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EFFECTS OF SHEAR DEFORMATION IN THE CORE OF A FLAT RECTANGULAR SANDWICH PANEL

1. Buckling Under Compressive End Load
2. Deflection Under Uniform Transverse Load

Information Reviewed and Reaffirmed

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UNITED STATES DEPARTMENT OF AGRICULTURE
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Madison 5, Wisconsin
In Cooperation with the University of Wisconsin

EFFECTS OF SHEAR DEFORMATION IN THE CORE OF A FLAT RECTANGULAR SANDWICH PANEL[1,2]

1. BUCKLING UNDER COMPRESSIVE END LOAD

2. DEFLECTION UNDER UNIFORM TRANSVERSE LOAD

By

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Introduction

Formulas for the buckling loads of flat sandwich panels under uniform compression along two opposite edges and subject to various edge conditions were presented in Forest Products Laboratory Report No. 1525. As stated in that report, the effect of shear deformation in the core was neglected in the derivation of the formulas, since they were an adaptation to sandwich panels of formulas applicable to panels of plywood in which the effect is usually negligible. The present report

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[1] This progress report is one of a series prepared and distributed by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics No. NBA-PO-NAer 00619, Amendment No. 1, and U. S. Air Force No. USAF-PO-(33-038)48-41E. Results here reported are preliminary and may be revised as additional data become available.

[2] Original report dated May 1948.

[3] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

is devoted to determining approximately the effect of the shear deformation of the core in two groups of problems: (1) the reduction in the compressive buckling load of rectangular panels subject to each of the four edge conditions treated in Report No. 1525; (2) the increase in the central deflection of rectangular panels under uniform transverse loads and subject to the conditions that all edges are simply supported or that they are all clamped.

The same underlying method of obtaining approximate results is used in both groups of problems. It is believed that a more satisfactory determination of the adequacy of the method can be obtained from the deflection of panels under uniform transverse load than from compressive buckling loads because the latter are usually not sharply defined, but are obscured to a greater or less extent by the initial lack of flatness of the panels.

The method employed in treating both groups of problems was used in several British reports concerned with the behavior of sandwich panels having isotropic cores and faces.[4]

The method is here extended to apply to sandwich constructions having orthotropic facings and cores. Two of the orthotropic axes of the core and facings in the rectangular panels are assumed to be parallel to the edges of the panels. The third orthotropic axis is then perpendicular to the facings of the panels. It is believed that the method may be expected to yield better approximations if the Young's moduli of the core in the two directions parallel to the edges of the panel are equal or nearly equal than if they are widely different. Hence, the method may be expected to apply reasonably well to panels with honeycomb cores and with cores of end-grain balsa, hycar, cellular cellulose acetate, and similar materials. Results for panels

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[4] Williams, D., Leggett, D. M. A., and Hopkins, H. G. "Flat Sandwich Panels Under Compressive End Loads," Report No. A. D. 3174 Royal Aircraft Extablishment, June 1941.

Leggett, D. M. A., and Hopkins, H. G., "Sandwich Panels and Cylinders Under Compressive End Loads," Report No. S. M. E. 3203, Royal Aircraft Establishment, Aug. 1942.

Hopkins, H. G., and Pearson, S. "The Behavior of Flat Sandwich Panels Under Uniform Transverse Loading," Report No. S. M. E. 3277, Royal Aircraft Establishment, March 1944.

with either isotropic facings or cores or with both facings and cores isotropic can be obtained at once from those for orthotropic facings and cores.

In order to obtain relatively simple approximate formulas, energy methods are used. As in the British reports to which reference is made in footnote 4, it is assumed that any line in the core that is initially straight and normal to the middle surface of the core will remain straight after the deformation of the panel, but that in general it will not be normal to the deformed middle surface but will deviate from this normal direction by an amount that is expressed by a parameter k. The parameter k is determined with the aid of energy methods.

In applying the formulas of Part I to the buckling panels whose isotropic facings are stressed beyond the proportional limit, it is recommended that E_f, the modulus of the facings, be replaced throughout by a reduced modulus. This procedure was followed in Forest Products Laboratory Reports Nos. 1525-A, B, C, and D, and was found to improve materially the agreement between the predicted and observed buckling loads of the panels with aluminum facings that were stressed beyond the proportional limit. The complete theoretical justification for the replacement of E_f by a reduced modulus has not been established. In the appendix to this report, however, it is shown that this procedure is correct for a panel acting as a column under a compressive load if the effect of shear deformation in the core is taken into account by the method used throughout the body of the report.

1. BUCKLING UNDER COMPRESSIVE END LOAD

Expressions for the Components of Displacement and Strain

As the xy plane, choose the undeformed middle surface of the core. The axes of x and y (fig. 1) are the intersections of this middle surface with the planes of two adjacent edges of the panel. The axis of z (fig. 2) is perpendicular to the undeformed middle surface. In the middle surface of the core, let the components of the displacement be

$$u = 0, \; v = 0, \; w = f(x, y) \tag{1}$$

where f(x, y) will be chosen to satisfy the boundary conditions in each of the cases to be considered.

Denote the components of the displacement at a point of the core whose ordinate is z by u_c, v_c, and w_c and assume that

$$u_c = -k\, z \frac{\delta w}{\delta x}, \quad v_c = -k\, z \frac{\delta w}{\delta y}, \quad w_c = w \tag{2}$$

where k is a parameter to be determined. It would probably be better to use different parameters k_1 and k_2 in the expressions for u_c and v_c, but the resulting analysis would be considerably more complicated than if the same parameter is used. Accordingly it seemed best to use but one parameter and to determine by test the adequacy of the resulting approximate formulas.

Denote by u_i, v_i, w_i (i = 1, 2) the components of the displacement at the upper and lower faces of the core. The subscript 1 will refer to the upper face, z = c/2, and the subscript 2 to the lower face, z = -c/2, where c is the thickness of the core. From (2) it follows that

$$u_i = (-1)^i \frac{kc}{2} \frac{\delta w}{\delta x}, \quad v_i = (-1)^i \frac{kc}{2} \frac{\delta w}{\delta y}, \quad w_i = w. \tag{3}$$

It will be assumed that the shear deformation in planes perpendicular to the panel may be neglected in the facings because of their relatively high shear moduli. Denote by f the thickness of the facings and by u_{fi}, v_{fi}, w_{fi} (i = 1, 2) the components of displacement in the middle surfaces of the upper (i = 1) and the lower (i = 2) facings, respectively, Then

$$u_{fi} = u_i + (-1)^i \frac{f}{2} \frac{\delta w}{\delta x} = \frac{(-1)^i}{2} (kc + f) \frac{\delta w}{\delta x}$$

$$v_{fi} = v_i + (-1)^i \frac{f}{2} \frac{\delta w}{\delta y} = \frac{(-1)^i}{2} (kc + f) \frac{\delta w}{\delta y} \tag{4}$$

$$w_{fi} = w.$$

Love's[5] notation will be used for the components of strain. Primed letters will denote components of strain in the core while unprimed letters will denote the corresponding components in the facings. Then from (2) it follows that:

$$
\begin{aligned}
e'_{xx} &= -k\,z\,\frac{\delta^2 w}{\delta x^2}, \quad e'_{yy} = -k\,z\,\frac{\delta^2 w}{\delta y^2}, \\
e'_{yz} &= (1-k)\,\frac{\delta w}{\delta y}, \quad e'_{zx} = (1-k)\,\frac{\delta w}{\delta x}, \\
e'_{xy} &= -2k\,z\,\frac{\delta^2 w}{\delta x \delta y}.
\end{aligned}
\tag{5}
$$

From (4) it follows that the membrane strains in the facings (strains in their middle surfaces) are:

$$
\begin{aligned}
e_{xx} &= \frac{(-1)^i}{2}(kc+f)\,\frac{\delta^2 w}{\delta x^2}, \quad e_{yy} = \frac{(-1)^i}{2}(kc+f)\,\frac{\delta^2 w}{\delta y^2}, \\
e_{xy} &= (-1)^i\,(kc+f)\,\frac{\delta^2 w}{\delta x \delta y}.
\end{aligned}
\tag{6}
$$

Superposed on this state of strain in the facings are the flexural strains arising from the bending of the facings about their middle surfaces. These strains in either facing are expressed by the equations:

$$
e_{xx} = -z'\,\frac{\delta^2 w}{\delta x^2}, \quad e_{yy} = -z'\,\frac{\delta^2 w}{\delta y^2}, \quad e_{xy} = -2z'\,\frac{\delta^2 w}{\delta x \delta y}, \tag{7}
$$

where z' is measured from the middle surface of the facing under consideration.

