A Polynomial Equation for the Natural Earth Projection

Bojan Šavrič, Bernhard Jenny, Tom Patterson, Dušan Petrovič, Lorenz Hurni

ABSTRACT: The Natural Earth projection is a new projection for representing the entire Earth on small-scale maps. It was designed in Flex Projector, a specialized software application that offers a graphical approach for the creation of new projections. The original Natural Earth projection defines the length and spacing of parallels in tabular form for every five degrees of increasing latitude. It is a pseudocylindrical projection, and is neither conformal nor equal-area. In the original definition, piece-wise cubic spline interpolation is used to project intermediate values that do not align with the five-degree grid. This paper introduces alternative polynomial equations that closely approximate the original projection. The polynomial equations are considerably simpler to compute and program, and require fewer parameters, which should facilitate the implementation of the Natural Earth projection in geospatial software. The polynomial expression also improves the smoothness of the rounded corners where the meridians meet the horizontal pole lines, a distinguishing trait of the Natural Earth projection that suggests to readers that the Earth is spherical in shape. Details on the least squares adjustment for obtaining the polynomial formulas are provided, including constraints for preserving the geometry of the graticule. This technique is applicable to similar projections that are defined by tabular parameters. For inverting the polynomial projection the Newton-Raphson root finding algorithm is suggested.

KEYWORDS: Projections, Natural Earth projection, Flex Projector

The Natural Earth Projection

The Natural Earth projection was developed by Tom Patterson in 2007 out of dissatisfaction with existing projections for displaying physical data on small-scale world maps (Jenny et al. 2008). Flex Projector, a freeware application for the interactive design and evaluation of map projections, was the means for creating the Natural Earth projection. The graphical user interface in Flex Projector allows cartographers to adjust the length, shape, and spacing of parallels and meridians of new projections in a graphical design process (Jenny and Patterson 2007).

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DOI: http://dx.doi.org/10.1559/15230406384363

The Natural Earth projection is an amalgam of the Kavraiskiy VII and Robinson projections, with additional enhancements (Figure 1). These two projections most closely fulfilled the requirement for representing small-scale physical data on world maps, but each had at least one undesirable characteristic (Jenny et al. 2008). The Kavraiskiy VII projection exaggerates the size of high latitude areas, resulting in oversized representation of polar regions. The Robinson projection, on the other hand, has a height-to-width ratio close to 0.5, resulting in a slightly too wide graticule with outward bulging sides and too much shape distortion near the map edges. Creating the Natural Earth projection required three major adjustments: Firstly, starting from the Robinson projection, its vertical extension was slightly increased to give it more height. Secondly, using the Kavraiskiy VII as a template, the parallels were slightly increased in length.

And thirdly, the length of the pole lines was decreased by a small amount to give the corners at pole lines a rounded appearance. Designing the Natural Earth projection in this way required trial-and-error experimentation and visual assessment of the appearance of continents in an iterative process (Jenny et al. 2008). The result of this procedure, the Natural Earth projection, is a true pseudocylindrical projection, i.e., a projection with regularly distributed meridians and straight parallels (Snyder 1993:189). As a compromise projection, the Natural Earth projection is neither conformal nor equal area, but its distortion characteristics are comparable to other well known projections (Jenny et al. 2008). All three projections exaggerate the size of high latitude areas (Figure 1). Appendix A provides further details about the distortion characteristics of the Natural Earth projection.

The shape of the graticule of any projection designed with Flex Projector is defined by tabular sets of parameters. For the Natural Earth projection, two parameter sets are used for specifying (1) the relative length of the parallels, and (2) the relative distance of parallels from the equator. Equation 1 defines the original Natural Earth projection, transforming spherical coordinates into Cartesian *X/Y* coordinates, and Table 1 provides the parameter values (Jenny et al. 2008; 2010):

$$X = R \cdot s \cdot l_\varphi \cdot \lambda \qquad l_\varphi \in [0, 1], \qquad \text{(Eq. 1)}$$
$$Y = R \cdot s \cdot d_\varphi \cdot k \cdot \pi \qquad d_\varphi \in [-1, 1],$$

where:

- X and Y are projected coordinates;
- R is the radius of the generating globe;
- $s = 0.8707$ is an internal scale factor;
- l_φ is the relative length of the parallel at latitude φ, with $\varphi \in [-\pi/2, \pi/2]$, $l_\varphi = 1$ for the equator and the slope of l_φ is 63.883° at the poles;
- d_φ is the relative distance of the parallel at latitude φ from the equator, with $\varphi \in [-\pi/2, \pi/2]$ and with $d_\varphi = \pm 1$ for the pole lines, and $d_\varphi = 0$ for the equator;
- λ is the longitude with $\lambda \in [-\pi, \pi]$; and
- $k = 0.52$ is the height-to-width ratio of the projection.

