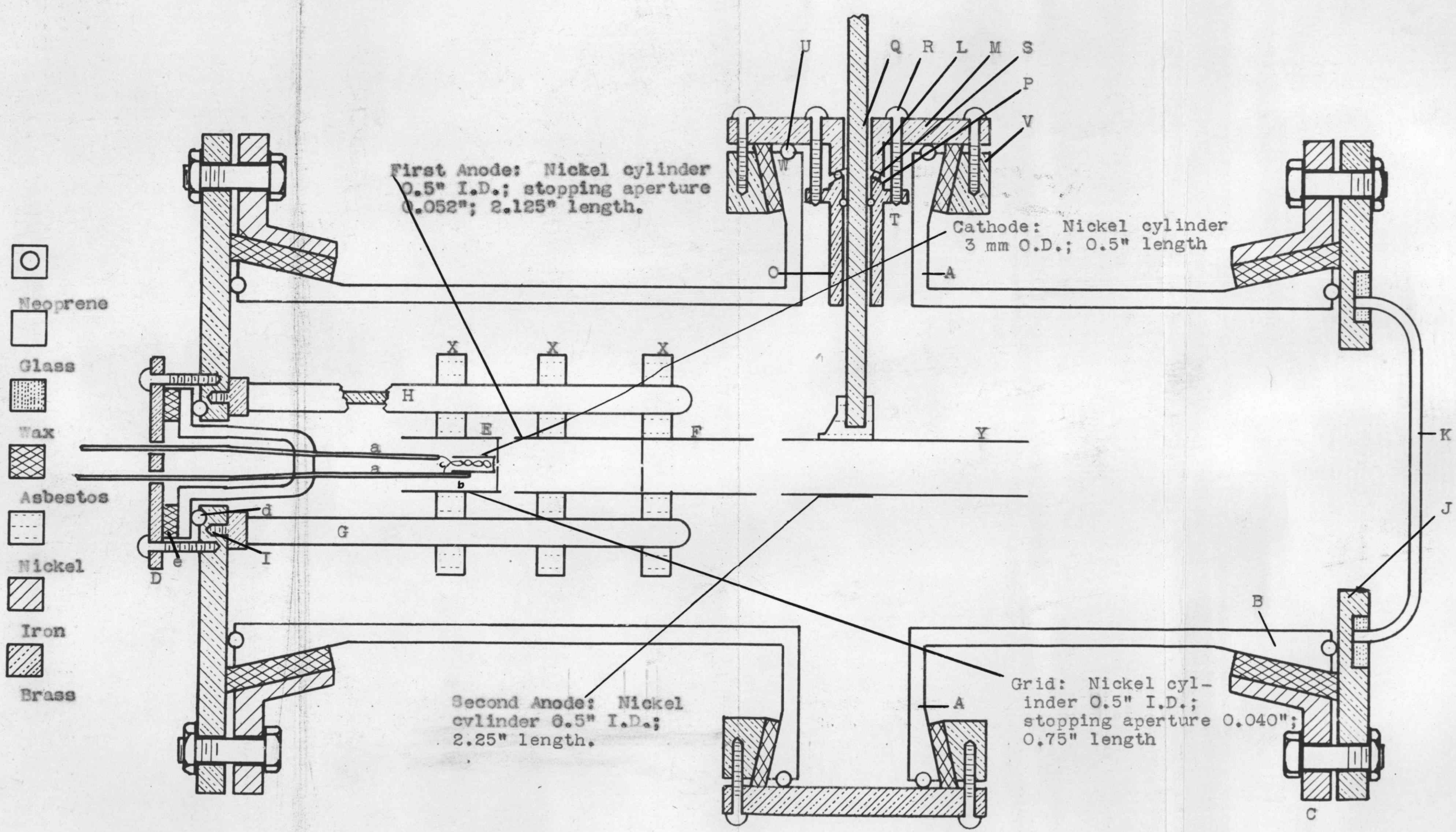


Figure 9. Full Size Cross Section of Apparatus.

ACTUAL CONSTRUCTION PROCEDURE

1. Electrodes. Telescoping brass tubing of approximately 1/2 inch diameter was cut into lengths indicated on page 40 to serve respectively for grid, first anode, and second anode. Sheet nickel was fitted tightly around the outside of the largest pieces of tubing with sufficient overlap to allow for spot welding. The spot welder was prepared by making a lower electrode of copper that extends along the brass tubing when set on end. The upper electrode was rounded to a rather blunt point.

2. Glass Tube. A 3-inch flanged Pyrex tube 10 inches long served as the main envelope. It was provided with O-ring grooves at the ends to facilitate sealing. An O-ring is a Neoprene washer.

Four Pyrex side arms of 1-inch I.D. were sealed to the main tube. (Figure 9, A)

3. End Plate, Electrode. (Figure 9) A 1/4-inch brass plate was turned to a diameter of 6 inches. To fasten the end plate to the tube, the glass was made slightly conical (B, Figure 9), a cast iron flanged ring (C, Figure 9) was packed with asbestos and fitted as shown, so that the end plate could be bolted to it. The cathode-heater assembly was inserted through a 9/16 inch hole at the center of the end plate. (See Section 9, Cathode Heater Assembly). This makes it possible to replace the cathode

or the heater. It will be noted that this construction simplifies the alignment of the coaxial cylinders. The cathode-filament assembly was fastened to the end plate by a 1/8-inch brass plate as shown in D, Figure 9. The end plate also carried a connection for exhausting, made of 3/8-inch brass tubing (not shown in diagram).

The anodes (E and F, Figure 9) were supported by two brass rods (G and H, Figure 9) protected by glass sheaths. Electrical connections were made through two Kovar seals (not shown), in the end plate.

4. Electrode Supports. (Figure 12) The supports (E and G Figure 9) were made of 1/2-inch brass rod provided with a silver-soldered flange and threads as shown in I, Figure 9. As the electrodes had to be insulated for 6000 volts, this mounting will be described in detail. Glass insulators (G and H, Figure 9) were placed over each brass rod and held in place by compressing the softened glass into a nick filed into each brass support near its distant end.

