AN ABSTRACT OF THE THESIS OF

Neil Kenneth Poulsen for the degree of Doctor of Philosophy in the Department of Statistics presented on April 27, 1982.

Title: <u>LARGE SAMPLE EFFICIENCIES OF INVARIANT QUADRATIC UNBIASED ESTIMATORS</u>

Redacted for privacy

Abstract approved ____________________________________

Justus F. Seely

This dissertation examines limiting efficiencies of quadratic unbiased estimators for the variance in the two variance component mixed model. The set of quadratic unbiased estimators considered includes the minimal complete class. A theorem is proved which shows that, in certain cases, a relatively simple expression converges to the same value to which the efficiency itself converges. The efficiency is a much more complex expression. Less general results are proved concerning limiting efficiencies, and tables of computed limiting efficiencies are provided for various behaviors of the eigenvalues. Based on these results and tables, recommendations are given for specific estimators.

LARGE SAMPLE EFFICIENCIES OF INVARIANT QUADRATIC UNBIASED ESTIMATORS

BY

Neil Kenneth Poulsen

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed April 27, 1982

Commencement June 1982

APPROVED:

Redacted for privacy

Professor of Statistics in charge of major

Redacted for privacy

Chairman of Department of Statistics

Redacted for privacy

Dean of Graduate School

Date thesis is presented April 27, 1982

Typed by Mary E. Rose for Neil Kenneth Poulsen

I would like to extend my warmest appreciation to Justus Seely for his help and support in completing my degree. His guidence made it possible to write this dissertation and his efforts contributed to and improved its contents.

I also wish to thank Dave Birkes for his comments and suggestions regarding several aspects of this dissertation.

To my parents, Mr. and Mrs. Wallace N. Poulsen, I extend my deepest gratitude for their moral and financial support in assisting me to obtain an education.

TABLE OF CONTENTS

I. Introduction	1
Model	3
Estimators of $\delta'\theta$	6
Assumptions	11
Uniqueness Property	14
II. Limiting Efficiencies	16
Efficiency	17
Main Result	30
III. Estimators of θ_2	48
The D_1 and D_2 Estimators	49
The D_w Estimator	52
IV. Behavior of Efficiencies - An Example	53
Tables of Limiting Efficiencies	54
Recommendations	73
V. Bibliography	76
VI. Appendix	77

LIST OF TABLES

Tables		Page
IV.1	Limiting Efficiencies of the (.16,.48,.36) Estimator	57
IV.2	Limiting Efficiencies When All Eigenvalues Converge to Zero.	59
IV.3	Limiting Efficiencies When the Eigenvalues Converge to Zero and to a Constant	61
IV.4	Limiting Efficiencies When All Eigenvalues Converge to a Constant	63
IV.5	Limiting Efficiencies When the Eigenvalues Converge to a Constant and to Infinity	65
IV.6	Limiting Efficiencies When All Eigenvalues Converge to Infinity	67
IV.7	Limiting Efficiencies When the Eigenvalues Converge to Zero and to Infinity	69
IV.8	Limiting Efficiencies When the Eigenvalues Converge to Zero, to a Constant, and to Infinity	71

Large Sample Efficiencies of Invariant Quadratic Unbiased Estimators

Chapter I

Introduction

This dissertation discusses properties of limiting efficiencies for estimators of variance in the two variance component mixed model. Results given herein apply to a fairly general class of limiting conditions, and the estimators considered are those characterized by Olsen, Seely, Birkes (1976). As shown in Olsen et al., these estimators contain the minimal complete class of all invariant quadratic unbiased estimators. After describing this class in detail and expressing the efficiency in a useful form, a result is presented which simplifies examination of limiting efficiencies. This result is applied to generalize results originally presented in Seely (1979). This dissertation then concludes with several examples of computed limiting efficiencies, and based on these examples, plus other results given in the text, makes some recommendations as to which estimators to use under various conditions.

In Chapter I is discussed the class of estimators to which results given in this dissertation will apply. Chapter I also enumerates the limiting conditions on which computation of the limiting efficiencies are based. Chapter II presents results relating to the general behavior of limiting efficiencies. Chapter III generalizes results given in Seely (1979) and Chapter IV contains examples of calculated limiting efficiencies and recommendations for specific estimators.

SECTION I.1: Model

Let the random vector $\underset{\sim}{Y}$ consist of n elements such that

$$\underset{\sim}{Y} = X\beta + A\underset{\sim}{a} + \underset{\sim}{e}$$

where X and A are matrices such that $X\varepsilon M(n,p)$[1] and $A\varepsilon M(n,t)$, and where β is an unrestricted parametric vector in R^p. The random vectors $\underset{\sim}{a}$ and $\underset{\sim}{e}$ are taken to be mutually independent, each following a multivariate normal distribution with mean vector zero, and having covariance structures $Cov(\underset{\sim}{a}) = \theta_2 I$ and $Cov(\underset{\sim}{e}) = \theta_1 I$ respectively, where $\theta = (\theta_1, \theta_2)\varepsilon\Omega$ with $\Omega = \{\theta : \theta > 0\}$[2]. The parameters θ_1 and θ_2 are unknown. Stated more concisely, the family of distributions $\mathscr{P}_Y$ associated with $\underset{\sim}{Y}$ is

$$\mathscr{P}_Y = \{N_n(X\beta, \theta_1 I + \theta_2 AA') : \beta\varepsilon R^p \text{ and } \theta\varepsilon\Omega\}.$$

All limiting processes considered will be sequences of families of distributions of $\underset{\sim}{Y}$ where n tends to

1. M(a,b) represents the set of all matrices with a rows and b columns.
2. $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 \neq 0$.

infinity.[3] Within any sequence, the components of the distribution which will change are the matrices X and A and the dimension n. In this regard, n and the contents of X and A will change, while p, t, and the ranks of X and A may or may not change. The parameters θ_1 and θ_2 will remain fixed. Therefore, any limiting process considered can be represented as a sequence of ordered triplets (X_k, A_k, n_k) for k = 1,2,... where n_k is an unbounded sequence of positive integers. We will express all limits as lim(•). This notation will be taken to mean the limit of the argument (•) as k tends to infinity.

For a matrix M, the notation $\underline{R}(M)$, $\underline{N}(M)$, $\underline{r}(M)$, $\underline{n}(M)$, M', det(M), and tr(M) denote the range, null space, rank, nullity, transpose, determinant and trace respectively of the matrix M. Further, P_M represents the orthogonal projection operator on the range of M, M^- represents any generalized inverse[4] of M, and M^+ represents the Moore-Penrose inverse of M.[5] In addition, for any subset of a k-dimensional Euclidian space R^k, let $\mathcal{B}^\perp$

---

3. That n tends to infinity follows from assumptions made later in Section I.3.
4. Any matrix M^- such that $MM^-M = M$.
5. The unique matrix M^+ such that $MM^+M = M$ and $M^+MM^+ = M^+$ with MM^+ and M^+M symmetric.

represent the orthogonal complement of $\mathscr{B}$ in R^k. If $\mathscr{D}$ is a subspace, define $\dim\{\mathscr{D}\}$ as its dimension.

Pertaining to the above model, define

$$f = \underline{r}(X,A) - \underline{r}(X)$$

$$r_0 = n - \underline{r}(X,A).$$

Considering the partitioned fixed linear model

$$\underset{\sim}{Y} = X\beta + A\alpha + \underset{\sim}{e}$$

with β and α unknown, one can see that f equals the degrees of freedom of the sum of squares for α adjusted for β, and r_0 equals the error degrees of freedom.

SECTION I.2: Estimators of $\delta'\theta$

This section describes the specific class of estimators $\mathscr{E}_{\delta'\theta}$ to which results given in this thesis apply. Generally speaking, these estimators are unbiased estimators of $\delta'\theta$ for some $\delta \varepsilon R^2$, and they have highly desirable variance characteristics. The remarks below are drawn chiefly from Olsen, Seely and Birkes (1976) in which the class $\mathscr{E}_{\delta'\theta}$ has been characterized in thorough detail. Note that differences exist between the notation used here and that found in Olsen et al. (1976). For this section assume that $f, r_0 > 0$.

Define $\mathscr{N}$ as the set of all symmetric quadratic forms invariant under the group of transformations $\mathscr{G} = \{g_x | x \varepsilon \underline{R}(X)\}$ where $g_x : y \to y+x$. That is, $\underset{\sim}{Y}'M\underset{\sim}{Y} \varepsilon \mathscr{N}$ if and only if $\underset{\sim}{Y}'M\underset{\sim}{Y} = (\underset{\sim}{Y}+u)'M(\underset{\sim}{Y}+u)$ for all $u \varepsilon \underline{R}(X)$. We wish to estimate the parametric function $\delta'\theta$ for some known $\delta \varepsilon R^2$. We will restrict attention to $\mathscr{N}_{\delta'\theta}$, the set of unbiased estimators of $\delta'\theta$ contained in $\mathscr{N}$.

To avoid the nuisance parameter β, consider the transformed random vectors of $\underset{\sim}{Y}$, $\underset{\sim}{U} = A'(I-P_X)\underset{\sim}{Y}$ and $\underset{\sim}{Z} = (I-P_{X,A})\underset{\sim}{Y}$ where $P_{X,A}$ is the orthogonal projection operator on the range of (X,A), the matrix formed by concatenating X and A. The family of distributions associated with $\underset{\sim}{U}$ is $\mathscr{P}_U = \{N_t(0, \theta_1 D + \theta_2 D^2) : \theta \epsilon \Omega\}$ with $D = A'(I-P_X)A$, and the family associated with $\underset{\sim}{Z}$ is $\mathscr{P}_Z = \{N_n[0, \theta_1(I-P_{X,A})]: \theta \epsilon \Omega\}$. D is a n.n.d.[6] matrix and thus has non-negative eigenvalues. Let $\lambda_1, \ldots, \lambda_m$ be the ordered distinct positive eigenvalues of D such that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m$ and define F_j for $j=1,2,\ldots,m$ as the orthogonal projection operator on the span of eigenvectors associated with λ_j.[7] From the Spectral Theorem we can decompose D in the following manner:

$$D = \lambda_1 F_1 + \lambda_2 F_2 + \ldots + \lambda_m F_m.$$

Finally, define $r_j = \underline{r}(F_j)$ for $j=1,2,\ldots,m$. The ranks $r_1, \ldots, r_m$ are the multiplicities of the eigenvalues $\lambda_1, \ldots, \lambda_m$ respectively.

We now define the following important quadratic forms in $\underset{\sim}{Z}$ and $\underset{\sim}{U}$:

$$T_0 = \underset{\sim}{Z}'\underset{\sim}{Z}/r_0$$

$$T_j = \underset{\sim}{U}'F_j\underset{\sim}{U}/r_j\lambda_j \qquad \text{for } j = 1,2,\ldots,m$$

6. non-negative definite.
7. The span of vectors e such that $De = \lambda_j e$.

As $\underset{\sim}{Z}'\underset{\sim}{Z} = \|(I-P_{X,A})Y\|^2$, T_0 is recognized to be the error mean square and, as described previously, r_0 is the corresponding error degrees of freedom from an analysis of variance in which the full model is $E(\underset{\sim}{Y}) = X\beta + A\alpha$ with α and β fixed and unknown. Let $\mathscr{T}$ be the set of all linear combinations of $T_0, T_1, \ldots, T_m$. The importance of $\mathscr{T}$ is that for any estimator in $\mathscr{N}\setminus\mathscr{T}$,[8] there exists an estimator in $\mathscr{T}$ with the same expectation and uniformly smaller variance. Therefore, in looking for estimators of $\delta'\theta$ in $\mathscr{N}$ we need only consider estimators in $\mathscr{T}_{\delta'\theta}$, the unbiased estimators for $\delta'\theta$ in $\mathscr{T}$. It is convenient to examine the estimators in $\mathscr{T}$ by way of a linear model. To this end, let $T = (T_0, T_1, \ldots, T_m)'$ and define

$$B = \begin{pmatrix} 1, & 1, \ldots, 1 \\ 0, & \lambda, \ldots, \lambda_m \end{pmatrix}'$$

$$V(\gamma) = (1-\gamma)^2 D_1 + 2\gamma(1-\gamma)D_2 + \gamma^2 D_3$$

where $\gamma = \theta_2/(\theta_1+\theta_2)$ and $\sigma^2 = \theta_1 + \theta_2$ for $\theta \epsilon \Omega$ and where

$$D_1 = \text{diag}\ (1/r_0, 1/r_1, \ldots, 1/r_m)$$
$$D_2 = \text{diag}\ (0, \lambda_1/r_1, \ldots, \lambda_m/r_m)$$
$$D_3 = \text{diag}\ (0, \lambda_1^2/r_1, \ldots, \lambda_m^2/r_m).$$

---

8. All estimators in $\mathscr{N}$ not contained in $\mathscr{T}$.

Let $\lambda_0 = 0$. Then the following statements can be made concerning T:

a) $E(T) = B\theta$

b) $Cov(T) = 2\sigma^4 \; V(\gamma)$

c) $r_j T_j/(\theta_1+\theta_2\lambda_j) \quad ChiSquare(r_j)$ for $j = 0,1,...,m$

d) $T_0, T_1, ..., T_m$ are mutually independent quadratic forms.

The quadratic forms T make it possible to characterize $\mathscr{E}_{\delta'\theta}$. Let $S = \{w\varepsilon R^3 : w_1+w_2+w_3 = 1 \text{ and } w > 0\}$, and for $w\varepsilon S$, let $\hat{\theta}(w)$ be the Gauss-Markov estimator for θ computed from the artificial model $E(T) = B\theta$ with $Cov(T) = V_w$ where $V_w = w_1D_1 + w_2D_2 + w_3D_3$. Since $\lambda_1 > 0$, $\underline{r}(B) = 2$ which implies $\delta'\theta$ is estimable for all $\delta\varepsilon R^2$. We define $\mathscr{E}_{\delta'\theta}$ as the set of estimators

$$\mathscr{E}_{\delta'\theta} = \{\delta'\hat{\theta}(w) : w\varepsilon S\}.$$

The artificial model described above has use only in so far as it helps to define $\mathscr{E}_{\delta'\theta}$. Previous comments show it to be the actual model for T only for the case in which $\sigma^2 = 1$ and $w = ((1-\gamma)^2, 2\gamma(1-\gamma), \gamma^2)'$.

