FIRST PASSAGE TIME DISTRIBUTIONS IN ELECTRONIC RECEIVERS

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ALBERT WILLIAM KRATZKE

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APPROVED:

Signature redacted for privacy.

Associate Professor of Mathematics

In Charge of Major

Signature redacted for privacy.

Acting Chairman of Department of Mathematics

Signature redacted for privacy.

Chairman of School Graduate Committee

Signature redacted for privacy.

Dean of Graduate School

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INTRODUCTION

The description of the behavior of physical systems in terms of probability distributions and the associated statistical parameters is necessary whenever noise is present. Especially is this true when a system is required to operate with inputs whose energy (or power) is of the same order of magnitude as that of the noise, that is, when a system is required to be very sensitive. The general problem considered is that of recognizing the presence of a weak signal when it is submerged in a background of noise. Considerable attention has been given to this problem and a brief description of the conventional means for such detection is given in the sequel.

The familiar way of accomplishing the detection of signals in noise is with a system comprising an IF filter, a quadratic envelope detector, and a low-pass postdetection filter. The first filter introduces spectral selectivity and enhances the signal relative to the noise. The quadratic detector (rectifier) squares the output of the first filter and extracts the envelope of the combined signal and noise thus generating a dc component. The postdetection filter transmits the dc component while rejecting as much fluctuation as possible consistent with the required system transient time, resulting in a smoothed output. The output of such a system is then presented to a threshold device whose function is to announce the presence of a signal if a certain pre-set threshold level is exceeded. Since the spectral composition of the signal is generally not known in advance, similar parallel channels are used for simultaneous search at different frequencies.

The output of such a threshold detector depends on the character of the random process introduced as input and on the response characteristic of the system itself. In principle, knowledge of these permits one to describe the output in terms of a distribution function. Let this function be $F(x_0;A)$ where x_0 is the threshold level and A is the signal-to-noise ratio at the IF stage. Let X be the output of system. Then

$$F(x_0;A) = Pr\{X<x_0;A\}, \tag{1.1}$$

that is, F is the probability that an instantaneous sampling of X is less than x_0 for a given signal-to-noise ratio. For the processes discussed here F will be differentiable and the probability density function, $f(x;A)$ will be used. The threshold level is determined by preassigning an allowable false signal probability. This is the probability that noise alone exceeds the threshold and is given by

$$(1.2) \qquad \Pr\{X>x_0;A=0\} = \int_{x_0}^{\infty} f(X;0)dx.$$

Solution of (1.2) for x_0 then determines the threshold level and the probability of detection is given by

$$(1.3) \qquad PD = \int_{x_0}^{\infty} f(X;A)dx$$

for given signal-to-noise ratio, A.

This skeletal description of a threshold detection system serves to introduce an allied problem, viz., the passage time problem. Since time is a prominent factor in many automatic control systems, it is important to know how long it takes for the output of a system to cross the threshold, given some initial condition at t=0. Consequently, the distribution function for the time to first passage of a threshold is worthy of study and several such distributions are derived in the sequel, corresponding to different physical processes.

Such problems are not new in the physical sciences. Chandrasekhar (3, p. 264) studied the rate of escape of stars from clusters by finding the probability that a star with a known initial velocity would reach an escape velocity in a time interval (t, t + dt). Schrödinger (9, p. 292) and Smoluchowski (11, P. 320) found the probability that a particle in Brownian motion reaches an absorbing barrier in a similar time interval. Stumpers (15, pp. 270-281) examined a passage time distribution for a simple R-C circuit, assuming a completely random output process. These are but a few examples of the classical absorbing barrier problem.

The particular system to be investigated here is the threshold detector described above. Sufficient generality obtains so that the results may be altered and used for other systems.

DISTRIBUTION FUNCTIONS

Before deriving expressions for the several first passage time probability density functions, it is expedient to introduce the hierarchy of joint probability densities and distribution functions of the output of the system described earlier. Also, it will be necessary to use the correlation function, which will be discussed in the next chapter.

The following notation will be used throughout:

$$F_1(x) = \Pr\{X(t)<x\}$$

$$F_2(x_1,x_2) = \Pr\{X(t_1)<x_1, X(t_2)<x_2\}$$

$$F_3(x_1,x_2,x_3) = \Pr\{X(t_1)<x_1, X(t_2)<x_2, X(t_3)<x_3\}$$

and so on. $X(t_k)$ is the value of the output of the system at time, t_k. Lower case letters f_1, f_2,.... will be used for the associated probability density functions.

This section leans heavily on papers by Kac and Seigert (6, pp. 383-397), Emerson (5, pp. 1168-1176), and Stone and Brock (14, pp. 65-69) for determining F_1 or f_1. The methods as well as pertinent results will be discussed below. Methods for obtaining higher order distribution functions are taken from papers by Krishnamoorthy and Parthasarathy (7, pp. 549-557) and Stone (12, pp. 1-5). For these also, the methods will be discussed and useful results cited.

First, the electronic receiver will be discussed in more detail than that given in the introductory sketch. Figure 2.1 is a block diagram of the receiver considered.

$$E_i(t) \rightarrow \boxed{\text{Filter}} \text{---} E(t) \rightarrow \boxed{\text{Square Law Detector}} \text{---} E^2(t) \rightarrow \boxed{\text{Filter}} \rightarrow E_0(t)$$

$f_1(t)$ $\quad$ $F_1(w)$ $\qquad\qquad$ $f_2(t)$ $\quad$ $F_2(w)$

Figure 1. Block diagram of receiver.

The IF filter is characterized by its voltage-frequency transfer function, $Y_{if}(\omega)$ or its Fourier transform, $y_{if}(t)$; similarly, the postdetection filter is characterized by $Y_a(\omega)$ or $y_a(t)$. The voltage, $E(t)$, applied to the quadratic detector is given by the convolution integral,

$$(2.1) \qquad E(t) = \int_{-\infty}^{\infty} y_{if}(t-x)E_i(x)dx,$$

where $E_i(t)$ is the input voltage. The quadratic detector squares $E(t)$, so that the input to the postdetection filter is $E^2(t)$, given by

$$(2.2) \qquad E^2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_{if}(t-x)\, E_i(x)E_i(y)y_{if}(t-y)dxdy.$$

The output voltage of the system is then

$$E_o(t) = \int_{-\infty}^{\infty} y_a(t-x)E^2(x)dx. \tag{2.3}$$

Substituting from equation (2.2) yields the following formula for the output voltage in terms of the input voltage:

$$E_o(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_i(t-u)g(u,v)E_i(t-v)dudv. \tag{2.4}$$

where,

$$g(u,v) = \int_{-\infty}^{\infty} y_{if}(u-z)y_a(z)y_{if}(v-z)dz. \tag{2.5}$$

Emerson (5, p. 1169) points out that the next step in the development is to expand the function g(u,v) into the uniformly convergent bilinear series,

$$g(u,v) = \sum_{j=1}^{\infty} \lambda_j h_j(u)h_j(v), \tag{2.6}$$

where the $h_j(x)$ and the λ_j are respectively the jth orthonormal function and the corresponding eigenvalue of the integral equation

$$\lambda h(x) = \int_{-\infty}^{\infty} g(x,y)h(y)dy. \tag{2.7}$$

For the receiver system studied here, Stone and Brock have derived an expression for the system kernel, g(u,v).

