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In this thesis consideration is given to generalizing the ideas of the intra-block estimator and the recovery of inter-block information. These ideas are generalized to any linear mixed model of the form

$$Y \sim N_n(X\beta, \pi\Sigma_\gamma)$$

where X is an $n \times p$ known matrix, $\beta \in R^p$, $\gamma \in [0,1]$, $\pi > 0$ and $\Sigma_\gamma = (1-\gamma)I_n + \gamma V$ where V is a known nonnegative definite matrix. This model contains, as a special case, the linear model of any incomplete block design.

Suppose that $\theta = \lambda'\beta$ is an estimable parametric function and that $\hat{\theta} = t'Y$ is, among all the best linear unbiased (blu) estimators of θ at $\gamma = 1$, the blu estimator for θ at $\gamma = 0$. Let $s = \underline{r}(X,V) - \underline{r}(X)$. When $s \geq 3$ this thesis shows that there is an unbiased estimator $\hat{\delta}$ for θ such that the variance of $\hat{\delta}$ is less than the variance of $\hat{\theta}$ for all $\gamma \in [0,1)$ and for all $\pi > 0$, i.e., the estimator $\hat{\delta}$ uniformly dominates the estimator $\hat{\theta}$. The estimator $\hat{\delta}$ is the generalization of the intra-block

estimator in an incomplete block design to the general model given above. And the estimator $\hat{\delta}$ is obtained from the general model in a manner similar to the recovery of inter-block information in an incomplete block design. For the special case of a connected incomplete block design, our condition on s is the same as the number of blocks being greater than or equal to four. This is the same condition given in the literature for the case of a proper binary equi-replicate incomplete block design for an estimable parametric function which is a contrast of the treatment effects.

The efficiency of several variations of $\hat{\delta}$ is computed with respect to the Cramer'-Rao lower bound. This has been done by a simulation study which illustrates the domination of $\hat{\delta}$ over $\hat{\theta}$.

Nonlinear Unbiased Estimators that Dominate the Intra-block Estimator

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Nonlinear Unbiased Estimators that Dominate the Intra-block Estimator

I. Introduction

The problem of combining the intra- and inter-block information in incomplete block designs has received a great deal of attention in the literature. Yates (1939, 1940) initiated this subject. Since then many authors have contributed to the area. All of the work to date has been limited to special structures of the incidence matrix of an incomplete block design.

The goal of this thesis is to study the problem of combining the intra- and inter-block information in a completely general setting. That is, with no particular structure on the incidence matrix. In fact, we generalize the idea of the intra-block estimator, as well as the recovery of inter-block information, to a completely general mixed model having two variance components. In particular, we identify for the general model a linear estimator that is the natural generalization of the intra-block estimator and then show that this estimator can, under mild conditions, be uniformly improved upon (as measured by the variance) by a nonlinear unbiased estimator. And that this improvement step can naturally be thought of as the recovery of inter-block information.

In Chapter II the general model is defined and the class of all unbiased linear estimators for any estimable parametric function is reduced to an admissible class via the work of Azzam, Birkes and Seely (1986). We then determine a canonical form for the general model and define the specific problem addressed by this thesis.

Chapter III contains a review of the literature. This review shows that in order to combine the intra- and inter-block information, conditions are needed on the number of treatments or blocks in the design. The weakest condition is given by Khatri and Shah (1974), whose results are restricted to a proper binary equi-replicate incomplete block design where the number of blocks exceeds three. Moreover, their results are applicable only to an estimable parametric function that is a contrast of the treatment effects.

Chapter IV is devoted to a general form of a nonlinear unbiased estimator which is uniformly better than the generalized intra-block estimator. This estimator is applicable to any incomplete block design. It does, however, require a condition similar to the number of blocks exceeding three. But it is not restricted to a contrast nor is it restricted to an incomplete block design. The exact variance of the estimator is given. Also, a truncated version of the estimator is given which has variance smaller than or equal to the variance of the orignal estimator. If the design is connected, then the estimator considered in Chapter IV can be obtained if the number of the blocks is greater than three. This is the same condition given by Khatri and Shah in a more restrictive setting.

Chapter V contains the results of a simulation study that concern several variations of our estimator and several different designs. This study shows that the estimator considered in Chapter IV is uniformly better than the intra-block estimator in each example and for the estimable parametric functions which were considered, some of which are not contrasts of the treatment effects.

Regarding notation, R^n is used to denote an n-dimensional Euclidean space. 1_n and I_n denote an $n \times 1$ vector of ones and an $n \times n$ identity matrix respectively. $\mathbf{J}$ is used to denote a matrix of ones. For a linear transformation or a matrix A the notation A', $\underline{R}(A)$, $\underline{N}(A)$ and $\underline{r}(A)$ denote the transpose, range, null space and rank of A respectively. The orthogonal complement of $\underline{R}(A)$ is denoted by $\underline{R}(A)^{\perp}$ and the orthogonal projection operator on $\underline{R}(A)$ is denoted by $\mathbf{P}_A$. For vectors $a, b \in R^n$, $a'b$ is used to denote the usual inner product. Further, $N_n(\mu, D)$ denotes an n-dimensional normal distribution with mean vector μ and covariance matrix D. Except for $\mathbf{I}_n$, $\mathbf{J}$, 1_n and $\mathbf{P}_A$, any other bold face letter is used to denote to a random variable.

II. The Model and Related Topics

2.1. The model.

The basic mixed linear model we consider in the following is

$$Y = X\beta + B\mathbf{b} + \mathbf{e}$$

where X is a known $n \times p$ matrix, β is a vector of p unknown parameters, B is a known $n \times b$ matrix, $\mathbf{b} \sim N_b(0, \sigma_{\mathbf{b}}^2 I_b)$ with $\sigma_{\mathbf{b}}^2 \geq 0$ and $\mathbf{e} \sim N_n(0, \sigma^2 I_n)$ with $\sigma^2 > 0$. We also assume $\mathbf{e}$ and $\mathbf{b}$ are independent random vectors.

The model has the following structure for its mean vector and covariance matrix:

$$E(Y) = X\beta \,, \quad \beta \in R^p,$$

and $Cov(Y) = \sigma^2 I_n + \sigma_{\mathbf{b}}^2 V$ where $V = BB'$.

It will be convenient for our purposes to consider a different parametrization for the covariance structure. Let $\pi = \sigma^2 + \sigma_{\mathbf{b}}^2$ and let $\gamma = \sigma_{\mathbf{b}}^2/\pi$. Then $\pi > 0$ and $\gamma \in [0,1)$. By letting the ratio $\sigma_{\mathbf{b}}^2/\sigma^2$ become arbitrarily large, we get $\gamma = 1$. For the purpose of this thesis, it will be covenient to allow $\gamma = 1$. Set $\Sigma_\gamma = (1-\gamma)I_n + \gamma V$. Then under this parametrization, we have the following:

$$E(Y) = X\beta, \quad \beta \in R^p,$$

$$Cov(Y) = \pi\Sigma_\gamma, \quad \gamma \in [0,1] \text{ and } \pi > 0.$$

This is the form of the model we will consider in the remainder of the thesis.

2.2. The admissible linear estimators.

Suppose that $\theta = \lambda'\beta$ is an estimable parametric function. For $a \in [0,1)$, let $G_a = \Sigma_a^{-1}$ and let G_1 be any g-inverse of $XX' + V$ which has the property that $\underline{R}(G_1) = \underline{R}(X,V)$. (For example, the Moore-Penrose inverse is a satisfactory choice.) For each $a \in [0,1]$, let $\hat{\theta}_a = \lambda'\hat{\beta}_a$ where $\hat{\beta}_a$ is any random vector satisfying $X'G_aX\hat{\beta}_a = X'G_aY$.

Azzam, Birkes and Seely (1986) characterized the class $\mathbb{C}$ of all admissible linear unbiased estimators (LUE's) for θ with respect to the class of all LUE's as

$$\mathbb{C} = \{\hat{\theta}_a : a \in [0,1]\}.$$

The class $\mathbb{C}$ is a minimal and complete class for θ among the class of all LUE's for θ.

2.3 The problem.

Let $a \in [0,1)$. Under normality $\hat{\theta}_a$ is the minimum variance unbiased estimator for θ with respect to the covariance structure $\pi\Sigma_a$. Because Σ_a is positive definite, it can be shown that $\hat{\theta}_a$ is admissible within the class of all unbiased estimators for θ. The argument used to justify this last statement cannot be used for the estimator $\hat{\theta}_1$ because $\Sigma_1 = V$ is not necessarily positive definite. In fact, in Chapter III we will see that at least in certin cases $\hat{\theta}_1$ is not admissible within the class of all unbiased estimators of θ. A review of the literature dealing with the inadmissibility of $\hat{\theta}_1$ is given in Chapter III.

The main thrust of this thesis is to investigate under what conditions the estimator $\hat{\theta}_1$ is not admissible within the class of all unbiased estimators for θ. Since we will, for the most part, be working with $\hat{\theta}_1$ we let $\hat{\theta} = \hat{\theta}_1$ to simplify the notation.