Olsen et al. (1976) have shown that $\mathscr{E}_{\delta'\theta}$ contains the minimal complete class of all estimators in $\mathscr{N}_{\delta'\theta}$. Indeed if $\mathscr{M}_{\delta'\theta}$ denotes the minimal complete class,[9] then

$$\mathscr{M}_{\delta'\theta} = \{\delta'\hat{\theta}(w) : w \epsilon S_c\}$$

with $S_c = \{w \epsilon S : 4w_1 w_3 \geq w_2{}^2\}$ and where $\hat{\theta}(w)$ has been previously defined.

In regard to estimators in $\mathscr{E}_{\delta'\theta}$, we will subscribe to the following notation. Let $D_w \epsilon \mathscr{N}_{\delta'\theta}$ denote the estimator $\delta'\hat{\theta}(w)$. For convenience, let D_1 denote the estimator D_w when $w_1 = 1$, let D_2 denote the estimator D_w when $w_2 = 1$, and let D_3 denote the D_w estimator when $w_3 = 1$. This notation, while convenient, might appear somewhat incomplete since D_w fails to specify the expectation of the estimator. As an example, D_1 could refer to an estimator in $\mathscr{N}_{\theta_1}$ or in $\mathscr{N}_{\theta_2}$. To avoid this problem, we will adopt the following convention: unless stated specifically to the contrary, any estimator denoted D_w will have expectation $\delta'\theta$. In all other cases, the expectation of D_w will be clear from the context of discussion.

---

9. See Appendix for a brief discussion of minimal complete classes.

SECTION I.3: Assumptions

Almost all propositions, theorems, and corollaries given in this dissertation will require one or more of the assumptions enumerated below. Each result will clearly state which of these assumptions hold, and only those stated will be required for the proof. Indeed, the following assumptions may be regarded as limiting conditions on the sequence of models (X_k, A_k, n_k):

A1: $\underline{r}(X) < \underline{r}(A,X) < n$

A2: $\lim f/r_0 = 0$

A3: $\lim tr(D^+D^+)/f = 0.$

Assumption A1 is included to prevent expressions discussed from becoming undefined. All results involving efficiencies will require this assumption. As an example, from the definitions of f and r_0, Assumption A1 immediately implies that f and r_0 are positive. The requirement $\underline{r}(X,A) > \underline{r}(X)$ ensures that $\underline{R}(A) \not\subset \underline{R}(X)$ and thus $A'(I-P_X) \neq 0$. This implies that $\underset{\sim}{U} = A'(I-P_X)\underset{\sim}{Y}$ cannot identically equal zero. Further, $A'(I-P_X) \neq 0$ also implies that $m \geq 1$ since $\underline{R}[A'(I-P_X)] = \underline{R}[A'(I-P_X)A]$. The following proposition summarizes these findings.

Proposition I.1: Assume A1. The following statements hold:

a) $m \geq 1$ b) $A'(I-P_X) \neq 0$ c) $f \geq 1$

d) $r_0 \geq 1$

Indeed, one can easily show that $f \geq 1$ and $r_0 \geq 1$ if and only if Assumption A1 holds.

Assumption A2 is the most dominant of the three assumptions. All results involving limits will require this assumption. Together, Assumptions A1 and A2 drive n to infinity. To see this, recall that Assumption A1 implies that $f \geq 1$. Thus, if $\lim f/r_0 = 0$, then $n - \underline{r}(X,A) \to \infty$; so $n \to \infty$. As an example, suppose (X_k, A_k, n_k) represents a sequence of balanced two-way additive mixed models where the number of levels in both the fixed and random factors remain constant. If r equals the number of replications within each cell, then $r \to \infty$ implies Assumption A2. Clearly, Assumption A1 is satisfied in this example since $f, r_0 \geq 1$ for each k. Note that, by itself, Assumption A2 places no bounds on the size of f or r_0 other than those imposed by Assumption A1.

Assumption A3 is necessary to the development of results in Chapter III. It is equivalent to the condition that

$$\lim \frac{1}{f} \left[\left(\frac{1}{\lambda_1}\right)^2 r_1 + \left(\frac{1}{\lambda_2}\right)^2 r_2 + \ldots + \left(\frac{1}{\lambda_m}\right)^2 r_m \right] = 0.$$

To see this, recall that $D = \lambda_1 F_1 + \lambda_2 F_2 + \ldots + \lambda_m F_m$ is the spectral decomposition of D. Therefore, one has

$$D^+ = (\frac{1}{\lambda_1})F_1 + (\frac{1}{\lambda_2})F_2 + \ldots + (\frac{1}{\lambda_m})F_m$$

which implies

$$D^+D^+ = (\frac{1}{\lambda_1})^2 F_1 + (\frac{1}{\lambda_2})^2 F_2 + \ldots + (\frac{1}{\lambda_m})^2 F_m.$$

As the F_j are orthogonal projection operators, $tr(F_j)=r_j$ for $j = 1,\ldots,m$ and so

$$tr(D^+D^+) = (\frac{1}{\lambda_1})^2 r_1 + (\frac{1}{\lambda_2})^2 r_2 + \ldots + (\frac{1}{\lambda_m})^2 r_m.$$

In fact, $\frac{1}{f}\, tr(D^+D^+)$ is a weighted average of $(1/\lambda_1)^2,\ldots,(1/\lambda_m)^2$ since

$$\begin{aligned} f &= r(X,A) - r(X) \\ &= r(D) \\ &= r_1 + r_2 + \ldots + r_m. \end{aligned}$$

Section I.4: Uniqueness Property

Through computation of Gauss-Markov estimators, the set S generates all estimators in $\mathscr{E}_{\delta'\theta}$. Olsen et al. (1976) have shown that, in fact, each $w\varepsilon S$ generates a unique estimator $D_w \varepsilon \mathscr{E}_{\delta'\theta}$, a property which is useful in deriving the variance of D_w. A formal statement and proof of this result follows.

<u>Proposition I.2</u>: Assume A1. Each element $w\varepsilon S$ generates a unique estimator in the set $\mathscr{E}_{\delta'\theta}$.

<u>Proof</u>: PART1. Show that $\underline{N}(V_w) \cap \underline{N}(B') = \{0\}$. Suppose $w_1 \neq 0$. As defined, $V_w = \text{diag}[w_1/r_0, (w_1+w_2\lambda_1+w_3\lambda_1^2)/r_1, \ldots, (w_1+w_2\lambda_m+w_3\lambda_m^2)/r_m]$. Since each diagonal element is positive, and since off-diagonal elements are zero, $\underline{N}(V_w) = \{0\}$ implying $\underline{N}(V_w) \cap \underline{N}(B') = \{0\}$. Suppose $w_1 = 0$. Let $a = (a_0, a_1, \ldots, a_m)'$ be an element in $\underline{N}(V_w) \cap \underline{N}(B')$. That $a\varepsilon\underline{N}(V_w)$ implies $a_1(w_2+w_3\lambda_1) = \ldots = a_m(w_2+w_3\lambda_m) = 0$ and thus, $a_1 = a_2 = \ldots = a_m = 0$. Since $a\varepsilon\underline{N}(B')$, then $a_0 + a_1 + \ldots + a_m = 0$ which implies $a_0 = 0$. Hence $\underline{N}(V_w) \cap \underline{N}(B') = \{0\}$. PART2: Let $t_1'T$ and $t_2'T$ be two estimators in $\mathscr{E}_{\delta'\theta}$ having the same expectation and generated from the same $w\varepsilon S$. This gives $t_1'B\theta = t_2'B\theta$ for all $\theta\varepsilon R^2$, which implies $B'(t_1-t_2) = 0$. From

Zyskind's Theorem (see Zyskind (1967)), it also implies $V_w t_j \varepsilon \underline{R}(B)$ for $j = 1,2$ and so there exists a v such that $V_w(t_1-t_2)=Bv$. This implies $(t_1-t_2)'V_w(t_1-t_2) = 0$. Since $(t_1-t_2)'V_w(t_1-t_2) = 0$ if and only if $(t_1-t_2)\varepsilon\underline{N}(V_w)$, and since $(t_1-t_2)\varepsilon\underline{N}(B')$, then $t_1-t_2 = 0$ by PART1. QED

CHAPTER II

Limiting Efficiencies

In this chapter we will discuss limiting efficiencies of the estimator D_w under Assumptions A1 and A2. When expanded, the efficiency function consists of a long and complicated expression. After a development of the efficiency to a useful form, a theorem will be presented which shows this efficiency to have the same limit as that of a much simpler expression. Practical as well as theoretical advantages derive from such a result. The complexity of the efficiency, as we will see later, makes it extremely difficult to examine the limiting efficiency from a theoretical point of view. The simpler expression provides a much more manageable base from which to predict the behavior of limiting efficiencies. On the practical side, this result simplifies the calculation of limiting efficiencies, in some cases to the extent that the limiting efficiency can be determined by inspection.

Section II.1: Efficiency

Let $\underset{\sim}{Y}'M\underset{\sim}{Y}$ be a symmetric quadratic form in the set $\mathscr{N}_{\delta'\theta}$. The efficiency of $\underset{\sim}{Y}'M\underset{\sim}{Y}$ is defined as the ratio of the minimum attainable variance[10] of all estimators in $\mathscr{N}_{\delta'\theta}$ to the variance of $\underset{\sim}{Y}'M\underset{\sim}{Y}$ itself. For the case in which the variance of $\underset{\sim}{Y}'M\underset{\sim}{Y}$ is zero, define the efficiency to be one. Since $\underset{\sim}{Y}'M\underset{\sim}{Y}\varepsilon\mathscr{N}_{\delta'\theta}$, the efficiency of $\underset{\sim}{Y}'M\underset{\sim}{Y}$ lies in the interval [0,1]. This efficiency has intuitive appeal in that among all unbiased estimators of the parameter, the estimator with the most desirable properties has the smallest variance. In this regard, the effectiveness of an estimator increases as its efficiency increases and attains its maximum effectiveness when the efficiency equals one. As mentioned in Section I.2, we will consider only the efficiencies of estimators in $\mathscr{E}_{\delta'\theta}$. Denote the efficiency of an estimator D_w as $E_{\delta'\theta}(w|\gamma)$ where $\delta'\theta$ identifies the expectation of D_w. Note that the efficiency will later be computed in terms of δ and not in terms of $\delta'\theta$. While the minimum attainable variance and the variance of D_w both depend upon σ^2 and γ, later development will show that the efficiency depends only upon γ.

10. Olsen et al. (1976) shows this minimum can be obtained within the set $\mathscr{N}_{\delta'\theta}$.

We introduce the following notation. Define $0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_f$ as the <u>non</u>-distinct ordered <u>positive</u> eigenvalues of $D = A'(I-P_X)A$.[11] That f of these eigenvalues exist follows from the fact that $f = r_1+r_2+\ldots+r_m$ which was shown in Section I.3. For all $x \geq 0$, for $\gamma\varepsilon[0,1]$, and for $w\varepsilon S$, define the functions ψ and ϕ as follows:

$$\psi(x|\gamma) = [1-\gamma+\gamma x]^2$$

$$\phi(x|w) = w_1+w_2x+w_3x^2.$$

As a function of these quantities define the summations given below:

$$c_k(\gamma) = \sum_{j=1}^{f} \mu_j^k/\psi(\mu_j|\gamma) \qquad \text{for } k = 0,1,2$$

$$h_k(w) = \sum_{j=1}^{f} \mu_j^k/\phi(\mu_j|w) \qquad \text{for } k = 0,1,2$$

$$g_k(\gamma,w) = \sum_{j=1}^{f} \mu_j^k\ \psi(\mu_j|\gamma)/[\phi(\mu_j|w)]^2 \qquad \text{for } k = 0,1,2$$

$$b_k(w) = \sum_{j=1}^{f} \mu_j^k/[\phi(\mu_j|w)]^2 \qquad \text{for } k = 0,1,2,3,4$$

In terms of these summations, construct the matrices $C(\gamma)$, $H(w)$ and $G(\gamma,w)$ in the following manner:

---

11. Recall that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m$ are the non-zero distinct ordered eigenvalues of D.

$$C(\gamma) = \begin{pmatrix} r_0/(1-\gamma)^2 + c_0(\gamma), & c_1(\gamma) \\ c_1(\gamma) & , c_2(\gamma) \end{pmatrix} \quad \text{for } \gamma\varepsilon[0,1)$$

$$H(w) = \begin{pmatrix} r_0/w_1 + h_0(w), & h_1(w) \\ h_1(w) & , h_2(w) \end{pmatrix} \quad \text{for } w_1 > 0$$

$$G(\gamma,w) = \begin{pmatrix} (1-\gamma)^2 r_0/w_1{}^2 + g_0(\gamma,w), & g_1(\gamma,w) \\ g_1(\gamma,w) & , g_2(\gamma,w) \end{pmatrix} \quad \text{for } w_1 > 0$$

In proofs it will be convenient to let

$$c_0^*(\gamma) = r_0/(1-\gamma)^2 + c_0(\gamma)$$

$$h_0^*(w) = r_0/w_1 + h_0(w)$$

$$g_0^*(\gamma,w) = (1-\gamma)^2 r_0/w_1^2 + g_0(\gamma,w).$$

When lack of space requires, the arguments of the above will be dropped leaving $c_0, h_0, g_0, b_0, c_1, \ldots$ etc. Each will still retain the same meaning. Finally, define the vectors $t_1, t_2 \varepsilon M(m+1,1)$ as follows:

$$t_1 = (1,0,\ldots 0)'$$

$$t_2 = (-h_1, r_1\lambda_1/\phi(\lambda_1|w), \ldots, r_m\lambda_m/\phi(\lambda_m|w))'/h_2$$

To obtain the efficiency of the D_w estimator will require determining individually an expression for the minimum attainable variance of all estimators in $\mathscr{N}_{\delta'\theta}$ and one for the variance of D_w itself. A derivation of the variance will follow later. Denote $\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2)$ as the minimum attainable variance of all estimators in $\mathscr{N}_{\delta'\theta}$. Finding an expression for $\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2)$ involves less trouble than one might expect. As $\mathscr{T}_{\delta'\theta}$ contains the minimal complete class, then $\mathscr{T}_{\delta'\theta}$ itself constitutes a complete class. Thus, finding the minimum variance of all estimators in $\mathscr{T}_{\delta'\theta}$ is equivalent to finding the expression for $\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2)$, and this may be accomplished by applying linear model theory to the random vector T.

