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Enrique Thomann

In the 1954 John Nash [1] showed, through use of an iterative scheme of approximate embedding maps, that the sphere $\mathbb{S}^2$ could be isometrically embedded into a ball of any radius by a C^1 map. In the 1980's M. Gromov [2] generalized Nash's work to the h-principal and convex integration. Recent research in fluid dynamics has used analogs to convex integration schemes to demonstrate nonuniqueness of weak solutions to both Euler and Navier-Stokes equations. In this paper we examine recent work by DeLellis and Székelyhidi Jr. [3] as well as Buckmaster and Vicol [4] on the nonuniqueness of Fluid Dynamics equations. Specifically we focus on solution to Reynolds Stress system which act as approximate solutions to Euler and Navier-Stokes equation. We also develop basic properties of the Dirichlet kernel used in the construction of intermittent Beltrami flows.

Reynolds Stress Tensor Systems and Application to Nonuniqueness of Weak Solutions to Fluid Dynamics Equations

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Patrick E. Donaghue, Author

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1. Introduction

In Nash's 1954 paper [1] the author introduced an iterative scheme to approximate *short* embeddings of $n-$dimensional Riemann manifolds in $\mathbb{R}^{2n+1}$ by C^1 isometric embeddings[1]. One consequence of this result was that any closed, orientable, Riemannian surface could be C^1 embedded into a ball in $\mathbb{R}^3$ of arbitrary radius $R > 0$ isometrically [5]. Expanding on the methods Nash employed in his paper Gromov introduce the concept of the *h-principle* and *convex integration* in his 1986 treatise Partial Differential Relations [2]. Currently a suitable variant of the $h-$principle is being used to show nonuniqueness of weak solutions to the equations of Fluid Equations.[2]

In this introduction we will give a flavor of Nash's paper original paper, consider a related example, and discuss the layout of the project.

1.1. **Nash's 1954 Paper.**

For this section we follow along with Nash's original paper [1], and the modern exposition of the paper given by [5] and [7]. As all math texts should, we begin with a definition.

Definition 1.1.1 (Short Map)**.** Let (M, g) be a Riemannian manifold. We say an embedding $u : M \to \mathbb{R}^n$ is *short* if for any tangent vector w

$$(u^*e)_{ij}\, w^i w^j \leq g_{ij} w^i w^j \tag{1.1.1}$$

where e is the Euclidean metric and u^*e is the pullback of the metric.

Casually, the above definition says a embedding is short if the length of every curve $\gamma \subset M$ is shorter when measured with the Euclidean metric in $\mathbb{R}^n$ on the embedding. Then by standard topology results we can always (smoothly) embed the m dimensional Riemannian manifold M in an appropriately high dimensional Euclidean space. Dividing the embedding by an appropriate constant we can produce short maps. We also say an embedding is *isometric* if the lengths of curves measured in the embedded space is the same as on the manifold. This is equivalent to strict equality in (1.1.1). With these definitions in hand we can now state Nash's 1954 result.

Theorem 1.1.2 (Nash's C^1 isometric embedding theorem, [5], 2.1.2)**.** *Let (M, g) be a smooth closed $m-$dimensional Riemannian manifold and $v : M \to \mathbb{R}^n$ a smooth C^∞ embedding with $n \geq m + 2$. Then for all $\epsilon > 0$ there is a C^1 isometric embedding $u : M \to \mathbb{R}^n$ such that $||u - v||_{L^\infty} < \epsilon$.*

Remark 1.1.3 (Immersions). In the above definitions and theorem we can replace 'embedding' with 'immersion' where a immersion is a local embedding. Throughout this section we will restrict our attention to embeddings.

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[1] Originally $\mathbb{R}^{2n+2}$ but the dimension was decreased by Kuiper a year later.

[2] As a brief aside, during a speech at the Balzan Prize Gromov said: '... the presence of the h-principle would invalidate the very idea of a physical law as it yields very limited global information effected by the infinitesimal data.'[6]

Remark 1.1.4. When applying the above theorem to the two sphere we can produce a family of C^1 embeddings that are distinct under the equivalence of rigid motions. This is vividly distinct from earlier work by Cohn and Vossen.

Theorem 1.1.5 (Cohn-Vossen (1927), [7], pg.284). *If (S^2, g) has positive Gauss curvature and $u \in C^2(S^2, \mathbb{R}^3)$ is an isometric immersion then $u(S^2)$ is determined up to a rigid motion.*

As alluded to earlier the construction of the isometric embedding u is a iterative scheme that produces a convergent sequence in the C^1 norm. We will assume that the embedding is strictly short for our ease. The main idea is as follows:

(1) Find a open covering of M where each open set is diffeomorphic to an m-ball and intersects a finite number of other open sets in the covering.
(2) Define a partition of unity based off the open covers.
(3) Let $\delta_{ij} = g_{ij} - (u^* e)_{ij}$
(4) Using the localization of the partition of unity increase the metric in each neighborhood by $\frac{1}{2}\delta_{ij}$.
(5) Take limit of iterate scheme above to get isometric embedding.

Full details of the scheme can be found in (the highly readable) [1] and the modern treatment in [5].

1.2. **A Basic Example.** Now we consider an example found in [6] to show how Nash's idea can be applied to other situations. Our goal is to develop and iterative scheme that produces functions $u_\infty : [0,1] \to \mathbb{R}$ with $|u_\infty| \overset{a.e.}{=} 1$ given appropriate initial function. Let u_0 be the given function with $|u_0| < 1$ and u_k be the function developed from u_0 after k iterates. Letting the auxiliary function

$$\rho_k(x) = 1 - u_k^2(x),$$

the oscillatory term be

$$s(x) = \begin{cases} 1 & 0 \le x \pmod 1 \le \frac{1}{2} \\ -1 & \frac{1}{2} < x \pmod 1 < 1, \end{cases}$$

and $\{\lambda_k\}$ a sequence to be deterimined later we define

$$u_{k+1}(x) = u_k(x) + \frac{1}{2}\rho_k(x) s(\lambda_k x).$$

Since $s(x)$ only takes values of 1 or -1 we have that at each point

$$u_{k+1} = u_k + \frac{1}{2} - \frac{1}{2}u_k^2$$

or

$$u_{k+1} = u_k - \frac{1}{2} + \frac{1}{2}u_k^2.$$

Then as the polynomial $p(x) = x + \frac{1}{2} - \frac{1}{2}x^2$ and $q(x) = x - \frac{1}{2} + x^2$ have $|p(x)| < 1$ and $|q(x)| < 1$ for $|x| < 1$ we see that if $|u_k| < 1$ then $|u_{k+1}| < 1$. Thus for all $k \in \mathbb{N}$ we have

$$\sup_x |u_k(x)| < 1$$

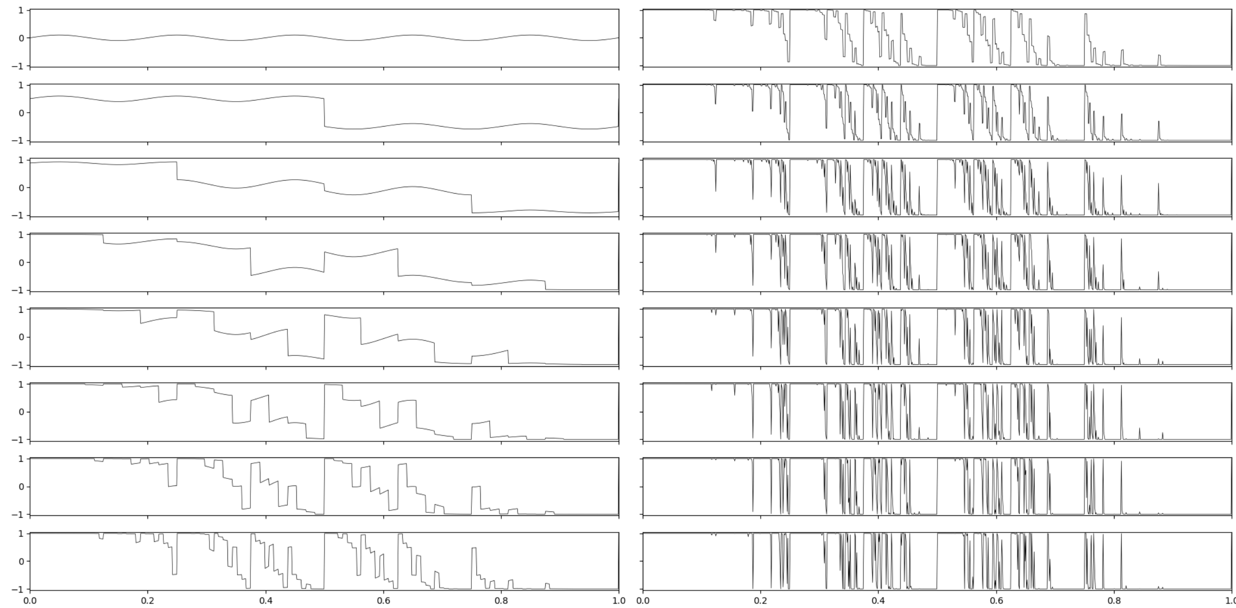

FIGURE 1. Iterative scheme from example applied to $u_0 = \frac{1}{10}\sin(10\pi x)$ for the first 15 iterations with $\lambda_k = 2^k$.

by choice of u_0. Next as $u_{k+1} - u_k = \frac{1}{2}(1-u_k^2)s(\lambda_k x)$ if we have that $u_k \to u_\infty$ in L^p then $|u_\infty| \overset{a.e.}{=} 1$. Using $p = 2$ we have

$$\int_0^1 |u_{k+1}|^2 \,\mathrm{dx} = \int_0^1 |u_k + \frac{1}{2}(1-u_k^2)s(\lambda_k x)|^2 \,\mathrm{dx} = \int_0^1 \left(u_k^2 + u_k(1-u_k^2)s(\lambda_k x) + \frac{1}{4}(1-u_k^2)^2\right) \mathrm{dx}\,.$$

Then as $s(\lambda_k x)$ oscillates faster with $\lambda_k \to \infty$ by Riemann-Lebesgue lemma we can choose λ_k large enough so

$$\int_0^1 |u_{k+1}|^2 \,\mathrm{dx} \geq \int_0^1 |u_k|^2 \,\mathrm{dx} + \frac{1}{8}\int_0^1 (1-u_k^2)^2 \,\mathrm{dx}\,.$$

Clearly, $||u_0||_{L^1} < ||u_1||_{L^2} < \cdots < ||u_k||_{L^2} < \cdots < 1$ and $\lim_{k\to\infty} ||u_k||_{L^2} = 1$.

1.3. **Layout.** Having seen a minor aspect of Nash's work and how it can be applied outside of embeddings we are ready to discuss the format of the paper and what it deals with. Briefly, in [6] the authors note that the h-principle as constructed by Nash has a analog to the *Euler-Reynolds system* (Definition 2.2.1) where the (symmetric, trace free) matrix (δ_{ij}) plays the role of the *Reynolds stress tensor*, $\mathring{R}$. That is $\mathring{R}$ is a measure of how far a solution to Euler-Reynolds equations is from being a solution to the Euler equations. More explicitly, the modifier of *Reynolds* appended to a system will refer to introduction of a forcing function of the form

$$\operatorname{div} \mathring{R}$$

to the system where $\mathring{R}$ is a trace free matrix.

With the above definition we note if we can construct consecutive (weak) solutions to the Reynolds Stress Tensor System with the norm of $\mathring{R}$ decreasing to zero then passing through the limit we will hope to retain a weak solution of the system. This idea is the guiding principle behind the organization of this paper with regard to recent treatment of nonuniqueness of fluid dynamic equations by [3] and [4]. In each section dealing with their respective works we will:

- Provide basic background information on the systems
- State inductive estimates we will use in convergence
- Build consecutive tuples
- Look at estimate associated with scheme convergence.

For the fourth item we will not attempt to provide all details due to the unenlightening nature of many estimates, but instead look at some important or representative ones that were special to the paper. Specific things not covered are the Reynolds stress tensor estimates due to the repetitive nature of calculations coming from their function as an error absorbing term for the systems.

Additional to the main parts of the paper we have also provide an appendix that deals with:

(1) a technical detail in the estimates of [4],
(2) an overview of some basic properties of the Dirichlet kernel.

2. Euler Equations

The $3D$ incompressible Euler equations are

$$\begin{cases} \frac{D}{Dt}v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} \tag{2.0.1}$$

where v is the fluid velocity, p is the scalar pressure, and

$$\frac{D}{Dt}v = \frac{\partial}{\partial t}v + \sum_{j=1}^{3} v^j \frac{\partial}{\partial x_j}v = \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v$$

is the convective derivative where-within v^j denotes the j^{th} component of v. Using the identity

$$\operatorname{div}(v \otimes v) = \operatorname{div}\left(vv^T\right) = (v \cdot \nabla)v + (\operatorname{div} v)v$$

we will write (2.0.1) as

$$\begin{cases} \frac{\partial}{\partial t}v + \operatorname{div} v \otimes v + \nabla p = 0 \\ \operatorname{div} v = 0. \end{cases} \tag{2.0.2}$$

Now given a velocity field v we also wish to be able to establish the scalar pressure from the velocity field. With this in mind we apply the divergence operator to (2.0.1) and retrieve,

$$\frac{\partial}{\partial t}(\operatorname{div} v) + \sum_{j=1}^{3}\left(\operatorname{div}\left(v^j \frac{\partial}{\partial x_j}v\right)\right) + \Delta p = 0.$$

Then as $\operatorname{div}\left(v^j \frac{\partial}{\partial x_j}v\right) = \left(\nabla v^j\right) \cdot v + v^j(\operatorname{div} v) = \nabla v^j \cdot v$ we have the elliptical equation

$$-\Delta p = \sum_{j=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}v^j\right)\left(\frac{\partial}{\partial x_j}v^i\right).$$

Inverting the Laplacian we recover that

$$p(x) = -\left(N * \sum_{i,\ j}\left(\frac{\partial}{\partial x_i}v^j\right)\left(\frac{\partial}{\partial x_j}v^i\right)\right)(x) \tag{2.0.3}$$

where[3] $N = \frac{-1}{4\pi|x|}$ and $*$ denotes convolution.

Finally as we are interested in *weak* solutions of the Euler equation on the torus ($\mathbb{T}^3$) we say that a pair (v, p) is a weak solution to (2.0.1) if

$$\int_0^1 \int_{\mathbb{T}^3} \left(\partial_t \varphi \cdot v + \nabla\varphi : v \otimes v + p \operatorname{div} \varphi\right) \mathrm{dx}\ \ \mathrm{d}t = 0$$

for a test function φ that is perodic and space and has compact support contained in $(0, 1)$ for time. Having developed basic facts about the Euler Equations we will now focus on a specific class of steady solutions.

[3] Here we use the letter N to signify the *Newtonian Potential* in following with [8].

2.1. **Beltrami Flows.**

Definition 2.1.1 (Beltrami Flow, [8])**.** A steady $3D$ flow is called a *Beltrami flow* if the vorticity $\omega = \nabla \times v$ satisfies the condition

$$\omega(x) = \lambda(x)v(x) \qquad (\lambda(x) \neq 0)$$

for all x. If $\lambda(x)$ is constant we call the flow a *strong Beltrami Flow.*

Now we would like to show that a Beltrami Flow solves the Euler equations. Before we undertake this we recall that we can reformulate (2.0.1) in terms of its *vorticity.* Specifically, the vorticity formulation of the Euler equations is

$$\frac{D}{Dt}\omega = \omega \cdot \nabla v,$$

for more details please see Proposition 2.21, pg.78 in [8]. For a steady flow the prior equations reduces to

$$(v \cdot \nabla)\,\omega = (\omega \cdot \nabla)\,v.$$

With this in hand we are now ready to prove that a incompressible Beltrami Flow solves the steady $3D$ Euler equations.

Proposition 2.1.2. *Let v be a incompressible Beltrami flow, then v solves* (2.0.1)*.*

Proof. Using that $\omega = \lambda v$ we have

$$(v \cdot \nabla)\,\omega = (v \cdot \nabla)\,\lambda v = (\lambda v \cdot \nabla)\,v + v \cdot (\nabla \lambda)\,v.$$

Next as

$$\operatorname{div} \omega = \operatorname{div} \nabla \times v = 0$$

we have

$$0 = \operatorname{div} \omega = \operatorname{div} \lambda v = (\nabla \lambda) \cdot v + \lambda\,(\operatorname{div} v) = (\nabla \lambda) \cdot v$$

where the final equality follows from incompressibility of v. Coupling this with the first equation we infer that

$$(v \cdot \nabla)\,\omega = (\lambda v \cdot \nabla)\,v = (\omega \cdot \nabla)\,v$$

as desired. Hence v solves the $3D$ steady Euler equations. $\square$

For the rest of this section we will restrict our attention to a specific class of (strong) Beltrami flows.

Proposition 2.1.3 (Beltrami Flows, [3], 3.1)**.** *Let $\lambda_0 \geq 1$ and let $A_k \in \mathbb{R}^3$ be such that*

$$A_k \cdot k = 0,\ |A_k| = \frac{1}{\sqrt{2}},\ A_{-k} = A_k$$

for $k \in \mathbb{Z}^3$ with $|k| = \lambda_0$. Furthermore, let

$$B_k = A_k + i\frac{k}{|k|} \times A_k.$$

Then for any choice of $a_k \in \mathbb{C}$ with $\overline{a_k} = a_{-k}$ the vector field

$$W(\xi) = \sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi}$$

is divergence free and satisfies

$$\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}.$$

Furthermore,

$$\fint W \otimes W \; d\xi = \frac{1}{2} \sum_{|k|=\lambda_0} |a_k|^2 \left(\mathrm{Id} - \frac{1}{|k|^2} k \otimes k \right)$$

.

Proof. By hypothesis we see that $a_{-k}B_{-k} = \overline{a_k B_k}$. Thus

$$\begin{aligned} W(\xi) &= \sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi} \\ &= \sum_{|k|=\lambda_0} \frac{1}{2} \left(a_k B_k e^{ik\cdot\xi} + \overline{a_k B_k e^{i\ k\cdot\xi}} \right) \\ &= \sum_{|k|=\lambda_0} \mathrm{Re}\left(a_k B_k e^{ik\cdot\xi} \right) \end{aligned} \tag{2.1.1}$$

is a real valued vector field. By direct calculation we have that

$$\operatorname{div} W(\xi) = \sum_{|k|=\lambda_0} \operatorname{div}\left(a_k B_k e^{ik\cdot\xi} \right) = \sum_{|k|=\lambda_0} a_k i \left(k \cdot B_k \right) e^{ik\cdot\xi} = 0 \tag{2.1.2}$$

where the final equality follows from $k, A_k, k \times A_k$ being a orthogonal set. Thus W is divergence free as desired.

Now using that W is divergence free we have that

$$\operatorname{div} W \otimes W = (W \cdot \nabla + \nabla \cdot W) W = (W \cdot \nabla) W = \nabla \frac{|W|^2}{2} - W \times (\nabla \times W).$$

Then since

$$\nabla \times B_k e^{ik\cdot\xi} = ik \times B_k e^{ik\cdot\xi}$$

and

$$ik \times B_k = ik \times \left(A_k + i \frac{k}{|k|} \times A_k \right) = |k| \left(A_k + i \frac{k}{|k|} \times A_k \right) = |k| B_k$$

we see

$$\nabla \times W = \lambda_0 W$$

and consequently

$$\operatorname{div} W \otimes W = \nabla \frac{|W|^2}{2}.$$

For the final part of the proposition we note that

$$W \otimes W = \sum_{|k|=|l|=\lambda_0} a_k a_l \left(B_k \otimes B_l \right) e^{i(k+l)}$$

which gives

$$\fint_{\mathbb{T}^3} W \otimes W = \sum_{|k|=\lambda_0} a_k a_{-k} B_k \otimes B_{-k} = \sum_{|k|=\lambda_0} |a_k|^2 \left(A_k \otimes A_k + \left(\frac{k}{|k|} \times A_k \right) \otimes \left(\frac{k}{|k|} \times A_k \right) \right).$$

Using the fact that

$$\mathrm{Id} = \frac{k}{|k|} \otimes \frac{k}{|k|} + 2A_k \otimes A_k + 2 \left(\frac{k}{|k|} \times A_k \right) \otimes \left(\frac{k}{|k|} \times A_k \right)$$

the result follow. $\square$

Remark 2.1.4. Coupling Proposition 2.1.3 and 2.1.2 we see that the constructed W is a solution to (2.0.2) with $p = -\frac{|W|^2}{2}$.