Arthur H. Robinson proposed the structure of Equation 1 and the associated graphical approach to the design of small-scale map projections when he developed his eponymous projection (Robinson 1974). In making the Natural Earth projection, Jenny et al. (2010) provide numerical values for the tabular parameters that define l_φ and d_φ in Equation 1 for every five degrees. For intermediate spherical coordinates that do not align with the five-degree grid, values for l_φ and d_φ need to be interpolated. The Flex Projector application uses a piece-wise cubic spline interpolation, with each piece of the spline curve covering five degrees. While this type of interpolation is rapid to evaluate, it is relatively intricate to program and requires a large number of parameters—factors that are likely to impede the widespread implementation of the Natural Earth projection in geospatial software. Seeking greater efficiency, the remainder of this paper discusses a compact analytical expression that approximates Equation 1 with two simple polynomial expressions.

Table 1. Parameters for the Natural Earth projection: Relative lengths of parallels and relative distance from the equator for every 5 degrees (after Jenny et al. 2008).

Latitude [degrees]	Relative length of parallels	Relative distance of parallels from equator
0	1	0
5	0.988	0.062
10	0.9953	0.124
15	0.9894	0.186
20	0.9811	0.248
25	0.9703	0.310
30	0.9570	0.372
35	0.9409	0.434
40	0.9222	0.4958
45	0.9006	0.5571
50	0.8763	0.6176
55	0.8492	0.6769
60	0.8196	0.7346
65	0.7874	0.7903
70	0.7525	0.8435
75	0.7160	0.8936
80	0.6754	0.9394
85	0.6270	0.9761
90	0.5630	1

Analytical Expressions for the Robinson Projection

Robinson and Patterson used an identical approach for the design of their pseudocylindrical projections. Both defined their projection by

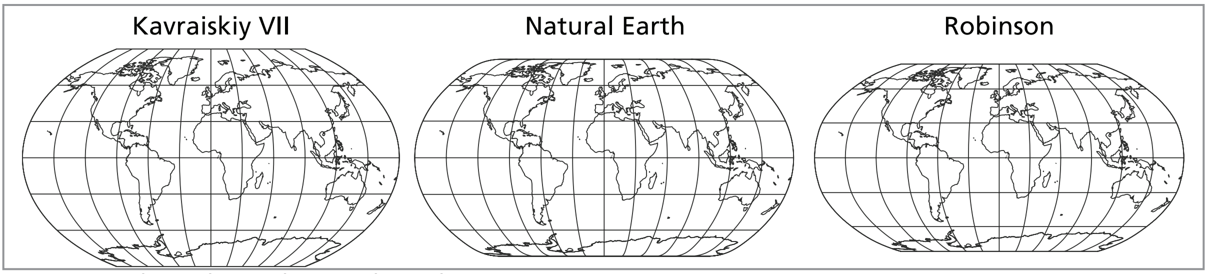

Figure 1: The polynomial Natural Earth projection compared to the Kavraiskiy VII and Robinson projections (after Jenny et al. 2008).

adjusting the appearance of the projected five-degree graticule in an iterative process—Robinson sketching the graticule with pen and paper, and Patterson fine-tuning it in Flex Projector. In the past, various authors have tackled the problem of finding an analytical expression for the Robinson projection. Since the two projections are closely related, this section reviews existing mathematical models of Robinson's projection. Polynomial approximation is recommended, which is applied to the Natural Earth projection in the next section.

Two general approaches exist for mathematically modeling graphically defined projections: (1) interpolation and (2) approximation. The Robinson projection has had both approaches applied.

Interpolating methods use a function that passes exactly through the reference points. Ipbüker (2004; 2005) presents a method based on multiquadric interpolation for the forward and the inverse projection. Others have used interpolating methods for finding continuous expressions of l_φ and d_φ in Equation 1. For example, Snyder (1990) applies the central-difference formula by Stirling; Ratner (1991), Bretterbauer (1994), and Evenden (2008) use cubic spline interpolation (which is also used in Flex Projector); and Richardson (1989) reports that Robinson applied the Aitken interpolation scheme. A disadvantage of the mentioned interpolating methods is the large number of parameters required (more than 40 for the Robinson projection), and their relatively difficult implementation. For these reasons they are not explored further here.