There are various ways to characterize the estimator $\hat{\theta}$. One description is given in the previous section. Another characterization that is sometimes convenient is the following: $\hat{\theta} = t'Y$ where t is the (unique) vector satisfying the three conditions:

(2.3.1) a) $X't = \lambda$,

b) $Vt \in \underline{R}(X)$,

and c) $t \in \underline{R}(X,V)$.

Another characterization that is also useful is that among all best linear unbiased estimators for $\theta = \lambda'\beta$ with respect to the covariance matrix V, $\hat{\theta}$ is the one that has minimum variance with respect to the identity covariance matrix.

Our primary goal in the following chapters will be to find under what conditions does there exist a nonlinear unbiased estimator $\hat{\delta}$ for θ satisfying

$$\mathrm{Var}(\hat{\delta}\,|\pi,\gamma,\beta) \leq \mathrm{Var}(\hat{\theta}\,|\pi,\gamma,\beta) \quad \text{for all } \pi,\ \gamma,\ \beta,$$

with strict inequality for some π, γ, β.

All of the estimators $\hat{\delta}$ that we shall consider will satisfy the following two properties:

(2.3.2) a) $\hat{\delta}(gy) = g\hat{\delta}(y)$ for all nonzero $g \in R$ and all $y \in R^n$.

b) $\hat{\delta}(y + X\beta) = \hat{\delta}(y) + \lambda'\beta$ for all $\beta \in R^p$ and all $y \in R^n$.

There are three consequences of these two properties that are useful to simplify our problem. These consequences are:

(2.3.3) a) Any such $\hat{\delta}$ is unbiased for θ provided that its expectation exists. This follows immediately from Seely and Hogg (1982).

b) $\hat{\delta}$ is of the form $\hat{\delta} = \hat{\theta} + \psi$ where ψ is a function of $\hat{e} = G'Y$ with G being any matrix satisfying $\underline{R}(G) = \underline{R}(X)^{\perp}$.

c) $\mathrm{Var}[\hat{\delta} \mid \pi, \gamma, \beta] = \pi \mathrm{Var}[\hat{\delta} \mid \pi = 1, \gamma, \beta = 0]$.

Because $\hat{\theta}$ has the properties (2.3.3), it is clear that for our purposes of variance comparison we may without loss in generality assume that $\pi = 1$ and $\beta = 0$ which we shall do throughout. The consequence of (2.3.3b) will be further explored in the next section where we set up a canonical structure for our problem.

2.4. The canonical form and estimator.

Consider the model and notation given previously in this chapter. Recall that any estimator we consider will be a function of $\hat{\theta}$ and $\hat{e} = G'Y$ where the range of G is the error space, i.e., the vector space $\underline{N}(X')$. In this section we describe a particular choice for G.

Set $q = n - \underline{r}(X,B)$ and let Q be an $n \times q$ matrix satisfying the following two conditions:

a) $\underline{R}(Q) = \underline{R}(X,B)^{\perp}$.

b) $Q'Q = I_q$.

Set $s = \underline{r}(X,B) - \underline{r}(X)$ and let L be an $n \times s$ matrix satisfying the following three conditions:

a) $\underline{R}(L) = \underline{R}(X)^{\perp} \cap \underline{R}(X,B)$.

b) $L'L = I_s$.

c) $L'VL = D$ where $D = \mathrm{diag}(d_1, \ldots, d_s)$.

A satisfactory choice for the matrix L is $L = AP$ where the columns of A form an orthonormal basis for $\underline{R}(X)^{\perp} \cap \underline{R}(X,B)$ and where P is an orthogonal matrix that diagonalizes $A'VA$. It can be checked that the matrix D has rank s and hence is a positive definite matrix. Therefore, it follows that each of the diagonal elements d_i of D are positive real numbers.

The matrices Q and L defined in the previous paragraph have the property that the columns of $G = (L,Q)$ form an orthonormal basis for $\underline{N}(X')$. Thus, the $\hat{\delta}$ estimators that we consider will be a function of $\hat{\theta}$, $Z = Q'Y$, and $U = L'Y$. The following distributional properties are straightforward to verify :

(2.4.1) a) Z is independent of $\hat{\theta}$ and U.

b) $Z \sim N_q(0, \pi(1-\gamma)I_q)$.

c) $U \sim N_s(0, \pi D_\gamma)$ where $D_\gamma = (1-\gamma)I_s + \gamma D$.

d) $\mathrm{Cov}(U, \hat{\theta}) = \pi(1-\gamma)c$ where $c = L't$.

We have included π in these expressions for completeness, even though we shall take $\pi = 1$ in the sequel.

The components of U are statistically independent among themselves, but are possibly correlated with $\hat{\theta}$. If it happens that $c = 0$, then $\hat{\theta}$ is the blue for θ and hence is the UMVUE for θ. Thus, the cases we shall be interested in are the situations where c is nonzero.

Let the elements of c be denoted by $c_1, \ldots, c_s$. Then any estimator $\hat{\delta}$ that we consider in this thesis will have the following structure:

$$\hat{\delta} = \hat{\theta} - \Sigma_i c_i \varphi_i U_i$$

where φ_i is a measurable function of R/W_{g_i} with $R = \|Z\|^2$ and $W_{g_i} = \Sigma_j g_{ij} U_j^2$ with $g_{i1}, \ldots, g_{is}$ all positive constants. The estimators will also be such that the variance of $\hat{\delta}$ exists. It can easily be checked that any function $\hat{\delta}$ that has this structure satisfies conditions (2.3.2).

III. Review of the Literature

3.1. Introduction.

Several authors have shown that improvement on $\hat{\theta}$ (when $\gamma < 1$) can be made when X and B have particular structures. This improvement on $\hat{\theta}$ is generally known as the recovery of inter-block information in Incomplete Block (IB) designs. Recovery of inter-block information is also related to the problem of combining two unbiased estimators for a common mean with different variances. In this chapter we will give a brief review of the literature for these two topics.

3.2. Combining two unbiased estimators for a common mean.

We review here the literature on the combining of two independent unbiased estimators. This problem, as originally set out by Graybill and Deal (1959), is not exactly the same as the problem of recovering inter-block information; but the two problems are closely related and the literature on the recovery of inter-block information frequently builds on the literature we discuss below.

The basic problem assumes that one has available the following statistics:

$$\bar{X}_1 \sim N(\zeta, \pi_1/n_1) \ ,$$
$$\bar{X}_2 \sim N(\zeta, \pi_2/n_2) \ ,$$
$$m_1 S_1/\pi_1 \sim \chi^2(m_1) \ ,$$

and $\quad m_2S_2/\pi_2 \sim \chi^2(m_2)$;

where $\bar{X}_1$, $\bar{X}_2$, S_1 and S_2 are mutually independent and where $m_i = n_i - 1$, $i = 1,2$. That is, $\bar{X}_i$ and S_i are like the sample mean and the sample variance based on a sample of size n_i from a $N(\zeta, \pi_i)$ distribution. The basic problem is to use these statistics to find an unbiased estimator for ζ that has smaller variance than $\bar{X}_1$ and/or $\bar{X}_2$.

If π_1 and π_2 are known, then the linear combined estimator for ζ which is unbiased and has uniformly minimum variance is

$$\hat{\zeta}(\pi_1, \pi_2) = [n_1\pi_2\bar{X}_1 + n_2\pi_1\bar{X}_2] / [n_2\pi_1 + n_1\pi_2]$$
$$= \bar{X}_1 + (n_2\pi_1 / [n_2\pi_1 + n_1\pi_2])F,$$

where $F = \bar{X}_1 - \bar{X}_2$.

In general π_1 and π_2 are not known so that $\hat{\zeta}(\pi_1, \pi_2)$ is not a statistic. As a result, the literature has given attention to estimators of the form

$$\tilde{\zeta} = \bar{X}_1 - \phi F$$

where ϕ is a measurable function of S_1, S_2, and F^2. The function ϕ is to be selected, if possible, so that $\tilde{\zeta}$ has smaller variance than $\bar{X}_1$ and/or $\bar{X}_2$.

Graybill and Deal (1959) used $\phi = n_2S_1/(n_2S_1 + n_1S_2)$ which is the same as $\hat{\zeta}(S_1, S_2)$. The corresponding combined estimator, say $\tilde{\zeta}_G$, in this case actually is the maximum likelihood estimator of ζ (see Sinha (1979)). Graybill and Deal proved that $\tilde{\zeta}_G$ is uniformly better than $\bar{X}_1$ and $\bar{X}_2$ if and only if n_1 and n_2

are both larger than ten.

Sinha and Mouqadem (1982) proved that $\tilde{\zeta}_G$ is admissible in the class

$$C_0 = \{ \tilde{\zeta} : \phi \equiv \phi(S_2/S_1)\}$$

if $n_1, n_2 \geq 2$. And they proved that $\tilde{\zeta}_G$ is extended admissible in the class

$$C = \{ \tilde{\zeta} : \phi \equiv \phi(S_1, S_2, F^2)\}$$

if $n_1, n_2 \geq 5$. However, Mehta and Gurland (1970) and Sinha (1979) proved that $\tilde{\zeta}_G$ is inadmissible when it is known that the variance of one specific population exceeds that of the other. We note that this situation is similar to the problem in the recovery of inter-block information.