5. End Plate, Screen End. (Figure 11) A 1/4-inch brass plate 6 inches in diameter was used and fastened as already described for the electrode end. The plate was really a ring with a central hole 2 1/4 inches in diameter. A groove 2 3/4 inches I.D. 1/2 inch wide was turned as shown at J, Figure 9. A Pyrex glass cap (K,

Figure 9) 3 inches in diameter was constructed on the glass lathe. Its appearance resembles the screen end of a 3-inch oscilloscope tube. Beryllium phosphor was scraped from a broken fluorescent lamp, the powder suspended in water and painted on the inside of the cap with a small camel's hair brush. Uniform distribution was aided by a jarring motion. After the cap was thoroughly dry the brass plate was placed on a ring stand and heated by a Bunsen burner until Cenco-seal stick was melted in the groove. The cap was carefully placed in the wax and cooled slowly. Care was taken not to overheat the wax and destroy its sealing properties. Experience shows that it is well, while tightening the bolts that clamp the end plate, to have the wax soft so as to prevent breaking the glass cap by strain due to slight bending of the brass parts.

6. Side Arm Assembly (Support and Adjustment of Second Anode). (Figure 9) The side arm assembly consists of a 1/2-inch rod (L, Figure 9) silver soldered to a 1/4-inch brass plate of 2 1/8 inches in diameter. (M, Figure 9). The end of the 1/2-inch brass rod was hollowed out, making it concave inward. (N, Figure 9). A 1-inch brass rod was turned to 1/2-inch in diameter except for a 1-inch sleeve 1/4-inch from the top end. (O, Figure 9). The top end (O, Figure 9) was made convex of the proper

contour to fit the concave section (L, Figure 9). This formed an adjustable ball and socket joint between the two pieces. A 3/8 inch brass rod was inserted through the entire assembly to support the second anode (Q, Figure 9). Three machine screws through the sleeve held the assembly together (R, Figure 9). Grooves were turned on the curved portions to fit a 1/4 inch I.D. O-ring (S, Figure 9). Another groove was placed as indicated (T, Figure 9). An O-ring between the brass plates and the flanged side arms forms the seal between them (U, Figure 9). The entire assembly was held in place by machine screws threaded into a brass ring which was constructed by silver soldering three sizes of brass tubing telescoped together (V, Figure 9). The ring was cut at the proper angle to coincide with the angle of the flanged Pyrex side arm (W, Figure 9).

7. Assembling First Anode and Grid. Two sheet nickel strips 1/4 inch wide were spot welded around the grid and first anode, leaving sufficient length to spot weld around the insulated mountings on the brass plate (X, Figure 9). Care was taken to center the electrode between the supports. To assure a coaxial system a piece of 1/2 inch I.D. brass tubing was placed through the grid and first anode during alignment.

8. Assembling Second Anode. The second anode

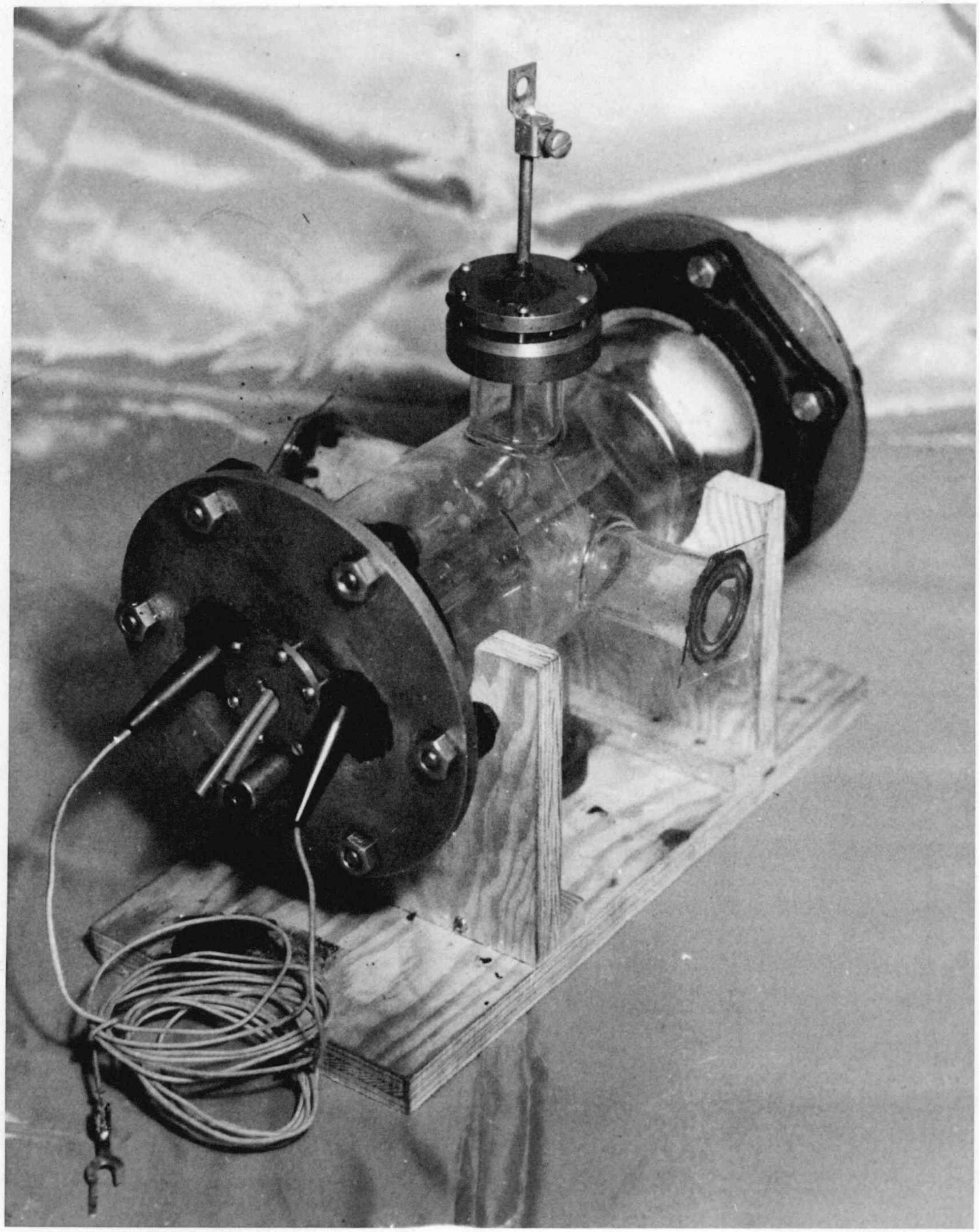
(Y, Figure 9) was fastened to the brass rod (Q, Figure 9) of the side arm assembly by spot welding a strip of nickel around the second anode and the brass rod. A piece of 1/2 inch I.D. brass tubing was inserted through the first and second anode to hold the cylinders in a coaxial position while adjustments were made in the side arm assembly.

