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The three important methods of approximation: interpolation, least-squares, and Chebyshev, are extended into bivariate approximations. A method of obtaining polynomial approximations for very general classes of bivariate samples is developed. Bivariate least-square approximations are reviewed and a method of developing bivariate orthogonal sequence is derived. A method of obtaining bivariate Chebyshev approximations is introduced with proofs that many of the important properties of the univariate Chebyshev approximation carry over into the bivariate case. Finally, a practical method of obtaining univariate or bivariate approximations to polynomials through the use of compiled tables of weights is developed, along with the necessary tables for Chebyshev or least-square approximations.
An Investigation of Bivariate Approximations

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Gary Royce Bills

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Typed by Clover Redfern for     Gary Royce Bills
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AN INVESTIGATION OF BIVARIATE APPROXIMATIONS

I. INTRODUCTION

The generation of approximating functions forms an important topic in numerical analysis. Until recently, little work was devoted to multivariate approximations as compared to that expended on functions of one variable. In particular, very little is known about classes of functions and generating algorithms. Bivariate approximations are, in fact, a logical extension of the classical methods employed in univariate cases. The relationship is not, however, always self-evident, and many algorithms become difficult to use when multiple variables are introduced. The advent of high-speed computers has now rendered it possible to evaluate lengthy numerical expressions previously impractical due to time considerations. Therefore, the topic of bivariate approximations has now come within the realm of investigation and application.

The greatest single limiting factor encountered in bivariate approximations is the sheer mass of information required to adequately describe a function of two independent variables. If data were to be taken at intervals normally used in univariate cases, the resulting mass of resultant data would be prohibitive. As a result, bivariate data is normally taken at greater intervals, and much higher-order interpolation formulas must be used to obtain acceptable results.
Apparently then, care must be exercised to obtain series approximations that are easy to evaluate.

The most general representation of a bivariate series is:

\[ \sum_{i} A_{i} \phi_{i}(x, y), \]

where \( \phi_{i}(x, y) \) is a real function of real variables, \( x \) and \( y \), defined over some domain of \( x \) and \( y \). Many approximations, however, may be formed by considering only series of the form:

\[ \sum_{i,j} A_{ij} U_{i}(x)V_{j}(y). \]

This form permits many factoring schemes, and avoids recomputation of values, particularly in the generation of tables. Some functions can be satisfactorily approximated by a separation of variables, where the series takes the form:

\[
\left[ \sum_{i} A_{i} U_{i}(x) \right] \left[ \sum_{j} B_{j} V_{j}(y) \right]
\]

Evidently, this form is a factoring of the general expression into two univariate series. While an expression of this form is relatively easy to evaluate, it is rather limited in its applications.
The subject of bivariate approximations may be conveniently divided into three general topics according to the criterion of fit. These are the same criteria commonly used in univariate expansions:

1. Interpolation, in which the approximating function must equal the generating function at prescribed points.

2. Least-squares, in which the sum of the squares of the error of the approximating function is minimized.

3. Chebyshev, or minimax, in which the maximum error of the approximating function is minimized.

While other criteria are possible, these are the most commonly used, and the thesis will be limited to these.
II. INTERPOLATION

Interpolation forms an important topic in the study of approximations not only because the theory is easy to follow, but also because interpolation gives an adequate solution to many problems. Before discussing interpolation, a few terms must be defined. A bivariate sample consists of the finite set \( \{(x_i, y_i, z_i): i = 1, 2, \ldots, N\} \), and sample points consist of the set \( \{(x_i, y_i): i = 1, 2, \ldots, N\} \). An approximation, \( U(x, y) \), interpolates over the bivariate sample \( \{(x_i, y_i, z_i): i = 1, \ldots, N\} \) if it satisfies the relation

\[
U(x_i, y_i) = z_i \quad \text{for} \quad i = 1, 2, \ldots, N.
\]

A number of methods have been devised for developing bivariate interpolating approximations. In particular, references (5), (6), and (7) suggest several such methods. A topic that has not been adequately investigated, however, is a unique method of developing an interpolating approximation from a given bivariate sample. In particular, suppose that it is desired to develop a series of the form

\[
\sum_{i=1}^{N} a_i \phi_i(x, y)
\]

that interpolates the bivariate sample \( \{(x_i, y_i, z_i): i = 1, \ldots, N\} \).

There are many choices for the sequence \( \{\phi_i(x, y)\} \), but
polynomials are a logical choice, not only because they are very easy to evaluate, but also because a great deal is known about their behavior. A common method of taking data of two variables is in "lines" of constant $x$ or $y$. Specifically, a line of constant $x$ consists of the sample points

$$\{(x_i, y_{ij}): j = 1, 2, \ldots, n_i\}.$$ 

A line of constant $x$ may consist of a single point. A unique interpolating polynomial may be developed for a bivariate sample of this type.

**Theorem 2-1:** For some choice of $\{a_{ij}\}$ and $\{d_i\}$, the polynomial

$$ M-1 \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} a_{ij} x^i y^j $$

interpolates the bivariate sample

$$ \{(x_i', y_{ij}', x_{ij}'): i = 1, 2, \ldots, M; j = 1, 2, \ldots, n_i\} $$

where the sample points $\{(x_i, y_{ij})\}$ are distinct.