<u>Proposition II.1</u>: Assume A1. The minimum attainable variance of all estimators in $\mathscr{N}_{\delta'\theta}$ is given by the following equation:

$$\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2) = \begin{cases} 2\sigma^4\delta' C(\gamma)^{-1}\delta & \gamma\varepsilon[0,1) \\ 2\sigma^4\ \delta_2^2/f & \gamma = 1. \end{cases}$$

<u>Proof</u>: CASE1: Suppose $\gamma = 1$. From Section I.2, $E(T) = B\theta$ and $Cov(T) = 2\sigma^4 V(1) = 2\sigma^4 diag(0,\lambda_1^2/r_1,\ldots,\lambda_m^2/r_m)$. Define $e_1, e_2 \varepsilon M(m+1,1)$ as

$$e_1 = (1,0,\ldots,0)'$$

$$e_2 = (-c_1(1), r_1/\lambda_1,\ldots,r_m/\lambda_m)'/f.$$

It is straightforward to show that $E(e_1'T) = \theta_1$ and $E(e_2'T) = \theta_2$. In addition, one can show

$$V(1)e_1 \varepsilon \underline{R}(B)$$

$$V(1)e_2 \varepsilon \underline{R}(B).$$

From Zyskind's Theorem, we can conclude that $e_1'T$ and $e_2'T$ are blue (best linear unbiased estimator) for θ_1 and θ_2. Linear combinations of blues are themselves blue, and so the estimator $(\delta_1 e_1+\delta_2 e_2)'T$ is blue for $\delta'\theta$. Since

$$\begin{aligned} \mathrm{Var}[(\delta_1 e_1+\delta_2 e_2)'T] &= 2\sigma^4(\delta_1 e_1+\delta_2 e_2)'V(1)(\delta_1 e_1+\delta_2 e_2) \\ &= 2\sigma^4\delta_2^2/f, \end{aligned}$$

the expression for $\gamma = 1$ is established. CASE2: Suppose $\gamma\varepsilon[0,1)$. For this case, $V(\gamma)$ is non-singular. Since B is of full column rank, $\hat{\theta} = [B'V(\gamma)^{-1}B]^{-1}B'V(\gamma)^{-1}T$ is blue for the estimable parameter θ. As $\mathrm{Cov}(\hat{\theta}) = 2\sigma^4[B'V(\gamma)^{-1}B]^{-1}$, and $B'V(\gamma)^{-1}B = C(\gamma)$, the validity of the proposition is established. QED

While Proposition II.1 proves that $\mathscr{L}_{\delta'\theta}(\delta,\sigma^2)$ is the minimum attainable variance of all estimators in $\mathscr{N}_{\delta'\theta}$, Seely (1972) has shown that indeed, $\mathscr{L}_{\delta'\theta}(\delta,\sigma^2)$ is the minimum attainable variance of all estimators invariant under the class of transformations $\mathscr{G}$ and having expectation $\delta'\theta$.

Based on Proposition II.1, we choose to express $\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2)$ in a slightly more useful, although more lengthy, form. Define

$$L(\delta,\gamma) = \frac{\delta_1^2(1-\gamma)^2 c_2 - 2\delta_1\delta_2(1-\gamma)^2 c_1 + \delta_2^2[r_0+(1-\gamma)^2 c_0]}{r_0 c_2 + (1-\gamma)^2[c_0 c_2 - c_1^2]} \qquad \text{II.1.1}$$

One obtains this expression by computing $C(\gamma)^{-1}$, pre- and post-multiplying the result by δ, and by multiplying both the numerator and the denominator of the resulting ratio by $(1-\gamma)^2$. Clearly for $\gamma\varepsilon[0,1)$, $L(\delta,\gamma) = \delta'C(\gamma)^{-1}\delta$. It is also true for $\gamma = 1$ that $L(\delta,1) = \delta_2^2/f$, since $c_2(1) = f$. Thus, we can express $\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2)$ as $\mathscr{L}_{\delta'\theta}(\gamma,\sigma^2) = 2\sigma^4\ L(\delta,\gamma)$ for all $\gamma\varepsilon[0,1]$.

Ideally, one is in a position to derive an expression for the efficiency of a D_W estimator. If $D_W = t'T$, then

$$E_{\delta'\theta}(w|\gamma) = L(\delta,\gamma)/t'V(\gamma)t. \qquad \text{II.1.2}$$

While technically correct, this expression falls short of addressing the case at hand, since given D_W, one must first find a t which depends on w. An expression of the variance of D_w as a function of w yields a much more useful means of determining efficiencies. Define $V_{\delta'\theta}(w|\gamma,\sigma^2)$ as the variance of the D_w estimator having expectation $\delta'\theta$.

Proposition II.2: Assume A1. If $\gamma\varepsilon[0,1]$ and $w\varepsilon S$, then the variance of the estimator D_w is given by the following equation:

$$V_{\delta'\theta}(w|\gamma,\sigma^2) = \begin{cases} 2\sigma^4 \ \delta' H(w)^{-1} \ G(\gamma,w) \ H(w)^{-1}\delta & w_1 \neq 0 \\ 2\sigma^4(\delta_1 t_1+\delta_2 t_2)'V(\gamma)(\delta_1 t_1+\delta_2 t_2) & w_1 = 0. \end{cases}$$

Proof: Proposition I.2 states that there exists a unique estimator D_w for each $w\varepsilon S$. As $D_w = \delta'\hat{\theta}(w)$ where $\hat{\theta}(w)$ is the Gauss-Markov estimator for θ in the artificial model $E(T) = B\theta$, for $\theta\varepsilon R^2$, and $Cov(T) = V_w$, finding the variance of $\delta'\hat{\theta}(w)$ determines the variance of D_w.

CASE1: Suppose $w_1 \neq 0$. Then V_w is non-singular so that $\hat{\theta}(w) = [B'V_w^{-1}B]^{-1}B'V_w^{-1}T$ implying

$$V_{\delta'\theta}(w|\gamma,\sigma^2) = 2\sigma^4\delta'[B'V_w^{-1}B]^{-1}B'V_w^{-1}V(\gamma)V_w^{-1}B[B'V_w^{-1}B]^{-1}\delta.$$

Through straightforward although tedious matrix operations and algebra, one can show that $H(w) = B'V_w^{-1}B$

and that $G(\gamma,w) = B'V_w^{-1}V(\gamma)V_w^{-1}B$ which gives the result.

CASE2: Suppose $w_1 = 0$. For this case $V_w = \text{diag}(0,\phi(\lambda_1|w)/r_1,\ldots,\phi(\lambda_m|w)/r_m)$. If one defines $e_1, e_2 \varepsilon M(m+1,1)$ as

$$e_1 = (1,0,\ldots,0)'$$

$$e_2 = (0,r_1\lambda_1/\phi(\lambda_1|w),\ldots,r_m\lambda_m/\phi(\lambda_m|w))' \ ,$$

then it's easily shown $V_w e_1 \varepsilon \underline{R}(B)$ and $V_w e_2 \varepsilon \underline{R}(B)$, implying $e_1'T$ and $e_2'T$ are blue with respect to the artificial model. Algebraically, it can be shown that

$$t_1 = e_1$$

$$t_2 = (e_2 - h_1(w)e_1)/h_2(w)$$

and that $E(t_1'T) = t_1'B\theta = \theta_1$ and $E(t_2'T) = t_2'B\theta = \theta_2$. As t_1 and t_2 are linear combinations of e_1 and e_2, $t_1'T$ and $t_2'T$ must be blue for θ_1 and θ_2 with respect to the artificial model. Therefore, by Proposition I.3 we have $D_w = (\delta_1 t_1+\delta_2 t_2)'T$ which implies $V_{\delta'\theta}(w|\gamma,\sigma^2) = 2\sigma^4(\delta_1 t_1+\delta_2 t_2)'V(\gamma)(\delta_1 t_1+\delta_2 t_2)$. QED

Combining equation II.1.1 with the results of Proposition II.2, one arrives at the following useful and applicable expression for the efficiency:

$$E_{\delta',\theta}(w|\gamma) = \begin{cases} L(\delta,\gamma)/\delta' H(w)^{-1} G(\gamma,w) H(w)^{-1}\delta & w_1 \neq 0 \\ L(\delta,\gamma)/(\delta_1 t_1 + \delta_2 t_2)' V(\gamma)(\delta_1 t_1 + \delta_2 t_2) & w_1 = 0 \end{cases} \qquad \text{II.1.3}$$

Proposition II.3 below expands this expression into a workable form. Because of the length of this expansion, it will need to be broken down into components. In this regard, define the ratio $\mathscr{R}$ as

$$\mathscr{R} = h_2^2(w)/c_2(\gamma)g_2(\gamma,w)$$

and define P_j for $j = 1,2,\ldots,7$ as follows: (These bear no relationship to the orthogonal projection operators discussed in Section I.2.)

$$P_1 = \delta_1^2(1-\gamma)^2 \frac{c_2}{f}\frac{f}{r_0} - 2\delta_1\delta_2 \frac{(1-\gamma)^2 c_1}{f}\frac{f}{r_0} + \delta_2^2[1 + \frac{(1-\gamma)^2 c_0}{f}\frac{f}{r_0}]$$

$$P_2 = 1 + (1-\gamma)^2 \frac{c_0}{f}(1 - \frac{c_1^2}{c_0 c_2})\frac{f}{r_0}$$

$$P_3 = \frac{c_2}{f}\frac{h_2^2}{c_2 g_2}\frac{g_0^*}{h_0^*}\frac{f}{h_0^*} - 2\frac{h_1}{f}\frac{g_1 h_2}{f\; g_2}(\frac{f}{h_0^*})^2 + (\frac{h_1}{f})^2(\frac{f}{h_0^*})^2$$

$$P_4 = \frac{g_1 h_2}{f\; g_2}\frac{f}{h_0^*} + \frac{h_1}{f}\frac{g_1 h_1}{f\; g_2}(\frac{f}{h_0^*})^2 - \frac{h_2 h_1}{f\; g_2}\frac{g_0^*}{h_0^*}\frac{f}{h_0^*} - \frac{h_1}{f}\frac{f}{h_0^*}$$

$$P_5 = 1 - 2\frac{h_1 g_1}{f\ g_2}\frac{f}{h_0^*} + \frac{h_1^2}{f\ g_2}\frac{g_0^*}{h_0^*}\frac{f}{h_0^*}$$

$$P_6 = \left(\frac{h_1^2}{f\ h_2}\right)^2 \frac{h_2^2}{g_2 c_2}\left(\frac{f}{h_0^*}\right)^2 - 2\frac{h_1^2}{f\ h_2}\frac{h_2^2}{g_2 c_2}\frac{f}{h_0^*}$$

$$P_7 = \delta_1^2(1-\gamma)^2\frac{h_2^2}{f\ g_2}\frac{f}{r_0} - 2\delta_1\delta_2(1-\gamma)^2\frac{h_1 h_2}{f\ g_2}\frac{f}{r_0}$$

$$+ \delta_2^2(1-\gamma)^2\frac{h_1^2}{f\ g_2}\frac{f}{r_0} + \delta_2^2$$

Proposition II.3: Assume A1. For all $\gamma\varepsilon[0,1]$ and all $w\varepsilon S$, the efficiency of D_w is given by the following equation:

$$E_{\delta,\theta}(w|\gamma) = \begin{cases} P_1(\mathscr{R}+P_6)/[P_2(\delta_1^2P_3+2\delta_1\delta_2P_4+\delta_2^2P_5] & w_1 \neq 0 \\ P_1\mathscr{R}/P_2P_7 & w_1 = 0 \end{cases}$$

Proof: The proof is primarily by substitution. Only an outline of the proof is provided. Those steps left out involve simple matrix operations and algebra. CASE1: Suppose $w_1 \neq 0$. From equation II.1.3, the efficiency is given by

$$E_{\delta,\theta}(\omega|\gamma) = \frac{P_1 c_2 g_2 (h_0^*)^2[\mathscr{R}+P_6]}{P_2 c_2 g_2 (h_0^*)^2[\delta_1^2P_3+2\delta_1\delta_2P_4+\delta_2^2P_5]}$$

since it can be shown that $L(\delta,\gamma) = P_1/c_2P_2$ and since

$$\delta'H(w)^{-1}G(\delta,w)H(w)^{-1}\delta = \frac{g_2(h_0^*)^2[\delta_1^2P_3+2\delta_1\delta_2P_4+\delta_2^2P_5]}{c_2g_2(h_0^*)^2[\mathscr{R}+P_6]} .$$

This establishes the result. CASE2: Let $w_1 = 0$. For this case the expression given for the efficiency by the proposition is $E_{\delta',\theta}(w|\gamma) = \mathscr{R}P_1c_2/P_2c_2P_7$. One can show that $L(\delta,\gamma) = P_1/c_2P_2$ and $P_7/\mathscr{R}c_2 = (\delta_1t_1+\delta_2t_2)'V(\gamma)(\delta_1t_1+\delta_2t_2)$. QED

We conclude this section with some observations and a proposition. To begin, it should be remarked that, strictly speaking, equations II.1.2, II.1.3 and Proposition II.3 are not correct for the case in which the variance of D_w equals zero. While the expressions given in II.1.2, II.1.3 and in Proposition II.3 are actually undefined, the efficiency was defined to be one when the variance equals zero. The reader is requested to take the efficiency as being one when indeed these expressions are undefined. It is not difficult to show that the only case for which the variance is zero is that in which $\delta_1 \neq 0$, $\delta_2 = 0$, $\gamma = 1$ and $w_1 = 0$.