9. Cathode-Filament Assembly. The filament assembly was constructed of uranium glass tubing with a flanged end extending to a diameter of 1 3/8 inches. (Z, Figure 9) The tungsten leads (a, Figure 9) were sealed into the end of the uranium glass tube by heating them to a white heat, forming an oxide which is essential for a satisfactory seal. A short sleeve of Nonex tubing (not shown) was fused to the tungsten leads by intense heat. The Nonex was then fused into the end of the uranium glass tubing. In making metal-to-glass seals, it is important to cool the glass slowly to avoid excessive strain. Since tungsten wire is frequently fibrous, having longitudinal channels which may leak if it is sealed into a vacuum apparatus, the tip of the tungsten was closed by fusing nickel wire over it. This nickel wire which was used to attach the cathode and heater also made spot welding a much easier operation.

The cathode was taken from a discarded cathode ray tube, while the heater was taken from a 2X2/879 rectifying tube. This heater was especially suitable since it operates at 2.5 volts and 1.75 amperes which is a close approximation to that of an ordinary cathode ray tube. Nickel wire was spot welded to the cathode and to one of the leads described above (b, Figure 9). The heater was inserted into cathode and spot-welded to each of the leads (c, Figure 9). An O-ring (d, Figure 9) was placed between the uranium glass flange and the brass electrode mounting plate. An asbestos washer (e, Figure 9) was placed over the outside of the glass as indicated in Figure 9.

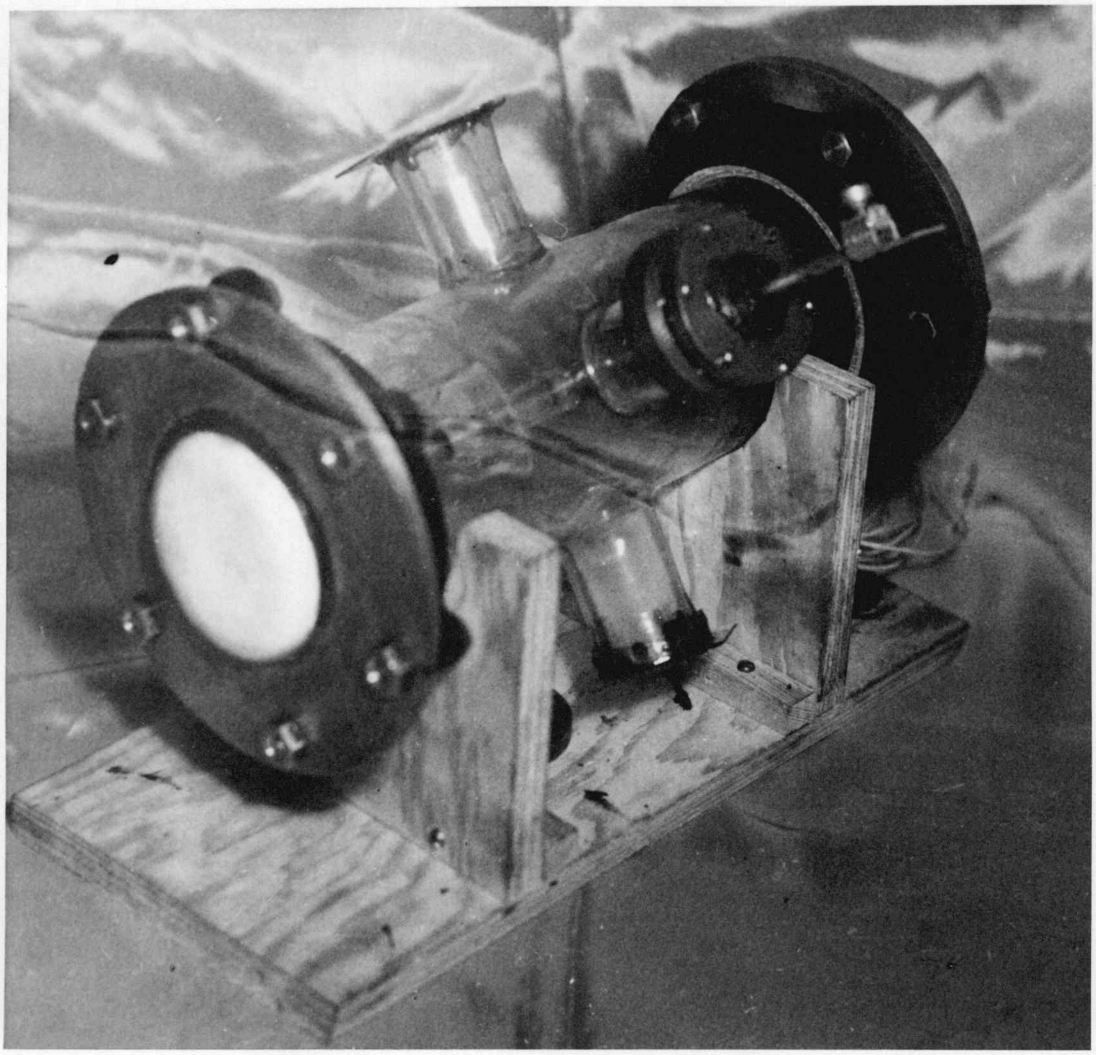
<u>10. General Assembly.</u> When tightening the brass plates to the O-rings to seal the side arms, one side arm was broken. Cracks appeared around other side arms as well as in various other places on the tube. This indicated excessive strain which was a result of improper annealing of the Pyrex. To avoid this an oven was prepared that would maintain temperatures above 400°C. The large tube was placed in the glass lathe and its temperature gradually increased to a working temperature. Especial care was necessary because of the previous annealing process. The damaged side arm was removed by heating the glass until it was soft enough to be removed by means of a glass rod. A new side arm was placed in position and