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[5] Love, A. E. H., "The Mathematical Theory of Elasticity." 1927.

Strain Energy In the Core and Facings

The strain energy in the core is expressed in terms of the strains by the following integral:[6]

$$U_c = \frac{1}{2\lambda'} \int_0^a \int_0^b \int_{-c/2}^{c/2} \Big[E'_x \, e'^2_{xx} + E'_y \, e'^2_{yy}$$

$$+ 2\, E'_x \sigma'_{yx} \, e'_{xx} e'_{yy} + \lambda' \, \mu'_{yz} \, e'^2_{yz} \tag{8}$$

$$+ \lambda' \, \mu'_{zx} \, e'^2_{zx} + \lambda' \, \mu'_{xy} \, e'^2_{xy} \Big] \, dzdydx$$

where $\lambda' = 1 - \sigma'_{xy}\sigma'_{yx}$, E'_x and E'_y are Young's moduli, μ'_{xy}, μ'_{yz}, and μ'_{zx} are moduli of rigidity, and σ'_{xy} and σ'_{yx} are Poisson's ratios.

After substituting the expressions for the strain components from equations (5) and performing the integration with respect to z equation (8) can be written in the form:

$$U_c = U_{cb} + U_{cs} \tag{9}$$

where

$$U_{cb} = \frac{c^3 k^2}{24\lambda'} \int_0^a \int_0^b \Big[E'_x \left(\frac{\delta^2 w}{\delta x^2}\right)^2 + E'_y \left(\frac{\delta^2 w}{\delta y^2}\right)^2 \tag{10}$$

$$+ 2E'_x \, \sigma'_{yx} \frac{\delta^2 w}{\delta x^2} \frac{\delta^2 w}{\delta y^2} + 4\lambda' \, \mu'_{xy} \left(\frac{\delta^2 w}{\delta x \delta y}\right)^2 \Big] \, dydx.$$

$$U_{cs} = \frac{c(1-k)^2}{2} \int_0^a \int_0^b \Big[\mu'_{yz} \left(\frac{\delta w}{\delta y}\right)^2 + \mu'_{zx} \left(\frac{\delta w}{\delta x}\right)^2 \Big] \quad dydx. \tag{11}$$

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[6] See for example U. S. Forest Products Laboratory Reports Nos. 1312 and 1503.

Of these two parts of the strain energy of the core the first, U_{cb}, will be called the strain energy of the core in bending and the second, U_{cs}, the strain energy of the deformation of the core in shear.

The strain energy of the two facings is the sum of the strain energies of the states of strain (6) and (7). Then

$$U_f = \frac{f(kc + f)^2}{4\lambda} \int_0^a \int_0^b \left[E_x \left(\frac{\delta^2 w}{\delta x^2}\right)^2 + E_y \left(\frac{\delta^2 w}{\delta y^2}\right)^2 \right.$$

$$\left. + 2E_x \sigma_{yx} \frac{\delta^2 w}{\delta x^2} \frac{\delta^2 w}{\delta y^2} + 4\lambda \mu_{xy} \left(\frac{\delta^2 w}{\delta x \delta y}\right)^2 \right] dydx$$

$$+ \frac{f^3}{12\lambda} \int_0^a \int_0^b \left[E_x \left(\frac{\delta^2 w}{\delta x^2}\right)^2 + E_y \left(\frac{\delta^2 w}{\delta y^2}\right)^2 \right.$$

$$\left. + 2E_x \sigma_{yx} \frac{\delta^2 w}{\delta x^2} \frac{\delta^2 w}{\delta y^2} + 4\lambda \mu_{xy} \left(\frac{\delta^2 w}{\delta x \delta y}\right)^2 \right] dydx.$$

On combining these two integrals it follows that:

$$U_f = \frac{[3(kc + f)^2 + f^2]f}{12\lambda} \int_0^a \int_0^b \left[E_x \left(\frac{\delta^2 w}{\delta x^2}\right)^2 \right.$$

$$\left. + E_y \left(\frac{\delta^2 w}{\delta y^2}\right)^2 + 2E_x \sigma_{yx} \frac{\delta^2 w}{\delta x^2} \frac{\delta^2 w}{\delta y^2} + 4\lambda \mu_{xy} \left(\frac{\delta^2 w}{\delta x \delta y}\right)^2 \right] dydx. \tag{12}$$

The elastic constants that appear in (12) have the same significance for the facings that those of (10) and (11) have for the core.

Case I. Panels With All Edges Simply Supported

For panels with all edges simply supported the function $w = f(x, y)$, the deflection of the middle surface, is chosen to be

$$w = C \sin \alpha x \sin \beta y \tag{13}$$

where

$$\alpha = \frac{\pi}{a}, \quad \beta = \frac{\pi}{b} \tag{14}$$

In choosing the form (13) it has been assumed that the ratio $\frac{b}{a}$ of the sides of the panel is such that the panel will buckle into a single half-wave. If there appears to be a possibility that the panel will buckle into more than one half-wave, β in (13) must be replaced by $n\beta = \frac{n\pi}{b}$ throughout and that value of the integer n chosen which leads to the smallest buckling load. This replacement can be made in the final formula for the buckling load.

On substituting (13) in (10) and (11), performing the integrations and introducing certain abbreviations it is found that

$$U_{cb} = \frac{C^2 ab\, k^2\, c^3}{96\lambda'} \alpha^2 \beta^2 H', \tag{15}$$

$$U_{cs} = \frac{C^2\, ab\, c\, (1-k)^2}{8} \beta^2 K', \tag{16}$$

where

$$H' = E'_x \frac{b^2}{a^2} + E'_y \frac{a^2}{b^2} + 2A'$$

$$A' = E'_x \,\sigma'_{yx} + 2\lambda' \mu'_{xy} \tag{17}$$

$$K' = \mu'_{yz} + \mu'_{zx} \frac{b^2}{a^2}$$

The substitution of (13) in (12) yields after some reduction:

$$U_f = \frac{C^2 ab f \left[3(kc + f)^2 + f^2\right] \alpha^2 \beta^2}{48\lambda} H \tag{18}$$

where

$$H = E_x \frac{b^2}{a^2} + E_y \frac{a^2}{b^2} + 2A \tag{19}$$

$$A = E_x \sigma_{yx} + 2\lambda\mu_{xy} \tag{20}$$

The work done by the compressive load P per inch of edge during buckling is found from the integral,

$$W = \frac{P}{2} \int_0^a \int_0^b \left(\frac{\delta w}{\delta y}\right)^2 dydx \tag{21}$$

to be

$$W = \frac{PC^2 ab \beta^2}{8} \tag{22}$$

The condition for instability of the panel is expressed by

$$W = U_{cb} + U_{cs} + U_f \tag{23}$$

After substituting (15), (16), (18), and (22) in (23) and solving for P it is found that

$$P = \frac{\pi^2 k^2 c^3 H'}{12\lambda' a^2} + (1-k)^2 cK' + \frac{\pi^2 f}{6\lambda a^2} \left[3(kc + f)^2 + f^2\right] H \tag{24}$$

In this expression k is to be chosen so that P is a minimum.