The salient features of that development are reproduced here for continuity.

Consider a receiver with simple RC (first order) filters as in Figure 2.2

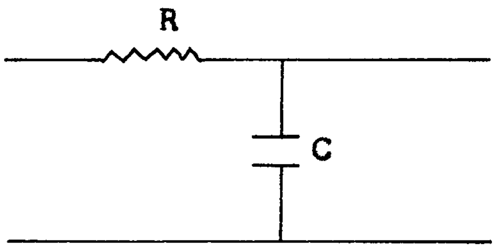

Figure 2.2. Filter diagram.

For the steady state situation

$$(2.8) \qquad \frac{E_o(\omega)}{E_i(\omega)} = \frac{1/i\omega C}{R+1/i\omega C} = \frac{1}{1+i\omega RC} = Y(\omega)$$

In the interest of symmetry, the concept of negative frequency is introduced and the IF and audio (postdetection) transfer functions then become

$$(2.9) \qquad Y_{if}(\omega) = \frac{1}{1+i(\omega-\omega_0)/\omega_1} + \frac{1}{1+i(\omega+\omega_0)/\omega_1}$$

$$(2.10) \qquad Y_a(\omega) = \frac{1}{1+i\ \omega/\omega_a}$$

where $\omega_1 = 1/R_1C_1$, $\omega_2 = 1/R_2C_2$, and ω_0 is the center frequency of the IF filter. ω_1 and ω_2 are proportional to the IF and audio filter band widths respectively. The Fourier transform of $Y_{if}(\omega)$ is

$$y_{if}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} Y_{if}(\omega)d\omega$$

$$(2.11) \qquad = \frac{\omega_1}{2\pi}\int_{-\infty}^{\infty} [e^{it(\omega_1 z+\omega_0)} + e^{it(\omega_1 z-\omega_0)}] \frac{dz}{1+iz}$$

$$= \frac{\omega_1}{\pi} \cos\omega_0 t \int_{-\infty}^{\infty} (\cos\omega_1 zt + i\sin\omega_1 zt)\frac{1 - iz}{1 - z^2} \; dz.$$

After some reduction, $y_{if}(t)$ is given by

$$(2.12) \qquad y_{if}(t) = \begin{cases} 2\omega_1 e^{-\omega_1 t}\cos\omega_0 t, & t \geq 0 \\ 0 & t < 0. \end{cases}$$

For the postdetection filter, which is low-pass,

$$(2.13) \qquad y_2(t) = \begin{cases} \omega_2 e^{-\omega_1 t}, & t \geq 0 \\ 0 \quad , & t < 0. \end{cases}$$

Substituting these into equation (2.5), g(u,v) becomes

$$(2.14)\, g(u,v) = 2\omega_1^2\omega_2 \cos\omega_0(u-v)\int_0^u e^{-\omega_1(u-z)-\omega_2 z-\omega_1(v-z)}dz. \; u \leq v.$$

Finally,

$$(2.15)\quad g(u,v)=\begin{cases}\dfrac{2\omega_1^2\omega_2}{2\omega_1-\omega_2}\cos\omega_0(u-v)e^{-\omega_1(u+v)}[e^{(2\omega_1-\omega_2)u}-1], & o\leq u\leq v.\\[2ex] \dfrac{2\omega_1^2\omega_2}{2\,\omega_1-\omega_2}\cos\omega_0(u-v)e^{-\omega_1(u+v)}[e^{(2\omega_1-\omega_2)v}-1], & o\leq v\leq u.\end{cases}$$

By noting particular identities among Bessel functions, Stone and Brock (14, p. 5) have succeeded in expressing g(u,v) as in equation (2.6) without actually solving the integral equation (2.7). The series is

$$g(u,v) = \frac{2\omega_1^2}{2\gamma-1}\cos\omega_0(u-v)e^{-\omega_1(u+v)}[e^{(2\omega_1-\omega_2)u}-1]$$

$$(2.16)\qquad =\sum_{j=1}^{\infty}\lambda_j[h_j^C(u)h_j^C(v) + h_j^S(u)h_j^S(v)] \qquad o \leq u \leq v$$

$$\lambda_j = \frac{4\omega_1\gamma}{r_j^2} \qquad \gamma = \frac{\omega_1}{\omega_2} \qquad J_{2\gamma-1}(r_j) = 0.$$

The orthonormal functions $h_j^C(u)$ and $h_j^S(u)$ are

$$(2.17)\qquad h_j^{C,S}(u)= \frac{\sqrt{2\omega_2}\, e^{\frac{-\omega_2 u}{2}}\, J_{2\gamma-1}(r_j e^{\frac{-\omega_2 u}{2}})}{J_{2\gamma}(r_j)}\begin{cases}\cos\omega_0 u\\ \sin\omega_0 u.\end{cases}$$

Having established a representation for output in terms of input, now consider a particular type of input, namely an additive mixture of signal and noise,

$$E_i(t) = S(t) + N(t), \tag{2.18}$$

where N(t) is purely random noise, normally distributed about zero. Now both the signal and noise are expanded in a series of eigenfunctions of the system operator. These functions are given by equation (2.17) and the series expansions are

$$S(t-u) = \sum_{j=1}^{\infty} [a_j h_j^C (u) + \beta_j h_j^S (u)] \tag{2.19}$$

for the signal, and

$$N(t-u) = \sum_{j=1}^{\infty} [\xi_j h_j^C (u) + \eta_j h_j^S (u)] \tag{2.20}$$

for the noise.

The coefficients of the two series are given by

$$\begin{aligned} a_j(t) &= \int_0^{\infty} S(t-u) h_j^C(u)\, du, \\ \beta_j(t) &= \int_0^{\infty} S(t-u) h_j^S(u)\, du, \\ \xi_j(t) &= \int_0^{\infty} N(t-u) h_j^C(u)\, du, \end{aligned} \tag{2.21}$$

$$\eta_j(t) = \int_0^\infty N(t-u)h_j^S(u)\,du.$$

Finally, $E_o(t)$ may now be written in terms of these coefficients and the $h_j^{C,S}$

$$E_o(t) = \int_0^\infty \int_0^\infty [S(t-u)+N(t-u)] \sum_{j=1}^\infty \lambda_j [h_j^C(u)h_j^C(v)+h_j^S(u)h_j^S(v)]$$

$$X[S(t-v) + N(t-v)\,dudv$$

(2.22)

$$= \sum_{j=1}^\infty \lambda_j [\xi_j + \alpha_j)^2 \;+\; (\eta_j + \beta_j)^2].$$

From the representation of the output in terms of the coefficients, ξ_j, η_j, α_j, and β_j, the characteristic function associated with the output is then determined. Since the distribution function associated with the noise coefficients is Gaussian, the characteristic function may be written immediately. This is carried out in both Emerson's paper (5, p. 1170) and Stone and Brock's paper (14, p. 10). The details of this as well as the inversion of the transform in order to obtain output probability density functions are omitted here but results useful to the first passage time distribution are cited below.

