AN ABSTRACT OF THE THESIS OF

Jack LeRoy Winchester for the M.S. in Civil Engineering
(Name (Degree) (Major)

Date thesis is presented May 5, 1966

Title PILE GROUP ANALYSIS CONSIDERING RIGID AND SEMIRIGID FOUNDATION CONDITIONS

Abstract approved [redacted]
(Major professor)

The structural behavior of pile groups (dolphins) is analyzed considering two types of boundary conditions. First the individual piles of the group are assumed to be rigidly embedded in the soil. In the second analysis the more realistic case is considered where the soil foundation is allowed to yield. The redistribution of forces caused by various degrees of rotational yielding of the embedded pile lengths is shown. The importance of allowing for some amount of yielding in the analysis of such a structure is thus illustrated.

The matrix analysis method of analyzing structures is utilized for both the rigid and semirigid cases. In addition, the method of sections is employed on the semirigid case. Each case is then programmed for solution on a high speed digital computer.

PILE GROUP ANALYSIS CONSIDERING RIGID AND SEMIRIGID FOUNDATION CONDITIONS

by

JACK LEROY WINCHESTER

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1966

ACKNOWLEDGMENT

The writer wishes to express sincere gratitude to all of his past and present instructors in the Department of Civil Engineering at Oregon State University. Particular recognition is given Dr. Harold I. Laursen whose guidance and encouragement during the writing of this thesis was invaluable.

APPROVED:

[redacted]

Associate Professor of Civil Engineering

In Charge of Major

[redacted]

Head of Department of Civil Engineering

[redacted]

Dean of Graduate School

Date thesis is presented May 5, 1966

Typed by Gail Dailey

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

PILE GROUP ANALYSIS CONSIDERING RIGID AND SEMIRIGID FOUNDATION CONDITIONS

INTRODUCTION

Pile groups or dolphins are relatively common waterfront structures utilized in the mooring and maneuvering of vessels in harbor areas. The structure itself consists of a varying number of closely spaced piles driven into the harbor bottom and rigidly connected above the water line by bolts, wire, or encasement in a concrete block as shown in Figure 1. There are two general types of loading conditions which the dolphin may be subjected to. First there is the steady pull of a moored ship, and secondly, the impact load associated with the collision of a maneuvering ship.

In general the mode of failure of the pile group is a plastic movement of the soil about individual piles. The piles simultaneously rotate and withdraw from the soil bed causing the entire structure to overturn. Because of the nature of the soil foundation medium it is apparent that any loading condition producing stresses in the piles will also cause yielding to some degree in the soil. It is also logical to assume that yielding of the soil medium will produce a redistribution of forces in the structure itself. This thesis will attempt to illustrate the effect on member forces and displacements by varying the degree of yielding, i.e the stiffness of the pile-soil connection.

It is readily apparent that the pile dolphin is a highly redundant structure. For this reason classical work and energy methods or iteration processes of analysis are impractical due to their complexity. This is especially true for the case where yielding of the soil is considered. Consequently in the past it has been the usual practice for design engineers to rely on simplified or empirical formulas as well as experience. However, with the advent of high speed digital computers the matrix analysis method of solution has become not only practical but highly convenient for problems of this nature. Therefore matrix analysis will be used in this study and all solutions will be arrived at through the use of a digital computer.

This thesis will be concerned primarily with the structural aspects of pile groups with rigid and semirigid foundations. To make the study more meaningful it is of interest to consider the actual mechanism through which this yielding occurs in the soil. The literature review will be devoted largely to the task of summarizing what has been done in this area. The remaining portion of the study will be divided into two parts. In the first part the pile group is analyzed considering the piles to be rigidly fixed in the soil foundation. In the second part the piles are allowed to yield in the soil as previously mentioned.

Because of the limited storage space in the computer used for this study and the computer time available, the structure used as an example in this thesis is made up of only four round piles subjected to one general loading condition. The semirigid analysis is carried out for only variations in the rotational stiffness at the pile bottoms. It is important to note, however, that the same analysis could be applied to any number of piles with various loads and various foundation stiffnesses. In essence the purpose of this thesis is then twofold. First the influence of soil yielding on the forces in a pile dolphin will be shown, and secondly, a method in which this highly indeterminate structure can be solved through matrix analysis is illustrated.

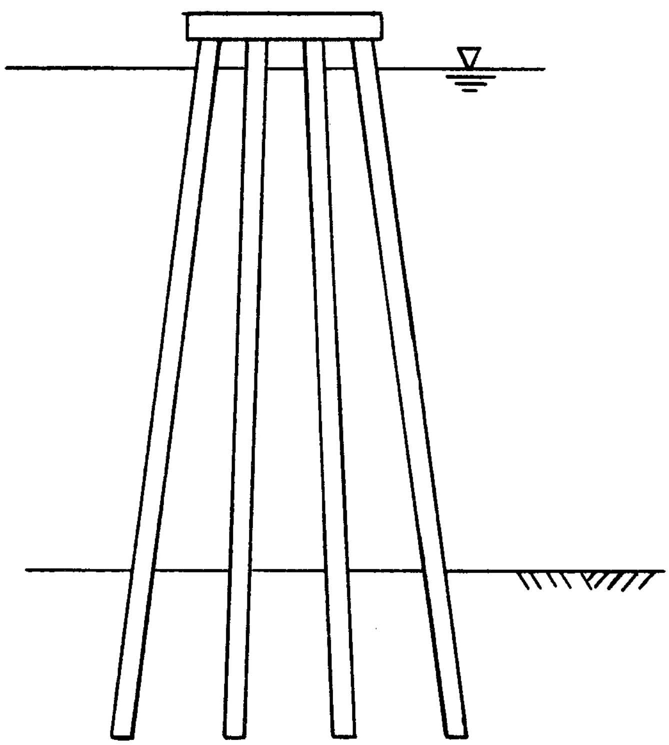

Figure 1. Typical pile group

REVIEW OF LITERATURE

The behavior of a pile embedded in an elastic soil medium and loaded horizontally above the ground level has long been a problem of interest in the field of soil mechanics. There are numerous cases appearing in literature where the problem is treated theoretically, by field testing, or by both methods. Perhaps the earliest recorded work was a field study conducted by Sandeman (2) in 1880. His study was concerned mainly with relating the ultimate load a pile could sustain to the depth of embedment.

The most significant early theoretical studies were performed by Seiler (9) and Drucker (3). Both of these investigators made attempts to describe the pressure distribution below ground level on a rigid vertical pile loaded above the ground level. From this pressure distribution Seiler found where the center of rotation of the embedded pile length was located as well as the resultants of the passive and active earth pressure forces applied to the pile. According to Seiler the location of the center of rotational yielding is at a distance 0.676 of the total embedded length from the ground surface. The latter was verified at a later date by Shilts, Graves, and Driscoll (10) in a series of actual field tests, and even later by Davisson and Prakash (2) in a more advanced theoretical study.

In a laboratory model study Williams (12) actually determined the portion of soil affected by the plastic yielding resulting when a pile overturns. His study showed definite shear failure planes on both the active and passive sides of the embedded pile.

It was evident to investigators that certain soil parameters are primarily responsible for governing the behavior of the pile-soil system. To illustrate the importance of soil properties Terzaghi (11) conducted a study on the reaction of cohesive and non-cohesive soils to similarly loaded piles. Because of its effect on the soil properties the method of driving a pile is also an important consideration. To illustrate this Meyerhoff (7) contrasted the vertical stiffness, or bearing capacity, of piles that were driven and ones that were buried. Chellis (1, p. 210) indicated that remolding of the soil surrounding piles by cyclic loading conditions could have a significant effect on the stiffness of the foundation.

To date perhaps the best analytical representation of soil-pile behavior is that by Davisson and Prakash (2). A subgrade modulus was defined which relates soil reaction to deflected position. The value of the modulus is both a function of the soil properties and of the depth along the embedded length. Assuming linear elastic soil behavior the authors developed equations from which the shear and moment along the embedded length as well as the rotation point could be calculated. It was found that the results depend additionally on one other soil parameter which varies between 0.15 for a cohesive soil and 1.00 for a granular soil. The effect of this factor is, however, small enough that a good determination of the rotation point can still be made. In virtually all theoretical analyses, including the latter, the stiffness of the pile is assumed to be so large in comparison to that of the soil stiffness that it may be considered rigid over the embedded length.

As far as practical design considerations are concerned Chellis (1, p. 151) has presented one of the most complete treatments of the problem. He illustrates empirical methods for evaluating the forces due to ship impact, waves, and current and relates them to the structure itself. Dunham (4) also presents a semi-empirical approach for design and analysis of pile groups acted on by horizontal forces. Elms and Schmid (5) were the only investigators found who dealt with the structure by using matrix analysis with the aid of a digital computer.

METHOD OF ANALYSIS

In the ensuing discussion it will be assumed that the reader is familiar with matrix algebra and also the basic concepts of matrix analysis of structures. The displacement method of analysis is used in this study because it is especially convenient for examining structures with a large number of members and relatively few nodal points (joints). The actual analysis is based on the method of virtual work and follows closely the presentation of Laursen (6, p. 52). In order to provide a common plane of reference for the forthcoming applications of the displacement method a short summary of the method will be presented here.

The Displacement Method of Matrix Analysis

The basic principle of the method of virtual work is that the work done by the external forces on a structure is equal to the work done by the internal member forces. The requirements of equilibrium and compatibility must also be satisfied. In the displacement method of matrix analysis the unknown external displacements are related to external loads by a stiffness matrix analagous to the manner in which the deformation of an elementary spring is related to its stiffness or spring constant. In matrix form this relationship is written

$$\underset{\sim}{Q} = \underset{\sim}{K}\underset{\sim}{D} \qquad (1)$$

where $\underset{\sim}{D}$ is a column matrix representing the external displacements, $\underset{\sim}{Q}$ is a column matrix representing the external loads, and $\underset{\sim}{K}$ is the stiffness of the structure. Symbols underlined with a wavy line denote matrices. The more often desired relationship of external loads and external displacements is

$$\underset{\sim}{D} = \underset{\sim}{F}\underset{\sim}{Q} \tag{2}$$

where

$$\underset{\sim}{F} = \underset{\sim}{K}^{-1} \tag{3}$$

The matrix $\underset{\sim}{F}$, defined as the flexibility of the structure, is equal to the inverse of the stiffness matrix $\underset{\sim}{K}$. The stiffness of a structure can furthermore by written

$$\underset{\sim}{K} = \underset{\sim}{A}'\underset{\sim}{k}\underset{\sim}{A} \tag{4}$$

The matrices $\underset{\sim}{k}$ and $\underset{\sim}{A}$ are obtained from the following relationships

$$\underset{\sim}{q} = \underset{\sim}{k}\underset{\sim}{d} \tag{5}$$

and

$$\underset{\sim}{d} = \underset{\sim}{A}\underset{\sim}{D} \tag{6}$$

The matrix $\underset{\sim}{k}$ relates the internal member forces $\underset{\sim}{q}$ to the internal displacements $\underset{\sim}{d}$ similar to the relationship of $\underset{\sim}{K}$ in Equation 1. The displacement transformation matrix $\underset{\sim}{A}$ relates the external structure displacements $\underset{\sim}{D}$ to the internal member displacements $\underset{\sim}{d}$. By combining Equations 2, 5, and 6 the expression for internal member forces can be written

$$\underset{\sim}{q} = \underset{\sim}{k}\underset{\sim}{A}\underset{\sim}{D}$$

or

$$\underset{\sim}{q} = \underset{\sim}{kAFQ} \qquad (7)$$

By using Equations 1 through 7 a structure can be solved by the displacement method. The primary task involved is that of formulating the $\underset{\sim}{A}$ and $\underset{\sim}{k}$ matrices and this problem will be discussed at some length in connection with the actual pile group analysis.