For $n_1 = n_2 = n$, Mehta and Gurland (1970) presented estimators for $n = 3,5,7,9$ and 11 which are more efficient than $\tilde{\zeta}_G$. They provided no justification for the choice of their estimators. However, Sinha and Mouqadem proved that the Mehta and Gurland estimators are similar to admissible estimators in the classes C_1 and C_2 described below:

$$C_1 = \{ \tilde{\zeta} : \phi \equiv \phi(S_1, S_2)\}$$

where any estimator corresponding to $\phi = (S_1 + \lambda_1)/(S_1 + S_2 + \lambda_2)$ with $\lambda_1 > 0$ and $\lambda_2 > 0$ is admissible. And

$$C_2 = \{ \tilde{\zeta} : \phi \equiv \phi(S_1/F^2,\ S_2/F^2)\}$$

where the estimator corresponding to $\phi = S_1/(S_1 + S_2)$ is admissible.

Zacks (1966) also studied the problem when $n_1 = n_2 = n$. His method is based on a two-sided F-test and, according to the value of the test, the decision to use $\phi = S_1/(S_1 + S_2)$ or $\phi = 1/2$ is taken. An explicit formula for the variance of his estimator is available if $n = 3$.

The class C_2 given above represents the class of equivariant estimators of ζ under the usual translation and scale transformation. This class was considered by Zacks (1970) who derived a general expression for an admissible Bayes estimator based on a proper prior distribution.

Cohen and Sachrowitz (1974) also studied the case of equal sample sizes. They offered a new form of ϕ for which the corresponding estimators are uniformly better than the sample mean based on only one of the populations for $n \geq 5$. They gave another form of an unbiased estimator for ζ which has the same property. A slight modification of one of their estimators is better than either sample mean simultaneously for $n \geq 10$.

Brown and Cohen (1974) provided another form of ϕ. The corresponding estimator is uniformly better than the sample mean of the first population provided $n_1 \geq 2$ and $n_2 \geq 3$.

Khatri and Shah (1974) also studied the estimator $\tilde{\zeta}$ using

$$\phi = S_1 /(S_1 + h_1 S_2 + h_2 F^2)$$

where h_1 and h_2 are constants to be suitably chosen. They proved that improvement over $\bar{X}_1$ is possible provided $n_2 \geq 3$ and that improvement over both $\bar{X}_1$ and $\bar{X}_2$ is possible when $(n_1 - 7)(n_2 - 7) \geq 16$.

3.3. Recovery of inter-block information.

An Incomplete Block design (IB) is a design for v treatments in b blocks. The i^{th} treatment appears in r_i plots and the j^{th} block contains k_j plots of which n_{ij} receive the i^{th} treatment. If all the blocks have the same number of plots, i.e., $k_j = k$ for all j, the design is called proper. If $r_i = r$ for each i, it is called equi-replicate. If n_{ij} takes only the values zero or one, it is called binary. In general let $Y_{ij\ell}$ denote the observable value at the ℓ^{th} of those plots of the j^{th} block which receive the i^{th} treatment. Then the assumed model is

$$Y_{ij\ell} = \mu + \tau_i + b_j + e_{ij\ell} ,$$

where $i = 1,\ldots,v$, $j = 1,\ldots,b$, $\ell = 1,\ldots,n_{ij}$ ($n_{ij} = 0$ means the i^{th} treatment does not occur in the j^{th} block). The treatment effects $\tau_1,\tau_2,\ldots,\tau_v$ are unknown constants; the block effects $b_1,b_2,\ldots,b_b$ are independent random variables distributed as $N(0,\sigma_b^2)$; and the $e_{ij\ell}$'s are independent random errors distributed as $N(0,\sigma^2)$. It is further assumed that the b_j's and $e_{ij\ell}$'s are independent.

The linear model for any IB design takes the form of the model in Chapter II. Yates (1939) initiated the recovery of inter-block information to increase the precision for estimating treatment effects. For any estimable parametric function of the treatment effects, say θ, the intra-block estimateor $\hat{\theta}$ is obtained from the usual intra-block analysis which depends on comparisons within the blocks. Yates obtained an inter-block estimator from an inter-block analysis which depends on comparisons between block totals. The basic idea in the recovery of inter-block information is to use the inter-block information to improve the intra-block estimator. In Yates' papers of (1939) and (1940), he showed how this can be done for a Cubic Lattice design and for a Balanced Incomplete Block (BIB) design respectively. Nair (1944) extended the method to Partially Balanced Incomplete Block (PBIB) designs and Rao (1947, 1956) gave the method for any proper IB design.

Since Yates' early papers, the problem of combining the intra- and inter-block estimators has received a great deal of attention in the literature. The goal has been to obtain an unbiased combined estimator for any treatment contrast that is uniformly better than the intra-block estimator in the sense of having smaller variance.

The early literature that rigorously establishes when it is possible to uniformly dominate the intra-block estimator with a nonlinear estimator concentrated on BIB designs. In these papers conditions were typically given in terms of the number of treatments v that were in the design. However, it should be realized that in the BIB design a condition on the number of treatments also imposes a condition on the number of blocks b because of Fisher's inequality (see Section 3.2 of John (1980)) which says that in a BIB design we must have $b \geq v$. We also mention that all papers in this area assumed the designs were connected.

Graybill and Weeks (1959) showed that Yates' combined estimator in a BIB design is based on a set of minimal sufficient statistics. Graybill and Seshadri (1960) showed that it is unbiased. Graybill and Deal (1959) claimed that in a BIB design the inter-block information should always be used provided either (i) $b - v \geq 10$ or (ii) $rv - b - v + 1 \geq 18$ and $b - v = 9$.

Roy and Shah (1962) studied the problem for a proper binary equi-replicate IB design under the assumption that the experimental design is connected. They gave an unbiased combined estimator for any treatment contrast based on a certain type of estimator $\hat{\rho}$ for ρ, where ρ is the ratio of inter- to intra-block variances, i.e, $\rho = (\sigma^2 + k\sigma_b^2)/\sigma^2$. They derived an expression for the increase in the variance of this estimator due to fluctuations of sampling in $\hat{\rho}$. Shah (1964a) proved that the combined estimator obtained by Roy and Shah's procedure has variance smaller than that of the corresponding intra-block estimator if ρ does not exceed 2.

Seshadri (1963a) showed that inter-block information should be used in BIB designs provided $v \geq 9$. Seshadri (1963b) proposed an unbiased estimator for any treatment effect in any BIB design which is uniformly better than either the intra- or inter-block estimators when $v > 3$.

Shah (1964b) showed that inter-block information should be recovered for the following designs: (i) A BIB design with $v \geq 6$. (ii) A simple Lattice design with $v \geq 16$. In this case it must be true that $b \geq 7$. (iii) A Triple Lattice design with $v \geq 9$ which implies that $b \geq 7$. Also, he showed for a Linked Block design with 4 or 5 blocks, that recovery of inter-block information by the Yates-Rao procedure may result in a loss of efficiency for large values of ρ.

Stein (1966) proved that in BIB designs recovery of inter-block information is possible with a guaranteed reduction in the variance of any treatment contrast estimator if the number of treatments is at least four.

Shah (1971) used a truncated estimator of the variance ratio ρ which belongs to the class of estimators given by Roy and Shah. He showed that truncation at any point less than the true value leads to a smaller variance for a combined estimator of a treatment contrast than the variance of the untruncated combined estimator. His result is applicable to any proper binary equi-replicate IB design, but there is the problem of proper choice of ρ_0, the truncation point of ρ.

Khatri and Shah (1974) showed for proper binary equi-replicate IB designs that improvement over the intra-block estimator is possible provided that the number of blocks exceeds three. Shah (1975) presented a method for evaluating the exact variance of a class of combined estimators which includes all the estimators proposed by Khatri and Shah, Graybill and Deal, Seshadri, Shah, and Yates-Rao. He compared the procedures given by Khatri and Shah with those given by both Yates and Rao for four BIB designs and for four PBIB designs. His comparison showed that the Yates-Rao procedure is very good when $\rho = 1$; and if $2 < \rho < 4$ that the two estimators are roughly equivalent, but that Khatri and Shah's estimator has slightly lower variance. When ρ is large, no procedure provides substantial improvement over the intra-block estimator. (This is to be expected because as ρ increases the intra-block estimator becomes the blue for its expecation.) Shah recommended the estimator given by Khatri and Shah if nothing is known about ρ.

Brown and Cohen (1974) also proved for BIB designs that inter-block information should be used provided $b \geq 4$. Their estimator does not offer improvement when $b = v = 3$.