Before presenting the resulting distribution function, it should be pointed out that the interesting problem from a physical viewpoint includes a randomly modulated signal.

The distribution from which the signal amplitude arises is the well-known Rayleigh distribution given by the density function

$$
\begin{aligned}
f(u) &= \frac{1}{z}\, u\, e^{-u^2/2z}, \qquad & u \geq 0. \\
f(u) &= 0 & u < 0,
\end{aligned}
\tag{2.23}
$$

where

$$z = E\,\{u^2/2\}$$

and 'E' is the expectation operator. Furthermore, the bandwidth ratio, γ, defined in equation (2.16) is usually very large in order to incorporate as much smoothing of the output as the allowable transient time of the system will allow. γ is taken to be of the order 100-1000. With these additional requirements the distribution function of the output is

$$F(x) = 1 - e^{-\frac{x-1}{z}}. \tag{2.24}$$

as defined in equation (2.23) is the average signal-to-noise ratio. Then the probability of the output exceeding a threshold level, x_0 is

$$
\begin{aligned}
\Pr\{X \geq x_0\} &= 1 - F(x_0) \\
&= e^{-\frac{x_0 - 1}{z}}.
\end{aligned}
\tag{2.25}
$$

This defines the first distribution function, F_1.

The higher order distribution functions necessary in the development of first passage time distributions must now be developed. Again only the salient features of the methods and the pertinent results are cited.

The joint probability density function of two correlated Gaussian variates is

$$(2.26) \qquad f(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1-\rho^2)}}$$

where ρ is the correlation coefficient

$$(2.27) \qquad \rho = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)\, dx_1 dx_2}{\sqrt{\sigma_1 \sigma_2}}.$$

and σ_i^2 is the variance of X_i and X_i is the random variable with marginal distribution function, $F_1(x)$. For two variates ξ_1 and ξ_2, the moment generating function (Laplace transform) of $f(\xi_1, \xi_2)$ is defined as

$$(2.28) \qquad G(s_1, s_2) = E\,\{e^{-(s_1\xi_1 + s_2\xi_2)}\}.$$

Now let

$$(2.29) \qquad \xi_i = \frac{1}{2}\, x_i^2,$$

where x_i have the joint density function given by (2.26). $G(s_1, s_2)$ may be evaluated immediately yielding

$$G(s_1, s_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left[\frac{x_1^2+x_2^2-2\rho x_1 x_2}{2(1-\rho^2)} - \frac{s_1 x_1^2}{2} - \frac{s_2 x_2^2}{2}\right]} dx_1\, dx_2$$

(2.30)

$$= \begin{vmatrix} 1+s_1 & s_2\rho \\ s_2\rho & 1+s_2 \end{vmatrix}^{-\frac{1}{2}} .$$

If the marginal density functions of the x_i are Gaussian with zero means and unit variances, then the probability density function associated with ξ_i is

$$f(\xi_i) = \begin{cases} \frac{1}{\sqrt{\pi\,\xi_i}}\, e^{-\xi_i} & \xi_i \geq 0 \\ 0 & \xi_i < 0. \end{cases} \tag{2.31}$$

The characteristic function (or Laplace transform since $f(\xi_i)$ is truncated at zero) of the distribution is

$$\varphi(s) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-(t+1)\xi_i}}{\sqrt{\xi_i}}\, d\xi_i = (t+1)^{\frac{1}{2}} . \tag{2.32}$$

The characteristic function associated with the sum of m such random variables, independent and identically distributed, is

$$\varphi_m(t) = \varphi^m(t) = (t+1)^{-\frac{m}{2}} \quad (2.33)$$

Inverting then yields

$$f(\xi_i) = \frac{\xi_i^{\frac{m}{2}-1} e^{-\xi_i}}{\Gamma(m/2)} \quad (2.34)$$

where

$$\xi_i = \xi_{i1} + \xi_{i2} + \xi_{i3} + \ldots + \xi_{im}. \quad (2.35)$$

Now specify that $m = 2p$, p an integer, so the probability density function associated with each of the composite Gamma distributed variables is

$$f(x_i) = \frac{x_i^{p-1} e^{-x_i}}{(p-1)} \quad . \quad (2.36)$$

Noting equation (2.28), it is seen that raising $G(s_1, s_2)$ to the 2p power is equivalent to defining each x_i to be the sum of 2p such variates.

$$\begin{aligned} G_p(s_1, s_2) &= G^p(s_1, s_2) = E\{e^{-(s_1 x_1 + s_2 x_2)}\} \\ &= \int_0^\infty \int_0^\infty f(x_1, x_2)\, e^{-(s_1 x_1 + s_2 x_2)} dx_1 dx_2 . \end{aligned} \quad (2.37)$$

$$G_p(s_1,s_2) = \frac{1}{(1+s_1)^p(1+s_2)^p}\left[1 - \frac{s_1 s_1 \rho^2}{(1+s_1)(1+s_2)}\right]^{-p}.$$

The $f(x_1,x_2)$ appearing in equation (2.37) is the joint density function of two correlated Gamma variables. W. M. Stone (12, p. 5) has extracted the form of $f(x_1,x_2)$. It is, for $p = 1$, (which is the interesting case for the output of first order filter systems)

$$f(x_1,x_2) = e^{-x_1-x_2}\sum_{n=0}^{\infty}\rho^{2n} L_n(x_1)L_n(x_2) \tag{2.38}$$

where

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}). \tag{2.39}$$

The infinite series in (2.38) may be written in terms of Bessel functions of imaginary arguments (16, p. 169),

$$\sum_{n=0}^{\infty}\rho^{2n}L_n(x_1)L_n(x_2) = I_0\left[\frac{2\sqrt{x_1x_2}\,\rho}{(1-\rho^2)}\right]\frac{e^{-(x_1+x_2)\rho^2/(1-\rho^2)}}{1-\rho^2} \tag{2.40}$$

so that equation (2.38) becomes

$$f(x_1,x_2) = I_0\left[\frac{2\sqrt{x_1x_2}\,\rho}{(1-\rho^2)}\right]\frac{e^{-(x_1+x_2)/(1-\rho^2)}}{1-\rho^2}. \tag{2.41}$$

This is the joint probability density function, f_2 in the hierarchy.

The generalization to n Gamma distributed variables is carried out in both the Stone (12, p. 6) and Krishnamoorthy and Parthasarathy (7, p. 554) papers. For the threshold detection problem, it is necessary to have only the first two distribution or density functions.

CORRELATION FUNCTIONS

The most important parameter in the second probability density function presented in the last chapter is the correlation coefficient, ρ. Since this parameter will depend in some way on time, it is natural to expect that it will have an important role in the development of first passage time distributions. Such, indeed, is the case and the correlation function is examined more closely in this chapter.

The correlation function is one of the average values or moments associated with a distribution. Specifically, it is the average of the product of two random variables. For the process described by the output of an electronic receiver, it is the autocorrelation function which is of interest. This function is a measure of the degree of dependence between a sample of the output and another sample at some other time. Only one process is involved but we shall be interested in the relation between the behavior of the output at different times.