It is appropriate at this time to define the sign convention and notation which will be used in the following discussion. To illustrate the sign convention for the case of internal forces and displacements the reader is referred to Figure 2(a). In Figure 2(a) the rotations, moments, axial force, and torque are all positive as represented. The double arrow representing torque is applied according to the right hand screw rule. Each member is given a number n and the ends of the member are designated by i and j. In referring to internal forces and displacements the member number appears as the subscript of the quantity under consideration. The superscripts, on the other hand, are comprised of first the member end, i or j, followed by the symbol for the axis about which rotation occurs. Not shown are the internal axial and torsional displacements which are represented by e_n and ψ_n respectively. These are positive when resulting from positive forces.

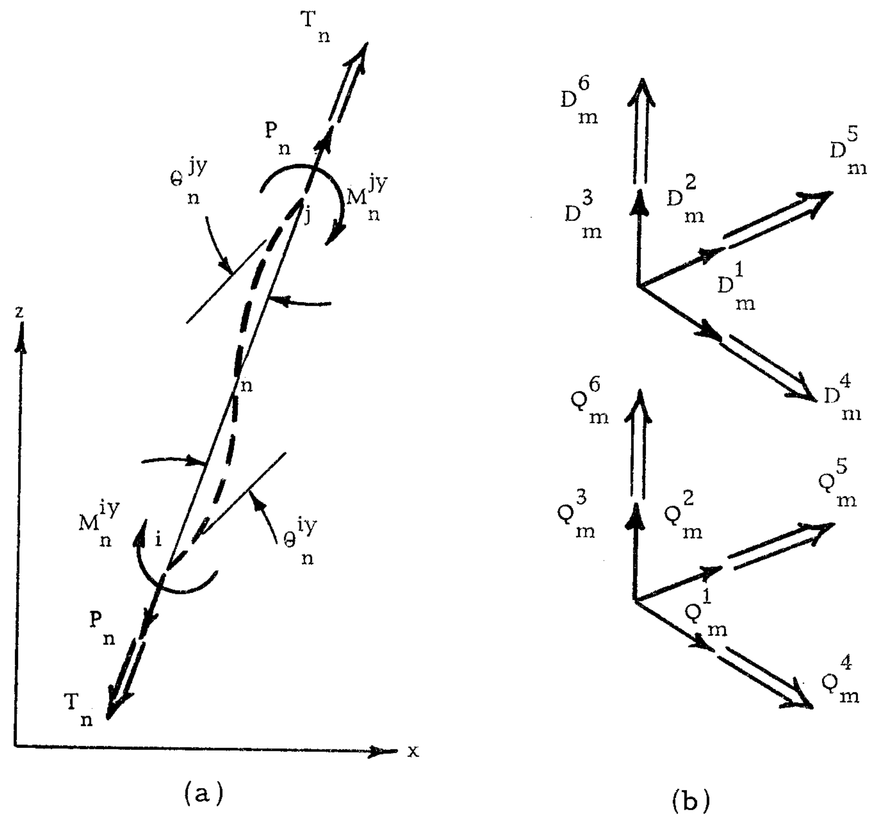

Figure 2. Sign convention and notation

The notation for external loads and displacements is shown in Figure 2(b). The loads and displacements as shown are considered positive. Again the double arrow implies rotation or moment in accordance with the right hand screw rule. For external load and displacement quantities the superscript indicates the direction as determined by the framework coordinate system, while the subscripts denote the nodal point under consideration.

Development of the Rigid Analysis

The first analysis performed is for the case of piles rigidly fixed in the soil foundation. The structure considered is shown in Figure 3. Though only four members were used here for simplicity, it will soon become evident that any number of members could have been analyzed.

The first step is to develop the relationship between the individual pile displacements and the pile forces. These are related by the pile stiffness $\underset{\sim}{k}_n$ where

$$\begin{bmatrix} M_n^{ix} \\ M_n^{jx} \\ M_n^{iy} \\ M_n^{jy} \\ P_n \\ T_n \end{bmatrix} = \underset{\sim}{k}_n \begin{bmatrix} \theta_n^{ix} \\ \theta_n^{jx} \\ \theta_n^{iy} \\ \theta_n^{jy} \\ e_n \\ \psi_n \end{bmatrix} \tag{8}$$

Beginning with the basic differential equation describing bending in a beam it can be shown (6) that the general relationship between moment and rotation is

$$\begin{bmatrix} M^i \\ M^j \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \theta^i \\ \theta^j \end{bmatrix}$$

where the stiffness of the $n^{\underline{th}}$ member due to bending is therefore represented as

$$\underset{\sim}{k}_n = \frac{EI_n}{L_n} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \tag{9}$$

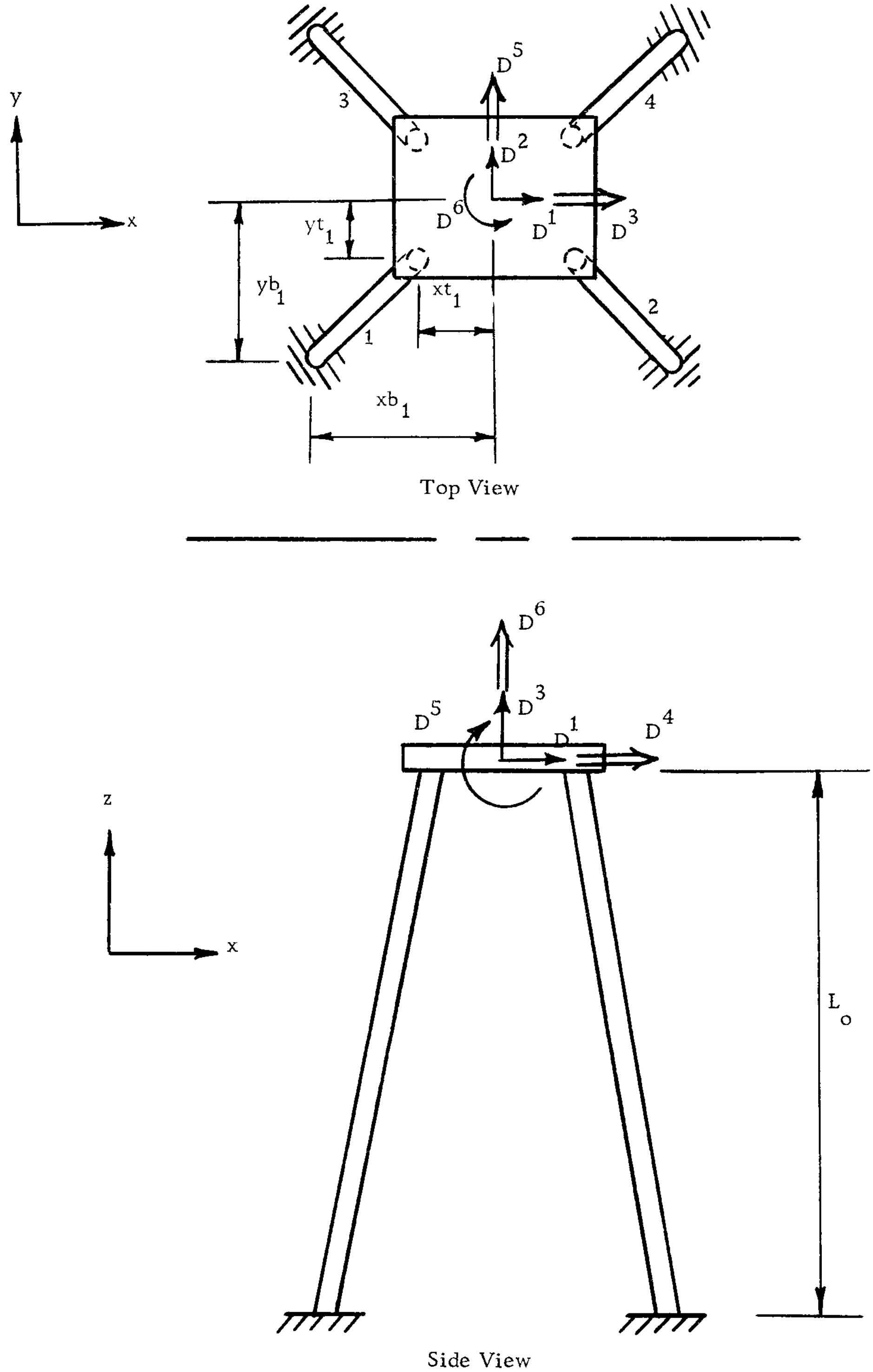

Figure 3. Pile group and notation used in the development

To maintain dimensionless quantities in the matrix operations it is convenient to express the stiffness of the member as a multiple of $\frac{EI_0}{L_0}$. I_0 is a reference moment of inertia and L_0 is a reference length. For the $n^{\underline{th}}$ member Equation 9 is therefore written

$$\underset{\sim}{k}_n = \frac{EI_0}{L_0}\left(\frac{L_0 I_n}{L_n I_0}\right)\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \tag{10}$$

where the dimensionless quantity in parenthesis is evaluated for member n and then combined with the matrix.

For axial deformation the general equation is

$$P = \frac{AE}{L} e \tag{11}$$

The stiffness due to axial load for the $n^{\underline{th}}$ member is therefore

$$\underset{\sim}{k}_n = \frac{A_n E}{L_n} \tag{12}$$

In order to maintain dimensional consistency in the $\underset{\sim}{q}$ and $\underset{\sim}{d}$ matrices Equation 11 is written

$$PL_0 = \frac{AEL_0^2}{L}\,\frac{e}{L_0} \tag{13}$$

To work in terms of the common premultiplier $\frac{EI_0}{L_0}$ Equation 12 is altered to give

$$\underset{\sim}{k}_n = \frac{EI_0}{L_0}\left(\frac{L_0^3 A_n}{I_0 L_n}\right) \tag{14}$$

The general relationship for twist in a member is written

$$T = \frac{JG}{L}\,\psi \tag{15}$$

where

$$\underset{\sim}{k}_n = \frac{J_n G}{L_n} \tag{16}$$

Again for dimensional consistency the torsional stiffness of Equation 16 is expressed as

$$\underset{\sim}{k}_n = \frac{EI_0}{L_0}\left(\frac{L_0 J_n G}{L_n I_0 E}\right) \tag{17}$$

By combining the various member stiffnesses as expressed in Equations 9, 14, and 17 the total member stiffness of the $n^{\underline{th}}$ member for use in Equation 8 is

$$\begin{bmatrix} M_n^{ix} \\ M_n^{jx} \\ M_n^{iy} \\ M_n^{jy} \\ P_n L_0 \\ T_n \end{bmatrix} = \frac{EI_0}{L_0}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{L_0^3 A_n}{I_0 L_n} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{L_0 J_n G}{L_n I_0 E} \end{bmatrix}\begin{bmatrix} \theta_n^{ix} \\ \theta_n^{jx} \\ \theta_n^{iy} \\ \theta_n^{jy} \\ e_n/L_0 \\ \psi_n \end{bmatrix} \tag{18}$$

The stiffness matrix $\underset{\sim}{k}$ relating all member displacements to member forces is found by combining the individual member stiffnesses in the form

$$\underset{\sim}{k} = \frac{EI_0}{L_0}\begin{bmatrix} \underset{\sim}{k}_1 & 0 & 0 & 0 \\ 0 & \underset{\sim}{k}_2 & 0 & 0 \\ 0 & 0 & \underset{\sim}{k}_3 & 0 \\ 0 & 0 & 0 & \underset{\sim}{k}_4 \end{bmatrix} \tag{19}$$

The concrete block which forms the pile group head is assumed to be rigid. It will therefore undergo displacements with no internal deformations.

The next step in the analysis is the formation of the displacement transformation matrix $\underset{\sim}{A}$. The $\underset{\sim}{A}$ matrix for the $n^{\underline{th}}$ member relates external and internal displacements in the form

$$\begin{bmatrix} \theta_n^{ix} \\ \theta_n^{jx} \\ \theta_n^{iy} \\ \theta_n^{jy} \\ e_n/L_0 \\ \psi_n \end{bmatrix} = \underset{\sim}{A}_n \begin{bmatrix} D^1/L_0 \\ D^2/L_0 \\ D^3/L_0 \\ D^4 \\ D^5 \\ D^6 \end{bmatrix} \tag{20}$$

Because of the nature of the pile group $\underset{\sim}{A}_n$ can be obtained by a matrix transformation process (8, p. 172). A typical pile is isolated and a three-dimensional Cartesian reference system is attached as indicated in Figure 4. It will be shown that displacements in the pile coordinate system (y_1, y_2, y_3) can be related to displacements in a general framework coordinate system (x_1, x_2, x_3) by a transformation matrix $\underset{\sim}{T}$

$$\underset{\sim}{y} = \underset{\sim}{T}\underset{\sim}{x} \qquad (21)$$

By correctly interpreting the elements of the transformation they may be used directly in the $\underset{\sim}{A}$ matrix.

