The use of efficient basis functions to model and represent flows with internal sharp velocity gradients, such as shocks or eddy microfronts, are investigated. This is achieved by analysing artificial data, observed atmospheric turbulence data and by the use of a Burgers' equation based spectral model. The concept of an efficient decomposition of a function into a basis set is presented and alternative analysis methods are investigated. The development of a spectral model using a generalized basis for the Burgers' equation is presented and simulations are performed using a modified Walsh basis and compared with the Fourier (trigonometric) basis and finite difference techniques.

The wavelet transform is shown to be superior to the Fourier transform or the windowed Fourier transform in terms of defining the predominant scales in time series of turbulent shear flows and in 'zooming in' on local coherent structures associated with sharp edges. Disadvantages are found to be its inability to provide clear information on the scale of periodicity of events. Artificial time series of varying amounts of noise added to structures
of different scales are analyzed using different wavelets to show that the technique is robust and capable of detecting sharp edged coherent structures such as those found in shear driven turbulence.

The Haar function is used as a wavelet to detect ubiquitous zones of concentrated shear in turbulent flows sometimes referred to as microfronts. The location and organization of these shear zones suggest that they may be edges of larger scale eddies. A wavelet variance of the wavelet phase plane is defined to detect and highlight events and obtain measures of predominant scales of coherent structures. Wavelet skewness is computed as an indicator of the systematic sign preference of the gradient of the transition zone. Inverse wavelet transforms computed at the dilation corresponding to the peak wavelet variance are computed and shown to contain a significant fraction of the total energy contained in the record. The analysis of data and the numerical simulation results are combined to propose that the sharp gradients normally found in shear induced turbulence significantly affect the nature of the turbulence and hence the choice of the basis set used for the simulation of turbulence.
Modelling and Analysis of Geophysical Turbulence:
Use of Optimal Transforms and Basis Sets

by

Nimal K.K. Gamage

A THESIS
submitted to
Oregon State University

in partial fulfillment of
the requirements for the
degree of
Doctor of Philosophy

Completed August 6, 1990
Commencement June 1991
APPROVED:

Redacted for privacy

Professor of Atmospheric Sciences in charge of major
Redacted for privacy

Head of Department of Atmospheric Sciences

Redacted for privacy

Dean of Graduate School

Date thesis is presented August 6, 1990

Typed by researcher for: Nimal K.K. Gamage
TO MY PARENTS
ACKNOWLEDGEMENTS

I am grateful to the guidance, teaching, support and friendship of my major advisor Prof. Larry Mahrt. The many hours of helpful discussions made this a significant learning experience for me. I would also like to thank Prof. Ron Guenther for his advice and critical comments on the thesis.

Thanks go to my colleagues, Mike Ek and Jin-Won Kim for many helpful comments, and, Wayne Gibson who provided computer programming and graphics assistance. Thanks are also due to Carl and Teri Hagelburg and the many Sri Lankan friends in Corvallis for their friendship, encouragement and moral support. Last but not least, I would like to thank my wife, Gayani, for her love, patience, understanding and tremendous encouragement.

Support for the research upon which this thesis is based was provided by the US Army Research Office (contract DAA-03089-k-0113) and the National Science Foundation (grant ATM-8521349). The National Center for Atmospheric Research is acknowledged for the Cray computer resources provided for computational work.
# TABLE OF CONTENTS

1. INTRODUCTION AND MOTIVATION .............................................. 1
   1.1 Gust and microfrontal nature of observed turbulence ............ 1
   1.2 Structural design requirements ........................................ 2
   1.3 Analysis using statistical approaches .............................. 3

2. THE AUTOCORRELATION FUNCTION AND FOURIER SPECTRA .............. 11
   2.1 Relationship between spectra and autocorrelation function .... 11
   2.2 The log-linear spectrum and its transforms ...................... 13
   2.3 Spectra and autocorrelation of randomly placed square pulses .. 15
   2.4 Criteria for determining the basis set ............................ 19

3. THE SQUARE PULSE AS A BASIS SET ......................................... 22
   3.1 Orthogonality relations .............................................. 22
   3.2 The constant coefficients .......................................... 23
   3.3 Energy (power) relations ............................................ 23
   3.4 The mean square error of a truncated decomposition .......... 23

4. NUMERICAL SOLUTIONS OF VISCOUS BURGERS' EQUATION ............... 25
   4.1 The Hopf-Cole analytic solution ................................... 26
   4.2 Spectral formulation of the Burgers' equation .................. 28
   4.3 Finite difference formulation ..................................... 31
   4.4 Comparison of results ............................................. 33
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>figure</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Streamlines of eddies as found in the NCAR DES model</td>
<td>4</td>
</tr>
<tr>
<td>1.2 Schematic view of the creation of a shear induced sharp edge</td>
<td>4</td>
</tr>
<tr>
<td>1.3 Spectra and structure functions of different artificial records</td>
<td>7</td>
</tr>
<tr>
<td>1.4 Performance of different basis sets for idealized test patterns</td>
<td>8</td>
</tr>
<tr>
<td>2.1 Spectra of velocity and temperature</td>
<td>12</td>
</tr>
<tr>
<td>2.2 Autocorrelation $R(\tau)$ and Power Spectral density $S(\omega)$ for exponential autocorrelation</td>
<td>12</td>
</tr>
<tr>
<td>2.3 Exponential autocorrelation for $\alpha = 1, 10, 100, 1000$ and a composite of all four plots</td>
<td>14</td>
</tr>
<tr>
<td>2.4 Square pulses of constant height and random width</td>
<td>18</td>
</tr>
<tr>
<td>2.5 Square pulses of random height and random width</td>
<td>18</td>
</tr>
<tr>
<td>2.6 Relationship between basis functions and record</td>
<td>20</td>
</tr>
<tr>
<td>4.1 Initial condition for the numerical solution $u(x, 0) = 1 + \sin 4\pi(x - 0.125)$ for $x \leq 0.5$</td>
<td>32</td>
</tr>
<tr>
<td>4.2 Available basis functions for spectral Models. (a) Element of the Trigonometric basis set. (b) Element of the Walsh ('top hat') basis set. (c) Element of the Modified Walsh basis set</td>
<td>34</td>
</tr>
<tr>
<td>4.3 The Hopf-Cole Solution at $t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.8$ and a composite of $t = 0.0$ to 1.0 in 0.1 steps</td>
<td>36</td>
</tr>
</tbody>
</table>
4.4 Comparison of solutions to Burgers’ equation.

(a) Hopf-Cole solution.
(b) Finite element solution.
(c) Spectral (trigonometric) Solution
(d) Spectral (Modified Walsh) Solution

5.1 A typical window function and its spectra as used in the windowed Fourier transform.

(a) A typical window function \( g(x) \)  
(b) The function \( g_{\omega_0,0}(x) \)
(c) The function \( g_{2\omega_0,0}(x) \)  
(a') The spectra of \( g(x) = \hat{g}(\omega) \)
(b') \( \hat{g}_{\omega_0,0}(\omega) \)  
(c') \( \hat{g}_{2\omega_0,0}(\omega) \)

5.2 Resolution of the Windowed Fourier transform phase space

(a) Minimum sampling of the phase space
(b) Domain of influence of each coefficient

5.3 Some elements of the Harr wavelet family

(a) \( h_{0,0} \)
(b) \( h_{-1,0} \): a dilation of \( h_{0,0} \)
(c) \( h_{-1,1} \): a translation of \( h_{-1,0} \)

5.4 (a) The effect of dilations of wavelet

(b) The Band-pass nature of the dilated wavelet spectrum

5.5 Resolution of the Wavelet Transform phase space

(a) Minimum sampling of the phase space
(b) Domain of influence of each coefficient

5.6 Over-sampling of the phase plane

(a) translation over sampled (redundant) phase plane
(b) dilation over sampled (redundant) phase plane
5.7 Some special wavelets
(a) The trigonometric wavelet (b) The ramp wavelet
(c) The step (Haar) wavelet (d) The Two-unit-impulse wavelet

6.1 Artificial record of randomly placed top-hats. The random
displacement from periodic midpoints is \( \zeta \) and the
periodic spacing is \( T_0 \).

6.2 Spectrum of top-hat function \( h(t) \). The width of the top-hat
is \( \tau \) and the amplitude \( C_k = 1 \).

6.3 Sampling of continuous spectrum by discrete function
when the product of the two are computed.

6.4 Discrete portion of Spectrum of randomly placed top-hats.
\( T_0 \) is the periodic spacing and \( \tau \) is the top hat width
(a) Periodic function with \( T_0 = 2\tau \)
(b) Periodic function with \( T_0 = 2\tau \)
(c) Periodic with small (10%) random displacement
(d) Periodic with small (50%) random displacement.

6.5 Spectra of ‘events’
(a) Event: top-hat, spectral minima: \( \tau \)
(b) Event: sine pulse, spectral minima: \( 1.5\tau \)
(c) Event: ramp, spectral minima: \( 2\tau \)

6.6 Record containing a single pulse of \( \tau \) width and
step weighting function as used in equations (6.16-6.22).

6.7 Wavelet transform of single top hat record for
(a) \( a/2 > \tau \) (b) \( a/2 = \tau \)
(c) \( \tau/2 \leq a/2 < \tau \) (d) \( a/2 < \tau/2 \)
6.8 Wavelet variance as a function of dilation \( \alpha \) for fixed event width \( \tau = 100 \)

(a) top hat record; Haar wavelet
(b) sine pulse record; Haar wavelet
(c) top hat record; sine wavelet
(d) sine pulse record; sine wavelet

6.9 Step wavelet variance as a function of dilation (in points) for

(a) periodic top-hats (top)
(b) sine pulse (middle) and
(c) ramps (bottom)

6.10 Wavelet variance for records with

(a) random displacement of events fixed spacing and event width (top)
(b) random variation of event width about a mean of 50 points (middle) and
(c) random additive white noise with periodic record of 50 point event width and 100 point spacing (bottom)

6.11 Wavelet skewness for records with periodic ramp events with no added noise (top) and, with varying amount of added white noise (bottom)

6.12 A record and its partial reconstruction using Wavelet inverse transform at a dilation of 50 points is shown (dashed line) along with the original record (solid line)

7.1 The three tower array at the Lammefjord site in Denmark used in LAMEX. The three dimensional sonic anemometer which provided the data used in this study is mounted on top of mast 1 (45m)
7.2 Wind speed and direction at 30m height during the 50 hr period analysed in this study ....... 95

7.3 Sample record of vertical velocity from HAPEX 19 May (top) with wavelet coefficients for a fixed dilation of 500m (middle) and wavelet phase plot (bottom). A step indicator function is used in the transform .................. 96

7.4 (a) Composite power spectra of \( u, v \)
        (b) composite Wavelet variance of \( u, v \) for HAPEX 19 May .... 99

7.5 (a) Composite power spectra of \( w, T \)
        (b) composite Wavelet variance of \( w, T \) for HAPEX 19 May .... 100

7.6 (a) Composite power spectra of \( u, v \)
        (b) composite Wavelet variance of \( u, v \) for HAPEX 25 May .... 101

7.7 (a) Composite power spectra of \( w, T \)
        (b) composite Wavelet variance of \( w, T \) for HAPEX 25 May .... 102

7.8 (a) Composite wavelet skewness of \( w, T \) for HAPEX 19 May
        (b) composite wavelet skewness of \( w, T \) for HAPEX 25 May .... 103

7.9 (a) Composite power spectra of \( u, v \)
        (b) composite Wavelet variance of \( u, v \) for SESAME 5 May .... 104

7.10 (a) Composite power spectra of \( W, T \)
        (b) composite Wavelet variance of \( W, T \) for SESAME 5 May .... 105

7.11 (a) Composite power spectra of \( u, v \)
        (b) composite Wavelet variance of \( u, v \) for SESAME 6 May .... 106

7.12 (a) Composite power spectra of \( w, T \)
        (b) composite Wavelet variance of \( w, T \) for SESAME 6 May .... 107

7.13 (a) Composite wavelet skewness of \( w, T \) for SESAME 5 May
        (b) composite wavelet skewness of \( w, T \) for SESAME 6 May .... 108

7.14 Composite power spectra of LAMEX 50 hours ......... 109
7.15 Composite wavelet variance for LAMEX 50 hours  

7.16 Composite wavelet skewness for LAMEX 50 hours  

7.17 Measure of the flatness of the wavelet variance peak is  
provided by $\delta a$. When $\delta a$ is large the curve is flat.  

7.18 Wavelet analysis of LAMEX data. Dilations at which  
wavelet variance reaches maximum. The computation is for  
a Haar wavelet computed for vertical winds  

7.19 Wavelet analysis of LAMEX data. Differences of  
dilations at which wavelet variance reaches 95% of maximum  
wavelet variance  

7.20 Wavelet analysis of aircraft data. Dilations at which wavelet  
variance reaches maximum and the corresponding variance.  
The computation is for a Haar wavelet computed for vertical  
winds. (a) HAPEX 19 May (b) HAPEX 25 May  
(c) SESAME 5 May (d) SESAME 6 May  

7.21 Wavelet analysis of aircraft data. Differences of dilations at  
which wavelet variance reaches 95% of maximum wavelet  
variance. The computation is for a Haar wavelet computed for  
vertical winds. (a) HAPEX 19 May (b) HAPEX 25 May  
(c) SESAME 5 May (d) SESAME 6 May  

7.22 19 May vertical velocity (solid) and wavelet inverse  
transform computed at a dilation of 200m (dashed).  
The wavelet inverse transform is for a Haar wavelet  

7.23 Sample of 2000 point segment from Segment of HAPEX  
19 May vertical velocity (solid) and wavelet inverse transform  
computed at a dilation of 200m (dashed). The wavelet inverse  
transform is for a Haar wavelet
<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>The aircraft observations used in the analysis.</td>
<td>91</td>
</tr>
<tr>
<td>7.2</td>
<td>Wavelet analysis of LAMEX data</td>
<td>112</td>
</tr>
<tr>
<td>7.3</td>
<td>Wavelet analysis of HAPEX and SESAME data</td>
<td>113</td>
</tr>
</tbody>
</table>
1. INTRODUCTION AND MOTIVATION

Coherent structures in turbulent flows play an important role in geophysical turbulence, in terms of transport and dissipation. Through the use of optimal basis functions and statistics of local transforms, this study attempts to isolate the physics of coherent structures in turbulence, beginning with the estimation of the scale of these randomly placed events.

1.1 Gust and Microfrontal nature of observed turbulence

Microfronts, or narrow zones of concentrated shear, are often observed in the heated atmospheric boundary layer. These are also observed as bursting events (Chen and Blackwelder 1978, Gibson et al. 1968) in laboratory studies. Microfronts are often considered to be the boundary of larger coherent structures such as ramp patterns (Antonia and Atkinson 1976; Antonia et. al. 1979, Schols 1984, Mahrt 1989 and others) or asymmetric top-hats (Mahrt 1989, Mahrt 1985; Kikuchi and Chiba 1985). These zones of concentrated shear, also called gust fronts, are modeled as vortex sheets (Corrsin 1962), or stretched vortices that concentrate shear across zones as thin as the Kolmogorov microscale (Tennekes 1968), and as regions of convergence associated with vortex stretching due to vertical acceleration of thermals (Kaimal and Businger 1970).

In laboratory studies, similar zones of concentrated shear or temperature gradients have been related to edges of large eddies associated with up lifting
of near surface fluid of weak momentum (Mahrt 1989 (survey), Willmarth 1975; Brown and Thomas 1977, Subramanian et al. 1982). These sharp edges have an important effect on the cascade of energy from the main eddy scales to smaller dissipation scales and may account for a large fraction of the dissipation. In addition, they may play an important role in the enhancement of mixing and transport, as noted in Schols (1984), Kikuchi and Chiba (1985) and Chen and Blackwelder (1978).

1.2 Structural design requirements

In many operational systems (for example: tall smoke stacks, electrical pylons, coastal structures, offshore oil platforms and bridges) the effects of wind and turbulence can have an important impact on both safety and/or performance. Although over-design is a possibility, it is not always practical. As noted by Panofsky and Dutton (1984), though it may be possible to manufacture an aircraft to withstand the strongest possible turbulent gusts, it may not be navigable. Hence, there is the need for designers to make intelligent design decisions on compromise. The difficulty of attempting to translate statistical information about wind and turbulence into statistical information about wind loading and structural response lead designers into the use of simulation methods. In the simplest simulation, a forcing function is applied to a system of differential equations describing the structure to obtain the full response function which provide information on other factors such as, natural frequencies and critical loading. However, this forcing function should be capable of simulating extreme events of low probability while maintaining the characteristics of the wind or turbulence it is intended to simulate. Microfronts or gusts are a major contributor to these extreme events and need to be well understood in order to be included in the design
1.3 Analysis using statistical approaches

An attempt to completely study the characteristics of microfronts and associated coherent structures would include two phases. In the first phase, one would analyse observed (measured) atmospheric data and try to ascertain the typical characteristics of the microfronts such as their magnitudes, frequency of occurrences and energetics. The second phase would attempt to simulate these microfronts using models.

In the analysis of measured turbulence data, computation of spectra plays an important role. In situations such as aircraft measured turbulence, spectral statistics are used to obtain an insight into the nature of the flow. In other applications such as advection and mixing of a pollutant, the spectra are used to measure the rate of spread of the patch of contaminant.

The aperiodic nature of the microfronts (Jensen and Lenschow 1978) and their localized sharp gradients make the use of trigonometric function based spectral methods unsatisfactory in the detection of characteristic sizes of the coherent structures. A large number of trigonometric Fourier coefficients are needed to adequately describe a sharp edge and thus spread the variance over a large number of wavenumbers. As an alternative, many previous researchers have made use of conditional sampling techniques (Mahrt and Frank 1988, Schols 1984 etc.,) in their analysis of these coherent events.

In an effort to quantify the scales of these coherent structures as well as their spacing, this study uses a (localized analysis) wavelet transform method along with spectral methods (global analysis). Initially, we examine simulated data sets where structures with compact spatial support, such as top-hats, sine pulses and ramps are placed with regular and random spacing.
fig 1.1 Streamlines of eddies as found in the NCAR DES model

fig 1.2 Schematic view of the creation of a shear induced sharp edge
In the wavelet transform one has the freedom to specify the form of the basis used for the decomposition. Our analysis uses different basis functions, chosen specifically to highlight certain properties such as gradients and coherent structures. These results are then used to quantify the observed atmospheric turbulence.