We shall abandon the notation of Chapter II and adopt the more conventional notation used for continuous processes. The correlation function is then defined

$$\psi(t_1,t_2) = E\{E_0(t_1)E_0(t_2)\}$$

(3.1)

$$= \iint x_1 x_2 f(x_1,x_2)dx_1 dx_2$$

where 'E' is the expectation operator and $x_1 = E_0(t_1)$ and $x_2 = E_0(t_2)$. $E_0(t)$ is, as defined earlier, the value of the output of the system at time, t. Note that the auto-correlation function defined by equation (3.1) is not the same as the correlation coefficient defined in equation (2.27) but differs by a normalizing constant.

For most problems in electronic detection, the output process enjoys a time-invariance which simplifies the mathematical model considerably. The random process is then said to be stationary. The important property of a stationary random process for development of correlation functions is that each probability distribution function of the process depends on time only through time differences. Explicitly, for each n

(3.2) $$F_n(x_1,t_1;x_2,t_2;\ldots;x_n,t_n)=F_n(x_1,0;x_2,t_2-t_1;\ldots;x_n,t_n-t_1).$$

Furthermore, every translation in time carries the set of functions comprising the process into itself so that all the statistical parameters remain unchanged. The autocorrelation function may then be written

$$(3.3) \qquad \psi(\tau) = E\{E_0(t)E_0(t+\tau)\}.$$

Another property often assumed in the analysis of stationary random processes is the ergodic property. In general terms, this property permits averaging over time rather than over ensembles. The expectation of any random variable is then equal to the average of that variable over all translations in time of a single function. The autocorrelation function, under the assumption of ergodicity, becomes

$$(3.4) \qquad \psi(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{T}^{T} E_0(t)E_0(t+\tau)dt.$$

It is for a process exhibiting the properties listed above that autocorrelation functions are developed in this chapter.

The autocorrelation function for the receiver described earlier will be discussed in general terms, that is, without specifying the particular filter transfer functions. However, in order to apply the results, the autocorrelation function for a receiver with first order filters will be cited. Stone and Brock (13, p. 31) have derived this function in another application.

Stationarity and ergodicity will be assumed. Then the averaging process is with respect to time and the expectation operator becomes a time averaging operator.

The definition of the autocorrelation function given in equation (3.4) was for a process with zero mean. In general, this is not the case and the correlation function should be

$$(3.5) \qquad \psi(\tau) = E[(E_0(t) - m)(E_0(t+\tau) - m)].$$

where 'E' is now taken to be a time averaging operator. Expansion of equation (3.5) yields

$$(3.6) \qquad \psi(\tau) = E[E_0(t)E_0(t+\tau)] - \{E[E_0(t)]\}^2.$$

Note that, because of stationarity,

$$(3.7) \qquad E[E_0(t)] = E[E_0(t+\tau)] = m.$$

Now in order to determine $\psi(\tau)$, it is necessary to perform the averaging process on $E_0(t)E_0(t+\tau)$ and $E_0(t)$. Recalling that $E_0(t)$ may be expressed in terms of $g(u,v)$, the system kernel, and the input process, the method requires that the time averaging operator operate on expressions like equation (2.4).

It is not the purpose of this chapter to present in detail such derivations, but rather to present the concept of autocorrelation and the methods for such derivations. Consequently, the pertinent result from reference 13 will be cited.

For an electronic detection (receiver) system characterized by first-order IF and audio filters as given by equations (2.9) and (2.19), the autocorrelation function is

$$\psi(\tau)=N_0^2\left[\frac{2e^{-\omega_2|\tau|}-e^{-2\omega_1|\tau|}}{4\gamma^2-1}+2z\,\frac{\gamma e^{-\omega_2|\tau|}-e^{-\omega_1|\tau|}}{\gamma^2-1}+z^2\right] \tag{3.8}$$

where

N_0 = Noise power admitted to the system

ω_1 = IF filter bandwidth

ω_2 = audio filter bandwidth

$\gamma = \dfrac{\omega_1}{\omega_2}$

z = average signal-to-noise ratio.

The input process used in deriving (3.8) is an additive mixture of Rayleigh distributed signal and white Gaussian noise.

FIRST PASSAGE TIME DISTRIBUTIONS

Consider a time interval, $t-t_o$ and a partition of $t-t_o$ by the sequence of time points, $\{t_k, k=1,\ldots n\}$. Further, consider a random sequence of events, $\{E_{t_i}, i=1,\ldots n\}$ where E_{t_k} is the event, "the output exceeds the threshold at time t_k," and $t_1<t_2<\ldots<t_n$. Note that the events are not necessarily independent nor is the output process necessarily stationary. Let $-E_{t_k}$ be the complementary event, "the output does not exceed the threshold at time t_k." Let B be a given initial event at $t_o<t_1$. Then the probability that the first time the threshold is exceeded is at time t_n is

$$Pr\{-E_{t_1}, -E_{t_2}, \ldots -E_{t_{n-1}}, E_{t_n} | B_{t_o}\}.$$

This is the joint probability that the threshold was not exceeded at $t_1, t_2, \ldots t_{n-1}$, but that it was exceeded at t_n, all conditioned on the event B at t_o. This is simply

$$Pr\{-E_{t_1}, -E_{t_2}, \ldots -E_{t_{n-1}}, E_{t_n} | B_{t_o}\} =$$

(4.1)

$$Pr\{-E_{t_1} | B\} \; Pr\{-E_{t_2} | B, -E_{t_1}\} \ldots Pr\{E_{t_n} | B, -E_{t_1}, -E_{t_2}, \ldots\}$$

$$\Pr(E_{t_n}|B)-\sum_{i=1}^{n-1}\Pr(E_{t_i},E_{t_n}|B)+\sum_{i=1}^{n-1}\sum_{k=i+1}^{n-1}\Pr(E_{t_i},E_{t_k},E_{t_n}|B)$$

$$-\sum_{i=1}^{n-1}\sum_{k=i+1}^{n-1}\sum_{m=k+1}^{n-1}\Pr(E_{t_i},E_{t_k},E_{t_m},E_{t_n}|B)+\dots\quad .$$

$\Pr(E_{t_i},E_{t_k},\dots E_{t_n}|B_{t_0})$ is the joint probability of having the threshold exceeded at $t_i,t_k,\dots t_n$ conditioned on B at t_0, regardless of events between t_i and t_k, etc. The output process is continuous so the discrete times t_k must be replaced by a continuous time, t. Now it must be assumed that the conditional probability of exceeding the threshold in $(t,t+\Delta t)$ is $p(t|t_0)\Delta t + o(\Delta t)$ where p is a probability density. Similarly, the probability of exceeding the threshold in $(t_1,t_1+\Delta t_1)$ and $(t_2,t_2+\Delta t_2)$ is $p(t_1,t_2|t_0)\Delta t_1\Delta t_2 + o(\Delta t_1\Delta t_2)$. Then passage to the limit as $n\to\infty$ and $||\Delta t||\to 0$ yields

$$\Pr\{X(t)\geq x_0 \text{ for the first time in } (t,t+dt)|B_{t_0}\}=P(t|t_0)=$$

$$(4.2)\quad p_1(t|t_0)-\int_{t_0}^{t}p_2(t_1,t|t_0)dt_1+\int_{t_0}^{t}\int_{t_1}^{t}p_3(t_1,t_2,t|t_0)dt_2dt_1$$

$$-\int_{t_0}^{t}\int_{t_1}^{t}\int_{t_2}^{t}p_4(t_1,t_2,t_3,t|t_0)dt_3dt_2dt_1+\dots \quad =$$

$$p_1(t|t_0) - \int_{t_0}^{t} p_2(t_1,t|t_0)dt_1 + \frac{1}{2!}\int_{t_0}^{t}\int_{t_0}^{t} p_3(t_1,t_2,t|t_0)dt_1dt_2$$

$$- \frac{1}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} p_4(t_1,t_2,t_3,t|t_0)dt_1dt_2dt_3 + \cdots$$

where

$p_1(t|t_0)dt$ = The probability of the output exceeding the threshold in $(t,t+dt)$, conditioned on some event at t_0.