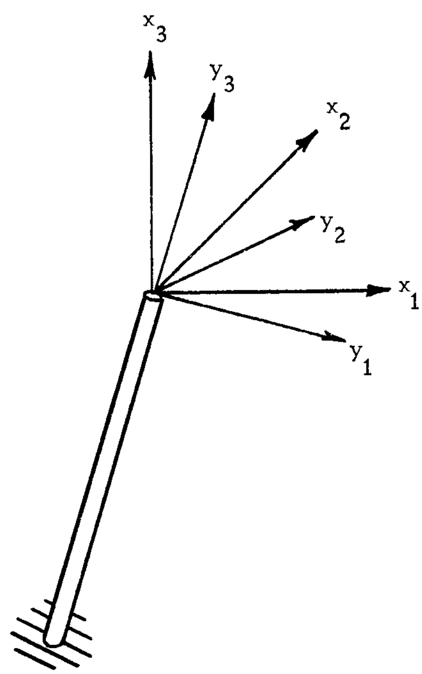

Figure 4. Coordinate transformation for typical pile

In order to develop the $\underset{\sim}{T}$ matrix the pile reference frame is first rotated about the x_1 axis an angle ϕ_1 yielding the relationship

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (22)$$

By similarly rotating the pile reference frame about the x_2 and x_3 axes Equations 23 and 24 are found to be

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (23)$$

and

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (24)$$

The total effect of three simultaneous rotations is equal to the product of the transformations in Equations 22, 23, and 24. This relationship is the desired matrix $\underset{\sim}{T}$ and can be written

$$\underset{\sim}{T} = \begin{bmatrix} C_2C_3 & C_2S_3 & -S_2 \\ S_1S_2C_3-C_1S_3 & C_1C_3+S_1S_2S_3 & S_1C_2 \\ S_1S_3+C_1S_2C_3 & C_1S_2S_3-S_1C_3 & C_1C_2 \end{bmatrix} \qquad (25)$$

where

$$C_i = \cos\phi_i \quad \text{and} \quad S_i = \sin\phi_i$$

To illustrate how $\underset{\sim}{T}$ can be used to develop $\underset{\sim}{A}$ consider the orientation of a typical pile to be described by the angles ϕ_1 and ϕ_2. Because the piles considered herein are nearly vertical the angles ϕ_1 and ϕ_2 are small. It is assumed therefore that $C_1 = C_2 = 1.0$. The values of S_1 and S_2 are also considered negligible except when they transform quantities of axial deformation y_3. This is due to the fact that the axial stiffness is much greater than that in bending. In addition, $C_3 = 1.0$ and $S_3 = 0.0$ because ϕ_3 is equal to zero.

By substituting the values of trigonometric functions into Equation 25 the final transformation $\underset{\sim}{T}$ can be written

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (xt_n - xb_n)/L_n & (yt_n - yb_n)/L_n & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (26)$$

where xt_n and yt_n are the coordinates of the top of the n^{th} pile and xb_n and yb_n are the coordinates of its bottom. All coordinates are taken with respect to the center of gravity of the pile tops in the horizontal plane. From Equation 26 the first three columns of the $\underset{\sim}{A}$ matrix can be directly written.

The last three columns of the $\underset{\sim}{A}$ matrix relating D^4, D^5 and D^6 to the internal displacements can be determined in the same manner. From compatibility requirements of the dolphin head these three external displacements are first transferred to the top ends of the pile members. For example if D^5 is applied to the structure a vertical displacement occurs at the pile top as shown in Figure 5. When D^4 is applied there will be a vertical displacement associated with the pile tops similar to the case of D^5.

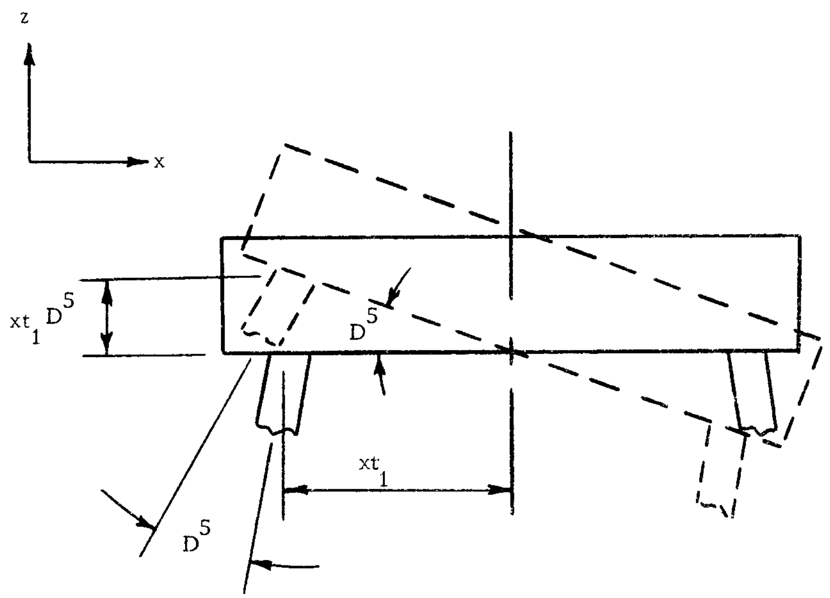

Figure 5. Rotation of pile head about y axis

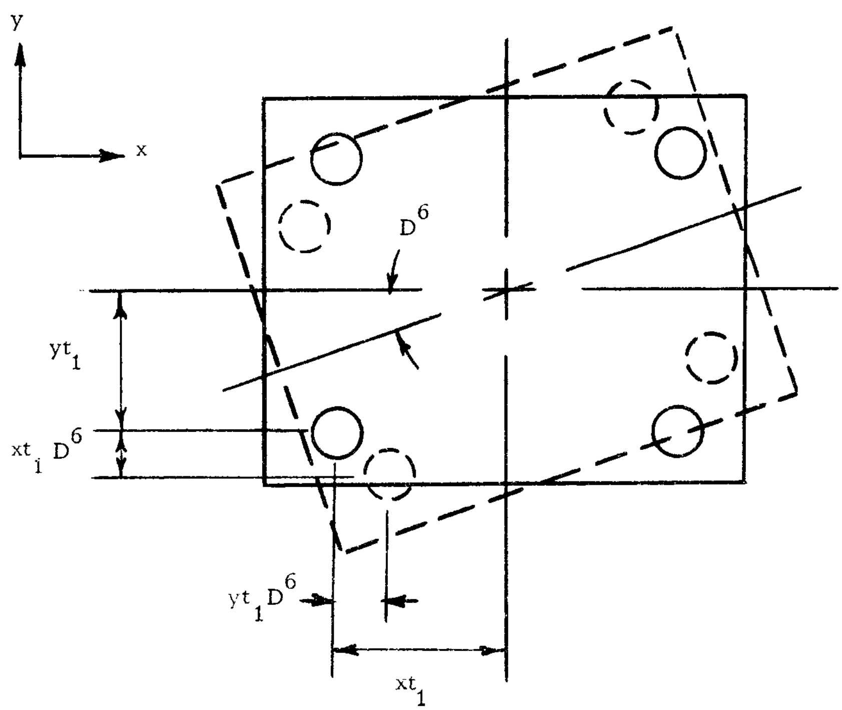

Figure 6. Rotation of pile head about z axis

The application of D^6 to the structure will cause two significant components of horizontal displacement at the pile top as indicated by Figure 6. Now that D^4, D^5, and D^6 have been transformed into displacements at the pile tops the last three columns of the $\underset{\sim}{A}$ matrix can be determined by using the $\underset{\sim}{T}$ matrix as before. If this is done the resulting $\underset{\sim}{A}$ matrix for a typical member with the proper sign convention can be written

$$\begin{bmatrix} \theta_n^{ix} \\ \theta_n^{jx} \\ \theta_n^{iy} \\ \theta_n^{jy} \\ e_n/L_0 \\ \psi_n \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \frac{-xt_n}{L_n} \\ 0 & -1 & 0 & -1 & 0 & \frac{-xt_n}{L_n} \\ -1 & 0 & 0 & 0 & 0 & \frac{yt_n}{L_n} \\ -1 & 0 & 0 & 0 & 1 & \frac{yt_n}{L_n} \\ \frac{xt_n-xb_n}{L_n} & \frac{yt_n-yb_n}{L_n} & 1 & \frac{yt_n}{L_n} & \frac{-xt_n}{L_n} & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D^1/L_0 \\ D^2/L_0 \\ D^3/L_0 \\ D^4 \\ D^5 \\ D^6 \end{bmatrix} \quad (27)$$

where

$$\alpha = \frac{(xt_n - xb_n)(yt_n) + (yb_n - yt_n)(xt_n)}{L_n^2}$$

For the total structure this relationship can be more concisely written as

$$\begin{bmatrix} \underset{\sim}{d}_1 \\ \underset{\sim}{d}_2 \\ \underset{\sim}{d}_3 \\ \underset{\sim}{d}_4 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}_1 \\ \underset{\sim}{A}_2 \\ \underset{\sim}{A}_3 \\ \underset{\sim}{A}_4 \end{bmatrix} \begin{bmatrix} D^1/L_0 \\ D^2/L_0 \\ D^3/L_0 \\ D^4 \\ D^5 \\ D^6 \end{bmatrix} \tag{28}$$

or

$$\underset{\sim}{d} = \underset{\sim}{A}\underset{\sim}{D} \tag{5}$$

Knowing the $\underset{\sim}{A}$ and $\underset{\sim}{k}$ matrices it is now a matter of simple matrix algebra to obtain the desired solution. The flexibility of the structure is obtained from Equations 3 and 4. The external displacements and member forces are determined according to Equations 2 and 7.

Development of the Semirigid Analysis

The second phase of the study is concerned with the same general structure but in this instance the soil foundation is allowed to yield about the individual piles. This analysis therefore attempts to more closely describe the actual behavior of a pile in the field. The method for including the semirigid condition is to introduce spring connections from the pile base to the subgrade as shown in Figure 7. In order to account for the six degrees of freedom present in a three-dimensional system a total of six springs are required at the base of each pile. For the three axial springs extension is considered to be positive, while an unwinding of the torsional springs is considered positive. The notation used in describing the springs is illustrated in Figure 7.

The addition of the springs increases the total number of members in the structure under consideration from four to twenty-eight. As a result, an analysis by the procedure used in the rigid case becomes awkward. The operations involved would remain unchanged but the sizes of the matrices would be greatly increased. For example the $\underset{\sim}{k}$ matrix for the structure would now be 48 by 48 while the $\underset{\sim}{A}$ matrix would be 48 by 12. In order to avoid calculations involving matrices of this size the method of sections is used to perform the analysis. In a pile group with a great number of members this approach would be a virtual necessity.

In applying the method of sections to this case each pile along with the six spring connections at its base is considered to be a separate structural section. In this manner the structure is divided into four sections plus the concrete head which is assumed to be rigid.