Shaarawi et al.(1975) gave two procedures based on marginal likelihoods for the variance components. They restricted attention to binary proper equi-replicate designs; and in partiucular to the BIB designs. They compared estimators by simulation between the procedures of Yates, Rao, Stein, Khatri and Shah and another two suggested procedures. Their comparisons, in the case of BIB designs, showed that Stein's procedure performs not too well because it ignores the error component of the between block analysis. The estimator given by Khatri and Shah performs well for smaller designs but not for the larger ones. Their procedures gave about the same performance as Yates' procedure.

Bhattacharya (1980) considered the estimators studied in Brown and Cohen and in Khatri and Shah as two parameter families of estimators. He accommodated these two families in a single two parameter family and unified their results.

Bhattacharya (1983) presented two modifications of the combined intra- and inter-block estimator when there is a priori knowledge of the lower limit on ρ, say ρ_o. A comparison was given between each of his new modifications and a modification based on the truncated estimator of ρ given by Shah (1964, 1971). Under the conditions given in his paper, all of these modifications are better than the intra-block estimator for $\rho \geq \rho_o$, but none of them is uniformly better than the other.

The methods given by Graybill and Deal(1959), Seshadri (1963b), Shah (1964b), Stein (1966) and Shah (1971) ignored some of the error component of the between block analysis.

3.4 The Khatri and Shah estimators.

The model and notation for any IB design was given in Section 3.3. Let us write the model of that section in matrix form as

$$Y = 1_n\mu + A\tau + Bb + e$$

where A, B, τ and b are defined in the obvious way. Let $X = (1_n, A)$ and let $\beta = (\mu, \tau_1, \ldots, \tau_v)'$. Then we can write the model in the form given in Chapter 2. That is, we have $Y = X\beta + Bb + e$. For reference purposes, let us observe the following facts about the matrices A and B:

$A'A = \text{diag}(r_1, \ldots, r_v)$,

$B'B = \text{diag}(k_1, \ldots, k_b)$,

and $A'B = (n_{ij})$.

Let $N = A'B$. This matrix is frequently called the incidence matrix and its structure is important in classifying various designs.

If the design is proper, notice that $B'B = kI_b$. If it is equi-replicate, notice that $A'A = rI_v$. And if the design is binary, notice that the incidence matrix is composed of zeros and ones.

The work given by Khatri and Shah (1974, 1975) to use the inter-block information in order to improve on the intra-block estimator is applicable to any proper binary equi-replicate IB design with more than three blocks. A formula for the exact variance of their estimator is available.

In the remainder of this section we desscribe the class of estimators given by Katri and Shah. Thus, we suppose we have a proper binary equi-replicate design that is connected. In this case the matrix NN' has one eigenvalue equal to rk and the other $f = v - 1$ are strictly smaller than rk. Let $\alpha_1, \ldots, \alpha_f$ denote the eigenvalues of NN', other than the value rk; and let P_i, $i = 1, \ldots, f$, be the corresponding eigenvectors. The P_i vectors are chosen to be orthonormal and each is orthogonal to 1_v which is the eigenvector corresponding to rk. Assume that

$P_1, \ldots, P_\upsilon$ are the eigenvectors corresponding to the non-zero eigen-values of NN'.

Following Roy and Shah (1962), define the f intra-block contrasts as

$$X_{1i} = P_i' A'(I - P_B)Y, \quad i = 1, \ldots, f.$$

And define the inter-block contrasts as

$$X_{2i} = P_i'(A'P_B - rJ/bk)Y, \quad i=1, \ldots, \upsilon,$$

where J denotes a matrix of ones of the proper size. For each $i = 1, \ldots, f$, it is easy to check that $X_{1i} \sim N(\zeta_i, a_i\sigma^2)$ where $a_i = k/(rk - \alpha_i)$ and $\zeta_i = P_i'\tau$. Also $X_{2i} \sim N(\zeta_i, b_i(\sigma^2 + k\sigma_b^2))$ where $b_i = k/\alpha_i$ for $i = 1, \ldots, \upsilon$.

Let S_1 denote the intra-block error sum of squares and let S_2 denote the inter-block error sum of squares. It follows that

$$S_1/\sigma^2 \sim \chi^2(m_1) \quad \text{where} \quad m_1 = bk - b - v + 1,$$

and $\quad S_2/(\sigma^2 + k\sigma_b^2) \sim \chi^2(m_2) \quad$ where $\quad m_2 = b - f.$

Also S_1, S_2, X_{1i} and X_{2i} are all mutually independent.

Any treatment contrast can be expressed as $\zeta = \Sigma_i \ell_i \zeta_i$. The combined estimator for ζ given by Khatri and Shah is

$$\hat{\zeta} = \Sigma_{i=1}^{\upsilon} \ell_i \{X_{1i} + \phi_i(X_{2i} - X_{1i})\} + \Sigma_{i=\upsilon+1}^{f} \ell_i X_{1i}$$

where

$$\phi_i = a_i S_1 / \{ h_1 a_i S_1 + h_2 b_i S_2 + \Sigma_{j=1}^{\upsilon} h_{j+2}(X_{1j} - X_{2j})^2\}.$$

Here the h_i are constants to be suitably chosen.

IV. A General Form for a Nonlinear Unbiased Estimator

4.1. Introduction.

In this chapter we introduce a general form for a nonlinear unbiased estimator $\hat{\delta}$ for θ. This estimator is uniformly better than $\hat{\theta}$ in the sense that it has smaller variance for all $\gamma \in [0,1)$. The exact variance of $\hat{\delta}$ is given. A truncated version of the estimator is introduced which has variance smaller than or equal to the variance of $\hat{\delta}$.

4.2. Preliminary facts.

In this section we investigate some properties of a quadratic form in multivariate normal random variables. To do this, it is convenient to introduce some notation and review some facts about gamma distributions with a scale parameter of 1/2.

For each positive number a, let G(a) denote a gamma distribution with scale parameter 1/2 and shape parameter a/2; and let $g(\cdot|a)$ denote the associated density function. From standard sources we can immediately conclude the following facts:

(4.2.1) a) $g(x|a) = x^{(a-2)/2}e^{-x/2}/2^{a/2}\Gamma(a/2)$, $x \geq 0$.

b) If a is an integer, then G(a) is a chi-squared distribution with a degrees of freedom.

c) If $X \sim G(a)$ and b is a real number, then $E(X^b)$ is finite if $b > -a/2$. (In particular, if d is positive, then $E(X^d)$ is finite for $d < a/2$.)

d) If $Q_1,\ldots,Q_m$ are independent $G(a_1),\ldots,G(a_m)$ respectively, then $S = Q_1+\ldots+Q_m$ has a $G(a_\bullet)$ distribution with $a_\bullet = a_1+\ldots+a_m$.

Now for each positive real number a and for any real number b for which $a+2b$ is positive, let

$$K(a,b) = 2^b\Gamma[(a+2b)/2]/\Gamma(a/2),$$

where $\Gamma(\cdot)$ is the usual gamma function. For reference purposes we state the following lemma, the proof of which is straightforward.

Lemma 4.2.2. Let $Q \sim G(a)$. Then $E(Q^b) = K(a,b)$ for all b such that $b > -a/2$.

Let m be a positive integer. For each $m \times 1$ vector $a > 0$ (i.e., all elements of a are positive), let Q_a denote an $m \times 1$ random vector composed of independent random variables $Q_1,\ldots,Q_m$ where $Q_i \sim G(a_i)$ for $i = 1,\ldots,m$. Further, suppose that $h : R^m \to R^1$ is a measurable function so that $h(Q_a)$ is a real-valued random variable.

Lemma 4.2.3. Let $P = h(Q_a)Q_1^{b_1}Q_2^{b_2}\ldots Q_m^{b_m}$ where b_i is such that $\alpha_i = a_i + 2b_i > 0$ for $i = 1,\ldots,m$. Let α denote the vector composed of $\alpha_1,\ldots,\alpha_m$. Then $E(P)$ is finite if and only if $E[h(Q_\alpha)]$ is finite; in which case

$$E(P) = K(a_1,b_1)\ldots K(a_m,b_m)E[h(Q_\alpha)]$$

Proof: Use the fact that

$$q_i^{b_i}g(q_i|a_i) = K(a_i,b_i)g(q_i|a_i+2b_i).$$

Then the result is straightforward since the $\mathbf{Q}_i$'s are independent.□

The functions $\Gamma(\cdot)$ and $K(\cdot,\cdot)$ occur frequently in the following sections. For this reason it is handy to consider some facts about these functions. In particular:

(4.2.4) a) $K(a,0) = 1$.

b) For k a positive integer, we have $K(a,k) = a(a+2)\ldots(a+2(k-1))$ and for $a-2k > 0$, we have $K(a,-k) = [(a-2)(a-4)\ldots(a-2k)]^{-1}$.

c) $\Gamma(\alpha)$ increasing for $\alpha \geq 2$.

Now we wish to study a particular form for the function h. Let

$$\mathbf{S} = \mathbf{Q}_1 + \ldots + \mathbf{Q}_m ,$$

$$\mathbf{W} = v_1\mathbf{Q}_1 + \ldots + v_m\mathbf{Q}_m$$

and $a_\bullet = a_1 + \ldots + a_m$;

where $0 < v_1 \leq \ldots \leq v_m$. Notice that $\mathbf{S} \sim G(a_\bullet)$.

