# Relay-MISO Channel Matrix and Application

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**Abstract**

The matrix inversion is an interesting topic in algebra mathematics. However, to determine an inverse matrix from a given matrix is required many computation tools and time resource if the size of matrix is huge. In this paper, we have shown an inverse closed form for an interesting matrix which has much applications in communication system. The matrix is called Relay-MISO channel matrix which is channel matrix of the Relay-MISON channel. Base on this inverse closed form, the channel capacity closed form of a this Relay-MISO channel can be determined as a function of the error rate parameter $\alpha$.

Keywords: Inverse matrix, convex optimization, channel capacity.

## I. Matrix Construction

In this paper, we will investigate the channel capacity for a class of channel named Relay-MISO (Relay - Multiple Input Single Output). Relay-MISO channel can be constructed by the combination of a Relay channel and a Multiple Input Single Output channel which is illustrated in Fig. 1.

In Relay-MISO channel, $n$ users want to transmit data, for example, synchronization data to a same server via $n$ relay base station nodes. The up-link of these users using wireless links, i.e, the GSM network, which are easy to vulnerable due to the transmission link attenuation by fog, rain, distance or shadow effect, etc. Each user can transmit bit "0" or "1" with the flip bit probability or the error probability is $\alpha$ and $0 \leq \alpha \leq 1$. For a simplicity analysis, we suppose that $n$ relay channels have the same error probability $\alpha$. Next, all of the relay base station nodes will relay the signal by a reliable channel such as optical fiber network to a same server. At the receiver side, server is simple adding total received signal to achieve a single output from multiple inputs signal.

We note that both the modulations and demodulations in the transmitter and receiver side of Relay-MISO channel are simple amplitude modulation such as PAM coding. Thus, from $n$ users which transmit only two states bit "0" or "1", server will receive and recognize total $n+1$ levels, i.e, $0, 1, 2, \dots, n$. Thus, the channel matrix of this Relay-MISO channel is a matrix size $(n+1) \times (n+1)$ which all entries $A_{ij}$ can be constructed a follows:

$$A_{ij} = \sum_{s=\max(i-j,0)}^{s=\min(n+1-j,i-1)} \binom{j-i+s}{n+1-i} \binom{s}{i-1} \alpha^{j-i+2s} (1-\alpha)^{n-(j-i+2s)} \tag{1}$$

These channel matrices have an inverse closed form for entries $A_{ij}^{-1}$ as follows. The closed form of both channel matrix and inverse channel matrix are proven at the Appendix.

$$A_{ij}^{-1} = \frac{(-1)^{i+j}}{(1-2\alpha)^n} \sum_{s=\max(n-j+1,i)}^{s=\min(0,i-j)} \binom{n-i}{j-i+s} \binom{s}{i-1} \alpha^{j-i+2s} (1-\alpha)^{n-(j-i+2s)} \tag{2}$$

For example, the channel matrix of a Relay-MISO channel with $n=3$ is given as follows:

$$\begin{bmatrix} (1-\alpha)^3 & 3(1-\alpha)^2\alpha & 3(1-\alpha)\alpha^2 & \alpha^3 \\ \alpha(1-\alpha)^2 & 2\alpha^2(1-\alpha)+(1-\alpha)^3 & 2(1-\alpha)^2\alpha+\alpha^3 & (1-\alpha)\alpha^2 \\ (1-\alpha)\alpha^2 & 2(1-\alpha)^2\alpha+\alpha^3 & 2\alpha^2(1-\alpha)+(1-\alpha)^3 & \alpha(1-\alpha)^2 \\ \alpha^3 & 3(1-\alpha)\alpha^2 & 3(1-\alpha)^2\alpha & (1-\alpha)^3 \end{bmatrix}$$

,

where $0 \leq \alpha \leq 1$. We note that this channel matrix is strictly diagonally dominant matrix for certain range of values of $\alpha$, specifically when $\alpha$ is close to 0 or $\alpha$ is close to 1. In the next part, the construction of channel matrix and inverse channel matrix will be characterized.

