Transmission Eigenvalues and Thermoacoustic Tomography

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Abstract

The spectrum of the interior transmission problem is related to the unique determination of the acoustic properties of a body in thermoacoustic imaging. Under a non-trapping hypothesis, we show that sparsity of the interior transmission spectrum implies a range separation condition for the thermoacoustic operator. In odd dimensions greater than or equal to three, we prove that the interior transmission spectrum for a pair of radially symmetric non-trapping sound speeds is countable, and conclude that the ranges of the associated theromacoustic maps have only trivial intersection.

Keywords: Interior Transmission Problem, Transmission Eigenvalues, Thermoacoustic Tomography, Hybrid Imaging Methods, Acoustics, Radial Symmetry

1 Introduction

The aims of this paper are to point out a connection between the interior transmission eigenvalue spectrum and a type of uniqueness question for sound speed in a wave equation and to use this connection to make some conclusions about the wave equation. The wave equation problem we consider arises in thermoacoustic tomography (TAT). In the standard model of TAT, a pressure wave is generated in a body $D \subset \mathbb{R}^d$

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whose sound speed is a pertubation of a constant background sound speed which we will take throughout to be unity. We take $u(x,t)$ to be the solution of the Cauchy problem

$$\begin{aligned} \partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ on } \mathbb{R}^d \times \mathbb{R}_+ \\ u(x,0) = f(x),\ \partial_t u(x,0) = 0 \text{ for } x \in \mathbb{R}^d \end{aligned} \tag{1.1}$$

for some initial pressure disturbance $f(x)$ supported in D. Data is measured on the boundary of the body

$$g(x,t) = u(x,t)|_{\partial D\times\mathbb{R}_+}. \tag{1.2}$$

The inverse TAT problem is to reconstruct $f(x)$ from data $g(x,t)$. To ensure that f might be determined by g it is essential that the solution $u(x,t)$ extend to a solution on all of $\mathbb{R}^d$.

It has become apparent that, to at least some extent, the measurements of an ultrasound field acquired in thermoacoustic tomography determine the acoustic properties of the body being imaged [21, 26, 10]. In [10] it was proved for example that the thermoacoustic data of a non-trivial source determines the speed from among its constant multiples. Numerical work trying to recover both f and c can be found in [11, 21, 22, 23, 24, 25, 26]. A more substantial theoretical result is due to Stefanov and Uhlmann [18], who proved that if the TAT data for two waves speeds and a very special common initial value agree, and if the speeds and domain satisfy some additional geometric hypotheses, then they must be equal. In this paper we are motivated to study a generalized interior transmission eigenvalue problem (ITP) by looking for conditions on two speeds that ensure the TAT measurements they generate are distinct.

To make the connection between the wave equation and the interior transmission problem, we take the temporal Fourier transform of the solution of the wave equation to get a family of Helmholtz equations. Since we need to appeal to analyticity of this Fourier transform, we must assume rapid decay in time of solutions of the wave equation. This will restrict the classes of sound speed we can treat, and will confine the discussion to odd dimensions.

Our uniqueness result asserts that, under suitable hypotheses on sound speeds, the ranges of two thermoacoustic maps have trivial intersection provided that the associated transmission eigenvalue problem has not too many solutions. In particular, discrete spectrum is sufficient. However, to date we know of no results in the literature which imply discrete transmission eigenvalue spectrum for smooth index of refraction without assuming constant sign for the contrast. For the special case of non-trapping radial sound speeds, we prove that in odd dimension at least three, the transmission eigenvalue spectrum for a distinct pair of sound speeds is discrete.

2 Background and Notation

Let u be the solution of (1.1) above.

Definition 2.1. *The* thermoacoustic map on D with sound speed $c(x)$ *is defined by*

$$\mathcal{L}_{c(x)} f(x) = u(x,t)|_{\partial D \times \mathbb{R}_+} \tag{2.1}$$

for all initial pressure disturbances $f(x) \in C_0^\infty(D)$ and $u(x,t)$ a solution of (1.1).

For a given domain D each sound speed defines a new thermoacoustic map. In this paper we only consider a special class of sound speeds.

Definition 2.2. *An* acoustic profile on a domain D *is a smooth function* $c(x) \in C^\infty(\mathbb{R}^d)$ *such that*

i*)* $0 < \sigma < c(x) < \infty$ *for* $x \in D$ *for some* $\sigma > 0$

ii*)* $\mathrm{supp}(1 - c(x)) \subset D$*.*

Properties of the range of $\mathcal{L}_{c(x)}$ are related to the unique determination of the acoustic speed $c(x)$ by which we mean the following. Given a non-zero function on ∂D which lies in the range of some thermoacoustic map, is there only one acoustic profile c for which it belongs to the range of $\mathcal{L}_c$? We do not address the question of recovering c in the case when it is unique, nor whether there is any stability estimate. In an analogous question for the linearized forward map, Stefanov and Uhlmann [17] have recently proved an instability result. We refine the question of unique determination by restricting the class of acoustic profiles.