**Proof:** The $\{d_i\}$ must be chosen in such a manner as to ensure the existence of a unique solution for $\{a_{ij}\}$. 
Let

\[ N_0 = \{ n_i : i = 1, 2, \ldots, M \}, \]

and

\[ N_{k+1} = N_k - \{ \max (n_i) \}, \quad \text{for } k = 0, 1, \ldots, M-1 \]

\[ n_i \in N_k \]

then define

\[ d_k = \max (n_i) - 1 \]

\[ n_i \in N_k \]

The resulting equations

\[
\sum_{i=0}^{M-1} \sum_{j=0}^{d_i} a_{ij} x^i y^j = z
\]

has a unique solution for \( \{a_{ij} \} \) if the corresponding homogeneous equation,

\[
\sum_{i=0}^{M-1} \sum_{j=0}^{d_i} b_{ij} x^i y^j = 0
\]

for \((x, y)\) contained in the set of sample points, does not have non-trivial solutions for \( \{b_{ij} \} \). The homogeneous equation may be re-written

\[
\sum_{i=0}^{M-1} \phi_i (y_j)^k x^i = 0 \quad \text{for } j = 1, 2, \ldots, M; \quad k = 1, 2, \ldots, n_j
\]
where \( \phi_i(y) \) is a polynomial. There are \( M \) distinct values of \( x \) for which

\[
\sum_{i=0}^{M-1} \phi_i(y)x^i = 0,
\]

therefore, \( \phi_i(y) = 0 \) for all \( \phi_i \), and all \( y_{jk} \) contained in the sample. But the degree of \( \phi_i(y) \) was chosen to be less than the number of sample points on at least one of the individual lines, therefore, the polynomial \( \phi_i(y) \) is identically zero, and the theorem if proven. Obviously the same development could be used to define and interpolate lines of constant \( y \).

This method may be applied to a wide variety of bivariate samples for, although the development is for a specific type of sample, the definition of this sample is so broad as to include all types of bivariate samples. Further, as soon as it is agreed upon whether a given sample is to be regarded as lines of constant \( x \), or lines of constant \( y \), the bivariate interpolating polynomial is uniquely determined.
III. LEAST-SQUARE APPROXIMATIONS AND FOURIER COEFFICIENTS

The principle of least-squares, in which the sum of the squares of the errors in the approximating series is minimized may be applied to bivariate approximations. In order to simplify notation, an inner-product may be developed for functions of two variables. Let $U(x, y)$ and $V(x, y)$ be two functions of $x$ and $y$ defined over some domain $z$. The inner product over $z$ is defined as:

$$(U, V)_z = \int \int_U U(x, y)V(x, y) dx dy .$$

If $z$ were a discrete set of sample points, $\{(x_i, y_i); \ i = 1, 2, \ldots, n\}$, then the definition would become

$$(U, V)_z = \sum_{i=1}^{n} U(x_i, y_i)V(x_i, y_i)$$

A bivariate approximation $U(x, y)$ is a least-square approximation to $f(x, y)$ over the set $\{(x, y)\} \in z$ if $(r, r)_z$ is minimized, where $r = U - f$.

The development of a bivariate least-square approximating series to fit a finite set of distinct points is very similar to the univariate development. If the function $U(x, y) = \sum_{i} a_i u_i (x, y)$ is to be a
least-square approximation to the function \( f(x, y) \) over some set, 
\[ z = \{(x_i, y_i)\}, \] then the expression \( (r, r)_z \) must be minimized. For any particular choice of data points and approximating sequence, the expression \( (r, r)_z \) may be regarded as a function of the coefficients, \( \{a_i\} \), in the series expansion of \( U(x, y) \). Therefore, as in calculus, to find the minimum of the expression, set

\[
\frac{\partial (r, r)_z}{\partial a_i} = 0
\]

\[
\frac{\partial (\sum a_j u_j - f, \sum a_k u_k - f)_z}{\partial a_i} = \sum_j \frac{\partial a_j u_j - f}{\partial a_i} + \sum_k \frac{\partial a_k u_k - f}{\partial a_i}
\]

From the definition of the inner product, it is a bilinear operator, therefore, the partial derivative operator may be distributed over the inner product.

\[
\frac{\partial (r, r)_z}{\partial a_i} = (u_i, U - f)_z + (U - f, u_i)_z
\]

\[
= 2(u_i, U - f)_z
\]

\[
= 2\left[(u_i, U)_z - (u_i, f)_z\right]
\]

Therefore:

\[
(u_i, U)_z = (u_i, f)_z
\]

Or, expressing the results explicitly:
\[
\sum_{j} a_j(u_j, u_i)_z = (f, u_i)_z
\]

The resulting set of equations defines the normal equations which may be solved for \( \{a_j\} \) if the corresponding homogeneous equations

\[
\sum_{j} b_j(u_j, u_i)_z = 0
\]

Do not have non-trivial solutions for \( \{b_j\} \).

Again making use of the bilinear property of the inner product,

\[
(\sum_{j} b_j u_j, u_i)_z = 0
\]

Now, multiply the \( i^{th} \) equation by \( b_i \), and sum over all the equations:

\[
\sum_{i} b_i (\sum_{j} b_j u_j, u_i)_z = 0
\]

and again distributing the sum over the inner product:

\[
(\sum_{j} b_j u_j, \sum_{i} b_i u_i)_z = 0
\]
This is true if and only if

$$\sum_{i} b_i u_i(x, y) = 0 \text{ for all } (x, y) \in z$$

Therefore, the homogeneous equations have non-trivial solutions for \(\{b_i\}\) if and only if \(\{u_i\}\) are linearly independent over the set \(z\).

The use of polynomials in least square approximations is of sufficient importance to require more investigation. In this case, \(u_i(x, y) = x^p y^q\), where \(\{p_i\}\) and \(\{q_i\}\) are nonnegative integers. The normal equations become:

$$\left(f(x, y), x^p y^q\right)_z = \sum_j a_j \left(x^j y^j, x^p y^q\right)_z$$

In section two, a method was introduced for finding a set \(\{p_i, q_i\}\) such that \(x^p, y^q\) would be linearly independent over the finite set \(z'\). This method may be used for developing least square polynomials by taking \(z' \subset z\).