From $\frac{\delta P}{\delta k} = 0$, it follows that

$$k = \frac{1 - \dfrac{\pi^2 f^2 H}{2\lambda a^2 K'}}{1 + \dfrac{\pi^2 c^2 H'}{12\lambda' a^2 K'} + \dfrac{\pi^2 c f H}{2\lambda a^2 K'}} \tag{25}$$

If this value of k is substituted in (24), an approximate formula will be obtained for P_{crs}, the buckling load per inch of edge when the effect of shear deformation of the core is taken into account. However, a further approximation is possible that leads to a much simpler formula for P_{crs}.

The second term in the denominator of (25) will normally be much smaller than the third term. The ratio of the second to the third term is found to be $\frac{1}{6} \frac{\lambda}{\lambda'} \frac{c}{f} \frac{H'}{H}$. This ratio will be small for sandwiches with weak cores and facings that are not too thin in comparison with the thickness of the core. For example, for a square panel whose facings and core are isotropic, $H' = 4E'$ and $H = 4E$ and λ may be taken equal to λ'. If $c = 0.5$, $f = 0.01$, $E' = 10^4$, $E = 2 \times 10^6$, the ratio in question is equal to $\frac{1}{6} \times 50 \times \frac{1}{200} = \frac{1}{24}$. If the sides of the panel are 10 inches long, the third term of the denominator of (25) is equal to 0.282. The value of the second term is then 0.0117. The change made in the denominator by neglecting this term is thus, in this case, about 1 percent. The term in question contains the factor H' and consequently arose from the term U_{cb} in equation (23). It will now be assumed that the term containing H' in equation (24) for P may also be neglected. This is equivalent to neglecting the term U_{cb} in equation (23). When this is done equations (24) and (25) take the following forms:

$$P = (1-k)^2 c K' + \frac{\pi^2 f}{6\lambda a^2} \left[3(kc + f)^2 + f^2\right] H, \tag{26}$$

$$k = \frac{1 - \frac{\pi^2 f^2 H}{2\lambda a^2 K'}}{1 + \frac{\pi^2 cf H}{2\lambda a^2 K'}} \tag{27}$$

When this value of k is substituted in (26) an approximate expression is obtained for the critical load P per inch of edge, as corrected for the effect of shear deformation in the core. This load has been denoted previously by P_{crs} to distinguish it from P_{cr}, the load obtained by the formulas of Forest Products Laboratory Report No. 1525 in which the effect of shear deformation is neglected.

Before this substitution is made, a change of notation will be adopted. This is done in order to express the results in a form that will be found for each of the other edge conditions, although the constants entering the formulas for the other conditions will have different meanings in each case.

In equations (26) and (27) let

$$R = \frac{\pi^2}{2\lambda a^2} H = \frac{\pi^2}{2\lambda a^2} (E_x \frac{b^2}{a^2} + E_y \frac{a^2}{b^2} + 2A). \tag{28}$$

Then

$$k = \frac{1 - \frac{f^2 R}{K'}}{1 + \frac{c f R}{K'}} \tag{29}$$

On substituting this value of k in (26) as modified by the introduction of R, the following equation is obtained for the buckling load P_{crs} corrected for shear:

$$P_{crs} = \frac{\left[3f(c + f)^2 + f^3 (1 + \frac{cfR}{K'})\right] R}{3(1 + \frac{cfR}{K'})} \tag{30}$$

Formula (30) can be put in a more significant form if it is noted that the numerator divided by 3 is approximately the buckling load per inch of edge of the panel when no allowance is made for the effects of shear deformation in the core. For if the effects of shear deformation of the core are neglected, as well as the direct contribution of the core through its bending stresses to the stiffness of the panel, the buckling load per inch of edge is given by the formula[7]

$$P_{cr} = \frac{\pi^2}{a^2} \frac{(h^3 - c^3)}{12\lambda} \left(E_x \frac{b^2}{a^2} + E_y \frac{a^2}{b^2} + 2A\right)$$

$$= \frac{(h^3 - c^3)R}{6} = \frac{\left[3f(c + f)^2 + f^3\right] R}{3} \tag{31}$$

Then, if the term, $\frac{c f^4 R^2}{K'}$, is neglected in the numerator of equation (30), this equation can be written in the form:

$$P_{crs} = \frac{P_{cr}}{1 + \eta} \tag{32}$$

where

$$\eta = \frac{c f R}{K'} \tag{33}$$

The term neglected will normally be small in comparison with the sum of the remaining terms in the numerator of (30). For

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[7] Formulas (6) and (9) of Report No. 1525 with n = 1. In obtaining this formula approximate expressions have been used for the constants D_1, D_2, and K as defined on page 3 of Report No. 1525. The approximations consist in neglecting the terms $(E_x/\lambda)_c c^3$, $(E_y/\lambda)_c c^3$ and $(A/\lambda)_c c^3$ from the respective numerators of the definitions of D_1, D_2, and K of Report No. 1525. This amounts to neglecting the contribution of the core to the stiffness of the panel in any other way than by merely separating the faces. Obviously this approximation is justified only for cores whose elastic constants are small in comparison with those of the faces.

$$\frac{\frac{c f^4 R}{K'}}{3f(c+f)^2 + f^3} < \frac{f^2}{3(c+f)^2}\,\eta .$$

Now η, which appears in the corrective factor $1/(1+\eta)$ in (32), may be expected to be less than unity in cases of practical interest. For such cases the error committed by neglecting the term in question is obviously small.

In obtaining (26) and (27), and (30) which results from combining them, the assumption was made that U_{cb}, the strain energy of the core in bending, could be neglected. A consideration of the magnitudes of the quantities involved in usual sandwich constructions indicates that the assumption is a plausible one. Numerical calculations of a number of special cases have shown satisfactory agreement between the results obtained by using (24) and (25) and those obtained by using (32), which is a close approximation to (30).

The approximate formula for P_{crs} is given by equation (32) with the constant η defined in equation (33). The quantities R and K' appearing in the definition of η are defined in equations (28) and (17), respectively.

Up to this point, it has been assumed that the panel will buckle in a single longitudinal half-wave. If it appears likely that the panel will buckle into n longitudinal half-waves, the letter b is to be replaced by b/n in all formulas. The smallest value of P_{crs} that is found for the various values of n is the buckling load per inch of edge.

Case II. Panels having the loaded edges simply supported and the remaining edges clamped

For panels having the loaded edges simply supported and the remaining edges clamped the procedure is exactly the same as for Case I, except that for a panel buckling in a single longitudinal half-wave the deflection of the middle surface is now chosen to be given by

$$w = C \sin^2 \alpha x \sin \beta y \tag{34}$$

instead of by (13).

As in Case I, the strain energy of the core in bending U_{cb} is neglected. The strain energy of the shear deformation of the core is found from (11) and (34) to be

$$U_{cs} = \frac{C^2\, a\, b\, c\, (1 - k)^2}{32} \left[3\mu'_{yz}\, \beta^2 + 4\mu'_{zx}\, \alpha^2\right]. \tag{35}$$

The strain energy in the facings, U_f, is found from (12) and (34) to be

$$U_f = \frac{C^2\, a\, b\, f \left[3(kc + f)^2 + f^2\right]}{12\lambda} \left[E_x\alpha^4 + \frac{3}{16} E_y\beta^4 + \frac{A}{2}\alpha^2\, \beta^2\right] \tag{36}$$

The work done by the compressive load P per inch of edge during buckling is found to be

$$W = \frac{3}{32}\, P\, C^2\, \beta^2\, a\, b \tag{37}$$

From the condition for instability of the panel

$$W = U_{cs} + U_f$$

it follows that

$$P = \frac{c(1 - k)^2}{3} K' + \frac{f\left[3(kc + f)^2 + f^2\right] R}{9} \tag{38}$$

where

$$K' = 3\mu'_{yz} + 4\mu'_{zx} \frac{b^2}{a^2}, \tag{39}$$

$$R = \frac{8\pi^2}{\lambda a^2} \left[E_x \frac{b^2}{a^2} + \frac{3}{16} E_y \frac{a^2}{b^2} + \frac{A}{2}\right] \tag{40}$$

The load P will be a minimum with respect to k if

$$k = \frac{1 - \frac{f^2 R}{K'}}{1 + \frac{c\,f\,R}{K'}} \tag{41}$$

This expression for k is the same in form as that in equation (29) but the quantities K' and R are defined by (39) and (40) instead of by (17) and (28).

