A Technique for the Solution of the Catenary Problem in Surveying

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ABSTRACT

A technique for solving efficiently the catenary problem encountered in surveying with tapes is presented. The theory of the catenary solution is outlined, and our technique is shown to solve correctly the catenary for all conditions. Analysis of error that compared the catenary correction to the more commonly used parabolic correction indicated that for slopes over 10 degrees and taped distances greater than 200 feet, systematic error inherent in the parabola may preclude the accuracy required for second- or even third-order surveys. An analysis of tension-gauge bias on taping errors is similar, but here the error is found to depend almost entirely upon taping distance.

INTRODUCTION

Taping, like all forms of measurement, is inexact. An important responsibility of the surveyor is to know the sources of errors that influence his measurements and to be familiar with procedures necessary for maintaining a required precision in the presence of such errors. We know that **accidental** errors in surveying are unimportant compared with **systematic** errors. For each measurement, the magnitude and algebraic sign of accidental errors are matters of chance and therefore cannot be computed [2]. Fortunately, such errors are also unbiased and thus tend to be compensating. Systematic errors are consistent and, therefore, noncompensating. Such errors always follow definite mathematical laws, however, and corrections can be determined and applied.

Procedures for correcting most systematic errors are straightforward and have been well documented. When error is introduced by the sag of a tape supported at the ends or at intervals rather than along its full length, however, the appropriate correction is not generally understood and has not been fully documented. When a tape sags between supports, it takes the form of a catenary [1, 2, 4, 7]. Thus, the measured distance is that of a catenary arc; the actual distance, that of the subtended chord. For the purpose of determining the difference between the arc and its chord, the assumption is usually made that the arc can be closely approximated by a parabola.

Recent work by Wood [7] suggests that although the parabolic correction gives reasonable results on level ground, it can lead to significant systematic errors when a slope is measured. Admittedly, recent advances in electronic equipment have resulted in decreased use of tapes for measuring distances precisely [3]. Many smaller surveying companies continue to use tapes, however, and higher order surveys commonly form an important part of their business. Wood [7] proposed that the catenary itself be solved whenever higher order surveys are conducted by tape, particularly in areas of steep terrain. His paper also compared several catenary solutions and the corresponding parabolic solutions, and he discussed the systematic error inherent in the latter. He did not document the mathematical procedure for solving the catenary, however, nor did he discuss the physical difficulties in such a solution.

METHODS

We have recently developed a computer program that is used by students in the Forest Engineering Department at Oregon State University to solve catenary problems for higher order surveys.

The Catenary with Level Supports

The simplest application of the catenary occurs when the supports at the two ends of the tape are level (Figure 1). In general, a catenary is fully defined by the parameter m (see Figure 1), which is numerically equal to the vertical distance from the origin of the x-axis (called the directrix) to the lowest point on the catenary [5]. By convention, the y-axis is usually taken to pass through this minimum point. The entire weight of the tape between supports (ws, where w is the unit weight of the tape, and s is the measured tape distance between supports) may be envisioned as being concentrated at the lowest point on the catenary. Thus, a free-body diagram of the tape shows that the tension (T) applied tangentially to the tape at one of the supports may be resolved into an unknown horizontal component (T_0) and a known vertical component equal to one-half of the total weight of the tape between supports (ws/2). When some known value of T is applied to the tape, its horizontal component can be computed by means of the Pythagorean theorem:

$$T_o = [T^2 - (w^2 s^2/4)]^{1/2} \quad 1$$

For $m = T_o/w$, the catenary equation for length takes the simple form [4]:

$s/2 = m \sinh HD/2m$,
where HD = horizontal distance between supports.

Solving this expression for HD, we have

$$HD = 2m \sinh^{-1} s/2m. \quad 2$$

For the catenary with level supports, the solution may therefore be found directly by means of equations 1 and 2.

The Catenary with Supports Not Level

Texts in engineering mechanics are generally content to cover only the solution of catenaries with level supports. The catenary with supports not level is somewhat more complicated and is shown diagrammatically in Figure 2. As before, we know the values of the parameters T, s, and w; in addition, we have measured the slope angle, ϕ. Here, however, the center of gravity of the tape is no longer at the minimum point of the catenary. In fact, as illustrated in Figure 2, the tape may not even intersect the minimum point of the catenary at all. Therefore, we do not have an expression for the vertical component of T and thus have no straightforward way of calculating T_o. Furthermore, we have two primary unknowns: DE (difference in elevation between supports) and HD. In the previous problem, DE was known to be zero.