It is remarked also that for a wide variety of cases, the efficiency is a continuous function of w given that γ, δ, and the eigenvalues $\mu_1,\mu_2,\ldots,\mu_f$ remain fixed.

The advantage of continuity is that "small" changes in an estimator (i.e. w) do not provoke "large" changes in the efficiency; that is, one can rely on a certain degree of stability when choosing an estimator. Consider the following proposition.

Proposition II.4: Assume A1. If $w \varepsilon S$ is such that $w_1 > 0$, then the efficiency $E_{\delta',\theta}(w|\gamma)$ is a continuous function of w.

Proof: The minimum attainable variance obviously cannot be a function of w. Thus, since $w_1 > 0$, by equation II.1.3 if one can show that the expression

$$\delta' H(w)^{-1} \; G(\gamma,w) \; H(w)^{-1}\delta \qquad \text{II.1.4}$$

is positive and is a continuous function of w for $w_1 > 0$, the proposition is proved. Since $\phi(\mu_j|w) = w_1 + w_2\mu_j + w_3\mu_j^2$ is a linear combination of w_1, w_2, and w_3, it is a continuous function of $w \varepsilon S$. As $\phi(\mu_j|w) > 0$ for all $w \varepsilon S$, g_0, g_1, g_2, h_0, h_1 and h_2 are all continuous functions of $w \varepsilon S$, and since $w_1 > 0$, h_0^* and g_0^* are also continuous functions of $w \varepsilon S$ such that $w_1 > 0$. We know that

$$\det[H(w)] = r_0 h_2/w_1 + h_2 h_0 - h_1^2 \geq r_0 h_2/w_1 > 0$$

as II.2.1 implies that $h_1^2 \leq h_0 h_2$. Since det[H(w)] is a continuous function of w for $w_1>0$, $H(w)^{-1}$ is a continuous function of w. Clearly, $G(\gamma,w)$ is also continuous in w for $w_1>0$ and therefore expression II.1.4 is continuous in w for $w_1>0$. Previous remarks have indicated that the variance of an estimator is positive except possibly at $w_1 = 0$, so expression II.1.4 must also be positive. QED

While it is convenient to know that the efficiency is continuous in w for $w_1>0$, one might ask when is the limiting efficiency continuous in w. To partially answer this question, note that over the models (X_k,A_k,n_k), one has a sequence of efficiencies which are a function of w. Since each of these efficiencies is a continuous function of $w\varepsilon S$ for $w_1>0$, it is a well known result that when these efficiencies converge uniformly to a limiting efficiency which is a function of w, then this limiting efficiency is continuous in $w\varepsilon S$ for $w_1>0$. Thus, when this uniform convergence occurs, one not only has a stability in the efficiency when choosing an estimator, one also has a stability in the limiting efficiency. Practically speaking, showing uniform convergence of efficiencies to a limiting efficiency might be extremely difficult.

Section II.2: Main Result

The Cauchy-Schwarz Inequality is one of the most useful of mathematical inequalities. Define ξ and η as two vectors $\xi,\eta\varepsilon R^k$. Let (ξ,η) be the usual Euclidean inner product, and define $\|\xi\|$ as the Euclidean norm where $\|\xi\|^2 = (\xi,\xi)$. The Cauchy-Schwarz Inequality states that

$$(\xi,\eta)^2 \leq \|\xi\|^2\|\eta\|^2$$

and that equality occurs if and only if the vectors ξ and η are proportional.[12] Applying this inequality to the case at hand, one can draw the following conclusion:

Result: Define $a_u = \sum_{j=1}^{f} \mu_j^u \alpha_j$ with $u\varepsilon R^1$ and where $\alpha_1,\ldots,\alpha_f$ are non-negative constant. If $v,r,s\varepsilon R^1$, then inequality II.2.1 holds.

$$a_v^2 \leq a_r \cdot a_s \qquad \text{for } 2v = r + s \qquad \text{II.2.1}$$

As an example, if $\alpha_j = [1-\gamma+\gamma\mu_j]^{-2}$ for $j=1,\ldots,f$, then inequality II.2.1 implies that $c_1^2(\gamma) \leq c_0(\gamma)c_2(\gamma)$. The Cauchy-Schwarz Inequality also establishes inequality

12. Two vectors are proportional when one is a scalar multiple of the other.

II.2.2 and inequality II.2.3.

$$h_v^2(w) \leq g_r(\gamma,w)c_s(\gamma) \text{ for } 2v = r + s \tag{II.2.2}$$

$$h_v^2(w) \leq f\, b_{2v}(w) \text{ for all } v\varepsilon R^1 \tag{II.2.3}$$

In particular, if $v = r = s = 2$, inequality II.2.2 shows that the ratio $\mathscr{R} = h_2^2(w)/c_2(\gamma)g_2(\gamma,w) \leq 1$.

In preface to the main result, we introduce Proposition II.5 and Lemmas II.7 and II.8. With the aid of Proposition II.5 and its corollary, Lemma II.7 establishes boundedness for several functions involving $c_0, c_1, c_2, \ldots, g_2$. Lemma II.8 gives conditions under which particular functions of these quantities and of r_0 converge to zero. These lemmas will be essential to the proofs of the main results which are Theorem II.9 and Proposition II.10. In following the proofs of these lemmas, the reader is reminded that $\gamma\varepsilon[0,1]$ and $w\varepsilon S$ remain constant within their respective sets.

<u>Proposition II.5</u>: Consider the polynomial $q(X) = \alpha_0 + \alpha_1 X + \ldots + \alpha_u X^u$ with $(\alpha_0, \alpha_1, \ldots, \alpha_u) > 0$, $X \geq 0$, and $u \geq 1$, and define $\pi(X) = X^k/q(X)$ for all X such that $q(X) > 0$ where $k \geq 0$. Then the statements below hold.

1. If $\alpha_0 > 0$ and $k = 0$, then $\pi(X)$ is a decreasing function of X and $0 \leq \pi(X) \leq \alpha_0^{-1}$.

2. If $\alpha_0, \alpha_u > 0$ and $1 \leq k < u$, there exists an $\bar{X} > 0$ such that $\pi(X)$ is strictly increasing on $[0,\bar{X}]$, strictly decreasing on $[\bar{X},\infty)$, and $0 \leq \pi(X) \leq \pi(\bar{X})$ for $X\varepsilon[0,\infty)$. Moreover, $\bar{X}$ is the unique solution to the equation $f(\bar{X}) = 0$ where $f(X) = kq(X) - Xq'(X)$.[13]

3. If $\alpha_u > 0$ and $k = u$, then $\pi(X)$ is an increasing function of X and $0 \leq \pi(X) < \alpha_u^{-1}$.

<u>Proof</u>: The first derivative of $\pi(X)$ with respect to $X\varepsilon(0,\infty)$ is given by $\pi'(X) = f(X)\ X^{k-1}/[q(X)]^2$. If $u = k + v$, then $f(X)$ is equal to

$$f(X) = k\alpha_0+(k-1)\alpha_1X+(k-2)\alpha_2X^2+...+\alpha_{k-1}X^{k-1}$$
$$-(\alpha_{k+1}X^{k+1}+2\alpha_{k+2}X^{k+2}+...+v\alpha_{k+v}X^{k+v}).$$

To avoid confusion, notice that no term can appear in which the α coefficient subscript is negative, nor can the subscript exceed u. One can establish that for all three cases mentioned in the proposition, $\pi(X)$ is continuous on $X\varepsilon(0,\infty)$. Since $X^{k-1}/[q(X)]^2 > 0$ for $X\varepsilon(0,\infty)$, the function $f(X)$ has the same sign as $\pi'(X)$. To prove part 1, observe that for $k = 0$, $f(X) \leq 0$ on $X\varepsilon(0,\infty)$ which, together with continuity of $\pi(X)$, implies

13. Take $q'(X)$ as the first derivative of $q(X)$ with respect to X.

$\pi(X)$ is decreasing on $X\varepsilon[0,\infty)$. Since $\pi(0) = \alpha_0^{-1}$, and since for $\alpha_0 > 0$ $\pi(X)$ is continuous on $[0,\infty)$, $0 \leq \pi(X) \leq \alpha_0^{-1}$. The proof of part 3 follows in a similar fashion. Since $k = u$, $f(X) \geq 0$ for $X\varepsilon(0,\infty)$ and therefore $\pi(X)$ must be increasing on $X\varepsilon(0,\infty)$. Since $\alpha_u > 0$, $\lim_{X\to\infty} \pi(X) = \alpha_u^{-1}$ which implies $0 \leq \pi(X) \leq \alpha_u^{-1}$. To prove part 2, consider the following argument. Since $1 \leq k < u$ and $\alpha_0,\alpha_u > 0$, the polynomial $f(X)$ has exactly one variation in sign -- it changes from positive to negative after the k^{th} term. By Descartes' rule of signs, this means $f(X) = 0$ has exactly one positive real root $\bar{X}\varepsilon(0,\infty)$. Since $\lim_{X\to 0} f(X) = k\alpha_0 > 0$, this implies $f(X) > 0$ for $X\varepsilon(0,\bar{X})$ and $f(X) < 0$ for $X\varepsilon(\bar{X},\infty)$. As $\alpha_0,\alpha_u > 0$, and since $\pi(X)$ is continuous, one has $\pi(X)$ strictly increasing on $(0,\bar{X}]$ and strictly decreasing on $[\bar{X},\infty)$. This implies $0 \leq \pi(X) \leq \pi(\bar{X})$ for $X\varepsilon(0,\infty)$. QED

<u>Corollary II.6</u>: Consider the polynomial $q(X) = \alpha_0 + \alpha_1 X + \ldots + \alpha_u X^u$ with $\alpha_0,\alpha_1,\ldots,\alpha_u \geq 0$, $\alpha_0,\alpha_u > 0$ and $u \geq 1$. Let $\pi_k(X) = X^k/q(X)$ for $X\varepsilon(0,\infty)$ and $0 \leq k \leq u$ and define

$$\pi(X) = \sum_{k=0}^{u} \beta_k \pi_k(X)$$

where $\beta_0, \beta_1, \ldots, \beta_u \geq 0$. Then there exists an M such that $0 \leq \pi(X) \leq M$ for all $X > 0$. Moreover, for each $k = 0, 1, \ldots, u$ one can find an $\bar{X}_k$ such that $0 \leq \pi_k(X) \leq \pi_k(\bar{X}_k)$ for $X > 0$ and one choice for M is $\beta_1\pi_1(\bar{X}_1)+\cdots+\beta_u\pi_u(\bar{X}_u)$.

Proof: From Proposition II.5.2, there exists an $\bar{X}_k$ such that $0 \leq \pi_k(X) \leq \pi_k(\bar{X}_k)$ for $k = 0,1,\ldots,u$. As $\beta_0, \beta_1, \ldots, \beta_u \geq 0$, then $0 \leq \pi(X) \leq M = \sum_{k=0}^{u} \beta_k \pi_k(\bar{X}_k)$. QED

Lemma II.7: Assume A1. For any $\mu_1, \ldots, \mu_f > 0$, the expressions below are bounded. The bounds are dependant only upon the value of $\gamma\varepsilon[0,1]$ and the value of $w\varepsilon S$.

For $\gamma\varepsilon(0,1]$ and $w\varepsilon S$:

a.) $(1-\gamma)^2 c_1(\gamma)/f$ b.) $c_2(\gamma)/f$

c.) $h_2^2(w)/fg_2(\gamma,w)$

For $\gamma\varepsilon[0,1)$ and $w\varepsilon S$:

d.) $c_0(\gamma)/f$ e.) $h_1^2(w)/fg_2(\gamma,w)$

For $\gamma\varepsilon[0,1]$ and $w\varepsilon S$ such that $0 < w_1 < 1$:

f.) $h_1(w)/f$ g.) $g_0(\gamma,w)/f$

For $\gamma\varepsilon[0,1]$ and $w\varepsilon S$ such that $w_1, w_3 \neq 0$:

h.) $g_2(\gamma,w)/h_2(w)$

For $\gamma\varepsilon(0,1]$ and $w\varepsilon S$ such $0 < w_1 < 1$:

i.) $g_1(\gamma,w)h_2(w)/fg_2(\gamma,w)$

For $\gamma\varepsilon[0,1]$ and $w\varepsilon S$ such that $w_1 \neq 0$:

j.) $h_0(w)/f$ k.) $h_1^2(w)/fh_2(w)$

l.) $h_1(w)g_1(\gamma,w)/fg_2(\gamma,w)$ m.) $h_1^2(w)g_0(\gamma,w)/f^2g_2(\gamma,w)$

Proof: Because of repetitive application, Proposition II.5 and Corollary II.6 will be abbreviated as PII.5 and CII.6 respectively. In the same fashion, abbreviate Inequalities II.2.1, II.2.2 and II.2.3 as IneqII.2.1, IneqII.2.2 and IneqII.2.3. The following argument will be implicit in proving boundedness for some of the expressions in this proposition: If one can bound each term $\mu_j^k\alpha_j$ for $j = 1,2,...,f$, then clearly $\frac{1}{f}\sum_{j=1}^{f} \mu_j^k\alpha_j$ must also be bounded. CASE1: Let $\gamma\varepsilon(0,1]$ and $w\varepsilon S$. If $q(X) = [1-\gamma+\gamma X]^2$, PII.5 implies c_1/f is bounded for $\gamma\varepsilon(0,1)$ and c_2/f is bounded for $\gamma\varepsilon(0,1]$. This proves a. and b. as $(1-\gamma)^2c_1/f = 0$ for $\gamma = 1$. To prove c., note that

$$g_2(\gamma,w) = (1-\gamma)^2b_2+2\gamma(1-\gamma)b_3+\gamma^2b_4$$

From IneqII.2.3, $h_2^2 \leq fb_4$ and thus $h_2^2 \leq f\gamma^{-2}g_2(\gamma,w)$. CASE2: Suppose $\gamma\varepsilon[0,1)$ and $w\varepsilon S$. If $q(X) = (1-\gamma+\gamma X)^2$, then PII.5.1 proves d. To prove e. one sees that by