The simultaneous solution of the normal equations becomes impractical in many instances. A more practical approach to the subject of least square approximation lies in the study of orthogonal series. The sequence of functions, \(\{u_i(x, y)\}\) are orthogonal over the set \(z\) if
\[(u_i, u_j)_z = c_i \delta_{ij} \quad c_i > 0\]

Where \(c_i\) is some constant and \(\delta_{ij}\) is the Kronecker delta.

This results in a diagonalization of the matrix

\[\left\{(u_i, u_j)_z\right\}\]

and the normal equations become

\[a_i (u_i, u_i)_z = (u_i, f)_z\]

A subject of particular interest is the generation of a bivariate orthogonal sequence from univariate orthogonal sequence. Univariate orthogonal sequences are defined in the usual manner, that is, the sequence \(\{u_i(x)\}\) is orthogonal over the set \(X\) if

\[< u_i, u_j>_X = c_i \delta_{ij}\]

where \(c_i\) is a constant, \(\delta_{ij}\) is the Kronecker delta, and the inner product, \(< u_i, u_j>_X\) is defined as:

\[< u_i, u_j>_X = \int_X u_i(x)u_j(x)dx\]

if \(X\) is real interval; or

\[= \sum_{x \in X} u_i(x)u_j(x)\]

if \(X\) is a finite set of points.
Since univariate orthogonal sequences form a familiar and well-developed topic, they provide an ideal basis for a practical approach to bivariate sequences.

**Theorem 3-1:** If \( \{v_i(x)\} \) is an orthogonal sequence over \( X \), \( \{w_j(y)\} \) is an orthogonal sequence over \( Y \), and \( u_{ij}(x, y) = v_i(x)w_j(y) \), then \( \{u_{ij}(x, y)\} \) is orthogonal over \( z = X \times Y \).

**Proof:** The proof follows directly from a property of the inner product:

\[
(v_i, w_k', v_j, w_l')_{X \times Y} = <v_i, v_j>_X <w_k', w_l'>_Y
\]

Since the univariate sequences \( \{v_i\} \) and \( \{w_j\} \) are orthogonal over the respective sets \( X \) and \( Y \),

\[
<v_i, v_j>_X = c_i \delta_{ij}
\]
\[
<w_k', w_l'>_Y = d_k \delta_{k\ell}
\]

\[
c_i d_k \delta_{ik} \delta_{\ell} = <v_i, v_j>_X <w_k, w_l>_Y
\]
\[
= (v_i, w_k', v_j, w_l')_{X \times Y}
\]
\[
= (u_{ij}, u_{ij})_z
\]

and therefore, \( \{u_{ij}\} \) is orthogonal over \( z = X \times Y \).

The root-mean-square error of a least square approximation
by orthogonal functions is easily obtained. \( e_{\text{rms}} = [(r, r)_z/(1, 1)_z]^{1/2} \)
is the root-mean-square error of the approximation where \( r \) is defined as above, since

\[
(r, r)_z = (f, f)_z - (U, U)_z
\]

\[
= (f, f)_z - \sum_i \sum_j a_i a_j (u_i, u_j)_z
\]

\[
= (f, f)_z - \sum_i a_i^2 (u_i, u_i)_z
\]

\[
e_{\text{rms}} = \left\{ \left[ (f, f)_z - \sum_i a_i^2 (u_i, u_i)_z \right]/(1, 1)_z \right\}^{1/2}
\]
IV. CHEBYSHEV POLYNOMIALS

While least-square series minimize the sum of the square of the error, isolated errors are allowed. Chebyshev polynomials do not necessarily minimize the square of the error, but rather the maximum error. The use of Chebyshev polynomials in univariate approximations is quite widespread, so it follows that a bivariate analogue could certainly find applications. The univariate Chebyshev polynomials consists of a sequence of polynomials, \( \{T_n(x)\} \), where \( T_n(x) = \cos (n \arccos (x)) \). The bivariate analogue consists of the sequence \( \{T_m(x) T_n(y)\} \), which has properties very similar to the classical Chebyshev polynomials.

**Theorem 4-1:** If \( T_n(\xi) = \cos (n \arccos (\xi)) \), \( P_{mn}(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j \) with \( a_{mn} = 1 \), and

\[
U_k = \begin{cases} 
2^{k-1} & \text{if } k \neq 0 \\
1 & \text{if } k = 0 
\end{cases}
\]

then

\[
\max \left[ \frac{1}{U_m U_n} \left| \frac{T_m(x)T_n(y)}{U_m U_n} \right| \right] \leq \max \left[ |P_{mn}(x, y)| \right] \text{ on } -1 \leq x \leq 1; -1 \leq y \leq 1
\]

**Proof:** The function \( T_n(\xi) \) is an equal-ripple function. That is, it alternately assumes its maxima and minima which are all equal in absolute value. Therefore, the polynomial \( T_m(x)T_n(y) \) assumes
its extrema of $\pm 1$ at the points $(x_i, y_j)$, where $x_i = \cos(\pi i/m)$; $i = 0, 1, \ldots, m$ and $y_j = \cos(\pi j/n)$; $j = 0, i, \ldots, n$. To simplify notation, let $R_{im}$ be the line segment defined by $x = \cos(\pi i/m)$, $-1 < y < 1$. Similarly, let $S_{jn}$ by the line segment defined by $y = \cos(\pi j/n)$, $-1 < x < 1$.