To solve this problem we will use the following catenary relations from Meriam [4]:

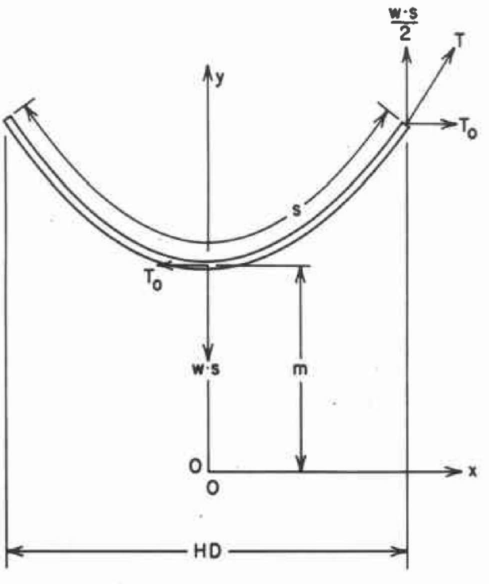

Figure 1. Catenary with level supports.

$$T = wm \cosh x/m, \quad (3)$$

$$y = m \cosh x/m, \quad (4)$$

$$s = m \sinh x/m. \quad (5)$$

First we derive an expression for s in terms of x_1, m, and HD:

$$s = s_1 - s_2$$

$$s = m \sinh x_1/m - m \sinh x_2/m$$

$$s = m \sinh x_1/m - m \sinh (x_1/m - HD/m).$$

Solving this expression for HD, we have

$$HD = m\left[(x_1/m) - \sinh^{-1}(\sinh (x_1/m) - s/m)\right]. \quad (6)$$

To solve equation 6, we will require an expression for x_1/m. Noting that the tension T is applied at x_1 and using equation 3 we have

$$T = wm \cosh x_1/m,$$

and therefore

$$x_1/m = \cosh^{-1} T/wm. \quad (7)$$

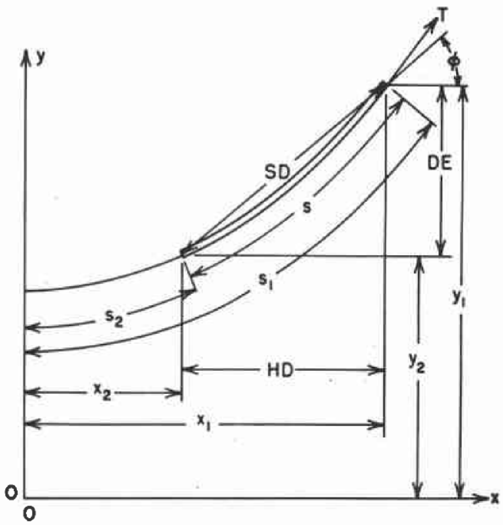

Figure 2. Catenary with supports not level.

Next we need an expression for DE. From Figure 2,

$$DE = y_1 - y_2 .$$

But from equation 4,

$$y_1 = m \cosh x_1/m.$$

Combining this expression with equation 3, we have

$$y_1 = T/w.$$

Now we can write

$$DE = T/w - y_2 .$$

Using equation 4, we have

$$y_2 = m \cosh (x_1/m - HD/m).$$

Therefore,

$$DE = T/w - m \cosh (x_1/m - HD/m). \qquad 8$$

Equations 6, 7, and 8 give us a means of finding the unique solution to the catenary problem with supports not level, if we know T, s, w, and ϕ. Notice, however, that all three of these equations show the catenary parameter m on the right-hand side. As the value of m is unknown, the problem will have to be solved by successive approximations.

Our method solves for the catenary by the secant method, a technique common for solving problems in numerical analysis [6]. The secant method is a two-point iteration method; that is, it extrapolates or interpolates to find an estimate of the solution value based upon two previous estimates. The procedure used in this application may be summarized as follows:

1. Initialize $T_o^{(1)} = T$, and $T_o^{(2)} = T \cos \phi$. These initial values are arbitrary; they were chosen simply because they can be determined easily. Our experience, however, suggests that they provide good initial estimates.