The spectral transform and the wavelet transform are then used to decompose and analyse the HAPEX, SESAME and LAMEX data. Decompositions using the Haar wavelet basis are computed, and the results are interpreted using the simulated data analysis. The wavelet basis functions have compact spatial support and thus have the ability to locally optimize the variance explained (Daubechies 1989) and helps interpret the wavelet transform results in terms of the sharp edges in the record. The wavelet transform and the spectral results are combined to characterize the observed data in terms of the sizes and spacing of the updrafts and downdrafts in the record. Information on the microfronts associated with these updrafts and downdrafts are also obtained.

In recent modelling studies of turbulence, spectral techniques have been used in Direct Eddy Simulation models (DES) where the practice has been to Fourier transform the Navier Stokes Equations and use a limited number of modes of the resulting transformed space for analysis. The number of such wave modes have mostly been dictated by available computing resources and has generally been limited to 32 or 64 modes (Herring and Metais 1986, Orzag et al. 1981 and others). In addition, these models may incorporate some form of hyperviscosity (Herring et al. 1986) to ensure that dissipation will prevent energy build up at the high wave number end. They also contain other forms of energy sinks to prevent build up of energy at the low wave
number end. These DES models directly compute all the terms of the Navier-Stokes equation in the spectral domain except for the non-linear term which is computed in the spatial domain to avoid the convolution problem (Riley et al. 1979 and others). The interaction between scales is limited by the number of wave modes in the model. For example, in Orzag and Patera (1981) only three modes are allowed in the transverse direction.

In our DES study of individual eddies, where the simulation results of the 64 mode DES model at the National Center for Atmospheric Research (NCAR) was examined (Herring and Metais 1986), we noticed that the individual eddies had relatively smooth edges as a result of the limited number of (smooth) trigonometric modes (fig. 1.1). A large number of wave modes would be needed to represent the observed sharp edges of the main eddies. In this study the ability of square pulse and other basis sets to decompose both artificial and observed fields are compared and the square pulse is used as a basis for a numerical simulation of the viscous Burgers’ equation. The Burgers’ equation is selected for the model simulation as the solutions yield shock waves which are similar to the microfronts in observed atmospheric data.

As in the trigonometric basis set, the square pulse basis consists of an odd periodic set of square pulses, an even periodic set of square pulses and a mean. Such a basis set is: (i) Sufficient to decompose any given field, (ii) Orthogonal, and (iii) The energy contained in the original field is totally contained in the coefficients of the decomposed modes (similar to Parsevals relation).
fig 1.3 Spectra and structure functions of different artificial records
fig 1.4 Performance of different basis sets for idealized test patterns
The advantage of selecting the square wave basis set is its ability to
describe a field with sharp gradients with a few modes. A series of tests were
performed on test data to study the behavior of this basis set (see fig. 1.3).
Naturally, this basis set efficiently decomposes a periodic square wave but
requires many modes to approximate a sine wave. In terms of total energy,
the primary square mode corresponding to the scale of the sine wave contains
\( \sim 60\% \) of the total energy of the original sine wave. This is similar to the
case of a square wave decomposed into a Fourier (trigonometric) basis where
a large number of sine modes are needed to approximate the square wave and
the primary mode contains \( \sim 60\% \) of the total energy of the square wave.
Figures (1.3) and (1.4) show the spectra of such decompositions for different
test records and the two (trigonometric and square pulse) bases.

The organization of the following chapters of this thesis is as follows. In
Chapter 2 the relationship between the autocorrelation function and spec-
trum is presented along with the non uniqueness properties of the spectrum.
These relationships are investigated for the exponential autocorrelation func-
tion. Methods of generating such an exponential autocorrelation function
are considered and a theoretical development of autocorrelation functions
for randomly placed top hat records are presented. It is shown that these
records exhibit similar characteristics to those of spectra of atmospheric tur-
bulence. In view of this, the influence of the sharp gradients on the spectra
are discussed. Chapter 3 gives a detailed description of the development of
a square wave basis set and also discusses its properties.

In Chapter 4 a numerical simulation of the viscous Burgers’ equation is
presented along with an investigation of how the choice of a basis set affects
the simulation of sharp gradients. Simulations using the Fourier trigonomet-
ric basis and the square wave basis are compared.
Chapter 5 presents the theoretical aspects of the wavelet transform method. The various formulations required to link this method to the traditional analysis methods such as, autocorrelation, structure function and spectra are presented.

In Chapter 6 the theory developed in Chapter 5 is used to perform a wavelet transform based decomposition of simulated data sets. Designed with the objective of mimicking the observed data, these data sets form a test suite for the theory developed in the previous chapter. Different wavelets are used in the decomposition and their properties with respect to their ability to decompose a turbulent flow are presented.

In Chapter 7 observed atmospheric data is analysed with the statistical tools presented in the earlier chapters. The results are used to define characteristic scales of the various features of the flow. Chapter 8 provides a summary of the results. Finally, in the Appendix, precise definitions of some of the measures, and terms used in the thesis are described.
2. THE AUTOCORRELATION FUNCTION AND FOURIER SPECTRA

In this chapter the relationships between spectra and autocorrelation function will be defined and their relationship to the original record from which they are generated will be presented. The properties of some special autocorrelation functions will be discussed and compared with observed atmospheric turbulence spectra.

2.1 Relationship between spectra and autocorrelation function

The autocorrelation function of $f(x)$ is defined as

$$ R_{xx}(\tau) = (f(x + \tau) \cdot f(x)) \quad (2.1) $$

where $\tau$ is the lag or separation distance and $\langle \rangle$ denotes the averaging operator (Monin and Yaglom 1975 or Guenther and Lee 1988 and references therein). The power spectral density of $f(x)$, denoted by $S_{xx}(\omega)$, is related to its autocorrelation function by the relationship,

$$ S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega \tau} d\tau \quad (2.2) $$

where $j = \sqrt{-1}$. By computing the inverse Fourier transform of (2.2) one obtains

$$ R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega \tau} d\omega \quad (2.3) $$

Equation (2.2) and (2.3) define an important relationship between the autocorrelation and the power spectral density, i.e., they are a Fourier transform pair. This implies that a unique relationship exists between an autocorrelation function and a power spectrum as stated in the following theorem.

**Bochner-Herglötz Theorem.** The correspondence between the autocorrelation function and the power spectral distribution are one to one, and the information either one provides about its generating process is the same.
fig 2.1 Spectra of velocity and temperature

fig 2.2 Autocorrelation $R(\tau)$ (left) and Power Spectral density $S(\omega)$ (right) for exponential autocorrelation.
Since the structure function (Monin and Yaglom 1975) and spectra are also uniquely related (for the relationships in 1-D homogeneous flow and 3-D isotropic flow see Mahrt and Gamage 1987), the specification of any one of the three quantities would suffice, although, none of the above three are uniquely related to the time series. This is due to the loss of phase information. Such relationships will find use in the next section. However, since the effect of some forms of contamination, such as trend, would have considerably different effects on the three quantities (Lumley & Panofsky 1964, Gamage 1986) they tend to provide independent information in analysis of observed turbulence.

2.2 The Log-Linear Spectrum and its Transforms

The spectrum of turbulent data (in log log coordinates) is typically characterized by a constant slope (fig. 2.1) particularly at scales smaller than the main eddies (Zhou et.al. 1985, Herring and Wyngaard 1986). This linear region is usually explained by certain physical reasoning attributed to Kolmogorov (1941) called the inertial sub-range energy cascade. Based on this linear spectrum (with the appropriate constant slope) some researchers have argued that all scales of motion exists between the largest (generating scale) and the smallest (dissipating) scale in the spectra and that there is a energy cascade through these successively smaller scales (Richardson 1948, Tennekes and Lumley 1971). This process is also called the Richardson energy cascade. A gap in the spectrum is thought to divide the turbulence from the large scale motions (Leyi and Panofsky 1983) and attempts have been made to predict its frequency range in Hogstrom and Hogstrom (1975) and later studies. A peak in the spectrum is sometimes interpreted as the most energetic scale or the generating scale.
fig 2.3 exponential autocorrelation for $\alpha = 1, 10, 100, 1000$ and a composite of all four plots.

In this study we examine the spectra in order to determine if the existence of the linear slope of the spectrum indeed implies the existence of a continuous cascade. Linear spectra referred to herein is on a log-log plot. As an example consider the autocorrelation function (see fig. 2.2)

$$R(\tau) = e^{-\alpha|\tau|}, \quad -\infty < \tau < \infty.$$  \hspace{1cm} (2.4)
Computing the power spectral density of the above autocorrelation function we obtain,

\[ S(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2}, \quad -\infty < \omega < \infty \]  

(2.5)

Now let us consider the above spectrum for different values of \( \alpha \) and \( \omega \) (See fig. 2.3). When \( \omega \) is much larger than \( \alpha \), the above spectrum will have a slope of -2. But as \( \alpha \) and \( \omega \) become of comparable magnitude, the slope of the above curve becomes less than -2 as seen in fig. (2.3). This can be shown by considering the following. Let \( \omega_1 \) be selected such that \( \omega_1 = \alpha + \omega \). Then we can find an index \( n \) such that

\[ \omega_1^n = \alpha^2 + \omega^2 \]  

(2.6)

where \( n < 2 \). Now equation (2.5) can be rewritten as

\[ S(\omega_1) = \beta \omega_1^{-n}, \quad -\infty < \omega < \infty \text{ where } \beta = f(\alpha). \]  

(2.7)

From the above analysis it is seen that if a function has a exponentially decaying autocorrelation function then its spectra will have a linear -2 region and also a linear region of slope less than -2. In the following discussion the above results will be used in conjunction with the Bochner-Herglòtz theorem to show that a spectrum of a given (specified) slope can be generated with a random square pulse train.

2.3 Spectra and autocorrelation of randomly placed square pulses

The purpose of this development is to show that a fixed slope spectrum, such as a -5/3 slope, does not necessarily imply a cascading process, or even a range of scales. As a counter example we will show that the existence of only a few intermediate scales can lead to a linear spectrum. For this purpose we
will use simulated data where top-hats of certain widths will be randomly placed in a record. Here, the scale of the motion is defined as the width of the square pulses. It should be noted that the selection of the square pulse example is for the purpose of illustration only and it does not necessarily represent a physical process. The autocorrelation function of a regularly placed (periodic) square pulse record is periodic. If the inter-pulse gap is gaussian, then its autocorrelation function is a decaying periodic function, the rate of decay being governed by the given standard deviation of the gap width. If the pulse locations are purely random in the record (Kharkevitch 1960, Larson and Shubert 1979) then the autocorrelation function is given by

\[ R(\tau) = a^2 (p_e - p_o) \]  

(2.8)

where \( p_e \) and \( p_o \) are the probability of having an even and odd number of zero crossings of the time series in \( \tau \) and \( a \) is the fixed height of the pulse(fig. 2.4). For the special case of the Poisson distribution this becomes a decaying function of the form

\[ R(\tau) = a^2 e^{-2\alpha|\tau|} \]  

(2.9)

where \( \alpha \) depends on the variance of the position of the pulses and \( \tau \) is the time or spatial lag.

If the magnitude as well as the location of the pulses are poisson random (fig. 2.5) then the autocorrelation function is given by

\[ R(\tau) = M(a) \exp(-\alpha|\tau|) \]  

(2.10)

where \( M(a) \) is a function of the random height distribution. This equation is obtained provided the randomness of the height of pulses are uncorrelated with the randomness of the widths and that the their variances of the poisson
processes are equal. For the more general case of unequal distributions the form of the autocorrelation still remains the same with only changes in the coefficients.

Since the autocorrelation function of a random pulse sequence decays exponentially (2.10), with its spectral slope depending on the relative magnitudes of the variance of the pulse position \( \alpha \), the spectral slope of such a process can be manipulated by the selection of a suitable variance of the underlying poisson generating process. A record including more than one pulse width would have an autocorrelation function given by,

\[
R(\tau) = e^{-\alpha_1|\tau|} + e^{-\alpha_2|\tau|} + e^{-\alpha_3|\tau|} + ... \tag{2.11}
\]

where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are the different decay rates. Here it is assumed that the different pulses of widths \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are placed independently of one another and that their cross correlation is zero. By changing the values of the position variance the slope of the spectrum can be changed.

As an idealized analogy to convectively driven turbulence where downdrafts are wider than updrafts, if one considers the possibility of having few pulse widths instead of the one pulse width as discussed above, then the slope of the spectra can be made linear and less than -2 for a larger frequency range. Considering the non-uniqueness of spectra with respect to the record as discussed earlier, this example shows that the observed sharp gradients in turbulence alone can cause the linear (on a log-log scale) spectra as found in many experiments. Another implication of this is that the inertial energy cascade as predicted by dimensional arguments may not be needed to explain the observed turbulence spectrum.
fig 2.4 Square pulses of constant height and random width

fig 2.5 Square pulses of random height and random width
2.4 Criteria For Determining the Basis Set

One could argue that the multiplicity of Fourier spectral scales associated with a cascading process is built into the square pulses. This is because a Fourier (trigonometric) spectral representation of a single square pulse train requires an infinite number of frequencies (fig. 1.4). However it can be described by a single square pulse scale given by the pulse width. A meaningful decomposition into basis functions should require only a limited number of functions for reasonable approximation, hence the trigonometric building blocks may be an inefficient basis. A (simple) extreme case is a perfectly periodic square pulse train, where one function in the square pulse basis completely describes the record while an infinite number of trigonometric basis functions are needed to do the same. In this example the trigonometric basis is very inefficient while the square pulse basis is ideal. Of course one could also consider the other extreme which is the decomposition of a sine wave into the two basis where the trigonometric basis is superior. Eddies in geophysical turbulence may be between square pulse and trigonometric and this thesis examines the efficiency of the use of different basis sets for the decomposition.

Although one might attempt to choose the basis set by inspecting the record, for example, as ramp functions, it is desirable to have an objective technique to determine the best basis set. While the traditional EOF analysis yields a basis set with the minimum number of modes for a approximation of a given record, the basis functions differ from application to application, and some of the details of the structure of even the leading eigen vector may be insignificant (fig. 2.6). The other extreme would be to use a Galerkin technique, such as with Chapau functions as the basis set, where a maximum number of modes (of the basis) are required to represent a given record.
fewer elements of basis set fully describes record

Dependence of basis set on record increases

Impulse functions as a basis
Chapau Functions as a basis
Sinusoidal (Fourier) Basis
Other Basis (Square, Triangular etc.) subjectively chosen
EOF (and variations) as a basis

fig 2.6 Relationship between basis functions and record.

One would rather have optimum basis functions for given class of related records (such as weak intermittent turbulence or convective thermals) which best approximate the desired characteristics (such as the main eddies). However there is apparently no 'automatic' way of selecting the best basis set.

In view of the fact that the spectra associated with the randomly placed square pulses duplicated the observed energy spectra in this study of geophysical turbulence, our assumption is that the sharp gradients observed in turbulence phenomena are associated with the maximum non-linearity (Mahrt 1988 and references therein) and, adequate representation of these sharp gradients are vital in the description of interaction between different
turbulent scales, instability, and cascade of energy to smaller scales. To
examine this assumption we try to reformulate the present spectral mod-
els in terms of a more efficient basis function containing sharper gradients,
which will better resolve the important physics even with a finite number of
modes. Since the traditional trigonometric decomposition with truncation
of the number of allowable modes results in inadequate representation of the
sharp gradients, in chapter 4 we attempt a spectral simulation of a simplified
version of the Navier Stokes equation, i.e., the viscous Burgers equation with
the use of a square pulse basis. The complete formulation of the square pulse
basis is presented in the next chapter.
3. THE SQUARE PULSE AS A BASIS SET

In this chapter, the theoretical aspects of defining a square pulse basis set will be developed. It will be shown that this set is orthogonal and that the total energy of the record is conserved in the decomposition. These derivations are similar to ones usually obtained for Fourier series decomposition and good descriptions are given in Harmouth (1970) and McGillem et al. (1974). It will also be shown that the coefficients of the decomposition $a_n$ and $b_n$ for a certain time series (these are the coefficients of the $n$th harmonic) will decay rapidly for increasing $n$ for certain types of test functions.

3.1 Orthogonality relations

A representation in terms of the square pulse basis for given record $f(x)$ is given by,

$$f(x) = \left(\frac{a_0}{2}\right) + \sum_{n=1}^{\infty} \left\{ a_n SQE \left(\frac{n\pi x}{L_x}\right) + b_n SQO \left(\frac{n\pi x}{L_x}\right) \right\} \quad (3.1)$$

where, $SQE() =$ Even periodic square pulse train, and $SQO() =$ Odd periodic square pulse train.

An even function is of type $f(x) = f(-x)$ and an odd function is of type $f(x) = -f(-x)$. The above is an orthogonal representation of the basis functions and the orthogonality relations for it are:

$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} SQO \left(\frac{n\pi x}{L_x}\right) \cdot SQO \left(\frac{m\pi x}{L_x}\right) \, dx = \begin{cases} 0 & m \neq n \\ L_x & m = n \end{cases} \quad (3.2)$$

$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} SQE \left(\frac{n\pi x}{L_x}\right) \cdot SQE \left(\frac{m\pi x}{L_x}\right) \, dx = \begin{cases} 0 & m \neq n \\ L_x & m = n \end{cases} \quad (3.3)$$

$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} SQO \left(\frac{n\pi x}{L_x}\right) \cdot SQE \left(\frac{m\pi x}{L_x}\right) \, dx = 0 \quad \forall \, m, n \quad (3.4)$$
3.2 The constant coefficients

The constant coefficients of the decomposition in (3.1) are,

\[ a_0 = \frac{1}{L} \int_0^L f(x) dx \]  \hspace{2cm} (3.5) \\
\[ a_n = \frac{1}{L} \int_0^L f(x) \cdot SQO \left( \frac{n \pi x}{L} \right) dx \]  \hspace{2cm} (3.6) \\
\[ b_n = \frac{1}{L} \int_0^L f(x) \cdot SQE \left( \frac{n \pi x}{L} \right) dx \]  \hspace{2cm} (3.7) \\

Where the orthogonality relations in equations (3.2), (3.3) and (3.4) have been used to determine \( a_n \) and \( b_n \).