Lemma 4.2.5. Let $\eta > 0$. Then $E(\mathbf{W}^{-\eta})$ is finite if $\eta < a_\bullet/2$; in which case

$$v_m^{-\eta}E(\mathbf{S}^{-\eta}) \leq E(\mathbf{W}^{-\eta}) \leq v_1^{-\eta}E(\mathbf{S}^{-\eta}).$$

Proof: Use (4.2.1c) and the fact that $v_1\mathbf{S} \leq \mathbf{W} \leq v_m\mathbf{S}$.□

Throughout the remainder of this section we will consider the special case where $\mathbf{Q}_i = \mathbf{X}_i^2$ and where $\mathbf{X}_1,\ldots,\mathbf{X}_m$ are independent $N(0,1)$ random variables. Notice for this special case that $a_1 = \ldots = a_m = 1$ and that $a_\bullet = m$.

Lemma 4.2.6. Let $\eta > 0$ and let $T_i = X_i W^{-\eta}$. Then the following statements are true:

(a) $T_i \overset{d}{=} -T_i$ for $i = 1,\dots,m$.

(b) $T_i T_j \overset{d}{=} -T_i T_j$ for $i \neq j = 1,\dots,m$.

(c) If $\eta < (m+2)/4$, then $Var(T_i)$ is finite.

(d) If $\eta < (m+2)/4$, then each T_i has expectation zero; and the covariance between T_i and T_j $(i \neq j)$ is also zero.

The following lemma is bssically Theorem 2 in Ruben (1962).

Lemma 4.2.7. Let p be any positive number. Then the density function of $\mathbf{W}$ can be expressed as:

$$f_{\mathbf{W}}(w) = \Sigma_{j=0}^{\infty}(\ell_j/p)g(w/p\,|\,m+2j),$$

where the ℓ_j are defined as follows:

$$\ell_0 = \Pi_{i=1}^{m}(p/v_i)^{1/2},$$

$$\ell_{j+1} = \Sigma_{r=0}^{j}\ \ell_{j-r}\ \upsilon_r/2(j+1)\ ,\ j = 0,1,2,\dots.$$

$$\upsilon_r = \Sigma_{i=1}^{m}(1-p/v_i)^{r+1}\ ,\ r = 0,1,2,\dots\quad .$$

If p satisfies $0 < p < 2v_1$, then the serise $\Sigma_{j=0}^{\infty}(\ell_j/p)g(w/p\,|\,m+2j)$ is uniformly convergent.

When some of the v_i's are equal, the expressions in Lemma 4.2.7 can be simplified by taking sums and/or products over equal v_i values.

Lemma 4.2.8. If $v_1 \geq v > 0$ and $w > 0$, then $p = 2(1/v+1/w)^{-1}$ satisfies $0 < p < 2v_1$.

Lemma 4.2.9. Let $h(\mathbf{W}) = \mathbf{W}^{-\eta}$ where $0 < \eta < m/2$ and let $0 < p < 2v_1$. Then

$$E[h(\mathbf{W})] = \Sigma_{j=0}^{\infty}(\ell_j/p)\int_0^{\infty} h(w)g(w/p\,|\,m+2j)dw.$$

Proof: According to Lemma 4.2.7 we can write

$$E[h(W)] = \int_0^\infty \Sigma_{j=0}^\infty (\ell_j/p)h(w)g(w/p|m+2j)dw.$$

The result follows from Fubini's Theorem (Royden (1963) page 233) if we show that $(\ell_j/p)h(w)g(w/p|m+2j)$ is jointly integrable with respect to j and w. By Tonelli's Theorem (Royden (1963) page 234), to show joint integrability it suffices to show that

$$\int_0^\infty \Sigma_{j=0}^\infty |\ell_j/p)h(w)g(w/p|m+2j)|dw < \infty$$

Kotz, Johnson and Boyd (1967, formula 92), showed that

$$\Sigma_{j=0}^\infty |\ell_j/p|g(w/p|m+2j) \le (A/p)g(w/p|m)e^{\epsilon w/2p}$$

where $\epsilon = \max_i |1-p/v_i|$ and $A = \Pi_{i=1}^m \sqrt{p/v_i}$. Therefore,

$$\Sigma_{j=0}^\infty \int_0^\infty |(\ell_j/p)h(w)g(w/p|m+2j)|dw$$

$$\le \int_0^\infty Ah(w)g(w/p|m)e^{\epsilon w/2p}dw$$

$$= Ap^m \int_0^\infty h(pz)g(z|m)e^{\epsilon z/2}dz \qquad (z = w/p)$$

$$= Ap^{-\eta}[\Gamma(m/2)2^{m/2}]^{-1}\int_0^\infty z^{(m-2\eta-2)/2}e^{-(1-\epsilon)z/2}dz$$

which is finite becasue $\epsilon < 1$ by the way p has been chosen and because $\eta < m/2$.□

Lemma 4.2.10. Let W and p be defined as before and let $0 < \eta < m/4$. Then

$$\kappa(2p)^{-\eta}E(W^{-\eta}) > E(W^{-2\eta})$$

where $\kappa = \max\{\kappa_1, \kappa_2, 1\}$ with

$$\kappa_1 = \Gamma[(m-4\eta)/2]/\Gamma[(m-2\eta)/2]$$

and $\kappa_2 = \Gamma[(m+2-4\eta)/2]/\Gamma[(m+2-2\eta)]$.

Proof: $E(\mathbf{W}^{-\eta}) = \int_0^\infty w^{-\eta} f_{\mathbf{W}}(w)dw$. Now using Lemma 4.2.9 we get,

$$E(\mathbf{W}^{-\eta}) = \Sigma_{j=0}^{\infty} \ell_j p^{-\eta} K(m+2j,-\eta).$$

Similarly,

$$E(\mathbf{W}^{-2\eta}) = \Sigma_{j=0}^{\infty} \ell_j p^{-2\eta} K(m+2j,-2\eta).$$

Now the result is straightforward because of the way κ was selected and (4.2.4c).□

4.3. A general form.

In this section we will study the properties of the estimator $\hat{\delta}$ which was introduced in Section 2.4. Recall that

(4.3.1) $$\hat{\delta} = \hat{\theta} - \Sigma_i c_i \phi_i U_i \ ,$$

where each ϕ_i is a measurable function of $\mathbf{R}/\mathbf{W}_{g_i}$ and $\mathbf{W}_{g_i} = \Sigma_j g_{ij} U_j^2$ with $g_{i1},\ldots,g_{is}$ all positive constants.

Theorem 4.3.2. Let $\hat{\delta}$ be defined as above and assume that the variance of $\phi_i U_i$ exists for each $i = 1,\ldots,s$. Then $\hat{\delta}$ is an unbiased estimator for θ and

$$\mathrm{Var}(\hat{\delta}) = \mathrm{Var}(\hat{\theta}) - 2\Sigma_i c_i^2\{(1-\gamma)E(\phi_i U_i^2)/\mathrm{Var}(U_i) - (1/2)E(\phi_i^2 U_i^2)\}.$$

Proof: Because $\phi_i U_i \overset{d}{=} -\phi_i U_i$, it follows that $E(\phi_i U_i) = 0$. Therefore $\hat{\delta}$ is unbiased for θ. Also,

$$\mathrm{Var}(\hat{\delta}) = \mathrm{Var}(\hat{\theta}) - 2\mathrm{Cov}\{\hat{\theta}\ ,\ \Sigma_i c_i \phi_i U_i\} + \mathrm{Var}\{\Sigma_i c_i \phi_i U_i\}.$$

First note that $\mathrm{Var}\{\Sigma_i c_i \phi_i U_i\} = \Sigma_i c_i^2 E(\phi_i^2 U_i^2)$; because $\phi_i \phi_j U_i U_j \overset{d}{=} -\phi_i \phi_j U_i U_j$ which implies that the covariance between $\phi_i U_i$ and $\phi_j U_j$ $(i \neq j)$ is zero.