$p_2(t_1,t|t_0)dt_1dt$ = The joint probability of the output exceeding the threshold in (t_1,t_1+dt_1) and $(t,t+dt)$, conditioned on the same event at t_0.

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$p_{k+1}(t_1,t_2,\ldots t_k,t|t_0)dt_1dt_2\ldots dt$ = The joint probability of the output exceeding the threshold in (t_1,t_1+dt_1), $(t_2,t_2+dt_2),\ldots(t_k,t+dt_k)(t,t+dt)$, conditioned on the same event at t_0.

A similar series is cited without derivation in Rice's paper (8, p. 70) in connection with the distribution of zeroes of a noisy process. It is equation (2.2) in one or the other of its forms from which the distributions for passage times in communication systems will be derived.

Before proceeding to apply (4.2) to various physical models, a closer scrutiny of the joint probability density functions, p_1, p_2,...p_k seems in order. Note that each of these density functions depends on a joint distribution function of amplitude of the output. Specifically, it is expected that the time density functions will depend on $F_1(x_0)$, $F_2(x_0,x_0)$, etc., where x_0 is the threshold level. The exact dependence of the p_k's on the F_k's will be presented in the next chapter. It suffices here to note that because of the dependence of the p_k's and consequently $P(t|t_0)$, on x_0, the first passage time density function requires a normalization constant depending on x_0. It is, of course, required that

$$(4.3) \qquad \int_0^\infty P(t|t_0)\, dt = 1.$$

It is obvious that this situation could not obtain if, for instance, x_0 were chosen to be infinite, for then the output voltage would never exceed x_0. Therefore, we must adjust $P(t|t_0)$ by multiplying it by a suitable normalizing constant for each process considered.

Equation (4.2) is cumbersome and not susceptible to easy solution as it stands. Therefore several cases of interest in electronic detection will be studied.

The first case to be treated is that of independent samples. In general, this situation is not very common. In order to be useful the operation of the receiver would have to be intermittent, with examinations separated by a time interval greater than or equal to the so-called Nyquist interval, $1/2\omega$, where ω is the effective spectral bandwidth of the output process. Nyquist's Theorem (10, p. 11) simply states that a function may be determined by its values at intervals of $1/2\omega$. If we denote by $\mu\,dt$ the probability of an event occurring in dt, then, for independent events and if μ is constant, the p_k's of equation (4.2) become

$$p_1(t|t_0) = \mu$$

$$p_2(t_1,t|t_0)=\mu^2$$

$$p_3(t_1,t_2,t|t_0) = \mu^3 \tag{4.4}$$

$$\vdots$$

$$p_{k+1}(t_1,t_2,\ldots p_k,t|t_0) = \mu^{k+1}$$

Then equation (4.2) reduces to

$$P(t) = \mu\, e^{-\mu(t-t_0)} \tag{4.5}$$

for the probability density function for first passage across a threshold. The constant, μ, must have the units of t^{-1} and may be interpreted as the average rate of the output exceeding the threshold, x_0. This result is certainly not new but has been included in the interest of completeness.

The next case to be examined is that of 'conditional independence.' In this case, sample values of the output depend only on the initial condition. Again there is considerable simplification in (4.2). The joint probability densities then take the form,

$$p_{k+1}(t_1, t_2, \ldots t_k, t|t_0) = p_1(t_1|t_0)p_1(t_2|t_0)\ldots p_1(t_k|t_0)p_1(t|t_0). \tag{4.6}$$

Equation (4.2) becomes

$$P(t|t_0) = p_1(t|t_0)\Big\{1 - \int_{t_0}^{t} p_1(t_1|t_0)dt_1 + \frac{1}{2!}\int_{t_0}^{t} p_1(t_1|t_0)dt_1 \int_{t_0}^{t} p_1(t_2|t_0)dt_2$$

$$- \frac{1}{3!}\int_{t_0}^{t} p_1(t_1|t_0)dt_1 \int_{t_0}^{t} p_1(t_2|t_0)dt_2 \int_{t_0}^{t} p_1(t_3|t_0)dt_3 + \ldots\Big\} \tag{4.7}$$

$$= p_1(t|t_0)\Big\{1 - \int_{t_0}^{t} p_1(x|t_0)dx + \frac{1}{2!}\left[\int_{t_0}^{t} p_1(x|t_0)dx\right]^2 - \frac{1}{3!}\left[\int_{t_0}^{t} p(x|t_0)dx\right]^3 + \ldots\Big\}$$

$$= p_1(t|t_0)e^{-\int_{t_0}^{t} p_1(x|t_0)dx} \, .$$

This expression requires a normalizing constant. Integrating the right side of (4.7) yields

$$\int_0^\infty p_1(t|t_0)\; e^{-\int_{t_0}^{t} p_1(x|t_0)\,dx}$$

$$\text{(4.8)} \qquad = e^{-\int_{t_0}^{0} p(x|t_0)\,dx} - e^{-\int_{t_0}^{\infty} p(x|t_0)\,dx}$$

$$= e^{\int_{0}^{t_0} p(x|t_0)\,dx} - e^{-\int_{t_0}^{\infty} p(x|t_0)\,dx}$$

Then the first passage time probability density for conditional independence becomes

$$\text{(4.9)} \qquad P(t|t_0) = \frac{p_1(t|t_0)\, e^{-\int_{t_0}^{t} p_1(x|t_0)\,dx}}{e^{\int_{0}^{t_0} p(x|t_0)\,dx} - e^{-\int_{t_0}^{\infty} p(x|t_0)\,dx}}$$

If we choose $t_0 = 0$, then

$$\text{(4.10)} \qquad P(t|t_0) = \frac{p_1(t|0)\; e^{-\int_{0}^{t} p_1(x|0)\,dx}}{1 - e^{-\int_{0}^{\infty} p(x|0)\,dx}}$$

It is interesting to note that (4.7) can be derived in another way which has, perhaps, a little more physical appeal. $P(t|t_0)dt$, the probability of first passage time in $(t,t + dt)$ may be expressed as the product of two probabilities, the probability that the event does not occur in the interval (t_0,t), conditioned on the initial condition at t_0, and the probability that the event does occur in $(t,t + dt)$, conditioned on the same initial event. Then

$$(4.11) \qquad P(t|t_0)\, dt = \left\{1-\int_{t_0}^{t} P(x|t_0)\, dx\right\} p(t|t_0)\, dt.$$

$$(4.12) \qquad \frac{P(t|t_0)}{p(t|t_0)} = 1 - \int_{t_0}^{t} P(x|t_0)\, dx.$$

Differentiating with respect to t,

$$(4.13) \qquad \frac{d}{dt}\left[\frac{P(t|t_0)}{p(t|t_0)}\right] = - P(t|t_0) = -p(t|t_0)\left[\frac{P(t|t_0)}{p(t|t_0)}\right]$$

The solution of (4.13) is

$$(4.14) \qquad P(t|t_0) = p(t|t_0)\, e^{-\int_a^t p(x|t_0)\, dx}$$

The constant, a, is evidently t_0 since

$$P(t|t_0) \to p(t|t_0) \text{ as } t \to t_0,$$

so that (4.14) is exactly the same as (4.7). Again, to complete the derivation, it is necessary to adjust the passage time density function with a normalizing constant.