Now, assume that

$$\max \left[ \frac{T_m(x)T_n(y)}{U_m U_n} \right] > \max \left[ |P_{mn}(x, y)| \right],$$

and consider the difference

$$\frac{T_m(x)T_n(y)}{U_m U_n} - P_{mn}(x, y).$$

Evidently, this difference may be written as a polynomial of degree $m$ in $x$, and $n$ in $y$. By proper grouping and factoring, the difference polynomial may be represented as

$$\sum_{i=0}^{m} \phi_i(y)x^i,$$

where $\phi_i(y)$ is a polynomial in $y$. For $0 \leq i \leq m-1$, $\phi_i(y)$ is of degree $n$ at most; $\phi_m(y)$, however, is of degree $n-1$ or less. This results from the requirement that the coefficient of
\(x^m y^n\) in the polynomial \(P_{mn}(x, y)\) be one. Since the coefficient of \(x^m y^n\) in \(T_m(x)T_n(y)/U_m U_n\) is also one, the corresponding coefficient in the different polynomial is zero, and hence, the difference polynomial does not include an \(x^m y^n\) term.

Now, consider the behavior of the difference polynomial along \(S_{kn}\). The polynomial \(T_m(x)T_n(y)/U_m U_n\) alternately assumes its maxima and minima along \(S_{kn}\) because \(T_n(y)\) is identically one or minus one, its extrema. Because \(P_{mn}(x, y)\) is less than \(T_m(x)T_n(y)/U_m U_n\) in absolute value, the difference polynomial must alternate in sign. Therefore, the difference polynomial must have \(m\) distinct zeros on \(S_{kn}\). These zeros can not occur at the extrema of \(T_m(x)T_n(y)\) which occur at the endpoints of \(S_{kn}\) and hence the zeros of the difference polynomial on \(S_{kn}\) may be represented as \(-1 < x_{k1} < x_{k2} < \ldots < x_{km} < 1\). The difference polynomial therefore is, except for an arbitrary constant, \(C_k\), uniquely determined on \(S_{kn}\) by

\[
\sum_{i=0}^{m} \phi_i(y_k) x_i = C_k \prod_{i=1}^{m} (x-x_{ki})
\]

By inspection, \(C_k = \phi_m(y_k)\). In a similar manner, \(T_m(x)T_n(y)\) alternately assumes its maxima and minima along \(R_{0m}(x=1, -1<y<1)\) at the intersections of \(R_{0m}\) with \(\{S_{kn}\}\). Therefore, the difference polynomial must alternate in sign, and \(\phi_m(y_k) \prod_{i=1}^{m} (1-x_{ki})\) must
alternate in sign as $k$ increases. The product $\prod_{i=1}^{m} (1-x_{ki})$ is positive which implies that $\phi_{m}(y_k)$ must alternate in sign. The polynomial $\phi_{m}(y_k)$ is of degree $n-1$, but has $n$ distinct zeros. Therefore, $\phi_{m}(y) = 0$. Contrary to the assumption

$$\max \left[ \frac{T_{m} (x) T_{n} (y)}{U_{m} U_{n}} \right] > \max |P_{mn} (x, y)|,$$

$P_{mn} (x, y) = T_{m} (x) T_{n} (y)/U_{m} U_{n}$ at its maximum, and the theorem is proven.

As a direct result of this theorem, it may be shown that the best uniform polynomial approximation of degree less than $m + n$ to $x^m y^n$ is a linear combination of bivariate Chebyshev polynomials.

**Theorem 4-2:** For some choice of coefficients $B_{ij}$, the sum

$$\sum_{(i,j) \in U} B_{ij} T_{i}(x)T_{j}(y) \quad U = \{(i,j): 0 \leq i \leq m; 0 \leq j \leq n; i+j \leq m+n-1\}$$

is the best polynomial approximation of degree $m + n - 1$ to the function $x^m y^n$ over the interval $-1 \leq x \leq 1, -1 \leq y \leq 1$, in the sense that it has the smallest maximum error.

**Proof:** Let
\[ \sum_{(i,j) \in U} C_{ij} P_{ij}(x, y) \]

be any other approximation to \( x^m y^n \), where

\[ P_{kl}(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} x^i y^j; \quad a_{k1} = 1. \]

The function \( x^m y^n \) may be represented exactly by the polynomial

\[ \sum_{i=0}^{m} \sum_{j=0}^{n} B_{ij} T_i(x) T_j(y). \]

Now, consider the approximation obtained by removing the last term of the series

\[ x^m y^n = \sum_{(i,j) \in U} B_{ij} T_i(x) T_j(y) + E_{mn}(x, y). \]

Where \( E_{mn}(x, y) \) is the error term. Evidently,

\[ E_{mn}(x, y) = B_{mn} T_m(x) T_n(y) \]

To obtain the initial requirement
\[ x^m y^n = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{ij} T_i(x) T_j(y), \]

\[ B_{mn} = \frac{1}{U m n}. \]

Therefore,

\[ E_{mn} = \frac{T_m(x)T_n(y)}{U m n}. \]

Now, consider the approximation

\[ x^m y^n = \sum_{(i, j) \in U} C_{ij} P_{ij}(x, y) + E'_{mn}(x, y). \]

The error term, \( E'_{mn} \), is evidently a polynomial:

\[ E'_{mn}(x, y) = x^m y^n - \sum_{(i, j) \in U} C_{ij} P_{ij}(x, y) \]

Theorem 4-1 may now be applied to obtain

\[ \max \left[ |E_{mn}(x, y)| \right] < \max \left[ |E'_{mn}(x, y)| \right] -1 \leq x \leq 1; \ -1 \leq y \leq 1 \]

This proves the theorem, a slightly more general theorem follows directly from Theorem 4-1.