fused into place. The glass was heated sufficiently to fuse together the cracks that had appeared during the heating as well as those described above. It was immediately placed into the oven that had been preheated to its maximum temperature of over 400°C. The tube was maintained at this temperature for four hours. A variac was used to reduce the temperature of the oven slowly over a period of four more hours, after which it remained closed for six hours to assure a completely uniform reduction in temperature. This process was evidently successful since no further cracking tendency was noted.



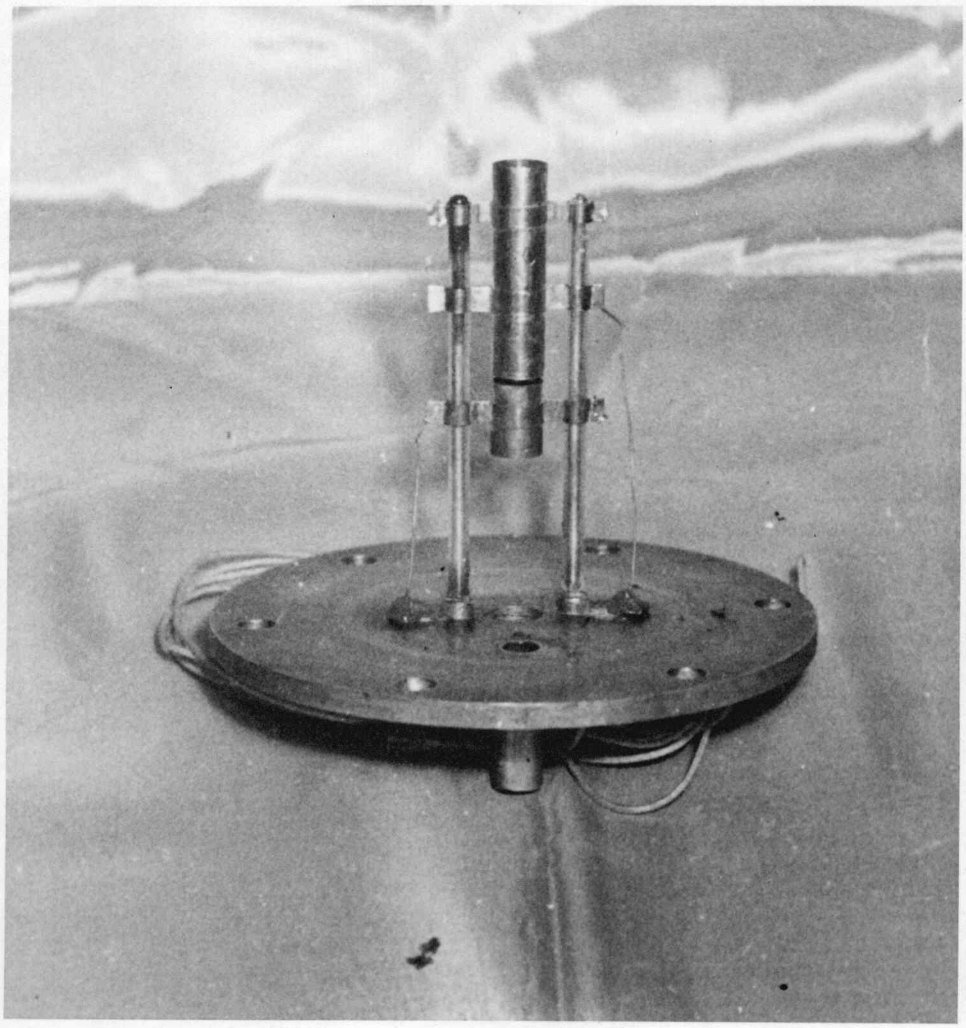
Electrode End Plate

Figure 10.



End Plate (Screen).

Figure 11.



Electrode and Electrode Supports.

Figure 12.