A striking application of the first passage time distribution is its usage in the 'confirmation search' problem for threshold detection. The problem is to find the distribution in time of the first time the threshold is exceeded, knowing that it was exceeded at time, t_0. The problem then becomes a recurrence time problem and the initial condition is simply the condition that the threshold was exceeded at t_0. This, of course, assumes that secondary (confirmation) search is made with a receiver identical to the one used for primary search. Here we shall specify that the p_k's enjoy the Markoffian property and that the process is stationary. Specifically,

$$(4.15) \qquad p_k(t_1,t_2,\ldots t_{k-1},t|t_0)$$

$$=p_1(t_1|t_0)p_1(t_2|t_1,t_0)\ldots p_1(t|t_{k-1},t_{k-2},\ldots t_0)$$

$$=p_1(t_1|t_0)p_1(t_2|t_1)\ldots p_1(t|t_{k-1}).$$

There is no loss of generality if t_0 is chosen to be zero. Then p_k becomes

$$(4.16)\qquad p_k(t_1,t_2\ldots t_{k-1},t|t_0)$$

$$=p_1(t_1|0)p_1(t_2-t_1|0)\ldots p_1(t-t_{k-1}|0)$$

Using the first form of (4.2), the first passage time (recurrence time) probability density function is then

$$(4.17)\qquad P(t|0)=p_1(t|0)-\int_0^t p_1(t_1|0)p_1(t-t_1|0)\,dt_1$$

$$+\int_o^t\int_{t_1}^t p_1(t_1|0)p_1(t_2-t_1|0)p_1(t-t_2|0)dt_2dt_1\ldots.$$

Note that the second term of (4.17) is the convolution integral or it is p_1 convolved with itself. Similarly, the k^{th} term of the series is the k-fold convolution of p_1. Writing (*) for the convolution operator, (4.17) may be expressed as

$$(4.18)\qquad P(t|0) = p_1(t|0)-p_1(t|0)*p_1(t|0)$$

$$+p_1(t|0)*p_1(t|0)*p_1(t|0)\ldots.$$

Having found $P(t|0)$ in terms of convolution integrals, the next step suggested is, of course, that of taking the Laplace transform of both sides of (4.18).

Let $\hat{P}(s|0)$ and $\hat{p}_1(s|0)$ denote the Laplace transform of $P(t|0)$ and $p_1(t|0)$ respectively. Taking the Laplace transform of both sides of (4.18) yields

$$\begin{aligned}\hat{P}(s|0) &= p_1(s|0)-[\hat{p}_1(s|0)]^2 + [\hat{p}_1(s|0)^3 \ldots \\ &= \frac{\hat{p}_1(s|0)}{1+\hat{p}_1(s|0)}\end{aligned} \tag{4.19}$$

Therefore, the probability density function for the recurrence time is

$$P(t|0) = \mathscr{L}^{-1}\left\{\frac{\hat{p}_1(s|0)}{1+\hat{p}_1(s|0)}\right\} \tag{4.20}$$

Where $\mathscr{L}^{-1}$ is the inversion operator for the Laplace transform operator, $\mathscr{L}$.

The recurrence time problem presented above is the most realistic for threshold detection since the physical problem provides an initial condition, namely that the threshold was exceeded at time, $t=0$. The threshold device is sensitive only to outputs greater than its setting; in other words, it is a 'yes-no' device. Consequently, problems requiring a knowledge of an initial condition depending on anything but the threshold setting are not practical from a physical standpoint. One could use for the initial condition, $X(0) = m_N$, where m_N is the mean thermal noise level, but this is not very realistic for there is little reason to believe that the output is exactly equal to m_N at the beginning of the operation. The other alternative is to use for an initial condition, $X(0) < x_o$, where x_o is the threshold setting. Then, the conditioning in (4.1) and (4.2) is that B_{t_o} is just $-E_{t_o}$. The statement of the problem, then, is, "what is the distribution in time to first exceeding of a threshold, knowing that the threshold was not exceeded at time, $t=0$?" The solution of the latter problem certainly does not give very precise information but this is not unexpected since the initial information is not very precise.

In each of the cases treated above for which there is sufficient information from the physical problem, $P(t)$ depends largely on p_1. Recall that the things available are the joint distribution functions for the amplitude of the output and the autocorrelation function of the output. From these p_1 must be developed. The next chapter deals with finding this dependence of p_1 on the amplitude distributions and the correlation function.

PROBABILITY DENSITIES OF TIME BETWEEN EVENTS

It has been noted that first passage time probability densities and recurrence densities depend on the hierarchy, $p_1, p_2 \ldots p_k \ldots$ It has also been pointed out that for cases of physical interest, the most important of these is p_1. To be sure, the elements of the hierarchy describe successively the output process in more detail. However, in the physical applications, there are very few examples where it is necessary to specify the output process in any more detail than that supplied by studying p_1. The present state of the art of threshold detection does not incorporate sufficient sensitivity in other parts of the system to warrant a more detailed specification of the output process. Therefore, we are justified in concentrating attention on p_1.

The probability density of the time between events was defined in Chapter IV. The first member of the hierarchy, p_1 is defined by the equation

(5.1) $p_1(t|0)dt = Pr\{X(s) \geq x_0, t < s < t+dt | X(0)\}$

= the probability that the output exceeds the threshold in $(t, t+dt)$, conditioned on some event at $t = 0$.

Here we have taken t_0 to be zero.