**Theorem 4-3:** If some function can be expressed as a bivariate Chebyshev polynomial times some constant term, say \( c T_m(x) T_n(y), \)
then any other error expression of the form \( b P_{mn}(x, y) \),
\[ b \geq c U_m U_n P_{mn}(x, y), \]
and \( U_k \) defined as above) will have a greater maximum absolute value in the range \(-1 \leq x \leq 1; \ -1 \leq y \leq 1\).

The proof follows trivially from Theorem 4-1.
V. A PRACTICAL MEANS OF APPROXIMATING POWER SERIES

Derivation

Let $F(x, y)$ be defined over some domain $U$, a subset of the finite plane, by the power series

$$
\sum_{i=0}^{M} \sum_{j=0}^{N} A_{ij} x^i y^j
$$

It is desired to approximate $F(x, y)$ by the finite power series:

$$
\sum_{i=0}^{m} \sum_{j=0}^{n} C_{ij} x^i y^j \quad m \leq M; \quad n \leq N
$$

The coefficients $C_{ij}$ may be determined to yield specific results, such as a Chebyshev fit over some domain $U' \subseteq U$. Specifically, let the sequence of functions $\{P_{ij}(x, y)\}$ be given such that for some choice of $\{B_{ij}\}$:

1. $P_{kl}(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} b_{ij} x^i y^j$ with $k! b_{kl} \neq 0$

2. $\sum_{i=0}^{M} \sum_{j=0}^{N} B_{ij} P_{ij}(x, y) = \sum_{i=0}^{M} \sum_{j=0}^{N} A_{ij} x^i y^j \quad (x, y) \in U$
Then the sum
\[
\sum_{i=0}^{M} \sum_{j=0}^{N} B_{ij} P_{ij}(x, y)
\]
may be truncated to yield
\[
\sum_{i=0}^{M} \sum_{j=0}^{K} B_{ij} P_{ij}(x, y).
\]

Finally, \(\{C_{ij}\}\) may be found such that
\[
\sum_{i=0}^{M} \sum_{j=0}^{K} C_{ij} x^i y^j = \sum_{i=0}^{M} \sum_{j=0}^{K} B_{ij} P_{ij}(x, y) \quad (x, y) \in U^1
\]

Examples of such sequences of polynomials are the Chebyshev polynomials discussed in section 3, and the orthogonal polynomials discussed in section 2.

Under the restriction \( i_a j_a \) does not equal zero, there exists a set \( \{k_{ij}\} \) such that
\[
x^k y^\ell = \sum_{i=0}^{k} \sum_{j=0}^{\ell} k_{ij} P_{ij}(x, y)
\]

Therefore, since
\[
\sum_{i=0}^{M} \sum_{j=0}^{N} B_{ij} P_{ij}(x, y) = \sum_{i=0}^{M} \sum_{j=0}^{N} A_{ij} x^i y^j,
\]

\[
\sum_{i=0}^{M} \sum_{j=0}^{N} B_{ij} P_{ij}(x, y) = \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \sum_{\ell=0}^{j} i_{ij} b_{\ell k} P_{\ell k}(x, y)
\]

and therefore:

\[
B_{\ell k} = \sum_{i=k}^{M} \sum_{j=\ell}^{N} A_{ij} i_{ij} b_{\ell k}
\]

Now, if the resulting series is truncated, the approximation becomes:

\[
\sum_{k=0}^{\eta} \sum_{\ell=0}^{\eta} \sum_{i=k}^{M} \sum_{j=\ell}^{N} A_{ij} i_{ij} b_{\ell k} P_{\ell k}(x, y)
\]

From the definition of \( P_{\ell k}(x, y) \), this equals

\[
\sum_{k=0}^{\eta} \sum_{\ell=0}^{\eta} \sum_{i=k}^{M} \sum_{j=\ell}^{N} \sum_{s=0}^{k} \sum_{t=0}^{\ell} A_{ij} i_{ij} b_{\ell k} \sum_{i=s}^{k} \sum_{j=t}^{\ell} a_{s t} x^y
\]

Or, regrouping the summation:

\[
\sum_{s=0}^{\eta} \sum_{t=0}^{\eta} \left[ \sum_{i=s}^{M} \sum_{j=t}^{N} A_{ij} \left( \sum_{k=s}^{r} \sum_{l=t}^{r} i_{ij} b_{\ell k} a_{s t} \right) \right] x^y
\]
where \( q = \min (i, \mathcal{N}) \); \( r = \min (j, \mathcal{N}) \).

This is of the desired form:

\[
\sum_{s=0}^{\mathcal{M}} \sum_{t=0}^{\mathcal{N}} C_{st} x^s y^t
\]

\[
C_{st} = \sum_{i=s}^{M} \sum_{j=t}^{N} A_{ij} \sum_{k=s}^{q} \sum_{\ell=t}^{r} i_{b_{kl}} a_{st}
\]

More significantly, for a given sequence of functions \( \{P_{ij}(x, y)\} \), the weights

\[
W_{ij}^{st} = \sum_{k=s}^{q} \sum_{\ell=t}^{r} i_{b_{kl}} a_{st}
\]

may be compiled so that the problem may be reduced to one suitable for solving on an ordinary desk calculator.