Examining the proof that W is a real vector field we note that if we let $a_k = r_k e^{i\theta_k}$ then we can express

$$\mathrm{Re}\left(a_k B_k e^{ik\cdot\xi}\right) = r_k A_k \,\mathrm{Re}\left(e^{i(k\cdot\xi+\theta_k}\right) - r_k \frac{k}{|k|} \times A_k \,\mathrm{Im}\left(e^{i(k\cdot\xi+\theta_k)}\right) = r_k A_k \cos(k\cdot\xi+\theta_k) - r_k \frac{k}{|k|} \times A_k \sin(k\cdot\xi+\theta_k)$$

and rewrite (2.1.1) as

$$W(\xi) = \sum_{|k|=\lambda_0} r_k \left(A_k \cos(k\cdot\xi+\theta_k) - \frac{k}{|k|} \times A_k \sin(k\cdot\xi+\theta_k) \right). \tag{2.1.3}$$

Let us now consider a simple example. Let $\lambda_0 = 1$ then
$\{k \in \mathbb{Z}^3 \mid |k| = \lambda_0\} = \{\pm e_1, \pm e_2, \pm e_3\}$. Setting

$$\begin{aligned} A_{\pm e_1} &= \frac{1}{\sqrt{2}} e_2 \\ A_{\pm e_2} &= \frac{1}{\sqrt{2}} e_3 \\ A_{\pm e_3} &= \frac{1}{\sqrt{2}} e_1 \end{aligned} \tag{2.1.4}$$

and applying the proposition we recover

$$\begin{aligned} B_{\pm e_1} &= \frac{1}{\sqrt{2}} (e_2 \pm i e_1 \times e_2) = \frac{1}{\sqrt{2}} (e_2 \pm i e_3) \\ B_{\pm e_2} &= \frac{1}{\sqrt{2}} (e_3 \pm i e_2 \times e_3) = \frac{1}{\sqrt{2}} (e_3 \pm i e_1) \\ B_{\pm e_3} &= \frac{1}{\sqrt{2}} (e_1 \pm i e_1 \times e_3) = \frac{1}{\sqrt{2}} (e_1 \mp i e_2). \end{aligned} \tag{2.1.5}$$

Next letting $a_{\pm e_j} = 1$ for $1 \le j \le 3$ and applying (2.1.3) we have that

$$W(\xi) = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 0 \\ \cos\xi_1 \\ -\sin\xi_1 \end{bmatrix} + \begin{bmatrix} -\sin\xi_2 \\ 0 \\ \cos\xi_2 \end{bmatrix} + \begin{bmatrix} \cos\xi_3 \\ -\sin\xi_3 \\ 0 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\xi_3 - \sin\xi_2 \\ \cos\xi_1 - \sin\xi_3 \\ \cos\xi_2 - \sin\xi_1 \end{bmatrix} \tag{2.1.6}$$

Remark 2.1.5 (ABC Beltrami Flows). Notice that the previous example is close to the celebrated Arnold-Beltrami-Childres (ABC) flows with form

$$v(\xi) = \begin{bmatrix} A\sin\xi_3 + C\cos\xi_2 \\ B\sin\xi_1 + A\cos\xi_3 \\ C\sin\xi_2 + B\cos\xi_1 \end{bmatrix}.$$

These flows were shown to have a diverse dynamics for different (A, B, C) ranging from helix particle trajectories to ergodic flows. Indeed examining the construct in [8] we see that we are interested in using eigenfunctions of the $(2D)$ Laplacian and then extending to $3D$ euler equations through stream functions. For the ABC construction the eigenfunctions chosen were

$$\psi_i(\xi) = a_i \sin\left(\xi_{i+2 \mod 3}\right)$$

in our example we choose

$$\psi_i(\xi) = a_i \sin\left(\xi_{i+1 \mod 3}\right).$$

Indeed, as we take different λ we see that W will become a sum of ABC (or closely equivalent) flows expressed in different basis.

The Beltrami flows developed in this section will form the basis of rest of the paper.

2.2. **The Iteration Scheme.** For the rest of this section we follow closely along with [3]. In subsequent subsections we will suppress proofs of standard estimates and predominately focus on the thematic elements. As such we start with a definition.

Definition 2.2.1 (Euler-Reynolds System)**.** Assume $v, p, \mathring{R}$ are smooth functions on $\mathbb{T}^3 \times [0,1]$ going to $\mathbb{R}^3, \mathbb{R}$, and $S_0^{3\times 3}$, the symmetric 3×3 trace free matrices, respectively. We say the triple solves the Euler-Reynolds system if

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R} \\ \operatorname{div} v = 0. \end{cases} \tag{2.2.1}$$

Proposition 2.2.2 ([3], 2.2)**.** *Let e be a smooth positive function from $[0,1]$ to $\mathbb{R}$. Then there exists positive constants η, M with the follow properties:*

Let $\delta < 1$ and $(v, p, \mathring{R})$ a solution to the Euler-Reynolds system such that

$$\frac{3}{4}\delta e(t) \le e(t) - \int |v|^2(x,t)\ \mathrm{d}x \le \frac{5}{4}\delta e(t) \tag{2.2.2}$$

. and

$$\sup_{x,t} |\mathring{R}(x,t)| \le \eta\delta. \tag{2.2.3}$$

Then there there exists a second triple (v_1, p_1, R_1) such that the following hold

$$\frac{3}{8}\delta e(t) \le e(t) - \int |v_1|^2(x,t)\ \mathrm{d}x \le \frac{5}{8}\delta e(t), \tag{2.2.4}$$

$$\sup_{x,t} |\mathring{R}(x,t)| \leq \frac{1}{2}\eta\delta, \tag{2.2.5}$$

$$\sup_{x,t} |v_1 - v| \leq M\sqrt{\delta}, \tag{2.2.6}$$

and

$$\sup_{x,t} |p_1 - p| \leq M\delta. \tag{2.2.7}$$

By iteratively applying the Proposition 2.2.2 we see that given a original solution to the Euler-Reynolds system satisfying the conditions then we can construct a sequence of triples $(v_n, p_n, \mathring{R}_n)$ that are Cauchy in the sup norm with the property that $\mathring{R}_n \to 0$, and v_n, p_n converge to continuous functions. That is to say that the pair (v_∞, p_∞) will solve the euler equation with

$$e(t) = \int |v_\infty|^2 (x,t) \, \mathrm{d}x.$$

Noting that the triple $(0,0,0)$ satisfy the hypothesis we have proven the following statement.

Theorem 2.2.3 ([3], 1.1)**.** *Let $e(t)$ be a smooth positive function from $[0,1]$ to $\mathbb{R}$. Then there exists a pair (v,p) that solves the Euler equations with the property*

$$e(t) = \int |v|^2(x,t) \, \mathrm{d}x.$$

The rest of this section will be dedicated to building the pair $(v_1, p_1, \mathring{R}_1)$ and showing they solve the estimates above. In our construction v_1 will be the sum of v and a 'modified' Beltrami flow $w = w_o + w_c$ where w_o is a highly oscillatory structure and w_c is a corrector to make w_o divergence free. p_1 will be the former pressure plus the pressure of the 'modified' Beltrami flow and finally $\mathring{R}_1$ will be a catch-all term defined by inverting the divergence operator.

2.3. **The Velocity Increment.** First we let λ, μ be large natural numbers with $\dfrac{\lambda}{\mu} \in \mathbb{N}$. These parameters will be choosen explicitly in later parts to aid in the convergence of the scheme.

Recalling that the h-principle is a local property we now wish to extend its use to the total space. As such we will build a partition of unity on the velocity space $\mathbb{R}^3$. Fix real numbers $\dfrac{\sqrt{3}}{2} < r_1 < r_2 < 1$. Then choose a smooth function $\varphi \in C_c^\infty(B_{r_2}(0))$ with the property that φ is identically 1 on $B_{r_1}(0)$. By construction we also have that the $\operatorname{Supp}\varphi \subset (-1,1)^3$. Next for $k \in \mathbb{Z}^3$ we define the translates of φ by k,

$$\varphi_k(x) = \varphi(x-k)$$

and the function

$$\psi = \sum_{k \in \mathbb{Z}^3} \varphi_k^2.$$

We observe that ψ is smooth since each ϕ_k is smooth and for each $x \in \mathbb{R}^3$, $0 < \psi(x) < 8$ by support of φ. Using these functions we define

$$\alpha_k = \frac{\varphi_k}{\sqrt{\psi}}$$

and

$$\phi_k^{(j)}(x,\tau) = \sum_{l \in C_j} \alpha_l(\mu x) e^{-i(k\cdot \frac{l}{\mu})\tau}$$

where C_j is a equivalence class of the quotient of $\mathbb{Z}^3$ by $2\mathbb{Z}^3$. Using the fact that α_k, α_l have disjoint supports for $k, l \in C_j$ we have

$$\left|\phi_k^{(j)}\right|^2 = \phi_k^{(j)}\, \overline{\phi_k^{(j)}} = \sum_{l \in C_j} \alpha_l^2(\mu x) + 2 \sum_{l \neq r \in C_j} \alpha_l(\mu x)\alpha_r(\mu x) = \sum_{l \in C_j} \alpha_l^2(\mu x). \tag{2.3.1}$$

Moreover by the definition of α_k we see that $\sum_{j=1}^{8} \left|\phi_k^{(j)}\right| = 1$. Similarly appealing to the disjoint supports of the α_k's we also deduce for all $x \in \mathbb{R}^3$ that

$$\left|D_x^m \phi_k^{(j)}(x,\tau)\right| = \mu^m \left| \sum_{l \in C_j} \alpha_l^{(m)}\Big|_{\mu x} e^{i\left(k\cdot \frac{l}{\mu}\right)\tau} \right| \leq C(m)\mu^m \tag{2.3.2}$$

for a constant $C(m)$ dependent on m. By direct computation we also have that

$$\partial_\tau \phi_k^{(j)}(x,\tau) = \frac{-i}{\mu} \sum_{l \in C_j} (k\cdot l)\, \alpha_l(\mu x) e^{-i(k\cdot \frac{l}{\mu})\tau} = \frac{-i}{\mu}\left(k \cdot \tilde{l}\right) \alpha_{\tilde{l}}(\mu x) e^{-i\left(k\cdot \frac{\tilde{l}}{\mu}\right)\tau}$$

where $\tilde{l}$ is the only element of C_j with the support of $\alpha_{\tilde{l}}$ containing μx. Furthermore by the support of $\alpha_{\tilde{l}}$ we have $|\mu x - \tilde{l}| < 1$. Hence in a neighborhood around any fixed (x,τ) we have the identity

$$\partial_\tau \phi_k^{(j)} + i\,(k\cdot x)\, \phi_k^{(j)} = ik \cdot \left(x - \frac{\tilde{l}}{\mu}\right) \phi_k^{(j)}. \tag{2.3.3}$$

Combining (2.3.2) and (2.3.3) we deduce

$$\sup \left| D_x^m \left(\partial_\tau \phi_k^{(j)} + i\,(k\cdot x)\, \phi_k^{(j)}\right)\right| \leq C(m,|k|)\mu^{m-1}. \tag{2.3.4}$$

Lemma 2.3.1 (Geometric Lemma, [3], 3.2)**.** *For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\lambda_0 > 1$ such that*

(i) There exists sets

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 \;:\; |k| = \lambda_0\}$$

which disjoint for $1 \leq j \leq N$ and $-\Lambda_j = \Lambda_j$.

(ii) There exists smooth functions $\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id}))$ for $1 \leq j \leq N$ where $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$, and satisfy the identity that

(iii) for each $R \in B_{r_0}(\mathrm{Id})$

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}(R)\right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right). \tag{2.3.5}$$

Having collected some basic estimates about the function $\phi_k^{(j)}$ we will defined our oscillatory term. Applying the Geometric Lemma 2.3.1 with $N = 8$ we find $\lambda_0 > 1$, $r_0 > 0$, and pairwise disjoint sets Λ_j with their smooth functions $\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id}))$. Then setting

$$\rho(t) = \frac{1}{3(2\pi)^3}\left(e(t)\left(1 - \frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |v|^2 (x,t)\,\mathrm{dx}\right)$$

and

$$R(x,t) = \rho(t)\,\mathrm{Id} - \mathring{R}(x,t) \tag{2.3.6}$$

we define the oscillatory component

$$w_o(x,t) = \sqrt{\rho(t)} \sum_{j=1}^{8} \sum_{k\in\Lambda_j} \gamma_k^{(j)} \left(\frac{R(x,t)}{\rho(t)} \right) \phi_k^{(j)} \left(v(x,t), \lambda t \right) B_k e^{i\lambda k\cdot x} \tag{2.3.7}$$

where B_k are defined as in Lemma 2.1.3.

We observe that w_o is only well defined for $\frac{R}{\rho} \in B_{r_0}(\mathrm{Id})$ from construction of $\gamma_k^{(j)}$ in the geometric lemma. Using hypothesis we see

$$\begin{aligned} \rho(t) &= \frac{1}{3(2\pi)^3} \left(e(t) - \int_{\mathbb{T}^3} |v|^2(x,t)\,\mathrm{dx} - e(t)\frac{\delta}{2} \right) \\ &\geq \frac{\delta}{12(2\pi)^3} e(t) \\ &\geq \frac{\delta}{12(2\pi)^3} \min_{t\in[0,1]} e(t) > 0. \end{aligned}$$

Then by construct we have

$$\left|\left| \frac{R}{\rho(t)} - \mathrm{Id} \right|\right| = \left|\left| \frac{\mathring{R}}{\rho(t)} \right|\right| = \frac{1}{|\rho(t)|} \left|\left| \mathring{R} \right|\right| \leq \frac{12(2\pi)^3}{\delta \min_{t\in[0,1]}} e(t)\eta\delta = \frac{12(2\pi)^3 \eta}{\min_{t\in[0,1]} e(t).}$$

Thus fixing $0 < \eta < \dfrac{\min_{t\in[0,1]} e(t)}{12(2\pi)^3}$ is sufficient to guarantee that (2.3.7) is well defined.

Next setting

$$a_k = \begin{cases} 0 & k \notin \bigcup_{j=1}^{8} \Lambda_j \\ \sqrt{\rho(t)} \sum_{j=1}^{8} \gamma_k^{(j)} \left(\frac{R(x,t)}{\rho(t)} \right) \phi_k^{(j)} \left(v(x,t), \lambda t \right) & k \in \bigcup_{j=1}^{8} \Lambda_j \end{cases}$$

we can also represent the oscillation term as

$$w_o(x,t) = \sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot(\lambda x)} \tag{2.3.8}$$

naturally leading to the characterization of w_o as *patched Beltrami Flows*. We also see that the 'patching' process used to create w_o means that w_o is no longer a eigenfunction of the curl operator. The following proposition characterize some aspects of w_o.

Proposition 2.3.2 (Patched Beltrami Flows)**.** *The patched Beltrami Flow w_o as defined in* (2.3.7) *satisfies the following identities,*

$$\nabla \times w_o = \sum_{|k|=\lambda_0} \nabla a_k \times B_k e^{i\lambda k\cdot x} + \sum_{|k|=\lambda_0} i\lambda a_k \left(k \times B_k \right) e^{i\lambda k\cdot x}, \tag{2.3.9}$$

$$w_o = \frac{1}{\lambda} \nabla \times \left(\sum_{|k|=\lambda_0} i a_k \frac{k \times B_k}{|k|^2} e^{i\lambda k\cdot x} \right) - \frac{1}{\lambda} \left(\sum_{|k|=\lambda_0} i \nabla a_k \times \frac{k \times B_k}{|k|^2} e^{i\lambda k\cdot x} \right), \tag{2.3.10}$$

and

$$w_o \otimes w_o = R(x,t) + \sum_{1\leq k\leq 2\lambda_0} U_k(x,t) e^{i\lambda k\cdot\xi} \tag{2.3.11}$$

where U_k *is smooth and* $U_k k = \frac{1}{2}\left(\operatorname{tr} U_k\right) k$.

Proof. First we will compute the curl of w_o. Specifically, we recall the vector identities

$$\nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f\left(\nabla \times \mathbf{v}\right)$$

and

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$

for $f, g \in C^1(\mathbb{R}^3, \mathbb{R})$ and constant vector $\mathbf{v} \in \mathbb{R}^3$. Then by direct computation

$$\begin{aligned}
\nabla \times w_o &= \nabla \times \left(\sum_{|k|=\lambda_0} a_k B_k e^{i\lambda k\cdot x} \right) \\
&= \sum_{|k|=\lambda_0} \left((\nabla a_k) + a_k(i\lambda k)\right) \times B_k e^{i\lambda k\cdot x} \\
&= \sum_{|k|=\lambda_0} \nabla a_k \times B_k e^{i\lambda k\cdot x} + \sum_{|k|=\lambda_0} i\lambda a_k \left(k \times B_k\right) e^{i\lambda k\cdot x}
\end{aligned}$$

as claimed. Inspecting (2.3.9) we also deduce (2.3.10). To see this we let $\tilde{w}_o = \sum_{|k|=\lambda_0} a_k \tilde{B}_k e^{i\lambda k\cdot x}$. Hence $\tilde{w}_o$ satisfies the equation

$$\nabla \times \tilde{w}_o = \sum_{|k|=\lambda_0} \nabla a_k \times \tilde{B}_k e^{i\lambda k\cdot x} + \sum_{|k|=\lambda_0} i\lambda a_k \left(k \times \tilde{B}_k\right) e^{i\lambda k\cdot x}.$$

Rearranging we have

$$\sum_{|k|=\lambda_0} a_k k \times \tilde{B}_k e^{i\lambda k\cdot x} = \frac{-i}{\lambda} \nabla \times \tilde{w}_0 + \frac{i}{\lambda} \sum_{|k|=\lambda_0} \nabla a_k \times \tilde{B}_k e^{i\lambda k\cdot x}.$$

Next using that $\{k, A_k, k \times A_k\}$ is an orthogonal basis and setting $\tilde{B}_k = \frac{k\times B_k}{|k|^2}$ we have the identity

$$k \times \frac{k \times B_k}{|k|^2} = -B_k$$

and consequently

$$w_o = \frac{1}{\lambda} \nabla \times \left(\sum_{|k|=\lambda_0} i a_k \frac{k \times B_k}{|k|^2} e^{i\lambda k\cdot x} \right) - \frac{1}{\lambda} \left(\sum_{|k|=\lambda_0} i \nabla a_k \times \frac{k \times B_k}{|k|^2} e^{i\lambda k\cdot x} \right)$$

as desired.

For (2.3.11) we appeal to the strong similarity of w_o to the Beltrami flows. In particular set

$$W(y, s, \tau, \xi) = \sqrt{\rho(s)} \sum_{j=1}^{8} \sum_{k\in\Lambda_j} \gamma_k^{(j)} \left(\frac{R(y,s)}{\rho(s)} \right) \phi_k^{(j)} \left(v(y, s), \tau\right) B_k e^{ik\cdot\xi} \quad = \sum_{|k|=\lambda_0} a_k(y, s, \tau) B_k e^{ik\cdot\xi}. \tag{2.3.12}$$

Then W is a Beltrami flow holding y, s, τ fixed, that is W satisfies (2.1.3). Next by direct assessment

$$\begin{aligned}
W \otimes W &= \sum_{|k|=\lambda_0} a_k a_{-k} B_k \otimes B_{-k} + \sum_{\substack{|k|=|l|=\lambda_0 \\ l\neq -k}} a_k a_l B_k \otimes B_l e^{i(k+l)\cdot\xi} \\
&:= U_0 + \sum_{1\leq |k|\leq 2\lambda_0} U_k e^{ik\cdot\xi}
\end{aligned} \tag{2.3.13}$$

Where

$$U_k = \sum_{k'+\ell'=k} a_{\ell'} a_{k'} B_{\ell'} \otimes B_{k'}$$

and $U_k = 0$ if $k \in \mathbb{Z}^3$ can not be decomposed as $k = k' + l'$ where $k',\ l' \in \mathbb{Z}^3$ with $|k'| = |l'| = \lambda_0$. Then using Proposition 2.1.3 we see

$$\begin{aligned}
U_0 &= \fint W \otimes W \mathrm{d}\xi \\
&= \frac{1}{2}\left(\sum_{|k|=\lambda_0} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)\right) \\
&= \frac{\rho}{2} \sum_{j=1}^{8} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}\left(\frac{R}{\rho}\right)\right)^2 \left|\phi_k^{(j)}(v,\tau)\right|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right) \\
&= \frac{\rho}{2} \sum_{j=1}^{8} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}\left(\frac{R}{\rho}\right)\right)^2 \left(\frac{1}{\psi(v)} \sum_{l \in C_j} \varphi_l^2(v)\right) \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right) \\
&= \frac{1}{\psi(v)} \sum_{j=1}^{8} \sum_{l \in C_j} \varphi_l^2(v) \rho \left[\frac{1}{2} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}\left(\frac{R}{\rho}\right)\right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)\right] \\
&= R \left[\frac{1}{\psi(v)} \sum_{j=1}^{8} \sum_{l \in C_j} \varphi_l^2(v)\right] \\
&= R.
\end{aligned}$$

Finally, again appealing to Proposition 2.1.3, we have

$$\operatorname{div}(W \otimes W) = \frac{1}{2} \nabla |W|^2.$$

Assessing the right hand side we have

$$|W|^2 = W \cdot W = \sum_{|k'|=|l|=\lambda_0} a_{k'} a_l (B_{k'} \cdot B_l) e^{i(k'+l)\cdot\xi} = \sum_{|k| \le 2\lambda_0} \operatorname{tr} U_k e^{ik\cdot\xi}$$

which gives

$$\frac{1}{2} \nabla |W|^2 = \sum_{1 \le |k| \le 2\lambda_0} \frac{1}{2} \operatorname{tr} U_k k e^{ik\cdot\xi}.$$

Investigating the left hand side,

$$\operatorname{div} W \otimes W = \sum_{1 \le |k| \le 2\lambda_0} \operatorname{div} U_k e^{ik\cdot\xi} = \sum_{1 \le |k| \le 2\lambda_0} U_k k e^{ik\cdot\xi}.$$

Comparing frequencies we obtain the desired result. $\square$

Examining (2.3.7) we see that w_o is not divergence free. To maintain the divergence free condition of the solution we introduce a corrector term w_c. We now introduce the Leray Projection of a vector onto its divergence free compont of zero mean to define w_c.