2. Compute $m_1 = T_o^{(1)}/w$, $m_2 = T_o^{(2)}/w$.

3. Compute x_1/m_1 and x_2/m_2 by means of equation 7.

4. Compute HD_1 and HD_2 from equation 6.

5. Compute DE_1 and DE_2 from equation 8.

6. Compute the error associated with the two estimates of T_o as follows:

$E_i = DE_i/HD_i - \tan \phi$ (for i = 1 and 2).

If $|E_2| < \epsilon$, where ϵ is some small constant that represents the accuracy desired, then the solution is complete (that is, we have found the unique catenary solution for HD and DE, given T, s, w, and ϕ), and we stop. Otherwise, we go on to step 7.

7. Determine a new estimate of T_o:

$$T_o^{(3)} = T_o^{(2)} - [(T_o^{(2)} - T_o^{(1)})/(E_2 - E_1)]\ E_2 .$$

Then, replace $T_o^{(1)}$ with $T_o^{(2)}$, and $T_o^{(2)}$ with $T_o^{(3)}$; go back to step 2.

Computational experience with this procedure indicates that convergence to an $\epsilon = 10^{-8}$ usually occurs within 5 to 10 iterations. Although most surveyors likely would not have the patience to work through these seven steps ten times for each leg of a traverse, the computational burden that this represents for a computer or even for many small, programmable calculators is insignificant.

Other Catenary Problems

The methodology described above was formulated to solve the catenary problem when the supports are not level and both supports are on the same side of the y-axis, as in Figure 2. Consider the slightly different problem posed in Figure 3, where the supports are on both sides of the y-axis. The question that has to be answered here is whether our procedure is general enough to solve this problem correctly without reformulation. Thus,

$$s = s_1 + s_2$$

$$s = m \sinh x_1/m + m \sinh x_2/m, \qquad 9$$

so that s is computed from a sum rather than from a difference as before. However, we note that because $x_2 < 0$, we still have

$$x_2 = x_1 - HD$$

because $x_1 < HD$. Therefore $x_2/m < 0$ and equation 9 may be rewritten as

$$s = m \sinh x_1/m - m \sinh |x_2/m|.$$

This is computationally equivalent to the expression we derived earlier when the supports were not level and both supports were on the same side of the y-axis. Therefore our procedure is sufficiently general to solve both problems.

All of our work to this point has assumed that the known tension is applied to the tape at the upper support. This is not always true. With level supports, tension at the two supports is equal. From equation 3, however, obviously this is not true if the supports are at different elevations. Then, tension at the lower support is related to tension at the upper support by the catenary relation.[1]

[1]Note that if T_{lower} and T_{upper} could both be measured accurately in the field, then DE (and consequently HD) could be computed directly from this relation.

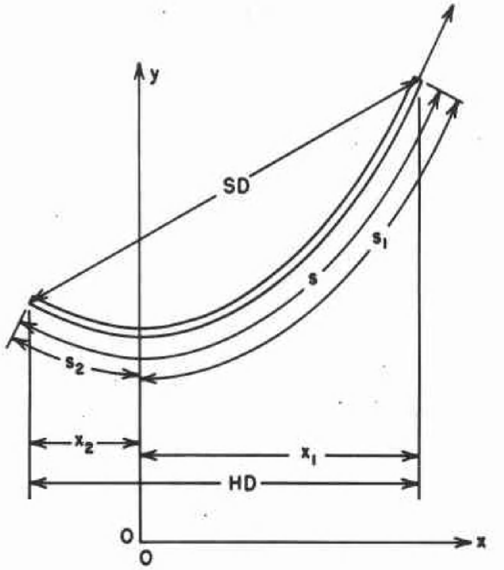

Figure 3. Supports on both sides of the y-axis.

$$T_{lower} = T_{upper} - (w)(DE).$$

Therefore, if the known tension is applied at the lower support as illustrated in Figure 4, then the resulting catenary will be different than if the same tension were applied at the upper support (for identical values of s, w, and ϕ). Typically, the head chainman is instructed to apply a known tension at his end of the tape. On any given traverse, he is likely to be at the uphill support some of the time and at the downhill support some of the time. Thus our procedure must be capable of computing both catenaries. When the known tension is at the downhill support,

$$s = s_2 - s_1 .$$

Carrying this relation through to its equivalent of equation 8, we find

$$HD = m\ [(x_1/m) - \sinh^{-1}(\sinh(x_1/m) + s/m)] . \qquad 10$$

Therefore, a convenient way of handling the problem when the known tension is at the lower support is to insert a logical check into the algorithm at step 4. If the known tension is at the uphill support (signalled in our computer program by a positive sign on the slope angle ϕ, then HD is computed from equation 6. On the other hand, if the known tension is at the downhill support (signalled by a negative sign on ϕ), then HD is computed from equation 10.