The substitution of this value of k in (38) yields, after some reduction,

$$P_{crs} = \frac{\left[3f(c + f)^2 + f^3 (1 + \eta)\right] R}{9(1 + \eta)} \tag{42}$$

where

$$\eta = \frac{cfR}{K'}. \tag{43}$$

After neglecting the term $f^3 \eta R$, equation (42) can be written

$$P_{crs} = \frac{P_{cr}}{1 + \eta}, \tag{44}$$

where P_{cr} is the buckling load per inch of edge without correction for shear deformation, for in accordance with equation (11) of Forest Products Laboratory Report No. 1525,

$$P_{cr} = \frac{16\pi^2}{3a^2}\left[D_1 \frac{b^2}{a^2} + \frac{3}{16} D_2 \frac{a^2}{b^2} + \frac{K}{2}\right] \tag{45}$$

$$= \frac{(h^3 - c^3)R}{18} = \frac{\left[3f(c + f)^2 + f^3\right]R}{9}.$$

In arriving at this form, the contribution of the core was neglected in calculating[7] D_1, D_2, and K. The justification for neglecting the term of $f^3 \eta R$ in the numerator of equation (42) is found in the discussion following equations (32) and (33).

The analysis just given assumes that the panel buckles into a single longitudinal half-wave. If there appears to be a possibility that the panel will buckle into more than one half-wave, β in (34) must be replaced by $n\,\beta = n\,\frac{\pi}{b}$ throughout and that value of the integer n chosen which leads to the smallest buckling load. As in Case I, this replacement can be made in the final formulas.

Case III. Panels having the loaded edges clamped and the remaining edges simply supported

The analysis used in the remaining cases is identical in plan with that used in the previous cases. In each instance, the buckling load per inch of edge is given by the formula

$$P_{crs} = \frac{P_{cr}}{1+\eta}, \tag{46}$$

where P_{cr} is the buckling load per inch of edge without correction for shear, as obtained from the approximate formulas of Forest Products Laboratory Report No. 1525, the core being neglected in calculating[7] D_1, D_2, and K and where

$$\eta = \frac{cfR}{K'} \tag{47}$$

with proper definitions for R and K' in each instance.

From this point on, only the form assumed for the deflected middle surface and the expressions for the quantities P_{cr}, R, and K' will be given.

For one longitudinal half-wave:

$$w = C \sin \alpha x \sin^2 \beta y,$$

$$K' = 4\mu'_{yz} + 3\mu'_{zx} \frac{b^2}{a^2}, \tag{48}$$

$$R = \frac{8\pi^2}{\lambda a^2} \left(\frac{3}{16} E_x \frac{b^2}{a^2} + E_y \frac{a^2}{b^2} + \frac{A}{2} \right), \tag{49}$$

$$P_{cr} = \frac{(h^3 - c^3)R}{24} \tag{50}$$

For two longitudinal half-waves:

$$w = C \sin \alpha x \sin \beta y \sin 2 \beta y. \tag{51}$$

(This choice is suggested by the fact that w vanishes along the nodal line, $y = b/2$, while the slope does not vanish along this line.)

$$K' = 5\mu'_{yz} + \mu'_{zx} \frac{a^2}{b^2} \tag{52}$$

$$R = \frac{\pi^2}{2\lambda a^2} \left(E_x \frac{b^2}{a^2} + 41\ E_y \frac{a^2}{b^2} + 10A \right), \tag{53}$$

$$P_{cr} = \frac{(h^3 - c^3)R}{30} \tag{54}$$

For three longitudinal half-waves:

$$w = C \sin \alpha x \sin \beta y \sin 3\beta y, \tag{55}$$

$$K' = 10\mu'_{yz} + \mu'_{zx} \frac{b^2}{a^2} \tag{56}$$

$$R = \frac{\pi^2}{2\lambda a^2} \left(E_x \frac{b^2}{a^2} + 136\ E_y \frac{a^2}{b^2} + 20A \right), \tag{57}$$

$$P_{cr} = \frac{(h^3 - c^3)R}{60} \tag{58}$$

Case IV. Panels having all edges clamped. For one longitudinal half-wave:

$$w = C \sin^2 \alpha x \sin^2 \beta y, \tag{59}$$

$$K' = \mu'_{yz} + \mu'_{zx} \frac{b^2}{a^2} \tag{60}$$

$$R = \frac{2\pi^2}{3\lambda a^2} (3E_x \frac{b^2}{a^2} + 3E_y \frac{a^2}{b^2} + 2A), \tag{61}$$

$$P_{cr} = \frac{(h^3 - c^3)R}{6} \tag{62}$$

For two longitudinal half-waves:

$$w = C \sin^2 \alpha x \sin \beta y \sin 2\beta y, \tag{63}$$

$$K' = 15\mu'_{yz} + 4\mu'_{zx} \frac{b^2}{a^2}, \tag{64}$$

$$R = \frac{\pi^2}{2\lambda a^2} (16E_x \frac{b^2}{a^2} + 123\ E_y \frac{a^2}{b^2} + 40A) \tag{65}$$

$$P_{cr} = \frac{(h^3 - c^3)R}{90} \tag{66}$$

For three longitudinal half-waves:

$$w = C \sin^2 \alpha x \sin \beta y \sin 3\beta y, \tag{67}$$

$$K' = \frac{15}{4}\mu'_{yz} + \frac{1}{2}\mu'_{zx}\frac{b^2}{a^2}, \tag{68}$$

$$R = \frac{\pi^2}{2\lambda a^2}\left(2E_x\frac{b^2}{a^2} + 51E_y\frac{a^2}{b^2} + 10A\right), \tag{69}$$

$$P_{cr} = \frac{2(h^3 - c^3)R}{45} \tag{70}$$

In all cases the constant R, which appears in (33) (also in (43) and (47)), the definition of η, may be expressed in terms of P_{cr} by solving the appropriate equation connecting P_{cr} and R.

2. DEFLECTION UNDER UNIFORM TRANSVERSE LOAD

I. Panels with simply supported edges

The increase in the central deflection associated with shear deformation of the core will be determined approximately by assuming that the displacements of points in the core are represented by equations (2). The form of the deflected middle surface of the panel is taken to be given by equation (13),

$$w = C \sin \alpha x \sin \beta y.$$

The coefficient C, the deflection at the center of the panel, and the parameter k of equations (2) will be determined to make a minimum the total potential energy of the system composed of panel and load.

A more general procedure would be to follow that used in the paper by Hopkins and Pearson[2] and take w to be given by

$$w = \Sigma_m \Sigma_n w_{mn}, \tag{71}$$

where

$$w_{mn} = C_{mn} \sin \alpha_m x \sin \beta_n y \tag{72}$$

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b} \tag{73}$$

The expressions (2) for the displacement would then be replaced by

$$u_c = -\Sigma_m \Sigma_n k_{mn} z \frac{\delta w_{mn}}{\delta x}, \quad v_c = -\Sigma_m \Sigma_n k_{mn} z \frac{\delta w_{mn}}{\delta y} \tag{74}$$

The use of the energy method will lead to pairs of equations, each pair containing only one of the coefficients C_{mn} and the associated parameter k_{mn}. For a panel that is square or nearly so, however, the assumption of a single term as in (13) of the double Fourier's series for w leads to quite accurate values of the central deflection for panels in which the effect of shear deformation can be neglected. The following analysis is based upon the use of equation (13).