Definition 2.3.3 (Leray Projection)**.** Let v be a $C^1(\mathbb{T}^3, \mathbb{R}^3)$ smooth function. We define

$$\mathcal{Q}v = \nabla\phi + \fint_{\mathbb{T}^3} v$$

where ϕ is the unique solution of

$$\Delta\phi = \operatorname{div} v$$

in $\mathbb{T}^3$ with zero average. Then the *Leray projection* of v onto divergence free fields of zero average is $\mathcal{P} = I - \mathcal{Q}$.

Now we take w_c to be the difference of the Leray projection of w_o and w_o. That is

$$w_c = \mathcal{P}w_o - w_o. \tag{2.3.14}$$

Then we define the field increment as

$$w = w_o + w_c. \tag{2.3.15}$$

and the new vector field as

$$v_1 = v + w. \tag{2.3.16}$$

2.4. **The Reynolds Stress and Pressure.** In keeping with the strong connection between w_o and Beltrami flows we define

$$p_1 = p - \frac{1}{2}|w_o|^2. \tag{2.4.1}$$

Then to define $\mathring{R}_1$ we recall that we need to satisfy the equality

$$\operatorname{div} \mathring{R}_1 = \partial_t v_1 + \operatorname{div} v_1 \otimes v_1 + \nabla p_1.$$

As such we introduce a new operator that acts as a right inverse of the divergence operator

Definition 2.4.1. Let v be a smooth vector field on the torus. We define the operator $\mathcal{R}$ by

$$\mathcal{R}v = \frac{1}{4}\left(\nabla\mathcal{P}u + (\nabla\mathcal{P}u)^T\right) + \frac{3}{4}\left(\nabla u + (\nabla u)^T\right) - \frac{1}{2}(\operatorname{div} u)\,\mathrm{Id} \tag{2.4.2}$$

where $u \in C^\infty\left(\mathbb{T}, \mathbb{R}^3\right)$ is the unique solution of

$$\Delta u = v - \fint v$$

with zero mean.

Proposition 2.4.2 ($\mathcal{R} = \operatorname{div}^{-1}$, [3], 4.3)**.** *For any $v \in C^\infty\left(\mathbb{T}^3, \mathbb{R}^3\right)$ we have the following properties*

(i) $\mathcal{R}v(x)$ *is a symmetric trace-free matrix for each* $x \in \mathbb{T}^3$*.*
(ii) $\operatorname{div} \mathcal{R}v = v - \fint v.$

Proof. We see the $\mathcal{R}v(x)$ is symmetric by construct. Next using the linearity of the trace operator and that $\operatorname{tr}\nabla\mathcal{P} = \operatorname{div}\mathcal{P} = 0$ we have

$$\operatorname{tr}\mathcal{R}v(x) = \frac{3}{4}\operatorname{tr}(\nabla u) + \operatorname{tr}(\nabla u)^T - \frac{3}{2}\operatorname{div} u = 2\cdot\frac{3}{4}\operatorname{div} u - \frac{3}{2}\operatorname{div} u = 0$$

as claimed by (i).

Next for (ii) using that $\operatorname{div}(\nabla\psi^T) = \nabla(\operatorname{div}\psi)$ for vector valued function ψ we have

$$\operatorname{div}\mathcal{R}v = \frac{1}{4}\Delta\mathcal{P}u + \frac{3}{4}\Delta u + \frac{1}{4}\nabla(\operatorname{div} u).$$

Recalling that $\mathcal{P}u = u - \fint u - \nabla\phi$ and that $\Delta\nabla = \nabla\Delta$ we see

$$\Delta\mathcal{P}u = \Delta u - \nabla\Delta\phi = \Delta u - \nabla(\operatorname{div} u)$$

and consequently

$$\operatorname{div}\mathcal{R}v = \Delta u = v - \fint v.$$

Thus $\mathcal{R}$ acts as the right inverse of the divergence operator on mean-free vector fields. $\square$

Using the fact that

$$\operatorname{div}(v_1 \otimes v_1 + p_1 \operatorname{Id}) = \operatorname{div}(v_1 \otimes v_1) + \nabla p_1$$

we have that $\operatorname{div}(v_1 \otimes v_1) + \nabla p_1$ is mean free. Similarly $\partial_t v = \operatorname{div}(v \otimes v + p \operatorname{Id} - \mathring{R})$ is mean free. Finally using that $w_1 = \mathcal{P}w_o$ is mean free we get $\partial_t v_1 = \partial_t v + \partial_t w_1$ is mean free and thus

$$\operatorname{div}\left(\mathcal{R}\left(\partial_t v_1 + v_1 \otimes v_1 + \nabla p_1\right)\right) = \partial_t v_1 + v_1 \otimes v_1 + \nabla p_1. \tag{2.4.3}$$

Hence we take $\mathrm{R}_1 = \mathcal{R}\left(\partial_t v_1 + v_1 \otimes v_1 + \nabla p_1\right)$ and have that the trio $(v_1, p_1, \mathring{R}_1)$ is a solution to Euler-Reynolds system.

2.5. **Estimates.** In this section we will give a sketch that shows the new triple satisfy the inductive estimates. First though we gather some standard Schauder estimates from [3].

Proposition 2.5.1 ([3], 5.1)**.** *For any $\alpha \in (0,1)$ and any $m \in \mathbb{N}$ there exists constants $C(\alpha, m)$ with the following properties. If $\phi, \psi : \mathbb{T}^3 \to \mathbb{R}$ are the unique solutions of*

$$\begin{cases} \Delta\phi = f \\ \\ \fint \phi = 0 \end{cases} \qquad\qquad \begin{cases} \Delta\psi = \operatorname{div} F \\ \\ \fint \psi = 0 \end{cases}$$

then

$$||\phi||_{m+2+\alpha} \leq C(m,\alpha)\,||f||_{m+\alpha}$$

and

$$||\psi||_{m+1+\alpha} \leq C(m,\alpha)\,||F||_{m+\alpha}\,.$$

More over we have the estimates

$$||\mathcal{Q}v||_{m+\alpha} \leq C(m,\alpha)\,||v||_{m+\alpha}$$

$$||\mathcal{P}v||_{m+\alpha} \leq C(m,\alpha)\, ||v||_{m+\alpha}$$

$$||\mathcal{R}v||_{m+1+\alpha} \leq C(m,\alpha)\, ||v||_{m+\alpha}$$

$$||\mathcal{R}(\operatorname{div} A)||_{m+\alpha} \leq C(m,\alpha)\, ||A||_{m+\alpha}$$

$$||\mathcal{R}\mathcal{Q}(\operatorname{div} A)||_{m+\alpha} \leq C(m,\alpha)\, ||A||_{m+\alpha}\,.$$

Proposition 2.5.2 ([3], 5.2)**.** *Let $k \in \mathbb{Z}^3 \setminus \{0\}$ and $\lambda \geq 1$ be fixed.*

(i) For any $a \in C^\infty(\mathbb{T}^3)$ and $m \in \mathbb{N}$ we have

$$\left| \int_{\mathbb{T}^3} a(x) e^{i\lambda k\cdot x}\, \mathrm{dx} \right| \leq \frac{[a]_m}{\lambda^m}.$$

(ii) Let $\phi_\lambda \in C^\infty(\mathbb{T}^3)$ be the solution of

$$\Delta \phi_\lambda = a(x)e^{i\lambda k\cdot x} - \fint_{\mathbb{T}^3} a(y)e^{i\lambda k\cdot y}\; \mathrm{d}y$$

with $\fint \phi_\lambda = 0$. Then for any $\alpha \in (0,1)$ and $m \in \mathbb{N}$ we have the estimate

$$||\nabla\phi_\lambda||_\alpha \leq \frac{C(\alpha,m)}{\lambda^{1-\alpha}}\, ||a||_0 + \frac{C(\alpha,m)}{\lambda^{m-\alpha}}[a]_m + \frac{C(\alpha,m)}{\lambda^m}[a]_{m+\alpha}.$$

Corollary 2.5.3 ([3], 5.3)**.** *Let $k \in \mathbb{Z} \setminus \{0\}$ be fixed and $\lambda \geq 1$. For a smooth vectorfield $a(x) \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ let $F(x) := a(x)e^{i\lambda k\cdot x}$. Then we have*

$$||\mathcal{R}(F)||_\alpha \leq \frac{C(\alpha,m)}{\lambda^{1-\alpha}}\, ||a||_0 + \frac{C(\alpha,m)}{\lambda^{m-\alpha}}[a]_m + \frac{C(\alpha,m)}{\lambda^m}[a]_{m+\alpha}.$$

With the above estimates we are now ready to start estimating the various terms in the scheme. In the following section the constant C will remain independent of λ and μ but can depend on things such as $e, v, \mathring{R}, \lambda_0, \alpha,$ and δ. Finally we will often make use of the inequality

$$1 \leq \mu < \lambda.$$

First we develop some estimates on the coefficient functions of a_k.

Proposition 2.5.4 ([3], 6.1)**.** *Let $a_k \in C^\infty(\mathbb{T}^3 \times [0,1] \times \mathbb{R})$ be given by (2.3.12). Then for any $r \geq 0$ the following estimates hold*

$$||a_k(\cdot, s, \tau)||_r \leq C\mu^r$$

$$||\partial_s a_k(\cdot, s, \tau)||_r \leq C\mu^{r+1}$$

$$||(\partial_\tau a_k + i(k\cdot v)a_k)(\cdot, s, \tau))||_r \leq C\mu^{r-1}$$

$$||\partial_\tau a_k(\cdot, s, \tau)||_r \leq C\mu^r.$$

Proof. Recall that (2.3.2) and (2.3.4) give

$$\sup_{v,\tau} \left| D_v^m \phi_k^{(j)}(v,\tau) \right| \leq C\mu^m$$

and

$$\sup_{v,\tau} \left| D_v^m \left(\partial_\tau \phi_k^{(j)} + i(k\cdot v)\phi_k^{(j)} \right) \right| \leq C\mu^{m-1}.$$

respectively. We only need to keep track of the μ and λ for these estimates as the constant can absorb the other dimensional factors and parameters. Hence we only need to see how often the derivative applies to ϕ_k with respect to v and we can use the above estimates. Thus we obtain

$$||a_k(\cdot,s,\tau)||_r \leq C\mu^r$$

$$||\partial_s a_k(\cdot,s,\tau)||_r \leq C\mu^{r+1}$$

directly. Similarly we also have

$$||(\partial_\tau a_k + i(k\cdot v)a_k)(\cdot,s,\tau))||_r \leq C\mu^{r-1}.$$

Then using triangle inequality we have

$$||\partial_\tau a_k(\cdot,s,\tau)||_r \leq ||(\partial_\tau a_k + i(k\cdot v)a_k)(\cdot,s,\tau))||_r + ||-i(k\cdot v)a_k)(\cdot,s,\tau))||_r \leq C\left(\mu^{r-1}+\mu^r\right) \leq C\mu^r.$$

completing the proposition. □

Corollary 2.5.5 ([3], 6.1)**.** *The functions U_k defined in* (2.3.13) *satisfy the bounds*

$$||U_k(\cdot,s,\tau)||_r \leq C\mu^r$$

$$||\partial_s U_k(\cdot,s,\tau)||_r \leq C\mu^{r+1}$$

$$||(\partial_\tau U_k + i(k\cdot v)U_k)(\cdot,s,\tau))||_r \leq C\mu^{r-1}$$

$$||\partial_\tau U_k(\cdot,s,\tau)||_r \leq C\mu^r.$$

Proof. Note each U_k is a finite sum of $a_k a_{k'}$ and apply the preceding proposition. □

Lemma 2.5.6 (Estimate on Corrector, [3], 6.2)**.**

$$||w_c||_\alpha \leq C\frac{\mu}{\lambda^{1-\alpha}} \tag{2.5.1}$$

Proof. By the characterization of the patched Beltrami flow and equation (2.3.10) we have the following identy for the corrector,

$$w_c = -\frac{1}{\lambda}\mathcal{Q}\left(\sum_{|k|=\lambda_0} i\nabla a_k \times \frac{k\times B_k}{|k|^2} e^{i\lambda k\cdot x} \right).$$

Then apply the derivatives to the coefficent function we have

$$\left|\left| \sum_{|k|=\lambda_0} i\nabla a_k \times \frac{k\times B_k}{|k|^2} e^{i\lambda k\cdot x} \right|\right|_\alpha \leq C\mu^{1+\alpha} \leq C\mu\lambda^\alpha$$

coupled with the Schuader estimates we recover

$$||w_c|| \leq C\frac{\mu}{\lambda^{1-\alpha}}$$

as desired. □

Lemma 2.5.7 (Energy Estimate, [3], 6.3)**.**

$$\left|e(t)(1-\frac{1}{2}\delta) - \int_{\mathbb{T}^3} |v_1|^2 \,\mathrm{dx}\right| \leq C\frac{\mu}{\lambda^{1-\alpha}}. \tag{2.5.2}$$

Proof. First we see that by applying the trace operator to (2.3.11) we have

$$|W|^2 - \operatorname{tr}\mathring{R}_1 = \sum_{1\leq k\leq 2\lambda_0} c_k$$

where each c_k (as sum of $a_k a_{k'}$) satisfies

$$||c_k(\cdot, s, \tau)||_r \leq C\mu^r.$$

Applying Proposition 2.5.2 with $m = 1$ we have

$$\left|\int_{\mathbb{T}^3} |W|^2 - \operatorname{tr}\mathring{R}_1 \,\mathrm{dx}\right| \leq C\frac{\mu}{\lambda}$$

where we also used $[a]_1 \leq ||a||_1$. Similarly since C can depend on v we also get from Proposition 2.5.2 with $m = 1$

$$\left|\int_{\mathbb{T}^3} v\cdot w_o \,\mathrm{dx}\right| \leq C\frac{\mu}{\lambda}.$$

Next we observe that

$$\begin{aligned}
|v_1|^2 - |v|^2 - |w_o|^2 &= |v_1|^2 - |v+w_o|^2 + 2v\cdot w_o \\
&= (v_1 + v + w_o)\cdot(v_1 - v - w_o) + 2v\cdot w_o \\
&= (v_1 + v + w_o)\cdot w_c + 2v\cdot w_o \\
&= (2v + 2w_o + w_c)\cdot w_c + 2v\cdot w_o \\
&= 2v\cdot w_c + 2w_o\cdot w_c + |w_c|^2 + 2v\cdot w_o.
\end{aligned}$$

Then using the boundedness of w_o and that $|w_c| \leq ||w_c||_\alpha$ we have

$$\int_{\mathbb{T}^3} |v_1|^2 - |v|^2 - |w_o|^2 \,\mathrm{dx} \leq C\left(\frac{\mu}{\lambda^{1-\alpha}} + \frac{\mu}{\lambda}\right) \leq C\frac{\mu}{\lambda^{1-\alpha}}.$$

Next by definition of R in (2.3.6) we have

$$\operatorname{tr}\mathring{R}_1 = 3\rho = \frac{1}{(2\pi)^3}\left(e(t)(1-\frac{1}{2}\delta) - \int_{\mathbb{T}^3} |v|^2 \,\mathrm{dx}\right).$$

Rearranging and adding zero (twice) we obtain

$$e(t)(1-\frac{1}{2}\delta) - \int_{\mathbb{T}^3} |v_1|^2 \,\mathrm{dx} = (2\pi)^3 \operatorname{tr}\mathring{R}_1 - \int_{\mathbb{T}^3} |w_o|^2 - |w_o|^2 - |v_1|^2 + |v|^2 \,\mathrm{dx}.$$

Factoring $\operatorname{tr}\mathring{R}_1$ into the integral and using triangle inequality with the above estimates the result follows. □

The rest of the proof deals with bookchecking the Reynolds Stress tensor by diving it up into easier to estimate parts and then using the Schauder estimates liberally. Ultimately we recover the estimate

$$\left|\left|\mathring{R}_1\right|\right|_\alpha \leq C\left(\lambda^{\alpha-\beta} + \lambda^{\alpha+2\beta-1} + \lambda^{2\alpha+\beta-1}\right)$$

when we set $\mu = \lambda^\beta$. Then taking $\alpha < \beta$ and $a + 2\beta < 1$ with a large enough λ we can assure the validity of the inductive estimates. From this we can construct a sequence that converges to a (weak) solution to the Euler equations with energy profile $e(t)$.

3. Navier-Stokes

The $3D$ incompressible Navier-Stokes equations are

$$\begin{cases} \frac{D}{Dt}\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} \\ \operatorname{div}\mathbf{v} = 0 \end{cases} \tag{3.0.1}$$

where $\mathbf{v}$, p, and ν are the velocity field, scalar pressure, and fluid viscosity as before. Similar to before we we say a pair (v, p) is a weak solution to Navier-Stokes if

$$\int_{\mathbb{R}^+}\int_{\mathbb{T}^3} v \cdot (\partial_t\varphi + (v\cdot\nabla)\varphi + \nu\Delta\varphi)\,\mathrm{dx}\ \ \mathrm{d}t = 0$$

for all test functions φ with compact support in space and time, $\mathbb{T}^3 \times (0,\infty)$. For the rest of this section we will take $\nu = 1$.

3.1. Iteration Scheme.

First we say that a triple $(v, p, \mathring{R})$ solves the Navier-Stokes-Reynolds (NSR) equations if

$$\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p = \Delta v + \operatorname{div}\mathring{R} \\ \operatorname{div} v = 0. \end{cases} \tag{3.1.1}$$

One of the main goals of [4] is to construct a weak solution to the Navier-Stokes equations that has arbitrary (nonnegative) energy profile $e(t)$. Towards this end [4] follows in the foot steps of [3] in creating a iterative scheme that produces a sequence of NSR solutions (v_q, p_q, R_q) with a limit of a weak Naiver-Stokes solution.

We define the following quantities

$$\lambda_q = a^{b^q}$$

$$\delta_q = \lambda_q^{3\beta}\lambda_q^{-2\beta}$$

with parameters $a >> 1$, $b \in 16\mathbb{N}$, $\beta \in (0,1)$ to be defined later and a sufficiently small ϵ_R. Then the following inductive estimates are used in the construction[4]:

$$||v_q||_{C^1_{x,t}} \le \lambda_q^4, \tag{3.1.2}$$

$$\left|\left|\mathring{R}_q\right|\right|_{L^1} \le \lambda_q^{\epsilon_R}\delta_{q+1}, \tag{3.1.3}$$

$$\left|\left|\mathring{R}_1\right|\right| \le \lambda_q^{10}. \tag{3.1.4}$$

Additionally for the energy profile we have for all time that

$$0 \le e(t) - \int_{\mathbb{T}^2} |v_q|^2\,\mathrm{dx} \le \delta_{q+1} \tag{3.1.5}$$

and if

$$e(t) - \int_{\mathbb{T}^3} |v_q|^2\,\mathrm{dx} \le \frac{\delta_{q+1}}{100} \tag{3.1.6}$$

[4]Here, we use $\left|\left|\mathring{R}\right|\right|_{L^p} = \left|\left|\mathring{R}\right|\right|_{L^p_x L^\infty_t}$ to denote the L^p norm in space and sup norm in time. In general we will take norms to be applied to the spacial component and a suprememum in time unless otherwise noted.

then $v_q(\cdot,t) \equiv 0$ and $\mathring{R}_q(\cdot,t) \equiv 0$.

3.2. **Intermittent Beltrami Waves.** Recall proposition 2.1.3 on the properties of the strong Beltrami flows used to construct the weak Euler-Reynolds sequence. Then using $\xi = \frac{k}{|k|}$ we have the following corollary.

Corollary 3.2.1 ([4], 3.2)**.** *Given $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$, let $A_k \in \mathbb{S}^2 \cap \mathbb{Q}^3$ be such that*

$$A_\xi \cdot \xi = 0, \qquad |A_\xi| = 1, \qquad A_{(-\xi)} = A_\xi.$$

Define

$$B_\xi = \frac{1}{\sqrt{2}} \left(A_\xi + i\xi \times A_\xi \right).$$

Then for any finite subset $\Lambda \subset \mathbb{Q} \cap \mathbb{S}^2$ with $\Lambda = -\Lambda$, $\lambda \in \mathbb{Q}$ with $\lambda\Lambda \subset \mathbb{Z}^3$, and $a_\xi \in \mathbb{C}$ with $\overline{a_\xi} = a_{(-\xi)}$ the vector field

$$W(x) = \sum_{\xi \in \Lambda} a_\xi e^{i\lambda\xi\cdot x}$$

is real valued and divergence free. More over $W(x)$ also satisfies the following properties

$$\operatorname{div}(W \otimes W) = -\frac{1}{2}\nabla |W|^2$$

and

$$\fint_{\mathbb{T}^3} W \otimes W = \frac{1}{2} \sum_{\xi \in \Lambda} |a_\xi|^2 \left(\operatorname{Id} - \xi \otimes \xi \right).$$

Proof. We note that $\lambda\Lambda$ is composed of finitely many elements and hence finitely many frequency shells. Applying the proposition to each shell we recover that W is real valued, divergence free and

$$\operatorname{div}(W \otimes W) = -\frac{1}{2}\nabla |W|^2.$$

For the final part we note that if $k - l \neq 0$ then

$$\fint_{\mathbb{T}^3} e^{i(k-l)\cdot x} = 0$$

along with that $\{\xi, A_\xi, \xi \times A_\xi\}$ is an orthonormal basis. □

Similarly we have the following counter part of the geometric lemma 2.3.1 from the Euler Equations.