The internal stiffness matrix $\underset{\sim}{k}$ for each section considering the spring stiffnesses as well as the pile stiffness expressed in Equation 18 can be written

$$\begin{bmatrix} M_n^{ix} \\ M_n^{ix} \\ M_n^{iy} \\ M_n^{jy} \\ P_n L_0 \\ T_n \\ F_n^x L_0 \\ F_n^y L_0 \\ F_n^z L_0 \\ F_n^{xx} \\ F_n^{yy} \\ F_n^{zz} \end{bmatrix} = \frac{EI_0}{L_0} \begin{bmatrix} 4 & 2 & & & & & & & & & & \\ 2 & 4 & & & & & & & & & & \\ & & 4 & 2 & & & & & & & & \\ & & 2 & 4 & & & & & & & & \\ & & & & \alpha & & & & & & & \\ & & & & & \beta & & & & & & \\ & & & & & & k_1 & & & & & \\ & & & & & & & k_2 & & & & \\ & & & & & & & & k_3 & & & \\ & & & & & & & & & k_4 & & \\ & & & & & & & & & & k_5 & \\ & & & & & & & & & & & k_6 \end{bmatrix} \begin{bmatrix} \theta_n^{ix} \\ \theta_n^{jx} \\ \theta_n^{iy} \\ \theta_n^{jy} \\ e_n/L_0 \\ \psi_n \\ X_n^x/L_0 \\ X_n^y/L_0 \\ X_n^z/L_0 \\ X_n^{xx} \\ X_n^{yy} \\ X_n^{zz} \end{bmatrix} \tag{29}$$

where

$$\alpha = \frac{L_0^3 A_n}{I_0 L_n} \quad \text{and} \quad \beta = \frac{L_0 J_n G}{L_n I_0 E}$$

and

$$k_1 = \frac{k_x L_0^3}{EI_0} \qquad k_4 = \frac{k_{xx} L_0}{EI_0}$$

$$k_2 = \frac{k_y L_0^3}{EI_0} \qquad k_5 = \frac{k_{yy} L_0}{EI_0}$$

$$k_3 = \frac{k_z L_0^3}{EI_0} \qquad k_6 = \frac{k_{zz} L_0}{EI_0}$$

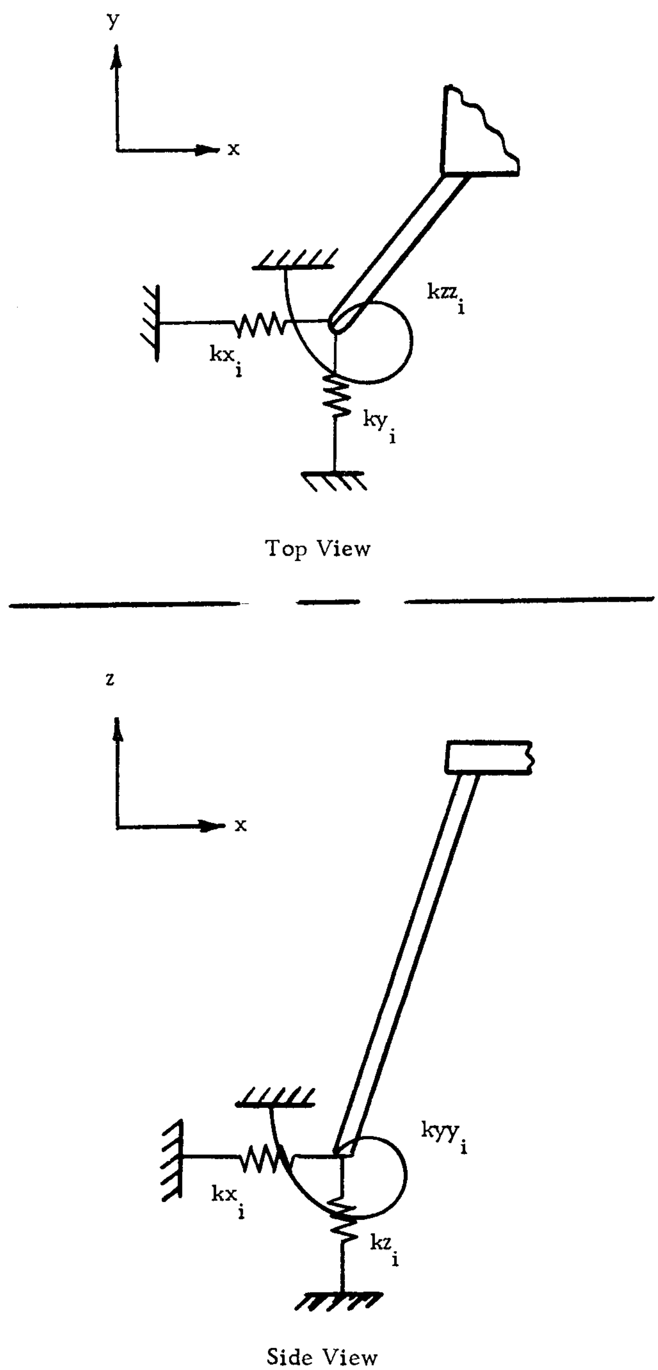

Figure 7. Semirigid condition at pile base

The transformation matrix $\underset{\sim}{A}_n$ for the $n^{\underline{th}}$ section can be expressed in the general form

$$\begin{bmatrix} \theta_n^{ix} \\ \theta_n^{jx} \\ \theta_n^{iy} \\ \theta_n^{jy} \\ e_n/L_0 \\ \psi_n \\ \hdashline X_n^x/L_0 \\ X_n^y/L_0 \\ X_n^z/L_0 \\ X_n^{xx} \\ X_n^{yy} \\ X_n^{zz} \end{bmatrix} = \underset{\sim}{A}_n \begin{bmatrix} D_n^1/L_0 \\ D_n^2/L_0 \\ D_n^3/L_0 \\ D_n^4 \\ D_n^5 \\ D_n^6 \\ \hdashline U_n^x/L_0 \\ U_n^y/L_0 \\ U_n^z/L_0 \\ U_n^{xx} \\ U_n^{yy} \\ U_n^{zz} \end{bmatrix} \tag{30}$$

Because there are no external loads associated with the external displacements $\underset{\sim}{U}$ the $\underset{\sim}{A}_n$ matrix can be partitioned with respect to $\underset{\sim}{D}$ and $\underset{\sim}{U}$ such that Equation 30 can be written

$$\underset{\sim}{d} = \left[\underset{\sim}{A}_1 \;\vdots\; \underset{\sim}{A}_2\right] \begin{bmatrix} \underset{\sim}{D} \\ \hdashline \underset{\sim}{U} \end{bmatrix} \tag{31}$$

The reason for partitioning the equation in this manner will soon become evident. The $\underset{\sim}{A}_1$ matrix can be obtained in the same manner as was the $\underset{\sim}{A}$ matrix for the rigid analysis. From Equation 27 we therefore have

$$\underset{\sim}{A}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \frac{-xt_n}{L_n} \\ 0 & -1 & 0 & -1 & 0 & \frac{-xt_n}{L_n} \\ -1 & 0 & 0 & 0 & 0 & \frac{yt_n}{L_n} \\ -1 & 0 & 0 & 0 & -1 & \frac{yt_n}{L_n} \\ \frac{xt_n - xb_n}{L_n} & \frac{yt_n - yb_n}{L_n} & 1 & \frac{yt_n}{L_n} & \frac{-xt_n}{L_n} & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (32)$$

where

$$\alpha = \frac{(xt_n - xb_n)(yt_n) + (yb_n - yt_n)(xt_n)}{L_n^{\,2}}$$

The $\underset{\sim}{A}_2$ matrix which relates the $\underset{\sim}{U}$ external displacements to internal displacements is described by Equation (33).

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (33)$$

By performing the appropriate matrix operations with Equations 29, 32, and 33 a reduced K matrix can be obtained for the section. The procedure (6, p. 73) is summarized by the following set of equations

$$C_{21} = A_2' k A_1 \qquad (34)$$

$$C_{22} = A_2' k A_2 \qquad (35)$$

$$\bar{A}_n = A_1 - A_2 C_{22}^{-1} C_{21} \qquad (36)$$

$$K_n = A_1 k \bar{A}_n \qquad (37)$$

By utilizing this procedure the size of the K matrix for this particular problem is reduced from a 12 by 12 to a 6 by 6. Thus the work involved in inverting the $\underset{\sim}{K}$ matrix is greatly decreased. It is important to remember that the reason this reduction could be made is that there are no external forces associated with the displacements $\underset{\sim}{U}$.

The next step in the analysis is to combine the different sections into one structure. For the combined structure a transformation relating external structure displacements to external section displacements is expressed in Equation 38.

$$\begin{bmatrix} D_I^1/L_0 \\ D_I^2/L_0 \\ D_I^3/L_0 \\ D_I^4 \\ D_I^5 \\ D_I^6 \\ D_{II}^1/L_0 \\ D_{II}^2/L_0 \\ D_{II}^3/L_0 \\ D_{II}^4 \\ D_{II}^5 \\ D_{II}^6 \\ D_{III}^1/L_0 \\ D_{III}^2/L_0 \\ D_{III}^3/L_0 \\ D_{III}^4 \\ D_{III}^5 \\ D_{III}^6 \\ D_{IV}^1/L_0 \\ D_{IV}^2/L_0 \\ D_{IV}^3/L_0 \\ D_{IV}^4 \\ D_{IV}^5 \\ D_{IV}^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D^1/L_0 \\ D^2/L_0 \\ D^3/L_0 \\ D^4 \\ D^5 \\ D^6 \end{bmatrix} \qquad (38)$$

The $\underset{\sim}{k}$ matrix for the total structure can be written

$$\underset{\sim}{k}_T = \begin{bmatrix} \underset{\sim}{K}_I & & & \\ & \underset{\sim}{K}_{II} & & \\ & & \underset{\sim}{K}_{III} & \\ & & & \underset{\sim}{K}_{IV} \end{bmatrix} \quad (39)$$

This is analogous to the form of $\underset{\sim}{K}$ for a structure of individual members. The flexibility of the total structure is then determined by

$$\underset{\sim}{K}_T = \underset{\sim}{A}'_T \underset{\sim}{k}_T \underset{\sim}{A}_T \quad (4)$$

and

$$\underset{\sim}{F}_T = \underset{\sim}{K}_T^{-1} \quad (3)$$

The external displacements resulting from external loads are found according to the expression

$$\begin{bmatrix} D^1/L_0 \\ D^2/L_0 \\ D^3/L_0 \\ D^4 \\ D^5 \\ D^6 \end{bmatrix} = \underset{\sim}{F}_T \begin{bmatrix} Q^1 L_0 \\ Q^2 L_0 \\ Q^3 L_0 \\ Q^4 \\ Q^5 \\ Q^6 \end{bmatrix} \quad (40)$$

By substituting this value for $\underset{\sim}{D}$ into Equation 38 the external section displacements can be determined. Using these external section displacements the internal section deformations can be found from

$$\begin{bmatrix} \underset{\sim}{d}_{I} \\ \underset{\sim}{d}_{II} \\ \underset{\sim}{d}_{III} \\ \underset{\sim}{d}_{IV} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}_{I} & & & \\ & \underset{\sim}{A}_{II} & & \\ & & \underset{\sim}{A}_{III} & \\ & & & \underset{\sim}{A}_{IV} \end{bmatrix} \begin{bmatrix} \underset{\sim}{D}_{I} \\ \underset{\sim}{D}_{II} \\ \underset{\sim}{D}_{III} \\ \underset{\sim}{D}_{IV} \end{bmatrix} \tag{41}$$

The internal section forces are then determined from the expression

$$\begin{bmatrix} \underset{\sim}{q}_{I} \\ \underset{\sim}{q}_{II} \\ \underset{\sim}{q}_{III} \\ \underset{\sim}{q}_{IV} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{k}_{I} & & & \\ & \underset{\sim}{k}_{II} & & \\ & & \underset{\sim}{k}_{III} & \\ & & & \underset{\sim}{k}_{IV} \end{bmatrix} \begin{bmatrix} \underset{\sim}{d}_{I} \\ \underset{\sim}{d}_{II} \\ \underset{\sim}{d}_{III} \\ \underset{\sim}{d}_{IV} \end{bmatrix} \tag{42}$$

APPLICATION OF THE METHOD

The Rigid Analysis

For the purpose of this study a pile group as shown in Figure 8 was analyzed. The diameter of all the circular piles was considered to have a constant value of 12 inches. An elastic modulus of 1.69×10^6 pounds per square inch and a torsional shear modulus of 0.103×10^6 pounds per square inch were chosen. For each of the six degrees of freedom of the pile head a positive unit force or moment, as appropriate, was applied. The problem was first solved for the rigid case. The method developed in the previous section was programmed in two parts for use in the IBM 1620 Data Processing System. The programs along with a sample of the computer output for the rigid case appear in Appendix A.