3.3 Energy (power) relationships

In the idealized case the total energy contained in the original record must be contained in the decomposed signal. If the energy is defined as \( \frac{1}{L} \int_0^L f(x)^2 dx \), using the decomposition in equation (2.1) and the orthogonality relations shown above, we obtain the following for the total energy,

\[ \frac{1}{L} \int_0^L f(x)^2 dx = \left[ a_0^2 + \sum_{k=1}^{\infty} (a_n^2 + b_n^2) \right] \]  \hspace{2cm} (3.8) \\

Therefore, for finite \( n \), we obtain (if the series is convergent)

\[ \frac{1}{L} \int_0^L f(x)^2 dx = \left[ a_0^2 + \sum_{k=1}^{n, n \to \infty} (a_n^2 + b_n^2) \right] \]  \hspace{2cm} (3.9) \\

In most situations the convergence requirement is stated as a rapid decay in the coefficients as

\[ a_n, b_n \overset{n \to \infty}{\longrightarrow} 0. \]  \hspace{2cm} (3.10) \\

3.4 The mean square error of a truncated decomposition

In decomposing the signal into its basis set, the coefficients of the expansion are selected so as to minimize the mean square deviation from the
original record. If some decomposition of the record $f(x)$ into the basis is denoted by $g_n(x)$, where,

$$g_n(x) = \left(\frac{a_0}{2}\right) + \sum_{n=1}^{n} \left\{ \alpha_n SQE \left(\frac{n\pi x}{L_x}\right) + \beta_n SQO \left(\frac{n\pi x}{L_x}\right) \right\}$$  \hspace{1cm} (3.11)

and $\alpha_n$ and $\beta_n$ are some arbitrarily chosen coefficients, the mean square deviation is defined as

$$E_n = \int_0^L (f(x) - g_n(x))^2 \, dx$$  \hspace{1cm} (3.12).

The condition for minimum (or maximum) error $(E_n)_{\text{min}}$ is obtained by differentiating (3.12) with respect to each $\alpha_n$ and $\beta_n$ and setting the resulting expression equal to zero as

$$\frac{\partial E_n}{\partial \alpha_0} = \frac{\partial}{\partial \alpha_0} \int_0^L (f(x) - g_n(x))^2 \, dx = 0$$

i.e $\alpha_0 = a_0$

and similarly $\alpha_n = a_n$, $\beta_n = b_n \forall n$  \hspace{1cm} (3.13)

Thus it is seen that the coefficients found in section 3.2 are the coefficients that decompose the signal with minimum mean square error. These results can also be extended to a modified square pulse basis where the sharp transition of the square pulse is changed to a transition with a finite gradient. In the numerical simulations carried out in Chapter 4, this modified basis is used where the transition is assumed to occur over one grid point in the $x$-axis direction. The advantage of this modification is that the basis is now once differentiable and can be used to spectrally solve the Burgers' equation.
4. NUMERICAL SOLUTION OF VISCOUS BURGERS’ EQUATION

The motivation for this part of the study is to study the effect of non-linearly on the choice of the basis set for decomposition of a flow. A simple one dimensional (1-D) nonlinear equation, in this case the 1-D viscous Burgers' equation is used to study the flow evolution for model solutions using different basis sets. This equation can be obtained from the Navier-Stokes equation by making certain assumptions about the flow characteristics. Although simpler in form, the Burgers’ equation still contains the nonlinear term and the viscous term as they appear in the original Navier-Stokes equation and has the advantage of yielding analytic solutions (Hopf 1950, Cole 1951). As discussed in the Introduction our study of individual eddies using a 64 mode 3-D spectral model (NCAR DES model) showed that the eddies had very smooth edges despite the relatively high resolution of the model. To study if this ‘smoothing’ was a result of spectral truncation, a spectral model based on an arbitrary global basis will be derived using the Burgers’ equation. An advantage of this approach is the ability to change the resolution and the basis of the spectral model and the ability to study a shock wave where the non linear term is thought to be important.

The decomposition of any given field (in this case velocity) into a basis set is said to be an efficient decomposition when the coefficients of the resulting set are significant only for limited number of modes. For example, if the variable of interest is \(bsin(kx)\) and if the basis is chosen to be the trigonometric sinusoidal basis then all coefficients of the basis elements are zero except for one corresponding to \((kx)\). However a square pulse basis set, as discussed in Chapter 3, for the above example would be an inefficient basis as a large number of nonzero coefficients are needed. Of course the above argument is based on the premise that such decomposition into basis sets
are used as an analysis tool to reduce the dimensionality of the problem. In particular if an \( n \) point observation field is written in terms of a basis set with \( n \) significant coefficients the analysis (or representation) might as well have been done with the original observed field. Another desirable feature is that the same basis elements be used to describe another unrelated field equally well and hence can be used in a numerical modelling situation with comparative ease.

In this chapter we will investigate the effect of the inefficiency of the basis on the representation of the dynamics. It should be noted that the decomposition into basis functions that we are considering in this chapter relate to global basis functions and not local basis functions. A global basis set contains functions that are (usually) periodic and span the whole domain of the flow. A familiar example for this is the trigonometric \( e^{ikx} \) basis. Local basis functions have compact spatial support and span a small fraction of the flow domain. An example for local basis functions are given by cubic splines which span one or two grid points when used in finite element methods.

4.1 The Hopf-Cole Analytic Solution

The general equation for conservation of momentum of a unidirectional fluid flow in a viscous, isotropic, homogeneous and pressure free medium is considered. This equation is called the viscous Burgers' equation and is given by,

\[
  u_t + uu_x = \nu u_{xx}
\]

(4.1)

where \( \nu \) is the viscosity of the fluid which is considered constant for this
study. The boundary conditions for (4.1) are

\[
\begin{aligned}
&u(0, t) = 0 \\
&u(1, t) = 0
\end{aligned}
\]  
(4.2)

which requires that the motion be between two fixed walls. The initial conditions are given by

\[
u(x, 0) = f(x) \quad 0 \leq x \leq 1.
\]  
(4.3)

where, \(f(x)\) is a positive, integrable curve with compatibility conditions of \(f(0) = 0\) and \(f(1) = 0\).

If \(U, L\) and \(T\) are the characteristic velocity, length and time scales then (4.1) can be written as

\[
u_t + uu_x = \frac{1}{R} u_{xx}.
\]  
(4.4)

\(R \equiv (\nu/LU)\) is known as the non-dimensional Reynolds number.

Equation (4.3) can be transformed using the Hopf-Cole transformation (Cole 1951, Hopf 1950), which reduces the nonlinear hyperbolic viscous Burgers' equation to a linear heat equation with analytic solutions. This is accomplished by the use of a nonlinear transformation. Let

\[
u = -2\nu \frac{W_x}{W}
\]  
(4.5)

where \(W \equiv W(x, t)\). Then, equation (4.1) can be written as

\[
W_t = \nu W_{xx}
\]  
(4.6)

The boundary conditions on \(W(x, t)\) now become

\[
\begin{aligned}
&W_x(0, t) = 0 \\
&W_x(1, t) = 0
\end{aligned}
\]  
(4.7)
The initial condition is obtained from

\[ f(x) = -2\nu \frac{W_x(x, 0)}{W(x, 0)} \]  \hspace{1cm} (4.8)

as

\[ W(x, 0) = \exp[-\frac{1}{2\nu} \int_0^x f(y)dy] \] \hspace{1cm} (4.9)

This system with the transformed boundary conditions and initial conditions can be solved using separation of variables to obtain the following solution for \( u(x, t) \)

\[ u = 2\pi \nu \sum_{n=1}^{\infty} n \int_0^1 \exp \left[ -\frac{1}{2\nu} \int_0^\eta f(\eta)d\eta \right] \cos(n\pi\eta) d\eta \ e^{-(n\pi)^2\nu t} \sin(n\pi x) \]

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \int_0^1 \exp \left[ -\frac{1}{2\nu} \int_0^\eta f(\eta)d\eta \right] \cos(n\pi\eta) d\eta \ e^{-(n\pi)^2\nu t} \cos(n\pi x) \] \hspace{1cm} (4.10)

where \( a_0 = 2 \int_0^1 \exp \left[ -\frac{1}{2\nu} \int_0^\eta f(\eta)d\eta \right] d\eta \). Solutions obtained from (4.10) will be called analytic solutions. This result will be used as the true solution as the summation in equation (4.10) can be carried out to a large number of terms, thus giving any desired level of accuracy.

4.2 Spectral formulation of the Burgers' equation

The second technique for solving (4.1) is a spectral method (Spectral solution). The Burgers' equation (4.1) can be formulated in a manner so as to accept any global orthogonal basis set. The solution of (4.1) at any point in space and time can be written as,

\[ u(x, t) = \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \]  \hspace{1cm} (4.11)

where \( \phi_k \) is the kth element of the basis set and \( a_k \) is its coefficient defined as

\[ a_k = \int_D (u(x, t) \cdot \phi_k) dx. \] \hspace{1cm} (4.12)
This expansion in a basis can be carried out for an orthogonal or non-orthogonal basis if the basis is complete. The total support of all elements of the basis should span $\mathbb{R}^n$, though individual elements may have compact spatial support. In this spectral formulation, we assume only orthogonal, global, complete basis sets. In addition we assume that each basis element is at least once differentiable. Substituting (4.11) into (4.1) gives

$$\left[ \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \right]_t + \left[ \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \right] \left[ \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \right]_x = \nu \left[ \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \right]_{xx}.$$

(4.13)

The second term on the l.h.s. is a convolution term and its computation is very expensive. This can be simplified by decomposing the $uu_x$ term in (4.1) also into the same basis as $u$ to give

$$uu_x = \sum_{k=1}^{\infty} b_k(t) \phi_k(x).$$

(4.14)

Substituting this result in (4.13) gives

$$\left[ \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \right]_t + \left[ \sum_{k=1}^{\infty} b_k(t) \phi_k(x) \right] = \nu \left[ \sum_{k=1}^{\infty} a_k(t) \phi_k(x) \right]_{xx}.$$

(4.15)

The above equation is called the spectral formulation of the Burgers' equation model. This formulation is a proper formulation if spectral solution (4.15) converges to the Analytic 'true' solution (4.1) at all time and space. This important requirement of any spectral formulation is discussed in detail for the Stokes equation and the Burgers’ equation in Constantin and Foias (1989). There are many aspects to this requirement: (a) each term of the equation in spectral form should converge to the corresponding term in the original equation. Constantin and Foias (1989) show that the spectral formulation (4.15) satisfies this requirement. However the non-linear term is
shown to be weakly convergent resulting in a weak solution to the Burgers' equation using spectral methods. (b) The rates of convergence of each term need not be equal. Hence, the slowest converging term will dominate the error of approximation of the spectral solution to the original solution for a fixed number of basis elements $k$. For spectral model (fixed $k$) this is an important consideration as the accuracy will be determined by the truncation error of the least convergent term. (c) Constantine and Foias (1989) show that the rate of convergence, which is dominated by the non-linear term, is dependent on the basis chosen for the spectral formulation. This implies that for a given number of modes the basis set that is most convergent leads to the best spectral solution. As an example of this Spiegel (1985) cites a spectral model based on the inverse scattering transform (IST) (Ablowitz and Segur, 1981) which is an analogue to the Fourier transform for non-linear conservative systems.

In this study we use a sinusoidal basis and a modified Walsh basis to study the evolution of a shock wave. Multiplication of (4.15) with $\phi_k(x)$ and integration over the flow domain using the boundary conditions of $\phi(0) = \phi(1) = 0$ gives

$$\frac{d a_k}{dt} \int_D (\phi_k \cdot \phi_k) dx + b_k \int_D (\phi_k \cdot \phi_k) dx = -\nu a_k \int_D ((\phi_k)_x \cdot (\phi_k)_x) dx \quad (4.16)$$

In the case of the trigonometric basis set, the integrals in equation (4.16) will be equal as the derivative of the basis element also lies in the same basis.

A modified top hat basis, with finite gradient transition zones at the edges of the top hat, such as the one described in Chapter 3 can also be used in (4.16) although the derivative of the basis elements are not in the same basis. This will cause the integrals in (4.16) to be unequal. For the top hat case it can be easily shown that the derivatives are members of an
orthogonal set and the integrals can be computed readily.

4.3 Finite difference formulation

Equation (4.1) can also be solved using an implicit finite difference scheme. This computation is performed for the purpose of providing a comparative benchmark for the spectral solutions. The (finite difference solution) of the Burgers' equation is presented in Sod (1985) and is given by

$$\frac{u_i^{n+1} - u_i^n}{k} = \nu D_+ D_- (u_i^{n+1}) + u_i^n D_0 (u_i^{n+1})$$

(4.17)

with initial condition

$$u_i^0 = f(ih) \quad \text{for} \quad 0 \leq i \leq 1$$

and boundary conditions

$$u_0^n = u_N^n \quad \text{for} \quad 0 \leq nk \leq t$$

where $0 \leq n \leq N$, and $h = \Delta x$ is the spatial grid resolution and $k = \Delta t$ the time step. $D_+$, $D_-$, and $D_0$ denote the forward, backward and centered difference operators respectively. The proof of convergence and stability of the scheme is shown in chapter 2 of Sod (1985). This scheme is shown to be accurate to $O(k) + O(h^2)$ and has the limitation that $k$ has to be chosen to be $O(h^2)$ specially for small values of the parameter $\nu$. 
fig 4.1 Initial condition for the numerical solution

\[ u(x, 0) = 1 + \sin 4\pi(x - 0.125) \quad \text{for} \quad x \leq 0.5. \]
4.4 Comparison of results

It has been shown (for example in Cole, 1955) that for a given initial condition the nonlinear equation dissipates more energy than the linear equation. This is because of the increased conversion of energy from the larger scales by the nonlinear term to the smaller scales where dissipation is more effective. Moore-Miller (1987) carried out a numerical simulation of the Burgers' equation using the Hopf-Cole solution and the finite difference solution for the viscous and non-viscous cases and confirmed the increased dissipation. In our simulations we use the Hopf-Cole solution as a benchmark to compare our spectral models with each other and with the finite difference scheme.

The viscosity of the flow is arbitrarily chosen to be \( \nu = 0.009 \) and the initial condition for our simulations is a modified sin wave (fig. 4.1) given by

\[
\begin{align*}
  u(x,0) &= \begin{cases} 
  0.5(1 + \sin(2\pi(x - 0.25))) & \text{for } 0 \leq x \leq 0.5 \\
  0.0 & \text{for } 0.5 < x \leq 1.0
  \end{cases}
\end{align*}
\]  

(4.17)

The simulations are carried out using the finite difference solution and the spectral solution techniques. For the spectral solution, the trigonometric (sine, cosine) basis and the modified square pulse basis (Walsh basis) are used in separate tests. The Walsh basis (a detailed description is provided in Harmuth 1970) has been modified with finite gradient transition zones at the edges to allow for the required derivatives to be defined and allow for applications with finite resolution.
fig 4.2 Available basis functions for spectral Models.  
(a) Element of the Trigonometric basis set.  
(b) Element of the Walsh (‘top hat’) basis set.  
(c) Element of the Modified Walsh basis set.
Fig. (4.2) shows an example of the modified Walsh basis as used in our study. The Hopf-Cole solution is computed at the same time steps as the numerical solutions with the added condition that the summation in equation (4.10) is carried out until the desired level of precision is achieved so that the residual is less than 0.001. This is then considered a true representation of the solution at that time step. The Hopf-Cole solution is shown in fig. (4.3). The finite difference solution and the two spectral solutions are shown in fig. (4.4) along with the Hopf-Cole solution.

For low viscosity flows, the initial field evolves into a shock wave and then starts dissipating. As the viscosity of the flow is increased, the shock wave formation is suppressed and the wave dissipates. The trigonometric basis set becomes inefficient when the shock wave begins to form and a large number of modes need to be used to maintain the same accuracy. With a fixed number of basis elements, as used in a modelling situation there is artificial damping of the shock wave introduced due to the inability of the basis to adequately represent the true solution. The resulting shock wave is damped at a faster rate than the actual solution (fig 4.4). When the modified top hat basis set is used, the numerical convergence is better than that obtained with trigonometric basis, as the solution approaches a shock. This is true even though the representation of the smooth initial field is poor. The finite difference scheme needs very small time steps to mimic the true solution and still dissipates more energy than the spectral method. The resulting shock is weaker than the true solution. The computational time for the finite difference solution was large.
fig 4.3 The Hopf-Cole Solution at $t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.8$ and a composite of $t = 0.0$ to 1.0 in 0.1 steps
In summary, the truncated trigonometric basis set is inefficient in the modelling of a sharp flow feature, even in the highly simplified flow. However for high viscosity fluids it will perform very well as the viscous dissipation will dampen the magnitude of the shock. With the modelling of planetary scale waves where the flow features are smooth the trigonometric set will perform adequately, not warranting the added complications of using basis sets which do not have derivatives which are members of the same basis. As can be easily shown, the main advantage of the spectral method is that the nonlinear term in the spectral representation, conserves energy. However it does this at the expense of aliasing all of the energy that cannot be represented by the limited number of modes, back into the finite number of modes used in the model. In the (extreme) case of a one mode Burgers’ equation with trigonometric basis the solution will never form a shock for any value of viscosity.

Of course the problem with the top hat basis is that it performs poorly in a smooth flow. In the modelling of a high Reynolds number turbulent fluid where a large amount of the flow field consists of such sharp edges (Frisch 1986) the top hat basis may perform better than the trigonometric basis. Indeed our analysis of observed turbulence shows that the rate of decay of the coefficients of the top hat basis is significantly faster than that of the trigonometric basis for this application. Consequently, the choice of the basis set plays an important role in adequately representing the solution in finite mode spectral solution.
fig 4.4 Comparison of solutions to Burgers' equation.  
(a) Hopf-Cole solution.  
(b) Finite element solution.  
(c) Spectral (trigonometric) Solution  
(d) Spectral (Modified Walsh) Solution
5. THE WAVELET TRANSFORM USED IN TURBULENCE DETECTION

The wavelet transform method of analysis can be efficiently used to detect and highlight events of a particular nature found in a data field. As discussed in the Introduction, turbulence contains numerous concentrated shear zones which probably contribute significantly to the dissipation of turbulent kinetic energy and characterize the nature of the flow. As will be shown in this chapter, global basis functions such as those used in the Fourier Analysis Method, usually are not very efficient as a decomposition tool, and cannot perform well in a situation where there are numerous local events. Hence, we look at a more localized (both spatially and spectrally) analysis technique such as the Wavelet Transform.

5.1 The windowed Fourier transform

The wavelet transform can be introduced in terms of a generalized windowed Fourier transform. This was first done by J. Morlet to compensate for the inconveniences of the windowed Fourier transform, and still provide good localization in both the spatial and spectral domains. In this section we will first investigate the windowed Fourier transform and then, in the following sections, extend it to obtain the more general wavelet transform.