## II. Formula of Relay-MISO channel matrix

**Proposition 1.** *For $n$ relay channels system, the transition matrix $A$ has size $(n+1) \times (n+1)$ and all entries $A_{ij}$ in row $i$ and column $j$ will be established as follows:*

$$A_{ij} = \sum_{s=\max(i-j,0)}^{s=\min(n+1-j,i-1)} \binom{j-i+s}{n+1-i} \binom{s}{i-1} \alpha^{j-i+2s} (1-\alpha)^{n-(j-i+2s)} \tag{3}$$

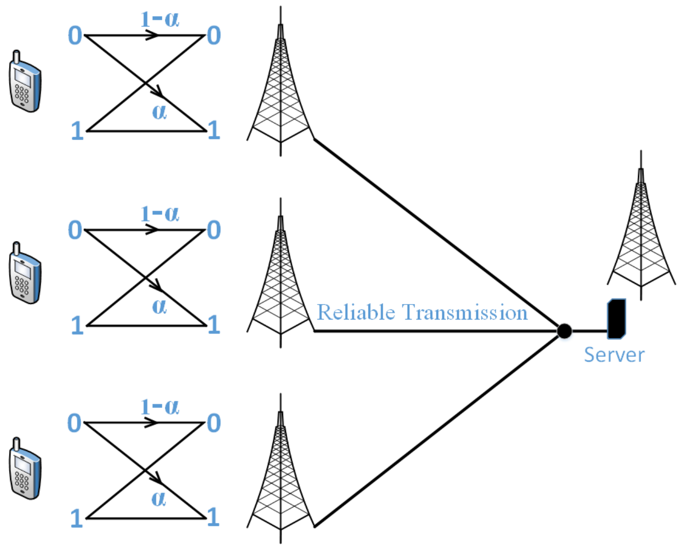


Figure 1. Relay-MISO channel for the number of user is three.

*Proof.* From the Relay-MISO channel definition, each the relay channel has only two states "1" (good channel) and "0" (bad channel) which is flipped at the probability $\alpha$. Therefore, $A_{ij}$ is the probability from state has $i-1$ "good" channels or $i-1$ bit "1" transfer to state has $j-1$ "good" channels or $j-1$ bit "1". Thus, suppose $s$ is the number channels in $i-1$ "good" channels that is flipped to "bad" channels after the transmission time $\sigma$ and $0 \leq s \leq i-1$. To maintaining $j-1$ "good" channels after the transmission time $\sigma$, the number of "bad" channels in $n+1-i$ "bad" channels must be flipped to "good" channels is:

$$(j-1)-((i-1)-s)=j-i+s$$

Therefore, the total number of channels are flipped their state after transmission time $\sigma$ is:

$$s+(j-i+s)=j-i+2s$$

and the total number of channels that preserves their state after transmission time $\sigma$ is $n-(j-i+2s)$ and $0 \leq s \leq i-1$. Similarly, the number of "bad" channels in $n+1-i$ "bad" channels must be flipped to "good" channels should be in $0 \leq j-i+s \leq n+1-i$. Thus:

$$\begin{cases} \max s = \min(n+1-j, i-1) \\ \min s = \max(0, i-j) \end{cases}$$

Therefore, $A_{ij}$ can be determined by below form:

$$A_{nij} = \sum_{s=\max(i-j,0)}^{s=\min(n+1-j,i-1)} \binom{j-i+s}{n+1-i} \binom{s}{i-1} \alpha^{j-i+2s} (1-\alpha)^{n-(j-i+2s)}$$