The Semirigid Analysis

For the semirigid analysis the same pile group was used (Figure 8). Again the structure was loaded with positive unit forces or moments for each degree of freedom. It was decided to vary the rotational stiffness at the pile-soil connection from a value of ten foot-kips per radian to infinity while holding the translational stiffnesses equal to infinity throughout the analysis. In this manner the redistribution of forces and moments in the pile group due to variation in rotational stiffness alone could be tabulated and plotted. The solution for the semirigid case was programmed into three parts for use in the computer. The programs and a sample of the computer output appear in Appendix B.

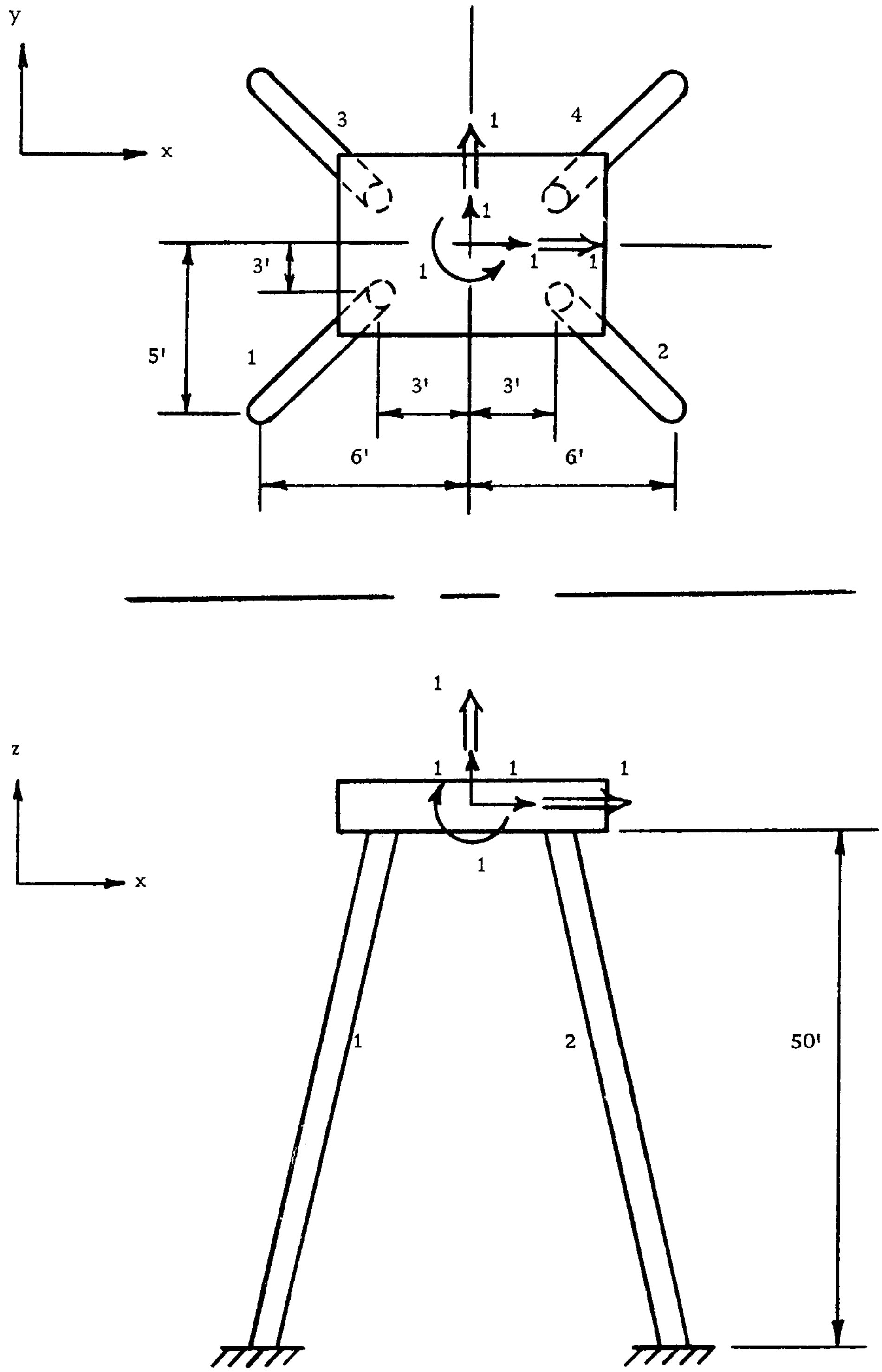

Figure 8. Loading and pile group dimensions used for example problems (all moments are in foot-kips and forces in kips)

DISCUSSION OF RESULTS

Results of Rigid Solution

The results of the rigid solution appear in Tables I and II. In Table I the member forces and moments are shown. It can be seen that the moments about the x axis are the same for piles one and three, also for members two and four. Similarly about the y axis members one and two have equal moments, likewise members three and four. These conditions arise from the symmetrical arrangement of the piles about their center of gravity. Because of the particular loading condition considered only pile number four has a compressive axial force. As would be expected the torque in all four piles, each being the same distance from the centroid of the pile group, is the same. For this loading condition it is quite small.

In Table II are summarized the pile group head displacements. The maximum horizontal displacement is seen to be 0.1253 feet in the direction of the y axis.

Results of Semirigid Solution

The results of the semirigid analysis are shown in Tables 3 and 4. In Table 3 the pile forces are given for each value of rotational stiffness considered. It is evident that as the stiffness increases the moments at the top ends of the piles decrease, while the moments at the bottom ends increase. This relationship has been illustrated by plotting (Figure 9) the moments for pile number two for the full range of stiffness considered. Pile number two was selected because it is subjected to the maximum moment components.

The axial force was found to increase in piles three and four with increasing stiffness, at the same time decreasing in piles one and two. This relationship has been plotted in Figure 10.

The torque in the piles increases rapidly at first with increasing stiffness but becomes virtually equal to the torque at infinite stiffness for a value of about 800 foot-kips per radian.

In Table 4 are summarized the head displacements for various rotational stiffnesses. As would be expected these values decrease with increasing stiffness. The two horizontal displacements, D^1 and D^2, which are of primary interest are plotted against stiffness in Figure 11. The vertical displacement remains unchanged because of the symmetrical configuration of this pile group about the centroid of the piles.

By using a value of infinity for the rotational stiffness in the semirigid analysis it can be seen that the results are identical to those for the rigid analysis. A partial check on the method of sections utilized in the semirigid analysis is thus provided.

Table 1. Pile forces for rigid analysis

Pile	M^{ix} (Ft -K)	M^{jx} (Ft -K)	M^{iy} (Ft -K)	M^{jy} (Ft -K)	Axial (Kip)	Torque (Ft -K)
1	-4.248	-4.977	-3.587	-4.423	2.662	.018
2	-4.357	-5.087	-3.587	-4.423	1.034	.018
3	-4.248	-4.977	-3.477	-4.314	.925	.018
4	-4.357	-5.087	-3.477	-4.314	-3.620	.018

Table 2. Head displacements for rigid analysis

Displacement	Distance (Feet)	Rotation	Angle (Radians)
D^1	.0945	D^4	.001531
D^2	.1253	D^5	.001755
D^3	.0659	D^6	0006396

Table 3. Pile forces for semirigid analysis

Stiffness[1]	Pile	M^{ix} (Ft-K)	M^{jx} (Ft-K)	M^{iy} (Ft-K)	M^{jy} (Ft-K)	Axial (Kips)	Torque (Ft-K)
10	1	-.114	-7.512	-.085	-6.121	4.001	.005
	2	-.115	-7.573	-.085	-6.121	1.374	.005
	3	-.114	-7.512	-.083	-6.060	.737	.005
	4	-.115	-7.573	-.083	-6.060	-5.110	.005
25	1	-.275	-7.415	-.205	-6.061	3.962	.009
	2	-.278	-7.476	-.205	-6.061	1.348	.009
	3	-.275	-7.415	-.202	-6.000	.739	.009
	4	-.278	-7.476	-.202	-6.000	-5.044	.009
50	1	-.517	-7.269	-.388	-5.970	3.895	.012
	2	-.523	-7.330	-.388	-5.970	1.318	.012
	3	-.517	-7.269	-.383	-5.909	.743	.012
	4	-.523	-7.330	-.383	-5.909	-4.954	.012
100	1	-.922	-7.023	-.701	-5.817	3.772	.015
	2	-.933	-7.087	-.701	-5.817	1.277	.015
	3	-.922	-7.023	-.690	-5.753	.763	.015
	4	-.933	-7.087	-.690	-5.753	-4.810	.015
200	1	-1.517	-6.663	-1.172	-5.585	3.582	.016
	2	-1.536	-6.730	-1.172	-5.585	1.227	.016
	3	-1.517	-6.663	-1.152	-5.518	.796	.016
	4	-1.536	-6.730	-1.152	-5.518	-4.604	.016
400	1	-2.237	-6.224	-1.766	-5.295	3.346	.017
	2	-2.270	-6.297	-1.766	-5.295	1.172	.017
	3	-2.237	-6.224	-1.732	-5.222	.837	.017
	4	-2.270	-6.297	-1.732	-5.222	-4.353	.017
800	1	-2.932	-5.797	-2.365	-5.005	3.113	.018
	2	-2.984	-5.879	-2.365	-5.005	1.123	.018
	3	-2.932	-5.797	-2.314	-4.923	.873	.018
	4	-2.984	-5.879	-2.314	-4.923	-4.107	.018
∞	1	-4.248	-4.977	-3.587	-4.423	2.662	.018
	2	-4.357	-5.087	-3.587	-4.423	1.034	.018
	3	-4.248	-4.977	-3.477	-4.314	.925	.018
	4	-4.357	-5.087	-3.477	-4.314	-3.620	.018

[1] (Ft-K/Rad)

Table 4. Head displacements for semirigid analysis

Stiffness (Ft-K/Rad.)	D^1 (Ft)	D^2 (Ft)	D^3 (10^{-4} Ft)	D^4 (Rad.)	D^5 (Rad.)	D^6 (Rad.)
10	.2165	.3211	.6592	.004067	-.004143	.0007061
25	.2122	.3135	.6592	+.003969	-.004061	.0006939
50	.2058	.3021	.6592	.003822	-.003934	.0006838
100	.1948	.2831	.6592	.003575	-.003720	.0006749
200	.1783	.2551	.6592	.003212	-.003396	.0006676
400	.1576	.2211	.6592	.002772	-.002990	.0006612
800	.1367	.1882	.6592	.002345	-.002581	.0006553
∞	.0945	.1253	.6592	.001531	-.001755	.0006396