POWER SUPPLY

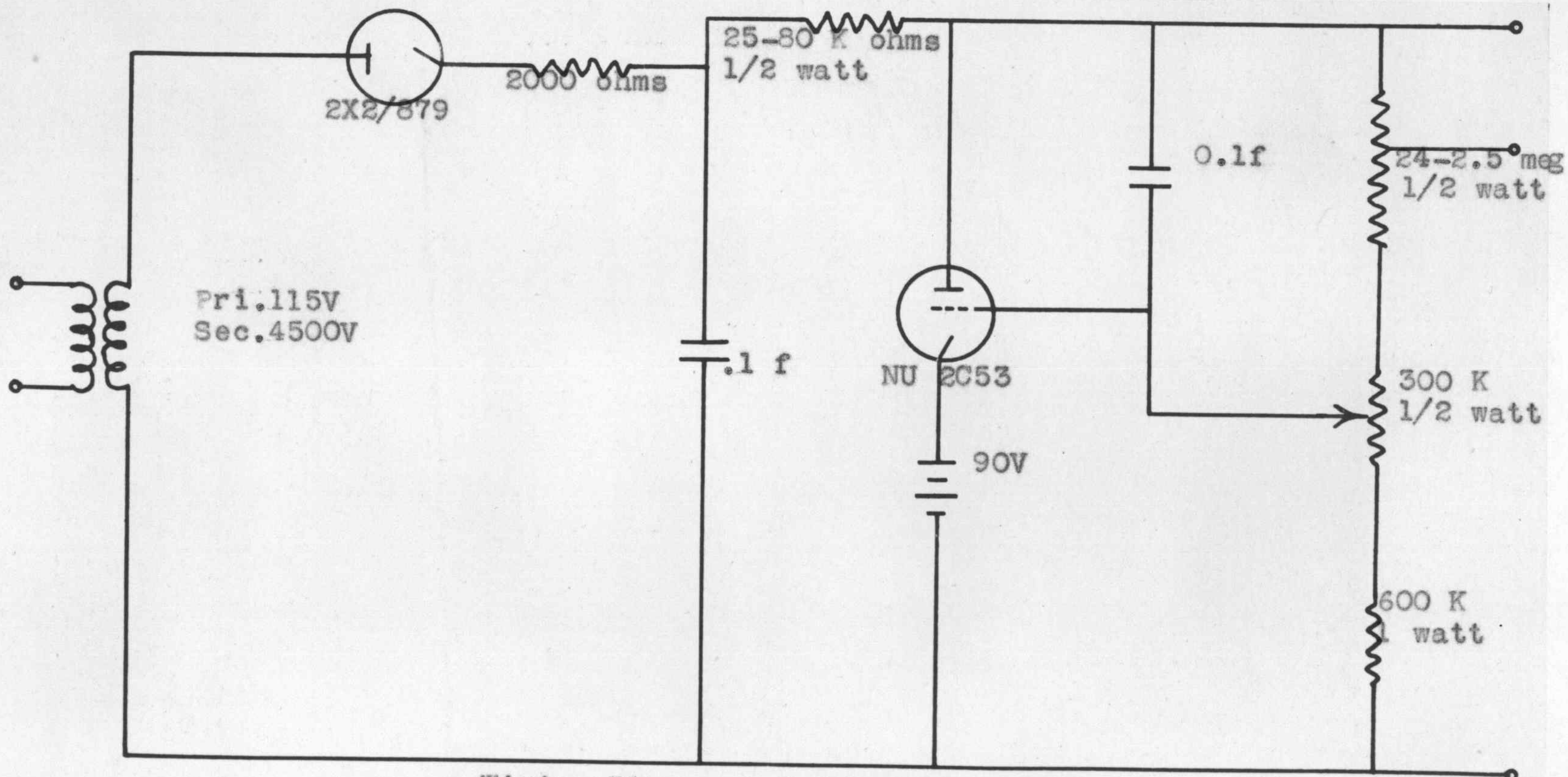
The power supply must be adjustable over a range from about 1000 to 6000 volts. It is necessary to secure two voltages from the power supply, one approximately four to six times the other since the voltage ratios have been chosen in this ratio during the design of the apparatus.

The total current (current drawn by first and second anodes) is the current in the beam that passes through the hollow anodes which act as electrostatic lenses and thus control the beam. The total current increases linearly with E_{p_1}. In most cases the total current remains below 1.5 milliamperes.

A wiring diagram of the apparatus is given in Figure (13) The circuit was mounted on plywood for convenience.

The 2X2/879 tube is a rectifier tube while the NU 2C53 tube is a voltage regulator. The circuit produces an output voltage substantially independent of the load impedance, since any fluctuation in the output voltage causes a change of the proper sign in the grid potential applied to the tube. The value of output voltage that the system attempts to maintain is determined by the setting of the potentiometer connected to the grid of the tube.

A second transformer (not indicated in Figure 13) consisted of three secondary windings of 2.5 volts. By connecting two in series with the center tap of the third it was possible to obtain 6.25 volts for the filament of the NU 2C53. The filament of the electron gun which operates at 2.5 volts was connected to one of the secondaries. A secondary of the transformer shown in Figure (9) was used for the 2.5 volts necessary for the filament of the 2X2/879.



Wiring Diagram of Power Supply.

Figure 13.

OPERATING PROCEDURE

<u>1. Activating Process.</u> A suspension of about 60 per cent barium carbonate (Ba CO_3) and 40 per cent strontium carbonate (Sr CO_3) was made in amyl acetate and painted on the cap of the cathode. Great care was necessary to get the cathode clean before painting since a very small amount of an impurity such as oil will make the phosphor inactive. The cathode-filament assembly was placed in position and the tube evacuated. A pressure of 0.2 micron was achieved; however, any pressure below which the discharge disappears in the tube would have been satisfactory. Several trials were necessary to achieve a sufficient vacuum. The O-rings had to be reseated several times since the correct sizes were not available. Three of the side arms had to be sealed with pieces of cover glass and a Cenco-seal stick. Small amounts of wax were needed around the adjustment rod and adjustment screws as described under Actual Construction Procedure, Section 6, page 44. Much work is to be expected in a vacuum system with several possibilities of leaks. After evacuating the tube it was ready for activation. A power supply adjusted to 10 volts and an ammeter were connected between the grid and the grounded side of the filament, the grid being made positive. The heater voltage was increased

slowly to about 2 volts and kept there for a few minutes for outgassing. The current was increased to about 2.5 amp. until a beam current was noted on the ammeter. It was necessary to reverse the leads for an instant, making the grid negative with respect to the filament, to start the beam current. After reversal to the original polarity, fifty volts positive were applied to both anodes, the current being kept at 2.5 amperes by controlling the filament voltage until the beam current ceased to increase. The filament current was then reduced to 1.75 amperes and the voltages removed from the gun elements for approximately one minute. To put the tube in operation, this procedure was necessary each time this tube was freshly evacuated. The emission current reached a maximum of 9 milliamperes in approximately one hour.