□

## III. Formula of Relay-MISO inverse channel matrix

**Proposition 2.** *All the entries of the inverse channel matrix $A^{-1}$ given in Proposition 1 can be determined via original transition matrix $A$ for $\forall\ \alpha \neq 0.5$ by:*

$$A_{ij}^{-1} = \frac{(-1)^{i+j}}{(1-2\alpha)^n} A_{ij}$$

*Proof.* To simplify our notation, the "good" and "bad" channel are represented by bit "1" and "0", respectively. Next, we will use the definition to show that:

$$AA^{-1} = I$$

If matrix $A^*$ is constructed by $A^*_{ij} = (-1)^{i+j} A_{ij}$, then we need to show that:

$$A_n A^*_n = B = (1-2\alpha)^n I$$

Firstly, we note that both $A_{ij}$ and $A^*_{ij}$ is only different by sign of the first index $(-1)^{i+j}$. Therefore, $B_{ij}$ which is computed by product of row $i$ in matrix $A_{ij}$ and column $j$ in matrix $A^*_{ij}$, can be computed by:

$$B_{ij} = \sum_{k=1}^{k=n+1} A_{ik} A^*_{kj}$$

Consider the entry $A_{ik}$ is the probability from state $i-1$ "good" channels ($i-1$ bit "1" and $n-i+1$ bit "0") to the intermediate state has $k-1$ "good" channels (with $k-1$ bit "1" and $n-k+1$ bit "0"). Moreover, if the sign is ignored, then $A^*_{kj}$ also is the probability going from intermediate state $k-1$ to state $j-1$, too. However, the state $k-1$ includes $C\binom{n}{k-1}$ sub-states which have a same number of "good" and "bad" channels. For example with $n=2$, state $k=2$ includes two sub-states that contains one "good" and one "bad" channels are "10" an "01". Therefore, the total number of sub-states while $k$ runs from 1 to $n$ is $\sum_{k=1}^{k=n+1} C\binom{n}{k-1} = 2^n$ sub-states. Let compute $B_{ij}$ by divided into two subsets:

**Compute $B_{ij}$ for i=j**: This means that $B_{ii}$ is the sum of the probability from state $i-1$ bit "1" go to the intermediate states has $k-1$ bit "1" then come back to state has $i-1$ bit "1". In $2^n$ sub-states, we can divide back to $n+1$ categories by the number of different position between $i$ and $k$.

- If all the bit in $i$ and $k$ are the same, then the probability is:

$$C\binom{n}{0}(1-\alpha)^n(1-\alpha)^n = C\binom{n}{0}(1-\alpha)^{2n}$$

- If all the bit in $i$ and $k$ different at only one position, then the probability is:

$$C\binom{n}{1}(1-\alpha)^{2(n-1)}(1-\alpha)^2$$

- If all the bit in $i$ and $k$ different at only two positions, then the probability is:

$$C\binom{n}{2}(1-\alpha)^{2(n-2)}(1-\alpha)^{2\times 2}$$

- If all the bit in $i$ and $k$ different at all positions, then the probability is:

$$C\binom{n}{n}(1-\alpha)^{2n}$$

Therefore, $B_{ii}$ can be determined by the probability of all $n+1$ categories such as:

$$B_{ii} = \sum_{t=0}^{t=n} C\binom{n}{t}\alpha^{2t}(1-\alpha)^{2n-2t} = ((1-\alpha)^2 - \alpha^2)^n = (1-2\alpha)^n$$

**Compute $B_{ij}$ for i$\neq$ j**: Let divide $A^*{}_{kj}$ into two subsets: $k+j$ is odd and $A^*_{kj} < 0$ or $k+j$ is even and $A^*_{kj} > 0$, respectively. Therefore, $B_{ij} = \sum_{k=1}^{k=n} A_{ik} A^*_{kj}$ also is distributed into the positive or negative subsets. Next, we will show that the positive subset in $B_{ij}$ equal the negative subset then $B_{ij} = 0$ for $i \neq j$.

Indeed, suppose that state $i$ with $i-1$ bit "1" go to state intermediate $k_1$ and then to back to state $j$ with $j-1$ bit "1" and $B_{ik_1}$ is positive value. Next, we will show that existence a state $k_2$ such that $B_{ik_2}$ is negative value and $B_{ik_1} = -B_{ik_2}$.