With no loss of generality, attention may be restricted to sequences of functions of the general form:

\[
P_{kl}(x, y) = \left( \sum_{i=0}^{k} \hat{a}_{ki} x^i \right) \left( \sum_{j=0}^{l} \hat{a}_{lj}^* y^j \right)
\]

Referring back to the previous notation, evidently

\[
k_{i}^l a_{ij} = \hat{a}_{ki} \hat{a}_{lj}^* \quad \text{and}
\]

\[
i_{j}^l b_{kl} = \hat{b}_{ij} \hat{b}_{j\ell}^*
\]
where

\[ x^i = \sum_{k=0}^{i} \sum_{j=0}^{k} \hat{b}_{ij} a_{kj} x^j \]  

and

\[ y^i = \sum_{k=0}^{i} \sum_{j=0}^{k} \hat{a}_{ik} a_{kj} y^j \]

Therefore,

\[ s^t W_{ij} = \sum_{k=s}^{q} \sum_{l=t}^{r} \hat{a}_{ks} \hat{a}_{lt} b_{ik} b_{jl} \]

\[ = \sum_{k=s}^{q} \sum_{l=t}^{r} \hat{a}_{ks} b_{ik} \hat{a}_{lt} b_{jl} \]

\[ = w_{si} w_{tj} \]

where

\[ w_{si} = \sum_{k=s}^{q} \hat{a}_{ks} b_{ik} \]

\[ w_{tj} = \sum_{l=t}^{r} \hat{a}_{lt} b_{jl} \]

This simplifies the compilation of the weights, since, for many problems, only two columns of weights need be referenced for the computation of a coefficient, rather than an entire page. The next two theorems further simplify the matter. The requirement that
implies that
\[ \sum_{k=i}^{j} \hat{b}_{jk} \hat{a}_{ki} = \delta_{ij} \text{ for } 0 \leq i \leq j \]

Which in turn implies that \( w_{ii} = 1 \) for \( 0 \leq i \leq m \), and \( w_{ij} = 0 \) for \( i < j \leq m \).

Therefore the formula for the coefficients of the approximating series may be reduced to

\[ C_{kl} = A_{kl} + \sum_{i=\kappa+1}^{N} w_{kl} A_{kj} + \sum_{i=\kappa+1}^{m} w_{ki} A_{il} + \sum_{i=\kappa+1}^{m} w_{ki} \sum_{j=\kappa+1}^{N} w_{lj} A_{ij} \]

The truncation error of the approximating series can also be obtained. If the least-square sum of orthogonal polynomials is truncated to

\[ \sum_{i=0}^{M} \sum_{j=0}^{N} B_{ij} P_{ij}(x, y) \]

is truncated to

\[ \sum_{i=0}^{\kappa} \sum_{j=0}^{\kappa} B_{ij} P_{ij}(x, y) \]
then the RMS error over the domain, \( z \), of the polynomials given by

\[
\left[ \sum_{i=m+1}^{N} \sum_{j=0}^{N} B_{ij}^2 (P_{ij}^*, P_{ij}) z + \sum_{i=0}^{N} \sum_{j=+1}^{N} B_{ij}^2 (P_{ij}^*, P_{ij}) z \right]^{1/2}
\]

Substituting for \( B_{ij} \):

\[
e_{\text{rms}} = \left[ \sum_{i=m+1}^{N} \sum_{j=0}^{N} (P_{ij}^*, P_{ij}) z \sum_{k=i \ell = j}^{i \ell} b_{k \ell} A_{k \ell} \right]^{1/2} + \left[ \sum_{i=0}^{m} \sum_{j=m+1}^{N} (P_{ij}^*, P_{ij}) z \sum_{k=i \ell = j}^{i \ell} b_{k \ell} A_{k \ell} \right]^{1/2} \frac{1}{(1, 1) z^{1/2}}
\]

The maximum error of a Chebyshev approximation to a power series may be obtained in a similar manner:

\[
e_{\text{max}} = \sum_{i=0}^{m} \sum_{j=0}^{N} \sum_{k=i \ell = j}^{i \ell} b_{k \ell} A_{k \ell} \left| \sum_{i=0}^{m} \sum_{j=0}^{N} \sum_{k=i \ell = j}^{i \ell} b_{k \ell} A_{k \ell} \right|
\]

**Use of Tables**

Table number one is a compilation of the weight \( m_{wij} \) for the shifted Chebyshev polynomials, \( T_j^*(x) \), which are defined over the
interval $0 \leq x \leq 1$. Table number two is a similar compilation for the polynomials $P_j(x)$, which are orthogonal over the interval $0 \leq x \leq 1$, and frequently used for generating least-square polynomials over this interval.

To generate the Chebyshev or least square approximating polynomial of degree $m$ over the interval $0 \leq x \leq 1$, the weights may be directly used by applying the formulas of the previous section. If the interval under investigation is other than that for which the tables were designed, they may still be used, but some additional work is required. Specifically, let the function

$$f(x) = \sum_{i=0}^{N} A_i x^i$$

be defined over some domain of $x$ which contain the real interval $a \leq x \leq b$. To generate the least-square or Chebyshev approximation

$$f(x) = \sum_{i=0}^{M} C_i x^i \quad a \leq x \leq b,$$

two linear transformations are required. In the first transformation, the interval $[a, b]$ is mapped onto $[0, 1]$. Therefore, let

$$x^* = \frac{(x-a)}{(b-a)}$$
with
\[ b \neq a \]

Substituting this into the defining power series,

\[ f(x) = \sum_{i=0}^{N} A_i \left( x^* + \frac{a}{b-a} \right)^i (b-a)^i \]

This corresponds to a translation and a magnification. Depending upon the problem at hand and the equipment available, one of two methods commonly used to simplify the expression will be more advantageous. The most familiar method is probably the direct expansion by the binomial theorem:

\[ f(x, y) = \sum_{i=0}^{N} A_i (b-a)^i \sum_{j=0}^{i} \binom{i}{j} (x^*)^j \left( \frac{a}{b-a} \right)^{i-j} \]

\[ = \sum_{j=0}^{N} (b-a)^j \sum_{i=j}^{N} A_i a^{i-j} \binom{i}{j} (x^*)^j \]

\[ = A_j^* (x^*)^j \]

where

\[ A_j^* = \sum_{i=j}^{N} A_i a^{i-j} \left( \frac{b-a}{a} \right)^j \]