In the domain \([0, L]\), the Fourier series coefficients of a function \(f(x)\), denoted by \(C_k\) are given by

\[
C_k = \int_0^L f(x) e^{-2\pi ikx} \, dx
\]  

(5.1)

and the Fourier series decomposition (also called the expansion in Fourier modes) is given by,

\[
f(x) = \sum C_k e^{2\pi ikx}.
\]

(5.2)
The Pasevals’ Identity is given by

\[ \sum |C_k|^2 = \int_0^L f(x)^2 \, dx. \]  \hfill (5.3)

For future use we denote the Hilbert space by \( H \), which is a complete inner product space where the inner product is defined as

\[ <f, g> = \int_0^L f(x)g(x) \, dx. \]  \hfill (5.4)

The Hilbert space of square integrable functions on \([0, L]\) are denoted as \( L^2[0, L] \), where the \( L^2 \) norm is defined as

\[ ||f||_2 = \left( \int_0^L |f(x)|^2 \, dx \right)^{\frac{1}{2}}. \]  \hfill (5.5)

The Fourier transform can be also be viewed as an isometric\(^1\) one-to-one mapping from \( L^2[0, L] \) on to \( l^2[\mathbb{Z}] \). Here \( l^2[\mathbb{Z}] \) denotes the space of square integrable sequences indexed on \( \mathbb{Z} \), where \( \mathbb{Z} \) is the set of all integers in \([-\infty, \infty]\). Note that this definition encompasses all positive and negative integers while the usual definition of \( l^2 \) includes the positive integers only. In this context the function being transformed \( f(x) \) is an element in \( L^2[0, L] \) while the Fourier series coefficients \( C_k \) are in \( l^2[\mathbb{Z}] \).

Although the Fourier transform has some advantages such as being a standard of comparison and having the property that derivatives of the basis also remain in the basis, the \( L^p \) norms (specially for \( p > 2 \)) are not well represented by the Fourier coefficients (Young (1980), Chapter 4 has a complete discussion of this topic). Also, other smoothness measures such as Hölder continuity\(^2\) are not well obeyed (Young 1980, Grossman 1988) by

---

\(^1\) See Appendix A section A.1

\(^2\) See Appendix A section A.2
this decomposition. In addition, the basic problem in Fourier analysis is poor localization. Although the Fourier transform of a function, \( \hat{f}(\omega) \), might indicate the presence of irregularities in the original function \( f(x) \), through the high frequency content, there is no information as to the spatial localization of the irregularity. A small local change (in space) in \( f \) changes all Fourier coefficients, thus affecting the Fourier series decomposition of the entire record.

In an effort to localize the information content of the Fourier transform the windowed Fourier transform is defined. First defined as the Gabor transform (Gabor 1946), this process involves spatially windowing the function to be transformed \( f(x) \), (such as observations of a geophysical process) with a function \( g(x) \) before computing the Fourier transform. The function \( g(x) \), centered at \( x = 0 \), has compact support in \( x \) and its support is unchanged during the whole computation. The windowing function \( g(x) \) is then translated to a position \( x = b \) to give the (definition of the) windowed Fourier transform at frequency \( \omega_0 \) of a function \( f(x) \in L^2(\mathbb{R}) \) as,

\[
F_{\omega_0,b}f(x) = \int_{-\infty}^{\infty} e^{i\omega_0 x} g(x - b) f(x) dx.
\]

This can also be written in the form of an inner product as

\[
F_{\omega_0,b}f(x) = \langle f(x), g_{\omega_0,b}(x) \rangle,
\]

where \( g_{\omega_0,b}(x) = e^{i\omega_0 x} g(x - b) \) and the Fourier transform of \( g_{\omega_0,b}(x) \) is given as \( \hat{g}_{\omega_0,b}(\omega) \). The windowing function \( g(x) \) can be viewed as a filter process, and is mostly selected to be an even real function of \( x \) with a low frequency Fourier spectrum. This type of window acts as a low pass filter\(^3\) (Mallet 1988). The function \( g_{\omega_0,b} \) would have a band pass nature with the pass band

\(^3\) See Appendix A section A.3
fig 5.1 A typical window function and its spectra as used in the windowed Fourier transform. (a) A typical window function $g(x)$, (b) The function $g_{\omega_0,0}(x)$, (c) The function $g_{2\omega_0,0}(x)$ (a') The spectra of $g(x) = \hat{g}(\omega)$, (b') $\hat{g}_{\omega_0,0}(\omega)$, (c') $\hat{g}_{2\omega_0,0}(\omega)$
frequency interval the same as that of $g(x)$ but centered at $\pm \omega_0$ as seen in fig. (5.1). It is also seen in fig. 5.1(b') and 5.1(c') that changes in $\omega_0$ causes a linear shift of the pass band along the $\omega$ axis. If the window function $g(x)$ has a standard deviation of $\sigma_x$ in the spatial domain and $\sigma_\omega$ in the frequency domain (see appendix A.4, A.5 for definition), using the Parseval's relation (in the generalized form for two functions) and equation (5.7) we obtain:

$$F_{\omega_0, b} f(x) = \int_{-\infty}^{\infty} g_{\omega_0, b}(x) f(x) dx = \int_{-\infty}^{\infty} \hat{g}_{\omega_0, b}(\omega) \hat{f}(\omega) d\omega. \quad (5.8)$$

This equation shows that the windowed Fourier transform $F_{\omega_0, b} f(x)$ describes $f(x)$ around $(x = b)$ with a standard deviation of $\sigma_x$ in the spatial domain, and around $(\omega = \omega_0)$ with a standard deviation of $\sigma_\omega$ in the frequency domain. This implies that the spatial resolution in the $(\omega, b)$ phase space is fixed for a given window type $g(x)$ due to its fixed spatial support (fig. 5.1.a,b,c) and the frequency resolution in the phase space is fixed by the fixed support in the frequency domain (fig. 5.1.a',b',c'). This fixed resolution in the phase space dictates the minimum sampling rate in an implementation of the discrete windowed Fourier transform. The minimum sampling rate is sometimes called the Nyquist frequency and determines the minimum translation distance of the window $b_0$ and, the minimum increment in $\omega$ according to the relation $b_0 = 2\pi$ (Duabechies 1988). The resulting sampling of the $(\omega, b)$ phase space is shown in fig. 5.2(a).
fig 5.2 Resolution of the Windowed Fourier transform phase space.

(a) Minimum sampling of the phase space (b) Domain of influence of each coefficient
From the preceding discussion we see that the windowed Fourier transform provides the 'localization' in x space. However the fixed phase space resolution due to the fixed spatial support of the window and the linear shift in the $\omega_0$ space (fig. 5.2.b) remains a drawback in trying to characterize micro front type irregularities in observed geophysical data. The linear shift in the frequency pass band occurs due to the exponential term introduced by the Fourier transform and also makes the resolution in the $\omega$ axis of the phase space constant.

5.2 The Continuous Wavelet Transform Method

The wavelet transform is defined as the decomposition of a signal on a family of functions (called wavelets due to their 'local' nature) which are the translates ($x \rightarrow x - b$) and dilates ($x \rightarrow x/a$) of a single function $I_{a,b}(x)$. The function $I_{a,b}(x)$ is called the basic wavelet and the corresponding family of wavelets are obtained as $I_{a,b}(x); a, b \in \mathbb{R}, a \neq 0$. In the most general sense $I_{a,b}$ is defined as,

$$I_{a,b}(x) = I\left(\frac{x-b}{a}\right)$$

(5.9)

An example for a wavelet family is the Haar Basis: \{1, h, h_{-j,k}\} where $j \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots 2^j - 1\}$ for $N = \{0, 1, 2, \ldots\}$. A few elements of $h_{j,k}$ are shown in fig. (5.3) Here $h$ is given by,

$$h = \begin{cases} 
1 & 0 \leq x \leq \frac{1}{2} \\
-1 & \frac{1}{2} \leq x \leq 1 \\
0 & \text{elsewhere}
\end{cases}$$

(5.10)

and

$$h_{mn} = h(2^{-m}x - n)$$

(5.11)

The parameters $a$ and $b$ in equation (5.9) may be chosen to vary continuously on their range $\mathbb{R}^* \times \mathbb{R}$ (where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$) to produce the continuous wavelet...
Fig 5.3 Some elements of the Harr wavelet family. (a) $h_{0,0}$ (b) $h_{-1,0}$: a dilation of $h_{0,0}$ (c) $h_{-1,1}$: a translation of $h_{-1,0}$
transform of the function \( f(x) \in L^2(\mathbb{R}) \) as,

\[
Wf(a, b) = \langle f, I_{a,b}(x) \rangle = \int_{-\infty}^{\infty} p(a) f(x) I\left(\frac{x-b}{a}\right) \, dx
\]  

(5.12)

In most cases function \( p(a) \) is chosen so that \( Wf(a, b) \) is normalized to be unity, when the function \( f \) perfectly matches the wavelet \( I_{a,b} \) within its region of spatial support. For example in the case of the Haar basis \( p(a) = |a|^{-1} \).

If \( I \) satisfies the 'admissibility condition' (Daubechies 1988, Grossman et.al 1985, and Mallet 1988) given by

\[
\frac{1}{K} = \int_{0}^{\infty} \frac{|\hat{I}(\omega)|^2}{\omega} \, d\omega < +\infty
\]  

(5.13)

then \( W \) is ( as defined by 5.12 ) an isometry (up to a constant) from \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}^* \times \mathbb{R}) \) (See Mallet 1988 and Daubechies 1988), i.e. it maps the function from 1-D space to 2-D phase space where the x-axis is \( b \) and the y-axis is \( (a) \). The constant \( K \) will be used in the formulation of the inverse wavelet transform. The Fourier transform of \( I(x) \), defined as \( \hat{I}(\omega) \) is obtained as,

\[
\hat{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int I(x) e^{-ix\omega} \, dx.
\]  

(5.14)

The 'admissibility condition' implies that \( I \) has mean zero, i.e \( \int I(x) \, dx = 0 \) (or \( \hat{I}(0) = 0 \)) and that \( \hat{I}(\omega) \) is small in the neighborhood of \( \omega = 0 \). The decay requirements of \( I \) are called one of 'compact support' in \( x \). The form of equation (5.12) is that of a convolution of \( f \) and \( I \). Hence the wavelet transform at point \([b]\) and dilation \([a]\) can be viewed as filtering \( f \) with a band pass filter \( I \). This band pass filter is different from the traditional high-pass or low-pass types where the weights for a group of frequencies are unity and the rest of the frequencies are weighted to be zero to define a pass band.
fig 5.4 (a) The effect of dilations of wavelet (b) The Band-pass nature of the dilated wavelet spectrum
In this description of the wavelet band-pass filter the weights assigned to the filter are determined by the shape of the wavelet and correspond to the Fourier transform of the wavelet itself.

Since the Fourier transform of the dilated function $I(\frac{x}{a})$ is given by (transform of scaled function) $a\hat{I}(a\cdot\omega)$, the convolution of $I$ and $f$ more specifically correspond to band-pass filtering of $f$ with frequency band of constant width on a logarithmic frequency scale (fig. 5.4). As shown in Mallet (1988) the resolution of the wavelet transform varies with the scale parameter.

If $\sigma_x$ and $\sigma_\omega$ are the respective standard deviations of the wavelet $I(x)$ in the spatial and frequency domains, Mallet (1988) shows that the wavelet $I(\frac{x-b}{a})$ has energy concentrated around $(\omega, b)$ with standard deviation $a\sigma_x$ in the spatial domain and standard deviation $\frac{a\sigma_\omega}{a}$ in the spectral domain. This shows that, depending on the dilation parameter $a$ the shape of the resolution cell varies on the phase plot as shown in fig. 5.5 (b). This in turn shows that when the wavelet is dilated the spatial resolution is coarse and the frequency resolution is fine and when the wavelet function is concentrated (i.e $a$ is small) in space, the frequency domain resolution is coarse. This is the main feature of the wavelet technique that allows it to zoom in and characterize local irregularities. If a local irregularity such as a sharp edge is considered it has small spatial support and large frequency support. This type of cell (in the phase space of frequency vs. space) is well resolved in the wavelet phase plane. However as shown in fig. 5.2 (a) the windowed Fourier transform maintains its resolution cell size in $(\omega, b)$ phase space and thus is unable to adjust the shape of the phase space cell to act in the same manner.

---

4 See Appendix A section A.4
5 See Appendix A section A.5
5.3 The discrete wavelet transform

In applications of signal analysis (analysis of observed data) the dilation and translation parameters, \((a, b)\) may have to be chosen to be in a discrete lattice due to the finite rate sampling of data. This discrete form of the wavelet transform should be a complete representation of the data. For example if the translation parameter is chosen to be large, certain observations may be skipped over and will not enter into the computation. This would result in an incomplete representation as these skipped observations could assume arbitrary values without affecting the wavelet transform. In order to build a complete representation one must sample the phase space with adequate resolution (Mallet 1988) to maintain the isometry property of the wavelet transform. As shown in fig. 5.5(a) the adequate sampling of the phase space entails complete coverage of the phase space by the chosen set of translations and dilations. In the case where one chooses \(a_0 > 1\) and \(b_0 \neq 0\) the family of wavelets of interest become, for \(m, n \in \mathbb{Z}\),

\[
I_{m,n}(x) = hI(a_0^{-m}x - nb_0)
\]  

(5.16)

where the choice of \(a = a_0^{-m}\) and \(b = nb_0a_0^{-m}\) has been used for the dilations and translations. The discrete wavelet transform associated with the discrete wavelet family map functions \(f \in L^2(\mathbb{R})\) into \(l^2(\mathbb{Z}^2)\). and is defined by

\[
W_d f(m,n) = \langle f, I_{m,n} \rangle.
\]  

(5.17)

This could also be written as

\[
W_d f(m,n) = \int_{-\infty}^{\infty} f(x) I(a_0^{-m}x - nb_0) \, dx
\]  

(5.18)
fig 5.5 Resolution of the Wavelet Transform phase space. (a) Minimum sampling of the phase space (b) Domain of influence of each coefficient
In order to reconstruct a function from its discrete wavelet transform (similar to the process of computing the inverse Fourier transform) the $W_d$ operator must be invertible on its range and must have a bounded inverse. Daubechies (1988) shows that if there exists an inverse then the set $\{h_{mn}; m, n \in \mathbb{Z}\}$ is a frame. The inverse transform can be defined for a tight wavelet frame as

$$f(x) = K \sum_{m} \sum_{n} W_d f(m,n) h(a_0^{-m}x - nb_0)$$  \hspace{1cm} (5.19)$$

Here $K$ is a constant scale factor as defined in (5.13).

The set of dilations $[ma_0]$ and translations $[nb_0]$ used in the construction of a wavelet frame can be highly redundant. The minimum set of $[ma_0, nb_0]$ required to have an invertible wavelet transform was shown in fig. (5.5.a). The dilations and translations can independently be over-sampled (i.e, certain data points influence more than one coefficient of the wavelet transform) and results in phase plots as shown in fig. (5.6). When the frame is redundant the wavelet coefficients in a neighborhood are correlated to each other. The degree of redundancy is proportional to the correlation distance of the coefficients. If the region of influence of the coefficients (as shown in fig. 5.6) do not overlap the coefficients are uncorrelated and the transform is orthogonal. When $[a_0, b_0]$ are chosen close to $[1, 0]$ respectively the resulting frame is very redundant and close to the continuous family of wavelets. The phase space created by a Fourier transform (where each observation influences all the coefficients) would contain a vertically aligned set of points each influenced by horizontal bands of constant width. Duabechies (1988) shows that over-sampling of the phase space helps in the reduction of computational noise in the computation of an inverse wavelet transform (also called image reconstruction in some applications).

\footnote{See Appendix A section A.6}
fig 5.6 Over-sampling of the phase plane (a) translation over sampled (redundant) phase plane (b) dilation over sampled (redundant) phase plane
5.4 Special Formulations of the wavelet transform method

The wavelet transform method as described in sections 5.2 and 5.3 (above) can be extended using special formulations to better suit turbulence time series analysis. These formulations will enable us to link spectra, autocorrelation function and structure functions to wavelet transforms and show that the wavelet transform encompasses all these forms of analysis. It also shows that the other analysis techniques are sub classes of the wavelet transform. The formulations can be broadly classified as

(a) Selection of different wavelets.

(b) Selection of special translation and dilation parameters.

(c) Further transformations of the wavelet phase plane.

5.4.1 Selection of different wavelets

The wavelet transform method was shown to have the characteristics of a band-pass filter where the filter weights are determined by the Fourier transform of the wavelet. The choice of the wavelet will thus affect the attributes of the wavelet phase plane. While certain wavelet would yield a bounded domain for the phase plane (and hence a tight frame) others will have an infinite domain, all for the same data set. Here we will describe 4 wavelets, and their special properties. These are:

i. The trigonometric wavelet: This wavelet (fig. 5.7.a) when extended periodically, gives the global basis set used by the Fourier transform and the windowed Fourier transform. It also admits all derivatives. However this implies that the wavelet is very smooth and may not be an efficient wavelet for the transformation of turbulence data.

ii. The ramp wavelet: This wavelet is as shown in fig. 5.7(b) and has been chosen because of its likeness to observed turbulence phenomena such as thermals influenced by wind shear (see Introduction and references
therein). With a mean shear, a strong gradient develops on the convergent upstream side of the draft with diffuse gradients occupying the other areas of it.

iii. *The step (Haar) wavelet*: This wavelet is as shown in fig. 5.7(c) and has been chosen because of its wide spread use in edge detection type situations (Marr and Hildreth 1980, Grossman and Morlet 1984). It is also thought to have special relevance in geophysical data analysis where the edges of coherent structures are being sought. The wavelet transform formulated with this wavelet will be used extensively in the following chapters.

iv. *The two-unit-impulse wavelet*: This wavelet is as shown in fig. 5.7(d) and has been chosen because of its transforms similarity to the mathematical operation done in the computation of moments such as the autocorrelation function and the structure function. These be discussed in more detail as further transformations of wavelet phase space.

5.4.2 Selection of special translation and dilation parameters

As discussed in section 5.3, the selection of the translation and dilation parameters play an important role in (i) the stability of the wavelet coefficients, (ii) the formulation of a tight frame and, (iii) the invertibility of the phase space. In practice it is not always possible to obtain a TIGHT\(^7\) frame in a finite number of translations and dilations (Daubechies 1989). The wavelets cannot be chosen so as to have a tight frame for all types of data. For example, while a top hat wavelet provides a tight frame for a data set containing a single top hat, it will only be a sung frame if the data contained a single sin pulse.

\(^7\) See Appendix A section A.7
fig 5.7 Some special wavelets: (a) The trigonometric wavelet (b) The ramp wavelet (c) The step (Haar) wavelet (d) The Two-unit-impulse wavelet
Daubechies (1989) explains that when a suitable wavelet has been chosen from considerations such as the nature of the expected coherent structures, the resulting frame might be a SNUG\textsuperscript{8} frame. In this type of frame the inversion of the truncated finite sized phase space results in a reconstruction of the original function to within finite error.