First, we examine the case where the condition is just that the threshold was exceeded at time $t = 0$. This situation finds application in the 'confirmation search' problem and reduces to a recurrence type problem. Then $p_1(t|0)$ is

$$p_1(t|0)dt = \Pr\left\{X(s) \geq x_0,\ t < s < t+dt \,|\, X(0) \geq x_0\right\}. \tag{5.2}$$

With this definition, construction of p_1, is quite straight-forward. Following the usual procedure for finding the density at a point, consider the difference quotient

$$\begin{aligned} &\frac{\Pr\left\{X(t) \geq x_0 | X(0) \geq x_0\right\} - \Pr\left\{X(t+\Delta t) \geq x_0 | X(0) \geq x_0\right\}}{\Delta t} \\ &= \frac{-\Pr\left\{X(t+\Delta t) \geq x_0 | X(0) \geq x_0\right\} - \Pr\left\{X(t) \geq x_0 | X(0) \geq x_0\right\}}{\Delta t}, \end{aligned} \tag{5.3}$$

where $X(s)$ is the value of the output voltage at time s. Taking the limit as $\Delta t \to 0$, we have

$$\begin{aligned} &\lim_{\Delta t \to 0} \left[\frac{-\Pr\left\{X(t+\Delta t) \geq x_0 | X(0) \geq x_0\right\} - \Pr\left\{X(t) \geq x_0 | X(0) \geq x_0\right\}}{\Delta t}\right] \\ &= -\frac{\partial}{\partial t}\left[\Pr\left\{X(t) \geq x_0 | X(0) \geq x_0\right\}\right] \end{aligned} \tag{5.4}$$

$$= -\frac{\partial}{\partial t}\left[\frac{\Pr\{X(t) \geq x_0,\ X(0) \geq x_0\}}{\Pr\{X(0) \geq x_0\}}\right]$$

$$= -\frac{\partial}{\partial t}\left[\frac{\int_{x_0}^{\infty}\int_{x_0}^{\infty} f_2[x_1, x_2, \psi(t)]\, dx_1 dx_2}{\int_{x_0}^{\infty} f_1(x_1)\, dx_1}\right],$$

where f_1 and f_2 are the marginal and joint probability density functions of the amplitude of the output defined in Chapter II and $\psi(t)$ is the correlation function of the output defined in Chapter III. The expression in (5.4) is a negative partial derivative because the first term on the right hand side of (5.3) is less than the second term since the conditional probability is a non-increasing function of time. Equation (5.4) is our expression for p_1. We shall need some properties of the parenthetical expression in (5.4).

First, consider the numerator. From (2.41) f_2 is

$$(5.5)\quad f_2[x_1, x_2, \psi_n(t)] = \frac{e^{-(x_1+x_2)/1-\psi_n^2}}{1-\psi_n^2}\ I_0\left[\frac{2\sqrt{x_1 x_2}}{1-\psi_n^2}\ \psi_n\right]$$

where ψ_n is the normalized autocorrelation function, i.e. $\psi_n(0)=1$. From the discussion in Chapter III, it is shown that for Markoffian type processes here considered, the normalized form of ψ is a non-increasing function of time and that $\psi_n(0) = 1$ and $\psi_n(\infty) = 0$. It is seen immediately that in the limit, as ψ_n goes to 0, (5.5) factors into $f_1(x_1)f_1(x_2)$ which assures us of the required independence as the time between examinations of the output becomes large. Furthermore, when $\psi_n = 1$, the examinations are simultaneous and since only one process is involved, $F_2[x_0,x_0,\psi_n(0)] = F_1(x_0)$. The denominator is not troublesome since it is merely $1-F_1(x_0)$. In the sequel we shall drop the subscript "n" in the normalized autocorrelation function, but it will be understood that ψ is normalized. Then the numerator may be written

$$\int_{x_0}^{\infty}\int_{x_0}^{\infty} f_2[x_1,x_2,\psi(t)]dx_1dx_2$$

$$= \int_{x_0}^{\infty}\left\{\int_0^{\infty} f_2[x_1,x_2,\psi(t)]dx_1 - \int_0^{x_0} f_2[x_1,x_2,\psi(t)]dx_1\right\}dx_2$$

(5.6)

$$= \int_0^{\infty}\int_0^{\infty} f_2[x_1,x_2,\psi(t)]dx_1dx_2 - \int_0^{\infty}\int_0^{x_0} f_2[x_1,x_2,\psi(t)]dx_1dx_2$$

$$- \int_0^{x_0}\int_0^{\infty} f_2[x_1,x_2,\psi(t)]dx_1dx_2 + \int_0^{x_0}\int_0^{x_0} f_2[x_1,x_2,\psi(t)]dx_1dx_2.$$

The first term of the last expression in (5.6) is obviously unity, by the definition of joint distribution functions. The second term is $-F_2(\infty, x_0)$, the third term is $-F_2(x_0, \infty)$ and the last term is $F_2[x_0, x_0, \psi(t)]$, also by the properties of distribution functions. But for the distribution defined by (5.5) in which x_1 and x_2 appear symmetrically,

$$(5.7) \qquad F_2(\infty, x_0) = F_2(x_0, \infty) = F_1(x_0).$$

Then (5.6) may be written as

$$(5.8) \qquad \int_{x_0}^{\infty}\int_{x_0}^{\infty} f_2[x_1, x_2, \psi(t)]dx_1 dx_2 = 1 - 2F_1(x_0) + F_2[x_0, x_0, \psi(t)].$$

Note that only the last term involves $\psi(t)$ so that $p_1(t|0)$ is, in terms of the distribution functions of the amplitude of the output,

$$(5.9) \qquad p_1(t|0) = -\frac{\partial}{\partial t}\left\{\frac{F_2[x_0, x_0, \psi(t)]}{1 - F_1(x_0)}\right\}.$$

Let

$$(5.10) \qquad \frac{F_2[x_0, x_0, \psi(t)]}{1 - F_1(x_0)} = G[x_0, \psi(t)].$$

Then (5.9) becomes

$$p_1(t|0) = -\frac{\partial}{\partial t} G[x_0, \psi(t)]$$

$$= -\frac{\partial G}{\partial \psi}\frac{\partial \psi}{\partial t}.$$

From Chapter III we have $\psi(t)$, the autocorrelation function of the output of a receiver with first order filters and a quadratic detector for an input consisting of an additive mixture of white Gaussian noise and a Rayleigh-modulated signal. It is

$$(5.12)\quad \psi(t) = CN_0^2\left[\frac{2\gamma e^{-\omega_2 t} - e^{-2\omega_1 t}}{4\gamma^2 - 1} + 2z\,\frac{\gamma e^{-\omega_2 t} - e^{-\omega_1 t}}{\gamma^2 - 1} + z^2\right]$$

where C is the normalizing constant, N_0 is the noise power admitted to the system, γ is the bandwidth ratio and z is the average signal to noise ratio at the IF stage. Differentiating (5.12),

$$(5.13)\qquad \frac{\partial \psi(t)}{\partial\ t} = CN_0^2\left[\frac{2\omega_1 e^{-2\omega_1 t} - 2\omega_2 \gamma e^{-\omega_2 t}}{4\gamma^2 - 1} + 2z\,\frac{\omega_1 e^{-\omega_1 t} - \omega_2 \gamma e^{-\omega_2 t}}{\gamma^2 - 1}\right].$$

Recall that the Laplace transform of the first passage time probability density for the recurrence type process is given by the ratio of terms in $\hat{p}_1(s|0)$, the Laplace transform of $p_1(t|0)$,

$$(5.14) \qquad \hat{P}(s|0) = \frac{\hat{p}_1(s|0)}{1+\hat{p}_1(s|0)} \,.$$

Writing G' for $\frac{\partial G}{\partial \psi}$ and $\hat{G}'$ for the Laplace transform of $\frac{\partial G}{\partial \psi}$, and substituting (5.11) and (5.13) into (5.14),

$$(5.15) \qquad \hat{P}(s|0) = \frac{K(s,\omega_1,\omega_2,\gamma,N_0)}{1+K(s,\omega_1,\omega_2,\gamma,N_0)}$$

$$K(s,\omega_1,\omega_2,\gamma,N_0) = N_0^2 \left\{ \frac{2\omega_2\gamma\hat{G}'(s+\omega_2) - 2\omega_1\hat{G}'(s+2\omega_1)}{4\gamma^2 - 1} \right.$$

$$\left. + 2z\,\frac{\omega_2\gamma\hat{G}'(s+\omega_2) - \omega_1\hat{G}'(s+\omega_1)}{\gamma^2 - 1} \right\}$$

This follows immediately from the fact that (5.13) contains only terms with exponential factors and the translation theorem for Laplace transforms. As has already been pointed out, a receiver used for threshold detection becomes more sensitive with large γ. The only restriction is that the bandwidth of the second filter must be compatible with the required system response time.