IneqII.2.3, $h_1^2 \leq fb_2$. From the above expression for g_2, $fb_2(1-\gamma)^2 \leq g_2$ and thus $(1-\gamma)^2 h_1^2 \leq fg_2$. CASE3: Let $\gamma \varepsilon [0,1]$ and $w \varepsilon S$ such that $0 < w_1 < 1$. If $q(X) = [w_1+w_2X+w_3X^2]$, then PII.5 proves f. and if $q(X) = [w_1+w_2X+w_3X^2]^2$, CII.6 proves g. This applies whether or not $w_3 = 0$. If $w_3 \neq 0$, the results follow directly from PII.5 and CII.6. If $w_3 = 0$, as $w_1+w_2+w_3 = 1$, $0 < w_1 < 1$ implies that $w_2 \neq 0$ and f. and g. still follow from PII.5 and CII.6. To prove h., CII.6 states that one can find an M such that $0 \leq [1-\gamma+\gamma\mu_j]^2/[w_1+w_2\mu_j+w_3\mu_j^2] \leq M$ for $w_3 \neq 0$ and all j. Thus, $g_2 \leq h_2 M$. CASE4: Let $\gamma \varepsilon (0,1]$ and $w \varepsilon S$ such that $0 < w_1 < 1$. From IneqII.2.1, $g_1^2 \leq g_0 g_2$ and so $g_1^2/fg_2 \leq g_0/f$. Therefore, g. implies boundedness for g_1^2/fg_2. This result plus c. proves i. since $h_2 g_1/fg_2 = (h_2^2/fg_2)^{1/2} \cdot (g_1^2/fg_2)^{1/2}$. CASE5: Let $\gamma \varepsilon [0,1]$ and $w \varepsilon S$ such that $w_1 \neq 0$. If $q(X) = w_1+w_2X+w_3X^2$, then PII.5 proves j. Part k. follows from j. since IneqII.2.1 implies $h_1^2 \leq h_0 h_2$ and so $h_1^2/fh_2 \leq h_0/f$. Now prove m. If $\gamma \varepsilon [0,1)$ and $w_1 < 1$, e. and g. prove m. For the case $w_1 = 1$ and $\gamma \varepsilon [0,1]$, define $d_k = \sum_{j=1}^{f} \mu_j^k$ for $k = 0,1,2,3,4$.

As $w_1 = 1$, $h_1 = d_1$ and

$$g_0 = (1-\gamma)^2 f + 2\gamma(1-\gamma)d_1 + \gamma^2 d_2 \qquad \text{II.2.4}$$

$$g_2 = (1-\gamma)^2 d_2 + 2\gamma(1-\gamma)d_3 + \gamma^2 d_4 \qquad \text{II.2.5}$$

Equations II.2.4 and II.2.5 yield the following inequalities,

$$\frac{h_1^2 g_0}{f^2 g_2} = (1-\gamma)^2 \frac{d_1^2 f}{f^2 g_2} + 2\gamma(1-\gamma) \frac{d_1^3}{f^2 g_2} + \gamma^2 \frac{d_1^2 d_2}{f^2 g_2}$$

$$\leq \frac{d_1^2 f}{f^2 d_2} + \frac{d_1^3}{f^2 d_3} + \frac{d_1^2 d_2}{f^2 d_4}$$

$$\leq \frac{d_1^2}{f d_2} + \frac{d_1^4}{f^2 d_2^2} \frac{d_2^2}{d_1 d_3} + \frac{d_1^2}{f d_2} \frac{d_2^2}{f d_4}$$

As $d_0 = f$, IneqII.2.1 gives $d_1^2 \leq f d_2$, $d_2^2 \leq d_1 d_3$ and $d_2^2 \leq f d_4$. Thus, $h_1^2 g_0 / f^2 g_2$ is bounded for $w_1 = 1$ and $\gamma_{\varepsilon}[0,1]$. The case $\gamma = 1$ and $w_1 \neq 0$ remains to be shown to prove m. One can show that for $\gamma = 1$, $g_0 = b_2$ and $g_2 = b_4$. Thus, one has

$$\frac{h_1^2 g_0}{f^2 g_2} = \frac{h_1^2 b_2}{f^2 b_4} = \frac{h_1^2}{f b_2} \frac{b_2^2}{f b_4}$$

IneqII.2.1 and IneqII.2.3 show this is bounded. This proves m. The proof of l. follows directly from m. and

IneqII.2.1 since $(h_1 g_1/fg_2)^2 = (h_1^2 g_0/f^2 g_2)(g_1^2/g_0 g_2)$. QED

Proposition II.5 and Corollary II.6 provide the means to determine numerical bounds for some of the expressions in Lemma II.7. As an example,

$$0 \leq c_0(\gamma)/f \leq (1-\gamma)^{-2} \qquad \text{for } \gamma\varepsilon[0,1)$$

$$0 < h_2(w)/f \leq w_3^{-1} \qquad \text{for } w\varepsilon S \text{ such that } w_3 \neq 0$$

and so on. Because later proofs do not require these bounds, they were not included in Lemma II.7 to avoid adding unnecessary arguments to the proof. The next lemma and later results will make extensive use of Lemma II.7. In most cases Lemma II.7 and Lemma II.8 will be abbreviated as LII.7 and LII.8 respectively.

Lemma II.8: Assume A1,A2. The following limits hold for constant γ and w in the given subsets of $\gamma\varepsilon[0,1]$ and $w\varepsilon S$:

a.) $\lim[f/h_0^*] = 0$ $\qquad \gamma\varepsilon[0,1]$ and $w_1 \neq 0$

b.) $\lim[g_0^*/h_0^*][f/h_0^*] = 0$ $\qquad \gamma\varepsilon[0,1]$ and $0<w_1<1$

c.) $\lim[h_1 h_2/fg_2][g_0^*/h_0^*][f/h_0^*] = 0$ $\qquad \gamma\varepsilon(0,1]$ and $0<w_1<1$

d.) $\lim[h_1^2/fg_2][g_0^*/h_0^*][f/h_0^*] = 0$ $\qquad \gamma\varepsilon[0,1]$ and $w_1 \neq 0$

e.) $\lim[r_0/h_0^*][f/h_0^*] = 0$ $\qquad \gamma\varepsilon[0,1]$ and $w_1 \neq 0$

Proof: Part a. is easily shown since $h_0^* = r_0/w_1 + h_0$ implies that $w_1 f/r_0 > f/h_0^* > 0$. Assumption A2 states $\lim (f/r_0) = 0$ which proves the result. To prove b., one can show that

$$\frac{g_0^*}{h_0^*} = \frac{(1-\gamma)^2 r_0/w_1 + g_0}{r_0/w_1 + h_0} = \frac{(1-\gamma)^2/w_1 + w_1(f/r_0)(g_0/f)}{1 + w_1(f/r_0)(h_0/f)} .$$

The ratio f/r_0 converges to zero. Since $0 < w_1 < 1$, LII.7.g and LII.7.j show that g_0/f and h_0/f are bounded. Thus, g_0^*/h_0^* must be bounded. This result together with a. proves b. Suppose $\gamma \varepsilon (0,1)$ and $0 < w_1 < 1$. For this case, LII.7.c, LII.7.e, and b. above prove c. If $\gamma = 1$, and $0 < w_1 < 1$, since $g_0 = b_2$ and $g_2 = b_4$, then

$$\frac{h_1 h_2}{f g_2} \frac{g_0^*}{h_0^*} \frac{f}{h_0^*} = \frac{h_1 h_2 g_0}{f^2 g_2} \left(\frac{f}{h_0^*}\right)^2$$

$$= \frac{h_1}{f} \left(\frac{h_2^2}{f b_4}\right)^{1/2} \left(\frac{b_2^2}{b_0 b_4}\right)^{1/2} \left(\frac{b_0}{f}\right)^{1/2} \left(\frac{f}{h_0^*}\right)^2 .$$

If $q(X) = [w_1 + w_2 X + w_3 X^2]^2$, Proposition II.5.1 implies b_0/f is bounded as $w_1 \neq 0$. The ratio h_1/f is bounded by LII.7.f, Inequality II.2.3 bounds $h_2^2/f b_4$ and Inequality II.2.1 bounds $b_2^2/b_0 b_4$. These observations along with a. above prove c. To prove d., by algebra one can establish the following inequality:

$$\frac{h_1^2}{fg_2}\ \frac{g_0^*}{h_0^*}\ \frac{f}{h_0^*} \le \frac{(1-\gamma)^2 h_1^2}{fg_2}\ \frac{1}{w_1}\ \frac{f}{h_0^*} + \frac{h_1^2 g_0}{f^2 g_2}\ \frac{f}{r_0}\ \frac{f}{h_0^*}$$

LII.7.e. bounds $(1-\gamma)^2 h_1^2/fg_2$ and LII.7.m bounds $h_1^2 g_0/f^2 g_2$ for $\gamma \varepsilon [0,1]$ and $w_1 \neq 0$. Thus, Assumption A2 and part a. above prove d. Part a. also proves e. since

$$\frac{r_0}{h_0^*} = \frac{r_0}{r_0/w_1+h_0} = \frac{w_1}{1+w_1(h_0/f)(f/r_0)} \quad .$$

By LII.7.j, h_0/f is bounded for $w_1 \neq 0$ which implies boundedness for r_0/h_0^*. QED

Theorem II.9: Assume A1,A2. Suppose $\gamma \varepsilon (0,1]$, and for $w \varepsilon S$ such that $w_1 \neq 1$, consider the estimator $D_w \varepsilon \mathscr{E}_{\delta',\theta}$ for $\delta \varepsilon R^2$ and $\delta_2 \neq 0$. Then, lim $\mathscr{R}$ exists if and only if lim $E_{\delta',\theta}(w|\gamma)$ exists, and if either exists the two limits are equal. Moreover, for the special case $\delta_1 = 0$ and for $\delta_2 \neq 0$, the above conclusion holds for all estimators $w \varepsilon S$ and for all $\gamma \varepsilon [0,1]$.

Proof: In PART1 through PART4, this proof determines the limits of $P_1,\ldots,P_7$ for appropriate subsets of $w \varepsilon S$ and $\gamma \varepsilon [0,1]$. PART5 applies these limits to prove the theorem. In PART1 through PART4, assume that in the sequence of models (X_k,A_k,n_k), k is large enough to bound those expressions whose limits are zero in Lemma II.8 for the given subsets of γ and w.

PART1: Let $\gamma\varepsilon(0,1]$ and $w\varepsilon S$. From Section II.1, P_1 equals the sum of three terms. LII.7.a and LII.7.b bound the first two terms and from LII.7.d, $(1-\gamma)^2 c_0/f$ is bounded for $\gamma\varepsilon[0,1]$. From Assumption A2, lim f/r_0 = 0 and so lim $P_1 = \delta_2^2$. If $\delta_1 = 0$, then one has lim $P_1 = \delta_2^2$ for all $\gamma\varepsilon[0,1]$ and $w\varepsilon S$. From IneqII.2.1, $c_1^2/c_0 c_2$ is bounded, thus lim $P_2 = 1$ for $\gamma\varepsilon[0,1]$ and $w\varepsilon S$. PART 2: Let $\gamma\varepsilon(0,1]$ and suppose $w\varepsilon S$ such that $0<w_1<1$. P_3 consists of three terms, the first bounded by II.2.2 and LII.7.b, the second by LII.7.f and LII.7.i, and the third by LII.7.f. LII.8.a and LII.8.b show that lim $P_3 = 0$. P_4 consists of four terms. The third term converges to zero by LII.8.c. The first term is bounded by LII.7.i, the second by LII.7.l and LII.7.f, and the fourth by LII.7.f. LII.8.a then shows that lim $P_4 = 0$. PART3: Let $\gamma\varepsilon[0,1]$ and $w\varepsilon S$ such that $w_1 \neq 0$. By LII.7.l, LII.8.a and LII.8.d, lim $P_5 = 1$. By LII.7.k, IneqII.2.2, and LII.8.a, lim $P_6 = 0$. PART4: Suppose $\gamma\varepsilon(0,1]$. In this case, LII.7.c and LII.7.e bound P_7. Assumption A2 then implies that lim $P_7 = \delta_2^2$ for $\gamma\varepsilon(0,1]$. If in addition, $\delta_1 = 0$, LII.7.e bounds $(1-\gamma)^2 h_1^2/fg_2$ so that Assumption A2 proves that lim $P_7 = \delta_2^2$ for all $\gamma\varepsilon[0,1]$. PART5: We can summarize the relevant results from PART1 through PART4 as follows below. Unless stated to the contrary, assume that $w \epsilon S$ and $\delta \epsilon R^2$.