Morlet in his early papers (1982, 1985) observed that a reduction in computational noise and an increase in reconstruction accuracy occurs when the phase space is over sampled (i.e. over-lapping non-orthogonal transformation as shown in fig.5.6) in the discretization process. Daubechies(1989) shows that a doubling of the sampling rate i.e smaller $\omega_0$ and smaller $b$, results in a near doubling of the accuracy of the representation in phase space. We also show that the choice of certain parameters for translation and dilation combined with special wavelets produce statistics (and decompositions) such as spectra, lagged autocorrelations and higher moments. A few examples are given below:

(a) Fourier coefficients: The Fourier transform uses $e^{i\omega x}$ as the basis set. Hence, let the wavelet be defined as

$$h(x) = \sin \left( \frac{2\pi (x - b)}{a} \right)$$

(5.20)

where the dilations are represented by $a$ and the translation is defined by $b$. Now select the special translation parameters as $b = mb_0, m = 1, 2, 3...$ and $b_0 = a/4$. With this special formulation the Fourier series coefficients are obtained as,

$$a_m = \sum_{n_1} W_d f(m, n_1) - \sum_{n_2} W_d f(m, n_2)$$

(5.21.a)

\textsuperscript{8} See Appendix A section A.8
where \( n_1 = 1 + 4 \cdot n \) and \( n_2 = 3 + 4 \cdot n \).

\[
b_m = \sum_{n_3} W_d f(m, n_1) - \sum_{n_4} W_d f(m, n_2)
\]

(5.21.b)

where \( n_3 = 2 + 4 \cdot n \) and \( n_4 = 4 + 4 \cdot n \).

\( n = 0, 1, 2, 3, 4, \ldots \)

\( W_d \) is the wavelet coefficient computed by the discrete wavelet transform of \( f(x) \). The special choice of \( b_0 = a/4 \) is selected to get the phase translation required in a Fourier decomposition. Once the Fourier series coefficients are known the spectrum can be obtained. Phase information can be extracted from the wavelet phase plane for the special formulation discussed above. In the case of the sinusoidal wavelet the phase angle at frequency \( m \) is \( \phi = \tan^{-1}(a_m/b_m) \). In other wavelets where a phase shift of the wavelet defines another member of the basis (for example the Radmacher basis or Haar basis) similar information can be obtained. Mallet (1989) shows that phase information can be computed for these cases if the wavelet is defined as a complex valued function. It should however be noted that phase information is related to the shape of the wavelet. In general a phase angle of \( \pi \) should be defined as a half width of the wavelet. Many admissible wavelets could exist where such a definition of phase is meaningless due to the wavelet being non-symmetric.

(b) Autocorrelation function: The Autocorrelation function of \( f(x) \), a record of length \( L \), is defined as

\[
R_{xx}(r) = 1/L \int_{-L}^{L} (f(x + r) - f(x))dx.
\]

(5.22)

If the special wavelet shown in fig. 5.7(d) is used for the wavelet transformation with the translation selected to be to every point in the record,
the resulting wavelet coefficient $W_d$, at a dilation of $r$ at spatial location $x$ in the phase plane, produces the difference in $f$ over the dilation distance. Hence, the average of the wavelet coefficients for a given dilation $r$ corresponds to the autocorrelation function at separation $r$. As discussed in chapter 3 the autocorrelation function and the spectrum form a transform pair and once either of them have been computed the other can be obtained.

5.4.3 Further transformations of the wavelet phase plane

A disadvantage of the wavelet transform method is the excessive amount of information that is contained in the wavelet phase plane coefficients. Hence it is useful to investigate possible transformations to the wavelet phase plane to collapse the information into more manageable statistics. The computation of variance and other moments of the wavelet coefficients for fixed spatial regions or fixed dilations can provide useful insight into the predominant scales of the coherent structures in the analyzed field in a localized region of localized frequency band. The process of further transforming the wavelet phase plane does lead to loss of some information. While the data and the wavelet phase plane are uniquely related (i.e. the phase plane can be inverted to produce the data) transformed phase plane statistics such as autocorrelations, spectra, structure functions etc., are non unique. However the information that these provide can be much more useful in understanding the underlying generating processes of the data.

The two special formulations described in section 5.4.2 are both transformations to the wavelet phase plane. They transform the 2-dimensional phase plane into 1-dimensional maps. The structure function of $f(x)$, a record of length $L$, is defined as
\[ D_2(r) = \frac{1}{L} \int_L (f(x + r) - f(x))^2 dx. \] 

(5.23)

If the special wavelet shown in fig. 5.7(d) is used for the wavelet transformation and the sum of the square of the coefficients \( W_d \), corresponding to a given dilation are computed the structure function at that dilation distance is obtained. Higher powers of \( W_d \) summed give corresponding higher moment.

In general, the wavelet variance defined as

\[ W^2(n) = \frac{1}{N} \sum_m [W_d f(m, n)]^2 \] 

(5.24)

can be computed and will provide information along the dilation axis. The structure function described above is exactly the wavelet variance with the special wavelet in fig (5.7.d). The wavelet variance will be used as the main analysis tool in the data analysis later in this thesis. The behavior of the variance for special wavelet and for certain types of data will be presented in the next chapter.
6. THEORETICAL STUDY OF SIMULATED DATA USING SPECTRA AND WAVELET TRANSFORMS

In this chapter, we will apply the wavelet transforms presented in chapter 5 to simulated data sets and study the behavior of the transforms. Our main goal is to test the ability of these transforms to identify characteristics of the coherent structures in the record. Once the characteristic behavior of the various transformations are known, we will apply the transformations to observed data and try to predict the underlying generating process, in later chapters.

The artificial record investigated here consists of localized events (such as top-hat functions) placed randomly in a record (see figure 6.1). These will be referred to as the building blocks of the record. The events are selected to be of width $\tau$ and placed a distance of $[T_0 + \zeta]$ apart. $T_0$ is fixed while $\zeta$ is a random number having a characteristic function of $\chi(\omega)$. For certain simulations, random noise $\eta(t)$ is also added to the record. The random displacement of events $\zeta$ and the random additive noise $\eta(t)$ are considered to be uncorrelated. The parameters $\tau, T_0, \zeta, \eta$ are independently varied to obtain different simulations. For most simulations the events are chosen to be top-hats. This is motivated by observations of top-hat like plumes or thermals (see Introduction for a comprehensive list). For some simulations ramp function events have been chosen to better emulate the observed structures in the presence of significant shear. For completeness sin building blocks will also be used in some simulations as a representation for smooth events.
6.1 Detailed description of the simulated data sets

A simulated record \( f(t) \) is constructed as follows (fig 6.1):

(i) Localized events such as top-hats are added to a blank field. At this stage the record is a linear combination of local building blocks.

(ii) The location of the \( k \)th top-hat in the record, is given by \((kT_0 + \zeta_k)\). Its magnitude is \(C_k\). The equation of the \( k \)th top hat can be obtained by translation of the top hat function \( h(t) \) from \( t = 0 \) to \( t = kT_0 + \zeta_k \). Thus the equation of the \( k \)th top hat is \( C_k h(t - kT_0 - \zeta_k) \). Here, \( \zeta_k \) is called the random displacement of the \( k \)th event from location \( kT_0 \).

(iii) At each instant an uncorrelated noise of \( \eta(t) \) is added.

(iv) To understand the meaning of the Fourier and wavelet transforms different analytical records are systematically generated. The parameter \( T_0 \) (referred to as the spacing scale or spacing size) can be changed for a given \( \tau \) (the event scale or event size). In a given simulation \( \zeta \) and \( \eta \) can be chosen to be on or off and their distributions can be changed.

Using the above mentioned properties the expression for the artificial record consisting of \( n + 1 \) events can be written as

\[
f(t) = \sum_{k=0}^{n} C_k h(t - kT_0 - \zeta_k) + \eta(t).
\] (6.1)

When the amplitude of events \( C_k \) is constant and the random displacement \( \zeta \), random noise \( \eta \) are set to zero the resulting record will be periodic with period \( T_0 \). Additionally when \( T_0 = 2\tau \) the record is completely described by a single scale (i.e. spacing and event width are same) periodic square pulse train. For the record to be composed of events, \( T_0 > \tau \) is a necessary condition.
fig 6.1 Artificial record of randomly placed top-hats. The random displacement from periodic midpoints is $\zeta$ and the periodic spacing is $T_0$.

Spectrum of $f(t)$

fig 6.2 Spectrum of top-hat function $h(t)$. The width of the top-hat is $\tau$ and the amplitude $C_k = 1$. 
6.2 Properties of the spectra of simulated data

The linearity and translation properties of the Fourier Transform can be used in the computation of the spectra of the simulated data $f(t)$. If $A(t) \Leftrightarrow A(\omega)$ and $B(t) \Leftrightarrow B(\omega)$ are Fourier transform pairs, then,

- **Linearity** - $A(t) + B(t) \Leftrightarrow A(\omega) + B(\omega)$.
- **Translation** - $A(t + \tau) \Leftrightarrow A(\omega) \cdot e^{i\omega \tau}$.

Using the definition of the Fourier transform (Chapter 3 equation 3.2), we can obtain an expression for the Fourier transform of $f(t)$ as,

$$f(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$ (6.2)

$$= \int_{-\infty}^{+\infty} \left( \sum_{k=0}^{n} C_k h(t - kT_0 - \zeta_k) + \eta(t) \right) e^{-i\omega t} dt.$$ (6.3)

$$= \int_{-\infty}^{+\infty} \sum_{k=0}^{n} C_k h(t - kT_0 - \zeta_k) e^{-i\omega t} dt + \int_{-\infty}^{+\infty} \eta(t) e^{-i\omega t} dt.$$ (6.4)

Since the additive noise $\eta$ is just a linear addition to the Fourier transform (form the linearity property) its effect depends solely on the type of noise and its Fourier transform. Therefore, for simpler bookkeeping, we will temporarily set the additive noise to zero. For simplicity, we will also fix the amplitude of the pulses at a constant value of unity. Using the linearity property given above, the Fourier transform of the summation in equation (6.4) can be written as a sum of the Fourier transforms to give

$$f(\omega) = \sum_{k=0}^{n} \int_{-\infty}^{+\infty} h(t - kT_0 - \zeta_k) e^{-i\omega t} dt.$$ (6.5)

Since the Fourier transform of an event $h(t)$ is given by,

$$h(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-i\omega t} dt$$ (6.6)
substituting (6.6) into (6.5) and using the translation properties of Fourier transforms we obtain,

\[ f(\omega) = \sum_{k=0}^{n} h(\omega) e^{-i\omega(kT_0 + \zeta_k)} \]  

(6.7)

Since \( h(\omega) \) is independent of \( k \), (6.7) becomes

\[ f(\omega) = h(\omega) \sum_{k=0}^{n} e^{-i\omega(kT_0 + \zeta_k)} . \]  

(6.8)

For an infinite length record the spectrum is \(|f(\omega)|^2\). The spectrum of a finite length random function can be computed from its Fourier transform following the method used in Kharkevich (1960) and Stark and Woods (1986). If \( f(\omega) \) is the Fourier transform of the random function \( f(t) \) then,

\[ |f(\omega)|^2 = \left( \int_{0}^{T} f(t) e^{-i\omega t} dt \right) \left( \int_{0}^{T} f(t) e^{-i\omega t} dt \right)^* \]

\[ = \int_{0}^{T} \int_{0}^{T} f(t_1) f(t_2)^* e^{-i\omega(t_1-t_2)} dt_1 dt_2 \]

where the \([*]\) denotes the complex conjugate. Dividing both sides of the above equation by the record length \( T \) and applying the expectation operator we obtain,

\[ \frac{1}{T} E \left[ |f(\omega)|^2 \right] = \int_{0}^{T} \int_{0}^{T} R_{xx}(t_1 - t_2) e^{-i\omega(t_1-t_2)} dt_1 dt_2 \]

\[ = \int_{-T}^{T} \left[ 1 - \frac{\tau}{T} \right] R_{xx}(\tau) e^{-i\omega(\tau)} d\tau \]

where \( R_{xx} \) denotes the autocorrelation of \( f(t) \) as defined in chapter 3.

Since the autocorrelation function and the spectrum are a Fourier transform pair, using equation (3.2) to transform the integral and considering the limiting case of \( T \rightarrow \infty \) we obtain the final result as

\[ F(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} E \left[ |f(\omega)|^2 \right] \]  

(6.9)
where $F(\omega)$ is the spectrum. The spectrum is a real, non-negative function. Now, substituting from the expression for $f(\omega)$ obtained in equation (6.7), we can compute $|f(\omega)|^2$ as

$$|f(\omega)|^2 = f(\omega)f(\omega)^*$$

$$= \left( h(\omega) \sum_{k=0}^{n} e^{-i\omega(kT_0 + \zeta_k)} \right) \left( h(\omega) \sum_{l=0}^{n} e^{-i\omega(lT_0 + \zeta_l)} \right)^*$$

$$= \left( |h(\omega)|^2 \sum_{k=0}^{n} \sum_{l=0}^{n} e^{-i\omega(kT_0 + \zeta_k)} e^{-i\omega(lT_0 + \zeta_l)} \right)$$

$$= \left( |h(\omega)|^2 \sum_{k=0}^{n} \sum_{l=0}^{n} e^{-i\omega T_0(k-l)} e^{-i\omega \zeta_k} e^{i\omega \zeta_l} \right)$$

Substituting (6.10) into (6.9) the spectrum of the function can be obtained as Kharkevich(1960),

$$F(\omega) = \frac{1}{T_0} |h(\omega)|^2 \left( 1 - |\chi(\omega)|^2 + |\chi(\omega)|^2 \sum_{m=-\infty}^{\infty} e^{-i\omega mT_0} \right)$$

where the function $\chi(\omega)$ is the characteristic distribution function of $\zeta$ defined as,

$$\chi(\omega) = E \left[ e^{-i\omega \zeta} \right]$$

Many steps are needed to proceed from equations (6.9) and (6.10) to (6.11) including a variable transformation of $(m = k - l)$. The summation term on the right side of (6.11) can be written as

$$\sum_{m=-\infty}^{\infty} e^{-i\omega mT_0} = 2\pi \left( \sum_{m=0}^{\infty} \delta(\omega - m\omega_0) \right)$$

where the $\delta(\omega)$ denotes the unit impulse function centered at zero and $\omega_0 = 2\pi/T_0$ is the frequency corresponding to the length scale of the periodic spacing of events. This series of unit impulse functions in (6.13) is a discrete line spectrum.
The spectrum (6.11) contains information on two distinct scales given by $\tau$ and $T_0$. The event width scale $\tau$ is embedded in $h(\omega)$ while the spacing scale $T_0$ is embedded in the discrete summation term. Equation (6.11) can be split into

(i) a continuous part: 
$$\frac{1}{T_0}|h(\omega)|^2 \left(1 - |\chi(\omega)|^2\right)$$

and

(ii) discrete part: 
$$\frac{1}{T_0}|h(\omega)|^2 \left(|\chi(\omega)|^2 \sum_{m=-\infty}^{\infty} e^{-i\omega m T_0}\right).$$

The characteristic function $\chi(\omega)$ of the distribution of the random displacement $\zeta$ affects the amplitude of both the continuous and the discrete part. If the random displacement is zero, corresponding to purely periodic events, then $|\chi(\omega)|^2 = 1$ and the spectrum of $F(\omega)$ is purely discrete. With nonzero random displacement $|\chi(\omega)|^2$ becomes smaller than unity and part of the spectrum $F(\omega)$ becomes continuous. Since geophysical and most other real time series are never purely periodic, actual spectra always contain a
continuous part. Most characteristic functions never reach zero, and hence 
$F(\omega)$ always contains a discrete part.

(i) The continuous portion is a product of two functions:

i.(a) The continuous function $|h(\omega)|^2$, is the spectrum of a single event 
and is shown in fig (6.2) for a single top-hat event. This spectrum 
has zero amplitudes at frequency intervals of $\frac{2\pi}{r}$, which is the 
frequency corresponding to the event width.

i.(b) The function $|1 - \chi(\omega)|^2$ acts to scale the amplitude of the spec-
trum of a single event described in (a) above. Here $\chi(\omega)$ is the 
characteristic function of the random displacement $\zeta$. For most sit-
uations, where the random displacement is small when compared 
to the event width, this scaling function is small compared to unity 
thus making the continuous part small.

(ii) The discrete portion is a product of three functions:

ii.(a) The spectrum of a single pulse $|h(\omega)|^2$.

ii.(b) The square of the characteristic function, $|\chi(\omega)|^2$, which decreases 
to less than unity for increasing random displacement and results 
in a smaller contribution by the discrete part of the spectrum $F(\omega)$.

ii.(c) The discrete function $\sum_{m=-\infty}^{\infty} e^{-i\omega m \varepsilon}$, which ‘samples’ the continu-
os functions at discrete points spaced at the multiple of the fund-
damental frequency $2\pi/T_0$. Fig (6.3) shows this ‘sampling effect’ 
graphically.
fig 6.4 Discrete portion of Spectrum of randomly placed top-hats. \( T_0 \) is the periodic spacing and \( \tau \) is the top hat width. (a) Periodic function with \( T_0 = 2\tau \). (b) Periodic function with \( T_0 = 2\tau \). (c) Periodic with small (10%) random displacement. (d) Periodic with small(50%) random displacement.
Therefore the discrete portion of the spectrum is the scaled and discretized (sampled) spectrum of a single pulse. Since the spectrum of a single pulse is large at low frequencies and decreases to zero at the frequency $2\pi/\tau$, the first sample due to discretization which is at $\omega_0 = 2\pi/T_0$ will be the largest. Thus the peak in the discrete spectrum will occur at $\omega_0$ which corresponds to the spacing of events. Therefore the spectral peak emphasizes the event spacing and not the event size.

Consequently for the purely periodic case, $\zeta = 0$, and the continuous part vanishes and discrete line spectrum becomes the spectra of a single event sampled at frequency multiples of the fundamental spacing frequency. Fig (6.4.a) shows the discrete spectrum for a top hat event simulation. It should be noted that although the spectrum for this case is discrete, the convention in the literature has been to draw a continuous line through the discrete points.

For the case of significant random displacement, $\zeta \neq 0$, and the continuous part becomes important. This spectrum is determined by the spectrum of a single event. For the case of a top hat, the spectrum will have zero amplitude at frequency multiples of the event size $\omega = \frac{2\pi}{\tau}$. Since the continuous part has a zero at $\frac{2\pi}{\tau}$ and the discrete part a peak at frequency $2\pi/T_0$ the total spectrum will still peak at frequency of $2\pi/T_0$. However the continuous contribution will broaden the peak of the total spectrum due to the large low frequency contribution. The frequency $\frac{2\pi}{\tau}$ will be a zero in the total spectrum. Fig. (6.4.b) shows a typical spectrum for this type of simulation. As shown if fig (6.5) the location of the zero in the continuous spectrum will depend on the building block being used in the record. This poses a problem in the use of spectral methods in the detection of the event widths as the event shape is unknown for all observed data. Furthermore the common
fig 6.5 Spectra of 'events' (a) Event: top-hat, spectral minima: $\tau$ (b) Event: sine pulse, spectral minima: $1.5\tau$ (c) Event: ramp, spectral minima: $2\tau$
practice is to infer the width of the main structures from the peak in the spectrum.