Our earlier discussion shows that this procedure is sufficiently general to solve the problem with known tension at the lower support with supports on both sides of the y-axis. The proof is similar to that advanced in the first part of this section and is not covered here because of limited space.

Before concluding our discussion of the general catenary formulation for taping, we should note that our supports-not-level algorithm will also solve the catenary when the supports are level. If the algorithm is to be implemented on a computer, however, coding the

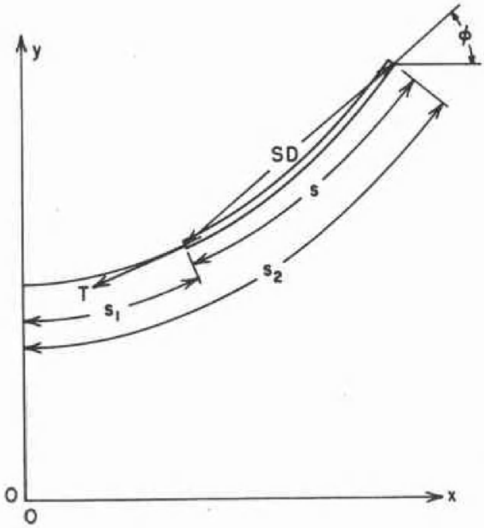

Figure 4. Known tension at the lower support.

supports-level formulation separately is usually worthwhile because of the significant saving in computing time that results from that formulation whenever the slope is flat. If the implementation is to be done on a small, programmable calculator with limited memory space, coding only the supports-not-level algorithm may be necessary.

ANALYSIS OF ERROR

Having developed the technique outlined above, we programmed it to operate on the Oregon State University CDC 3300 computer, on a Hewlett-Packard Model 9830 programmable calculator operated by the OSU Forest Engineering Department, and on a Hewlett-Packard Model 65 hand-held programmable calculator[2,3]. We discuss briefly the implications of tests we have made using these programs.

Catenary Compared with Parabolic Correction

Table 1 is a summary of systematic errors that would result from the application of no correction and the parabolic correction, for representative values of ϕ and s. We have used equations presented by Colcord and Chick [1] to compute the parabolic corrections for slope taping. Table 1 confirms Wood's [7] hypothesis that the parabolic correction on steeper slopes introduces systematic errors that may preclude higher order traverses. The effect is more pronounced for longer distances, which are often dictated by practical considerations in steep topography, and when the known tension is at the upper support. For taping on level ground,

[2] Interested readers may obtain a listing of the FORTRAN program (for the CDC3300), the BASIC program (for the HP-9830), or the HP-65 machine-language program by writing to Dennis P. Dykstra, Forest Engineering Department, Oregon State University, Corvallis, Oregon 97331.

[3] The use of trade names in this paper is for information only and does not constitute endorsement by Oregon State University.

Table 1. Comparison of Systematic Taping Errors Introduced by Sag (T = 20 lbs, w = 0.015 lbs/ft).

		No correction		Parabolic correction		Catenary correction	
ϕ	s	HD	error	HD	error	HD	DE
Deg	*Ft*	*Ft*		*Ft*		*Ft*	*Ft*
0	300.00	300.00	1/500	299.37	1/66,500	299.36	0
10	300.00	295.44	1/500	294.84	1/10,200	294.81	51.98
20	300.00	281.91	1/500	281.38	1/5,900	281.34	102.40
30	300.00	259.81	1/600	259.40	1/4,800	259.34	149.73
0	200.00	200.00	1/1,100	199.81	1/333,000	199.81	0
10	200.00	196.96	1/1,100	196.78	1/36,400	196.78	34.70
20	200.00	187.94	1/1,100	187.78	1/21,300	187.77	68.34
30	200.00	173.21	1/1,300	173.08	1/17,000	173.07	99.92
-10	300.00	295.44	1/500	294.83	1/12,200	294.86	51.99
-20	300.00	281.91	1/600	281.37	1/6,400	281.42	102.43
-30	300.00	259.81	1/700	259.39	1/5,100	259.44	149.79
-10	200.00	196.96	1/1,100	196.78	1/41,900	196.79	34.70
-20	200.00	187.94	1/1,300	187.78	1/22,100	187.79	68.35
-30	200.00	173.21	1/1,500	173.08	1/17,800	173.09	99.93