Approximating curves with parametric expressions that do not exactly replicate the original projection are an acceptable alternative, if deviations to the approximated values are small. Canters and Decleir (1989) present two polynomial equations for approximating the Robinson projection (Equation 2). For the X coordinates they use even powers up to the order four, and for the Y coordinates odd powers up to the order five. Each expression contains three coefficients, and the constants k, s and π of Equation 1 are integrated with l_φ and d_φ. Their solution contains only six parameters, and is fast and simple to compute.

$$X = R \cdot \lambda \cdot (A_0 + A_2 \cdot \varphi^2 + A_4 \cdot \varphi^4) \qquad \text{(Eq. 2)}$$
$$Y = R \cdot (A_1 \cdot \varphi + A_3 \cdot \varphi^3 + A_5 \cdot \varphi^5)$$

where:

X and Y are projected coordinates;
φ and λ are the latitude and longitude;
R is the radius of the generating globe;
A_0 = 0.8507;
A_1 = 0.9642;
A_2 = -0.1450;
A_3 = -0.0013;
A_4 = -0.0104; and
A_5 = -0.0129.

A similar approach is proposed by Beineke (1991; 1995). For l_φ he suggests a polynomial with even degrees up to the sixth order, and for d_φ he proposes an exponential approximation with a real number exponent (Beineke 1991). This approach uses a total of eight parameters to approximate Robinson's projection. However, evaluating an exponential function with a real number exponent is slow. A test with the Java programming language, for example, shows that Beineke's exponential approximation is more than ten times slower to evaluate than a polynomial, such as the one by Canters and Decleir.

The approximating curves by Canters and Decleir, as well as Beineke, use a smaller number of parameters, and are considerably simpler to program than the interpolating methods. Polynomial equations are best in terms of computation speed and code simplicity, but higher-order terms might be necessary to minimize deviations from the original curve. Polynomial approximations, however, sometimes suffer from undulations if the maximum degree is too high, which must be avoided for a graticule to appear smooth. Another potential drawback of polynomial equations is the difficulty of finding inverse equations that transform from projected *X/Y* coordinates to spherical coordinates. Indeed, an analytical inverse does not generally exist for higher-order polynomial equations. To solve for spherical coordinates, numerical approximation methods are necessary, such as the bisection or the Newton-Raphson root finding algorithm.

A Polynomial Approximation for the Natural Earth Projection

In a trial-and-error process, a polynomial approximation with a minimum number of terms was determined for the original Natural Earth projection. Polynomials of varying degrees and different number of terms were selected and their coefficients computed using the method of least squares with constraints. Two criteria were used to evaluate variants developed with this iterative trial-and-error procedure: First, the number of polynomial terms and the number of multiplications required to evaluate the equation need to be minimized. This criterion is important for simplifying the programming of the equations. It is also relevant for accelerating computations, for example, for web mapping applications that project maps on the fly using JavaScript or other interpreted programming languages that are comparatively slow. The second criterion aims at minimizing the absolute differences between the original projection and the approximated projection. Differences should be minimal throughout the entire projection.

When designing the original Natural Earth projection, special focus was given to the smoothness of the rounded corners where the bounding meridians meet the horizontal pole lines. It was found that the graphical tools and the cubic spline interpolation in Flex Projector do not provide sufficient control for defining rounded corners with adequate smoothness. The development of a polynomial approximation provided the possibility to further improve this distinguishing characteristic of the Natural Earth projection. The new polynomial form of the projection therefore deliberately deviates from the original projection by adding curvature to the corners. The changes to the smoothness of the corners were entirely esthetic and done to satisfy the authors' sensibilities. They result in a subjective improvement that cannot be evaluated with objective criteria. Nor were they applied for improving the projection's distortion characteristics.