Let call $s$ is the number of positions where state $i$ and $j$ have a same bit. Obviously that $s \leq n-1$ due to $i \neq j$. For example if $n=4$ and $i=1111$ and $j=0001$, we have $s=1$ because $i$ and $j$ share a same bit "1" in the positions fourth. Suppose that an arbitrary state $k_1$ are picked, we will show how to chose the state $k_2$ with $B_{ik_1} = -B_{ik_2}$. Consider the two follows cases:

- If $(n-s)$ is odd, $k_2$ is constructed by maintain $s$ position of $k_1$ where $i$ and $j$ have same bit and flip bit in the $n-s$ rest positions.
- If $(n-s)$ is even, $k_2$ is constructed by maintain $s+1$ position of $k_1$ where $s$ position are $i$ and $j$ have a same bit and one position where $i$ and $j$ have a different bit, next $n-s-1$ rest positions will be flipped. Note that since $s \leq n-1$ then we are able to flip $n-s-1$ rest positions.

We obviously can see that $k_1$ and $k_2$ satisfied the probability condition $|B_{ik_1}| = |B_{ik_2}|$ due to the number of flipped bit between $i$ and $k_1$ equals the number of flipped bit between $k_2$ and $j$ and the number of flipped bit between $j$ and $k_1$ equals the number of flipped bit between $k_2$ and $i$.

Next, we will prove that $k_1$ and $k_2$ make $B_{ik_1}$ and $B_{ik_2}$ in different positive and negative subsets. Indeed, consider the number of bit "1" in $k_1$ is $b_1$, number of bit "1" in $k_2$ is $b_2$, number of bit "1" in $s$ bit same of $i$ and $j$ is $b_s$, respectively. Therefore, the number of bit "1" of $k_1$ in $(n-s)$ rest positions is $(k_1 - k_s)$, the number of bit "1" of $k_2$ in $(n-s)$ rest positions is $(k_2 - k_s)$.

• If $(n-s)$ is odd. Since all bit in $(n-s)$ rest positions of $k_1$ is flipped to create $k_2$, then total number of bit "1" in $n-s$ bit of $k_1$ and $k_2$ is $(k_1 - k_s + k_2 - k_s = n - s)$ is odd. So, $(k_1 + k_2)$ should be an odd number. That said $(k_1 - k_2)$ is odd or $(k_1 + j) - (k_2 + j)$ is odd. Therefore, $B_{ik_1}$ and $B_{ik_2}$ bring the contradict sign.

• If $(n-s)$ is even. Because, we fix one more position to create $k_2$, then number of flipped bit $(n-s-1)$ is odd number. If one more bit is fixed in $k_1$ is "0", we have a same result with case $(n-s)$ is odd. If fixed bit is "1", similarly in first case $(k_1 - k_s - 1) + (k_2 - k_s - 1) = n - s - 1$ is odd number, therefore $(k1 + k2)$ is odd number. That said $(k_1 - k_2)$ is odd or $(k_1 + j) - (k_2 + j)$ is odd. Therefore, $B_{ik_1}$ and $B_{ik_2}$ bring the contradict sign.

Therefore, the state $k_2$ always can be created from a random state $k_1$ and $B_{ik_1}$ and $B_{ik_2}$ bring a contradict sign. That said for $i \neq j$, $B_{ij} = 0$. Therefore:

$$B = (1 - 2\alpha)^n I$$

The Proposition 2, therefore, are proven. □

## IV. Conclusion

In this paper, our contributions are twofold: (1) establish a channel matrix closed form for Relay-MISO channel based on the error probability $\alpha$; (2) figure out the closed form for inverse channel matrix in Relay-MISO channel. This result can be used to obtain a closed form or achieve a tight upper bound of Relay-MISO channel capacity.