Note: the errors listed in this table were computed from horizontal distances carried to four decimal places.

dictated by practical considerations in steep topography, and when the known tension is at the we have little motivation to use the catenary correction rather than the more easily computed parabolic correction.

Readers familiar with the paper by Wood [7] may note that although Table 1 is based upon the identical conditions considered in that article, our results for HD and DE with the catenary correction agree with his results only at $\phi = 0$. Our tests indicate that Wood used a catenary formulation that assumes

$$DE = HD_o \sin \phi \qquad 11$$

where HD_o = the horizontal distance between supports that would be computed by applying the catenary correction when the supports are level. This procedure easily can be shown to give incorrect results for $\phi \neq 0$. As an example, for Course 4 in Wood's paper (T = 20 lbs, w = 0.015 lbs/ft, s = 300.00 ft, and ϕ = -30 degrees[4]), DE was found to be 149.68 ft, and HD, 259.4953 ft. Therefore DE/HD = 0.57681, which should be equal to tan 30°. But tan 30° = 0.57735, and the solution is obviously in error. The fault in this formulation is that equation 11 constrains the solution in an unnatural way. The methodology presented in our paper uses the known value of ϕ to converge on the correct values of HD and DE and thus does not suffer from this defect.

[4] Although no mention is made of it in his paper, all of Wood's solutions are for known tension at the lower support.

Table 2. Errors Introduced by a 1-pound Error in Applied Tension at the Upper Support (Prescribed Tension is 20 lbs, w = 0.015 lbs/ft).

ϕ	s	Error when actual T=19 lbs	Error when actual T=21 lbs
Deg	*Ft*		
10	100.00	1/39,800	1/46,300
20	100.00	1/42,800	1/50,000
30	100.00	1/49,500	1/57,800
10	200.00	1/9,700	1/11,200
20	200.00	1/10,200	1/11,900
30	200.00	1/11,600	1/13,600
10	300.00	1/4,200	1/4,900
20	300.00	1/4,300	1/5,000
30	300.00	1/4,800	1/5,600

Tension-Gauge Bias

Our experience has been that the tension gauges commonly used in taping are often poorly calibrated. To investigate the influence of a biased tension gauge on distance errors, we computed the error in horizontal distance that would result, after the catenary correction is applied, from a 1-pound error in applied tension at the upper support. This analysis is summarized in Table 2. Note that, unlike the error that results from failure to apply the catenary correction (Table 1), error from bias in the tension gauge depends most strongly upon s rather than ϕ. In fact, as the slope increases, the error is reduced. Note also that the effect of tension bias is nonlinear; a positive 1-pound error has a smaller effect than a negative error of the same magnitude. Bias in either direction could destroy the precision necessary for higher order surveys when longer tape distances are necessary or desirable. Thus, insuring that tension gauges have been properly calibrated and are being used correctly is essential.

Combined Corrections

As a final comment, we note that whenever corrections for several effects are to be applied simultaneously, they must be made before determining the catenary correction for sag. Higher order surveys commonly require, for example, that corrections be applied for standardization, temperature, tension, and occasionally alignment and wind [1]. All of these corrections affect the measured length, s, along the catenary. Thus, if they are not applied before calculation of the catenary correction, the wrong s will be used, and it follows that the wrong catenary will be computed.

CONCLUDING REMARKS

This paper has presented a technique that efficiently solves the catenary problem encountered when surveying with tapes. Although the difference between the catenary correction and the commonly used parabolic correction may be significant for higher order surveys, we do not anticipate a full-scale rush to employ the catenary correction on ordinary surveys. We have taken the trouble to develop this methodology because we feel it can be of some importance in certain applications. In addition, our presentation is also intended to encourage an appreciation of catenary problems in general.

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