The polynomial expression for the Natural Earth projection is given in Equations 3 and 4. The polynomials are of higher degrees than those by Canters and Decleir (1989) for the Robinson projection. Higher degrees are required for the Natural Earth projection to smoothly model the curved corners connecting the meridian lines to the horizontal pole line.

$$X = R \cdot \lambda \cdot (A_1 + A_2 \cdot \varphi^2 + A_3 \cdot \varphi^4 + A_4 \cdot \varphi^{10} + A_5 \cdot \varphi^{12}) \quad \text{(Eq. 3)}$$

$$Y = R \cdot (B_1 \cdot \varphi + B_2 \cdot \varphi^3 + B_3 \cdot \varphi^7 + B_4 \cdot \varphi^9 + B_5 \cdot \varphi^{11}) \quad \text{(Eq. 4)}$$

where:

X and *Y* are the projected coordinates;

φ and λ are the latitude and longitude in radians;

R is the radius of the generating globe, and

A_1 to A_5 and B_1 to B_5 are coefficients given in Table 2.

Equation 3 replaces both l_φ and the factor *s* in Equation 1 with a polynomial expression (with $A_1 = s$). The polynomials in Equation 4 include the constant factors *s*, *k*, π, and d_φ for reducing the number of required multiplications and accelerating calculations.

For developing Equations 3 and 4, the following considerations were taken into account: (1) The Natural Earth projection is symmetrical about the x and y-axis; (2) it has straight but not equally spaced parallels; and (3) the parallels are equally divided by meridians. Due to these characteristics, Equation 3 contains only even powers of φ that are multiplied by λ, and Equation 4 only consists of odd power terms of φ (Canters, 2002, p. 133 ff.). For the purpose of accelerating computations, the number of polynomial terms has been reduced. Equation 3 has no terms with degree 6 and 8, and Equation 4 has no term with degree 5.

When estimating the polynomial coefficients with the method of least squares, two additional constraints were added to bring the polynomial graticule to the exact same size as the original graticule. In Equation 3, the first coefficient A_1 was forced to equal the value of the internal scale factor s (Equation 1). This is to ensure that the length of the equator remains the same. In Equation 4, the distance of the pole line from the equator was forced to the original value by introducing a second constraint.

Two additional measures were required to increase the smoothness of the rounded corners between the meridians and the pole lines. For Equation 4, an additional constraint was added to the method of least squares, fixing the slope of the polynomial to 7 degrees at the poles. The second measure for improving the smoothness of the corners involved slightly reducing the length of the pole line before computing the polynomial coefficients (Figure 2). The result is a new polynomial Natural Earth projection that deliberately deviates from the original projection near the poles. Appendix B provides details on the application of the method of least squares, including the technique for integrating the additional constraints, which should allow the reader to apply this technique to other similar projections.

Inverting the Polynomial Natural Earth Projection

The inverse of a map projection transforms Cartesian coordinates into spherical coordinates. To determine the inverse of the polynomial Natural Earth projection, Equations 3 and 4 must be inverted. The system defined by these two polynomials has two known variables (the Cartesian coordinates X and Y) and two unknowns (the spherical coordinates φ and λ). The system is solved by first finding the latitude φ in Equation 4, and then solving Equation 3 for the unknown longitude λ.

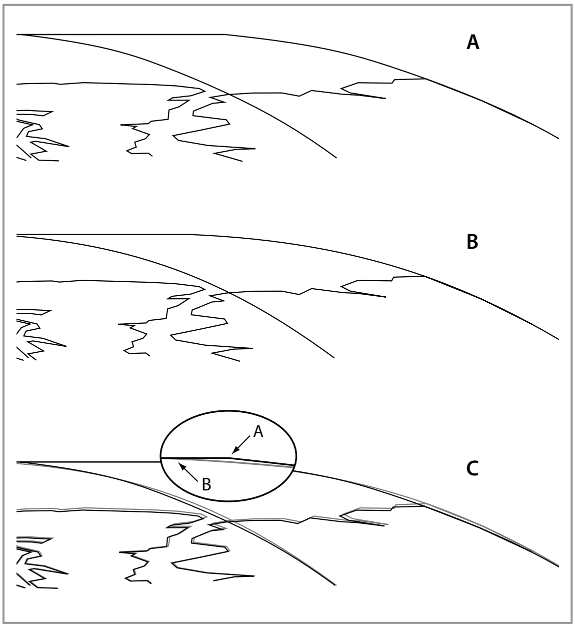

Figure 2: The original (A) and polynomial (B) Natural Earth projection are overlaid in (C). Arrows indicate changes in smoothness at the end of the pole line, which is shortened in (B).