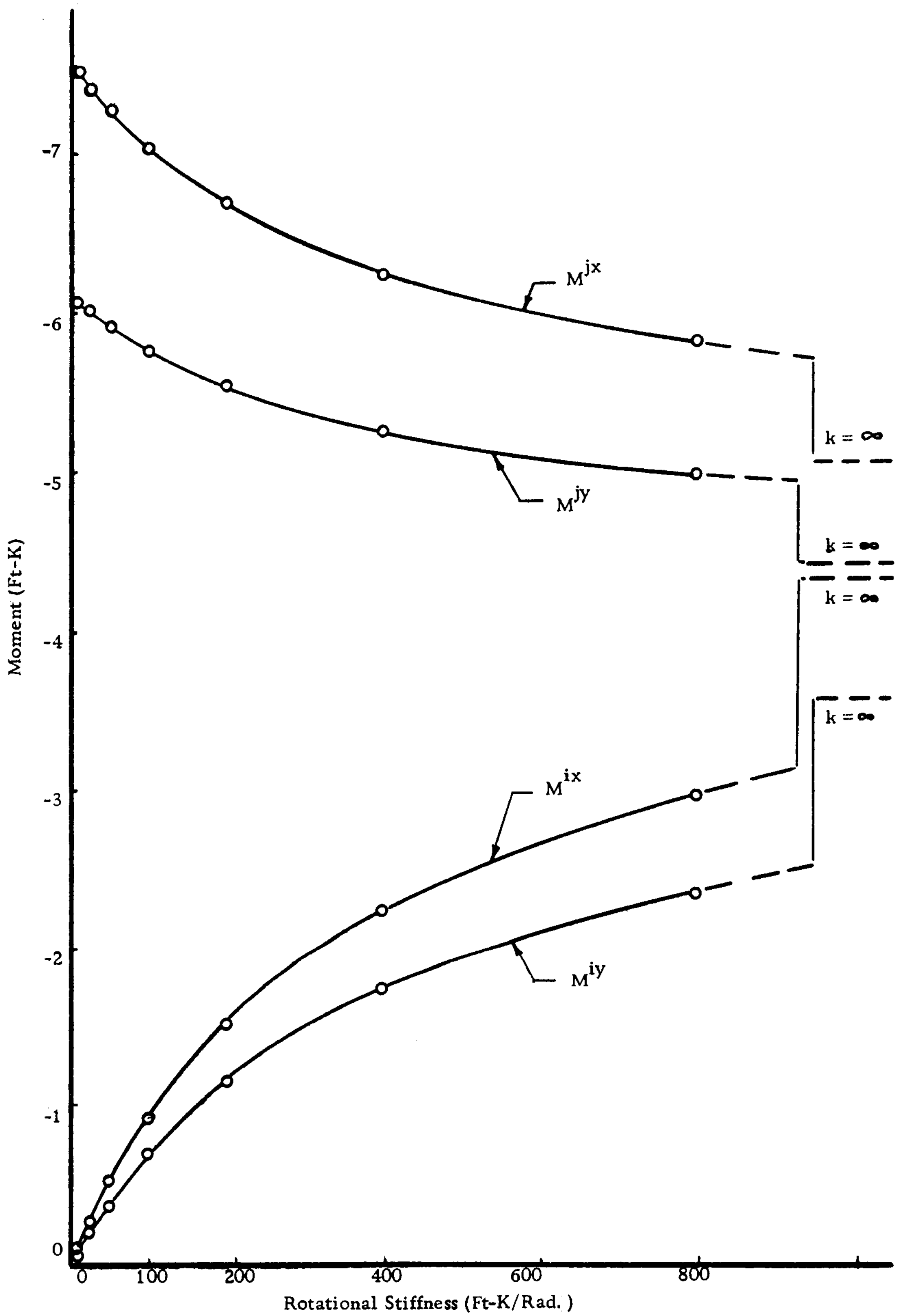

Figure 9. Moment in pile two versus rotational stiffness

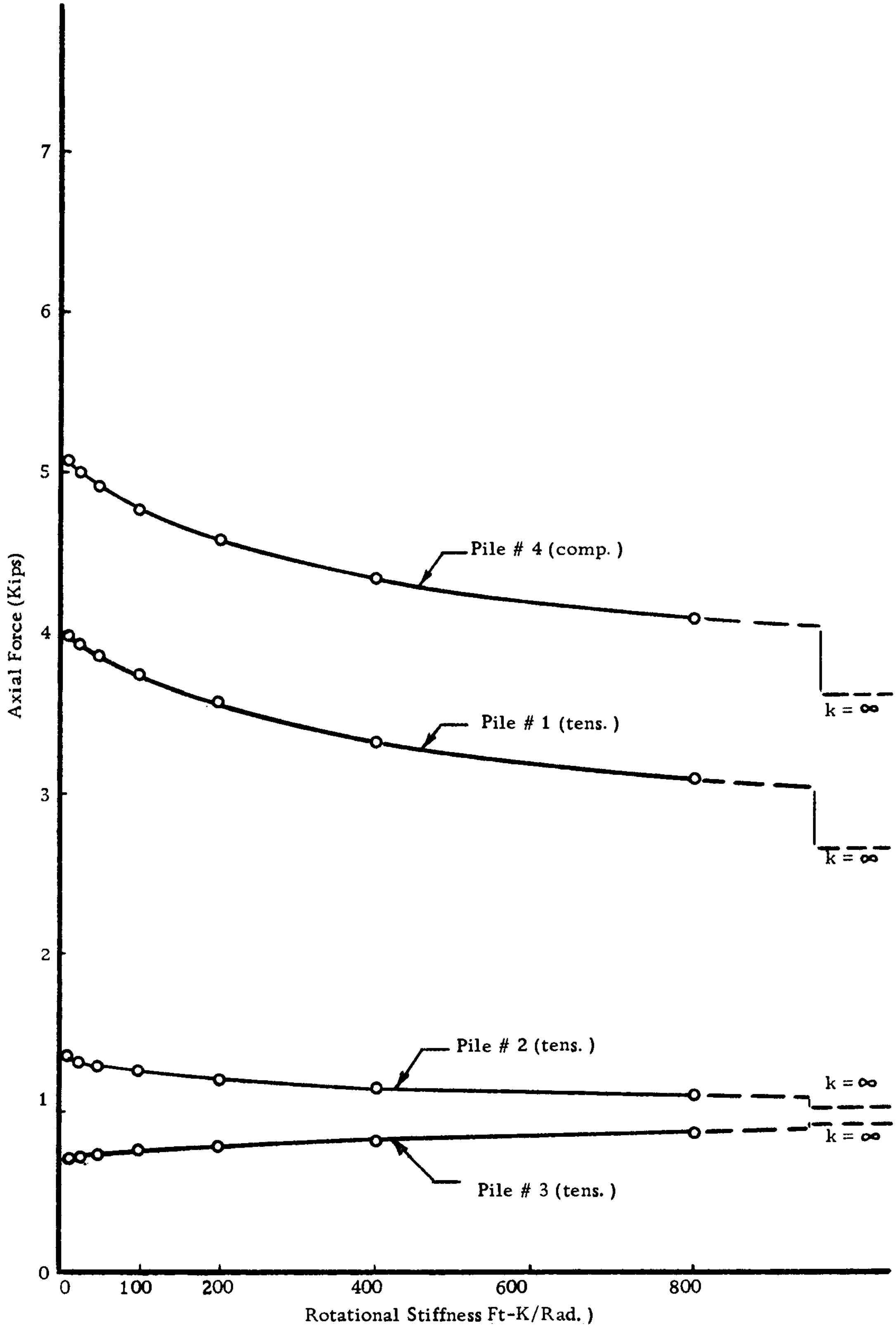

Figure 10. Axial force in piles versus rotational stiffness

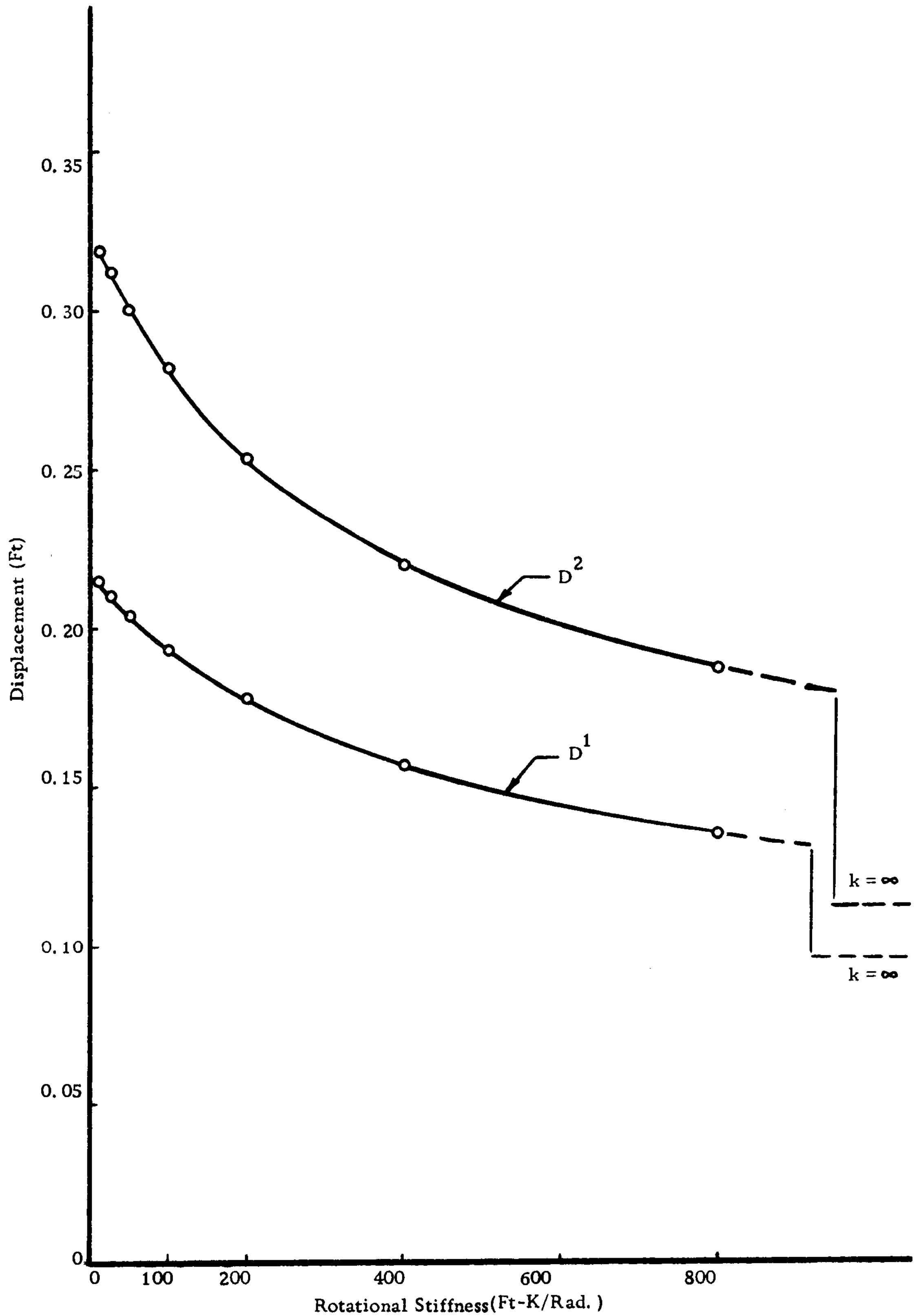

Figure 11. Horizontal head displacements versus rotational stiffness

CONCLUSION

A pile group, although difficult to analyze by classical methods, is easily handled by matrix analysis with the aid of a computer. It has long been apparent to engineers that piles are often not rigidly embedded in the soil foundation. The methods developed herein can be used to analyze either the rigid or semirigid foundation case. It has been shown by varying the rotational stiffness at the pile-soil connections that there is a significant redistribution of forces in individual piles when yielding of the soil occurs.

It should be emphasized that the redistribution may be quite different for other conditions of external loads than those used here. The method presented can, however, be used for studies of other conditions of loading. The methods developed for solution of both cases were programmed for use on a high speed computer. Though only a relatively simple pile group was chosen for analysis, it was illustrated that the same method could be expanded to encompass much larger structures. The only limitations placed on the number of piles in the group analyzed is the amount of storage available in the computer.

BIBLIOGRAPHY

1. Chellis, Robert D. Pile foundations. New York, McGraw-Hill, 1961. 704 p.

2. Davisson, M. T. and Shamsher Prakash. A review of soil-pole behavior. In: Stresses in soils and layered systems: Reports presented at the 42nd annual meeting of the Highway Research Board of the National Research Council. Washington, D. C., 1963. p. 25-48. (Highway Research Record no. 39)

3. Drucker, M. A. Embedment of poles, sheeting and anchor piles. Civil Engineering 4:622-626. 1934.

4. Dunham, Clarence W. Foundations of structures. New York, McGraw-Hill, 1962. 722 p.

5. Elms, David G. and Werner E. Schmid. Structural action of pile cluster dolphins. Proceedings of the American Society of Civil Engineers, Journal of the Waterways and Harbors Division 91:11-30. 1965.