2. Tube Operation.

a. Trial 1. With the heater adjusted to 2.5 volts and 1.75 amperes, the voltage on the grid, first anode and second anode was varied. The grid voltage E_g was kept constant at -10 volts and the first anode voltage E_{p_1} was kept constant at 300 volts. The second anode was varied between limits of about 800 and 5000 volts. Observations were taken on the spot dimensions and focus. These are given under Trial I, page 58.

b. Trial 2. Trial 2 consisted of increasing

E_g to -1 volt and keeping E_{p_1} constant at 300 volts.

c. Trial 3. Trial 3 changed E_g to -20 volts with other values remaining as before but varying E_{p_2} over its range.

d. Trial 4. Trial 4 consisted of E_g = -10 and E_{p_2} = 1000 volts and kept constant while E_{p_1} was varied. Pertinent information was taken under "Spot Size" and "Remarks."

e. Trial 5. Trial 5 consisted of raising E_{p_1} to 520 volts with E_g = -10 volts. E_{p_2} was varied again. This rise of voltage caused an increase in the light intensity from the filament and thus more reflection of light on the screen without an increase in spot intensity. Therefore, observation was more difficult.

f. Trial 6. Trial 6 changes E_g to -15 volts with E_{p_1} = 300 volts.

DATA

E_g	E_{p_1}	E_{p_2}	Spot Size	Remarks
Trial 1				
-10	300	960	1 mm	Spot appears.
-10	300	1600	2 mm	Best focus.
-10	300	1850	4 mm	Starts to appear fuzzy.
-10	300	4970	8 mm	More diffused.
Trial 2				
-1	300	200	0 mm	No spot.
-1	300	850	1 mm	Spot appears.
-1	300	1450	2 mm	Focused.
-1	300	2350	5 mm	Starts focusing.
-1	300	4950	8 mm	Diffused.
Trial 3				
-20	300	900	1 mm	Spot appears.
-20	300	1500	2 mm	Focused.
-20	300	3960	10 mm	Fuzzy spot.
Trial 4				
-10	210	1000	Point	
-10	235	1000	1.5 mm	Focused.
-10	335	1000	1.5 mm	Diffused spot.

E_g	E_{p_1}	E_{p_2}	Spot Size	Remarks
Trial 5				
-10	520	1560		Point appears.
-10	520	2800	3 mm	Focused.
-10	520	3960	8 mm	Very diffused (faint).
Trial 6				
-15	300	500	1 mm	Spot appears.
-15	300	1450	2 mm	Best focus.
-15	300	4900	8 mm	Large diffused spot.

CONCLUSIONS

The data indicate that the voltage ratio for each trial is as follows:

Trial 1	E_{p_2}/E_{p_1} = 1600/300 =	5.3
Trial 2	E_{p_2}/E_{p_1} = 1450/300 =	4.83
Trial 3	E_{p_2}/E_{p_1} = 1500/300 =	5.00
Trial 4	E_{p_2}/E_{p_1} = 1000/235 =	4.25
Trial 5	E_{p_2}/E_{p_1} = 2800/520 =	5.38
Trial 6	E_{p_2}/E_{p_1} = 1550/300 =	4.83
	Average	4.93

The average voltage ratio obtained from the experimental data was 4.93 while in the design the ratio 4.5 was used. This discrepancy may be due to the inability to determine the exact focus when taking the data. The voltage can be varied through a rather large range with a relatively small change in the appearance of the spot on the screen. The agreement between the voltage ratios measured and computed is quite satisfactory considering all the possible sources that may cause deviation.

The grid aperture is 0.040 inch. If the equation ($M = k\,\frac{q}{p}$) is considered it is found that the magnification $M = 0.80\,\frac{6}{2.5} = 1.92$. The spot size should equal the magnification multiplied by the grid aperture. This leads to a spot diameter of 1.96 mm. An average of the focused spot size is 2.1 mm, which is as close as could be expected.

It was noted that the spot was deflected in an arc path as E_{p_2} or E_{p_1} was increased. This was in all probability due to imperfect alignment of the second anode with the first anode.

It is noted that varying the grid voltage has no effect on the focusing characteristic of the tube. An attempt was made to operate without the E_g voltage and was unsuccessful; this indicates the necessity of negative potential on the grid. It is noted also that it makes no difference as to which voltage (E_{p_1} or E_{p_2}) is varied in determining the voltage ratio, as would be expected. Although tubes were not constructed according to the other three designs, it may be reasonably assumed that results similar to the above would be obtained.

BIBLIOGRAPHY

1. Epstein, D. W. Electron optical systems of two cylinders as applied to cathode ray tubes. Institute of radio engineers proceedings 24:1095-1139, Aug. 1936.

2. Maloff, I. G. and Epstein, D. W. Electron optics in television. New York, McGraw-Hill Book Co., 1938. 299 p.

3. Maloff, I. G. and Epstein, D. W. Theory of electron guns. Institute of radio engineers Proceedings 22:1386-1411, Dec. 1934.

4. Morton, G. A. Electron guns for television application. Review of modern physics 18: 947-955 July 1946.

5. Strong, John. Procedure in experimental physics. New York, Prentice-Hall, Inc., 1938. 642p.

6. Terman, Frederick Emmons. Radio engineers' handbook. New York, McGraw-Hill Book Co., 1943. 1018p.

7. Zworykin, V. K. and Morton, G. A. Applied electron optics. Journal of the optical society of America 26:181-189, April 1936.

8. Zworykin, V. K. and Morton, G. A. Television. New York, John Wiley and sons, Inc., 1940. 646p.

9. Zworykin, V. K.; Morton, G. A.; Ramberg, E. G.; Hillier, J.; and Vance, A. W. Electron optics and the electron microscope. New York, John Wiley and sons, Inc., 1945. 766p.

APPENDIX

1. Analogy in General Electrostatic Case. The analogy in general electrostatic case may be seen from the principles of least time for geometric optics and of least action of electron optics.