An analytical expression of the inverse of the polynomial Equation 4 does not exist, but a large number of methods are available for polynomial system solving (Elkadi and Mourrain 2005). The Newton-Raphson algorithm is a numerical method for finding successively better approximations to the roots or zeros of a real-valued function, and is commonly used for the numerical solving of nonlinear equations. The Newton-Raphson root finding algorithm was chosen for inverting the Natural Earth projection, because it converges rapidly, is easy to compute, and requires only one initial guess. Equation 5 shows the general form of the Newton-Raphson algorithm.

$$x_{n+1} = x_n - F'(x_n)^{-1} \cdot F(x_n) \longrightarrow F(x_n) = 0 \qquad \text{(Eq. 5)}$$

where:

$F(x_n)$ and $F'(x_n)$ are a given function and its derivative;

x_n and x_{n+1} are the previous and the next

Table 2. Coefficients for the polynomial expression of the Natural Earth projection.

Coefficients for Equation 3		Coefficients for Equation 4	
A_1	0.870700	B_1	1.007226
A_2	-0.131979	B_2	0.015085
A_3	-0.013791	B_3	-0.044475
A_4	0.003971	B_4	0.028874
A_5	-0.001529	B_5	-0.005916

solution of the given function; and

n and $(n+1)$ the steps of the iterative process.

The function $F(x_n)$ is formed by converting Equation 4 to Equation 6:

$$F(x_n) = B_1 \cdot \varphi + B_2 \cdot \varphi^3 + B_3 \cdot \varphi^7 + B_4 \cdot \varphi^9 + B_5 \cdot \varphi^{11} - Y \cdot R^{-1} = 0 \quad \text{(Eq. 6)}$$

The iterative approximation is repeated until a sufficiently accurate solution is reached. Convergence to the solution is quadratic for Equation 6, since the derivative $F'(x_n)$ is positive for all $\varphi \in [-\pi/2, \pi/2]$, and $F(x_n)$ has therefore no local minimum or maximum in the valid range of φ. The closest local extremum is at $\varphi = \pm 1.59$, which is outside the valid range of φ. The quotient $Y \cdot R^{-1}$ can be used as an initial guess for the Newton-Raphson algorithm, as it is in the range of the latitude φ, and does not have any local extremum in this range (Equation 7).

$$Y \cdot R^{-1} \in [-s \cdot k \cdot \pi, s \cdot k \cdot \pi] \in [-\pi/2, \pi/2] \quad \text{(Eq. 7)}$$

Applying the inverse projection of the polynomial Natural Earth projection consists of the following steps:

(1) The initial guess for the unknown latitude: $\varphi_0 = Y \cdot R^{-1}$;
(2) With the Newton-Raphson approximation method an improved latitude φ is calculated: $\varphi_{n+1} = \varphi_n - F'(\varphi_n)^{-1} \cdot F(\varphi_n)$, where $F(\varphi_n)$ is the function from Equation 6, $F'(\varphi_n)$ its derivative, and $n = 0, 1, 2, \ldots, m$. At step m the iteration stops if $|\varphi_{m+1} - \varphi_m| < \varepsilon$, where ε is a sufficiently small positive quantity, typically close to the maximum precision of floating point arithmetic;
(3) The final latitude: $\varphi = \varphi_{m+1}$; and
(4) The final longitude: $\lambda = X \cdot R^{-1} \cdot (A_1 + A_2 \cdot \varphi^2 + A_3 \cdot \varphi^4 + A_4 \cdot \varphi^{10} + A_5 \cdot \varphi^{12})^{-1}$

The Newton-Raphson method is only applied to compute the latitude φ in step (2); the longitude λ can be computed in step 4 by inverting Equation 3. The Newton-Raphson method converges quickly with Equation 6. On average, less than four iterations are needed when transforming a regularly spaced graticule with 15 degrees resolution covering the whole sphere (with $\varepsilon = 10^{-11}$).

An alternative general method for inverting arbitrary map projections without explicit inverse expressions was described by Ipbüker and Bildirici (2002). They utilize the two forward expressions to calculate the geographical coordinates φ and λ using Jacobian matrices. For the Natural Earth projection, this method based on Jacobian matrices results in the same values as the Newton-Raphson approach presented here. For both methods, an equal number of iterations is required (with an identical ε). However, the Newton-Raphson method is faster, as it involves fewer calculations, and is algorithmically simpler.