Finally for the case when the random white noise $\eta(t) \neq 0$, the effect is to add a constant value to the whole spectrum. This will not change the location of the peaks in the spectrum though it converts the zero's of the spectrum to relative minima and will hamper the ability to define the event width scale.

In conclusion, the spectral analysis of the above simulated data indicates that the spectra identifies the spacing scale of events in the form of a peak in the spectrum. While this 'spacing peak' is the strongest one lesser peaks occur associated with the modulation role of the spectrum of a single event of the record building block, which is a periodic decaying function. The relative minima in the spectrum roughly corresponds to the event width of the building blocks depending on the type of building block. However in the presence of random noise these relative minima do not stand out in the spectrum and is hard to detect for observed data. The random displacement of the events cause a smoothing and broadening of the spectral peak corresponding to the event spacing.
6.3 The wavelet transform of simulated data

In this section we will apply the wavelet transform method developed in Chapter 5 to the simulated data sets. The wavelet transform allows for the choice of the basis (or indicator) function \( I_{a,b}(t) \) such that the desired features of the record can be emphasized. Since the Fourier basis functions are smooth we anticipate that the wavelet transform may be most useful as complementary to the Fourier spectrum for the detection of sharp edges. Indeed this is compatible with the physical goal of identifying microfronts in turbulent flows, as discussed in the introduction.

As shown in (5.12) the wavelet transform \( W_d \) of function \( f(t) \) is defined as

\[
W_d(a, b) = \frac{1}{a} \int_{-\infty}^{\infty} f(t)I_{a,b}(t)dt
\]  

(6.12)

where \([a, b]\) are the translation and dilation as discussed in section (5.2). This study evaluates the performance of the step function (Haar function)

\[
I_{a,b}(t) = \begin{cases} 
-1, & \text{if } b-a/2 < t \leq b; \\
+1, & \text{if } b < t \leq b + a/2. 
\end{cases}
\]  

(6.13)

for detecting the predominant scales associated with sharp gradients. A full wave of a sine is also used as an indicator function in some simulations for the purpose of comparison with spectra. Since \( I_{a,b} \) has compact support in \( t \), the limits of the integral in (12) can now be changed to \([b-a/2, b+a/2]\) to give

\[
W_d(a, b) = \frac{1}{a} \int_{b-a/2}^{b+a/2} f(t)I_{a,b}(t)dt.
\]  

(6.14)

For the artificial record of a linear combination of top-hat functions (6.1), equation (6.14) can be written as,

\[
W_d(a, b) = \frac{1}{a} \int_{b-a/2}^{b+a/2} \sum_{k=0}^{n} h(t - kT_0 - \zeta_k)I_{a,b}(t)dt.
\]  

(6.15)
For a single top-hat of width $\tau$ (figure 6.6) the wavelet transform at some translation $b = x$, and dilation $a$ is can be written as,

$$W_d(a, x) = \frac{1}{a} \int_{x-a/2}^{x+a/2} h(t) I_{a,x}(t) dt$$  \hspace{1cm} (6.16)$$

Since $h(t)$ is symmetric [$h(t) = h(-t)$] and, for the Haar basis $I(t)$ is skew symmetric [$I(t) = -I(-t)$], the wavelet transform is skew symmetric. Hence computation of the transform for $x < 0$ is sufficient. If the half-width of the wavelet basis function $I_{a,b}(t)$ is larger than the width of the top-hat ($a/2 \geq \tau$), equation (6.16) for $x < 0$ becomes,

$$W_d(a, x) = \begin{cases} 
0; & x < -(\frac{a}{2} + \frac{\tau}{2}) \\
\frac{1}{a} (\frac{a}{2} + \frac{\tau}{2} + x); & -(\frac{a}{2} + \frac{\tau}{2}) \leq x \leq -(\frac{a}{2} - \frac{\tau}{2}) \\
\frac{1}{a} (\frac{a}{2} - \frac{\tau}{2} - x); & -(\frac{a}{2} - \frac{\tau}{2}) \leq x \leq 0; \\
\frac{1}{a} (-2x); & -\frac{\tau}{2} \leq x \leq 0
\end{cases}$$ \hspace{1cm} (6.17)$$

For the case where the half-width of the wavelet basis is smaller than the width of the top hat, ($a/2 < \tau$), the wavelet transform for $x < 0$ becomes,

$$W_d(a, x) = \begin{cases} 
0; & x < -(\frac{a}{2} + \frac{\tau}{2}) \\
\frac{1}{a} (\frac{a}{2} + \frac{\tau}{2} + x); & -(\frac{a}{2} + \frac{\tau}{2}) \leq x \leq -(\frac{a}{2} - \frac{\tau}{2}) \\
\frac{1}{a} (\frac{a}{2} - \frac{\tau}{2} - x); & -(\frac{a}{2} - \frac{\tau}{2}) \leq x \leq -(\frac{a}{2} - \frac{\tau}{2}) \\
\frac{1}{a} (-2x); & -\frac{\tau}{2} \leq x \leq 0
\end{cases}$$ \hspace{1cm} (6.18)$$
fig 6.7 Wavelet transform of single top hat record for

(a) \( a/2 > \tau \)  
(b) \( a/2 = \tau \)  
(c) \( \tau/2 \leq a/2 < \tau \) and (d) \( a/2 < \tau/2 \)
The wavelet transform (for a few dilations) is shown in figure (6.7) as a function of the translation. It is seen that the wavelet transform for basis width less than the width of the event (fig 6.7.d), isolates the edges of the event. When the basis width increases beyond the event width (fig 6.7.a), the response is smoothed, but still peaks at the event width. It should also be noted that the transform separates the positive as well as the negative edges of the events as positive and negative peaks. As discussed in the previous chapter the wavelet transform is a mapping from 1-dimensional space (space or time) into a 2-dimensional space (translation and dilation), and hence provides a large amount of localized information. However this large information content turns out to be a disadvantage for the purpose of identifying the statistical event width of the record.

We therefore compute the wavelet variance for the record as defined in Section (5.4) as

\[ W^2(a) = \frac{1}{L} \int_{b} [W_d(a, x)]^2 dx \]  \hspace{1cm} (6.19)

or, in discrete form,

\[ W^2(a) = \frac{1}{N} \sum_{N} [W_d(a, x)]^2 \]  \hspace{1cm} (6.20)

where \( N \) is the number of points in the record at which the wavelet coefficients were computed at dilation \( a \). In general, \( N \) is a function of \( a \), as the translation \( (b) \) is determined by the dilation \( (a) \).

In order to determine if the wavelet variance can determine the width of the events, we find the value of \( [a] \) for which the wavelet variance is a maximum, i.e., find \( [a] \) such that,

\[ \frac{d}{da} (W^2(a)) = 0. \]  \hspace{1cm} (6.21)
For the wavelet coefficients obtained from equation (6.16), using equation (6.19), we obtain an expression for $W^2(a)$ as,

$$W^2(a) = \begin{cases} \frac{1}{\pi}(3\frac{\pi}{a}(1 - \frac{\pi}{a})); & (\frac{\pi}{2}) \geq -\tau \\
\frac{1}{\pi}(\frac{\pi}{2\tau} - [1 - \frac{\pi}{a}]^3); & (\frac{\pi}{2}) \leq -\tau \end{cases}$$  \hspace{1cm} (6.22)

Applying (6.21) to (6.22) we obtain the dilation corresponding to the maximum in the wavelet variance as $a/2 = \tau$. Fig. (6.8.a) shows a plot of $W^2(a)$ vs. $[a/\tau]$ for the case of a single top-hat given above. The non zero values of variance at dilations other than the event width is due to leakage caused by the non orthogonal decomposition by the basis and phase errors. Therefore, for this test case, the wavelet variance maxima identifies the event width, as defined by the width of the top-hat.

The same analysis can be carried out for a record with a single sine pulse. Applying (6.21) the dilation corresponding to the peak wavelet variance for a step wavelet transform of a sine pulse is given by the solution (fig 6.8.b) to

$$\cos(\pi \frac{a}{\tau}) + 2\pi \frac{a}{\tau} \sin(\pi \frac{a}{\tau}) = 1.$$  \hspace{1cm} (6.23)

where, $\tau$ is the wavelength of the sine pulse and $a$ is the dilation of the step wavelet. The peak in the wavelet spectrum obtained from equation (6.23) is about 80% of the width of the sine pulse, because the coherent event scale as determined by the step wavelet transform is smaller than the sine pulse width. This analysis can be further extended to include the sine function instead of the step function as the wavelet basis. Fig. (6.8.c) and (6.8.d) show the wavelet variance for the top hat record and sine pulse record computed using the sine indicator function. Fig. (6.8.a) and (6.8.c) show that the dilation corresponding to the peak wavelet variance coincides with the event width, when the data record is a single square pulse, for both step function and sine function wavelets. The peak value of the wavelet variance using the
sine wavelet is less than that using the Haar wavelet, for the record of top hat building blocks, since the shape of the basis set is less similar to the event shape. The wavelet variance peak is also broader than the Haar wavelet variance due to the same reason. Therefore the corresponding wavelet frame will be larger than that of a step basis. As discussed earlier a smaller wavelet frame (in the case of the Fourier transform this implies a faster decay rate for the magnitude of the coefficients) is always more efficient.

When the data is a sine pulse the location of the maximum wavelet variance is shifted to smaller dilations even when the wavelet is a sine function. The variance at the peak dilation is largest in the top hat record (fig. 6.8.a) where the record has the sharpest gradient. The rate of decay of the wavelet variance is steepest when the wavelet basis is similar to the building block used in the record. This will be discussed in more detail in section (6.5) where the inverse wavelet transform of a pulse is presented.
fig 6.8 Wavelet variance as a function of dilation \( a \) for fixed event width \( \tau = 100 \). (a) top hat record; Haar wavelet (b) sine pulse record; Haar wavelet (c) top hat record; sine wavelet (d) sine pulse record; sine wavelet.
fig 6.9 Step wavelet variance as a function of dilation (in points) for (a) periodic top-hats (top) (b) sine pulse (middle) and (c) ramps (bottom). The event width is 50 points and spacing varies from 200 points to 350 points in steps of 25. In all cases the largest peak corresponds to the smallest spacing and decreases with increasing spacing. See text for details.
fig 6.10 Wavelet variance for records with (a) random displacement of events fixed spacing and event width (top), (b) random variation of event width about a mean of 50 points (middle) and (c) random additive white noise with periodic record of 50 point event width and 100 point spacing (bottom).
The above analysis can be extended to simulations of data where more than one event is placed in a record, such as described by the function (6.1). Here again for simplicity we will consider cases where the amplitude of the event is constant throughout the record. Fig.(6.9) shows a wavelet variance plot where the events are top-hats (top), sine pulses (middle) and ramps (bottom) for cases with zero random displacement \([\zeta = 0]\), event width of 50 points and spacing of events \((T_0)\) between 200 to 350 points. Regardless of spacing, the wavelet variance indicates a peak at the approximate event width, although in the case of ramp events, the peak slightly under estimates the event size defined as the sloped part of the ramp. The location of the peak in the wavelet variance is unaffected by the spacing of events, as long as the spacing is larger than the event width. If the spacing is less than the event width, for this artificial data set, the role of event size and spacing are reversed. The magnitude of the peak in the wavelet variance decreases with increasing spacing due to the reduction in the number of events in a fixed length record.

Random displacement of event location has little effect on the variance peak (fig. 6.10.a), although it contains more variance at larger scales. When the width of the events are randomized about a mean width (fig. 6.10.b) the variance peak still corresponds to the mean event width, though the randomness broadens the peak in the wavelet variance. Random noise (fig. 6.10.c) which leads to variance on all scales has negligible effect on the maximum of the wavelet variance, even when the signal to noise ratio is 1 : 10. Thus randomness of event width and spacing and additive noise all act to broaden the peak of the wavelet variance, but do not significantly alter the dilation scale of the peak.
6.4 Computation of higher moment statistics of the phase plane.

The second moment of the wavelet phase plane coefficients is defined (equation 6.19) as a function of the dilation \((a)\) as,

\[
W^2(a) = \frac{1}{L} \int_b W_d(a, x)^2 dx.
\]

Similarly, the third moment of the phase plane can be defined as,

\[
W^3(a) = \frac{1}{L} \int_b W_d(a, x)^3 dx.
\]

Using the above two definitions, the skewness of the wavelet phase plane for fixed dilations can be defined as,

\[
W_s(a) = \frac{\frac{1}{L} \int_b [W_d(a, x)]^3 dx}{\left[\frac{1}{L} \int_b [W_d(a, x)]^2 dx\right]^{\frac{3}{2}}}
\]  
(6.23)

The skewness provides information that is dependent on the sign of the coefficient and hence may be used to determine the preferred direction of the transition zone, or edge, in a record. The skewness, computed for ramp building block records where the slope of the sharp transition zone has been varied (fig 6.11), indicates its ability to detect the direction of the sharp transition and the scale of the transition which occurs at the inflection point of the skewness plot. The scale of the sharp transition zone is better identified by the derivative of the skewness (fig 6.11.b). The skewness is sensitive to added random noise (fig 6.11.c), and reduces its ability to detect the scale of the sharp transition zone (fig 6.11.d).
fig 6.11 Wavelet skewness for records with periodic ramp events with no added noise (top) and, with varying amount of added white noise (bottom).
6.5 The image reconstruction using inverse wavelet transform

The inverse wavelet transform is defined (equation 5.19) as

\[ f(x) = K \sum_m \sum_n W_d f(m, n) h(a_0^{-m} x - nb_0). \] (6.24)

Where the constant \( K \) is given by

\[ \frac{1}{K} = \int_0^\infty \frac{|\tilde{f}(\omega)|^2}{\omega} d\omega \] (6.25)

and \( m, n = 1, \ldots, \infty \).

The wavelet coefficients obtained for the single top-hat shown in fig (6.6) can be used to reconstruct the top-hat using the inverse wavelet transform defined above. The accuracy of reconstruction of the image will depend on the snugness of the wavelet phase plane as discussed in chapter 5. Recall that if bounds on \( m \) and \( n \) can be found, such that the wavelet transform coefficients vanish for all \( m, n \) larger than the bounds, then the phase plane is tight and precisely invertible. If the coefficients do not vanish, but remain arbitrarily small, then the phase plane is snug and is invertible with small error. If the resulting tight or snug frame is small, i.e. the number of elements in the bounded set of \( m, n \) is small, then the transform is efficient.

For the single top-hat record, a tight frame would include all dilations with non zero wavelet variance (fig. 6.8). Since the rate of decay of the variance for the step indicator function is faster than that of the sine indicator function for the top hat record, the error of the reconstructed image for a fixed frame size (i.e., fixed upper bounds of \( m, n \)) is smaller for the step basis.

Fig (6.12) shows the original top-hat record and the reconstructed record for a special frame. The boundaries of this frame have been chosen to include only the dilation corresponding to the maxima in the wavelet variance peak.
and all possible translations (maximum over sampling of the translations). This reconstruction shows that the width of the event is correctly reconstructed though the sharp edges are poorly captured. If a series of events with different amplitudes of top-hats with the same width are used as the record, the reconstructed image would capture the width of events as well as the amplitude of the individual events. In other tests on artificial data made of two sine modes (not shown here) the inverse transform successfully isolates each mode when the inverse was computed at the dilation corresponding to the wavelength of each mode. Also, where the data consists of a packet of localized waves, the inverse transform accurately reconstructs the data. This is due to the local nature of the wavelet transform. As an analogy, if this exercise were performed with Fourier spectra, the reconstructed image would be a single sine or cosine mode with a fixed amplitude and ‘spacing’ for the entire record.

Hence it is seen that the wavelet inverse transform computed at the dilation corresponding to the peak of the wavelet variance reconstructs an image with accurate representation of event width and event amplitude. The addition of few more dilations to this special frame, reduces the side lobes of the reconstruction (similar to Gibbs phenomenon) of the top hat and provides a modest improvement in the edge representation.

6.6 Summary of results from simulated data analysis.

Simulated data sets constructed from ‘events’ placed in a record were analyzed using the Fourier transforms and wavelet transforms. The Fourier spectrum detects the event periodicity of the records in the form of a variance (or energy) peak. The event width may be a relative minima in the spectra depending on the event shape and the randomness of the spacing.
fig 6.12 A record and its partial reconstruction using Wavelet inverse transform at a dilation of 50 points is shown (dashed line) along with the original record (solid line).
For example, in the case of top-hat events, the spectral minima occurs at the width of the top-hat while for ramp events, the minima occurs at twice the ramp width. The spectral shape is also dependent on the signal to noise ratio.

The wavelet variance computed from the wavelet transform for a step basis function detects the approximate event width for all simulations considered here. The wavelet variance is relatively unaffected by the noise due to the band-pass nature of the transform. Addition of random components to the spacing and event width broadens the peak with no significant change of dilation scale corresponding to the energy peak. Higher moment statistics of the wavelet phase plane such as wavelet skewness provide information on the predominant direction of the sharp transition zone and provide approximate estimates on the size of the transition zone. However, the sensitivity of the wavelet skewness to noise makes the identification of sharp transition scales difficult.

The combined spectral and wavelet analyses of data is able to provide information on event width and in the case of periodic events the spacing. This information can now be used in the analysis of observed data. The inverse wavelet transform will be used to reconstruct an image of the original record with emphasis on the predominant scale associated with the wavelet variance peak. The inverse transform then functions as a band pass filter to extract events that are local in space.
7. ANALYSIS OF OBSERVED TURBULENCE

Microfronts or concentrated zones of horizontal gradients of velocity and temperature, are often located at the upstream edges of updrafts in the presence of mean shear (Kaimal and Businger 1970, Schols 1984, Bergstrom and Hogstrom 1989 and others). With mean shear, rising elements are characterized by weaker horizontal momentum leading to horizontal convergence and formation of sharp gradients at the upstream edges of the updrafts. Microfronts and associated coherent structures, are frequently viewed as ramp patterns (Antonia and Atkinson 1976; Antonia et al. 1979, Schols 1984, Mahrt 1989 and others) or asymmetric top-hats (Mahrt 1989, Mahrt 1985; Kikuchi and Chiba 1985). These zones of concentrated shear are posed as vortex sheets in Corrsin (1962), as stretched vortices with concentrated shear zones as thin as the Kolmogorov microscale in Tennekes (1968), and as regions of convergence associated with vortex stretching due to vertical acceleration of thermals in Kaimal and Businger (1970).