The approximating series
may now be developed using the relationship

\[
c_j^* = A_j^* + \sum_{i=M+1}^{N} M_{wj} A_i^*
\]

The final series may be left in the form

\[
\sum_{j=0}^{M} C_j^* \left( \frac{x-a}{b-a} \right)^j
\]

or the series may be rearranged in powers of \( x \) through the relationship

\[
\sum_{j=0}^{M} C_j^* \left( \frac{x-a}{b-a} \right)^j = \sum_{i=0}^{M} \left[ \sum_{j=1}^{M} C_j^* (b-a)^{-j} (-a)^{j-i} (i \ i \ x^i \right]
\]

\[
= \sum_{i=0}^{M} C_i x^i
\]

where

\[
C_i = \sum_{j=1}^{M} \binom{j}{i} C_j^* (b-a)^{-j} (-a)^{j-i}
\]
Horner's method is an alternate means of expanding the polynomial

\[ \sum_{i=0}^{N} A_i(x-a)^{n-i}. \]

Horner's method of expanding the polynomial consists of the algorithm

\[
\begin{align*}
R_{1,1} &= A_0 a \\
R_{n+1,1} &= R_{n,1} + A_0 a \quad \text{for } 1 \leq n \leq m \\
R_{n+1,j} &= R_{n,j} + (R_{n+1,j-1})a \quad \text{for } 1 \leq n \leq m+1-j, \quad 1 \leq j \leq m \\
\sum_{i=0}^{N} A_i(x-a) &= A_0 + \sum_{j=1}^{N} (R_{m+1-j,j})x^j
\end{align*}
\]

A careful examination will show that this method is really the binomial theorem. This algorithm, however, allows a very convenient computation scheme of the form
Returning to the bivariate case, the polynomial

\[ P = \sum_{i=0}^{M} \sum_{j=0}^{N} A_{ij} x^i y^j \]

may be approximated over the domain \( a \leq x \leq b; \ c \leq y \leq d \) in a similar manner. The first step is to establish the mapping

\[ x^* = \frac{(x-a)}{(b-a)} \]

\[ y^* = \frac{(y-c)}{(d-c)} \]

which maps the domain onto the unit square: \( 0 \leq x \leq 1; \ 0 \leq y \leq 1 \).

As in the univariate case, the polynomial is expressed in the form:
The series
\[
\sum_{k=0}^{M} \sum_{\ell=0}^{N} A_{k\ell}^* (x^*)^k (y^*)^\ell
\]
\[
= \sum_{i=k}^{M} \binom{i}{k} (b-a)^k a^{-i} \sum_{j=\ell}^{N} \binom{j}{\ell} (d-c)^\ell c^{-j} A_{ij}
\]

is now used to generate the approximating polynomial

\[
\sum_{k=0}^{M} \sum_{\ell=0}^{N} C_{k\ell}^*(x^*)^k (y^*)^\ell
\]
\[
= \sum_{i=k}^{M} \sum_{j=\ell}^{N} w_{ki} w_{\ell j} A_{ij}
\]

through the relationship

Again, the approximating series may be left in the form

\[
P = \sum_{i=0}^{\mathfrak{m}} \sum_{j=0}^{\mathfrak{n}} C_{ij}^* [(x-a)/(b-a)]^i [(y-c)/(d-c)]^j
\]
or expanded into a power series in \( x \) and \( y \):

\[
P = \sum_{i=0}^{m} \sum_{j=0}^{n} C_{ij} x^i y^j
\]

where

\[
C_{k\ell} = \sum_{i=k}^{m} \left\{ (b-a)^{-i} \binom{i}{k} (-a)^{i-k} \sum_{j=\ell}^{n} [(d-c)^{(j)}] (-c)^{j-1} C_{ij} \right\}
\]

**Examples**

The methods developed in this section may be further illustrated by a few examples:

Suppose that it is desired to approximate the function

\[ f(x, y) = e^{x-y} \]

by a polynomial of the form

\[
U(x, y) = A_{00} + A_{01} x + A_{02} x^2 + A_{10} y + A_{11} xy + A_{12} x^2 y
\]

\[ 0 \leq x \leq 1; \ 0 \leq y \leq 1 \]

Specifically, let \( U(x, y) \) be a Chebyshev approximation. Using the methods of section four, a bivariate Chebyshev approximation could be developed directly. Such an approximation would be:

\[
U(x, y) = B_{00} T_0^*(x) T_0^*(y) + B_{01} T_0^*(x) T_1^*(y) + B_{02} T_0^*(x) T_2^*(y) + B_{10} T_1^*(x) T_0^*(y) + B_{11} T_1^*(x) T_1^*(y) + B_{12} T_1^*(x) T_2^*(y)
\]
Where

\[ T_n(\xi) = \cos \left( n \arccos \frac{2\xi - 1}{1} \right) \]

\[ B_{ij} = C_{ij} \int_0^1 \int_0^1 \frac{T_i(x)T_j(y)e^{x-y}}{4(1-x^2)(1-y^2)} \, dx \, dy \]

\[
C = \begin{cases} 
2/\pi^2 & \text{if } i \neq 0 \neq j \\
4/\pi^2 & \text{if } i = j = 0 \\
2/\pi^2 & \text{if } i \text{ or } j = 0 \text{ but not both}
\end{cases}
\]