Conclusion

The Natural Earth projection expressed by the polynomial Equations 3 and 4 slightly deviates from Patterson's original projection by adding additional curvature to meridians where they meet the horizontal pole line. The curved corners are smoother than in the original design, which improves the visual appearance of the graticule. This enhancement was developed in collaboration with Tom Patterson, the author of the original Natural Earth projection. The polynomials are easy to code and fast to compute as only seven multiplications are required for each polynomial if factorized appropriately. The Newton-Raphson method for inverting the projection converges quickly, with only a few iterations required. The scale distortion index, the areal distortion index, as well as the mean angular deformation index (Canters and Decleir, 1989) of the polynomial approximation of the Natural Earth projection are identical to those of the original projection. The areal distortion and maximum angular distortion are similar to those of other pseudocylindrical projections (Appendix A). For these reasons, the authors recommend using the polynomial equation of the Natural Earth projection.

This article presents the development of polynomial expressions for the Natural Earth projection, which is one specific projection designed with the graphical approach offered by Flex Projector. Details on the least squares adjustment with constraints for obtaining the polynomial formulas are provided (Appendix B) to allow others to apply this technique to similar projections defined by tabular parameters. It remains to be explored how the polynomial approximation method can be generalized for any projection designed with Flex Projector.

ACKNOWLEDGEMENTS

This research was funded by the Erasmus student exchange program, and was carried out at ETH Zürich. The authors thank the anonymous reviewers for their valuable comments.

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Appendix A: Distortion Characteristics of the Natural Earth Projection

As a compromise projection, the Natural Earth projection is neither conformal nor equal area, but its distortion characteristics are comparable to other well known projections. Its distortion values fall somewhere between those of the Kavraiskiy VII and Robinson projections, which were used in the design procedure.

Figure 3 shows Tissot's indicatrices for every 30 degrees. With increasing distance from the equator the area of indicatrices increases, indicating that the size of high latitude areas is exaggerated. Figure 3 omits indicatrices along pole lines, since they are of infinite size. The axes of the indicatrices do not coincide with the directions of parallels and meridians, except at the equator and the central meridian. Jenny et al. (2008) present isocols of areal distortion for the Natural Earth projection. Areal distortion increases with latitude and does not change with longitude (Table 3). All isocols of areal distortion are therefore parallel to the equator. Areal distortion is computed with $\sigma = a_j \cdot b_j$ with a_j and b_j the scale factors along the principal directions at position j on the sphere.

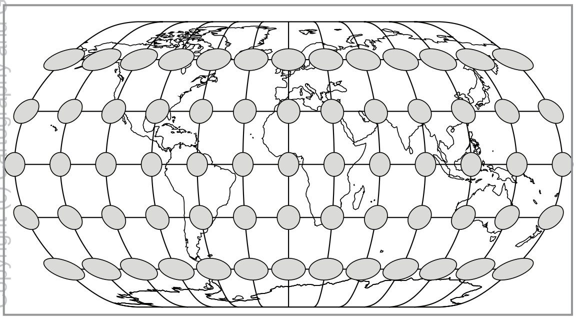

Figure 3. Tissot's indicatrices for the polynomial Natural Earth projection.

Table 3. Areal distortion increases with latitude.

Latitude φ [°]	Area Scale
85	3.28
60	1.31
30	0.98
0	0.88

Angular distortion is moderate near the equator and increases towards the edges of the graticule (Table 4). Jenny et al. (2008) provide isocols of maximum angular distortion. The values of maximum angular distortion ω_j are constant along the equator. Equation 8 computes ω_j:

$$\omega_j = 2 \arcsin \frac{a_j - b_j}{a_j + b_j} \qquad \text{(Eq. 8)}$$

where:

a_j and b_j are the scale factors along the principle directions at position j on the sphere.

Table 4. Maximum angular distortion for every 30 degrees of increasing latitude and longitude. All values are in degrees.

φ \ λ	0	30	60	90	120	150	180
0	8.3	8.3	8.3	8.3	8.3	8.3	8.3
30	3.0	5.4	9.3	13.6	17.9	22.1	26.3
60	25.0	26.2	29.5	34.1	39.6	45.4	51.3
85	115.37	115.44	115.67	116.05	116.56	117.20	117.96

Appendix B:
Least Squares Adjustment for Curve Fitting with Additional Constraints

This appendix presents the approximation method using least squares adjustment (LSA) with additional constraints. The presented approach is a modified LSA of indirect observations with functionally dependent parameters (Mikhail and Ackerman 1976). It is hoped that the details provided here will allow others to find similar polynomial expressions for other projections.