In laboratory studies, similar zones of concentrated shear or temperature gradients have been related to edges of large eddies associated with up lifting of near surface fluid of weak momentum and sometimes have been referred to as bursting events (Willmarth 1975; Brown and Thomas 1977, Chen and Blackwelder 1978, Gibson et al. 1968, Subramanian et al. 1982, see Mahrt 1989 for a brief survey of additional studies). These sharp edges apparently have a role in the cascade of energy from the eddy scales to smaller dissipation scales. In addition they play an important role in the enhancement of mixing and turbulent transport as noted in Schols (1984), Kikuchi and Chiba (1985) and Chen and Blackwelder (1978). This chapter reports on attempts to isolate these coherent structures with the use of localized statistics and study
their characteristics under different stability conditions.

Toward this goal, we analyze atmospheric turbulence data from three different boundary layer field experiments. The wavelet transform statistics will be used to provide estimates of the scale and asymmetry of the main coherent structures and associated microfronts, while the global spectral statistics will be used to estimate the spatial periodicity of such structures which includes the spacing between events.

7.1 Description of the data

We will analyze data from two types of experiments, (a) aircraft based measurements and (b) tower measurements. The aircraft speed is much faster than the measured velocities of turbulence allowing for the assumption of 'frozen' turbulence. However, the tower measurements are to be interpreted only as time based records. This is due to the possibility of significant flow evolution while passing the instrumented tower. This phenomena especially affects the larger scale motions such as boundary layer scale eddies. Advantages of tower based measurements include immunity from contamination due to movement of the sensors (which occur during aircraft turns) and the possibility of very long continuous high resolution samples.

The primary aircraft data set being analysed is from the fast response instrumentation on the NCAR Queen Air research aircraft used in the observation of sheared heated atmospheric boundary layers during the Special Observation Period (SOP) of the Hydrological and Atmospheric Pilot Experiment (HAPEX).
Table 7.1 The aircraft observations used in the analysis. The data resolution is 4m and the average aircraft speed is 80m/s. H denotes the heading of the aircraft and HGM is the height above ground. U (+ east), V (+ north) are the mean values of the measured velocities.

<table>
<thead>
<tr>
<th>Program</th>
<th>Leg</th>
<th>Length (km)</th>
<th>H</th>
<th>HGM (m)</th>
<th>u (m/s)</th>
<th>v (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SESAME</td>
<td>L1</td>
<td>20.48</td>
<td>S</td>
<td>67.75</td>
<td>5.06</td>
<td>-0.79</td>
</tr>
<tr>
<td>5/5/79</td>
<td>L2</td>
<td>6.56</td>
<td>S</td>
<td>58.41</td>
<td>2.85</td>
<td>-0.60</td>
</tr>
<tr>
<td></td>
<td>L3</td>
<td>11.28</td>
<td>N</td>
<td>54.08</td>
<td>1.27</td>
<td>-1.01</td>
</tr>
<tr>
<td></td>
<td>L4</td>
<td>10.08</td>
<td>N</td>
<td>52.92</td>
<td>0.72</td>
<td>-0.81</td>
</tr>
<tr>
<td></td>
<td>L5</td>
<td>12.88</td>
<td>N</td>
<td>17.66</td>
<td>0.46</td>
<td>-0.62</td>
</tr>
<tr>
<td>SESAME</td>
<td>L1</td>
<td>14.88</td>
<td>S</td>
<td>19.79</td>
<td>-1.31</td>
<td>3.89</td>
</tr>
<tr>
<td>5/6/79</td>
<td>L2</td>
<td>12.08</td>
<td>N</td>
<td>18.71</td>
<td>-1.65</td>
<td>5.08</td>
</tr>
<tr>
<td></td>
<td>L3</td>
<td>11.68</td>
<td>N</td>
<td>31.72</td>
<td>-0.79</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>L4</td>
<td>10.64</td>
<td>N</td>
<td>166.95</td>
<td>3.65</td>
<td>10.17</td>
</tr>
<tr>
<td></td>
<td>L5</td>
<td>15.28</td>
<td>S</td>
<td>31.33</td>
<td>2.66</td>
<td>7.65</td>
</tr>
<tr>
<td>HAPEX</td>
<td>L1</td>
<td>120</td>
<td>W</td>
<td>127.97</td>
<td>-2.64</td>
<td>3.32</td>
</tr>
<tr>
<td>5/19/86</td>
<td>L2</td>
<td>124.8</td>
<td>E</td>
<td>124.62</td>
<td>-2.83</td>
<td>2.75</td>
</tr>
<tr>
<td></td>
<td>L3</td>
<td>115.2</td>
<td>W</td>
<td>120.53</td>
<td>-2.40</td>
<td>2.33</td>
</tr>
<tr>
<td></td>
<td>L4</td>
<td>118.4</td>
<td>E</td>
<td>117.42</td>
<td>-2.31</td>
<td>2.15</td>
</tr>
<tr>
<td></td>
<td>L5</td>
<td>118.8</td>
<td>W</td>
<td>117.27</td>
<td>-2.18</td>
<td>2.34</td>
</tr>
<tr>
<td></td>
<td>L6</td>
<td>80</td>
<td>E</td>
<td>129.55</td>
<td>-2.04</td>
<td>2.43</td>
</tr>
<tr>
<td>HAPEX</td>
<td>L1</td>
<td>118</td>
<td>W</td>
<td>144.45</td>
<td>-2.99</td>
<td>-1.68</td>
</tr>
<tr>
<td>5/25/86</td>
<td>L2</td>
<td>122.4</td>
<td>E</td>
<td>139.54</td>
<td>-2.94</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>L3</td>
<td>118.88</td>
<td>W</td>
<td>226.49</td>
<td>-3.43</td>
<td>-0.76</td>
</tr>
<tr>
<td></td>
<td>L4</td>
<td>123.2</td>
<td>E</td>
<td>228.56</td>
<td>-3.66</td>
<td>-0.84</td>
</tr>
<tr>
<td></td>
<td>L5</td>
<td>112.96</td>
<td>W</td>
<td>221.85</td>
<td>-2.97</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>L6</td>
<td>124</td>
<td>E</td>
<td>222.62</td>
<td>-3.17</td>
<td>-0.50</td>
</tr>
</tbody>
</table>
This experiment was conducted in 1986 over nearly flat pine forest in the Southwest of France (Andre 1988, Pinty et al. 1989, Noilhan and Planton 1989, Mahrt 1989). The aircraft instrumentation is similar to that discussed in Wyngaard et. al. (1978). The aircraft flew at an average height of 150m above ground. Occasional forest clearings contribute slightly to the surface inhomogeneity. The scale of such inhomogeneity is probably comparable to or smaller than the scale of the main boundary layer eddies and therefore is expected to be of minimal importance. The forest is mainly of a single species of pine grown on relatively uniform sandy soil. The data analysed here consists of two sets of six east-west flight legs (table 7.1), each approximately 120km long, collected on the two fair weather days, 19 and 25 May 1986. This data is the largest existing sample size of low level aircraft measured data over relatively homogeneous land. These days were characterized by significant surface heating and the average wind at 150m height was 3-4m/s from the Southeast.

The second data set was collected during synoptically quiet periods of the Severe Environmental Storms and Mesoscale Experiment (SESAME) conducted in 1979 over rolling grassland in Oklahoma. The data from this experiment include both windy conditions with weak stratification and weak airflow with strong stratification sampled on 5 and 6 May 1979, respectively. On each day, five north-south flight legs (table 7.1) were selected. On 5 May the five legs were collected at the top of a strong inversion layer about 50m deep. On this day intermittent turbulence is driven by modest shear at the top of the inversion layer while strong stratification prevents the development of any significant turbulence within the layer itself. On 6 May, strong winds of nearly 10-15 m/s maintain a boundary layer of 300m depth despite strong surface radiative heat losses (Lenschow et. al. 1988).
The aircraft time was converted to distance using a constant aircraft speed of 80m/s for both SESAME and HAPEX data. Using this constant value could result in slight over or under estimation of the actual speed of the aircraft on the order of 10% or less. The data has been collected at 20 samples per second resulting in an approximate interval of 4m between adjacent samples. Further on site processing of data prior to storage leads to an effective resolution closer to 8m. Segmentation of flights into legs has been carried out to avoid regions of sharp turns or aircraft banking.

The tower data is from the Lammejord experiment (LAMEX) conducted over a 12 month period from June 1987 to May 1988 by the Risø National Laboratory in Denmark. The site was chosen on the Danish island of Zealand and is a reclaimed flat-bottomed fjord converted to agricultural land (mainly root crops). There were no undulations greater than 1m for at least 3km upwind of the instruments with the prevailing southwesterly wind direction for the data analyzed here.

The data was recorded from an array of instruments mounted on 3 separate masts oriented such that the prevailing winds cut the plane of the mast array perpendicularly (Fig. 7.1). Both wind speed and wind direction are measured at 8 different locations using fast responding (8 Hz) cup anemometers and wind vanes. A detailed description of the site and instrumentation is given in Courtney (1987). In addition, a Kaijo Denki DAT 300 omnidirectional sonic anemometer is located at 45m above ground on top of the highest mast, providing all 3 components of the velocity. Calibration of the sonic instrument was performed in wind tunnel experiments prior to installation on site (Mortensen et.al. 1987). The sonic data is collected at a rate of 16 Hz and corrected for instrument response errors as it is collected by a correcting algorithm based on the measured response characteristic.
For this research we analyze 50 hours of the LAMEX sonic data collected during a period of relatively strong uniform wind in June 1987. Fig. (7.2) shows that the mean wind direction was relatively constant during this period and the horizontal speed at 30m height linearly decreased from about 13 m/s to about 9m/s during the 50 hour period.

7.2 The Analysis of data.

The aircraft data has been analysed using spectral statistics and wavelet transform statistics, as discussed in chapters 5 and 6. The spectra for each variable for each leg has been computed and the results averaged over legs for a given day to obtain the composite spectra for a given variable on a given day. The spectral estimates are computed to a wavenumber corresponding to about $\frac{1}{10} - \frac{1}{5}$ of the record length to reduce the instability of spectral estimates. This corresponds to spectral estimates at wavelengths up to 1.3km for SESAME and up to 10km for HAPEX.

The wavelet transform of the data can be plotted on a wavelet phase plot of dilation verses translation as discussed in chapter 5. Fig. (7.3)(top) shows a 1600m sub-segment of vertical wind velocity for leg 1 HAPEX 19 May, with the corresponding Haar wavelet transform for 500m dilation (center) and the wavelet phase plot(bottom). The contours on the phase plot suggests three (with some subjectivity) three different scales for coherent sharp horizontal gradients. The smallest or resolved scale (4-12m), corresponds to fine scale turbulence. The intermediate 50-80m scale probably corresponds to microfront structure, while the largest coherent spatial scale of 200-250m corresponds to the coherent eddies. Elongation of the contours in the dilation direction at certain locations in the data (translation axis) correspond to locations of sharp changes in the velocity.
fig 7.1 The three tower array at the Lammejord site in Denmark used in LAMEX. The three dimensional sonic anemometer which provided the data used in this study is mounted on top of mast 1 (45m).

fig 7.2 Wind speed and direction at 30m height during the 50 hr period analysed in this study.
fig 7.3 Sample record of vertical velocity from HAPEX 19 May (top) with wavelet coefficients for a fixed dilation of 500m (middle) and wavelet phase plot (bottom). A step indicator function is used in the transform.
In order to reduce the subjectivity and the amount of information contained in the wavelet transform, the wavelet variance and wavelet skewness are computed. The wavelet variance (chapter 6) provides information on the scale of the main coherent event (eddy), and wavelet skewness provides information on the smaller microfront scale, as shown in the previous chapter. The wavelet variance and skewness have been computed using the Haar function for the wavelet as this analysis attempts to study the characteristics of eddies from the point of view of spatial coherency of horizontal changes of velocities. The properties of the wavelet transform using the Haar wavelet has been presented in the previous chapter. The same type of compositing as applied to the spectra has been applied to the wavelet variances. The spectra vs. wavelength and wavelet variance and skewness vs. dilation (horizontal scale) for HAPEX and SESAME data are shown in fig. (7.4-7.13).

In computing the skewness alternating legs are flipped to align the legs with respect to the mean shear. The special choice of wavelength for the spectral x-axis facilitates easy comparison between of spectra and wavelet statistics.

To analyze the LAMEX data, we partition the 50 hour sample into segments of approximately 10 minute duration corresponding to a translation distance of approximately 6-7km. Fig (7.14-16) shows the 50 hour composite plots of spectra, wavelet variance and skewness for this data. The wavelet variance and spectral plots for individual segments are not shown due to the large number of legs.

The wavelet variances computed for each segment of LAMEX data were sorted and the dilation corresponding to the maximum wavelet variance extracted. The bandwidth of dilations at which the variance equals or exceeds 95% of the maximum variance were also extracted and are summarized in table (7.2). Table (7.3) shows the same information for HAPEX and SESAME
data even though there were a smaller number of legs for these experiments. The width of the region of dilations exceeding the 95% level provides a measure of the flatness of the wavelet variance peak, or the uncertainty of the scale of maximum variance (fig 7.17). If the events are very similar to one another, then the wavelet variance plot is very peaked and the difference between the two values at which the variance exceeds 95% is small. When the wavelet plot does not have a well defined peak, the range of scales above 95% of maximum wavelet variance is large. This information provides a measure of the similarity of individual events that pass the sensor.

The dilation corresponding to the maximum wavelet variance and the corresponding wavelet variance are plotted against time in fig (7.18) and the range of dilations exceeding 95% of maximum variance is plotted in fig (7.19). Points corresponding to HAPEX and SESAME are also shown on fig (7.20) and (7.21). These plots indicate the variation of the nature of turbulent air passing through the tower array with time. The value of the wavelet variance at the peak dilation is an indication of the number of coherent events of a given scale occurring in a 10 minute segment. Larger values of maximum wavelet variance correspond to more frequent and organized structures at the corresponding scale.
HAPEX 19th May (SPECTRA)

HAPEX 19th May (WAVELET)

fig 7.4 (a) Composite power spectra of u(solid), v(dashed) (top) and (b) composite Wavelet variance of u(solid), v(dashed) (bottom) for HAPEX 19 May.
fig 7.5 (a) Composite power spectra of w(solid), T(dashed) (top) and (b) composite Wavelet variance of w(solid), T(dashed) (bottom) for HAPEX 19 May.
fig 7.6 (a) Composite power spectra of u(solid), v(dashed) (top) and (b) composite Wavelet variance of u(solid), v(dashed) (bottom) for HAPEX 25 May.
fig 7.7 (a) Composite power spectra of $w$(solid), $T$(dashed) (top) and (b) composite Wavelet variance of $w$(solid), $T$(dashed) (bottom) for HAPEX 25 May.
fig 7.8 (a) Composite wavelet skewness of w(solid), T(dashed) (top) for HAPEX 19 May and (b) composite wavelet skewness of w(solid), T(dashed) (bottom) for HAPEX 25 May.
fig 7.9 (a) Composite power spectra of u(solid), v(dashed) (top) and (b) composite Wavelet variance of u(solid), v(dashed) (bottom) for SESAME 5 May.
fig 7.10 (a) Composite power spectra of $W$(solid), $T$(dashed) (top) and (b) composite Wavelet variance of $w$(solid), $T$(dashed) (bottom) for SESAME 5 May.
fig 7.11 (a) Composite power spectra of \(u\) (solid), \(v\) (dashed) (top) and (b) composite Wavelet variance of \(u\) (solid), \(v\) (dashed) (bottom) for SESAME 6 May.
fig 7.12 (a) Composite power spectra of w(solid), T(dashed) (top) and (b) composite Wavelet variance of w(solid), T(dashed) (bottom) for SESAME 6 May.
fig 7.13 (a) Composite wavelet skewness of $w$(solid), $T$(dashed) (top) for SESAME 5 May and (b) composite wavelet skewness of $W$(solid), $T$(dashed) (bottom) for SESAME 6 May.
fig 7.14 Composite power spectra of LAMEX 50 hours.
fig 7.15 Composite wavelet variance for LAMEX 50 hours.
fig 7.16 Composite wavelet skewness for LAMEX 50 hours.
<table>
<thead>
<tr>
<th>Wavelet dilations</th>
<th>Correlation</th>
<th>Variance</th>
<th>95% band</th>
<th>Wavelet dilations</th>
<th>Correlation</th>
<th>Variance</th>
<th>95% band</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.55</td>
<td>0.65</td>
<td>0.28</td>
<td>34</td>
<td>0.42</td>
<td>0.28</td>
<td>34</td>
<td>0.42</td>
</tr>
<tr>
<td>0.50</td>
<td>0.62</td>
<td>0.38</td>
<td>48</td>
<td>0.47</td>
<td>0.21</td>
<td>112</td>
<td>0.62</td>
</tr>
<tr>
<td>0.50</td>
<td>0.62</td>
<td>0.35</td>
<td>48</td>
<td>0.47</td>
<td>0.21</td>
<td>112</td>
<td>0.62</td>
</tr>
<tr>
<td>0.60</td>
<td>0.65</td>
<td>0.46</td>
<td>10</td>
<td>0.60</td>
<td>0.65</td>
<td>0.46</td>
<td>10</td>
</tr>
<tr>
<td>0.70</td>
<td>0.65</td>
<td>0.42</td>
<td>72</td>
<td>0.60</td>
<td>0.65</td>
<td>0.42</td>
<td>72</td>
</tr>
<tr>
<td>0.80</td>
<td>0.65</td>
<td>0.40</td>
<td>100</td>
<td>0.60</td>
<td>0.65</td>
<td>0.40</td>
<td>100</td>
</tr>
<tr>
<td>0.90</td>
<td>0.65</td>
<td>0.38</td>
<td>126</td>
<td>0.60</td>
<td>0.65</td>
<td>0.38</td>
<td>126</td>
</tr>
<tr>
<td>1.00</td>
<td>0.65</td>
<td>0.36</td>
<td>149</td>
<td>0.60</td>
<td>0.65</td>
<td>0.36</td>
<td>149</td>
</tr>
<tr>
<td>1.10</td>
<td>0.65</td>
<td>0.34</td>
<td>172</td>
<td>0.60</td>
<td>0.65</td>
<td>0.34</td>
<td>172</td>
</tr>
<tr>
<td>1.20</td>
<td>0.65</td>
<td>0.32</td>
<td>196</td>
<td>0.60</td>
<td>0.65</td>
<td>0.32</td>
<td>196</td>
</tr>
<tr>
<td>1.30</td>
<td>0.65</td>
<td>0.30</td>
<td>221</td>
<td>0.60</td>
<td>0.65</td>
<td>0.30</td>
<td>221</td>
</tr>
<tr>
<td>1.40</td>
<td>0.65</td>
<td>0.28</td>
<td>247</td>
<td>0.60</td>
<td>0.65</td>
<td>0.28</td>
<td>247</td>
</tr>
<tr>
<td>1.50</td>
<td>0.65</td>
<td>0.26</td>
<td>274</td>
<td>0.60</td>
<td>0.65</td>
<td>0.26</td>
<td>274</td>
</tr>
<tr>
<td>1.60</td>
<td>0.65</td>
<td>0.24</td>
<td>302</td>
<td>0.60</td>
<td>0.65</td>
<td>0.24</td>
<td>302</td>
</tr>
<tr>
<td>1.70</td>
<td>0.65</td>
<td>0.22</td>
<td>331</td>
<td>0.60</td>
<td>0.65</td>
<td>0.22</td>
<td>331</td>
</tr>
<tr>
<td>1.80</td>
<td>0.65</td>
<td>0.20</td>
<td>361</td>
<td>0.60</td>
<td>0.65</td>
<td>0.20</td>
<td>361</td>
</tr>
<tr>
<td>1.90</td>
<td>0.65</td>
<td>0.18</td>
<td>393</td>
<td>0.60</td>
<td>0.65</td>
<td>0.18</td>
<td>393</td>
</tr>
<tr>
<td>2.00</td>
<td>0.65</td>
<td>0.16</td>
<td>426</td>
<td>0.60</td>
<td>0.65</td>
<td>0.16</td>
<td>426</td>
</tr>
</tbody>
</table>

Table 7.2 Wavelet analysis of LAMEX data: The Haar wavelet variance of variable $w$ is computed. The dilation corresponding to Max variance is column (2). Column (3) shows the correlation coefficient between the original data and the inverse wavelet transform at peak variance dilation. Column (4) is the % of variance explained by the inverse. Column (5) shows the difference in dilations for 95% of maximum variance.
Table 7.3 Wavelet analysis of HAPEX and SESAME data: The Haar wavelet variance of variable $w$ is computed. The dilation corresponding to Max variance is column (2). Column (3) shows the maximum value of variance. Column (4) shows the difference in dilations for 95% of maximum variance.