Practical values of γ are 100-1000. Since then $\gamma^2 >> 1$, some simplification of (5.15) is possible. Remembering also that $\gamma = \frac{\omega_1}{\omega_2}$, $K(s,\omega_1,\omega_2,\gamma,N_0)$ becomes approximately

$$K(s,\omega_1,\omega_2,\gamma,N_0)$$

(5.16)
$$= \frac{2N_0^2\omega_1}{\gamma^2}\left[\hat{G}'(s+\omega_2)(\tfrac{1}{4} + z) - \frac{\hat{G}'(s+2\omega_1)}{4} - z\,\hat{G}'(s+\omega_1)\right] .$$

This is still cumbersome but the number of terms in (5.15) has at least been reduced by one.

The preceding development has led to an expression for the Laplace transform of the first passage time probability density in terms of the Laplace transform of G'. Unfortunately, little more can be done without evaluating G' for a particular system characterized by the parameters, ω_1, ω_1, and x_0. Writing that part of the explicit form of G which depends on $\psi(t)$, we find that the Laplace transform of the following function must be found:

(5.17)
$$G'[\psi(t)] = \frac{\partial}{\partial\psi}\int_0^{x_0}\int_0^{x_0} \frac{e^{\frac{-x_1+x_2}{1-\psi^2}}}{1-\psi^2}\, I_0\left[\frac{2\sqrt{x_1x_2}\,\psi}{1-\psi^2}\right] dx_1\,dx_2$$

where the dependence on t is through $\psi(t)$ and it is given by (5.12) I_0 is the Bessel function with imaginary argument. Equation (5.17) can be evaluated numerically rather easily since the U. S. Bureau of Mines has published a table of the temperature distribution function for heat exchange between a fluid and a porous solid (1, p. 1) which tabulates the function

$$\varphi(x,y) = e^{-y} \int_0^x e^{-t} I_0[(2\sqrt{y\,t})]dt. \tag{5.18}$$

To see how the evaluation of (5.17) may be accomplished by use of these tables let

$$\begin{aligned} x_1 &= u_1(1 - \psi^2) \\ x_2 &= u_2(1 - \psi^2) \end{aligned} \tag{5.19}$$

in the integral of (5.17). Then the integral becomes

$$\begin{aligned} &\int_0^{x_0}\int_0^{x_0} \frac{e^{\frac{-x_1+x_2}{1-\psi^2}}}{1-\psi^2} \quad I_0\left[\frac{2\sqrt{x_1 x_2}}{1-\psi^2}\,\psi\right] dx_1\,dx_2 \\ &= \frac{(1-\psi^2)^2}{(1-\psi^2)} \int_0^{\frac{x_0}{1-\psi^2}} \int_0^{\frac{x_0}{1-\psi^2}} e^{-(u_1+u_2)}\, I_0\,[2\sqrt{u_1(u_2\psi^2)}]du_1\,du_2 \end{aligned} \tag{5.20}$$

Now we make the further substitution

$$(5.21) \qquad u_2\psi^2 = w$$

and drop the subscript on u_1. We denote the integral by I and substituting (5.21) into (5.20),

$$(5.22) \qquad I = \frac{1-\psi^2}{\psi^2} \int_0^{\frac{x_0}{1-\psi^2}} e^{-\frac{w}{\psi^2}} \int_0^{\frac{x_0\psi}{1-\psi^2}} e^{-u}\, I_0\,[2\sqrt{uw}]\, du\, dw$$

$$= \frac{1-\psi^2}{\psi^2} \int_0^{x_0/1-\psi^2} e^{-w\left(\frac{1-\psi^2}{\psi^2}\right)}\, \Theta\,[w, x_0, \psi(t)]\, dw$$

where

$$(5.23) \qquad \Theta\,[w, x_0, \psi(t)] = e^{-w} \int_0^{\frac{x_0\psi}{1-\psi^2}} e^{-u}\, I_0[2\sqrt{uw}]\, du.$$

Comparing (5.23) with (5.18), we see that

$$(5.24) \qquad \Theta\,[w, x_0, \psi(t)] = \varphi\left(\frac{x_0\psi}{1-\psi^2}\,,\, w\right).$$

Thus we see that evaluation of the double integral in (5.17) can be reduced to evaluating the single integral of (5.22). One factor of the integrand of (5.22) will be a tabular function which necessitates numerical calculation for $G[\psi(t)]$. Recall that the expression we seek is $\frac{\partial}{\partial\psi} G[\psi(t)]$. This, of course, requires further numerical calculation. Finally, we must calculate the Laplace transform of $\frac{\partial}{\partial\psi} G[\psi(t)]$,

$$(5.25) \qquad \hat{G}(s) = \int_0^{\infty} e^{-st} \frac{\partial}{\partial\psi} G[\psi(t)]\, dt,$$

This, too, must be done numerically. However, it is conceivable that by a curve-fitting procedure an approximating function may be found for $\frac{\partial}{\partial\psi} G[\psi(t)]$ whose Laplace transform can be found in closed form.

We have shown that, in principle $p_1(t|0)$ can be found for a reasonable physical model. The particular model with first order filters was chosen since it constitutes the most general realizable system. Extension to other filter types is immediate if the filter transfer functions can be written in closed form. Stone and Brock (13, pp. 40-57) have developed the system kernel, $g(u,v)$ for first order filters, second order filters, and Gaussian filters. The last type of filter is, of course, not realizable. With $g(u,v)$ the distribution functions of the output as well as the autocorrelation function of the output can be found quite easily.

The final form for P(t) can then be found simply by substituting $p_1(t|0)$ in the proper equation, depending on the type of output process that has been assumed and then multiplying by a suitable normalizing constant found by integration of P(t) over the range of t.

Finally, we consider briefly the most general case. It is well known that the output process may be described in more detail by specifying higher order amplitude distribution functions. Similarly, the first passage time probability density function may be specified in more detail with a knowledge of higher order p_k's. It is, of course, to be expected that the higher order p_k's will depend on the F_k's described in Chapter II and on the autocorrelation function of the output. The exact dependence of p_k on the hierarchy, $F_n, n = 1,2,\ldots$ will be through a mixed partial derivative with respect to the various times intervals. However, there seems to be no way of summing the series (4.2) in closed form.

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