6. Laursen, Harold I. Matrix analysis of structures. New York, McGraw-Hill, 1966. 155 p.

7. Meyerhoff, G. G. The ultimate bearing capacity of foundations. Geotechnique 2:312-331. 1951.

8. Pipes, Louis A. Matrix methods for engineering. New Jersey, Prentice-Hall, 1963. 427 p.

9. Seiler, J. F. Effect of depth of embedment on pole stability. Wood Preserving News 10:152-162. 1932.

10. Shilts, Walter L., LeRoy D. Graves and George D. Driscoll. A report of field and laboratory tests on stability of posts against lateral loads. In: Proceedings of the Second International Conference on Soil Mechanics and Foundation Engineering, Rotterdam, 1948. Vol. 5 n.d. p. 107-124.

11. Terzaghi, Karl. Evaluation of coefficients of subgrade reaction. Geotechnique 5:297-326. 1954.

12. Williams, T.E.H. Soil failure during overturning of piles. Engineering 173:134-136. 1952.

Appendix A. Computer programs with output sample for rigid analysis

```
C      RIGID ANALYSIS          PROGRAM  1
C      N IS NUMBER OF PILES IN GROUP
C      SMOD IS SHEAR MODULUS
C      EMOD IS ELASTIC MODULUS
C      DIAM IS MEAN DIAMETER OF PILES
C      H IS HEIGHT OF PILE GROUP
C      XT AND YT ARE COORDINATES OF PILE TOPS
C      XB AND YB ARE COORDINATES OF PILE BOTTOMS
       DIMENSION SMK(36,36),A(36,6),XT(6),YT(6),XB(6),YB(6),DIFX(4),DIFY(
      14),XLEN(6)
     2 READ 80,N,SMOD,EMOD,DIAM,H
       READ 81,(XT(I),YT(I),I=1,N)
       READ 81,(XB(I),YB(I),I=1,N)
       AREA=(3.1416*DIAM**2.)/4.
       XMOM=(3.1416*DIAM**4.)/64.
       PMOM=2.*XMOM
       M=6*N
       ND=6
       DO 3 I=1,M
       DO 3 J=1,M
     3 SMK(I,J)=0.0
    70 DO 74 I=1,N
       DIFX(I)=XT(I)-XB(I)
       IF(DIFX(I))71,72,72
    71 DIFX(I)=-DIFX(I)
    72 DIFY(I)=YT(I)-YB(I)
       IF(DIFY(I))73,74,74
    73 DIFY(I)=-DIFY(I)
    74 XLEN(I)=SQRTF(DIFX(I)**2.+DIFY(I)**2.+H**2.)
       J=1
       L=1
       MY=L+3
     5 DO 8 I=L,MY
     8 SMK(I,I)=(4.*H)/XLEN(J)
       IF(N-J)14,14,10
    10 L=L+6
       MY=L+3
       J=J+1
       GO TO 5
    14 MEM=M-5
       DO 12 I=1,MEM,6
       SMK(I,I+1)=.5*SMK(I,I)
       SMK(I+1,I)=.5*SMK(I,I)
       SMK(I+2,I+3)=.5*SMK(I,I)
    12 SMK(I+3,I+2)=.5*SMK(I,I)
    15 I=5
```

```
      J=1
   16 SMK(I,I)=(H**3.*AREA*144.)/(XMOM*XLEN(J))
   17 SMK(I+1,I+1)=(H*PMOM*SMOD)/(EMOD*XMOM*XLEN(J))
      IF(N-J)25,25,18
   18 J=J+1
      I=I+6
      GO TO 16
   25 DO 30 I=1,M
      DO 30 J=1,6
   30 A(I,J)=0.0
      J=1
      DO 35 I=1,N
      A(J+2,1)=-1.
      A(J+1,2)=-1.
      A(J+3,1)=-1.
      A(J,2)=-1.
      A(J+1,4)=-1.
      A(J+3,5)=1.
      A(J,6)=-XT(I)/XLEN(I)
      A(J+1,6)=-XT(I)/XLEN(I)
      A(J+2,6)=YT(I)/XLEN(I)
      A(J+3,6)=YT(I)/XLEN(I)
      A(J+5,6)=1.0
      A(J+4,1)=(XT(I)-XB(I))/XLEN(I)
      A(J+4,2)=(YT(I)-YB(I))/XLEN(I)
      A(J+4,3)=1.
      A(J+4,4)=YT(I)/XLEN(I)
      A(J+4,5)=-XT(I)/XLEN(I)
      A(J+4,6)=((XT(I)-XB(I))*YT(I)-(YB(I)-YT(I))*XT(I))/(H*XLEN(I))
      IF(M-J-5)40,40,32
   32 J=J+6
   35 CONTINUE
   40 PUNCH 83,M,ND
      DO 45 I=1,M
   45 PUNCH 82,(SMK(I,J),J=1,M)
      DO 50 I=1,M
   50 PUNCH 82,(A(I,J),J=1,6)
      GO TO 2
   80 FORMAT(I3,F8.0,F8.0,F8.3,F6.3)
   81 FORMAT(12F6.3)
   82 FORMAT(6E12.0)
   83 FORMAT(2I3)
      END
```

```
C     RIGID ANALYSIS          PROGRAM  2
      DIMENSION SMK(36,36),A(36,6),BGK(6,6),B(6,1),D(6),Q(36),F(36,6),AD
     1(36)
    3 READ 50,M,N
      DO 5 I=1,M
    5 READ 51,(SMK(I,J),J=1,M)
      DO 6 I=1,M
    6 READ 51,(A(I,J),J=1,N)
      DO 7 I=1,M
      DO 7 J=1,N
```

```
      F(I,J)=0.0
      DO 7 K=1,M
    7 F(I,J)=F(I,J)+SMK(I,K)*A(K,J)
      DO 8 I=1,N
      DO 8 J=1,N
      BGK(I,J)=0.0
      DO 8 K=1,M
    8 BGK(I,J)=BGK(I,J)+A(K,I)*F(K,J)
      CALL MATINV(BGK,N,B,0,DETERM)
    9 READ 58,(Q(I),I=1,N)
      DO 10 I=1,N
      D(I)=0.0
      DO 10 J=1,N
   10 D(I)=D(I)+BGK(I,J)*Q(J)
      DO 11 I=1,M
      AD(I)=0.0
      DO 11 J=1,N
   11 AD(I)=AD(I)+A(I,J)*D(J)
      DO 17 I=1,M
      Q(I)=0.0
      DO 17 J=1,M
   17 Q(I)=Q(I)+SMK(I,J)*AD(J)
      PUNCH 52
      PUNCH 56
      DO 12 I=1,N
   12 PUNCH 55,I,D(I)
      PUNCH 53
      MN=M/N
      DO 33 LX=1,N
      LY=LX+N*MN-N
   33 PUNCH 57,(AD(I),I=LX,LY,6)
      PUNCH 54
      PUNCH 60
      IXN=6*(N-1)+5
      DO 90 I=5,IXN,6
   90 Q(I)=Q(I)/50.
      I=1
      IX=1
      NX=IX+5
   20 PUNCH 61,I,(Q(J),J=IX,NX)
      IF(MN-I)22,22,21
   21 I=I+1
      IX=IX+6
      NX=IX+5
      GO TO 20
   22 GO TO 3
   50 FORMAT(2I3)
   51 FORMAT(6E12.0)
   52 FORMAT(16X,57HRESULTS OF DOLPHIN PILE ANALYSIS WITHOUT YIELDING AL
     1LOWED///)
   53 FORMAT(//12X,26HMEMBER DISPLACEMENTS *L/EI//)
   54 FORMAT(//12X,12HMEMBER LOADS//)
   55 FORMAT(13X,1HD,I1,2X,E14.7)
   56 FORMAT(//12X,24HHEAD DISPLACEMENTS *L/EI//)
   57 FORMAT(14X,4E15.7)
   58 FORMAT(6F12.4)
   60 FORMAT(12X,4HPILE,6X,6HMOM IX,3X,6HMOM JX,3X,6HMOM IY,3X,6HMOM JY,
     14X,5HAXIAL,4X,6HTORQUE//)
   61 FORMAT(I15,5X,6F9.3)
      END
```

MATRIX INVERSION SUBROUTINE

```
  SUBROUTINE MATINV(A,N)
  DIMENSION A(15,15)
  DO 5 K = 1,N
  COM = A(K,K)
  A(K,K) = 1.0
  DO 2 J = 1,N
2 A(K,J) = A(K,J) / COM
  DO 5 I = 1,N
  IF (I-K) 3,5,3
3 COM = A(I,K)
  A(I,K) = 0.0
  DO 4 J = 1,N
4 A(I,J)= A(I,J) - COM * A(K,J)
5 CONTINUE
  RETURN
  END
```

RESULTS OF DOLPHIN PILE ANALYSIS WITHOUT YIELDING ALLOWED

HEAD DISPLACEMENTS *L/EI

D1	4.5166625E-01
D2	5.9871590E-01
D3	3.1494079E-04
D4	3.6567197E-01
D5	-4.1933939E-01
D6	1.5279789E-01

MEMBER DISPLACEMENTS *L/EI

-5.8802110E-01	-6.0630934E-01	-5.8802110E-01	-6.0630934E-01
-9.5369307E-01	-9.7198131E-01	-9.5369307E-01	-9.7198131E-01
-4.5964055E-01	-4.5964055E-01	-4.4135231E-01	-4.4135231E-01
-8.7897994E-01	-8.7897994E-01	-8.6069170E-01	-8.6069170E-01
3.3374172E-03	1.2970668E-03	1.1600088E-03	-4.5379928E-03
1.5279789E-01	1.5279789E-01	1.5279789E-01	1.5279789E-01

MEMBER LOADS

PILE	MOM IX	MOM JX	MOM IY	MOM JY	AXIAL	TORQUE
1	-4.248	-4.977	-3.587	-4.423	2.662	.018
2	-4.357	-5.087	-3.587	-4.423	1.034	.018
3	-4.248	-4.977	-3.477	-4.314	.925	.018
4	-4.357	-5.087	-3.477	-4.314	-3.620	.018