The original Natural Earth projection is defined by 37 control points distributed over the complete range of possible latitude values with $\varphi \in [-\pi/2, \pi/2]$ and a control point every 5 degrees. For both equations, polynomial expressions with five terms were chosen. The functional model of the LSA is given in Equation 9, which is derived from Equations 1, 3, and 4:

$$A_1 + A_2 \cdot \varphi_i^2 + A_3 \cdot \varphi_i^4 + A_4 \cdot \varphi_i^{10} + A_5 \cdot \varphi_i^{12} = s \cdot l_{\varphi\, i} \qquad (i, j = 1, \ldots 37)$$
$$B_1 \cdot \varphi_j + B_2 \cdot \varphi_j^3 + B_3 \cdot \varphi_j^7 + B_4 \cdot \varphi_j^9 + B_5 \cdot \varphi_j^{11} = s \cdot k \cdot d_{\varphi j} \cdot \pi \qquad \text{(Eq. 9)}$$

The coefficients of the two polynomial expressions in Equation 9 are unknown parameters, and the polynomial powers of latitude are known coefficients for the LSA. Equation 9 can be expressed in matrix form (Equation 10), with n denoting the number of rows in the coefficients matrix $\boldsymbol{A}$, and u the number of parameters ($n = 37$, $u = 5$). The first row of matrix $\boldsymbol{A}$ is $[1, \varphi_1^2, \varphi_1^4, \varphi_1^{10}, \varphi_1^{12}]$ for the first polynomial, and $[\varphi_1, \varphi_1^3, \varphi_1^7, \varphi_1^9, \varphi_1^{11}]$ for the second polynomial. Vector $\boldsymbol{x}$ contains the parameters, i.e. the unknown polynomial coefficients. In this particular case, no parameters are included in matrix $\boldsymbol{A}$, which results in a linear system that can be solved using the well known method of least squares in Equation 11. The vector $\boldsymbol{v}$ represents the minimized residuals after adjustment, for evaluating the standard deviation of the model, and the differences between the original and the approximated graticule. The vector $\boldsymbol{l}$ includes l_φ and d_φ, multiplied with the constant factors s, k and π as in Equation 9. Since this is a linear model, no initial guess is required for the parameters.

$$\boldsymbol{A}_{n\times u} \cdot \boldsymbol{x}_{u\times 1} = \boldsymbol{l}_{n\times 1} \qquad \text{(Eq. 10)}$$
$$\boldsymbol{l} + \boldsymbol{v} = \boldsymbol{A}\cdot\boldsymbol{x} \qquad \boldsymbol{v}^{\mathrm{T}} \cdot \boldsymbol{v} \longrightarrow \min \qquad \boldsymbol{x} = (\boldsymbol{A}^{\mathrm{T}}\cdot\boldsymbol{A})^{-1} \cdot \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{l} \qquad \boldsymbol{v} = \boldsymbol{A}\cdot\boldsymbol{x} - \boldsymbol{l} \qquad \text{(Eq. 11)}$$

To initiate the constraints, an additional linear matrix equation is added to this model. For the Natural Earth projection three constraints were imposed: (1) the length of the parallel l_φ at 0 degrees must be 1; (2) the relative distance between the equator and the parallel d_φ at 90 degrees must be 1; and (3) the slope of the polynomial in Equation 4 is fixed to 7 degrees at the pole line. All three constraints are expressed with parameters, which is possible because the expressions in Equation 9 are linear.

For computing the polynomial coefficients of the relative length l_φ 37 control points are used covering the whole range of possible latitude values between $-\pi/2$ and $+\pi/2$ with a distance of 5 degrees between each pair of control points. The symmetrical arrangement of the control points around the equator guarantees a continuously differentiable function.