$\lim P_1 = \delta_2^2$	for $\gamma \epsilon (0,1]$	II.2.6.a
$\lim P_1 = \delta_2^2$	for $\gamma \epsilon [0,1]$ when $\delta_1 = 0$	II.2.6.b
$\lim P_2 = 1$	for $\gamma \epsilon [0,1]$	II.2.6.c
$\lim P_3 = 0$	for $\gamma \epsilon (0,1]$ and $0 < w_1 < 1$	II.2.6.d
$\lim P_4 = 0$	for $\gamma \epsilon (0,1]$ and $0 < w_1 < 1$	II.2.6.e
$\lim P_5 = 1$	for $\gamma \epsilon [0,1]$ and $w_1 \neq 0$	II.2.6.f
$\lim P_6 = 0$	for $\gamma \epsilon [0,1]$ and $w_1 \neq 0$	II.2.6.g
$\lim P_7 = \delta_2^2$	for $\gamma \epsilon (0,1]$	II.2.6.h
$\lim P_7 = \delta_2^2$	for $\gamma \epsilon [0,1]$ when $\delta_1 = 0$	II.2.6.i

Suppose $0 < w_1 < 1$ and $\delta_2 \neq 0$. From Proposition II.3, one has

$$E_{\delta,\theta}(w|\gamma) = P_1(\mathscr{R}+P_6)/[P_2(\delta_1^2 P_3 + 2\delta_1\delta_2 P_4 + \delta_2^2 P_5)]$$

If $\gamma \epsilon (0,1]$, provided $\lim \mathscr{R}$ exists, II.2.6.a,c-g imply $\lim E_{\delta,\theta}(w|\gamma) = \lim \mathscr{R}$. Vice versa, by simple algebra, one can conclude that if $\lim E_{\delta,\theta}(w|\gamma)$ exists, then $\lim \mathscr{R}$ also exists and the two limits are equal. If in addition $\delta_1 = 0$, then

$$E_{\delta,\theta}(w|\gamma) = P_1(\mathscr{R}+P_6)/[P_2\delta_2^2 P_5]$$

and II.2.6.b,c,f,g give this same result for $\gamma\varepsilon[0,1]$ when $w\varepsilon S$ such that $w_1 \neq 0$. Suppose now that $w_1 = 0$. From Proposition II.3,

$$E_{\delta'\theta}(w|\gamma) = P_1\mathscr{R}/P_2P_7.$$

In the same manner as described above, if $w_1 = 0$ and $\gamma\varepsilon(0,1]$, II.2.6.a,c,h imply that if either lim $\mathscr{R}$ or lim $E_{\delta'\theta}(w|\gamma)$ exist then the other exists and the two limits are equal. If $\delta_1 = 0$, results II.2.6.b,c,i imply the same is true for $\gamma\varepsilon[0,1]$ and $w_1 = 0$. QED

Table IV.6 in Chapter IV provides counter-examples for several cases not included in Theorem II.9. As an example, if $\gamma = 0$, $\mathscr{R}$ has a different limit[14] than the efficiency of all but one of the D_w estimators given in the table for estimating $\theta_1+\theta_2$. In addition, when $w_1 = 1$, $\gamma\varepsilon(0,1]$ and $\delta'\theta = \theta_1+\theta_2$ or $\delta'\theta = \theta_1$, Table IV.6 shows that $\mathscr{R}$ has a different limit than the true limiting efficiency.

From Theorem II.9, one can see that the ratio $\mathscr{R}$ can be effectively used to examine the behavior of large sample efficiencies. For this reason, there exist characteristics of $\mathscr{R}$ which merit comment.

---

14. From Theorem II.9, lim $\mathscr{R}$ = lim $E_{\theta_2}(w|\gamma)$.

The ratio $\mathscr{R}$ clearly serves as an approximation for $E_{\delta,\theta}(w|\gamma)$ at large sample sizes for the subsets $\delta\varepsilon R^2$, $\gamma\varepsilon[0,1]$ and $w\varepsilon S$ indicated in Theorem II.9. It is comforting, as mentioned previously in this section, that $0 \leq \mathscr{R} \leq 1$. Naturally, if $\mathscr{R}$ identically equals one for cases considered in Theorem II.9 and for all models (X_k, A_k, n_k), then $\lim E_{\delta,\theta}(w|\gamma) = 1$. This leads one to wonder whether $\mathscr{R} = 1$ if and only if $E_{\delta,\theta}(w|\gamma) = 1$. Algebraically, one can show that in general this does not hold.

It is interesting that $c_2(\gamma)$, $h_2(w)$, and $g_2(\gamma,w)$ need not have limits, nor even to be bounded, for $\mathscr{R}$ to have a limit which exists. To see this, consider the case when $\gamma = w_2 = 1$, $\lim f = \infty$, and $\mu_j = j$ for $j = 1,2,\ldots,f$. For this case,

$$c_2(\gamma) = f$$

$$h_2(w) = 1+2+3+\ldots+f = f(f+1)/2$$

$$g_2(\gamma,w) = 1+4+9+\ldots+f^2 = f(f+1)(2f+1)/6,$$

and so $\mathscr{R} = 3(f+1)/2(2f+1)$. This implies $\lim \mathscr{R} = 3/4$. Indeed, even $h_2(w)/f$ and $g_2(\gamma,w)/f$ are unbounded for this case. However, from Hardy, Littlewood and Polya (1934), if $c_2(\gamma)/f$ and $g_2(\gamma,w)/f$ converge, then $h_2(w)^2/f$ converges which implies $\mathscr{R}$ converges. In the same sense, if $c_2(\gamma)$ and $g_2(\gamma,w)$ converge, then so must $h_2^2(w)$ and $\mathscr{R}$.

It would be informative if one could show that $\mathscr{R}$ consistently over-estimates or under-estimates the efficiency for those cases considered in Theorem II.9. Unfortunately, this is not the case. Examples can be found in which for one case $\mathscr{R} < E_{\delta,\theta}(w|\gamma)$ and in which for another case $E_{\delta,\theta}(w|\gamma) < \mathscr{R}$.

Theorem II.9 concentrates on the behavior of the efficiency when $\delta_2 \neq 0$. Proposition II.10 below examines what happens to the efficiency when $\delta_2 = 0$ and $\delta_1 \neq 0$.

<u>Proposition II.10</u>: Assume A1,A2. Let $\gamma\varepsilon[0,1)$ and $w\varepsilon S$ where $w_3\neq 0$. Consider the estimator $D_w \varepsilon \mathscr{E}_{\delta,\theta}$ in which $\delta_1 \neq 0$ and $\delta_2 = 0$. Then, $\lim E_{\delta,\theta}(w|\gamma) = 1$.

<u>Proof</u>: CASE1: Let $w\varepsilon S$ such that $w_1, w_3 \neq 0$ and $\gamma\varepsilon[0,1)$. From Proposition II.3, one can show algebraically that the efficiency is given by expression II.2.7 below.

$$\frac{1 - 2\dfrac{h_1^2}{fh_2}\dfrac{f}{h_0^*} + \dfrac{h_1^4}{f^2h_2^2}\left(\dfrac{f}{h_0^*}\right)^2}{P_2(1-\gamma)^{-2}\left[\dfrac{g_0^*}{h_0^*}\dfrac{r_0}{h_0^*} - 2\dfrac{h_1 g_1}{f\, g_2}\dfrac{g_2}{h_2}\dfrac{r_0}{h_0^*}\dfrac{f}{h_0^*} + \dfrac{h_1^2}{fh_2}\dfrac{g_2}{h_2}\dfrac{r_0}{h_0^*}\dfrac{f}{h_0^*}\right]} \qquad \text{II.2.7}$$

By LII.7.k and LII.8.a, the limit of the numerator of II.2.7 is one. By II.2.6.c from the proof of Theorem

II.9, $\lim P_2 = 1$. To prove that $\lim E_{\delta,\theta}(w|\gamma) = 1$ for this case, it remains to show that the sum in the denominator of II.2.7 also has a limit of one. By LII.7.1, LII.7.h, LII.7.k and LII.8.e, the expression

$$- 2 \frac{h_1 g_1}{f} \frac{g_2}{g_2} \frac{r_0}{h_2} \frac{f}{h_0^*} \frac{}{h_0^*} + \frac{h_1^2}{fh_2} \frac{g_2}{h_2} \frac{r_0}{h_0^*} \frac{f}{h_0^*}$$

has a limit of zero. Applying simple algebra, it can be shown that

$$(1-\gamma)^{-2} \frac{g_0^*}{h_0^*} \frac{r_0}{h_0^*} = \frac{1+(1-\gamma)^{-2} w_1 (g_0/f)(f/r_0)}{[1+w_1^2(f/r_0)(h_0/f)]^2} \qquad \text{II.2.8}$$

By LII.7.g, g_0/f is bounded, and by LII.7.j, h_0/f is bounded. Assumption A2 then implies that expression II.2.8 converges to one which implies expression II.2.7 also converges to one. CASE2: Let $\gamma\varepsilon[0,1)$, $w\varepsilon S$ such that $w_1 = 0$ and suppose $\delta\varepsilon R^2$ where $\delta_1 \neq 0$ and $\delta_2=0$. It can be shown that $E_{\delta,\theta}(w|\gamma) = 1/P_2$. II.2.6.c from the proof of Theorem II.9 also implies that the $\lim P_2 = 1$ for this case as well. QED

Tables IV.2 through IV.8 in Chapter IV show that Proposition II.10 fails to hold for any D_w estimator among those given when $\gamma = 1$ and $w_1 > 0$. Tables IV.6 through IV.8 show that Proposition II.10 fails to hold for the estimator D_w with $w_1 = 1$ estimating $\theta_1 + \theta_2$ when $\gamma > 0$. Note that for $\delta_1 \neq 0$ and $\delta_2 = 0$, the D_2 and D_3 estimators are the same. Thus, Proposition II.10 also holds for the D_2 estimator.

We conclude this chapter with the following observation based on Theorem II.9. It is interesting to note that, except for cases when δ_1 and δ_2 change from zero to a non-zero value, the limiting efficiency rarely depends upon δ. One can see this since $\mathscr{R}$ does not depend upon δ, and yet $\mathscr{R}$ has the same limit as the efficiency. Thus, it can be shown that the limiting efficiency for the D_w estimators (where $w_1 \neq 1$) estimating $\delta'\theta$ with $\delta_2 \neq 0$ does not depend upon δ when $\gamma \varepsilon (0,1]$.

CHAPTER III

Estimators of θ_2

With the aid of Theorem II.9, one can draw definite conclusions about estimators in $\mathscr{E}_{\theta_2}$. The results presented in this chapter were originally given in Seely (1979). We have only generalized these results to a broader class of cases. In addition to Assumptions A1 and A2, we will invoke Assumption A3. Seely (1979) has assumed conditions given by Hartley et al. (1978) as the limiting process in connection with the one-way randomized design. Assumptions A1, A2 and A3 comprise a weaker collection of limiting conditions and yet still encompass those models covered by the Hartley et al. conditions. To simplify notation, define D_k in $\mathscr{E}_{\theta_2}$ as the estimator such that $w \in S$ where $w_k = 1$ for $k = 1,2,3$, and define $E_2(D_k|\gamma)$ as its efficiency. Also, recall that in Section II.2 we defined $d_k = \mu_1^k + \mu_2^k + \cdots + \mu_f^k$ for $k = 1,2,3,4$.

Section III.1: The D_1 and D_2 Estimators

Several useful results follow from making the additional Assumption A3. Define

$$s_k = \sum_{j=1}^{f} 1/\mu_j^k \quad \text{for } k = 1,2.$$

Since r_j is the multiplicity of the distinct positive eigenvalue λ_j, from Section I.3 it is easy to see that Assumption A3 is equivalent to the condition that

$$\lim s_2/f = 0.$$

From Inequality II.2.1, one can conclude inequality III.1.1.

$$\left(\frac{s_2}{f}\right)^{\frac{1}{2}} \geq \frac{s_1}{f} \geq \frac{f}{d_1} \geq \frac{d_1}{d_2} \geq \ldots \geq \frac{d_k}{d_{k+1}} \geq 0 \qquad \text{III.1.1}$$

Thus under Assumption A3 all of these ratios converge to zero. In addition, recalling that $d_0 = f$, one can also conclude that for $k \geq 0$ and $i \geq 1$,

$$\lim d_k/d_{k+i} = 0. \qquad \text{III.1.2}$$

To see this, consider the special case in which $k = 1$ and $i = 2$. Limit III.1.2 states that $\lim d_1/d_3 = 0$.

This is indeed true under Assumption A3 since by inequality III.1.1,

$$\left(\frac{f}{d_1}\right)^2 \geq \left(\frac{d_1}{d_2}\right)^2 \geq \frac{d_1}{d_2} \cdot \frac{d_2}{d_3} = \frac{d_1}{d_3} \geq 0.$$

By inequality III.1.1, we can further conclude from Assumption A3 that

$$\lim s_1/f = 0. \qquad \text{III.1.3}$$

By the Schwarz Inequality applied to $f = \sum_{i=1}^{f} x_i/x_i$ with $x_i = \mu_i/[\phi(w|\mu_i)]^{1/2}$ and by Proposition II.4, one can show for $w_3 \neq 0$ that

$$[w_1 s_2/f + w_2 s_1/f + w_3]^{-1} \leq h_2(w)/f \leq w_3^{-1} .$$

Thus, under Assumption A3 one has

$$\lim h_2(w)/f = w_3^{-1} , \qquad \text{III.1.4}$$

and for the case $w = [(1-\gamma)^2, 2\gamma(1-\gamma), \gamma^2]'$, if $\gamma \neq 0$, one has the additional result that

$$\lim c_2(\gamma)/f = \gamma^{-2}. \qquad \text{III.1.5}$$

Consider the following propositions.

Proposition III.1.1: Assume A1, A2, A3. If $\gamma \varepsilon (0,1]$ then $\lim E_2(D_1 | \gamma)$ exists if and only if $\lim d_2^2/fd_4$ exists, and if either exists, the two limits are equal.

Proof: For $w = (1,0,0)'$, one can show that

$$\mathscr{R} = d_2^2/\{[c_2(\gamma)/f][(1-\gamma)^2 d_2 + 2\gamma(1-\gamma)d_3 + \gamma^2 d_4]f\}.$$

The proof follows directly from Theorem II.9 and from III.1.2 and III.1.5. QED

Proposition III.1.2: Assume A1, A2, A3. If $\gamma \varepsilon (0,1]$ then $\lim E_2(D_2 | \gamma)$ exists if and only if $\lim d_1^2/fd_2$ exists, and if either exists, the two limits are equal.