Next consider

$$\mathrm{Cov}\{\hat{\theta}\ ,\ \Sigma_i c_i \phi_i U_i\} = \Sigma_i c_i \mathrm{Cov}\{\hat{\theta}\ ,\ \phi_i U_i\} = \Sigma_i c_i E\{\hat{\theta}\phi_i U_i\}.$$

Now note that $\hat{\theta}\phi_i U_i$ is a function of $(\hat{\theta}, U, Z)$ which is jointly normal. Conditional on $U = u$, $\hat{\theta}$ and Z are independent and $\hat{\theta}$ has mean $\theta + (1-\gamma)c' D_\gamma^{-1} u$. Hence

$$\begin{aligned} E(\hat{\theta}\phi_i U_i) &= E[E(\hat{\theta}\phi_i U_i\,|U)] = E[\phi_i U_i E(\hat{\theta}|U)] \\ &= \theta E(\phi_i U_i) + (1-\gamma)\ \Sigma_j c_j E[\phi_i U_i U_j/\mathrm{Var}(U_j)] \\ &= (1-\gamma)c_i E[\phi_i U_i^2]/\mathrm{Var}(U_i). \end{aligned}$$

The result is now easy to obtain.□

4.4. The main result.

In the following we will study the properties of the estimator $\hat{\delta}$ in (4.3.1) with ϕ_i defined by

(4.4.1) $$\phi_i = h_i \varphi\ ,\ i = 1,\ldots,s,$$

where $\varphi = \{R/(g_o R + W_g)\}^\lambda$ with $g_o \geq 0$, $\lambda > 0$, $g = (g_1,\ldots,g_s) > 0$ and $h_1,\ldots,h_s$ are arbitrary real numbers. Our purpose is to put conditions on the choice of $\lambda, h_1,\ldots,h_s$ in terms of $g_o, g_1,\ldots,g_s$ to make $\mathrm{Var}(\hat{\delta}) \leq \mathrm{Var}(\hat{\theta})$. It is easy to see that

$$\varphi \overset{d}{=} (1-\gamma)^\lambda Q_o^\lambda \ /\ \{\sum_{i=0}^{s} v_i Q_i\}^\lambda$$

where $Q_o = R/(1-\gamma) \sim \chi^2(q)$, $Q_i = U_i^2/\mathrm{Var}(U_i) \sim \chi^2(1)$, $i = 1,\ldots,s$, $v_o = g_o(1-\gamma)$ and $v_i = g_i \mathrm{Var}(U_i)$ for $i = 1,\ldots,s$.

Let $W_1 = \sum_{i=0}^{s} v_i Q_i$. Notice that $v_i = g_i[(1-\gamma)+\gamma d_i]$, $i = 1,\ldots,s$, is strictly positive for all $\gamma \in [0,1]$ and that

$\varphi^2 U_i^2 \le R^{2\lambda} U_i^2 (W_g)^{-2\lambda}$. As a result, we know from Lemma 4.2.1c and 4.2.6 that the variance of $\phi_i U_i$ is finite for all possible $\gamma \in [0,1]$ whenever $\lambda < (s+2)/4$.

Theorem 4.4.2. Let $\hat{\delta}$ be as given by 4.3.1 where ϕ_i is given by 4.4.1. Let $0 < \lambda < (s+2)/4$. Then the variance of $\hat{\delta}$ exists and is given by

$$\mathrm{Var}(\hat{\delta}) = \mathrm{Var}(\hat{\theta}) - 2(1-\gamma)^{2\lambda} \sum_{i=1}^{s} c_i^2 B_i(\gamma),$$

where

$B_i(\gamma) = (1-\gamma)^{1-\lambda} K(q,\lambda) h_i E(W_{1i}^{-\lambda}) - (K(q,2\lambda)/2)[1+\gamma(d_i-1)] h_i^2 E(W_{2i}^{-2\lambda})$

with $W_{1i} \overset{d}{=} v_o Q_o^* + W_{gi}^*$ where $Q_o^* \sim G(q+2\lambda)$ and

$W_{gi}^* \sim v_i \chi^2(3) + \Sigma_{j\neq i} v_j \chi^2(1)$;

with $W_{2i} \overset{d}{=} v_o Q_o^{**} + W_{gi}^*$ where $Q_o^{**} \sim G(q+4\lambda)$;

and with the random variables in Q_o^*, Q_o^{**} and W_{gi}^* all independent.

Proof: In this case we have

$$E(\phi_i U_i^2) = h_i (1-\gamma)^{\lambda} \mathrm{Var}(U_i) E\{Q_o^{\lambda} Q_i W_1^{-\lambda}\}.$$

Using Lemma 4.2.3 we get

$$E(\phi_i U_i^2) = h_i (1-\gamma) \mathrm{Var}(U_i) K(q,\lambda) E(W_{1i}^{-\lambda}).$$

Similarly,

$$E(\phi_i^2 U_i^2) = h_i^2 (1-\gamma)^{2\lambda} K(q,2\lambda) \mathrm{Var}(U_i) E(W_{2i}^{-2\lambda}).$$

Now the result is easy to obtain.□

Let us now examine $B_i(\gamma)$ with the intention of choosing the h_i so that $B_i(\gamma) > 0$ for all $\gamma \in [0,1)$. We need some preliminary lemmas.

Lemma 4.4.3. Assume $1 \le \lambda < (s+2)/4$. Then for all $\gamma \in [0,1)$ we have

$$E(W_{1i}^{-\lambda}) \ge B_1^{-\lambda} K(m_1, -\lambda)$$

where $m_1 = s + 2$ if $g_o = 0$, $m_1 = s + q + 2(\lambda+1)$ if $g_o > 0$, and $B_1 = \max\{g_o, g_1, \ldots, g_s, g_1 d_1, \ldots, g_s d_s\}$.

Proof: First apply Lemma 4.2.5 depending upon whether or not $g_o = 0$. Then note that $\max_i\{v_i\} \le B_1$ for all choices of $\gamma \in [0,1)$.□

Lemma 4.4.4. Assume $1 \le \lambda < (s+2)/4$. Then for all $\gamma \in [0,1)$ we have

$$E(W_{2i}^{-2\lambda}) \le B_2^{-2\lambda} K(m_2, -2\lambda)$$

where $m_2 = s+2$ and $B_2 = \min\{ g_1, \ldots, g_s, g_1 d_1, \ldots, g_s d_s\}$.

Proof: Note that $W_{2i} \ge W_g^*$. Now use the same idea as in the proof of Lemma 4.4.3 and note that $\min\{v_1, \ldots, v_s\} \ge B_2$ for all $\gamma \in [0,1)$.□

We are now in a position to answer the question raised above about the choice of $h_1, \ldots, h_s$. Let B_1 and m_1 be defined as in Lemma 4.4.3. Set

$$A_1 = K(q,\lambda) K(m_1, -\lambda)/B_1^{\lambda}$$

and for each $i = 1, \ldots, s$ set

$$d_{1i} = \max\{1, d_i\}$$

$$A_{2i} = d_{1i} K(q, 2\lambda) K(m_2, -2\lambda)/B_2^{2\lambda}$$

$$\text{and} \quad b_i = A_1/A_{2i}$$

where B_2 and m_2 are defined in Lemma 4.4.4. Then:

Lemma 4.4.5. Suppose $1 \leq \lambda < (s+2)/4$. For each $i = 1,\ldots,s$ we have that

$$B_i(\gamma) \geq b(h_i), \text{ all } \gamma \in [0,1),$$

where $b(h_i) = h_i A_1 - h_i^2 A_{2i}/2$. Furthermore, the function $b(h_i)$ is strictly positive whenever h_i satisfies $0 < h_i < 2b_i$ and takes its maximium value at $h_i = b_i$.

Proof: Let $i \in \{1,\ldots,s\}$ and let $\gamma \in [0,1)$. Because $(1-\gamma)^{1-\lambda} \geq 1$ and because $1+\gamma(d_i-1) \leq d_{1i}$ we have

$$B_i(\gamma) \geq h_i K(q,\lambda)E(W_{1i}^{-\lambda}) - h_i^2 d_{1i} K(q,2\lambda)E(W_{2i}^{-2\lambda})/2.$$

Then using Lemmas 4.4.3 and 4.4.4 we get $B_i(\gamma) \geq b(h_i)$. A little algebra will convince the reader of the validity of the remaining statements of the Lemma.□

Remark. Let $\hat{\delta}_o = \hat{\theta} - \Sigma_i c_i h_i \varphi_o U_i$ where $\varphi_o = \{R/W_g\}^{\lambda}$. That is, φ_o is a special case of φ when $g_o = 0$. In this case we have

$$\mathrm{Var}(\hat{\delta}_o) = \mathrm{Var}(\hat{\theta}) - 2(1-\gamma)^{2\lambda}\Sigma_i c_i^2 A_i(\gamma)$$

where

$$A_i(\gamma) = (1-\gamma)^{1-\lambda}K(q,\lambda)h_i E(W_{gi}^{-\lambda}) - K(q,2\lambda)[1+\gamma(d_i-1)]h_i^2 E(W_{gi}^{-2\lambda})/2$$

with $W_{gi} \sim v_i\chi^2(2) + W_g$. Also notice that for every $i = 1,\ldots,s$ we have

$$g_i[1+\gamma(d_i-1)] \geq \min\{g_1,\ldots,g_s,g_1d_1,\ldots,g_sd_s\} = B_2$$

$$\text{and } g_i[1+\gamma(d_i-1)] \leq \max\{g_1,\ldots,g_s,g_1d_1,\ldots,g_sd_s\} = B_1.$$

Using Lemma 4.2.10 we get

$$\kappa(2p)^{-\lambda}E(W_{gi}^{-\lambda}) > E(W_{gi}^{-2\lambda}),$$

where $p = 2(1/B_1 + 1/B_2)^{-1}$ and κ is as given in Lemma 4.2.10

with $k = s + 2$. As befor we can see that

$$A_i(\gamma) \geq a_i(\gamma)$$

where

$$a_i(\gamma) = h_i(1-\gamma)^{1-\lambda}K(q,\lambda)E(W_{gi}^{-\lambda}) - h_i^2 d_{1i}(2p)^{-\lambda}K(q,2\lambda)E(W_{gi}^{-\lambda})/2.$$

With the assumption that $1 \leq \lambda < (s+2)/4$, $a_i(\gamma)$ is strictly positive whenever $0 < h_i \leq b_{oi}$ where
$b_{oi} = (2p)^{\lambda}K(q,\lambda)/d_{1i}\kappa K(q,2\lambda)$.