<table>
<thead>
<tr>
<th>Leg</th>
<th>Max dilation</th>
<th>Variance</th>
<th>95%(up-low)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hapex 19-May</td>
<td>1</td>
<td>448</td>
<td>0.206</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>448</td>
<td>0.213</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>296</td>
<td>0.223</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>448</td>
<td>0.228</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>368</td>
<td>0.283</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>368</td>
<td>0.271</td>
</tr>
<tr>
<td>Hapex 25-May</td>
<td>1</td>
<td>536</td>
<td>0.206</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>536</td>
<td>0.207</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>632</td>
<td>0.298</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>968</td>
<td>0.332</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>632</td>
<td>0.310</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>632</td>
<td>0.277</td>
</tr>
<tr>
<td>Sesame 5-May</td>
<td>1</td>
<td>968</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>16</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>296</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>536</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>848</td>
<td>0.001</td>
</tr>
<tr>
<td>Sesame 6-May</td>
<td>1</td>
<td>16</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>56</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>56</td>
<td>0.071</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>88</td>
<td>0.084</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>176</td>
<td>0.087</td>
</tr>
</tbody>
</table>
fig 7.17 Measure of the flatness of the wavelet variance peak is provided by $\delta a$. When $\delta a$ is large the curve is flat.
fig 7.18 Wavelet analysis of LAMEX data. Dilations at which wavelet variance reaches maximum. The computation is for a Haar wavelet computed for vertical winds.
fig 7.19 Wavelet analysis of LAMEX data. Differences of dilations at which wavelet variance reaches 95% of maximum wavelet variance. The computation is for a Haar wavelet computed for vertical winds. The second graph (dashed line) shows the correlation between data and inverse wavelet transform at maximum dilation.
fig 7.20 Wavelet analysis of aircraft data. Dilations at which wavelet variance reaches maximum and the corresponding variance. The computation is for a Haar wavelet computed for vertical winds. (a) HAPEX 19 May (b) HAPEX 25 May (c) SESAME 5 May (d) SESAME 6 May
fig 7.21 Wavelet analysis of aircraft data. Differences of dilations at which wavelet variance reaches 95% of maximum wavelet variance. The computation is for a Haar wavelet computed for vertical winds. (a) HAPEX 19 May (b) HAPEX 25 May (c) SESAME 5 May (d) SESAME 6 May
7.3 Inference of principle characteristics and scales

To compare spectra and wavelet statistics recall that the scale of maximum spectral energy is expected to be most well defined with events which are periodic with a relatively constant wavelength or frequency. The scale of such periodicity is the distance from the beginning of one event to the beginning of the next event (denoted by $T_0$ in fig. 5.1). As discussed in the previous chapters, the peak in the spectrum for almost periodic events corresponds to the approximate scale of periodicity of the events. However a peak in the Haar-wavelet variance corresponds to a scale twice the event width, since it measures the distance over which the change is coherent. When the events are periodically spaced such that the event width is half the periodicity (i.e. event width = spacing between events), the peak in the Haar-wavelet variance and spectra should theoretically coincide. However since the wavelet variance is (formally) unaffected by the spacing of the events, due to loss of information of phase of each event in the record, the spacing, in general, cannot be inferred from the wavelet transform.

The wavelet variance of vertical motion ($w$) for HAPEX 19 May (fig 7.5) indicates a well defined peak at 400m scale implying an 'average' eddy updraft width of 200m. The wavelet variance of temperature ($T$) shows a peak at the same scale, although the peak is more diffuse, possibly due to thermal inhomogeneity of the updrafts. The wavelet skewness for both $w$ and $T$ (fig 7.8) reaches a positive maximum at 100m apparently responding to sharper changes at scales smaller than the wavelet variance. While the skewness is difficult to interpret the 100m peak is undoubtedly influenced by the smaller scale microfronts. The positive skewness at small scales suggests a ramp like $w$ signal, consistent with asymmetric eddies with leading microfront edges and weaker gradients of opposite sign occurring in the ramp (diffuse wake).
part of the structure. The small positive gradient at larger scales may be
due to nonlinear wave activity or other unexplained ‘noise’.

The variance peak in the $v$-component of the horizontal velocity (defined
to be normal to the flight path) is probably due to vorticity created by
convergence in the microfront region such as frontogenesis or generation of
$\frac{\partial w}{\partial z}$ due to horizontal convergence. Gradients of longitudinal velocity are
constrained by mass continuity and therefore do not show such peaks.

The spectrum of $w$ and $v$ for 19 May HAPEX (fig 7.4 and 7.5) suggest a
periodicity of 1200m implying that the spacing between the 200m drafts is on
the order of 1km. The positive skewness in $w$ at the smallest scales indicate
that the predominant event is associated with a ‘front’. This interpretation
is discussed in terms of the inverse wavelet transforms, later in this chapter.

The HAPEX 25 May data suggest results similar to that of 19 May.
However the peak in the wavelet variance of $w$ (fig 7.7) is at slightly larger
scales and is more diffuse, suggesting a wider distribution of scales associated
with the eddies. The slightly higher elevation of the flights (average height
of 150m instead of 120m) and the greater alignment of the wind shear along
the flight path may have contributed to this spreading. The wide region
of positive skewness at smaller scales on the 25 May (fig 7.8) also suggest
that the microfront is diffuse or occur at a variety of scales. The coincidence
of skewness peak and variance peak suggest more symmetric events with a
slight preference in sign for positive gradients. The spectra suggests an event
periodicity of 1000m for this day. The weak spectral peak of $w$ also suggest
a more aperiodic spacing of the events.

The above inferences are based on experience with application of spec-
tral and wavelet transforms to artificial data. While there is no proof that
such inferences can be made, the existence of such a proof is probably lim-
ited by Heisenberg's uncertainty principle. However, additional supporting evidence is provided by wavelet inverse transform at a given scale, which offer information on event location and overall asymmetry of the events. The wavelet variance peak detected in HAPEX vertical motion suggested eddies on the 200-250m scale. An inverse wavelet transform computed at the dilation corresponding to this peak is shown in fig (7.22) and (7.23). Mathematically, the inverse transform provides a 'decomposition' of the record into that scale corresponding to the maximum wavelet variance. In addition to event location, the magnitude of the inverse transform provides a measure of the coherency of the event. Although the inverse transform corresponds to a single dilation, it is not periodic or symmetric.

The analysis of the SESAME data provides a different physical situation. On 5 May the wavelet variance of vertical velocity (fig 7.10) suggests weak turbulence with scale of few 10's of meters. The skewness peak at 50-60m in $w$, $T$ and $u$ may be interpreted as a preference for asymmetry of the main eddies. The negative skewness at smaller scales agrees with the concept of concentration of gradients by mean shear and the low boundary layer thickness of 20m on this day. The large wavelet variance and high skewness over a broad range of scales is probably due to shear modified wave activity or breaking waves.

The spectra (fig 7.9) increases with scale for all velocity components possibly due to the large scale wave activity and non-periodic turbulent updrafts/downdrafts. That is, the turbulent drafts seem to be more randomly distributed through the record or are of variable scale as implied by the flatness in the $w$ Haar wavelet variance. These patches of turbulence seem to be different from the microfronts observed in HAPEX in that those microfronts were associated with a larger more regular structure instead of isolated events.
which have little tendency to be periodic; this may be one factor preventing a turbulence scale peak in the spectrum.

The results on SESAME 6 May indicate the presence of sheared eddies of about 80-100m width. The sharp peak in the wavelet variance as seen from the width of dilations at 95% of wavelet peak variance (fig 7.21) hints that eddies are well organized with preferred scale, probably due to the strong shear generation of turbulence on this day. The spectra suggests that the eddies are placed 0.4-0.5km apart, which is the same scale as the maximum value of the wavelet skewness. The sign of the skewness also agrees with the expected concentration of horizontal gradients by the mean shear for this day. Thus, for this case, we imagine shear driven updrafts and downdrafts of comparable width, in contrast to shear modified thermals (HAPEX case) where the spacing is much larger than the updraft width.

The LAMEX data analysis shows spectra which has minimum variability between records for the whole experiment, while the wavelet variance indicates a changing velocity field. This is possibly a good example of the robust character and no unique aspect of spectra as discussed in chapter 3 and chapter 4. While the spectra indicated an approximate -5/3 law over all the ten minute data segments, the wavelet variance changed from very peaked variance spectrum to diffuse variance spectrum over time (shown in fig 7.17). The measure of this effect is seen in the plot of the dilation range exceeding 95% of maximum wavelet variance in fig (7.18). The flow seem to have periods of 'chaotic' flow followed by periods of organization at a fixed scale. These periods of organized eddy activity seem to last for about an hour and are generally separated by about 3 hour intervals. The organized activity is also evident in the correlation plot in fig (7.19). For the computation of these correlations the inverse Haar wavelet transforms are computed.
at the dilation corresponding to the peak wavelet variance and correlated
with the original data. When the wavelet variance plot is very peaked, it is
associated with well defined (not necessarily periodic) events. The inverse
transform at this peak should recover these events and thus yield a high cor-
relation with the data. This concept seems to be supported by the LAMEX
data. As seen in fig (7.18) when the 95% variance is sharply peaked the cor-
relation coefficient is a local maxima. The variance explained by the single
mode inverse is approximately 40%, which is high when compared with typ-
ical values obtained with other decomposition methods such as eigen vector
decomposition. The wavelet inverse transform when correlated with the raw
record provides a measure of the variance over the whole record. Eigen value
methods provide a measure of variance from a sample of the whole record,
making the wavelet correlation a better global measure of variance explained
by the main mode. Inexplicably, the skewness of the wavelet transform does
not indicate a preferred sign at small scales.
fig 7.22 Sample of 2000 point segment from HAPEX 19 May vertical velocity (solid) and wavelet inverse transform computed at a dilation of 200m (dashed). The wavelet inverse transform is for a Haar wavelet.
fig 7.23 Sample of 2000 point segment from Segment of HAPEX 19 May vertical velocity (solid) and wavelet inverse transform computed at a dilation of 200m (dashed). The wavelet inverse transform is for a Haar wavelet.
8. SUMMARY

In the simulation of flows with internal sharp velocity gradients, such as shocks or eddy microfronts, the choice of the basis function can affect the efficiency of the decomposition of the flow into global orthogonal modes. In a modelling context, efficiency is defined as the ratio of energy contained in a fixed number of modes of the global basis set to the total energy at a given time step. Although the Fourier trigonometric basis forms a universal basis set, our simulations using the Burghers' equation (Chapter 4) show that a square pulse basis set (Walsh basis) is more efficient in representing the evolution of the shock front in the flow. In spectral models, where the number of modes are fixed a priori, the appropriate choice of the basis functions can affect the simulated flow. For example, if the flow is a 1-dimensional shock front, the Walsh basis is more suited for use in the spectral model. However the ability to extend use of this basis to higher dimensions remains uncertain.

In flows where such sharp edges are randomly distributed, analysis based on spectral decomposition may provide ambiguous results. These flows can yield spectra similar to spectra predicted by Kolmogorov energy cascade theorems even though the underlying physics is completely different (Chapters 2, 3). Hence, it is suggested that the existence of a certain spectral shape, such as a -5/3 slope, does not provide unambiguous information on the energy transfer characteristics of the flow. Spectral decompositions provide information only on the periodic spacing of coherent events (Chapter 6), and does not provide reliable information on the scales of the randomly placed coherent events. Local wavelet transforms provide better information on the scale of the coherent event (Chapters 5, 6) but does not provide clear information on the scale of the periodicity. The wavelet variance for the entire
record detects the approximate width of the dominant coherent events. For the artificial and observed data analysed in this study, the exact form of the basis function was not crucial. The skewness of the wavelet coefficients for the record identifies the systematic sign preference of the gradients across the transition zone.

The inverse wavelet transform can be used to reconstruct efficient approximations to the record. Unlike the Fourier inverse transform, where the whole record is reconstructed, the local nature of the wavelet basis allows reconstruction of flow associated with certain local features, such as sharp edges or coherent events. Furthermore, the wavelet inverse transform, computed at the dilation corresponding to the wavelet variance peak, provides an efficient representation of the coherent events. Such a single mode inverse is not constrained to be a periodic wave of constant amplitude as in the Fourier inverse transform.

The combined analysis of observed atmospheric data using the wavelet transform and spectral methods provides better and more complete information about the nature of the flow compared to the use of spectra alone. In the heated atmospheric boundary layer observed in HAPEX, the turbulence seems to be characterized by coherent events that are distributed randomly about periodic mid-points. The spacing of these events seem to be larger than the event scale itself.

In the neutral, shear driven boundary layer observed in SESAME, the updrafts and downdrafts are of similar scale. The spectral peak and the wavelet variance peak coincide in scale, suggesting that the events occur in updraft-downdraft pairs. When the shear is weak, the coherent structures seem to be less organized, as seen from the weak wavelet variance.

The spectra for the LAMEX data does not change with time, while
the wavelet variance suggests a changing turbulent flow. The flow seem to alternate between periods of organized eddy activity and periods of 'chaotic' flow. The scale of the eddies decreases with decreasing wind velocity. Based on the inverse wavelet transform, computed at dilation corresponding to the peak wavelet variance, the eddies seem to be randomly distributed spatially and account for about 40% of the energy contained in the flow. The -5/3 slope of the spectra seem to be unaffected by the changing of the nature of the flow. This indicates that the -5/3 spectral slope is robust but not very indicative of the physics of the flow.
REFERENCES


Daubechies, I., 1988: Orthonormal basis of compactly supported wavelets.


Hopf, E., 1950: The partial differential equation, $u_t + u u_x = \nu u_{xx}$, *Comm. on pure and appl. math.*, 3, 201-230.

Jensen, N. O. and D. H. Lenschow, 1978: An observational investigation of


Mahrt, L., 1985: Vertical structure and turbulence in the very stable bound-


APPENDIX

In this appendix the definitions of some items described in the text are given along with a somewhat exhaustive list of the notation. While most symbols are selected to agree with literature on this subject some times a modified notation has been used to clarify the development. While many books on real analysis provide ‘text book’ definitions of standard items this section follows “Introductory Real Analysis” by Kolmogorov and Fomin (1970) and “Partial Differential Equations of Mathematical Physics and Integral Equations” by Guenther and Lee (1988).

(A.1)

Definition 1. A metric space is a pair \((X, \rho)\) consisting of a set \(X\) and a distance \(\rho\) (a single valued and non negative function of \(x, y \in X\)) such that

1. \(\rho(x, y) = 0\) iff \(x = y\);
2. \(\rho(x, y) = \rho(y, x)\);
3. \(\rho(x, z) \leq \rho(x, y) + \rho(y, z)\).

Definition 2. Let \(f\) be a mapping of one metric space \(X\) into another metric space \(Y\), so that \(f\) associates an element \(y = f(x) \in Y\) with each element \(x \in X\). Then a one-to-one mapping of one metric space \(R = (X, \rho)\) onto another metric space \(R' = (Y, \rho')\) is said to be an isometry if \(\rho(x_1, x_2) = \rho'(f(x_1), f(x_2))\) for all \(x_1, x_2 \in R\).

(A.2)

The Hölder continuity measure is usually given as,

\[ |f(x) - f(y)| \leq c|x - y|^\alpha. \]
A low pass filter is a function which has weights of 1 in the low frequency part of the spectrum and weights of zero beyond some pre-selected (cut-off) frequency. The convolution of such a function with the function to be filtered would result in an output that has a frequency content only in the low frequency 'pass band'. A band pass filter is a function which has a pass band in some frequency interval, and is obtained by filtering the record two times with the two cut-off frequencies.

\[ \sigma_x^2 = \int_{-\infty}^{\infty} x^2 |g(x)|^2 \, dx. \]

\[ \sigma_\omega^2 = \int_{-\infty}^{\infty} \omega^2 |\hat{g}(\omega)|^2 \, d\omega. \]

**Definition** A sequence of distinct vectors \([f_1, f_2, f_3, \ldots]\) belonging to a Hilbert space \(H\) is said to be a Frame if there exists positive constants \(A\) and \(B\) such that \(A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2\) for every \(f \in H\). The numbers \(A\) and \(B\) are called the frame Bounds.

The frame is called a TIGHT frame if the numbers \(A\) and \(B\) called the frame Bounds are equal.

The frame is called a SNUG frame if the numbers \(A\) and \(B\) called the frame Bounds are such that \((1 - A/B)\) is small.