Such a method of evaluating \( B_{ij} \) is, at best, difficult. A much simpler method is to expand \( f(x, y) \) into a power series and use the methods of section five to develop an approximation to the power series.

\[
f(x, y) = \sum_{i=0}^{4} \sum_{j=0}^{4} (-1)^{i+j} x^i y^j / i! \, j!
\]

Now, the coefficients of the Chebyshev approximation may be determined through the relation:

\[
A_{kl} = a_{kl} + \sum_{i=\ell+1}^{4} 2w_{ki}a_{ki} + \sum_{i=k+1}^{4} 1w_{li}a_{il} + \sum_{i=k+1}^{4} \sum_{j=\ell+1}^{4} 2w_{ki}w_{lj}a_{ij}
\]

Where \( \{a_{ij}\} \) are the coefficients from the power series.
\[ a_{ij} = (-1)^j i! j! \]

and the weights are taken from the table:

\[
\begin{array}{c|cccc}
  & i = 2 & 3 & 4 \\
\hline
 0 & -0.125 & -0.150 & -0.164 \\
 1 & 1.000 & 0.938 & 0.547 \\
\end{array}
\]

\[
\begin{array}{c|cc}
  & i = 3 & 4 \\
\hline
 0 & 0.031 & 0.055 \\
 1 & -0.562 & -0.876 \\
 2 & 1.500 & 0.500 \\
\end{array}
\]

the resulting approximation is

\[ e^{x-y} = 1.000 + 0.838x + 0.700x^2 - 0.649y - 0.543xy - 0.506x^2y \]

Using the same method but substituting the weights for a least-square approximation, the resulting least-square approximation is

\[ e^{x-y} = 0.942 + 0.923x + 0.684x^2 - 0.630y - 0.617xy - 0.458x^2y \]

As a final example, the method of weights may be used to
approximate \( f(x, y) = 6 + 3xy - 2x^2y + .5y^2 - 4x^2 + x^3 \) for
\[
2 \leq x \leq 4 \\
1 \leq y \leq 2
\]

First the mapping
\[
x^* = \frac{x-2}{4-2} \\
y^* = \frac{y-1}{2-1}
\]

along with the inverse
\[
x = 2(x^* + 1) \\
y = y^* + 1
\]

is established which maps the domain of the approximation onto the unit square, \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \). Substituting the inverse function into \( f(x, y) \), and expanding the function as a polynomial in \( x^* \) and \( y^* \),
\[
f(x, y)^* - 3.5 - 18x^* + 8x^3 - y^* + 0.5y^2 - 10x^*y^* - 8x^2y^*
\]
The coefficients \( \{A_{ij}\} \) may be determined from the table for
\[
A_{00} = -3.5 + (-0.125)(0.5) + (-0.150)(8) \\
\quad = -4.81 \\
A_{01} = -1 + (1.000)(0.5) + (-0.125)(-8) \\
\quad = 0.5 \\
A_{10} = -18 + (8)(0.938) \\
\quad = -10.5 \\
A_{11} = -10 + (-8)(1.000) \\
\quad = -18
Therefore, the resulting Chebyshev approximation is

\[-4.71 - 10.5x^* + 0.5y^* - 18x^* y^*\]

Upon substituting \(x\) and \(y\) back into the polynomial, the approximation becomes

\[-12.8 - 3.8x + 18.5y - 9xy\]

The maximum error of the approximation is 1.31, 9.7\%. 
VI. CONCLUSION

Bivariate approximations form a logical extension of univariate approximations. Indeed, many of the concepts carry through to the bivariate case with only slight modification. In particular, the three general classes of approximations, interpolation, least-squares, and Chebyshev, have logical extensions into bivariate approximations.

While interpolation in two variables is more complex than univariate interpolation, it is never-the-less possible to develop polynomial approximations which interpolate any given set of sample points. There are, in general, more than one such polynomial approximation, but it is possible to define a process which will arrive at a unique interpolating approximation to any such set of sample points.

Similarly, least-square approximations of functions of two variables are very similar to the classical approach to least-square approximations of univariate functions. In particular, it is possible to develop a bivariate analogy of orthogonal functions that allow the development of Fourier coefficients, and hence, a convenient method of expanding functions in least square approximations.

Chebyshev or minimax approximations also have an analogue in bivariate approximations. As stated in Theorem 4-1, the sequence of functions \( \{T_m(x)T_n(y)\}, \ T_n(\xi)\cos(n\arccos \xi) \), has the same property on the square \(-1 \leq x \leq 1; -1 \leq y \leq 1\) that \( T_n(\xi) \) has over
the interval \(-1 \leq \xi \leq 1\). That is, for any polynomial of the form

\[
P_{mn}(x, y) = \sum_{i=0}^{1} \sum_{j=0}^{1} A_{ij} x^i y^j ;
\]

\[
A_{mn} = 1, \quad P_{mn} = T_m T_n / U_m U_n \quad \text{minimizes} \quad \max |P_{mn}(x, y)| \quad -1 \leq x \leq 1; \quad -1 \leq y \leq 1,
\]

where \(U_k = 2^{k-1}\) if \(k > 0\); \(U_0 = 1\). This allows a uniform polynomial approximation to \(x^m y^n\) that has minimum maximum error over the indicated square, and provides a sound basis for bivariate Chebyshev approximations.

Finally, the use of the table of weights affords a practical method of developing least-square or Chebyshev approximations to power series. By using the algorithms developed in section five, the compilation of extensive tables is avoided, and yet, the computations involved are suitable for the desk calculator.
BIBLIOGRAPHY


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### Chebyshev Polynomials

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