Appendix B. Computer programs with output sample for semirigid analysis

```
C      SEMIRIGID ANALYSIS          PROGRAM  1
C      N IS NUMBER OF PILES IN GROUP
C      SMOD IS SHEAR MODULUS
C      EMOD IS ELASTIC MODULUS
C      DIAM IS MEAN DIAMETER OF PILES
C      H IS HEIGHT OF PILE GROUP
C      XT AND YT ARE COORDINATES OF PILE TOPS
C      XB AND YB ARE COORDINATES OF PILE BOTTOMS
C      SPRH IS HORIZONTAL SPRING STIFFNESS
C      SPRV IS VERTICAL SPRING STIFFNESS
C      SPRR IS ROTATIONAL SPRING STIFFNESS
       DIMENSION SMK(12,12),BGK(6,6),XB(4),XT(4),YB(4),YT(4),KI(4),F(6,12
      1),C21(6,6),C22(6,6),A1(12,6),C(6,6),ABAR(12,6),BARK(6,6),D(12,6),A
      22(12,6),DIFX(4),DIFY(4),XLEN(4)
       M=0
     2 READ 80,N,SMOD,EMOD,DIAM,H
       READ 81,(XT(I),YT(I),I=1,N)
       READ 81,(XB(I),YB(I),I=1,N)
     4 READ 85,SPRH,SPRV,SPRR
       AREA=(3.1416*DIAM**2.)/4.
       XMOM=(3.1416*DIAM**4.)/64.
       PMOM=2.*XMOM
    70 DO 74 I=1,N
       DIFX(I)=XT(I)-XB(I)
       IF(DIFX(I))71,72,72
    71 DIFX(I)=-DIFX(I)
    72 DIFY(I)=YT(I)-YB(I)
       IF(DIFY(I))73,74,74
    73 DIFY(I)=-DIFY(I)
    74 XLEN(I)=SQRTF(DIFX(I)**2.+DIFY(I)**2.+H**2.)
     1 DO 3 I=1,12
       DO 3 J=1,12
     3 SMK(I,J)=0.0
       M=M+1
       J=1
     5 DO 8 I=1,4
     8 SMK(I,I)=(4.*H)/XLEN(M)
       I=1
       SMK(I,I+1)=.5*SMK(I,I)
       SMK(I+1,I)=.5*SMK(I,I)
       SMK(I+2,I+3)=.5*SMK(I,I)
    12 SMK(I+3,I+2)=.5*SMK(I,I)
    15 I=5
    16 SMK(I,I)=(H**3.*AREA*144.)/(XMOM*XLEN(M))
    17 SMK(I+1,I+1)=(H*PMOM*SMOD)/(EMOD*XMOM*XLEN(M))
       XTX=(H**3.*1728.)/(EMOD*XMOM)
```

```
      SMK(7,7)=SPRH*XTX
      SMK(8,8)=SPRH*XTX
      SMK(9,9)=SPRV*XTX
      SMK(10,10)=(SPRR*XTX)/(H**2.*12.)
      SMK(11,11)=SMK(10,10)
      SMK(12,12)=SMK(10,10)
      IN=M
   19 DO 20 I=1,12
      DO 20 J=1,6
      A1(I,J)=0.0
   20 A2(I,J)=0.0
      A1(1,2)=-1.0
      A1(2,2)=-1.0
      A1(3,1)=-1.0
      A1(4,1)=-1.0
      A1(2,4)=-1.0
      A1(4,5)=1.0
      A1(1,6)=-XT(IN)/XLEN(IN)
      A1(2,6)=-XT(IN)/XLEN(IN)
      A1(3,6)=YT(IN)/XLEN(IN)
      A1(4,6)=YT(IN)/XLEN(IN)
      A1(5,1)=(XT(IN)-XB(IN))/XLEN(IN)
      A1(5,2)=(YT(IN)-YB(IN))/XLEN(IN)
      A1(5,3)=1.
      A1(5,4)=YT(IN)/XLEN(IN)
      A1(5,5)=-XT(IN)/XLEN(IN)
      A1(5,6)=((XT(IN)-XB(IN))*YT(IN)-(YB(IN)-YT(IN))*XT(IN))/(H*XLEN(IN
     1))
      A1(6,6)=1.0
   21 A2(1,2)=1.0
      A2(1,4)=1.0
      A2(2,2)=1.0
      A2(3,1)=1.0
      A2(3,5)=1.0
      A2(4,1)=1.0
      A2(5,3)=-1.0
      A2(6,6)=-1.0
   22 DO 23 I=1,6
   23 A2(I+6,I)=1,0
      DO 25 I=1,6
      DO 25 J=1,12
      F(I,J)=0.0
      DO 25 K=1,12
   25 F(I,J)=F(I,J)+A2(K,I)*SMK(K,J)
      DO 27 I=1,6
      DO 27 J=1,6
      C21(I,J)=0.0
      C22(I,J)=0.0
      DO 27 K=1,12
      C21(I,J)=C21(I,J)+F(I,K)*A1(K,J)
   27 C22(I,J)=C22(I,J)+F(I,K)*A2(K,J)
      CALL MATINV(C22,6,B,0,DETERM)
      DO 30 I=1,6
      DO 30 J=1,6
      C(I,J)=0.0
      DO 30 K=1,6
   30 C(I,J)=C(I,J)+C22(I,K)*C21(K,J)
      DO 32 I=1,12
      DO 32 J=1,6
      D(I,J)=0.0
```

```
      DO 32 K=1,6
   32 D(I,J)=D(I,J)+A2(I,K)*C(K,J)
      DO 34 I=1,12
      DO 34 J=1,6
   34 ABAR(I,J)=A1(I,J)-D(I,J)
      DO 36 I=1,6
      DO 36 J=1,12
      F(I,J)=0.0
      DO 36 K=1,12
   36 F(I,J)=F(I,J)+A1(K,I)*SMK(K,J)
      DO 38 I=1,6
      DO 38 J=1,6
      BARK(I,J)=0.0
      DO 38 K=1,12
   38 BARK(I,J)=BARK(I,J)+F(I,K)*ABAR(K,J)
      DO 40 I=1,12
   40 PUNCH 82,(ABAR(I,J),J=1,6)
      DO 42 I=1,6
   42 PUNCH 82,(BARK(I,J),J=1,6)
      DO 43 I=1,12
   43 PUNCH 82,(SMK(I,J),J=1,12)
      IF(N-M)44,44,1
   44 IF(SENSE SWITCH 1)2,45
   45 PUNCH 81,(XT(I),YT(I),I=1,N)
      PUNCH 81,(XB(I),YB(I),I=1,N)
   80 FORMAT(I3,F8.0,F8.0,F8.3,F6.3)
   81 FORMAT(12F6.3)
   82 FORMAT(6E12.0)
   85 FORMAT(3F20.5)
      END


C     SEMIRIGID ANALYSIS          PROGRAM  2
      DIMENSION A(24,6),F(6,24),TOTK(24,24),BGK(6,6),XT(4),XB(4),YT(4),Y
     1B(4)
    2 READ 50,N,H
      M=6*N
      DO 7 I=1,M
      DO 7 J=1,M
    7 TOTK(I,J)=0.0
      MX=1
      NX=MX+5
   10 DO 12 I=MX,NX
   12 READ 51,(TOTK(I,J),J=MX,NX)
      IF(M-NX)15,15,13
   13 MX=MX+6
      NX=MX+5
      GO TO 10
   15 READ 52,(XT(I),YT(I),I=1,N)
      READ 52,(XB(I),YB(I),I=1,N)
      DO 18 I=1,M
      DO 18 J=1,6
   18 A(I,J)=0.0
      I=1
      DO 24 J=1,N
      A(I,1)=1.0
      A(I+1,2)=1.0
      A(I+2,3)=1.0
      A(I+3,4)=1.0
```

```
      A(I+4,5)=1.0
      A(I+5,6)=1.0
      IF(M-I-5)30,30,23
   23 I=I+6
   24 CONTINUE
   30 DO 32 I=1,6
      DO 32 J=1,M
      F(I,J)=0.0
      DO 32 K=1,M
   32 F(I,J)=F(I,J)+A(K,I)*TOTK(K,J)
      DO 35 I=1,6
      DO 35 J=1,6
      BGK(I,J)=0.0
      DO 35 K=1,M
   35 BGK(I,J)=BGK(I,J)+F(I,K)*A(K,J)
      CALL MATINV(BGK,6,B,0,DETERM)
      DO 40 I=1,M
   40 PUNCH 51,(A(I,J),J=1,6)
      DO 45 I=1,6
   45 PUNCH 51,(BGK(I,J),J=1,6)
   50 FORMAT(I2,F8.2)
   51 FORMAT(6E12.0)
   52 FORMAT(12F6.3)
      END
```

```
C     SEMIRIGID ANALYSIS          PROGRAM  3
      DIMENSION Q(6),BGD(6),BGK(6,6),DX(24),A(24,6),ABAR(48,6),D(48),QIN
     1(48),SMK(48,12)
    2 READ 50,N
      NM=12*N
      M=6*N
      DO 3 I=1,NM
      DO 3 J=1,6
    3 ABAR(I,J)=0.0
      DO 5 I=1,NM
    5 READ 51,(ABAR(I,J),J=1,6)
   15 DO 16 I=1,NM
      DO 16 J=1,12
   16 SMK(I,J)=0.0
      DO 17 I=1,NM
   17 READ 51,(SMK(I,J),J=1,12)
      DO 18 I=1,M
   18 READ 51,(A(I,J),J=1,6)
      DO 20 I=1,6
   20 READ 51,(BGK(I,J),J=1,6)
   36 READ 54,(Q(I),I=1,6)
      DO 38 I=1,6
      BGD(I)=0.0
      DO 38 J=1,6
   38 BGD(I)=BGD(I)+BGK(I,J)*Q(J)
      DO 40 I=1,M
      DX(I)=0.0
      DO 40 J=1,6
   40 DX(I)=DX(I)+A(I,J)*BGD(J)
      IX=0
      NX=1
      MX=12
   41 DO 42 J=NX,MX
```

```
      D(J)=0.0
      DO 42 I=1,6
      IXY=IX+I
   42 D(J)=D(J)+ABAR(J,I)*DX(IXY)
      IF(NM-MX)44,44,43
   43 IX=IX+6
      NX=NX+12
      MX=NX+11
      GO TO 41
   44 IX=0
      NX=1
      MX=12
   45 DO 46 J=NX,MX
      QIN(J)=0.0
      DO 46 I=1,12
      IXY=IX+I
   46 QIN(J)=QIN(J)+SMK(J,I)*D(IXY)
      IF(NM-MX)48,48,47
   47 IX=IX+12
      NX=NX+12
      MX=NX+11
      GO TO 45
   48 PUNCH 62
   49 PUNCH 102
      DO 85 I=1,6
   85 PUNCH 103,I,BGD(I)
      PUNCH 101
      DO 72 LX=1,12
      LY=LX+36
   72 PUNCH 55,(D(I),I=LX,LY,12)
      PUNCH 104
      PUNCH 60
      DO 90 I=5,41,12
   90 QIN(I)=QIN(I)/50.
      I=1
      IX=1
      NX=IX+5
   70 PUNCH 61,I,(QIN(J),J=IX,NX)
      IF(N-I)73,73,71
   71 I=I+1
      IX=IX+12
      NX=NX+12
      GO TO 70
   73 GO TO 2
   50 FORMAT(I2)
   51 FORMAT(6E12.0)
   52 FORMAT(12F6.3)
   54 FORMAT(6F12.4)
   55 FORMAT(14X,4E15.7)
   60 FORMAT(12X,4HPILE,6X,6HMOM IX,3X,6HMOM JX,3X,6HMOM IY,3X,6HMOM JY,
     14X,5HAXIAL,4X,6HTORQUE//)
   61 FORMAT(I15,5X,6F9.3)
   62 FORMAT(16X,54HRESULTS OF DOLPHIN PILE ANALYSIS WITH YIELDING ALLOW
     1ED///)
  100 FORMAT(6E12.5)
  101 FORMAT(//12X,26HMEMBER DISPLACEMENTS *L/EI//)
  102 FORMAT(//12X,24HHEAD DISPLACEMENTS *L/EI//)
  103 FORMAT(13X,1HD,I1,2X,E14.7)
  104 FORMAT(//12X,12HMEMBER LOADS//)
      END
```

RESULTS OF DOLPHIN PILE ANALYSIS WITH YIELDING ALLOWED

ROTATIONAL STIFFNESS = INFINITY

HEAD DISPLACEMENTS *L/EI

D1	4.5167401E-01
D2	5.9871704E-01
D3	3.1494050E-04
D4	3.6567633E-01
D5	-4.1934815E-01
D6	1.5279800E-01

MEMBER DISPLACEMENTS *L/EI

-5.8802223E-01	-6.0631049E-01	-5.8802223E-01	-6.0631049E-01
-9.5369856E-01	-9.7198682E-01	-9.5369856E-01	-9.7198682E-01
-4.5964830E-01	-4.5964830E-01	-4.4136004E-01	-4.4136004E-01
-8.7899645E-01	-8.7899645E-01	-8.6070819E-01	-8.6070819E-01
3.3371395E-03	1.2969115E-03	1.1601645E-03	-4.5377185E-03
1.5279800E-01	1.5279800E-01	1.5279800E-01	1.5279800E-01
7.0888289E-17	7.0888289E-17	6.8951361E-17	6.8951361E-17
8.1642181E-17	8.3579109E-17	8.1642181E-17	8.3579109E-17
1.1781501E-15	4.5787621E-16	4.0956041E-16	-1.6019979E-15
1.1278159E-12	1.1568697E-12	1.1278159E-12	1.1568697E-12
9.5229491E-13	9.5229491E-13	9.2324099E-13	9.2324099E-13
4.9314943E-15	4.9314943E-15	4.9314943E-15	4.9314943E-15

MEMBER LOADS

PILE	MOM IX	MOM JX	MOM IY	MOM JY	AXIAL	TORQUE
1	-4.248	-4.977	-3.587	-4.423	2.662	.018
2	-4.357	-5.087	-3.587	-4.423	1.034	.018
3	-4.248	-4.977	-3.477	-4.314	.925	.018
4	-4.357	-5.087	-3.477	-4.314	-3.620	.018