Proof: For $w = (0,1,0)'$, one can show that

$$\mathscr{R} = d_2^2/\{[c_2(\gamma)/f][(1-\gamma)^2 f + 2\gamma(1-\gamma)d_1 + \gamma^2 d_2]f\}.$$

The proof follows from Theorem II.9, III.1.2 and III.1.5. QED

Section III.2: The D_w estimator

In Section III.1, it has been shown that under Assumption A3, the efficiencies of the D_1 and D_2 estimators converge to the same limit as that of certain expressions provided the limits of those expressions exist. For most estimators in $\mathscr{E}_{\theta_2}$ we can show that under Assumption A3 the efficiency actually converges to one.

<u>Proposition III.3</u>: Assume A1, A2, A3. Suppose $\gamma\varepsilon(0,1]$ and consider the D_w estimator such that $w\varepsilon S$ and $w_3 \neq 0$. Then, $\lim E_2(D_w|\gamma) = 1$.

<u>Proof</u>: Since $\gamma, w_3 \neq 0$, from inequality II.2.2 and Proposition II.4, one has

$$[h_2(w)/f]^2 \leq [c_2(\gamma)/f][g_2(\gamma,w)/f] \leq w_3^{-2}.$$

Thus, $\lim \mathscr{R} = 1$ by III.1.4 and the proof follows from Theorem II.9. QED

Propositions III.1, III.2 and III.3 generalize the results of Seely (1979) pertaining to estimators of θ_2. Those results pertaining to estimators of θ_1 generalize directly from Proposition II.10. The same type of results for estimators in $\mathscr{E}_{\delta'\theta}$ with $\delta\varepsilon R^2$ can be proved in a similar fashion to those for estimators of θ_2.

CHAPTER IV

Behavior of Efficiencies - An Example

One can gain useful information from computing limiting efficiencies over different sequences of models and for different estimators. From such examples can be obtained an intuitive idea of how limiting efficiencies behave under varied situations. Computation of limiting efficiencies can also provide a basis on which to make recommendations about specific estimators. In particular, we hope to make some recommendations concerning the D_1, D_2 and D_3 estimators, about Rao's MINQUE estimator with equal priory weights given in Rao (1972), and a few others. We mention that the D_1 estimator is Rao's MINQUE0 estimator (see Searle (1979)), the D_2 estimator is the Henderson III estimator (see Searle (1969)) and the D_3 estimator for θ_2 is the ANOVA estimator for θ_2 based on unweighted means, (see Henderson (1978)). The efficiencies computed in this chapter also demonstrate some cases in which the results of Theorem II.9 and Proposition II.10 fail to hold.[15]

---

15. See Section II.2

Section IV.1: Tables of Limiting Efficiencies

From equation II.1.3, one can show that, given values for γ, w and δ, the calculation of the efficiency of an estimator $D_w \varepsilon \mathscr{E}_{\delta ' \theta}$ depends upon the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and their corresponding multiplicities r_1, $r_2, \ldots, r_m$ plus n and $\underline{r}(X)$. This follows since $r_0 = n - \underline{r}(X) - f$ and because $f = r_1 + r_2 + \cdots + r_m$. In this section we will calculate limiting efficiencies for various values of γ, w and δ when $n \to \infty$ and where $\underline{r}(X) = 5$, $r_j = \zeta_j \sqrt{n}$ for $j=1,2,\ldots,9$ and

$$\begin{array}{lll} \lambda_1 = 1/\sqrt{n} & \lambda_4 = 4 & \lambda_7 = 7\sqrt{n} \\ \lambda_2 = 2/\sqrt{n} & \lambda_5 = 5 & \lambda_8 = 8\sqrt{n} \\ \lambda_3 = 3/\sqrt{n} & \lambda_6 = 6 & \lambda_9 = 9\sqrt{n} \ . \end{array}$$

Let the constants $\zeta_1, \zeta_2, \ldots, \zeta_9 \varepsilon \{0,1\}$. In this way, the eigenvalues are indeed distinct, and as n tends to infinity, some of the eigenvalues converge to zero, some remain constant, and the remainder of the eigenvalues tend to infinity. All of the multiplicities tend to infinity. By appropriate selection of $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_9)$, one has at hand a fairly wide variety of different behaviors among the eigenvalues. For instance, if $\zeta = (1,1,1,0,0,0,0,0)$, one has a case in which all eigenvalues tend to zero, and if $\zeta = (1,1,1,0,0,0,1,1,)$, one has a case in which the eigenvalues converge to zero or to infinity, but in which none of the eigenvalues remain constant. For the general case, we can establish that

$$f = \sqrt{n}\ (\zeta_1 + \zeta_2 + \cdots + \zeta_9)$$

$$r_0 = n - 5 - \sqrt{n}\ (\zeta_1 + \zeta_2 + \cdots + \zeta_9).$$

For $n \geq 100$, $f, r_0 \geq 1$ and thus Assumption A1 holds. Assumption A2 also holds, and Assumption A3 holds if all eigenvalues tend to infinity. The limiting efficiencies were computed by calculating the efficiency directly from equation II.1.3 for successively larger n until the efficiency converged to a specific value. In no case did the efficiency fail to converge. The computed limiting efficiencies appear in Table IV.1 through Table IV.8. Each table contains the value of the limiting efficiency for a different choice of ζ and for all possible combinations of γ, δ and w in the following sets.

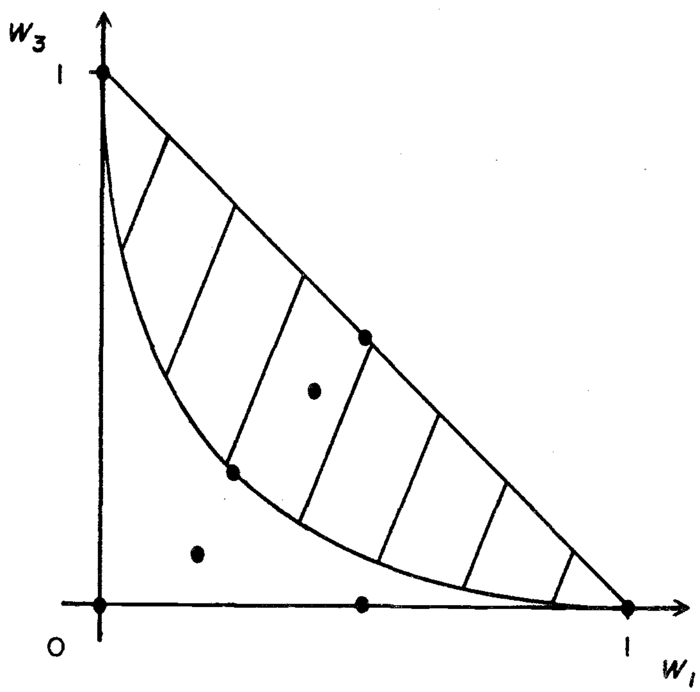
$$w \in \left\{ \begin{pmatrix} .25 \\ .50 \\ .25 \end{pmatrix}, \begin{pmatrix} .4 \\ .2 \\ .4 \end{pmatrix}, \begin{pmatrix} .2 \\ .7 \\ .1 \end{pmatrix}, \begin{pmatrix} .5 \\ 0 \\ .5 \end{pmatrix}, \begin{pmatrix} .5 \\ .5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\gamma \in \{0, .001, .01, .1, .25, .5, .75, .9, .99, .999, 1\}$$

$$\delta \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

To better understand the choices of w included in the tables consider Figure IV.1, recall that once one has chosen w_1 and w_3 for $w \in S$, then w_2 is determined and $w_2 = 1 - w_1 - w_3$. Thus, all points $w \in S$ can be represented as an ordered pair (w_1, w_3) on the triangle in Figure IV.1. The shaded region represents those points for which $w \in S_c$. Each plotted point on the graph is an estimator included in the tables. The point at the origin is the D_2 estimator since $w_1 = w_3 = 0$ implies $w_2 = 1$. The criteria was to choose w's to represent each different area on the graph.

Figure IV.1



In computing the limiting efficiencies of several estimators of $\delta'\theta$ where $\delta_1, \delta_2 \neq 0$, it was interesting to note that the maximum efficiency occurred at a finite value of n and not at $n = \infty$. Ordinarily, one might think that a bigger n is a better n! At least in regard to the limiting efficiency, this is not always the case. Considering the estimator $w = (.16,.48,.36)'$, estimating $\theta_1 + \theta_2$ when $\gamma = 1$, if $\zeta=(0,0,0,0,0,1,1,1,1)$, then Table IV.1 shows the maximum efficiency does not occur at $n = \infty$, but must occur at some finite value of n.

Table IV.1
Limiting Efficiencies of the
(.16,.48,.36) Estimator

n	$E_{\delta'\theta}(D_w \mid \gamma)$
10	.992623590
10^4	.992597468
10^5	.992594858
10^6	.992594597
10^7	.992594571
10^8	.992594568
10^9	.992594568

The limiting efficiencies given in Tables IV.2 through IV.8 were rounded to three decimal places. Limiting efficiencies which round to 0 but do not equal 0 are expressed as "0+", and similarly, limiting efficiencies which round to 1 but do not equal 1 are expressed as "1-". Note that by Theorem II.9, for all cases $\lim E_{\theta_2}(w|\gamma) = \lim \mathscr{R}$ so that the tables also give the limit of $\mathscr{R}$. In addition, except at $w_1 = 1$, the D_w estimator has the same limiting efficiency when estimating $\theta_1 + \theta_2$ as it has when estimating $\delta_1\theta_1 + \delta_2\theta_2$ provided $\delta_1, \delta_2 \neq 0$. This result follows directly from Theorem II.9. Thus, except at $w_1 = 1$ or at $\gamma = 0$, the tables provide the limiting efficiencies of the given estimators for all $\delta \varepsilon R^2$.

TABLE IV.2

Limiting Efficiencies When All Eigenvalues Converge to Zero

$\zeta = (1,1,1,0,0,0,0,0,0)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
$\gamma = .001$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
$\gamma = .01$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
$\gamma = .1$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
$\gamma = .25$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
$\gamma = .50$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
$\gamma = .75$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472

TABLE IV.2 (continued)

Limiting Efficiencies When All Eigenvalues Converge to Zero

ζ = (1,1,1,0,0,0,0,0,0)

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
γ = .9								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
γ = .99								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
γ = .999								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	.857	.472
θ_2	1	1	1	1	1	1	.857	.472
γ = 1.000								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	.667	.667	.667	.667	.667	.667	.857	1
θ_2	.667	.667	.667	.667	.667	.667	.857	1

TABLE IV.3

Limiting Efficiencies When The Eigenvalues Converge To Zero And To A Constant

$\zeta = (1,1,1,1,1,1,0,0,0)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.928	.913	.951	.905	.982	1	.487	0
θ_2	.928	.913	.951	.905	.982	1	.487	0
$\gamma = .001$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.929	.914	.952	.906	.983	1-	.490	0
θ_2	.929	.914	.952	.906	.983	1-	.490	0
$\gamma = .01$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.936	.921	.958	.914	.986	1-	.513	0
θ_2	.936	.921	.958	.914	.986	1-	.513	0
$\gamma = .1$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.976	.966	.988	.961	1-	.988	.707	0
θ_2	.976	.966	.988	.961	1-	.988	.707	0
$\gamma = .25$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.995	.990	1-	.987	.996	.963	.876	0
θ_2	.995	.990	1-	.987	.996	.963	.876	0
$\gamma = .50$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	.999	.998	.998	.982	.935	.962	0
θ_2	1	.999	.998	.998	.982	.935	.962	0
$\gamma = .75$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.999	1-	.993	1-	.972	.919	.976	0
θ_2	.999	1-	.993	1-	.972	.919	.976	0

TABLE IV.3 (continued)

Limiting Efficiencies When The Eigenvalues Converge To Zero And To A Constant

$\zeta = (1,1,1,1,1,1,0,0,0)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
γ = .9								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.998	1-	.991	1-	.968	.912	.976	0
θ_2	.998	1-	.991	1-	.968	.912	.976	0
γ = .99								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.997	.999	.989	1-	.965	.908	.974	0
θ_2	.997	.999	.989	1-	.965	.908	.974	0
γ = .999								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.997	.999	.989	1-	.965	.908	.974	0
θ_2	.997	.999	.989	1-	.965	.908	.974	0
γ = 1.000								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	.498	.500	.495	.500	.483	.454	.487	1
θ_2	.498	.500	.495	.500	.483	.454	.487	1

TABLE IV.4

Limiting Efficiencies When All
Eigenvalues Converge to A Constant
$\zeta = (0,0,0,1,1,1,0,0,0)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.928	.913	.951	.905	.982	1	.974	.897
θ_2	.928	.913	.951	.905	.982	1	.974	.897
$\gamma = .001$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.929	.914	.952	.906	.983	1-	.975	.898
θ_2	.929	.914	.952	.906	.983	1-	.975	.898
$\gamma = .01$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.936	.921	.958	.914	.986	1-	.979	.906
θ_2	.936	.921	.958	.914	.986	1-	.979	.906
$\gamma = .1$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.976	.966	.988	.961	1-	.988	.998	.955
θ_2	.976	.966	.988	.961	1-	.988	.998	.955
$\gamma = .25$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.995	.990	1-	.987	.996	.963	.998	.984
θ_2	.995	.990	1-	.987	.996	.963	.998	.984
$\gamma = .50$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	.999	.998	.998	.982	.935	.989	.997
θ_2	1	.999	.998	.998	.982	.935	.989	.997
$\gamma = .75$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.999	1-	.993	1-	.972	.919	.980	1-
θ_2	.999	1-	.993	1-	.972	.919	.980	1-

TABLE IV.4 (continued)

Limiting Efficiencies When All Eigenvalues Converge To A Constant

$\zeta = (0,0,0,1,1,1,0,0,0)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
γ = .9								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.998	1-	.991	1-	.968	.912	.976	1-
θ_2	.998	1-	.991	1-	.968	.912	.976	1-
γ = .99								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.997	.999	.989	1-	.965	.908	.974	1-
θ_2	.997	.999	.989	1-	.965	.908	.974	1-
γ = .999								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	.997	.999	.989	1-	.965	.908	.974	1-
θ_2	.997	.999	.989	1-	.965	.908	.974	1-
γ = 1.000								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	.997	.999	.989	1-	.965	.908	.974	1
θ_2	.997	.999	.989	1-	.965	.908	.974	1