Corollary 3.2.2 ([4], 3.3)**.** *For every $N \in \mathbb{N}$ there is $\epsilon > 0$, and $\lambda > 1$ with the following properties.*

(i) There exist finite disjoint subsets $\Lambda_j \subset \mathbb{S}^2 \cap \mathbb{Q}^2$ for $j \in \{1, \dots, N\}$ with $\lambda\Lambda_j \subset \mathbb{Z}^3$ and $-\Lambda_j = \Lambda_j$.

(ii) For each $\xi \in \Lambda_j$ we have a smooth positive function

$$\gamma_\xi^{(j)} \in C^\infty \left(B_\epsilon(\operatorname{Id}) \right)$$

with

$$\gamma_{-\xi}^{(j)} = \gamma_{(\xi)}^{(j)}.$$

(iii) For all $R \in B_\epsilon(\operatorname{Id})$ we have the identity

$$R = \frac{1}{2} \sum_{\xi \in \Lambda_j} \left(\gamma_\xi^{(j)}(R) \right)^2 \left(\operatorname{Id} - \xi \otimes \xi \right). \tag{3.2.1}$$

Proof. Based on proposition 2.3.1 the only thing that needs to be shown is that there exists λ with $\lambda\Lambda_j \subset \mathbb{Z}^3$. This follows given that $|\Lambda_j| < \infty$ for each j. □

Remark 3.2.3. Given the above the finite nature of the sets constructed we can find universal constants N_Λ and c_Λ with the properties that

$$\{\xi, A_\xi, \xi \times A_\xi\} \subset \frac{1}{N_\Lambda}\mathbb{Z}^3$$

and if $\xi \neq \xi'$ then

$$|\xi - \xi'| > c_\Lambda$$

for all $\xi \in \cup\Lambda_j$.

Next we recall that the Dirichlet kernel is defined as

$$D_r(x) = \sum_{\xi=-r}^{r} e^{i\xi x} =: D_r^{(1)}(x)$$

and for $p > 1$ grows like

$$cr^{1-\frac{1}{p}} \leq ||D_r|| \leq Cr^{1-\frac{1}{p}}$$

where the constants are independent of r as shown in Lemma B.2. We extend this construction to $\mathbb{R}^n$ by letting

$$\Omega_r^n = \{z \in \mathbb{Z}^n \ : \ ||z||_\infty \leq r\}$$

and

$$D_r^{(n)}(x) = \frac{1}{(2r+1)^{\frac{n}{2}}} \sum_{\xi \in \Omega_r^{(n)}} e^{i\xi \cdot x}.$$

Clearly we have that

$$D_r^{(n)}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \prod_{i=1}^{n}\left(\frac{1}{(2r+1)^{\frac{1}{2}}} D_r^{(1)}(x_i)\right)$$

and by Fubini with B.13,

$$\left|\left|D_r^{(n)}\right|\right| = (2\pi)^n \tag{3.2.2}$$

and

$$\left|\left|D_r^{(n)}\right|\right|_{L^p} \leq C(p,n) r^{\frac{n}{2}-\frac{n}{p}}. \tag{3.2.3}$$

Next we define

$$\eta_{(\xi)}(x,t) = D_r^{(n)}\left(\lambda\sigma\begin{bmatrix} N_\Lambda \xi \cdot x + \mu t \\ N_\Lambda A_\xi \cdot x \\ N_\Lambda \xi \times A_\xi \cdot x \end{bmatrix}\right) \tag{3.2.4}$$

where $\xi \in \Lambda_j^+$ and Λ_j^+ is one of the equivalence class of Λ_j under the identification $x \sim -x$. Then for $\xi \in \Lambda_j \setminus \lambda_j^+$ we set $\eta_{(\xi)} = \eta_{(-\xi)}$. The parameters λ, σ, r, and μ will be chosen later. For now we will assume that $\sigma << 1$, $r >> 1$,

$$\sigma r \leq \frac{c_\Lambda}{10 N_\Lambda}, \tag{3.2.5}$$

λ is a multiple of N_Λ, $\lambda\sigma \in \mathbb{N}$, and $\mu \in (\lambda, \lambda^2)$.

Then as $\{N_\Lambda \xi, N_\Lambda A_\xi, N_\Lambda \xi \times A_\xi\} \subset \mathbb{Z}^3$ for all ξ we have that $\eta_{(\xi)}(x,t)$ is $2\pi\lambda\sigma$ periodic in every coordinate by 2π periodicity of $D_r^{(3)}$. Similarly it also inherits the equality

$$\fint_{\mathbb{T}^3} \eta_{(\xi)}(x,t)\,\mathrm{dx} = 1 \tag{3.2.6}$$

and the bound

$$\left|\left|\eta_{(\xi)}\right|\right|_{L^p} \leq C(p) r^{\frac{3}{2}-\frac{3}{p}}$$

from the multidimensional Dirichlet kernel. With this in hand we define the *intermittent Beltrami wave* $\mathbb{W}_{(\xi)}$ as

$$\mathbb{W}_{(\xi)}(x,t) = \eta_{(\xi)}(x,t) B_{(\xi)} e^{i\lambda\xi\cdot x}. \tag{3.2.7}$$

An analogue ot Corollary 3.2.1 for intermittent Beltrami flows is given next.

Proposition 3.2.4 ([4], 3.4)**.** *Let $\mathbb{W}_{(\xi)}$ be defined as before and $\Lambda_j, \epsilon, \gamma_\xi^{(j)}$ be as in Corollary 3.2.2. Then if $a_\xi \in \mathbb{C}$ are constants chosen such that $\overline{a_\xi} = a_{(-\xi)}$, the vector field*

$$v = \sum_j \sum_{\xi \in \Lambda_j} a_\xi \mathbb{W}_{(\xi)}$$

is real valued. More over, for each $R \in B_\epsilon(\mathrm{Id})$ the identities

$$\sum_{\xi\in\Lambda_j} \left(\gamma_\xi^{(j)}(R)\right)^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)}\,\mathrm{dx} = R \tag{3.2.8}$$

and

$$\sum_{\xi\in\Lambda_j} \left(\gamma_\xi^{(j)}(R)\right)^2 B_\xi \otimes B_{-\xi} = R \tag{3.2.9}$$

hold.

Proof. We see that v is real valued as

$$\begin{aligned}
\sum_j \sum_{\xi\in\Lambda_j} a_\xi \mathbb{W}_{(\xi)} &= \sum_j \sum_{\xi\in\Lambda_j^+} \left(a_\xi \mathbb{W}_{(\xi)} + a_{-\xi}\mathbb{W}_{(-\xi)}\right) \\
&= \sum_j \sum_{\xi\in\Lambda_j^+} \left(a_{(\xi)}\eta_\xi(x,t) B_\xi e^{i\xi\cdot x} + \overline{a_\xi}\,\eta_\xi(x,t)\,\overline{B_\xi}\,e^{-\xi\cdot x}\right) \\
&= \sum_j \sum_{\xi\in\Lambda_j^+} \eta_{(\xi)}(x,t) 2\,\mathrm{Re}\left(a_\xi B_\xi\right)\cos\left(\xi\cdot x\right).
\end{aligned}$$

Then for (3.2.8) we note that

$$\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} = \eta_{(\xi)}\eta_{(-\xi)} B_{(\xi)} \otimes B_{(-\xi)} = \eta_{(\xi)}^2 B_\xi \otimes \overline{B_\xi}$$

which when coupled with (3.2.6) gives

$$\fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)}\,\mathrm{dx} = B_\xi \otimes \overline{B_\xi}\,.$$

Hence

$$\begin{aligned}\sum_{\xi\in\Lambda_j} \left(\gamma_\xi^{(j)}(R)\right)^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)}\,\mathrm{dx} &= \sum_{\xi\in\Lambda_j} \left(\gamma_\xi^{(j)}(R)\right)^2 B_\xi \otimes \overline{B_\xi} \\ &= \sum_{\xi\in\Lambda_j^+} \left(\gamma_\xi^{(j)}(R)\right)^2 \left(B_\xi \otimes \overline{B_\xi} + \overline{B_\xi} \otimes B_\xi\right) \\ &= \sum_{\xi\in\Lambda_j^+} \left(\gamma_\xi^{(j)}(R)\right)^2 2\,\mathrm{Re}\left(B_\xi \otimes B_{(-\xi)}\right).\end{aligned}$$

By definition of B_ξ direct computation shows

$$\mathrm{Re}\left(B_\xi \otimes B_{(-\xi)}\right) = \frac{1}{2}\left(A_\xi \otimes A_\xi + (\xi \times A_\xi) \otimes (\xi \times A_\xi)\right) = \frac{1}{2}\left(\mathrm{Id} - \xi \otimes \xi\right).$$

Consequently,

$$\sum_{\xi\in\Lambda_j^+} \left(\gamma_\xi^{(j)}(R)\right)^2 2\,\mathrm{Re}\left(B_\xi \otimes B_{(-\xi)}\right) = \sum_{\xi\in\Lambda_j^+} \left(\gamma_\xi^{(j)}(R)\right)^2 2\,\mathrm{Re}\left(\mathrm{Id} - \xi \otimes \xi\right) = R$$

where the last equality follows from Corollary 3.2.2. Thus (3.2.8) and (3.2.9) hold. ☐

One of the key difference between Corollary 3.2.2 and Proposition 3.2.4 is that the intermittent Beltrami flow is *not* divergence free or a eigenfunction of the curl. Recalling the Bernstein inequality we will be able to bound these quantities.

Lemma 3.2.5 (Bernstein Inequality)**.** *Suppose that $f \in L^1 + L^2$ and $\hat{f}$ is supported in $B_r(0)$. Then for any α and $p \in [1,\infty)$ we have that*

$$\|D^\alpha f\|_{L^p} \le C r^{|\alpha|} \|f\|_{L^p} \tag{3.2.10}$$

Proposition 3.2.6 ([4], 3.5)**.** *Let $\mathbb{W}_{(\xi)}$ be defined as before. Then the bounds*

$$\left\|\nabla^N \partial_t^K \eta_{(\xi)}\right\|_{L^p} \le C(N,K,p)\,(\lambda\sigma r)^N (\lambda\sigma\mu r)^K r^{\frac{3}{2}-\frac{3}{p}}. \tag{3.2.11}$$

and

$$\left\|\nabla^N \partial_t^K \mathbb{W}_{(\xi)}\right\|_{L^p} \le C(N,K,p)\lambda^N (\lambda\sigma\mu r)^K r^{\frac{3}{2}-\frac{3}{p}} \tag{3.2.12}$$

hold.

Proof. Let

$$W_{(\xi)}(x) = B_\xi e^{i\lambda\xi\cdot x}$$

for the Beltrami wave component of $\mathbb{W}_{(\xi)}$. Then we see $\mathbb{W}$ is supported in the closed ball of radius λ in the frequency domain. Similarly $\eta_{(\xi)}$ is supported in the closed ball of radius $2\lambda\sigma r N_\Lambda < \lambda$. Hence $\mathbb{W}$ has frequency support in the $B_{2\lambda}(0)$. These supports give that

$$\left\|\nabla \mathbb{W}_{(\xi)}\right\|_{L^p} \le C(p)\lambda \left\|\mathbb{W}_{(\xi)}\right\|_{L^p}.$$

and

$$\left|\left|\nabla \eta_{(\xi)}\right|\right|_{L^p} \leq C(p)\lambda\sigma r \left|\left|\eta_{(\xi)}\right|\right|_{L^p}.$$

Similarly treating the time domain in terms of frequencies we have that the support of $\eta_{(\xi)}$, and by implication $\mathbb{W}_{(\xi)}$, is contained in $B_{cr\sigma\lambda\mu N_\Lambda}(0)$ for $c > 1$. Hence

$$\left|\left|\partial_t \eta_{(\xi)}\right|\right|_{L^p} \leq C(p) r\sigma\lambda\mu \left|\left|\eta_{(\xi)}\right|\right|$$

and

$$\left|\left|\partial_t \mathbb{W}_{(\xi)}\right|\right|_{L^p} \leq C(p) r\sigma\lambda\mu \left|\left|\mathbb{W}_{(\xi)}\right|\right|.$$

Finally the inequality

$$|\mathbb{W}_{(\xi)}| \leq |\eta_{(\xi)}|$$

in conjunction with (3.2.3) and induction give

$$\left|\left|\nabla^N \partial_t^K \eta_{(\xi)}\right|\right|_{L^p} \leq C(N,K,p) \left(\lambda\sigma r\right)^N \left(\lambda\sigma\mu r\right)^K r^{\frac{3}{2}-\frac{3}{p}}$$

and

$$\left|\left|\nabla^N \partial_t^K \mathbb{W}_{(\xi)}\right|\right|_{L^p} \leq C(N,K,p) \lambda^N \left(\lambda\sigma\mu r\right)^K r^{\frac{3}{2}-\frac{3}{p}}$$

as desired. □

Using this we can quantify how far the the intermittent Beltrami flows fail from being divergence free and eigenfunctions of curl. We summarize this result in the follow Corollary for the L^2 norm.

Corollary 3.2.7 ([4]). *For $\mathbb{W}_{(\xi)}$ defined as above we have that*

$$\left|\left|\operatorname{div} \mathbb{W}_{(\xi)}\right|\right|_{L^2} \leq C\lambda\sigma r \tag{3.2.13}$$

and

$$\left|\left|\nabla \times \mathbb{W}_{(\xi)} - \lambda \mathbb{W}_{(\xi)}\right|\right|_{L^2} \leq C\lambda\sigma r \tag{3.2.14}$$

Proof. Direct computation gives

$$\operatorname{div} \eta_{(\xi)} W = \eta \left(\operatorname{div} W\right) + \nabla\eta \cdot W = \nabla \eta \cdot W$$

and

$$\nabla \times \eta W = \left(\nabla \eta\right) \times W + \eta \left(\nabla \times W\right) = \nabla\eta \times W + \lambda \eta W.$$

Rearranging and using Corollary 3.2.7 we have

$$\left|\left|\operatorname{div} \mathbb{W}_{(\xi)}\right|\right|_{L^2} = \left|\left|\nabla \eta_{(\xi)} \cdot W_\xi\right|\right|_{L^2} \leq \left|\left|\nabla \eta_{(\xi)}\right|\right|_{L^2} \leq C\lambda\sigma r$$

and

$$\left|\left|\nabla \times \mathbb{W}_{(\xi)} - \lambda \mathbb{W}_{(\xi)}\right|\right|_{L^2} = \left|\left|\nabla \eta_{(\xi)} \times W_\xi\right|\right|_{L^2} \leq \left|\left|\nabla \eta_{(\xi)}\right|\right|_L^2 \leq C\lambda\sigma r$$

which complete the proof. □

Consequently of $\mathbb{W}_{(\xi)}$ not being a eigenfunction of the curl we lose the Beltrami identity relating divergence of the tensor to the gradient. Namely,

$$\operatorname{div}(W_\xi \otimes W_\xi) = \nabla \frac{1}{2}|W_\xi|^2.$$

As such we state the following proposition.

Proposition 3.2.8 ([4]). *Let $\mathbb{W}_{(\xi)}$ and $\mathbb{W}_{(\zeta)}$ be intermittent Beltrami waves. Then*

$$\begin{aligned}\operatorname{div}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} + \mathbb{W}_{(\zeta)} \otimes \mathbb{W}_{(\xi)}\right) &= \left(\left(W_{(\xi)} \cdot \nabla\right)\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right) W_{(\zeta)} + \left(\left(W_{(\zeta)} \cdot \nabla\right)\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right) W_{(\xi)} \\ &\quad + \eta_{(\xi)}\eta_{(\xi)} \nabla\left(W_{(\zeta)} \cdot W_{(\xi)}\right)\end{aligned} \tag{3.2.15}$$

and for $\zeta = -\xi$,

$$\operatorname{div}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)} \otimes \mathbb{W}_{(\xi)}\right) = \nabla \eta_{(\xi)}^2 - \left(\left(\xi \cdot \nabla\right)\eta_{(\xi)}^2\right)\xi = \nabla \eta_{(\xi)}^2 - \frac{1}{\mu}\partial_t \eta_{(\xi)}^2 \xi. \tag{3.2.16}$$

Proof. Direct computation gives

$$\operatorname{div}(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} + \mathbb{W}_{(\zeta)} \otimes \mathbb{W}_{(\xi)}) = (W_\xi \otimes W_\zeta + W_\zeta \otimes W_\xi)\nabla(\eta_{(\xi)}\eta_{(\zeta)}) + \eta_{(\xi)}\eta_{(\zeta)}(\operatorname{div}(W_\xi \otimes W_\zeta + W_\zeta \otimes W_\xi)).$$

Then using the fact $\operatorname{div}(W_\xi \otimes W_\zeta) = (W_\xi \cdot \nabla)W_\zeta + (\nabla \cdot W_\zeta)W_\xi = (W_\xi \cdot \nabla)W_\zeta$ and the identity

$$(u \cdot \nabla)v + (v \cdot \nabla)u = \nabla(u \cdot v) - u \times (\nabla \times v) - v \times (\nabla \times u)$$

we see

$$\begin{aligned}\operatorname{div}(W_\xi \otimes W_\zeta + W_\zeta \otimes W_\xi) &= \nabla(W_\xi \cdot W_\zeta) - W_\xi(\nabla \times W_\zeta) - W_\zeta(\nabla \times W_\xi) \\ &= \nabla(W_\xi \times W_\zeta) - \lambda W_\xi \times W_\zeta - \lambda W_\zeta \times W_\xi \\ &= \nabla(W_\xi \cdot W_\zeta)\end{aligned}$$

for the final equality we used the antisymmetry of the cross product. Similarly one shows that

$$(W_\xi \otimes W_\zeta + W_\zeta \otimes W_\xi)\nabla(\eta_{(\xi)}\eta_{(\zeta)}) = \left(\left(W_\zeta \cdot \nabla\right)\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right) W_\xi + \left(\left(W_\xi \cdot \nabla\right)\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right) W_\zeta.$$

Combining these last two identities gives the first equality of the proposition directly. Then letting $\zeta = -\xi$ we have $W_\xi \cdot W_{-\xi} = 1$ so $\nabla\left(W_\xi \cdot W_{(-\xi)}\right) = 0$. That is,

$$\operatorname{div}(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)} \otimes \mathbb{W}_{(\xi)}) = \left(\left(W_{(-\xi)} \cdot \nabla\right)\left(\eta_{(\xi)}^2\right)\right) W_\xi + \left(\left(W_\xi \cdot \nabla\right)\left(\eta_{(\xi)}^2\right)\right) W_{(-\xi)}.$$

Next using that $W_{\pm\xi} = B_{\pm\xi}e^{\pm i\lambda\xi \cdot x}$ with $B_{\pm\xi} = \frac{1}{\sqrt{2}}\left(A_\xi \pm i\xi \times A_\xi\right)$ we have

$$\left(\left(W_{(-\xi)} \cdot \nabla\right)\left(\eta_{(\xi)}^2\right)\right) W_\xi + \left(\left(W_\xi \cdot \nabla\right)\left(\eta_{(\xi)}^2\right)\right) W_{(-\xi)} = \left((A_\xi \cdot \nabla)\eta_{(\xi)}^2\right) A_{(\xi)} + \left((\xi \times A_\xi \cdot \nabla)\eta_{(\xi)}^2\right) \xi \times A_{(\xi)}.$$

Finally as $\{\xi, A_\xi, \xi \times A_\xi\}$ form a orthonormal basis we have the identity

$$x = (\xi \cdot x)\xi + (A_\xi \cdot x)A_\xi + ((\xi \times A_\xi) \cdot x)\xi \times A_\xi$$

which gives

$$\operatorname{div}(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)} \otimes \mathbb{W}_{(\xi)}) = \nabla \eta_{(\xi)}^2 - \left((\xi \cdot \nabla)\eta_{(\xi)}^2\right)\xi.$$

Then as

$$\partial_t \eta_{(\xi)} = \mu(\xi \cdot \nabla)\eta_{(\xi)}$$

we have

$$\partial_t \eta^2_{(\xi)} = \mu(\xi \cdot \nabla)\eta^2_{(\xi)}$$

completing the proof. □

3.3. **The Velocity Increment.** Now we will build the tools to define the velocity increment

$$w_{q+1} = v_{q+1} - v_1 := w^{(p)}_{q+1} + w^{(c)}_{q+1} + w^{(t)}_{q+1}. \tag{3.3.1}$$

Here we have:

- $w^{(p)}_{q+1}$ is the *principal* part that adds the majority of the energy at a given iteration,
- $w^{(c)}_{q+1}$ is the *incompressibility corrector* that makes w_{q+1} divergence free, and
- $w^{(t)}_{q+1}$ is the *temporal corrector* that is used to cancel out slow frequency terms upon differentiation.