The principle of least time states that the path of a ray of light from point A to point B is always such as to make the integral

$$\int_A^B \mu(\nu, x, y, z)\, ds \qquad \begin{array}{l} u = \text{index of refraction} \\ \nu = \text{frequency of light} \\ ds = \text{element of path} \end{array} \qquad (1)$$

an extreme (usually a minimum) with respect to all neighboring paths for rays of the same frequency. The principle is usually stated as

$$\delta \int_A^B \mu(\nu, x, y, z)\, ds = 0 \qquad (\nu = \text{constant}) \quad (2)$$

The principle of least action for electron velocities less than 1/10 the velocity of light states that an electron of total energy E, kinetic energy T and mass m moves through an electrostatic field with potential energy V(x,y,z) in such a way as to make the action integral

$$s = \int_A^B 2T dt = \int_A^B \left[2m(E - V)\right]^{\frac{1}{2}} ds$$

over the actual path between the two points A and B an extreme as compared with its value for all adjacent paths for the same value of E. As the integrand $\left[2m\,(E-V)\right]^{\frac{1}{2}}$ is identical with the absolute value of the momentum p which the electron would assume at (x, y, z), the principle may be stated as

$$\delta \int_A^B p(E, x,y,z)\, ds = 0 \ (E = \text{constant}) \qquad (3)$$

A comparison of Equations 2 and 3 shows that the path of an electron in an electrostatic field may be identical with the rays of light in geometrical optics if the index of refraction is chosen to be

$$\mu = k'(E - V)^{\frac{1}{2}} = k''p = kv \qquad (4)$$

where k is a constant of proportionality and v is the speed of the electron. So the index of refraction at any point of an electrostatic field is proportional to the speed of the electron at the point. If the index of refraction is taken as a pure numeric, then k must have the dimension 1/v. The value assigned to k is of no importance since only the ratio of μ at two different places is used, so that if μ_1 and μ_2 are the indices of refraction at two different places, the relative index of refraction is

$$\mu = \frac{\mu_2}{\mu_1} = \frac{kv_2}{kv_1} = \frac{v_2}{v_1} \qquad (5)$$

Since the potential function V(x,y,z) is a continuous scalar function of position, it follows from (4) that the index of refraction of an electrostatic field is also a continuous function of position. This means that an electrostatic field constitutes an isotropic non-homogeneous medium for electrons.

2. Equations of Motion of Electrons. The equations of motion of an electron moving in a meridian plane are

$$m\,\frac{d^2z}{dt^2} = e\,\frac{\partial V}{\partial z} \tag{6}$$

$$m\,\frac{d^2r}{dt^2} = e\,\frac{\partial V}{\partial r} \tag{7}$$

In terms of the axial distribution of potential these equations become from

$$m\,\frac{d^2z}{dt^2} = e\Big[V_o'(z) - \frac{r^2}{2^2}\,V_o^{(3)}(z) + \cdots + \frac{(-1)^n r^{2n}}{2^2\cdot 4^2\cdots\cdots(2n)^2}\,V_o^{(2n+1)}(z) + \cdots\Big] \tag{8}$$

$$m\,\frac{d^2r}{dt^2} = e\Big[\frac{r}{2}V_o''(z) + \frac{r^3}{2^2\cdot 4}\,V_o^{(4)}(z) + \cdots + \frac{(-1)^n r^{2n-1}}{2^2\cdot 4^2\cdots\cdots(2n-1)^2\cdot 2n}\,V_o^{(2n)} + \cdots\Big] \tag{9}$$

3. Energy Equation. Adding the two equations (6) and (7), the following is obtained:

$$m\left[\frac{dz}{dt}\,d\left(\frac{dz}{dt}\right) + \frac{dr}{dt}\,d\left(\frac{dr}{dt}\right)\right] = e\left[\frac{\partial V}{\partial z}\,dz + \frac{\partial V}{\partial r}\,dr\right] = e\,dV \tag{10}$$

The solution of (10) is exact, therefore

$$\frac{1}{2}\,mv^2 = \frac{1}{2}\,m\left[\left(\frac{dz}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right] = eV + C \tag{11}$$

If the velocity of the electron is zero when $V = 0$, then $C = 0$ and

$$\frac{1}{2}\, mv^2 = \frac{1}{2}\, m \left[\left(\frac{dz}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 \right] = eV \qquad (11)$$

and,

$$v = \sqrt{2\, \frac{e}{m}\, V} = 5.95 \times 10^7 \sqrt{V} \text{ volts cm/sec} \qquad (12)$$

If $v = v_o$ when $V = 0$ then $c = \frac{1}{2} m v_o^2$ and

$$\tfrac{1}{2} m (v^2 - v_o^2) = eV$$

If the velocity v_o be given in equivalent volts then $\frac{1}{2} m v_o^2 = e(V_o/300)$ and

$$v_o = \sqrt{2\, \frac{e}{m}\, \frac{(V + V_o)}{300}} = 5.95 \times 10^7 \sqrt{(V + V_o)} \text{ volts cm/sec} \qquad (13)$$

4. Differential Equation of Trajectory of Electron.

It will be shown that the trajectory of an electron traversing an axially symmetric electrostatic field described by the potential function $V(r,z)$ satisfies the following differential equation

$$\frac{d^2 r}{dz^2} + \frac{\left[1 + \frac{dr}{dz}^{2}\right]}{2V} \frac{\partial V}{\partial z} \frac{dr}{dz} - \frac{\left[1 + \frac{dr}{dz}^{2}\right]}{2V} \frac{\partial V}{\partial r} = 0 \qquad (14)$$

To show this note that

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{dz}{dt}\, \frac{d}{dz}\left(\frac{dr}{dz}\, \frac{dz}{dt}\right)$$

$$= \left(\frac{dz}{dt}\right)^2 \frac{d^2 r}{dz^2} + \frac{dr}{dz}\, \frac{dz}{dt}\, \frac{d}{dz}\left(\frac{dz}{dt}\right) \qquad (15)$$

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{dz}{dt}\, \frac{d}{dz}\left(\frac{dz}{dt}\right) \qquad (16)$$

$$\frac{1}{2}m\left(\frac{dz}{dt}\right)^2\left[1+\frac{dr}{dz}^{\,2}\right] = eV \tag{17}$$

From (15), (16) and (17) it follows that

$$\frac{d^2r}{dt^2} = \frac{2\,\frac{e}{m}\,V}{\left[1+\left(\frac{dr}{dz}\right)^2\right]}\;\frac{d^2r}{dz^2} + \frac{dr}{dz}\,\frac{d^2z}{dt^2} \tag{18}$$

Inserting into (18) the values of d^2r/dt^2 and d^2z/dt^2 as given by (6) and (7), there results (14).