Definition 2.3. *Let $\mathscr{D}$ be a set of acoustic profiles on some domain D. We say that a profile $c(x) \in \mathscr{D}$ is uniquely determined in $\mathscr{D}$ by thermoacoustic data if and only if*

$$\mathcal{R}g(\mathcal{L}_{c(x)}) \cap \mathcal{R}g(\mathcal{L}_{b(x)}) = \{0\}$$

for every $b(x) \in \mathscr{D}$ with $b(x) \neq c(x)$.

The goal is to find conditions on $c(x)$ and $b(x)$ such that

$$\mathcal{R}g(\mathcal{L}_{c(x)}) \cap \mathcal{R}g(\mathcal{L}_{b(x)}) = \{0\}. \tag{2.2}$$

To find these we prove results about the analyticity of the temporal Fourier transform over the range of $\mathcal{L}_{c(x)}$.

Definition 2.4. *The* temporal Fourier transform *of a function $u(x,t)$, with support in $\mathbb{R}^d \times \mathbb{R}_+$, is defined by*

$$\hat{u}(x,k) = \frac{1}{2\pi} \int_0^\infty u(x,t) e^{ikt}\, dt.$$

The next proposition follows directly from analyticity in a strip of the Fourier transform of an exponentially decaying function.

Proposition 2.5. *If the solution $u(x,t)$ of the forward thermoacoustic problem in domain $D \subset \mathbb{R}^d$ has exponential decay in time uniformly over the closure of D, then there is an open strip in $\mathbb{C}$ containing $\mathbb{R}_+$ such that for each fixed $x \in D$ the real part of the temporal Fourier transform, $\mathcal{R}e(\hat{u})(x,k)$, is the restriction to $\mathbb{R}_+$ of a function analytic in the strip.*

Some sufficient conditions for exponential decay are known.

Definition 2.6. *An acoustic profile $c(x)$ is said to be* non-trapping *if solutions, the* bicharacteristics, *to*

$$\begin{cases} \dot{x} = c^2(x)\xi \\ \dot{\xi} = -\frac{1}{2}\nabla(c^2(x))|\xi|^2 \\ x(0) = x_0,\ \xi(0) = \xi_0, \end{cases} \tag{2.3}$$

in $\mathbb{R}^{2n}_{x,\xi}$ have projections, rays, *in $\mathbb{R}^d_x$ tending to infinity as $t \to \infty$ as long as $\xi_0 \neq 0$.*

If this holds, one has the following theorem of Vainberg. [19, 20].

Theorem 2.7. *If the non-trapping condition is satisfied, then for any multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)$, the following estimate holds for solutions of the thermoacoustic forward problem:*

$$\left|\partial^{\alpha}_{(t,x)} u(x,t)\right| \leq C\eta(t)\|f\|_{L^2},\ x \in D.$$

Here the function $\eta(t)$ that characterizes the decay is $t^{-d-\alpha_0+1}$ when the dimension d is even and $e^{-\epsilon t}$ when $d \geq 3$ is odd.

By the theorem of Vainberg, analyticity of the temporal Fourier transform in a neighborhood of the positive real axis will hold for non-trapping speeds in odd dimensions greater than one. In the remainder of this paper, we shall implicitly assume all acoustic profiles are non-trapping. Non-trapping also ensures that for a single sound speed c, the thermoacoustic map $\mathcal{L}_c$ is injective, [7, 16].

3 Relation of TAT to the Interior Transmission Problem

The following relation of the wave equation to the Helmholtz equation is standard.

Proposition 3.1. *Let $u(x,t)$ satisfy*

$$\begin{aligned} &\partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ in } \mathbb{R}^d \times \mathbb{R}_+ \\ &u(x,0) = f(x),\ \partial_t u(x,t) = 0 \text{ on } \mathbb{R}^d \end{aligned} \tag{3.1}$$

and set $n(x) = \frac{1}{c^2(x)}$*. For* $k \in \mathbb{R}_+$ *the temporal Fourier transform* $\hat{u}(x,k)$ *satisfies*

$$\Delta \hat{u}(x,k) + k^2 n(x)\hat{u}(x,k) = \frac{ik}{2\pi} n(x) f(x) \text{ for } x \in \mathbb{R}^d \tag{3.2}$$

and $U(x,k) = \mathcal{R}e(\hat{u})(x,k)$ *satisfies*

$$\Delta U(x,k) + k^2 n(x) U(x,k) = 0 \text{ for } x \in \mathbb{R}^d. \tag{3.3}$$