It will be assumed that sides of the rectangular panels considered are nearly equal. On the basis of the behavior of panels of isotropic materials in which correction for shear deformation is not necessary, it appears that satisfactory results may be obtained by the use of a single term of (71) if the ratio[8] of the longer to the shorter side is less than 1.4. This statement applies only to the calculation of the central deflection. More terms are necessary for the calculation of bending moments and shearing forces.

By using equations (13) and (2) the expressions for U_{cs}, the strain energy of the deformation of the core in shear, and U_f, the strain energy of the facings, are given by (16) and (18), respectively. As

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[8]For orthotropic panels, the ratio in question is $\frac{b}{a} \frac{E_1}{E_2}^{1/4}$ where E_1 associated with the direction x parallel to the side a is defined by the equation $E_1 = \frac{E_x (h^3 - c^3)}{h^3}$ and in like manner $E_2 = \frac{E_y (h^3 - c^3)}{h^3}$, E_x and E_y being the moduli of the facings. In case the ratio as determined in this way turns out to be less than unity, interchange b and a, and E_1 and E_2 to determine the approximate range within which restriction to a single term of the double Fourier's series may be considered to be adequate.

in the case of buckling, the strain energy of the core in bending will be neglected. The work W_l done by the uniform load p per unit area is

$$W_l = p\int_0^a \int_0^b w \, dydx = \frac{4pabC}{\pi^2} \tag{75}$$

The total potential energy of the system is then

$$W = U_{cs} + U_f - W_l \tag{76}$$

In the expression (18) for U_f introduce the quantity R, which will be defined in terms of H by the equation

$$R = \frac{\pi^2}{2\lambda a^2} H = \frac{a^2 H}{2\lambda} \tag{77}$$

Then (76) becomes

$$W = \frac{C^2 \, abc}{8} (1 - k)^2 \, \beta^2 K' + \frac{C^2 abf\left[3(kc + f)^2 + f^2\right] \beta^2 R}{24} - \frac{4pabC}{\pi^2} \tag{78}$$

In this equation K' is defined by (17) and H, which occurs in the definition of R, is defined by (19).

The quantities C and k are to be found from the equations

$$\frac{\delta W}{\delta C} = 0 \text{ and } \frac{\delta W}{\delta k} = 0$$

$$\frac{\delta W}{\delta C} = \frac{C \, abc}{4} (1 - k)^2 \, \beta^2 K' + \frac{C \, abf\left[3(kc + f)^2 + f^2\right]\beta^2 R}{12} - \frac{4pab}{\pi^2} = 0 \tag{79}$$

$$\frac{\delta W}{\delta k} = \frac{-C^2 \, abc \, (1 - k) \, \beta^2 K'}{4} + \frac{C^2 \, abcf \, (kc + f) \, \beta^2 R}{4} = 0 \tag{80}$$

From (80) it follows that

$$k = \frac{1 - \frac{f^2R}{K'}}{1 + \frac{cfR}{K'}} \tag{81}$$

On substituting this value of k in (79) it is found that (79) reduces to the following equation:

$$\frac{C\beta^2}{12} \frac{\left[3f(c + f)^2 + f^3\right] R + \frac{cf^4R^2}{K'}}{1 + \frac{cfR}{K'}} = \frac{4p}{\pi^2} \tag{82}$$

Let

$$\frac{cfR}{K'} = \eta \tag{83}$$

and note that

$$3f(c + f)^2 + f^3 = \frac{h^3 - c^3}{2} \tag{84}$$

It follows from (82) that

$$C = \frac{96p}{\pi^2\beta^2R\ (h^3 - c^3 + 2f^3\eta)} (1 + \eta) \tag{85}$$

In accordance with the discussion following equations (32) and (33) the term $2f^3\eta$ in the denominator will be neglected in comparison with $(h^3 - c^3)$. Then (85) becomes

$$C = \frac{96p\ (1 + \eta)}{\pi^2\beta^2R\ (h^3 - c^3)} \tag{86}$$

For a panel in which no correction is to be made for shear deformation in the core, the parameter k of equations (2) is to be replaced by unity. At the same time replace C in (13) by C_o so that C_o is the central deflection of a panel for which no correction for shear deformation is necessary. The expression for the total potential energy then becomes:

$$W = \frac{C_o^2\, ab(h^3 - c^3)\, \beta^2 R}{48} - \frac{4p\, ab\, C_o}{\pi^2} \tag{87}$$

if use is made of (84).

It follows immediately from the equation

$$\frac{\delta W}{\delta C_o} = 0$$

that

$$C_o = \frac{96p}{\pi^2 \beta^2 R(h^3 - c^3)} \tag{88}$$

The presence of the factor $(h^3 - c^3)$ in equation (88) is to be attributed to the fact that the contribution of the core to the bending stiffness of the panel has been neglected.

Equation (86) may then be written

$$C = C_o \ (1 + \eta) \tag{89}$$

This equation states that the central deflection of a panel as corrected for the effect of shear deformation in the core is to be found by multiplying by the factor $(1 + \eta)$, the central deflection as calculated for the panel without allowance for the effect of shear deformation. The quantity η is defined by equation (83). Written out in full

$$\eta = \frac{\pi^2 cf(E_x \frac{b^2}{a^2} + E_y \frac{a^2}{b^2} + 2A)}{2\lambda a^2 \ (\mu'_{yz} + \mu'_{zx} \frac{b^2}{a^2})}, \tag{90}$$

where the primed letters for the elastic constants refer to the core and the unprimed letters for such constants refer to the facings.

II. Edges clamped

The form of the deflected surface will be chosen to be

$$w = C \sin^2 \alpha x \sin^2 \beta y \tag{91}$$

where

$$\alpha = \frac{\pi}{a}, \ \beta = \frac{\pi}{b}$$

This choice of the form of the deflected surface leads to satisfactory results for the central deflection of isotropic panels in which shear deformation can be neglected if the ratio of the larger to the smaller side of the panel is less than 1.4. For the corresponding ratio in the case of orthotropic panels see footnote . By using equations (11) and (12), the strain energy U_{cs} of the shearing deformation of the core and U_f, the strain energy of the facings, are found to be, respectively:

$$U_{cs} = \frac{3C^2abc}{32}(1 - k)^2 \beta^2 K', \tag{92}$$

$$U_f = \frac{C^2abf\left[3(kc + f)^2 + f^2\right]\beta^2 R}{32}, \tag{93}$$

where

$$K' = \mu'_{yz} + \mu'_{zx}\frac{b^2}{a^2}, \tag{94}$$

$$R = \frac{2\pi^2}{3\lambda a^2}\left(3E_x \frac{b^2}{a^2} + 3E_y \frac{a^2}{b^2} + 2A\right) \tag{95}$$

The work W_l done by the load is readily found.

$$W_l = p\int_0^a \int_0^b w \, dydx = p\,C \int_0^a \int_0^b \sin^2 \alpha x \sin^2 \beta y \, dydx = \frac{pCab}{4} \tag{96}$$

The total potential energy W of the system is therefore

$$W = U_{cs} + U_f - W_l$$

$$W = \frac{3C^2abc}{32}(1 - k)^2 \beta^2 K' + \frac{C^2abf\left[3(kc + f)^2 + f^2\right]\beta^2 R}{32} - \frac{pCab}{4} \quad (97)$$

The values of k and C that render W a minimum are to be obtained from the equations

$$\frac{\delta W}{\delta C} = \frac{3Cabc}{16}(1 - k)^2 \beta^2 K' + \frac{Cabf\left[3(kc + f)^2 + f^2\right]\beta^2 R}{16} - \frac{pab}{4} = 0 \quad (98)$$

and

$$\frac{\delta W}{\delta k} = -\frac{3C^2abc}{16}(1 - k)\beta^2 K' + \frac{3C^2abcf\,(kc + f)\,\beta^2 R}{16} = 0 \quad (99)$$