Next we introduce the C^∞ mollifiers ϕ and φ with compact support of radius 1 on $\mathbb{R}^3$ (space) and $\mathbb{R}$ (time) respectively. Further we define $\phi_\ell(x) = \ell\phi\left(\frac{x}{\ell}\right)$ and $\varphi_\ell(x) = \ell\phi\left(\frac{x}{\ell}\right)$ to be the mollifications at length ℓ. Now we will smooth the velocity field v_q and $\mathring{R}_q$. That is we let

$$v_\ell = (v_q *_x \phi_\ell) *_t \varphi_\ell \tag{3.3.2}$$

and

$$\mathring{R}_\ell = \left(\mathring{R}_q *_x \phi_\ell\right) *_t \varphi_\ell. \tag{3.3.3}$$

We have the identity

$$\begin{aligned}((v \otimes v) *_x \phi_l) *_t \varphi_\ell &= v_\ell \otimes v_\ell - (v_\ell \otimes v_\ell - ((v \otimes v) *_x \phi) *_t \varphi) \\ &= v_\ell \otimes v_\ell - \left(v_\ell \mathring{\otimes} v_\ell - \left((v \mathring{\otimes} v) *_x \phi_\ell\right) *_t \varphi_\ell\right) - \frac{1}{3}\left(|v_\ell|^2 - \left((|v|^2) *_x \phi_\ell\right) *_t \varphi_\ell\right) \mathrm{Id}\end{aligned}$$

where $a \mathring{\otimes} b = a \otimes b - \frac{1}{3} a \cdot b \,\mathrm{Id}$ denotes the traceless component of $a \otimes b$. Using the fact

$$\operatorname{div}(|v|^2 \,\mathrm{Id}) = \nabla |v|^2$$

we see the pair $(v_\ell, \mathring{R}_\ell)$ satisfies the Navier-Stokes-Reynolds equation

$$\partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell = \Delta v_\ell + \operatorname{div}\left(\mathring{R}_\ell + R_{com}\right), \tag{3.3.4}$$

$$\operatorname{div} v_\ell = 0 \tag{3.3.5}$$

with

$$\begin{aligned} p_\ell &= (p_q *_x \phi_\ell) *_t \varphi_\ell - \frac{1}{3}(|v_\ell|^2 - (|v_q|^2 *_x \phi_\ell) * \varphi_\ell) \\ R_{com} &= (v_\ell \mathring{\otimes} v_\ell) - ((v_q \mathring{\otimes} v_q) *_x)\phi *_t \varphi. \end{aligned} \tag{3.3.6}$$

Having smoothed the vector field and Reynolds stress we now will define a family of *stress cutoff* functions which will have the same role as the partition of unity in the coefficients of the Euler paper. Here our stress cutoff functions will depend on the Reynolds stress which leads to the term stress cutoff function. Towards this end let $\tilde{\chi}_0$ and $\tilde{\chi}$ be bump functions on the intervals $[0,4]$ and $[\frac{1}{4},4]$ respectively with

$$1 \equiv \tilde{\chi}_0^2(x) + \tilde{\chi}^2\left(\frac{1}{4}(x-1)\right) + \sum_{i=2}^{\infty} \tilde{\chi}^2\left(\frac{x}{4^i}\right)$$

for all $x \geq 1$. Then we introduce the stress component and define

$$\chi_{(0)}(x,t) = \tilde{\chi}_0\left(\left(1 + \left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|^2\right)^{\frac{1}{2}}\right), \tag{3.3.7}$$

$$\chi_{(1)}(x,t) = \tilde{\chi}\left(\frac{1}{4}\left(\left(1 + \left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|^2\right)^{\frac{1}{2}} - 1\right)\right) \tag{3.3.8}$$

and,

$$\chi_{(i)}(x,t) = \tilde{\chi}\left(\frac{1}{4^i}\left(1 + \left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|^2\right)^{\frac{1}{2}}\right) \tag{3.3.9}$$

for $i > 2$. Here we have used that $|A|$ denotes the Euclidean norm of a matrix and ϵ_R is a small constant to be chosen later.

Remark 3.3.1. It is important to note that $\chi_{(1)}$ is shifted right to allow us to achieve a desired estimate on $\int \chi_{(0)}^2 \,\mathrm{dx}$. By shifting $\chi_{(1)}$ we have that

$$\operatorname{Supp}\left(\sum_{i\geq 1} \chi_{(i)}\right) \subseteq \{x \;:\; \mathring{R}_\ell \geq (\sqrt{3})100\delta_{q+1}\lambda_1^{-\epsilon R}\} \subset \{x \;:\; \mathring{R}_\ell \geq 2\left|\mathbb{T}^3\right|^{-1}\delta_{q+1}\lambda_1^{-\epsilon R}\}.$$

Thus with Chebyshev we have

$$\int_{\mathbb{T}^3} \sum_{i\geq 1} \chi_{(i)}^2 \,\mathrm{dx} \leq \left|\{x \;:\; \mathring{R}_\ell \geq 2\left|\mathbb{T}^3\right|^{-1}\delta_{q+1}\lambda_1^{-\epsilon R}\}\right| \leq \frac{|\mathbb{T}^3|\left\|\mathring{R}_\ell\right\|_{L^1}}{2\lambda_q^{-\epsilon_R}\delta_{q+1}} \leq \frac{|\mathbb{T}^3|\left\|\mathring{R}_q\right\|_{L^1}}{2\lambda_q^{-\epsilon_R}\delta_{q+1}} \overset{(3.1.4)}{\leq} \frac{|\mathbb{T}^3|}{2}.$$

Consequently,

$$\int_{\mathbb{T}^3} \chi_{(0)}^2 \,\mathrm{dx} = |\mathbb{T}^3| - \int_{\mathbb{T}^3} \sum_{i\geq 1} \chi_{(i)}^2 \,\mathrm{dx} \geq |\mathbb{T}^3| - \frac{|\mathbb{T}^3|}{2} = \frac{|\mathbb{T}^3|}{2}. \tag{3.3.10}$$

With the stress functions on hand we are ready to define the coefficients of the intermittent Beltrami flow that will be the principal component in the velocity iterate. For $i \geq 1$ let

$$a_{(\xi)} = \sqrt{\rho_i}\chi_{(i)}\gamma_{(\xi)}\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_i}\right) \tag{3.3.11}$$

and for $i \geq 1$ where $\rho_i = \lambda_q^{-\epsilon_R}\delta_{q+1}4^{i+c_0}$ and $c_0 \in \mathbb{N}$ is taken so $\dfrac{\mathring{R}_\ell}{\rho_i}$ is close enough to the Id to apply (3.2.1). Now to motivate the definition of ρ_0 we consider the (formal) calculation

$$\begin{aligned}
\sum_{i\geq 1}\int_{\mathbb{T}^3}\left|\sum_{\xi\in\Lambda_{(i)}} a_{(\xi)}\mathbb{W}_{(\xi)}\right|^2 \mathrm{dx} &= \sum_{i\geq 1}\sum_{\xi,\zeta\in\Lambda_{(i)}}\int_{\mathbb{T}^3} a_{(\xi)}a_{(\zeta)}\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(-\zeta)}\,\mathrm{dx} \\
&= \sum_{i\geq 1}\sum_{\xi\in\Lambda_{(i)}}\int_{\mathbb{T}^3} a_{(\xi)}^2\left(\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(-\xi)}\right)\mathrm{dx} + (\text{mixed terms}) \\
&= \sum_{i\geq 1}\sum_{\xi\in\Lambda_{(i)}}\int_{\mathbb{T}^3} a_{(\xi)}^2\,\mathrm{tr}\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}\right)\mathrm{dx} + (\text{mixed terms}) \\
&= \sum_{i\geq 1}\sum_{\xi\in\Lambda_{(i)}}\int_{\mathbb{T}^3} a_{(\xi)}^2\,\mathrm{tr}\left(\fint_{\mathbb{T}^3}\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}\right)\mathrm{dx} + (\text{mixed terms}) \\
&\qquad + (\text{mean zero symmetric}) \\
&= \sum_{i\geq 1}\int_{\mathbb{T}^3}\rho_{(i)}\chi_{(i)}^2\,\mathrm{tr}\left(\sum_{\xi\in\Lambda_{(i)}}\gamma_{(\xi)}\left(\mathrm{Id}-\frac{\mathring{R}_\ell}{\rho_i}\right)\fint_{\mathbb{T}^3}\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}\right)\mathrm{dx} + (\text{error}) \\
&= \sum_{i\geq 1}\int_{\mathbb{T}^3}\rho_{(i)}\chi_{(i)}^2\,\mathrm{tr}\left(\mathrm{Id}-\frac{\mathring{R}_\ell}{\rho_i}\right)\mathrm{dx} + (\text{error}) \\
&= 3\sum_{i\geq 1}\rho_{(i)}\int_{\mathbb{T}^3}\chi_{(i)}^2\,\mathrm{dx} + (\text{error})
\end{aligned}$$

with the set $\Lambda_{(i)} = \Lambda_{(i \mod 2)}$ where the set $\Lambda_{(0)}$ and $\Lambda_{(1)}$ are defined by taking $N = 2$ in the Geometric Lemma, mixed terms are sum of $\xi - \zeta \neq 0$, and the mean zero symmetric are of the form

$$\sum\int a_{(\xi)}^2\left(W_{(\xi)}\otimes\mathbb{W}_{(-\xi)} - \fint W_{(\xi)}\otimes\mathbb{W}_{(-\xi)}\right).$$

Now in the calculation above we expect the error terms to be negligible [5] due to the high frequencies of $\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\zeta)}$ compared to the $a_{(\xi)}a_{(\zeta)}$ with $\xi \neq \zeta$, similar with the mean zero symmetric terms. This is similar to the Riemann-Lebesgue lemma. Then as we want the principal component to encode the energy difference of the current iterate and the energy profile we wish to design ρ_0 so that

$$e(t) - \int_{\mathbb{T}^3}|v_q|^2\,\mathrm{dx} \approx \sum_{i\geq 0}\int_{\mathbb{T}^3}\left|\sum_{\xi\in\Lambda_{(0)}} a_{(\xi)}\mathbb{W}_{(\xi)}\right|^2 \mathrm{dx} \approx 3\sum_{i\geq 0}\rho_{(i)}\int_{\mathbb{T}^3}\chi_{(i)}^2\,\mathrm{dx}\,.$$

Rearranging we see that we want

$$\rho_{(0)} \approx \frac{1}{3\int_{\mathbb{T}^3}\chi_{(0)}^2\,\mathrm{dx}}\left(e(t) - \int_{\mathbb{T}^3}|v_q|^2\,\mathrm{dx} - 3\sum_{i\geq 1}\rho_{(i)}\int_{\mathbb{T}^3}\chi_{(i)}^2\,\mathrm{dx}\right).$$

[5] Indeed, they less then $C\ell^{\frac{1}{2}}$.

Since we need to be able to correct the energy on future iterates as we drive the Reynolds stress norm down we let

$$\rho(t) = \max\left(\frac{1}{3\int_{\mathbb{T}^3}\chi_0^2\,\mathrm{dx}}\left(e(t) - \int_{\mathbb{T}^3}|v_q|^2\,\mathrm{dx} - 3\sum_{i\geq 1}\rho_{(i)}\int_{\mathbb{T}^3}\chi_{(i)}^2\,\mathrm{dx} - \frac{\delta_{q+2}}{2}\right),\ 0\right). \tag{3.3.12}$$

Then to ensure that $\sqrt{\rho_{(0)}}$ is sufficiently smooth we let

$$\rho_{(0)} = \left(\sqrt{\rho} *_t \varphi\right)^2. \tag{3.3.13}$$

With $\rho_{(0)}$ we let (3.3.11) be defined for $i \geq 0$ and define the principal component as

$$w_{q+1}^{(p)} = \sum_{i\geq 0}\sum_{\xi\in\Lambda_{(i)}} a_{(\xi)}\mathbb{W}_{(\xi)}. \tag{3.3.14}$$

We note that $w_{q+1}^{(p)}$ is not divergence free so we introduce a corrector term. Inspecting the proof of Corollary 3.2.7 we deduce

$$\nabla\times\left(w_{q+1}^{(p)}\right) = w_{q+1}^{(p)} + \sum_{i\geq 0}\sum_{\Lambda_{(i)}}\nabla\left(a_{(\xi)}\eta_{(\xi)}\right)\times W_{(\xi)}.$$

Hence choosing

$$w_{q+1}^{(c)} = \sum_{i\geq 0}\sum_{\Lambda_{(i)}}\nabla\left(a_{(\xi)}\eta_{(\xi)}\right)\times W_{(\xi)} \tag{3.3.15}$$

we have

$$\operatorname{div}\left(w_{q+1}^{(p)} + w_{q+1}^{(c)}\right) = 0.$$

To finish the definition we need to define the *temporal corrector*. To motivate the definition we note that if we expand

$$\operatorname{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)})$$

we will end up with mixed terms and symmetric terms. The mixed terms we will expect to be controllable due to frequency differences. For the symmertic terms we will have things of the form

$$\operatorname{div}\left(a_{(\xi)}^2\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)}\otimes\mathbb{W}_{(\xi)}\right)\right).$$

Expanding the divergence we recover a term of the form

$$a_{(\xi)}^2\operatorname{div}\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)}\otimes\mathbb{W}_{(\xi)}\right)$$

which by proposition 3.2.16 is

$$a_{(\xi)}^2\left(\nabla\eta_{(\xi)}^2 - \frac{1}{\mu}\partial_t\eta_{(\xi)}^2\xi\right).$$

Rewriting again we see that we have a component ([4] , eqn 5.13) equal to

$$-\frac{1}{\mu}\partial_t\mathcal{P}\left(a_{(\xi)}^2\eta_{(\xi)}^2\xi\right)$$

which has too slow of a frequency to provide a good estimate with the iteration scheme. As such the temporal corrector is designed to convert this term into a gradient to be put in with the pressure. Recalling the definition of the Leray Projector we see $\mathcal{Q} = I - \mathcal{P}$ is a gradient. Thus if

$$w_{q+1}^{(t)} = \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}} \mathcal{P}\left(a_{(\xi)}^2 \eta_{(\xi)}^2 \xi\right) \tag{3.3.16}$$

then we have

$$\partial_t w_{q+1}^{(t)} - \sum_i \sum_{\xi \Lambda_{(i)}} \frac{1}{\mu} \partial_t \mathcal{P}\left(a_{(\xi)}^2 \eta_{(\xi)}^2 \xi\right) = \sum_i \sum_{\xi \in \Lambda_{(i)}} \frac{1}{\mu} \mathcal{Q}\left(\partial_t a_{(\xi)}^2 \eta_{(\xi)}^2 \xi\right) \tag{3.3.17}$$

which is a gradient. Having defined all the components of the iterate we will now clean up some technical details of the construction. Specifically we will (i) show the the indexing variable i can be taken as finite and (ii) that $a_{(0)}$ is well defined.

For the finiteness of i we recall the inductive estimate (3.1.4),

$$\left|\left|\mathring{R}_q\right|\right|_{C^1} \leq C\lambda_q^{10}.$$

Coupling this with the fact that

$$\mathrm{Supp}(\chi_{(i)}) \subseteq \left(4^{i-1}, 4^{i+1}\right)$$

we see that we want to show that there exists i_{max} such that

$$\left(1 + \left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|^2\right)^{\frac{1}{2}} \leq 4^{i_{max}}$$

for all (x,t). Noting $\delta << 1$ and $\lambda_q >> 1$ we see

$$\left(1 + \left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|.$$

Then estimating $\mathring{R}_\ell$ from above we have

$$|\mathring{R}_\ell| \leq \left|\left|\mathring{R}_\ell\right|\right|_\infty \leq \left|\left|\mathring{R}_\ell\right|\right|_{C^1} \leq ||\phi||_{L^1}\, ||\varphi||_{L^1} \left|\left|\mathring{R}_q\right|\right|_{C^1}$$

where last inequality follows from Young's inequality. Then using the inductive estimate we have that

$$\left(1 + \left|\frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon_R}\delta_{q+1}}\right|^2\right)^{\frac{1}{2}} \leq C\frac{\left|\left|\mathring{R}_q\right|\right|_{C^1}\lambda_q^{\epsilon_R}}{\delta_{q+1}} \leq \frac{\lambda_q^{11}}{\delta_{q+1}}$$

hence if we may choose

$$i_{max} = \min\left(i \geq 0 : 4^{i-2} \geq \lambda_q^{11}\delta_{q+1}^{-1}\right) \tag{3.3.18}$$

then we see $\chi_{(j)} \equiv 0$ for all $j > i_{max}$ which lets the formal computation above precede with out stronger tools.

Now for the well definedness of $a_{(0)}$ we first note that we have the estimate ([4])

$$e(t) - \int_{\mathbb{T}^3} |v_q|^2 \geq \frac{\delta_{q+1}}{200}$$

which by the definition of ρ gives

$$\rho \geq \frac{1}{3|\mathbb{T}^3| \int_{\mathbb{T}^3} \chi_{(0)}^2 \,\mathrm{dx}} \left(\frac{\delta_{q+1}}{200} - \frac{\delta_{q+2}}{2} \right) \geq \frac{\delta_{q+1}}{1500|\mathbb{T}^3| \int_{\mathbb{T}^3} \chi_{(0)}^2 \,\mathrm{dx}} \overset{(3.3.10)}{\geq} \frac{\delta_{q+1}}{750|\mathbb{T}^3|^2}.$$

Thus,

$$\rho_0 \geq \frac{\delta_{q+1}}{750|\mathbb{T}^3|^2}$$

Finally on the support of $\chi_{(0)}$ we have $|\mathrm{R}_\ell| \leq 400\lambda_q^{-\epsilon_R}\delta_{q+1}$ so

$$\left|\left| \frac{\mathring{R}_\ell}{\rho_0} \right|\right|_{L^\infty(\mathrm{Supp}\,\chi_{(0)})} \leq (3)10^5|\mathbb{T}^3|^2\lambda_q^{-\epsilon_R} \leq \epsilon. \tag{3.3.19}$$

This gives us that $\gamma_{(0)}$ argument is in the domain.

Remark 3.3.2. To achieve the above equation we need to take $a >> 1$ very large. In fact the upper bound presented above is a few magnitudes higher then in [4].