We now suppose that c and b are two acoustic profiles, and f_1, f_2 two initial conditions supported in D such that $\mathcal{L}_{c(x)}(f_1) = \mathcal{L}_{b(x)}(f_2)$. Let u, v denote the respective solutions of (1.1). Then as $c = b = 1$ and $f_1 = f_2 = 0$ in $\mathbb{R}^d \setminus D$, u, v are solutions of the (same) wave equation in the exterior domain, with the same boundary value on $\partial D \times \mathbb{R}_+$ and with the same (zero) initial conditions. Since the exterior initial boundary-value problem is well-posed, the solutions are equal in the exterior domain. Let $\hat{u}$ and $\hat{v}$ be the temporal Fourier transforms. This implies that $\hat{u}(x,k) = \hat{v}(x,k)$ in $\mathbb{R}^d \setminus D$ for all k so their normal derivatives on ∂D are equal. Since by hypothesis, $u = v$ on $\partial D \times \mathbb{R}_+$, then $\hat{u}(x,k) = \hat{v}(x,k)$ for $x \in \partial D$. Then $U = \Re(\hat{u})$ and $V = \Re(\hat{v})$ are solutions of the following problem.

Definition 3.2. *A wavenumber* $k \geq 0$ *is called a* transmission eigenvalue *if there exists an non-trivial pair* $(u, v) \in H^2(D) \times H^2(D)$ *solving the* interior transmission problem *(ITP) relative to the acoustic profiles* $c(x)$ *and* $b(x)$ *in* D *if*

$$\begin{aligned} \Delta u + k^2 n_c(x) u &= 0 \text{ in } D \\ \Delta v + k^2 n_b(x) v &= 0 \text{ in } D \\ u = v,\ \partial_\nu u &= \partial_\nu v \text{ on } \partial D. \end{aligned} \tag{3.4}$$

Here, ∂_ν *represents the outward normal derivative on* ∂D*,* $n_c(x) = c^{-2}(x)$ *and* $n_b(x)$ *is defined similarly.*

Definition 3.3. *The* real transmission eigenvalue spectrum *is the set of non-negative real transmission eigenvalues.*

Remark 3.4. The interior transmission problem arose in scattering theory. In that setting, one of the sound speeds is usually taken to be constant, but here it is natural to assume that both are variable. Researchers in scattering theory have profitably considered complex transmission eigenvalues, but they play no role here.

The next theorem shows range separation conditions may be derived from sparseness of the interior transmission spectrum.

Theorem 3.5. *Let* $c(x)$ *and* $b(x)$ *be non-trapping acoustic profiles in a domain* D*. If the complement of the interior transmission spectrum has a finite cluster point in* $\mathbb{R}_+$ *then the intersection of the range of the thermoacoustic operators,* $\mathcal{L}_{c(x)}$ *and* $\mathcal{L}_{b(x)}$ *reduces to zero.*

Proof. Let u, v be the solutions corresponding to f_1, f_2 with $\mathcal{L}_c f_1 = \mathcal{L}_b f_2$. We have already observed that $U(x,k) = \Re(\hat{u})$ and $V(x,k) = \Re(\hat{v})$ satisfy (3.4) for every $k \in \mathbb{R}_+$. If k is not a transmission eigenvalue, then the only solution to (3.4) is the zero function, hence for such k, $U(x,k) = V(x,k) = 0$ in D. Since by lemma 2.5 both U and V are real analytic on the positive real axis, if they are zero on set with a finite accumulation point, they are identically zero. Then u, v must be zero. □

Therefore, any result which implies that the real transmission spectrum associated to the pair n_c, n_b is discrete, or has a positive lower bound will imply that the ranges of the corresponding thermoacoustic operators have only trivial intersection. If this holds for every pair in a class $\mathcal{D}$, then acoustic profiles are uniquely determined within the class.

On a domain D, let n_c and n_b be associated to acoustic profiles c and b. Define the *contrast* to be the difference:

$$m_{cb}(x) = n_c(x) - n_b(x)$$

and note $supp(m_{cb}) \subset \bar{D}$. There are many results in the literature giving conditions on m_{cb} which are sufficient to guarantee discrete spectrum, or a spectral gap. These are usually insufficient for our purposes however, since in most cases there is an assumption that m_{cb} is discontinuous at some interface, while we require that m_{cb} be smooth.

One easy result is the following theorem, which is similar to the lower bound in [6], for the case when m_{cb} is either non-negative or non-positive. Let λ_0 be the first eigenvalue of the Dirichlet Laplacian in D and set $n_i^* = \sup_D n_i(x)$ for $i = c, b$.