From (99)

$$k = \frac{1 - \frac{f^2R}{K'}}{1 + \frac{cfR}{K'}} \quad (100)$$

By substituting this value of k in (98) the following equation is obtained:

$$\frac{C\beta^2}{16}\;\frac{\left[3f(c + f)^2 + f^3\right]R + \frac{cf^4R^2}{K'}}{1 + \frac{cfR}{K'}} - \frac{p}{4} = 0 \quad (101)$$

Let

$$\eta = \frac{cfR}{K'} \quad (102)$$

and make use of equation (84). It follows from (101) that:

$$C = \frac{8p(1 + \eta)}{\beta^2 R(h^3 - c^3 + 2f^3\eta)} \tag{103}$$

If $2f^3\eta$ is neglected in comparison with $h^3 - c^3$, equation 103 becomes:

$$C = \frac{8p(1 + \eta)}{\beta^2 R(h^3 - c^3)} \tag{104}$$

If C_o denotes the central deflection of a panel when no allowance is made for shear deformation, it is easily found from (97) by setting $k = 1$ and $C = C_o$, that

$$C_o = \frac{8p}{\beta^2 R(h^3 - c^3)} \tag{105}$$

It is readily checked that, for a panel of isotropic facings and core in which the contribution of the core to the bending stiffness is neglected, equation (105) gives the central deflection with an error of less than 4 percent if the ratio of the longer side to the shorter side of the panel is 1.4 or less.

In accordance with equations (104) and (105) the correction of the central deflection for shear deformation in the core is to be obtained from the equation

$$C = C_o\ (1 + \eta), \tag{106}$$

where η is defined by equation (102). Written out in full

$$\eta = \frac{2\pi^2 cf(3E_x \frac{b^2}{a^2} + 3E_y \frac{a^2}{b^2} + 2A)}{3\lambda a^2(\mu'_{yz} + \mu'_{zx} \frac{b^2}{a^2})} \tag{107}$$

The argument for neglecting the term of $2f^3\eta$ in the denominator of equation (103) is identical with that following equations (32) and (33).

Notation

a, b	dimensions of the panel with the sides b parallel to the line of action of the compressive load.
c	thickness of the core.
f	thickness of each facing.
$h = c + 2f$	thickness of the panel.
$e_{xx} \dots e_{xy}$	components of strain in facings.
$e'_{xx} \dots e'_{xy}$	components of strain in core.
u_c, v_c, w_c	components of displacement in core.
u, v, w	components of displacement at center of core.
u_i, v_i, w_i $(i = 1, 2)$	components of displacement at upper and lower faces of core, respectively.
u_{fi}, v_{fi}, w_{fi} $(i = 1, 2)$	components of displacement in middle of upper and lower facings, respectively.
E_x, E_y	Young's moduli of facings.
E'_x, E'_y	Young's moduli of core.
K'	a constant in the formula $\eta = cfR/K'$. A different definition in each case.
p	transverse load per unit area.
P_{crs}	compressive buckling load per inch of edge corrected for the effect of shear deformation.
P_{cr}	compressive buckling load per inch of edge when effect of shear deformation is neglected.
R	a constant in the formula $\eta = cfR/K'$. A different definition in each case.
U_c	strain energy of deformation of the core, associated with buckling.
U_{cb}	flexural strain energy of the core.
U_{cs}	strain energy of shear deformation of the core.
U_f	strain energy of deformation of the facings.
W	work done by the load P per inch of edge of sandwich. It is associated with the shortening of the panel in bending to the form $w = cf(x, y)$.
$\alpha = \frac{\pi}{a}$	
$\beta = \frac{\pi}{b}$	

$\lambda = 1 - \sigma_{xy}\,\sigma_{yx}$ for the faces.

$\lambda' = 1 - \sigma'_{xy}\,\sigma'_{yx}$ for the core.

μ_{xy}	shear modulus of facings in plane xy corresponding to the orthotropic axes x and y.
μ'_{yz}, μ'_{zx}, μ'_{xy}	shear moduli of core.
σ_{xy}, σ_{yx}	Poisson's ratios of facings.
σ'_{xy}, σ'_{yx}	Poisson's ratios of core.

APPENDIX[9]

Use of the Reduced Modulus in the Buckling Formulas for Isotropic Facings When the Facings Are Stressed Beyond the Proportional Limit

In Forest Products Laboratory Reports Nos. 1525-A, B, C, and D the buckling stress for the sandwich panels with aluminum facings was usually above the proportional limit. In such cases the formula for calculating the buckling stress was modified by replacing the modulus of the facings by a reduced modulus. A theoretical justification of this procedure is not known, but it leads to results that are considered to be in satisfactory agreement with the results of tests.

Some support can be found for this procedure by applying the method of Williams, Leggett, and Hopkins[4] that was applied in the present report to the buckling of sandwich panels, to the buckling of a sandwich column whose isotropic facings are stressed beyond the proportional limit. It will be found that the formula for buckling below the proportional limit stress will apply if the Young's modulus of the facings is replaced by a reduced modulus, E_r.

In figure 3, A, is shown a section of the column made by a plane perpendicular to the facings and parallel to the direction of loading and in figure 3, B, is shown a part of this section drawn to a larger scale. It is assumed that, at the instant when buckling begins, the modulus of the facing that is beginning to be convex outward becomes, because of the diminishing stress in that facing, the usual modulus of the material of the facings below the proportional limit. In the facing that is concave outward the modulus will be the tangent modulus of the material at the stress at which buckling begins. As a consequence, the neutral axis will be displaced by an amount that is to be determined in the course of the analysis.

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[9]This appendix was prepared from the notes of C. B. Smith, formerly mathematician at the Forest Products Laboratory.

Let the x-axis be chosen to coincide with the neutral axis at the instant at which buckling begins. In figure 3, B, the facing that is to be convex outward (the left-hand facing) is taken to be at a distance t from the x-axis, while the other face is at a distance c-t. The modulus in the left-hand facing, $z > t$, will be E_f, the modulus of the facings at a stress below the proportional limit, while that in the right-hand facing will be E_t, the tangent modulus of the material of the facings at the stress in question.

Denote the core, the left-hand and the right-hand facings by the subscripts c, 1, and 2, respectively. Denote components of the displacement parallel to the x and z axes by u and w, respectively, and assume that w is independent of z.

In the core let

$$u_c = -k\, z \frac{\delta w}{\delta x} \tag{108}$$

Then at the inner boundary $z = t$ of the left-hand facing

$$u_1 = -k\, t \frac{\delta w}{\delta x} \tag{109}$$

And at the inner boundary $z = -(c - t)$ of the right-hand facing

$$u_2 = k(c - t) \frac{\delta w}{\delta x} \tag{110}$$

The effect of shear deformation in the facings will be neglected. Hence, at the center, $z = t + \frac{f}{2}$, of the left-hand facing,

$$u_{f1} = u_1 - \frac{f}{2}\frac{\delta w}{\delta x} = -(k\, t + \frac{f}{2}) \frac{\delta w}{\delta x}\, ; \tag{111}$$

and at the center, $z = -(c - t) - \frac{f}{2}$ of the right-hand facing.

$$u_{f2} = u_2 + \frac{f}{2}\frac{\delta w}{\delta x} = \left[k(c - t) + \frac{f}{2}\right]\frac{\delta w}{\delta x} \tag{112}$$

The expressions for the strains in core and facings follow:

In the core the shearing strain is

$$(e_{xz})_c = \frac{\delta u_c}{\delta z} + \frac{\delta w}{\delta x} = (1 - k) \frac{\delta w}{\delta x} \tag{113}$$

The energy of the core that is associated with the direct strains $(e_{xx})_c$ that are developed in bending will be neglected. Consequently, an expression for this component of strain will not be written down.