3.4. **The Reynolds Stress and Pressure.** Having defined the velocity increment we now wish to construct a pressure and stress tensor for $v_{q+1} = v_\ell + w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}$ so that (3.1.1) is satisfied. As such we write

$$\begin{aligned}
\partial_t v_{q+1} + \mathrm{div}\,(v_{q+1} \otimes v_{q+1}) - \Delta v_{q+1} &= \partial_t w_{q+1} + \mathrm{div}\,(w_{q+1} \otimes w_{q+1}) - \Delta w_{q+1} \\
&\quad + \mathrm{div}\,(w_{q+1} \otimes v_\ell + v_\ell \otimes w_{q+1}) \\
&\quad + \partial_t v_\ell + \mathrm{div}\,(v_\ell \otimes v_\ell) - \Delta v_\ell \\
&= \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \mathrm{div}\,(w_{q+1} \otimes v_\ell + v_\ell \otimes w_{q+1}) - \Delta w_{q+1} \\
&\quad + \mathrm{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \right) \\
&\quad + \mathrm{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}) + \partial_t w_{q+1}^{(t)} + \mathrm{div}\left(\mathring{R}_\ell + R_{com} \right) - \nabla p_\ell \\
&= \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \mathrm{div}\,(w_{q+1} \otimes v_\ell + v_\ell \otimes w_{q+1}) - \Delta w_{q+1} \\
&\quad + \mathrm{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \right) \\
&\quad + \mathrm{div}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell \right) + \partial_t w_{q+1}^{(t)} \\
&\quad + \mathrm{div}\,(R_{com}) - \nabla p_\ell.
\end{aligned}$$

Motivated by the above decomposition we define

$$R_{linear} = \mathcal{R}\left(\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \mathrm{div}\,(w_{q+1} \otimes v_\ell + v_\ell \otimes w_{q+1}) - \Delta w_{q+1} \right)$$

and

$$R_{corrector} = \mathcal{R}\left(\mathrm{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \right) \right).$$

For the third line we expand the divergence and use that $\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\zeta)}$ is mean free for $\xi \neq \zeta$ we have that

$$\begin{aligned}
\operatorname{div}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell\right) &= \sum_i \sum_{\xi,\zeta\in\Lambda_{(i)}} \operatorname{div}\left(a_{(\xi)}a_{(\zeta)}\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\right) + \operatorname{div}\mathring{R}_\ell \\
&= \sum_i \sum_{\xi,\zeta\in\Lambda_{(i)}} \operatorname{div}\left(a_{(\xi)}a_{(\zeta)}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} - \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx}\right)\right) \\
&\qquad + \operatorname{div}\left(\sum_i \rho_{(i)}\chi_{(i)}^2 \operatorname{Id}\right) \\
&= \sum_i \sum_{\xi,\zeta\in\Lambda_{(i)}} \operatorname{div}\left(a_{(\xi)}a_{(\zeta)}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} - \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx}\right)\right) + \nabla\left(\sum_i \rho_{(i)}\chi_{(i)}^2\right)
\end{aligned}$$

where for the second equality we have used that for $\xi + \zeta \neq 0$

$$\fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx} = 0$$

and

$$\sum_{\xi\in\Lambda_{(i)}} a_{(\xi)}a_{(-\xi)} \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)} \overset{(3.3.11)}{=} \rho_i\chi_{(i)}^2 \sum_{\xi\in\Lambda_{(i)}} \gamma_{(\xi)}^2\left(\operatorname{Id} - \frac{\mathring{R}_\ell}{\rho_i}\right) \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}\,\mathrm{dx} \overset{(3.2.8)}{=} \chi_{(i)}^2\left(\rho_{(i)}\operatorname{Id} - \mathring{R}_\ell\right).$$

Here we set $P_1 = \nabla\left(\sum_i \rho_{(i)}\chi_{(i)}^2\right)$. Next by the fact that

$$\operatorname{div}(fu\otimes w) = (u\otimes w)\nabla f + f\operatorname{div}(u\otimes v)$$

we see a generic term in the sum of divergences is

$$\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} - \fint \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx}\right)\nabla\left(a_{(\xi)}a_{(\zeta)}\right) + a_{(\xi)}a_{(\zeta)}\operatorname{div}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} - \fint \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx}\right)$$

Set $T_1 := \sum_i\sum_{\xi,\zeta}\left(\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} - \fint \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx}\right)\nabla\left(a_{(\xi)}a_{(\zeta)}\right)\right)$. For the second term let us consider the case of $\xi+\zeta\neq 0$ and $\xi+\zeta=0$ individually. First for the case $\xi+\zeta\neq 0$ we note that

$$a_{(\xi)}a_{(\zeta)}\operatorname{div}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} - \fint \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\,\mathrm{dx}\right) = a_{(\xi)}a_{(\zeta)}\operatorname{div}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)}\right).$$

Next combining conjugate terms and using Proposition 3.2.8 to expand the divergence we compute

$$\begin{aligned}
a_{(\xi)}a_{(\zeta)}\operatorname{div}\left(\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\zeta)} + \mathbb{W}_{(\zeta)} \otimes \mathbb{W}_{(\xi)}\right) &= a_{(\xi)}a_{(\zeta)}\left(\left(\left(W_{(\xi)}\cdot\nabla\right)\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right)W_{(\zeta)}\right. \\
&\qquad \left.+\left(\left(W_{(\zeta)}\cdot\nabla\right)\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right)W_{(\xi)} + \eta_{(\xi)}\eta_{(\xi)}\nabla\left(W_{(\zeta)}\cdot W_{(\xi)}\right)\right).
\end{aligned}$$

Then using the fact that

$$\left[W_{(\zeta)}\cdot\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right]W_{(\xi)} + \left[W_{(\xi)}\cdot\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right]W_{(\zeta)} = \left(W_{(\xi)}\otimes W_{(\zeta)} + W_{(\zeta)}\otimes W_{(\xi)}\right)\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)$$

and since $\mathbb{W}_{(\xi)} = \eta_{(\xi)}W_{(\xi)}$ we have

$$\begin{aligned}
\nabla\left(a_{(\xi)}a_{(\zeta)}\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(\zeta)}\right) &= \left(\nabla\left(a_{(\xi)}a_{(\zeta)}\right)\right)\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(\zeta)} + a_{(\xi)}a_{(\zeta)}\left(\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right)W_{(\xi)}\cdot W_{(\zeta)} \\
&\qquad + a_{(\xi)}a_{(\zeta)}\eta_{(\xi)}\eta_{(\zeta)}\left(\nabla\left(W_{(\xi)}\cdot W_{(\zeta)}\right)\right)
\end{aligned}$$

which imply that

$$a_{(\xi)}a_{(\zeta)}\operatorname{div}\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(\zeta)}+\mathbb{W}_{(\zeta)}\otimes\mathbb{W}_{(\xi)}\right)=a_{(\xi)}a_{(\zeta)}\left(W_{(\xi)}\otimes W_{(\zeta)}+W_{(\zeta)}\otimes W_{(\xi)}\right)\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)$$
$$-\left(\nabla\left(a_{(\xi)}a_{(\zeta)}\right)\right)\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(\zeta)}-a_{(\xi)}a_{(\zeta)}\left(\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right)W_{(\xi)}\cdot W_{(\zeta)}+\nabla\left(a_{(\xi)}a_{(\zeta)}\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(\zeta)}\right).$$

Now set

$$T_2=a_{(\xi)}a_{(\zeta)}\left(W_{(\xi)}\otimes W_{(\zeta)}+W_{(\zeta)}\otimes W_{(\xi)}\right)\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)-\left(\nabla\left(a_{(\xi)}a_{(\zeta)}\right)\right)\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(\zeta)}$$
$$-a_{(\xi)}a_{(\zeta)}\left(\nabla\left(\eta_{(\xi)}\eta_{(\zeta)}\right)\right)W_{(\xi)}\cdot W_{(\zeta)}$$

and

$$P_2=\nabla\left(a_{(\xi)}a_{(\zeta)}\mathbb{W}_{(\xi)}\cdot\mathbb{W}_{(\zeta)}\right).$$

Now for $\xi+\zeta=0$ we have

$$\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}-\fint_{\mathbb{T}^3}\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}\,\mathrm{dx}=\mathbb{P}_{\neq0}\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}$$

where $\mathbb{P}_{\neq0}$ is the projection on mean free fields. Using Proposition 3.2.8 one has

$$\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\operatorname{div}\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}+\mathbb{W}_{(-\xi)}\otimes\mathbb{W}_{(\xi)}\right)\right)=\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\nabla\eta_{(\xi)}^2-\frac{1}{\mu}\partial_t\eta_{(\xi)}^2\xi.\right)\tag{3.4.1}$$

Then using that

$$\mathbb{P}_{\neq0}\left(\nabla\left(a_{(\xi)}^2\eta_{(\xi)}^2\right)\right)=\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\left(\nabla\eta_{(\xi)}\right)+\left(\nabla a_{(\xi)}^2\right)\eta_{(\xi)}^2\right)$$

and

$$\frac{1}{\mu}\partial_t\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\eta_{(\xi)}^2\xi\right)=\frac{1}{\mu}\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\partial_t(\eta_{(\xi)}^2)\xi+\eta_{(\xi)}^2\partial_t(a_{(\xi)}^2)\xi\right)$$

we see

$$\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\operatorname{div}\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}+\mathbb{W}_{(-\xi)}\otimes\mathbb{W}_{(\xi)}\right)\right)=\mathbb{P}_{\neq0}\nabla\left(a_{(\xi)}^2\eta_{(\xi)}^2\right)-\frac{1}{\mu}\partial_t\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\eta_{(\xi)}^2\xi\right)$$
$$-\mathbb{P}_{\neq0}\left(\eta_{(\xi)}^2\nabla\left(a_{(\xi)}^2\right)\right)+\frac{1}{\mu}\mathbb{P}_{\neq0}\left(\eta_{(\xi)}^2\partial_t a_{(\xi)}^2\xi\right).$$

Summing the above equation over i and ξ, adding $\partial_t w_{q+1}^{(t)}$ and using (3.3.17) we see that

$$\partial_t w_{q+1}^{(t)}+\sum_i\sum_{(\xi)}\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\operatorname{div}\left(\mathbb{W}_{(\xi)}\otimes\mathbb{W}_{(-\xi)}+\mathbb{W}_{(-\xi)}\otimes\mathbb{W}_{(\xi)}\right)\right)$$
$$=\left(\sum_i\sum_{(\xi)}\left(\mathbb{P}_{\neq0}\nabla\left(a_{(\xi)}^2\eta_{(\xi)}^2\right)\right)\right)+\mathcal{Q}\left(\frac{1}{\mu}\sum_i\sum_{(\xi)}\partial_t a_{(\xi)}^2\eta_{(\xi)}^2\xi\right)$$
$$-\sum_i\sum_{(\xi)}\mathbb{P}_{\neq0}\left(\eta_{(\xi)}^2\nabla\left(a_{(\xi)}^2\right)\right)+\sum_i\sum_{(\xi)}\frac{1}{\mu}\mathbb{P}_{\neq0}\left(\eta_{(\xi)}^2\partial_t a_{(\xi)}^2\xi\right).$$

Using the above we define

$$T_3=\sum_i\sum_{(\xi)}\mathbb{P}_{\neq0}\left(\eta_{(\xi)}^2\nabla\left(a_{(\xi)}^2\right)-\frac{1}{\mu}\left(\eta_{(\xi)}^2\partial_t a_{(\xi)}^2\right)\right)$$

and

$$\nabla P_3=\left(\sum_i\sum_{(\xi)}\left(\mathbb{P}_{\neq0}\nabla\left(a_{(\xi)}^2\eta_{(\xi)}^2\right)\right)\right)-\mathcal{Q}\left(\frac{1}{\mu}\sum_i\sum_{(\xi)}\partial_t a_{(\xi)}^2\eta_{(\xi)}^2\xi\right).$$

Collect all the T_i and P_i above we define

$$R_{osc} = \mathcal{R}\left(T_1 + \frac{1}{2}T_2 + \frac{1}{2}T_3\right)$$

and

$$P = P_1 + P_2 + P_3$$

then taking

$$p_{q+1} = p_\ell - P - \Delta^{-1} \operatorname{div}\operatorname{div}\left(R_{linear} + R_{corrector} + R_{osc} + R_{com}\right)$$

and

$$\mathring{R}_{q+1} = \mathcal{R}\left(\mathcal{P} \operatorname{div}\left(R_{linear} + R_{corrector} + R_{osc} + R_{com}\right)\right)$$

we see the triple $(v_{q+1}, p_{q+1}, \mathrm{R}_{q+1})$ is a solution to the NSR equations.

3.5. **Convergence of the Velocity Sequence.** In this section we wish to show that velocity increments are forming a bounded sequence in L^2. Specifically we will gain the estimate

$$\left|\left|v_q - v_{q+1}\right|\right|_{L^2} \leq M\delta_{q+1}^{\frac{1}{2}}.$$

Since $v_{q+1} - v_q = w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + (v_\ell - v_q)$ we will wish to retrieve L^2 estimates for the four quantities. For the iterate terms we will need a few estimates on the coefficient terms first. Additionally at this time we explicitly fix ℓ, r, μ, σ as

$$\ell = \lambda_q^{-20}$$

$$r = \lambda_{q+1}^{\frac{3}{4}}$$

$$\sigma = \lambda_{q+1}^{\frac{-15}{16}}$$

and

$$\mu = \lambda_{q+1}^{\frac{5}{3}}$$

to make these estimations possible. Additionally we will let C denote an arbitrary constant independent of q. Next we have by properties of convolution that

$$\begin{aligned}
\left|\left|f * \phi_\ell\right|\right|_{C^N} &= \sum_{0\leq\alpha\leq N} \left|\left|D^\alpha (f * \phi_\ell)\right|\right|_{L^\infty} \\
&= \sum_{0\leq\alpha\leq k} \left|\left|(D^\alpha f) * \phi_\ell\right|\right|_{L^\infty} + \sum_{1\leq\alpha\leq N-k} \ell^{-\alpha} \left|\left|(D^k f * D^\alpha(\phi)\right|\right|_{L^\infty} \\
&\leq C\ell^{k-N} \left|\left|f\right|\right|_{C^k}
\end{aligned}$$

for all $0 \leq k \leq N$. Specifically for $k = N-1$ we obtain

$$\left|\left|\mathring{R}_\ell\right|\right|_{C^N_{x,t}} \leq C\ell^{1-N} \left|\left|\mathring{R}_\ell\right|\right|_{C^1_{x,t}} \leq C\ell^{1-N} \left|\left|\mathring{R}_q\right|\right|_{C^1_{x,t}}$$

where we have used $||f * \phi||_{L^\infty} \leq ||f||_{L^\infty} ||\phi||_{L^1}$. Finally by calculation we also have that

$$\begin{aligned}||f - f * \phi_\ell||_{L^\infty} &= \sup \left| \int_{\mathbb{T}^3} \left(\frac{1}{||\phi||_{L^1}} f(x) - f(x-y) \right) \phi(y) \, \mathrm{d}y \right| \\ &\leq \sup \int |y| \, |Df(c)| \, |\phi| \, \mathrm{d}y \\ &\leq \ell \, |Df| \, ||\phi||_{L^1} \\ &\leq C\ell \, ||f||_{C^1}\end{aligned}$$

where we have used the generalized mean value theorem, for instance see [9]. Applying this to v_q we have

$$||v_q - v_\ell||_{L^\infty} \leq C\ell \, ||v_q||_{C^1_{x,t}}$$

and furthermore

$$||v_q - v_\ell||_{L^2} \leq C\ell \, ||v_q||_{C^1_{x,t}} \leq \ell \lambda_q^4 \leq \delta_{q+1}^{\frac{1}{2}}.$$

Proposition 3.5.1 ([4], 4.2)**.** *Let $0 \leq i \leq i_{max}$ and $N \geq 1$. Then the following identities hold*

$$\left|\left|\chi_{(i)}\right|\right|_{L^2} \leq C 2^{-i}, \tag{3.5.1}$$

and

$$\left|\left|\chi_{(i)}\right|\right|_{C^N_{x,t}} \leq C\ell^{-N}. \tag{3.5.2}$$

Proof. Recall that by interpolation

$$||f||_{L^2} \leq ||f||_{L^1}^{\frac{1}{2}} \, ||f||_{L^\infty}^{\frac{1}{2}}$$

for $f \in L^1 \cap L^\infty$. Then as $\chi_{(i)}$ is a bounded function we just need to show that $\left|\left|\chi_{(i)}\right|\right|_{L^1} \leq C 4^{-i}$. For $i = 0, 1$ we have $\left|\left|\chi_{(i)}\right|\right|_{L^1} \leq |\mathbb{T}^3| \leq C4^{-i}$. Then since $\left|\left|\chi_{(i)}\right|\right|_{L^\infty} \leq 1$ we have

$$\left|\left|\chi_{(i)}\right|\right|_{L^1} \leq \sup_t \left|\operatorname{Supp}\left(\chi_{(i)}\right)\right| \leq \sup_t \left| \left\{ 4^{i-1} \leq \left\langle \frac{\lambda_q^{\epsilon_R} \mathring{R}_\ell}{100\delta_{q+1}} \right\rangle \right\} \right|.$$

Next using Chebyshev's inequality we have

$$\begin{aligned}\left| \left\{ 4^{i-1} \leq \left\langle \frac{\lambda_q^{\epsilon_R} \mathring{R}_\ell}{100\delta_{q+1}} \right\rangle \right\} \right| &= \left| \left\{ 16^{i-1} - 1 \leq \left| \frac{\lambda_q^{\epsilon_R} \mathring{R}_\ell}{100\delta_{q+1}} \right|^2 \right\} \right| \\ &\leq \sup_t \left| \left\{ \frac{4^{i-2} 100 \delta_{q+1}}{\lambda_q^{\epsilon_R}} \leq \left| \mathring{R}_\ell \right| \right\} \right| \\ &\leq \frac{\lambda_q^{\epsilon_R}}{(100) 4^{i-2} \delta_{q+1}} \left|\left| \mathring{R}_\ell \right|\right|_{L^1}.\end{aligned}$$

Then by the inductive estimate for $\left|\left| \mathring{R}_q \right|\right|_{L^1}$ we see

$$\left|\left|\chi_{(i)}\right|\right|_{L^1} \leq C \frac{\lambda_q^{\epsilon_R}}{4^i \delta_{q+1}} \frac{\delta_{q+1}}{\lambda_q^{\epsilon_R}} \leq C 4^{-i}$$

which gives the desired result. To obtain (3.5.2) we observe that for $\alpha \in \mathbb{N}$

$$D^\alpha (f \circ g) = D^{\alpha-1} \left(f'(g) g'\right) = \sum_{0 \leq i \leq \alpha - 1} \binom{\alpha - 1}{i} \left(D^{\alpha - i - 1} f'(g)\right) \left(D^i g'\right).$$

If we next assume that $||f||_{C^N} < \infty$ then by triangle inequality

$$||D^\alpha(f \circ g)||_{L^\infty} \leq \sum_{1 \leq i \leq \alpha} \binom{\alpha - 1}{i - 1} ||f||_{C^N} \left|\left|D^i g\right|\right|_{L^\infty} \leq C \, ||g||_{C^N} \, .$$

Applying this to $\chi_{(i)}$, using the inductive estimate and the fact $\langle \cdot \rangle \geq 1$ we get

$$\left|\left|\chi_{(i)}\right|\right|_{C^N} = \sum_{0 \leq \alpha \leq N} \left|\left|D^\alpha \chi_{(i)}\right|\right|_{L^\infty} \leq C \left|\left| \left\langle \frac{\lambda_{q+1}^{\epsilon_R} \mathring{R}_\ell}{100 \delta_{q+1}} \right\rangle \right|\right|_{C^N_{x,t}} \leq C \ell^{1-N} \frac{\lambda_{q+1}^{\epsilon_R}}{\delta_{q+1}} \left|\left|\mathring{R}_q\right|\right|_{C^1_{x,t}} \leq C \ell^{-N}.$$

□

With bounds on the stress cutoff functions we can now bound the coefficient functions $a_{(\xi)}$.

Proposition 3.5.2 ([4], 4.4)**.** *For all $0 \leq i \leq i_{max}$ and $N \geq 1$ we have the following bounds,*

$$\left|\left|a_{(\xi)}\right|\right|_{L^2} \leq C \rho_i^{\frac{1}{2}} 2^{-i} \leq C \delta_{q+1}^{\frac{1}{2}}, \tag{3.5.3}$$

$$\left|\left|a_{(\xi)}\right|\right|_{C^N_{x,t}} \leq C \ell^{-N} \lambda_q^{10}. \tag{3.5.4}$$

Proof. We recall that for $i \geq 1$ we have

$$a_{(\xi)} = \rho_i^{\frac{1}{2}} \chi_{(i)} \gamma_{(\xi)} \left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_i} \right) = \left(\lambda_q^{-\epsilon_R} \delta_{q+1} 4^{i+c_0} \right)^{\frac{1}{2}} \chi_{(i)} \gamma_{(\xi)} \left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_i} \right).$$

Then using (3.5.1) we have

$$\left|\left|a_{(\xi)}\right|\right|_{L^2} \leq C \rho_{(i)}^{\frac{1}{2}} \left|\left|\chi_{(i)}\right|\right|_{L^2} \leq C \rho_{(i)}^{\frac{1}{2}} 2^{-i} \leq C \delta_{q+1}^{\frac{1}{2}}.$$

For $i = 0$ the estimate is slightly more involved due to the definition of $\rho_{(0)}$. Using the definition of $\rho(t)$ we have for all t that

$$|\rho(t)| \leq C \left(e(t) - \int_{\mathbb{T}^3} |v_q| \, \mathrm{dx} \right) + C \left(\sum_{i \geq 1} \rho_{(i)} \int_{\mathbb{T}^3} \chi_{(i)}^2 \, \mathrm{dx} \right) + C \frac{\delta_{q+2}}{2}.$$

By inductive estimates we have that

$$e(t) - \int_{\mathbb{T}^3} |v_q| \, \mathrm{dx} \leq C \delta_{q+1}.$$

For the second part we use that $|\rho_{(i)}| \leq \delta_{q+1}$ to write

$$\sum_{i \geq 1} \rho_{(i)} \int_{\mathbb{T}^3} \chi_{(i)}^2 \, \mathrm{dx} \leq C \delta_{q+1} \left(\int_{\mathbb{T}^3} 1 \, \mathrm{dx} - \int_{\mathbb{T}^3} \chi_{(0)}^2 \, \mathrm{dx} \right) \leq C \frac{|\mathbb{T}^3|}{2} \delta_{q+1} = C \delta_{q+1},$$

where we have used that $\int \chi_{(0)}^2 \, \mathrm{dx} \geq \dfrac{|\mathbb{T}^3|}{2}$. Finally using that $\delta_{q+2} << \delta_{q+1}$ we obtain (3.5.3) upon integrating.