4.5. A truncated estimator.

Recall the general form of the estimator $\hat{\delta}$. In particular:

$$\hat{\delta} = \hat{\theta} - \Sigma_i c_i \phi_i U_i,$$

The variance of this estimator can be expresed as:

$$\begin{aligned} \mathrm{Var}(\hat{\delta}) &= \mathrm{Var}(\hat{\theta}) - 2(1-\gamma)\Sigma_i c_i^2 h_i^2 E(\phi_i U_i^2)/h_i \mathrm{Var}(U_i) + \Sigma_i c_i^2 h_i^2 E(\phi_i^2 U_i^2). \\ &= \mathrm{Var}(\hat{\theta}) + A(\gamma), \end{aligned}$$

where

$$\begin{aligned} A(\gamma) &= \Sigma_i c_i^2 E\{U_i^2 \phi_i^2 - 2(1-\gamma)U_i^2 \phi_i / \mathrm{Var}(U_i)\} \\ &= \Sigma_i c_i^2 E\{U_i^2 \phi_i [\phi_i - 2\,(1-\gamma)/((1-\gamma)+\gamma d_i)]\}. \end{aligned}$$

For each $\gamma \in [0,1)$ let,

$$K_i(\gamma) = (1-\gamma)/[(1-\gamma)+\gamma d_i] = [1+d_i\gamma/(1-\gamma)]^{-1}.$$

Then it is easy to see that $0 \leq K_i(\gamma) \leq 1$ for all $\gamma \in [0,1)$.

<u>Theorem 4.6.2.</u> Define;

$$\begin{aligned} \phi_i^+ &= \phi_i && \text{if } \phi_i \leq 2 \\ &= 2 && \text{if } \phi_i > 2 \end{aligned}$$

Then the truncated estimator $\hat{\delta}^+$ given by;

$$\hat{\delta}^+ = \hat{\theta} - \Sigma_i c_i \phi_i^+ U_i$$

is unbiased and has variance less than or equal to the variance of $\hat{\delta}$.

<u>Proof:</u> Unbaisedness is easy. Now consider

$$\begin{aligned} \mathrm{Var}(\hat{\delta}) - \mathrm{Var}(\hat{\delta}^{+}) &= \Sigma_i c_i^2 E\{U_i^2 \phi_i [\phi_i - 2K_i(\gamma)]\} \\ &\quad - \Sigma_i c_i^2 E\{U_i^2 \phi_i^{+} [\phi_i^{+} - 2K_i(\gamma)]\} \\ &= \Sigma_i c_i^2 E\{U_i^2 \Psi(\phi_i, \phi_i^{+})\}, \end{aligned}$$

where

$$\begin{aligned} \Psi(\phi_i, \phi_i^{+}) &= \phi_i(\phi_i - 2K_i(\gamma)) - \phi_i^{+}(\phi_i^{+} - 2K_i(\gamma)) \\ &= \{\phi_i(\phi_i - 2K_i(\gamma)) - 2(2 - 2K_i(\gamma))\} I_{\{\phi_i > 2\}}, \end{aligned}$$

and where $I_{\{\phi_i > 2\}} = 1$ if $\phi_i > 2$, and $= 0$ if $\phi_i \leq 2$.

Therefore $\mathrm{Var}(\hat{\delta}) \geq \mathrm{Var}(\hat{\delta}^{+})$. □

V. A Simulation Study

5.1. Introduction.

In this part of the thesis we present a simulation study for the $\hat{\delta}$ estimator. In this study we consider three types of IB designs: two for connected designs and the other for a disconnected design. The efficiency of the various estimators with respect to the Cramer'-Rao lower bound is given. The main purpose of this study is to see how the estimators behave; but not to recommend particular values for the elements of the vector h. This will be the subject of another study.

5.2. The efficiency.

Recall that $\hat{\theta} = t'Y$ and $U = L'Y$ where t and L are selected as in Section 2.3. Also recall that

$$\hat{\theta} \sim N(\theta,(1-\gamma)t't + \gamma t'BB't) \ ;$$

$$U \sim N_s(0,D_\gamma) \ , \quad D_\gamma = (1-\gamma)I + \gamma D \ ;$$

and $\quad \mathrm{Cov}(\hat{\theta},U) = (1-\gamma)c' = c_\gamma' \ , \quad c = L't.$

If γ is known, then the combined estimator between $\hat{\theta}$ and U which has minimum variance is of the form $\hat{\theta}_\gamma = \hat{\theta} + f'U$ where $\mathrm{Cov}(\hat{\theta} + f'U \ , \ U) = 0$. This implies that $f = -D_\gamma^{-1}c_\gamma$. Therefore the minimum variance unbiased estimator is

$$\hat{\theta}_\gamma = \hat{\theta} - c_\gamma' D_\gamma^{-1} U.$$

And its variance is

$$\mathrm{Var}(\hat{\theta}_\gamma) = \mathrm{Var}(\hat{\theta}) + \mathrm{Var}(c_\gamma' D_\gamma^{-1} U) - 2\mathrm{Cov}(\hat{\theta}, c_\gamma' D_\gamma^{-1} U)$$
$$= \mathrm{Var}(\hat{\theta}) - c_\gamma' D_\gamma^{-1} c_\gamma .$$

We define the efficiency of any $\hat{\delta}$ estimator previously discussed as

$$eff(\hat{\delta}|\gamma) = \mathrm{Var}(\hat{\theta}_\gamma|\gamma)/\mathrm{Var}(\hat{\delta}|\gamma).$$

Under normality we know that the variance of the estimator $\hat{\theta}_\gamma$ at γ attains its Cramer'-Rao lower bound. Thus our efficiency is the ratio of the Cramer'- Rao lower bound to the actual variance of the estimator.

5.3. The estimators.

In this section we describe the $\hat{\delta}$ estimators used in our simulation study. We use three versions of the estimator $\hat{\delta}$ given by 4.3.1 with ϕ_i as given by 4.4.1.

Let $W = U'U$. In all of our $\hat{\delta}$ estimators we use $W_g = W$. That is, we have $g_1 = \ldots = g_s = 1$. Furthermore, we have $g_o = 0$ in $\hat{\delta}_1$ and $\hat{\delta}_2$ and $g_o = 1$ in $\hat{\delta}_3$. Under these conditions B_1 and B_2 defined in the previous chapter reduce to

$$B_1 = \max\{\ 1\ ,\ d_1\ ,\ldots,\ d_s\ \}$$

$$\text{and}\quad B_2 = \min\{\ 1\ ,\ d_1\ ,\ldots,\ d_s\ \}.$$

Using these general facts, we now describe the three $\hat{\delta}$ estimators used in our simulation study.

$\hat{\delta}_1$: The estimator obtained by choosing $\varphi = (R/W)^\lambda$ and

$$h_i = K(q,\lambda)(2p)^\lambda\ /\ \kappa K(q,2\lambda)B_1$$

for $i = 1,\ldots,s$. The value of κ is obtained from Lemma 4.2.10 with $m = s + 2$ and $p = 2[1/B_1 + 1/B_2]^{-1}$. This estimator is based on the remark at the end of Section 4.4. Notice that these h_i satisfy the required conditions in the remark because $B_1 \geq d_{1i}$ for $i = 1, \ldots, s$.

$\hat{\delta}_2$: The estimator obtained by choosing $\varphi = (R/W)^{\lambda}$ and $h_i = (B_2)^{2\lambda} K(q, \lambda) K(m_1, -\lambda) \;/\; (B_1)^{\lambda} d_{1i} K(q, 2\lambda) K(m_2, -2\lambda)$ for $i = 1, \ldots, s$. In this expression $m_1 = m_2 = s + 2$. This estimator is based on Lemma 4.4.5 with $g_o = 0$ and $h_i = b_i$ for $i = 1, \ldots, s$.

$\hat{\delta}_3$: The estimator obtained by choosing $\varphi = [R/(R + W)]^{\lambda}$ and the h_i exactly the same as the h_i in the $\hat{\delta}_2$ estimator except that $m_1 = s + q + 2(\lambda + 1)$. This estimator is based on Lemma 4.4.5 using $g_o = 1$.

The simulation study covered the above three estimators with different λ values and the associated truncated estimator for each one. The truncated estimator of $\hat{\delta}_1$ gave a slight improvement over the untruncated estimator. However, the efficiency for both estimators was essentially the same. For both $\hat{\delta}_2$ and $\hat{\delta}_3$ the truncated estimators did not provide any improvement over the untruncated ones. For these reasons, we have not given the efficiencies of the truncated versions of our estimators.