It is noted that e/m does not appear in (14), signifying that the trajectory is the same for any charged particle. Also, it is noted that (14) is homogeneous in V so that if the voltages on the electrodes are all increased by a constant factor the trajectory of the electrons will remain unaltered. Equation (14) is also homogeneous in r,z, that if all dimensions are increased by a constant factor then the trajectory is also increased by the same factor.

5. Paraxial Electrons. The focusing action of an electrostatic field is described to a first approximation by considering only paraxial electrons. Paraxial electrons are characterized by the fact that in calculating their paths it is assumed that their distance from the axis, r, and their inclinations toward the axis, dr/dz, are so small that the second and higher powers of r and dr/dz are negligible.

For the case of paraxial electrons,

$$V(r,z) = V_0(z) \tag{19}$$

$$m \frac{d^2r}{dt^2} = -e \frac{r}{2} V_o''(z)$$

$$m \frac{d^2z}{dt^2} = eV_o'(z) \qquad (20)$$

$$\frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{dz}{dt}\right)^2 = eV_o(z) \qquad (21)$$

The differential equation for the trajectory traversed by a paraxial electron becomes from (14)

$$\frac{d^2r}{dz^2} + \frac{V_o'}{2V_o} \frac{dr}{dz} + \frac{V_o''}{4V_o} r = 0 \qquad (22)$$

Equation (22) is the fundamental equation of electron optics.

<u>6. The Two Fundamental Trajectories.</u> Multiplying (22) by $\sqrt{V}$ there results the self-adjoint equations

$$L(r) = V \frac{d^2r}{dz^2} + \frac{V'}{2\sqrt{V}} \frac{dr}{dz} + \frac{V''}{4\sqrt{V}} r$$

$$L(r) = \frac{d}{dz} \sqrt{V} \frac{dr}{dz} + \frac{V''}{4\sqrt{V}} r = 0 \qquad (23)$$

Let $r_1(z)$ and $r_2(z)$ be two independent solutions of (22) representing the trajectories of two electrons then,

$$r_2L(r_1) - r_1L(r_2) = \frac{d}{dz}\left[\sqrt{V}\left(r_2 \frac{dr_1}{dz} - r_1 \frac{dr_2}{dz}\right)\right] = 0 \qquad (24)$$

Integrating this equation between the limits a and b results that[1]

$$\int_a^b \left[r_2 L\,(r_1) - r_1 L(r_2)\right] dz = \left[\sqrt{V}\left(r_2 \frac{dr_1}{dz} - r_1 \frac{dr_2}{dz}\right)\right]_a^b \qquad (25)$$

substituting the limits,

$$\sqrt{V(b)}\left[r_2(b)r_1'(b) - r_1(b)r_2'(b)\right]$$
$$= \sqrt{V(a)}\left[r_2(a)r_1'(a) - r_1(a)r_2'(a)\right] \qquad (26)$$

In particular let $r_1(z)$, $r_2(z)$, $r'(z)$, and $r_2'(z)$ assume the following values at a and b:

$$r_1(a) = h_1 \qquad r_1(b) = 0$$
$$r_1'(a) = 0 \qquad r_1(b) = \tan\beta_2$$
$$r_2(a) = 0 \qquad r_2(b) = -h_2$$
$$r_2'(a) = \tan\beta_1 \qquad r_2'(b) = 0 \qquad (27)$$

then (26) reduces to

$$\sqrt{V(b)}h_2 \tan\beta_2 = \sqrt{V(a)}h_1 \tan\beta_1 \qquad (28)$$

The two trajectories $r_1(z)$ and $r_2(z)$ satisfying (27) will be called the two fundamental trajectories.

Figure 4 shows two fundamental trajectories. Only two independent trajectories may be taken as the fundamental pair. This particular case is chosen because by means of this pair the usual optical relations are easily obtained.

[1] This discussion is limited to electrostatic fields having finite extension, i.e., $V = V(z)$ for $\alpha \leq z \leq \beta$ and $V =$ constant for $\alpha \leq z \leq \beta$ and further $a \leq \alpha$ and $b \geq \beta$.

Let f_1 and f_2 be the focal lengths of the focusing system then

$$f_1 = -\frac{h_2}{\tan\beta_1} \quad \text{and} \quad f_2 = \frac{h_1}{\tan\beta_2} \tag{29}$$

by definition. Inserting (29) in (28) there results that

$$\frac{f_2}{f_1} = -\sqrt{\frac{V(b)}{V(a)}} \tag{30}$$

Further let

$$X_1 = \frac{h_1}{\tan\beta_1} \quad \text{and} \quad X_2 = -\frac{h_2}{\tan\beta_2} \tag{31}$$

then from (31) and (29) it follows that

$$X_1 X_2 = f_1 f_2 \tag{32}$$

X_1 is the distance between an object and the first focal point and X_2 is the distance between the image and the second focal point, if h_1 is the height of the object and h_2 is the height of the image. The magnification is

$$m = \frac{h_2}{h_1} = -\frac{f_1}{X_1} = -\frac{X_2}{f_2} \tag{33}$$