The first constraint (1) for the length l_φ can be derived from Equation 9:

$$\varphi = 0 \longrightarrow l_\varphi = 1 \longrightarrow A_1 = s \qquad \text{(Eq. 12)}$$

The constraints (2) and (3) are applied to the relative distance d_φ. The fixed distance of the parallel at 90 degrees is expressed in a similar way as the constraint (1) in Equation 12. For the slope — constraint (3)—a derivative of Equation 9 is used to express it. Both conditions are described in Equation 13.

$$\varphi = \pi/2 \longrightarrow d_\varphi = 1 \longrightarrow B_1 \cdot \pi/2 + B_2 \cdot \pi^3/8 + B_3 \cdot \pi^7/128 + B_4 \cdot \pi^9/512 + B_5 \cdot \pi^{11}/2048 = s \cdot k \cdot \pi$$
$$\varphi = \pi/2 \longrightarrow B_1 + 3 \cdot B_2 \cdot \pi^2/4 + 7 \cdot B_3 \cdot \pi^6/64 + 9 \cdot B_4 \cdot \pi^8/256 + 11 \cdot B_5 \cdot \pi^{10}/1024 = \tan(7°) \qquad \text{(Eq. 13)}$$

The additional constraints can be written in matrix form (Equation 14), where the vector $\boldsymbol{x}$ is the same as in Equations 10 and 11. p is the number of additional constraints, which must be less than the number of unknowns ($p < u$), as the model would otherwise become under-determined. The first constraint for the lengths is represented in the matrix $\boldsymbol{C}$ and vector $\boldsymbol{g}$ by Equation 15. Equation 16 shows the two matrixes for the distance constraints.

$$\boldsymbol{C}_{p\times u} \cdot \boldsymbol{x}_{u\times 1} = \boldsymbol{g}_{p\times 1} \tag{Eq. 14}$$

$$\boldsymbol{C} = [1 \quad 0 \quad 0 \quad 0 \quad 0],\ \boldsymbol{g} = [\,s\,] \tag{Eq. 15}$$

$$\boldsymbol{C} = \begin{bmatrix} \pi/2 & \pi^3/8 & \pi^7/128 & \pi^9/512 & \pi^{11}/2048 \\ 1 & 3\cdot\pi^2/4 & 7\cdot\pi^6/64 & 9\cdot\pi^8/256 & 11\cdot\pi^{10}/1024 \end{bmatrix} \tag{Eq. 16}$$

$$\boldsymbol{g} = \begin{bmatrix} s\cdot k\cdot \pi \\ \tan(7°) \end{bmatrix}$$

Equation 14 expresses the relationship between the functionally dependent parameters that must be included in the LSA (Equation 11). This step is presented in Equation 17, where the two systems in 10 and 14 are solved together. The first row in Equation 17 is a normal LSA of indirect observations. The results are parameters not including the constraints. In the second row of Equation 17, the corrections $\boldsymbol{dx}$ for the parameters are calculated. On the third row, the vector $\boldsymbol{x}$ is computed containing the coefficients of the polynomial approximation. And finally, the vector of residuals $\boldsymbol{v}$ is computed. The polynomial in $\boldsymbol{x}$ fulfills all additional constraints expressed in Equation 14, and minimizes the deviations from the curve defined by the 37 control points.

$$\boldsymbol{l} + \boldsymbol{v} = \boldsymbol{A}\cdot\boldsymbol{x} \qquad \boldsymbol{N} = \boldsymbol{A}^{\mathrm{T}}\cdot\boldsymbol{A} \qquad \boldsymbol{x_0} = \boldsymbol{N}^{-1}\cdot\boldsymbol{A}^{\mathrm{T}}\cdot\boldsymbol{l} \tag{Eq. 17}$$

$$\boldsymbol{C}\cdot\boldsymbol{x} = \boldsymbol{g} \qquad \boldsymbol{M} = \boldsymbol{C}\cdot\boldsymbol{N}^{-1}\cdot\boldsymbol{C}^{\mathrm{T}} \qquad \boldsymbol{dx} = \boldsymbol{N}^{-1}\cdot\boldsymbol{C}^{\mathrm{T}}\cdot\boldsymbol{M}^{-1}\cdot(\boldsymbol{g} - \boldsymbol{C}\cdot\boldsymbol{x}_0)$$

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{dx} \qquad \boldsymbol{v} = \boldsymbol{A}\cdot\boldsymbol{x} - \boldsymbol{l}$$

As Equation 9 is linear, no iterations are needed to solve the functional model, and no initial guesses are required for the unknown parameters. All constraints can be expressed with functionally dependent parameters. The three constraints for the Natural Earth projection are linear, but non-linear constraints could also be used. In this case, matrix $\boldsymbol{C}$ would contain partial derivatives of the constraints equations with respect to all parameters in vector $\boldsymbol{x}$, calculated from parameter values in vector $\boldsymbol{x_0}$. However, non-linear constraints can only partially be fulfilled.