The membrane strains e_{xx} in the facings are:

$$e_{xx} = -(k\ t + \frac{f}{2}) \frac{\delta^2 w}{\delta x^2}, \text{ in the left-hand facing;} \tag{114}$$

$$e_{xx} = \left[k(c - t) + \frac{f}{2} \right] \frac{\delta^2 w}{\delta x^2}, \text{ in the right-hand facing.} \tag{115}$$

The flexural strains of the facings associated with bending about the middle planes of the facings will be expressed as usual:

$$e_{xx} = -z' \frac{\delta^2 w}{\delta x^2}$$

where z' is a coordinate perpendicular to each of the respective middle planes of the facings.

Let

$$\mu'_{xz} = \mu' \tag{116}$$

be the modulus of rigidity of the core in a plane perpendicular to the facings and parallel to the direction of loading.

As usual, assume that the lateral deflection of the column as buckling begins is given by

$$w = A \sin \alpha x \tag{117}$$

where

$$\alpha = \frac{\pi}{L}, \tag{118}$$

and L is the length of the column.

The strain energy W_c of the core per unit width associated with this lateral deflection is, since the energy of the core in bending is neglected,

$$W_c = \frac{1}{2} \int_o^L \int_{t-c}^{t} \mu' (e_{xz})_c^2 \, dzdx \tag{119}$$

$$= \frac{\mu'}{2} \int_o^L \int_{t-c}^{t} (1-k)^2 \left(\frac{\delta w}{\delta x}\right)^2 dzdx$$

$$= \frac{\mu' (1-k)^2 A^2 \alpha^2 Lc}{4}$$

The strain energy W_f of the facings per unit width is made up of two parts for each facing, one part being associated with the membrane stresses, the other with the flexural stresses. It is to be noted that in one facing $E = E_f$, while in the other $E = E_t$. Since w is positive in the direction of z positive, the left-hand facing of the column in figures 3, A, and 3, B, will be considered to be convex outward. Consequently, E_f is to be associated with the left-hand facing and E_t with the right-hand facing. Accordingly with subscripts 1 and 2 referring to the left- and right-hand facings, respectively,

$$W_f = \frac{f}{2\lambda} \int_0^L E_f(e_{xx})_1^2 \, dx + \frac{f}{2\lambda} \int_0^L E_t(e_{xx})_2^2 \, dx$$

$$+ \frac{f^3}{24\lambda} \int_0^L E_f(\frac{\delta^2 w}{\delta x^2})^2 \, dx + \frac{f^3}{24\lambda} \int_0^L E_t \, (\frac{\delta^2 w}{\delta x^2})^2 \, dx \qquad (120)$$

$$= \frac{E_f f}{2\lambda} (kt + \frac{f}{2})^2 \int_0^L (\frac{\delta^2 w}{\delta x^2})^2 \, dx + \frac{E_t f}{2\lambda} \left[k(c-t) + \frac{f}{2}\right]^2 \int_0^L (\frac{\delta^2 w}{\delta x^2})^2 \, dx$$

$$+ \frac{E_f f^3}{24\lambda} \int_0^L (\frac{\delta^2 w}{\delta x^2})^2 \, dx + \frac{E_t f^3}{24\lambda} \int_0^L (\frac{\delta^2 w}{\delta x^2})^2 \, dx$$

After entering the value of w from (117) it is found after some reduction that

$$W_f = \frac{A^2 a^4 fL}{48\lambda} \left\{ 12E_f \, (kt + \frac{f}{2})^2 + 12E_t \left[k(c-t) + \frac{f}{2}\right]^2 \right. \qquad (121)$$

$$\left. + (E_f + E_t) \, f^2 \right\}$$

The work W_l done by the compressive load P per unit width in buckling to the form (117) is given by

$$W_l = \frac{P}{2} \int_0^L (\frac{\delta w}{\delta x})^2 \, dx \qquad (122)$$

Then

$$W_l = \frac{P A^2 a^2 L}{4} \qquad (123)$$

From the condition for instability

$$W_l = W_c + W_f \tag{124}$$

it is readily found that

$$P = \mu'(1 - k)^2 c + \frac{f\alpha^2}{12\lambda}\left\{12E_f \left(kt + \frac{f}{2}\right)^2 + 12E_t \left[k(c - t) + \frac{f}{2}\right]^2 + (E_f + E_t) f^2 \right\} \tag{125}$$

The parameters k and t are to be chosen to make P a minimum.

From $\frac{\delta P}{\delta k} = 0$, it follows that

$$f\alpha^2 \left\{E_f t\left(kt + \frac{f}{2}\right) + E_t(c-t)\left[k(c-t) + \frac{f}{2}\right]\right\} - \lambda\mu'(1-k)c = 0 \tag{126}$$

From $\frac{\delta P}{\delta t} = 0$ it follows that

$$E_f k\left(kt + \frac{f}{2}\right) - E_t k\left[k(c-t) + \frac{f}{2}\right] = 0. \tag{127}$$

If equations (126) and (127) are solved for k and these values of k are equated, an equation is obtained that can be solved for t. It is found that

$$t = \frac{\lambda\mu'\left[2E_t c - (E_f - E_t)f\right] - cf^2 \alpha^2 E_f E_t}{2\left[\lambda\mu'(E_f + E_t) - f^2 \alpha^2 E_f E_t\right]} \tag{128}$$

On substituting this value of t in the equation expressing k in terms of t that is obtained from (127) it is found that

$$k = \frac{\lambda\mu' (E_f + E_t) - f^2 a^2 E_f E_t}{\lambda\mu' (E_f + E_t) + cf a^2 E_f E_t} \tag{129}$$

After substituting (128) and (129) in (125) and making rather extensive reductions it is found that

$$P_{crs} = a^2 \frac{6E_r f(c+f)^2 + f^3 (E_f + E_t) \left[1 + \dfrac{cf a^2 E_r}{2\lambda\mu'}\right]}{12\lambda \left[1 + \dfrac{cf a^2 E_r}{2\lambda\mu'}\right]} \tag{130}$$

where

$$E_r = \frac{2E_f E_t}{E_f + E_t} \tag{131}$$

In equation (130) it will be convenient to introduce the abbreviation

$$\eta = \frac{cf a^2 E_r}{2\lambda\mu'} = \frac{\pi^2 cf E_r}{2\lambda L^2 \mu'} \tag{132}$$

If, further, the term $2E_r f^3$ is added and subtracted in the numerator of equation (130) and the relation,

$$h^3 - c^3 = 6f(c+f)^2 + 2f^3,$$

is used, this equation can be written:

$$P_{crs} = \frac{\pi^2 E_r (h^3 - c^3)}{12\lambda L^2 (1 + \eta)} \left\{ 1 + \frac{f^3 \left[(E_f + E_t)(1 + \eta) - 2E_r\right]}{E_r \; 6f (c + f)^2 + 2f^3} \right\} \tag{133}$$

By reference to equation (32) of the body of the report it is easy to identify the coefficient of the expression in brackets with the buckling load per unit width of a very wide ($a - \infty$) isotropic sandwich panel of height L whose faces have the modulus E_r. This expression also includes the correction for shear deformation in the factor $1 + \eta$ in the denominator, for it follows from (31) with $b = L$ and $a = \infty$, and with $E_x = E_y = A = E_f$ for isotropic facings that

$$P_{cr} = \frac{\pi^2(h^3 - c^3)\, E_f}{12\lambda\, L^2} \qquad (134)$$

and from (33), (28) and (17) under the same circumstances that

$$\eta = \frac{\pi^2\, cf\, E_f}{2\lambda\, L^2\, \mu'} \qquad (135)$$

It may be noted that in the expression (132) for η the modulus E_f is replaced by the modulus E_r.