To obtain (3.5.4) for $i \geq 1$ we distribute derivatives, collect terms, use the boundedness of χ and γ (similar to estimates of the χ) to find

$$\left|\left|a_{(\xi)}\right|\right|_{C^N_{x,t}} \leq \rho_{(i)}^{\frac{1}{2}} \left(\left|\left|\chi_{(i)}\right|\right|_{L^\infty} \left|\left| \gamma \left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_{(i)}} \right) \right|\right|_{C^N_{x,t}} + \left|\left|\chi_{(i)}\right|\right|_{C^N_{x,t}} \left|\left| \gamma \left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_{(i)}} \right) \right|\right|_{L^\infty} \right).$$

Then by (3.5.1) we see

$$||\chi_{(i)}||_{C^N_{x,t}} \left|\left|\gamma\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_{(i)}}\right)\right|\right|_{L^\infty} \leq C\ell^{-N}.$$

Distributing derivatives, the boundedness of γ, and inductive hypothesis we also have

$$\left|\left|\gamma\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_{(i)}}\right)\right|\right|_{C^N_{x,t}} \leq C\left(\left|\left|\frac{\mathring{R}_\ell}{\rho_{(i)}}\right|\right|_{C^N_{x,t}} + \left|\left|\frac{\mathring{R}_\ell}{\rho_{(i)}}\right|\right|^N_{C^1_{x,t}}\right) \leq C\rho_{(i)}^{-1}\left(\ell^{1-N}\lambda_q^{10} + \lambda_q^{10N}\right) \leq C\rho_{(i)}^{-1}\ell^{1-N}\lambda_q^{10}.$$

Combining these facts we obtain the bound

$$||a_{(\xi)}||_{C^N_{x,t}} \leq C\left(\rho_{(i)}^{-\frac{1}{2}}\ell^{1-N}\lambda_q^{10} + \rho_{(i)}^{\frac{1}{2}}\ell^{-N}\right).$$

Now recalling the definition of i_{max} give by (3.3.18) we see

$$\rho_{(i)} \leq C\lambda_q^{11}\delta_{q+1}^{-1} \leq C\ell^{-1}$$

where the second equality follows from the choice of βb in the inductive estimate section. Similarly by definition of $\rho_{(i)}$ given after (3.3.11) we also have

$$\rho_{(i)} \geq \lambda_q^{-\epsilon_R}\delta_{q+1} \geq \lambda_q^{-11}\delta_{q+1} \geq C\ell.$$

Then as $\ell = \lambda_q^{-20}$ we conclude that

$$||a_{(\xi)}||_{C^N_{x,t}} \leq C\lambda_q^{10}\ell^{-N}.$$

For the case $i = 0$ we need to distribute derivatives onto $\rho_{(0)}$. We use the smoothing to gain a power of ℓ^{-1} for each derivative and the bounds above in the sup norm to finalize (3.5.4). □

Now with bounds on the coefficient functions we can prove the following proposition.

Proposition 3.5.3 ([4], 4.5)**.** *The principal, incompressibility and temporal corrector have the bounds*

$$\left|\left|w_{q+1}^{(p)}\right|\right|_{L^2} \leq \frac{M}{2}\delta_{q+1}^{\frac{1}{2}}, \tag{3.5.5}$$

$$\left|\left|w_{q+1}^{(c)}\right|\right|_{L^2} + \left|\left|w_{q+1}^{(t)}\right|\right|_{L^2} \leq Cr^{\frac{3}{2}}\mu^{-1}\ell^{-1}\delta_{q+1}^{\frac{1}{2}} \tag{3.5.6}$$

.

Proof. Applying Lemma 3.7 (‘L^p decorrelation’) from [4] we have

$$||a_{(\xi)}\mathbb{W}_{(\xi)}||_{L^2} \leq C\rho_i^{\frac{1}{2}}2^{-i} \leq C\delta_{q+1}^{\frac{1}{2}}2^{-i}.$$

Summing over i and using the triangle inequality we have

$$\left|\left|w_{q+1}^{(p)}\right|\right|_{L^2} \leq C\delta_{q+1}^{\frac{1}{2}}.$$

Using the (universal) finiteness of the various sets and parameters we can find a bounding M independent of all parameters to satisfy (3.5.5).

Now we consider a generic term of $w_{q+1}^{(c)}$. By direct calculation we have,

$$\left|\left|\frac{1}{\lambda_q}\nabla(a_{(\xi)}\eta_{(\xi)})\times W_\xi\right|\right|_{L^2} \leq C\frac{1}{\lambda_q}\left|\left|(\nabla a_{(\xi)})\eta_{(\xi)} + a_{(\xi)}(\nabla\eta_{(\xi)})\right|\right|_{L^2}$$
$$\leq C\frac{1}{\lambda_q}\left(\left|\left|\nabla a_{(\xi)}\right|\right|_{L^\infty}\left|\left|\eta_{(\xi)}\right|\right|_{L^2} + \left|\left|a_{(\xi)}\right|\right|_{L^\infty}\left|\left|\nabla\eta_{(\xi)}\right|\right|_{L^2}\right).$$

Using that $\left|\left|a_{(\xi)}\right|\right|_{L^\infty} \leq C\delta_{q+1}^{\frac{1}{2}}2^i$, (3.5.4), and Proposition 3.2.6 we have

$$\frac{1}{\lambda_{q+1}}\left(\left|\left|\nabla a_{(\xi)}\right|\right|_{L^\infty}\left|\left|\eta_{(\xi)}\right|\right|_{L^2} + \left|\left|a_{(\xi)}\right|\right|_{L^\infty}\left|\left|\nabla\eta_{(\xi)}\right|\right|_{L^2}\right) \leq C\frac{1}{\lambda_{q+1}}\left(\ell^{-1}\lambda_q^{10} + 2^i\delta_{q+1}^{\frac{1}{2}}\lambda_q\sigma r\right).$$

Next by selection of parameters we have that $\ell^{-1} \leq \lambda_q\delta_{q+1}^{\frac{1}{2}}\sigma r$ and we obtain

$$\left|\left|\frac{1}{\lambda_{q+1}}\nabla(a_{(\xi)}\eta_{(\xi)})\times W_\xi\right|\right|_{L^2} \leq C\delta_{q+1}^{\frac{1}{2}}\lambda_q^{10}2^i\sigma r.$$

Summing over i and ξ and absorbing any q constants with a power of $\ell^{-\frac{1}{2}}$ we have

$$\left|\left|w_{q+1}^{(c)}\right|\right|_{L^2} \leq C\frac{\delta_{q+1}^{\frac{1}{2}}\sigma r}{\ell^{\frac{1}{2}}}.$$

Similarly for $w_{q+1}^{(t)}$ we have

$$\left|\left|w_{q+1}^{(t)}\right|\right|_{L^2} = \left|\left|\frac{1}{\mu}\mathcal{P}(a_{(\xi)}^2\eta_{(\xi)}^2\xi)\right|\right|_{L^2} \leq C\frac{1}{\mu}\left|\left|a_{(\xi)}^2\right|\right|_{L^\infty}\left|\left|\eta_{(\xi)}\right|\right|_{L^4}^2 \leq C\frac{\delta_{q+1}4^ir^{\frac{3}{2}}}{\mu}.$$

Summing over i and ξ,

$$\left|\left|w_{q+1}^{(t)}\right|\right| \leq C\frac{\delta_{q+1}4^ir^{\frac{3}{2}}}{\ell\mu}.$$

Adding the two estimates and using that $\ell\mu\sigma \leq r$ (by choice of β) we obtain (3.5.6) as desired. □

Directly applying the proposition above we deduce that

Corollary 3.5.4 ([4], 4.6)**.**

$$||v_q - v_{q+1}||_{L^2} \leq M\delta_{q+1}^{\frac{1}{2}}. \tag{3.5.7}$$

Proof. Note $r^{\frac{3}{2}}\mu^{-1}\ell^{-1} = \lambda_q^{-20}\lambda_{q+1}^{\frac{-1}{8}} < 1$ as b was taken sufficiently large. Thus we have

$$||v_{q+1} - v_q||_{L^2} \leq \left|\left|w_{q+1}^{(p)}\right|\right|_{L^2} + \left|\left|w_{q+1}^{(t)}\right|\right|_{L^2} + \left|\left|w_{q+1}^{(c)}\right|\right|_{L^2} + ||v_q - v_\ell||_{L^2} \leq M\delta_{q+1}^{\frac{1}{2}}$$

as desired. □

Finally noting that $\delta_a^{\frac{1}{2}} = \lambda_1^{\frac{3}{2}\beta}\left(a^{-\beta}\right)^{b^a}$ with $a, b >> 1$ and $0 < \beta$ we have

$$\sum_{a\geq 1}\delta_a^{\frac{1}{2}} = \lambda_1^{\frac{3}{2}\beta}\sum_{a\geq 1}\left(a^{-\beta}\right)^{b^a} \leq \lambda_1^{\frac{3}{2}\beta}\sum_{a\geq 1}\left(a^{-\beta}\right)^a = \lambda_1^{\frac{3}{2}\beta}\frac{1}{1-a^{-\beta}} < \infty.$$

Letting $i < j$ be natural numbers and using the triangle inequality we obtain

$$||v_i - v_j||_{L^2} \leq \sum_{k=j+1}^{i}||v_k - v_{k-1}|| \leq \sum_{k=j+1}^{i}\delta_k^{\frac{1}{2}}.$$

Hence $\{v_q\}$ is a Cauchy sequence in the L^2 norm and we define $v = \lim_{n\to\infty} v_n$. Then from the other inductive estimates stated above and proved in [4] we see that v is a weak solution to (3.0.1) with arbitrary nonnegative energy profile. Using this we can construct velocity fields v that disappear for a finite time then reappear, giving non-uniqueness of weak solutions.

4. Conclusion

Nash shows in [1] that highly surprising examples and results could be built using iterative schemes to gradually increase a desired quantity to a specific bound. In [6] DeLellis and Székelyhidi Jr extended an analogue of the Nash's method to the equations of Fluid Dynamics. Applying this analogous method to a special class of stationary solutions of the Euler equations DeLellis and Székelyhidi Jr produced an iterative scheme that locally increased the energy of a approximate solution while making the approximate solution closer to a 'true' solution. Taking the limit of the approximate solutions we recovery a weak solution to the Euler equations with an arbitrary (positive, smooth) energy profile. Similarly, Buckmaster and Vicol in [4] built an iterative scheme using modified Beltrami flows giving rise to a weak solution of the Navier-Stokes equations with arbitrary energy profile .

So thematically, this paper has explored the notion of relaxing the idea of solution (or, equivalently, using approximate solutions) and building a sequence of relaxed solutions that converge to a strict solution. For John Nash this was using short embeddings, for the fluid dynamic papers it was introducing a Reynolds System.

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Appendix A. A Remark on Estimates of [4]

One of the crucial estimates for the velocity iterates inductive estimates is Lemma 4.4 in [4] that states the following bound,

$$||a_{(\xi)}||_{C^N_{x,t}} \leq C\ell^{-N}. \tag{A.1}$$

To derive this bound the authors use the bound

$$||\chi_{(i)}||_{C^N_{x,t}} \leq C\lambda_{q+1}^{10}\ell^{1-N}$$

as stated in Lemma 4.2. In the proof provided of the lemma a key dependence on q is lost in the final equations. Specifically it is claimed,

$$\left|\left|\left\langle \frac{\lambda_q^{\epsilon_R}\mathring{R}_\ell}{100\delta_{q+1}} \right\rangle\right|\right|_{C^N_{x,t}} + \left|\left|\left\langle \frac{\lambda_q^{\epsilon_R}\mathring{R}_\ell}{100\delta_{q+1}} \right\rangle\right|\right|^N_{C^1_{x,t}} \leq C\left(\ell^{1-N}\left|\left|\mathring{R}_\ell\right|\right|_{C^1_{x,t}} + \left|\left|\mathring{R}_\ell\right|\right|^N_{C^1_{x,t}}\right) \tag{A.2}$$

which has lost the term $\dfrac{\lambda_q^{\epsilon_R}}{\delta_{q+1}}$ which has q dependence. Restoring this term and using the inequality $\dfrac{\lambda_{q+1}^{\epsilon_R}}{\delta_{q+1}} < \lambda_{q+1}^{9+\epsilon_R}$ we see that (A.2) can be replaced with

$$\left|\left|\left\langle \frac{\lambda_q^{\epsilon_R}\mathring{R}_\ell}{100\delta_{q+1}} \right\rangle\right|\right|_{C^N_{x,t}} + \left|\left|\left\langle \frac{\lambda_q^{\epsilon_R}\mathring{R}_\ell}{100\delta_{q+1}} \right\rangle\right|\right|^N_{C^1_{x,t}} \leq C\left(\lambda_{q+1}^{9+\epsilon_R}\ell^{1-N}\left|\left|\mathring{R}_\ell\right|\right|_{C^1_{x,t}} + \left(\lambda_{q+1}^{9+\epsilon_R}\right)^N\left|\left|\mathring{R}_\ell\right|\right|^N_{C^1_{x,t}}\right). \tag{A.3}$$

Substituting this in the proof we gain the following modified lemma.

Lemma A.1 (Lemma 4.2, [4])**.** *Let $0 \leq i \leq i_{max}$. Then we have*

$$||\chi_{(i)}||_{L^2} \leq C2^{-i},$$

$$||\chi_{(i)}||_{C^N_{x,t}} \leq C\lambda_{q+1}^{19+\epsilon_R}\ell^{1-N} \leq C\ell^{-N}.$$

As mentioned this also affect the crucial Lemma 4.4 which becomes

Lemma A.2 (Lemma 4.4, [4])**.** *For all $N \geq$ and $0 \leq i \leq i_{max}$ we have he bounds*

$$||a_{(\xi)}||_{L^2} \leq C\rho_i^{\frac{1}{2}}2^{-i} \leq C\delta_{q+1}^{\frac{1}{2}},$$

$$||a_{(\xi)}||_{L^\infty} \leq C\rho_i^{\frac{1}{2}} \leq C\delta_{q+1}^{\frac{1}{2}}2^i,$$

$$||a_{(\xi)}||_{C^N_{x,t}} \leq C\ell^{-N} \leq \ell^{-N}\lambda_{q+1}^{9+\epsilon_R}. \tag{A.4}$$

Here (A.4) has been modified from the original paper to gain a factor of $\lambda_{q+1}^{9+\epsilon_R}$.

The error introduced in Lemma 4.2 then ends in Proposition 4.5 due to an over estimate for i_{max}. It is helpful to examine the calculation of $\left|\left|w_{q+1}^{(p)}\right|\right|_{W^{1,p}}$ in equations (4.45) of [4]. Specifically they state,

$$\begin{aligned} \left|\left|w_{q+1}^{(p)}\right|\right|_{W^{1,p}} &\leq C\left(\sum_i \sum_{\xi\in\Lambda_{(i)}} \left|\left|a_{(\xi)}\right|\right|_{C^1_{x,t}} \left|\left|\mathbb{W}_{(\xi)}\right|\right|_{W^{1,p}}\right) \\ &\leq C\left(\sum_i \sum_{\xi\in\Lambda_{(i)}} \ell^{-1}\lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}\right) \\ &\leq C\left(\ell^{-2}\lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}\right) \end{aligned} \tag{A.5}$$

where we use Proposition 3.2.6, Lemma 4.4, and that 'summing over i and ξ loses a factor of ℓ^{-1}'. This final line is a consequence of i being a finite index with $i_{max} = \min\{i \ : \ 4^{i-2} \geq \lambda_q^{11}\delta_{q+1}^{-1}\}$ as

$$i_{max} \leq C\ln\left(\lambda_{q+1}^{20}\right) = C\ln\left(\ell^{-1}\right) << C\ell^{-1}.$$

Clearly we have a lot of room in this estimate. Indeed for large enough a we have

$$i_{max} \leq C\lambda_{q+1}^{10\frac{3}{4}}.$$

Then as $\epsilon_R < \frac{1}{4}$ we have that

$$\begin{aligned} \left|\left|w_{q+1}^{(p)}\right|\right|_{W^{1,p}} &\leq C\left(\sum_i \sum_{\xi\in\Lambda_{(i)}} \left|\left|a_{(\xi)}\right|\right|_{C^1_{x,t}} \left|\left|\mathbb{W}_{(\xi)}\right|\right|_{W^{1,p}}\right) \\ &\leq C\left(\sum_i \sum_{\xi\in\Lambda_{(i)}} \ell^{-1}\lambda_{q+1}^{9+\epsilon_R}\lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}\right) \\ &\leq C\left(\ell^{-1}\lambda_{q+1}^{19\frac{3}{4}+\epsilon_R}\lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}\right) \\ &\leq C\left(\ell^{-2}\lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}\right). \end{aligned} \tag{A.6}$$

Thus the omission of the q dependence does not change the estimate. In the rest of the paper we see this same behavior appear. Whenever a $C^N_{x,t}$ estimate is needed for the coefficent functions $a_{(\xi)}$ we are summing over our various families. The factor introduced in the sum can be taken to be smaller then in the original paper to restore the desired estimates. Consequently the inductive estimates also hold.

Appendix B. Dirichlet Kernel L^p norms

Here we develop some basic properties of the N^{th} order Dirichlet kernel D_N.

Lemma B.1 (Equivalent Definitions). *The following definitions of the Dirichlet Kernel are equivalent*

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}, \tag{B.1}$$

$$D_N(x) = \frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin\left(\frac{1}{2}x\right)}, \tag{B.2}$$

$$D_N(x) = 1 + 2\sum_{k=1}^{N} \cos(kx), \tag{B.3}$$

$$D_N(x) = \left(e^{-ix}\right)^N \prod_{k=1}^{2N} \left(e^{ix} + \xi_{2N+1}^k\right) \tag{B.4}$$

where $\xi_m = e^{i\frac{2\pi}{m}}$ is a m^{th} root of unity.

Proof. Let us first show that (B.1) and (B.3) are equivalent. By reordering the sum in (B.1) we have

$$\sum_{k=-N}^{N} e^{ikx} = 1 + \sum_{k=1}^{N}\left(e^{ikx} + e^{-ikx}\right) = 1 + 2\sum_{k=1}^{N}\cos(kx)$$

which shows that (B.1) and (B.3) are equivalent.

Next we will show that (B.1) and (B.2) are equivalent. Using that the exponential form of the Dirichlet kernel is a geometric sum we write

$$\begin{aligned}
\sum_{k=-N}^{N} e^{ikx} &= e^{-iNx}\left(\frac{1-e^{i(2N+1)x}}{1-e^{ix}}\right) \\
&= \left(\frac{e^{-i\frac{x}{2}}}{e^{-i\frac{x}{2}}}\right)\left(\frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ikx}}\right) \\
&= \frac{e^{-i\frac{2N+1}{2}x} - e^{i\frac{2N+1}{2}x}}{e^{-i\frac{1}{2}x} - e^{i\frac{1}{2}x}} \\
&= \frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin\left(\frac{1}{2}x\right)}
\end{aligned} \tag{B.5}$$

which gives the desired equivalence.

Finally to complete the proof we will show that (B.1) and (B.4) are equal. To this end observe

$$1 + x + x^2 + \cdots + x^{m-1} = \prod_{k=1}^{m-1}\left(x + \xi_m^k\right).$$

Hence

$$\sum_{k=-N}^{N} e^{ikx} = e^{-iNx}\sum_{k=0}^{2N}(e^{ix})^k = e^{-iNx}\prod_{k=1}^{2N}\left(e^{ix} + \xi_{2N+1}^k\right)$$

which completes the proof. ∎

Using this we can now prove the following classical result.

Lemma B.2 (Dirichlet L^p growth)**.** *For fixed $1 < p \leq \infty$ there exists positive constants $c(p), C(p)$ independent of N such that*

$$c(p)\,(2N+1)^{1-\frac{1}{p}} \leq ||D_N||_{L^p} \leq C(p)\,(2N+1)^{1-\frac{1}{p}}\,. \tag{B.6}$$

Proof. First for the case $p = \infty$ using (B.3) and triangle inequality we immediately get

$$|D_N(x)| \leq 2N+1 \qquad \forall x \in \mathbb{R}.$$

Then evaluating $D_N(0)$ using (B.3) we have equality and deduce

$$||D_N||_\infty = 2N+1$$

which gives the result with $c(\infty) = C(\infty) = 1$.