TABLE IV.5

Limiting Efficiencies When The Eigenvalues Converge to a Constant and to Infinity

$\zeta = (0,0,0,1,1,1,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	0	0	0	0	1	1	1	0
θ_2	0	0	0	0	.586	1	.495	0
$\gamma = .001$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	0+	0+	0+	0+	.990	.171	.990	0+
θ_2	0+	0+	0+	0+	.990	.961	.990	0+
$\gamma = .01$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.013	.010	.027	.009	.987	.170	.987	.009
θ_2	.013	.010	.027	.009	.987	.958	.987	.009
$\gamma = .1$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.525	.433	.765	.400	.878	.151	.878	.385
θ_2	.525	.433	.765	.400	.878	.852	.878	.385
$\gamma = .25$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.921	.846	1-	.813	.713	.123	.713	.796
θ_2	.921	.846	1-	.813	.713	.693	.713	.796
$\gamma = .50$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	.986	.941	.972	.586	.101	.586	.964
θ_2	1	.986	.941	.972	.586	.569	.586	.964
$\gamma = .75$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.986	1-	.883	.998	.528	.091	.528	.995
θ_2	.986	1-	.883	.998	.528	.512	.528	.995

TABLE IV.5 (continued)

Limiting Efficiencies When The Eigenvalues Converge To a Constant and to Infinity

$\zeta = (0,0,0,1,1,1,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
γ = .9								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.974	.998	.857	1-	.506	.087	.506	.999
θ_2	.974	.998	.857	1-	.506	.491	.506	.999
γ = .99								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.967	.995	.845	1-	.496	.086	.496	1-
θ_2	.967	.995	.845	1-	.496	.481	.496	1-
γ = .999								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.966	.995	.844	.999	.495	.085	.495	1-
θ_2	.966	.995	.844	.999	.495	.480	.495	1-
γ = 1.000								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	.966	.995	.844	.999	.495	.085	.495	1
θ_2	.966	.995	.844	.999	.495	.480	.495	1

TABLE IV.6

Limiting Efficiencies When All Eigenvalues Converge To Infinity

$\zeta = (0,0,0,0,0,0,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	1	1
θ_2	.959	.959	.959	.959	.990	1	.990	.959
$\gamma = .001$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .01$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .1$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .25$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .50$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .75$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1

TABLE IV.6 (continued)

Limiting Efficiencies When All Eigenvalues Converge To Infinity

$\zeta = (0,0,0,0,0,0,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = .9$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .99$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = .999$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1
$\gamma = 1.000$								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	1
θ_2	1	1	1	1	.990	.961	.990	1

TABLE IV.7

Limiting Efficiencies When The Eigenvalues Converge to Zero and to Infinity

$\zeta = (1,1,1,0,0,0,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	1	1	1	1	1	1	1	0
θ_2	.003	.003	.013	.003	.990	1	.495	0
$\gamma = .001$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
$\gamma = .01$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
$\gamma = .1$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
$\gamma = .25$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
$\gamma = .50$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
$\gamma = .75$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0

TABLE IV.7 (continued)

Limiting Efficiencies When The Eigenvalues Converge to Zero and to Infinity

$\zeta = (1,1,1,0,0,0,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
γ = .9								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
γ = .99								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
γ = .999								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	1	1	1	.990	.171	.990	0
θ_2	1	1	1	1	.990	.961	.990	0
γ = 1.000								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	.5	.5	.5	.5	.495	.085	.495	1
θ_2	.5	.5	.5	.5	.495	.480	.495	1

TABLE IV.8

Limiting Efficiencies When The Eigenvalues Converge to Zero, to a Constant, and to Infinity

$\zeta = (1,1,1,1,1,1,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
$\gamma = 0$								
θ_1	1	1	1	1	1	1	1	1
$\theta_1+\theta_2$	0	0	0	0	1	1	1	0
θ_2	0	0	0	0	.586	1	.330	0
$\gamma = .001$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	0+	0+	0+	0+	.990	.171	.990	0
θ_2	0+	0+	0+	0+	.990	.961	.990	0
$\gamma = .01$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.013	.010	.027	.009	.987	.170	.987	0
θ_2	.013	.010	.027	.009	.987	.958	.987	0
$\gamma = .1$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.525	.433	.765	.400	.878	.151	.878	0
θ_2	.525	.433	.765	.400	.878	.852	.878	0
$\gamma = .25$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.921	.846	1-	.813	.713	.123	.713	0
θ_2	.921	.846	1-	.813	.713	.693	.713	0
$\gamma = .50$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	1	.986	.941	.972	.586	.101	.586	0
θ_2	1	.986	.941	.972	.586	.569	.586	0
$\gamma = .75$								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.986	1-	.883	.998	.528	.091	.528	0
θ_2	.986	1-	.883	.998	.528	.512	.528	0

TABLE IV.8 (continued)

Limiting Efficiencies When The Eigenvalues Converge to Zero, to a Constant, and to Infinity

$\zeta = (1,1,1,1,1,1,1,1,1)$

w_1 =	.25	.4	.2	.5	.5	1	0	0
w_2 =	.50	.2	.7	0	.5	0	1	0
w_3 =	.25	.4	.1	.5	0	0	0	1
γ = .9								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.974	.998	.857	1-	.506	.087	.506	0
θ_2	.974	.998	.857	1-	.506	.491	.506	0
γ = .99								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.967	.995	.845	1-	.496	.086	.496	0
θ_2	.967	.995	.845	1-	.496	.481	.496	0
γ = .999								
θ_1	1	1	1	1	1	0	1	1
$\theta_1+\theta_2$	.966	.995	.844	.999	.495	.085	.495	0
θ_2	.966	.995	.844	.999	.495	.480	.495	0
γ = 1.000								
θ_1	0	0	0	0	0	0	1	1
$\theta_1+\theta_2$	.644	.663	.563	.666	.330	.057	.330	1
θ_2	.644	.663	.563	.666	.330	.320	.330	1

Section IV.2: Recommendations

Given Tables IV.2-IV.8, one is in a position to make some recommendations concerning the choice of estimators for various cases. It should be remembered that, while these tables cover many cases, they cannot hope to be entirely representative. If any one point is clear, it is that the best approach to choosing an estimator is to determine how the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and their multiplicities behave and to compute the limiting efficiencies of the estimators under consideration over $\gamma\varepsilon[0,1]$.

In estimating θ_1, the D_3 estimator (the standard ANOVA estimator) appears to be an excellent choice. For all behaviors of the eigenvalues and for all values of $\gamma\varepsilon[0,1]$ given above, the limiting efficiency equals one. This recommendation corresponds to that given by Seely (1979). In addition, this estimator is readily computed and has been discussed extensively in the statistical literature. Note that for estimating θ_1, the D_3 estimator equals the D_2 estimator. For cases in which some or all of the eigenvalues tend to infinity, the D_1 estimator constitutes an especially poor choice for estimating θ_1. This estimator is also known as Rao's MINQUE0 estimator. In all such cases given in the tables, for $\gamma > 0$, its limiting efficiency is zero.

From Proposition II.10, all estimators other than the D_1 estimator have a limiting efficiency of one for $\gamma\varepsilon[0,1)$. These estimators would also be acceptable choices so long as $\gamma \neq 1$. Among the estimators considered, only the D_3 estimator has a non-zero limiting efficiency at $\gamma = 1$.

When estimating θ_2, we can make the following observations based on the tables and on the results of this dissertation. When all eigenvalues converge to zero or converge to a constant greater than zero, or both, those D_w estimators for which $w_1 > 0$ appear to be good choices. When $\gamma\varepsilon(.25,1)$, Rao's MINQUE is also a good choice. When the eigenvalues converge both to infinity and to a constant greater than zero, for $\gamma\varepsilon(0,.1)$ the estimator where $w = (.5,.5,0)'$ shows good performance. For $\gamma\varepsilon(.1,1]$, one would avoid this estimator and benefit more by choosing the MINQUE or possibly the D_3 estimator. In fact, at γ near one, the D_3 estimator would be the best choice. When all eigenvalues converge to infinity, the estimators for which $w_3 > 0$ seem best. Preference is again given to the D_3 estimator. In studying the one-way randomized design in which all eigenvalues converge to infinity, Seely (1979) also gives the D_3 estimator high marks. When the eigenvalues converge both to zero and to infinity, for $\gamma\varepsilon(0,1)$, the best choices appear to be those for which

$w_1, w_3 > 0$. In this case, the D_3 estimator is the worst possible choice except at $\gamma = 1$. At $\gamma = 0$, the best choice is the D_1 estimator. When the eigenvalues converge to zero, to a constant, and to infinity, at γ near zero, the estimator with $w = (.5,.5,0)'$ is a good choice; however, as γ grows larger the estimator with $w = (.5,0,.5)'$ improves on the $(.5,.5,0)'$ estimator. Provided the eigenvalues don't converge to a constant while also converging to zero or to infinity, the D_2 estimator is a decent estimator. In fact, it also shows good performance when all eigenvalues converge to a constant. One can show that when estimating θ_2, for $\gamma = 0$, the efficiency for the D_1 estimator equals one, and for $\gamma = 1$, the efficiency of the D_3 estimator equals one. Thus, for $\gamma = 0$ use the D_1 estimator and for $\gamma = 1$ use the D_3 estimator.

In recommending estimators of $\delta'\theta$ such that $\delta_1, \delta_2 \neq 0$, one can refer to recommendations made for estimators of θ_2 unless the D_1 estimator provides a better choice. This observation follows from Theorem II.9 which shows that, except at $\gamma = 0$ or for the D_1 estimator, the limiting efficiency is independent of δ so long as $\delta_2 > 0$. At $\gamma = 1$, the D_3 estimator is the best choice since it necessarily must have an efficiency of one.

BIBLIOGRAPHY

Hardy, G. and Littlewood J. and Polya, G. (1934). Inequalities, Cambridge University Press, New York.

Hartley, H. O., Rao, J. N. K., and LaMotte, L. R. (1978). A simple "synthesis" based method of variance component estimation. Biometrics 34, 233-242.

Henderson, C. R. (1978). Variance-covariance matrix of estimators of variances in unweighted means ANOVA. Biometrics 34, 463-468.

Olsen, A. and Seely, J. and Birkes, D. (1976). Invariant quadratic unbiased estimation for two variance components. Annals of Statistics 4, 878-890.

Rao, C. R. (1972). Estimating variance and covariance components in linear models. Journal of the American Statistical Association 67, 112-115.

Searle, S. R. (1969). Another look at Henderson's methods of estimating variance components. Biometrics 24, 749-778.

Searle, S. R. (1979). Maximum Likelyhood and Minimum Variance Estimation of Variance Components. Variance components and Animal Breeding, eds. L. D. Van Vleck and S. R. Searle, Ithica: Cornell University.

Seely, J. (1972). Completeness for a family of multivariate normal distributions. The Annals of Mathematical Statistics 43, 1644-1647.

Seely, J. (1979). Large sample properties of invariant quadratic unbiased estimators in the random one way model. Variance Components and Animal Breeding, eds. L. D. Van Vleck and S. R. Searle, Ithica: Cornell University.

Zyskind, G. (1967). On canocical forms, non-negative covariance matrices and best and simple least squares linear estimators in linear models. Annals of Mathematical Statistics 38 1092-1109.

APPENDIX

Complete Classes

Let $\underset{\sim}{z} \varepsilon R^m$ be a random vector having covariance structure $Cov(\underset{\sim}{z}) = \Sigma$ for all $\Sigma \varepsilon V$ and consider the set of estimators $\{b'\underset{\sim}{z} | b \varepsilon B\}$ where B is any non-empty set of coefficient vectors to $\underset{\sim}{z}$. Then, one can define the following terms.

1. An estimator $b'\underset{\sim}{z}$ is <u>as good as</u> an estimator $a'\underset{\sim}{z}$ when $b'\Sigma b \leq a'\Sigma a$ for all $\Sigma \varepsilon V$.

2. An estimator $b'\underset{\sim}{z}$ is <u>better than</u> an estimator $a'\underset{\sim}{z}$ if $b'\Sigma b \leq a'\Sigma a$ for all $\Sigma \varepsilon V$ and if there exists a $\Sigma_0 \varepsilon V$ such that $b'\Sigma_0 b < a'\Sigma_0 a$.

3. An <u>essentially complete class</u> in B is a subset $C_e \subset B$ such that for all $b \varepsilon B$ there exists an $a \varepsilon C_e$ where $a'\underset{\sim}{z}$ is as good as $b'\underset{\sim}{z}$.

4. A <u>complete class</u> in B is a subset $C \subset B$ such that for all $a \varepsilon B \backslash C$ there exists a $b \varepsilon C$ where $b'\underset{\sim}{z}$ is better than $a'\underset{\sim}{z}$.

5. A <u>minimal complete class</u> is a complete class which is a subset of every other complete class.

6. Given that $a \varepsilon B$, $a'\underset{\sim}{z}$ is an <u>admissable estimator</u> in $\{d'\underset{\sim}{z} | d \varepsilon B\}$ if there exists no $b \varepsilon B$ such that $b'\underset{\sim}{z}$ is is better than $a'\underset{\sim}{z}$. Correspondingly, the <u>admissable class</u> is the set of all $a \varepsilon B$ such that $a'\underset{\sim}{z}$ is an admissable estimator.

In regard to the above definitions, it is noted that a complete class is also an essentially complete class. One can also conclude from these definitions that an admissable class is a subset of every complete class, and thereby of the minimal complete class provided a minimal complete class exists. Thus, one can show that the admissable class is an essentially complete class if and only if it is a complete class if and only if it is a minimal complete class. A minimal complete class is unique, since if there were two different minimal complete classes, they would necessarily be a subset of each other.