For cases of practical interest the second term in the brackets in equation (133) can be neglected in comparison with unity. Then equation (133) becomes

$$P_{crs} = \frac{\pi^2(h^3 - c^3)\, E_r}{12\lambda\, L^2\, (1 + \eta)} = \frac{\pi^2(h^3 - c^3)\, E_r}{12\lambda\, L^2\, \left(1 + \dfrac{\pi^2\, cf\, E_r}{2\lambda\, L^2\, \mu'}\right)} \qquad (136)$$

Now from (134) and (135) when the buckling stress is below the proportional limit

$$P_{crs} = \frac{\pi^2(h^3 - c^3) E_f}{12\lambda\, L^2\, (1 + \eta)} = \frac{\pi^2(h^3 - c^3)\, E_f}{12\lambda\, L^2\, \left(1 + \dfrac{\pi^2\, cf\, E_f}{2\lambda\, L^2\, \mu'}\right)} \qquad (137)$$

It is evident that (136) is obtained from (137) by replacing E_f throughout by E_r. This fact is adduced to give some support to the procedure of replacing E_f throughout by E_r in the formulas for the buckling loads of panels with various edge conditions if the isotropic faces are stressed beyond the proportional limit.

Except for very high stresses for which the tangent modulus E_t is a small fraction of the modulus E_f, the reduced modulus E_r, as defined by equation (131), agrees closely with usual reduced modulus defined by the equation

$$E_r = \frac{4E_f\, E_t}{(\sqrt{E_f} + \sqrt{E_t})^2}$$

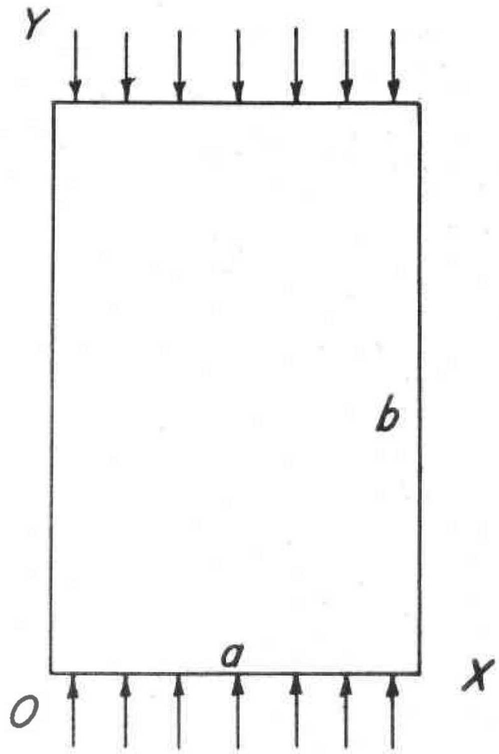

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Figure 1.--Flat sandwich panel in compression.

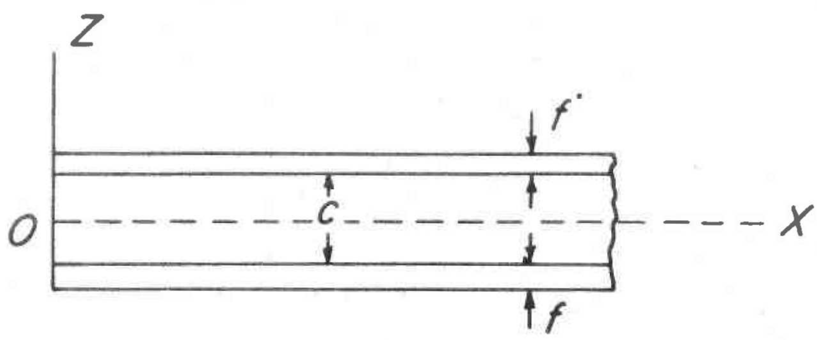

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Figure 2.--Cross section of sandwich panel.

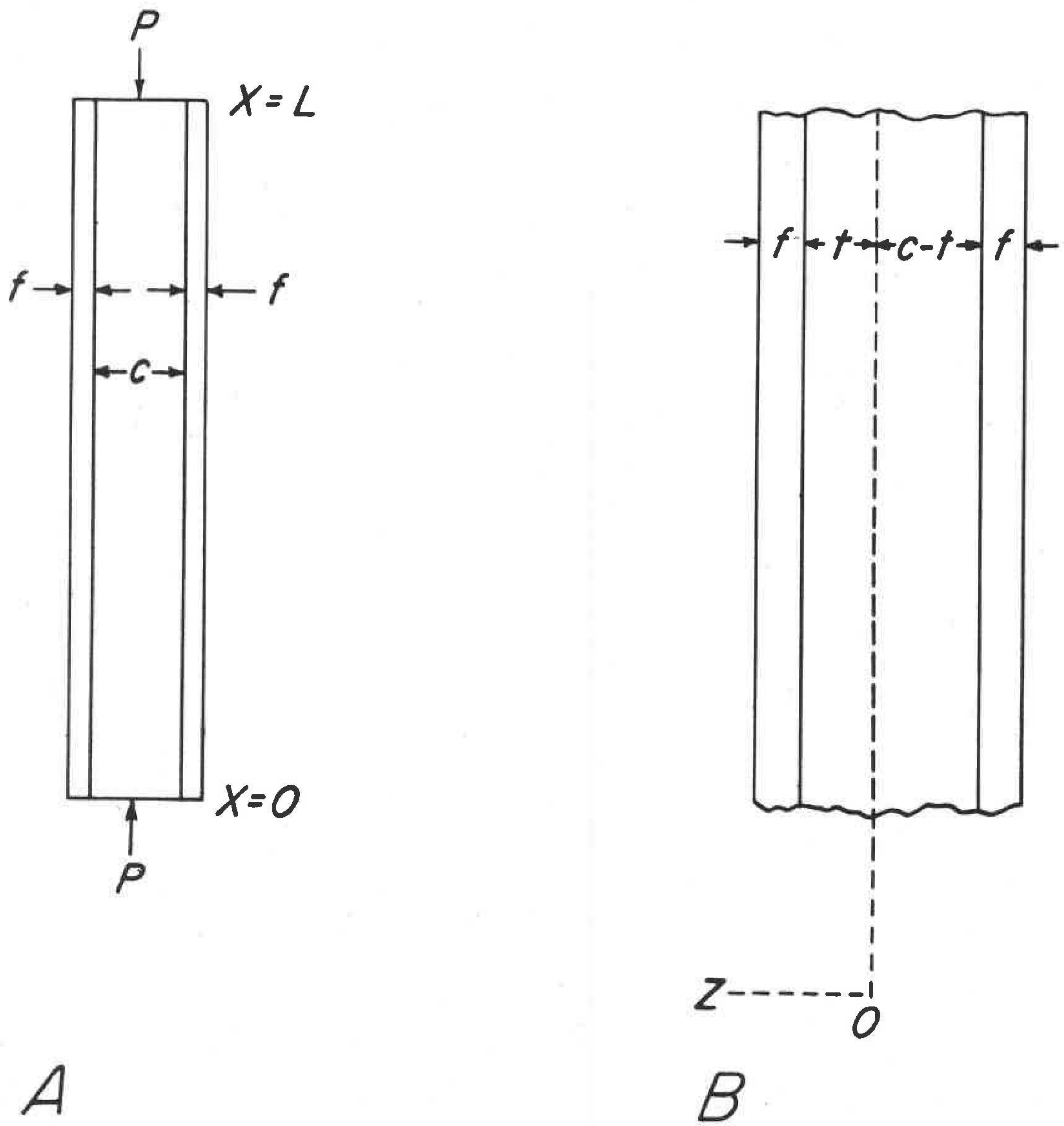

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Figure 3.--Cross sections of sandwich column. A, entire column; B, portion of column showing position of neutral axis.