Now for fixed $1 < p < \infty$ we will instead bound $||D_N||_{L^p}^p = 2\int_0^\pi |D_N|^p$ for our ease. Recalling the elementary inequality

$$\frac{2}{\pi}x \leq \sin x \leq x$$

for $x \in \left[0, \frac{\pi}{2}\right]$ we deduce

$$2^p \int_0^\pi \left|\frac{\sin\left(\frac{2N+1}{2}x\right)}{x}\right|^p \,\mathrm{d}x \leq \int_0^\pi \left|\frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin\left(\frac{1}{2}x\right)}\right|^p \,\mathrm{d}x \leq \pi^p \int_0^\pi \left|\frac{\sin\left(\frac{2N+1}{2}x\right)}{x}\right|^p \,\mathrm{d}x.$$

Next changing variables we have

$$\int_0^\pi \left|\frac{\sin\left(\frac{2N+1}{2}x\right)}{x}\right|^p \,dx = \left(\frac{2N+1}{2}\right)^{p-1} \int_0^{\frac{2N+1}{2}\pi} \left|\frac{\sin u}{u}\right|^p \,\mathrm{d}u$$

which in combination with the inequalities

$$\int_0^{\frac{\pi}{2}} \left|\frac{\sin u}{u}\right|^p \,\mathrm{d}u \leq \int_0^{\frac{2N+1}{2}\pi} \left|\frac{\sin u}{u}\right|^p \,\mathrm{d}u \leq \int_0^1 \,\mathrm{d}u + \int_1^\infty \frac{\mathrm{d}u}{u^p} = \frac{p}{p-1}$$

gives

$$4\left(\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,\mathrm{d}x\right)(2N+1)^{p-1} \leq ||D_N||_p^p \leq 4\left(\frac{p}{p-1}\right)\left(\frac{\pi}{2}\right)^p (2N+1)^{p-1}\,.$$

Thus using $c(p) = \left(4\left(\int_0^{\frac{\pi}{2}} \left|\frac{\sin x}{x}\right|^p \,\mathrm{d}x\right)\right)^{\frac{1}{p}}$ and $C(p) = \left(4\left(\frac{p}{p-1}\right)\left(\frac{\pi}{2}\right)^p\right)^{\frac{1}{p}}$ the result follows. □

Seeing that $||D_N||_p^p$ grows like N^{p-1} it is natural to ask if for integer p, $||D_N||_p^p$ is a polynomial of degree $p-1$ in N. To approach this result we will pass through the complex plane to derive a series identity for the integral

$$\int_{-\pi}^{\pi} |D_N(x)|^p \,\mathrm{d}x$$

where p is a positive rational number. To this end we first observe that D_N is a real valued function so we have that

$$|D_N(x)| = \operatorname{sign}(D_N(x))\, D_N(x),$$

where the sign of $D_N(x)$ is easily computed from (B.2). Combining this with the fact that e^{ix} parameterizes the unit circle for $-\pi \leq x \leq \pi$ we deduce for $p \notin 2\mathbb{N}$ that

$$\int_{-\pi}^{\pi} |D_N|^p \, \mathrm{d}x = \sum_{j=-N}^{N} \int_{\gamma_j} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz}$$

with γ_j being the following parameterized paths:

($j > 0$) $\gamma_j = \{e^{it} \ : (j-1)\frac{2\pi}{2N+1} \leq t \leq j\frac{2\pi}{2N+1}\}$ with counterclockwise orientation if j is even and clockwise if j is odd;

($j < 0$) $\gamma_j = \overline{\gamma_{-j}}$ with counterclockwise orientation if j is even and clockwise if j is odd;

($j = 0$) $\gamma_j = \{e^{it} \ : m\frac{2\pi}{2N+1} \leq t \leq (m+1)\frac{2\pi}{2N+1}\}$ with counterclockwise orientation if m is even and clockwise if m is odd.

Next letting $\gamma_{j,r}(t) = r\gamma_j(1-t)$ be the r^{th} dilate with reversed orientation we define the straight line parameterized paths

$$\delta_{j,r,+} = (1-t)\gamma_j(1) + t\gamma_{j,r}(0),$$

$$\delta_{j,r,-} = (1-t)\gamma_{j,r}j(1) + t\gamma_j(0).$$

Thus for any r the parameterized path $\gamma_j \circ \delta_{j,r,+} \circ \gamma_{j,r} \circ \delta_{j,r,-}$ is a closed path enclosing no singularities of D_N. From this we infer that

$$\int_{-\pi}^{\pi} |D_N(x)|^p \, \mathrm{d}x = \sum_{j=-N}^{N} \left\{ \int_{\gamma_{j,r}} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz} + \int_{\delta_{j,r,+}} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz} + \int_{\delta_{j,r,-}} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz} \right\}.$$

Remark B.3. Here it is important to note that for rational p the subdivision of the real integral above when treated as a complex integral may cross the 'cuts' of the root function. To overcome this obstical we may either take N large so the root is well defined on a nieghborhood of the arc of the circle or create a futher subdivision. In the case of the further subdivision the arguement below proceeds naturally upon noting the orientation introduced for each sub arc will provide cancelation among the radial lines. Then expansion into the power series allows us to 'reglue' the radius r subarc back together to obtain the original arc.

Then by (B.4) and the definition of the $\delta_{j,r,\pm}$ we have

$$|D_N(x)| \leq \frac{2^{2N-1}}{r^N} \left| e^{ix} - \xi_{2N+1}^k \right| \qquad 1 \leq k \leq 2N+1$$

and

$$\left| \int_{\delta_{j,r,\pm}} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz} \right| \leq \frac{2^{2N-1}}{r^N} \int_0^{1-r} t dt = \frac{2^{2N-2}}{r^{N+1}} (1-r)^2$$

which imply that

$$\int_{-\pi}^{\pi} |D_N(x)|^p \, \mathrm{d}x = \lim_{r \to 1} \left(\sum_{j=-N}^{N} \int_{\gamma_{j,r}} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz} \right). \tag{B.7}$$

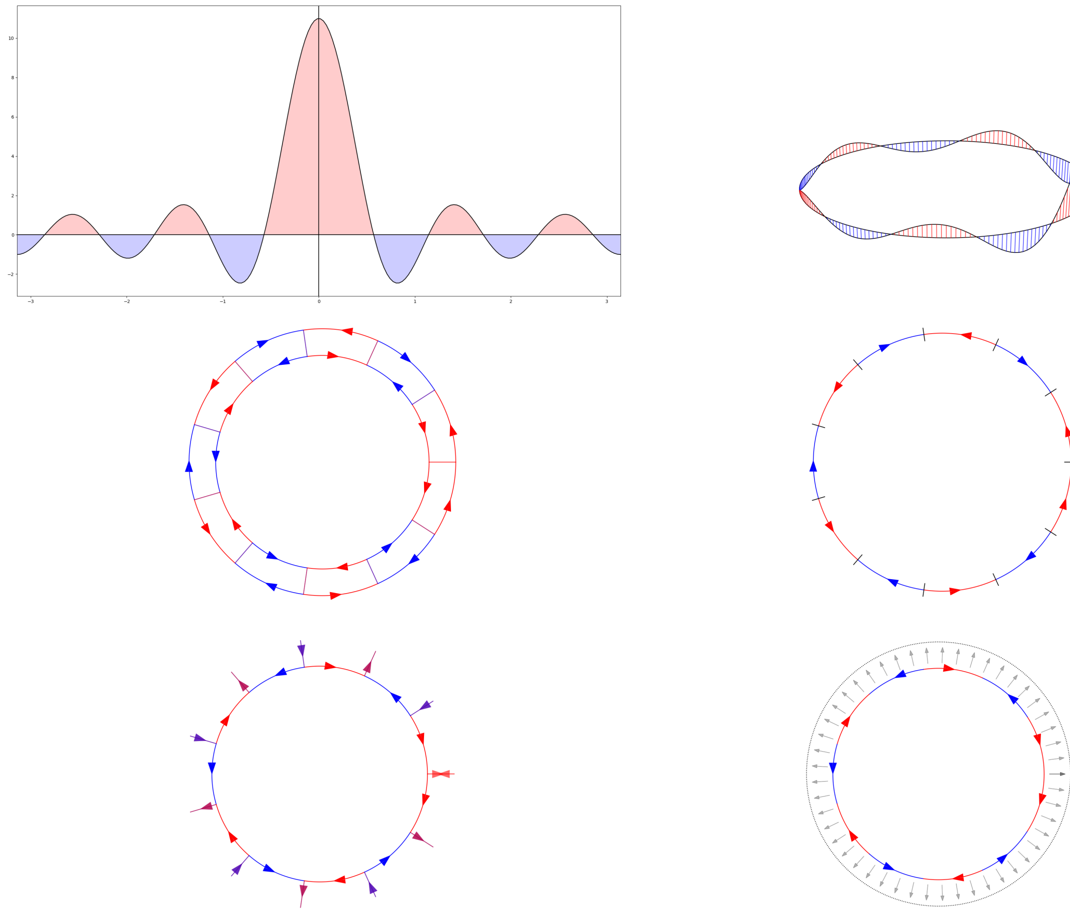

FIGURE 2. Visualization of integration technique for D_5. Start at top left then go right, down, left, down, right for order of steps.

Next as $|z| = r < 1$ in the above identity we can use the geometric series and binomial theorem to write

$$\begin{aligned}\int_{\gamma_{j,r}} \left(\sum_{k=-N}^{N} z^k\right)^p \frac{\mathrm{d}z}{iz} &= \frac{1}{i}\int_{\gamma_{j,r}} \frac{1}{z^{Np+1}}\frac{1}{(1-z)^p}\left(1-z^{2N+1}\right)^p \;\mathrm{d}z\\ &= \frac{1}{i}\int_{\gamma_{j,r}} \frac{1}{z^{Np+1}}\left(\sum_{k=0}^{\infty}\binom{k+p-1}{k} z^k\right)\left(\sum_{k=0}^{\infty}(-1)^k\binom{p}{k} z^{k(2N+1)}\right)\;\mathrm{d}z\\ &= \frac{1}{i}\int_{\gamma_{j,r}} \frac{1}{z^{Np+1}}\left(\sum_{k=0}^{\infty}\tilde{c}_k z^k\right)\;\mathrm{d}z\\ &= \sum_{k=0}^{\infty}\frac{\tilde{c}_k}{i}\int_{\gamma_{j,r}} z^{k-Np-1}\;\mathrm{d}z\end{aligned}$$

where

$$\tilde{c}_k = \sum_{l=0}^{\lfloor \frac{k}{2N+1} \rfloor} (-1)^l \binom{(k-l(2N+1))+p-1}{(k-l(2N+1))}\binom{p}{l}.$$

Then since

$$\int_{\gamma_{j,r}} z^{k-Np-1}\ \mathrm{d}z = \begin{cases} \frac{1}{k-Np} z^{k-Np}\Big|_{\gamma_{j,r}(0)}^{\gamma_{j,r}(1)} & k \neq Np \\ \ln z\big|_{\gamma_{j,r}(0)}^{\gamma_{j,r}(1)} & k = Np \end{cases}$$

we naturally split into the cases that Np is integer[6] and Np is not an integer. As such define

$$c_{Np} = \begin{cases} 0 & Np \notin \mathbb{Z} \\ \tilde{c}_{Np} & Np \in \mathbb{Z} \end{cases}$$

then

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\tilde{c}_k}{i} \int_{\gamma_{j,r}} z^{k-Np-1}\ \mathrm{d}z &= (-1)^{\lfloor \frac{|j|}{2} \rfloor} \frac{2\pi}{2N+1} c_{Np} + \sum_{\substack{k=0 \\ k\neq Np}}^{\infty} \frac{1}{i} c_k \left((r\gamma_j(1))^{k-Np} - (r\gamma_j(1))^{k-Np} \right) \\ &= (-1)^{\lfloor \frac{|j|}{2} \rfloor} \frac{2\pi}{2N+1} c_{Np} + \frac{1}{r^{Np}} \sum_{\substack{k=0 \\ k\neq Np}}^{\infty} c_k r^k \frac{1}{i} \left(\gamma_j(1)^{k-Np} - \gamma_j(0)^{k-Np} \right). \end{aligned}$$

Thus we have

$$\int_{\gamma_{j,r}} \left(\sum_{k=-N}^{N} z^k \right)^p \frac{\mathrm{d}z}{iz} = (-1)^{\lfloor \frac{|j|}{2} \rfloor} \frac{2\pi}{2N+1} c_{Np} + \frac{1}{r^{Np}} \sum_{\substack{k=0 \\ k\neq Np}}^{\infty} c_k r^k \frac{1}{i} \left(\gamma_j(1)^{k-Np} - \gamma_j(0)^{k-Np} \right). \tag{B.8}$$

. Then observing we can rewrite γ_j as

$$\gamma_j(t) = \begin{cases} \exp\left(i \frac{2\pi}{2N+1} \left[j - \frac{1}{2} - (-1)^{\lfloor \frac{|j|}{2} \rfloor} \left(\frac{1}{2} - t \right) \right] \right) & j \neq 0 \\ \exp\left(i \frac{2\pi}{2N+1} \left[N - \frac{1}{2} - (-1)^{\lfloor \frac{|N|}{2} \rfloor} \left(\frac{1}{2} - t \right) \right] \right) & j = 0 \end{cases}$$

we see that if we fix k and sum over j,

$$\begin{aligned} \sum_{j=-N}^{N} \frac{1}{i} \left(\gamma_j(1)^{k-Np} - \gamma_j(0)^{k-Np} \right) &= \frac{1}{i} \left(\gamma_0^{k-Np}(1) - \gamma_0^{k-Np}(0) \right) \\ &\quad + \frac{1}{i} \sum_{j=1}^{N} \left(\left[\gamma_j^{k-Np}(1) - \gamma_j^{k-Np}(0) \right] + \left[\gamma_{-j}^{k-Np}(1) - \gamma_{-j}^{k-Np}(0) \right] \right) \\ &= \frac{1}{i} (-1)^{N+1} \left(\left(\xi_{2N+1}^{N} \right)^{k-Np} - \left(\xi_{2N+1}^{-N} \right)^{k-Np} \right) \\ &\quad + \frac{1}{i} \sum_{j=1}^{N} (-1)^{j+1} \left(\left[\left(\xi_{2N+1}^{j} \right)^{k-Np} - \left(\xi_{2N+1}^{j-1} \right)^{k-Np} \right] \right. \\ &\quad \left. + \left[\left(\xi_{2N+1}^{1-j} \right)^{k-Np} - \left(\xi_{2N+1}^{-j} \right)^{k-Np} \right] \right) \\ &= 2 \sum_{j=1}^{N} (-1)^{j+1} \sin\left(\left(\frac{2\pi}{2N+1} \right) (k-Np)\, j \right). \end{aligned}$$

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[6] Here we have used that we can find an antiderivative for $\frac{1}{z}$ in a neighborhood of the path.

Now for ease let $\eta = k - Np$, and $\xi := \xi_{2N+1}$. Then as Im is a linear operator over $\mathbb{C}$ viewed as a $\mathbb{R}$ vector space we have

$$\begin{aligned}\sum_{j=1}^{N}(-1)^{j+1}\sin\left(\frac{2\pi}{2N+1}\eta j\right) &= -\operatorname{Im}\left(\sum_{j=1}^{N}(-\xi^{\eta})^{j}\right)\\ &= \operatorname{Im}\left(\frac{\xi^{\eta}+(-\xi^{\eta})^{N+1}}{1+\xi^{\eta}}\right)\\ &= \operatorname{Im}\left(\frac{\xi^{\eta}+(-1)^{N+2}\xi^{\frac{\eta}{2}}}{1+\xi^{\eta}}\right)\\ &= \operatorname{Im}\left(\frac{\xi^{\frac{\eta}{2}}\pm 1}{\xi^{-\frac{\eta}{2}}+\xi^{\frac{\eta}{2}}}\right)\\ &= \frac{1}{\cos\left(\frac{\pi}{2N+1}\eta\right)}\operatorname{Im}\left(\frac{\xi^{\frac{\eta}{2}}\pm 1}{2}\right)\\ &= \frac{1}{2}\tan\left(\frac{\pi}{2N+1}\eta\right).\end{aligned}$$

That is,

$$\sum_{j=-N}^{N}\frac{1}{i}\left(\gamma_j(1)^{k-Np}-\gamma_j(0)^{k-Np}\right) = \tan\left(\frac{\pi}{2N+1}(k-Np)\right). \tag{B.9}$$

Collecting (B.7), (B.8), and (B.9) we deduce that

$$\int_{-\pi}^{\pi}|D_N(x)|^p \;\mathrm{d}x = \frac{2\pi}{2N+1}c_{Np} + \lim_{r\to 1}\left(\frac{1}{r^{Np}}\sum_{\substack{k=0\\k\neq Np}}^{\infty}c_k r^k\tan\left(\frac{2\pi}{2N+1}(k-Np)\right)\right). \tag{B.10}$$

Then for an odd integer p we see that $c_k \equiv 0$ for $k \geq (2N+1)(p+1)$ which collapses the infinite sum to a finite sum. Evaluating the limit for odd p we have

$$\int_{-\pi}^{\pi}|D_N(x)|^p \;\mathrm{d}x = \frac{2\pi}{2N+1}c_{Np} + \sum_{\substack{k=0\\k\neq Np}}^{(2N+1)(p+1)-1}c_k\tan\left(\frac{2\pi}{2N+1}(k-Np)\right). \tag{B.11}$$

Unfortunately, for even p the above technique is more painful to make rigorous as the complex function will be integrated around a singularity. Ultimately though the only thing that is left is the integral $\frac{1}{i}\int_{S^1}c_{Np}z^{-1}\,\mathrm{d}z$ which gives

$$||D_N||_p^p = 2\pi c_{Np}.$$

Instead we will treat even p as a special case and develop a polynomial identity to pickout 'zero frequencies' of the $|D_N|^p$. First a telling example to our approach.

Example B.4. For any natural number N,

$$\int_{-\pi}^{\pi}|D_N(x)|^2\;\mathrm{d}x = 2\pi(2N+1)$$

Proof. Recall that D_N is a real valued function so $|D_N(x)|^2 = (D_N(x))^2$. Then using (B.1) we compute

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_N(x)|^2 \, \mathrm{d}x &= \int_{-\pi}^{\pi} \left(\sum_{k=-N}^{N} e^{ikx} \right)^2 \, \mathrm{d}x \\
&= \sum_{-N \le k=j \le N} \int_{-\pi}^{\pi} \mathrm{d}x + \sum_{-N \le k \ne j \le N} \int_{-\pi}^{\pi} e^{i(k-j)x} \, \mathrm{d}x \\
&= 2\pi(2N+1)
\end{aligned}$$

which finishes the example. $\square$

Building off this example our goal is prescribe coefficients in the following equality,

$$\left(\sum_{k=-N}^{N} e^{ikx} \right)^p = \sum_{-Np}^{Np} a_k e^{ikx}$$

as this implies that

$$||D_N||_p^p = 2\pi a_0.$$

To this end we prove the following proposition.

Proposition B.5. *For any $p, M \in \mathbb{N}$ the following identity holds*

$$\left(\sum_{k=0}^{M} x^k \right)^p = \sum_{k=0}^{Mp} m_k x^k \tag{B.12}$$

where

$$m_k = \sum_{l=0}^{\lfloor \frac{k}{M+1} \rfloor} (-1)^l \binom{(k - l(M+1)) + p - 1}{p-1} \binom{p}{l}.$$

Proof. Much as before we use the geometric sum identity, (formal) binomial expansions and geometric series. That is,

$$\left(\sum_{k=0}^{M} x^k \right)^p = \frac{1}{(1-x)^p} \left(1 - x^{M+1}\right)^p = \left(\sum_{k=0}^{\infty} \binom{k+p-1}{k} x^k \right) \left(\sum_{k=0}^{p} \binom{p}{k} (-1)^k x^{k(M+1)} \right).$$

Expanding the product, collecting terms, and noting the left hand side is a polynomial of degree Mp we deduce that

$$m_k = \sum_{l=0}^{\lfloor \frac{k}{M+1} \rfloor} (-1)^l \binom{(k - l(M+1)) + p - 1}{p-1} \binom{p}{l}$$

and $m_k = 0$ for $k > Mp$. This completes the proposition. $\square$

Remark B.6. Observing that if $N = 2N$ then $m_k = \tilde{c}_k$. As such since $m_k = 0$ for $k > Mp$ we that the finite sum in (B.11) in fact has the index $0 \le k \le 2Np$.

Now using B.12 we have that for any integer p,

$$\left(\sum_{k=-N}^{N} e^{ikx}\right)^p = e^{-iNpx}\left(\sum_{k=0}^{2N} e^{ikx}\right)^p$$
$$= e^{-iNpx}\left(\sum_{k=0}^{2Np} \tilde{c}_k e^{ikx}\right)$$
$$= \sum_{k=0}^{2Np} \tilde{c}_k e^{i(k-Np)x}.$$

Hence for even p,

$$||D_N||_p^p = 2\pi c_{Np}. \tag{B.13}$$

Finally we collect (B.10), (B.11), and (B.13) in the following proposition.

Proposition B.7. *Let p be a positive number, N a natural number, and D_N denote the order N Dirichlet kernel.. Then the following identities hold*

$$||D_N||_p^p = \frac{2\pi}{2N+1}c_{Np} + \lim_{r\to 1}\left(\frac{1}{r^{Np}}\sum_{\substack{k=0\\k\neq Np}}^{\infty} c_k r^k \tan\left(\frac{2\pi}{2N+1}(k-Np)\right)\right),$$

$$||D_N||_p^p = \frac{2\pi}{2N+1}c_{Np} + \sum_{\substack{k=0\\k\neq Np}}^{2Np} c_k \tan\left(\frac{2\pi}{2N+1}(k-Np)\right) \qquad (p \text{ odd}),$$

and

$$||D_N||_p^p = 2\pi c_{Np} \qquad (p \text{ even})$$

where for $k \neq Np$

$$c_k = \frac{1}{k-Np}\sum_{j=0}^{\lfloor \frac{k}{2N+1}\rfloor} (-1)^j \binom{(k-j(2N+1))+p-1}{(k-j(2N+1))}\binom{p}{j}$$

and

$$c_{Np} = \sum_{j=0}^{\lfloor \frac{Np}{2N+1}\rfloor} (-1)^j \binom{(Np-j(2N+1))+p-1}{(Np-j(2N+1))}\binom{p}{j}.$$

Remark B.8. Here we have used the continuity of the L^p norm.

Now we return to our original motivation. Is $||D_N||_p^p$ a polynomial in N for integers p? Surprisingly, *yes* if p is even. No if p is not. Finally we remark that the c_k are related to bounded affine linear subspaces in $\mathbb{R}^p$, and hence the L^p norm of the Dirichlet kernel is adding up weighted sums of the integer lattice points where weights come from the affine subspaces.