Each of next three sections is devoted to a simulation study for a particular example. We mention that all d_i values in the examples are greater than one. This means $d_{1i} = d_i$ for $i = 1,\ldots,s$ and that $B_2 = 1$.

5.4. Example I.

For this example we take an incomplete block design as discussed in Section 3.3 with the following incidence matrix:

$$N = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

This is a proper binary equi-replicate design with $v = b = 9$ and $r = k = 4$. (The design is in fact a PBIB design with parameters $\lambda_1 = 1$, $\lambda_2 = 2$ and $n_1 = n_2 = 4$ where the λ_i and n_i are defined as in John (1980).) For this example we have $\underline{r}(X) = 9$, $\underline{r}(X,B) = 17$, $s = 8$ and $q = 19$. The D matrix is given by

$$D = \text{diag}(\ 15/4\ ,\ 15/4\ ,\ 15/4\ ,\ 15/4\ ,\ 3\ ,\ 3\ ,\ 3\ ,\ 3\ ).$$

We consider first the estimable parametric function

$$\theta_{11} = 18\mu + \tau_1 + \tau_4 + \tau_6 + \tau_7 + \tau_8 + \tau_9 + 4(\tau_2 + \tau_3 + \tau_5).$$

Let $\hat{\theta}_{11}$ be the intra-block estimator for θ_{11}. Then we have $\text{Var}(\hat{\theta}_{11}) = 15(1-\gamma) + 36\gamma$

and $\quad c = (\ 0,\ 0\ ,\ 0\ ,\ 0\ ,\ 3/2\sqrt{2}\ ,\ -3/2\sqrt{30}\ ,\ -3/2\sqrt{30}\ ,\ -3/4\sqrt{3}\ )'.$

The following table shows the efficiencies of our three estimators for some different values of γ and λ. It also gives the efficiency of the intra-block estimator.

Table 5.4.1. Efficiencies for θ_{11}.

γ	λ	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_3$	$\hat{\theta}_{11}$
0	0.5	.930	.929	.911	.903
	1.0	.918	.913	.903	.903
	1.5	.912	.905	.903	.903
	2.0	.908	.903	.903	.903
.2	0.5	.978	.978	.970	.975
	1.0	.971	.969	.966	.975
	1.5	.968	.966	.965	.975
	2.0	.966	.965	.965	.975
.4	0.5	.993	.993	.990	.988
	1.0	.990	.989	.988	.988
	1.5	.988	.988	.988	.988
	2.0	.988	.988	.988	.988
.6	0.5	.997	.997	.996	.996
	1.0	.996	.996	.996	.996
	1.5	.996	.996	.996	.996
	2.0	.996	.996	.996	.976
.8	0.5	1.00	1.00	1.00	.999
	1.0	.999	.999	.999	.999
	1.5	.999	.999	.999	.999
	2.0	.999	.999	.999	.999

The above table seems to suggest the following conclusions:

(5.4.1) a) For the range of γ values indicated in the table, the choice $\lambda = .5$ appears to provide better efficiency than the choices for $\lambda \geq 1$.

b) At each value of γ and for $\lambda \geq 1$, the efficiencies of $\hat{\delta}_1$ and $\hat{\delta}_2$ decrease as λ increases; while $\hat{\delta}_3$ maintains the same efficiency over the different values of λ.

c) The efficiency of all the estimators increases as γ increases.

d) As γ increases, the above estimators tend to have the same efficiency regardless of the value of λ.

5.5. Example II.

For this example we considered a disconnected design having the following incidence matrix:

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This incidence matrix gives $\underline{r}(X) = 9$, $\underline{r}(X,B) = 14$, $s = 5$ and $q = 13$. The D matrix is given by

$$D = \text{diag}(\ 3\ ,\ 3\ ,\ 10/3\ ,\ 4\ ,\ 14/3\ ).$$

We considered first the following estimable parametric function:

$$\theta_{21} = 16\mu + 12\tau_1 + \tau_3 + \tau_4 + \tau_5 + \tau_9.$$

Let $\hat{\theta}_{21}$ be the intra-block estimator for θ_{21}. Then we have $\text{Var}(\hat{\theta}_{21}) = 68(1-\gamma) + 64\gamma$

and $\quad c = (\ 0\ ,\ 0\ ,\ 10/\sqrt{60}\ ,\ 1\ ,\ 0\ )'$.

The following table shows the efficiencies of our three $\hat{\delta}$ estimators and the intra-block estimator for different values of γ and λ. The value $\lambda = 2$ does not occur here because $(s + 2)/4 = 1.75$ (i.e., see Lemma 4.4.5).

Table 5.5.1. Efficiencies for θ_{21}.

γ	λ	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_3$	$\hat{\theta}_{21}$
0	0.5	.972	.969	.963	.961
	1.0	.969	.963	.961	.961
	1.5	.964	.961	.961	.961
.2	0.5	.990	.988	.985	.983
	1.0	.987	.984	.983	.983
	1.5	.984	.983	.983	.983
.4	0.5	.996	.995	.994	.993
	1.0	.994	.993	.993	.993
	1.5	.993	.993	.993	.993
.6	0.5	.999	.999	.999	.997
	1.0	.998	.998	.997	.997
	1.5	.998	.997	.997	.997
.8	0.5	1.00	1.00	1.00	.999
	1.0	1.00	.999	.999	.999
	1.5	.999	.999	.999	.999

From this table we can draw the same conclusions as in (5.4.1).

Now we consider the following estimable parametric function:

$$\theta_{22} = -7\tau_1 + 4(\tau_2 - \tau_3) + 5(2\tau_4 + \tau_8) + 3(2\tau_5 + \tau_6 + \tau_7 - \tau_9).$$

Let $\hat{\theta}_{22}$ be the intra-block estimator for θ_{22}. Then we have

$$\mathrm{Var}(\hat{\theta}_{22}) = 96(1-\gamma) + 84\gamma$$

and $\quad c = (0 , 0 , -10/\sqrt{60} , 0 , -14/\sqrt{84})'$.

The efficiencies of $\hat{\delta}_1$, $\hat{\delta}_2$, $\hat{\delta}_3$ and the intra-block estimator are shown in the following table.

Table 5.5.2. Efficiencies for θ_{22}.

γ	λ	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_3$	$\hat{\theta}_{22}$
0	0.5	.969	.966	.960	.958
	1.0	.967	.961	.958	.958
	1.5	.962	.959	.958	.958
.2	0.5	.990	.987	.985	.983
	1.0	.987	.984	.983	.983
	1.5	.984	.983	.983	.983
.4	0.5	.996	.995	.994	.993
	1.0	.994	.993	.993	.993
	1.5	.993	.993	.993	.993
.6	0.5	.999	.999	.999	.997
	1.0	.998	.998	.997	.997
	1.5	.998	.997	.997	.997
.8	0.5	1.00	1.00	1.00	.999
	1.0	1.00	.999	.999	.999
	1.5	.999	.999	.999	.999

The same conclusions can be drawn as in (5.4.1).

5.6. Example III.

Here we considered an incomplete block design with the following incidence matrix:

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This is a BIB design with $v = 4$, $r = b = 6$ and $k = 3$ with an associate parameter equal to 4. For this design we have $\underline{r}(X) = 4$, $\underline{r}(X,B) = 11$, $s = 7$ and $q = 13$. The D matrix is given by

$$D = \text{diag}(\ 3\ ,\ 3\ ,\ 3\ ,\ 3\ ,\ 8/3\ ,\ 8/3\ ,\ 8/3\ ,\ 8/3\ ).$$

To illustrate, we consider the estimable parametric function

$$\theta_{31} = 3\mu + \tau_1 + \tau_2 + \tau_4.$$

Let $\hat{\theta}_{31}$ be the intra-block estimator for θ_{31}. Then we have $Var(\hat{\theta}_{31}) = (33/64)(1-\gamma) + (9/8)\gamma$

and $c = (0 , 0 , 0 , 0 , \sqrt{6}/16 , 1/2\sqrt{18} , 1/24)'$.

The following table shows the efficiencies of our three estimators, as well as the intra-block estimator, for the same values of γ and λ as in Example II.

Table 5.6.1. Efficiencies for θ_{31}.

γ	λ	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_3$	$\hat{\theta}_{31}$
0	0.5	.949	.948	.933	.924
	1.0	.940	.934	.925	.924
	1.5	.934	.927	.924	.924
.2	0.5	.984	.983	.976	.971
	1.0	.977	.975	.971	.971
	1.5	.974	.971	.971	.971
.4	0.5	.995	.994	.992	.989
	1.0	.991	.990	.990	.989
	1.5	.990	.989	.989	.989
.6	0.5	.999	.999	.998	.996
	1.0	.997	.997	.997	.996
	1.5	.997	.997	.996	.996
.8	0.5	1.00	1.00	1.00	.999
	1.0	.999	.999	.999	.999
	1.5	.999	.999	.999	.999

The above table to suggests the same conclusions as in (5.4.1).

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