Theorem 3.6. *If $k \in \mathbb{R}_+$ is a transmission eigenvalue and $m(x) = n_c(x) - n_b(x) \geq 0$ then*

$$k \geq \sqrt{\frac{\lambda_0}{n_c^*}}.$$

If $m(x) = n_c(x) - n_b(x) \leq 0$ then

$$k \geq \sqrt{\frac{\lambda_0}{n_b^*}}.$$

We omit the proof.

Theorem 3.7. *Consider two profiles, $c(x)$ and $b(x)$, relative to a domain D. If*

$$c(x) - b(x) \geq 0 \text{ or } c(x) - b(x) \leq 0$$

in D then the thermoacoustic data generated by the domain D from the acoustics $c(x)$ cannot be generated by the domain D with the profile $b(x)$. That is, the intersection of the ranges of the operators $\mathcal{L}_{c(x)}$ and $\mathcal{L}_{b(x)}$ is zero for any two acoustic profiles whose difference does not change signs in D.

Proof. The proof follows by noticing that $c(x) - b(x)$ not changing signs in D is equivalent to $m(x)$ not changing signs in D. Then one appeals to theorem 3.6 and theorem 3.5. □

Corollary 3.8. *Suppose thermoacoustic data $h(x,t)$ on $\partial D \times \mathbb{R}_+$ is generated by an acoustic profile in some set $\mathscr{D}$. Assume also that for every pair $c(x)$, $b(x) \in \mathscr{D}$*

$$c(x) - b(x) \geq 0 \text{ or } c(x) - b(x) \leq 0$$

on D. Then the acoustic profile generating data $h(x,t)$ is determined uniquely in $\mathscr{D}$.

4 Radially Symmetric ITP

In this section we restrict to the class $\mathcal{D}_S$ of radially symmetric acoustic profiles in a ball. We prove sparsity of the transmission spectrum for a distinct pair from $\mathcal{D}_S$.

We assume $d \geq 3$ is odd, and assume that D is the unit ball B_1. Let S_1 be the unit sphere. The first step in the study is to reduce the problem to an ordinary differential equation. Following [4, 13, 12, 3] the Helmholtz equation with radial sound speed can be separated in spherical coordinates as a sum of products of radial functions with spherical harmonics, where the radial components satisfy an ordinary differential equation depending on the degree of the spherical harmonic. Using the independence of spherical harmonics, a pair of solutions corresponding to two radial acoustic profiles satisfies (3.4) for some k^2 if and only if the radial components corresponding to some spherical harmonic have the same Cauchy data at $r = 1$. It is convenient to make a Liouville transformation on the resulting radial differential equations. We summarize with the following lemmas.

Lemma 4.1. *For a radially symmetric, smooth, refractive index $n(r)$ a solution $w(r,\theta)$ of*

$$\Delta w + k^2 n(r) w = 0 \tag{4.1}$$

is given by

$$w(r,\theta) = \sum_{j=0}^{\infty} \sum_{l=0}^{M} f_{jl}(r) r^j Y_{jl}(\theta). \tag{4.2}$$

The coefficient functions, $f_{jl}(r)$, satisfy the Sturm-Liouville equation

$$\partial_r (r^\gamma \partial_r f_{jl}) + k^2 n(r) r^\gamma f_{jl} = 0 \text{ on } [0,1], \tag{4.3}$$

where $\gamma = d + 2j - 1$. The $f_{jl}(r)$ are bounded at $r = 0$ and so are constant multiples of single such f_j. Set $m = \frac{\gamma}{2} - 1 = j + \frac{d-3}{2}$. Defining X_m by the Liouville transformation,

$$\eta = \int_0^r \sqrt{n(\sigma)}\, d\sigma,\ X_m(\eta) = [r^{2\gamma} n(r)]^{1/4} f_j(r) \tag{4.4}$$

then $X_m(\eta)$ *satisfies*

$$-\ddot{X}_m(\eta) + \left(\frac{m(m+1)}{\eta^2} + p_m(\eta)\right) X_m(\eta) = k^2 X_m(\eta) \text{ for } \eta \in [0, C], \tag{4.5}$$

where

$$p_m(\eta) = \frac{1}{4}\frac{\ddot{n}(r)}{(n(r))^2} - \frac{5}{16}\frac{(\dot{n}(r))^2}{(n(r))^3} + m(m+1)\left(\frac{1}{r^2 n(r)} - \frac{1}{\eta^2}\right) \tag{4.6}$$

and $C = \int_0^1 \sqrt{n(s)}\, ds$.

Note that, in the summation (4.2), the index M depends on the order j and dimension d. The specific value of M is not important to our results, it is the dimension of the space of spherical harmonics of order j in dimension d.

Proof. Everything is standard, except perhaps the form $f_{jl}(r)r^j$ which results from the smoothness of w and the orthogonality of the spherical harmonics to lower degree terms in the Taylor expansion of w. The independence (up to constant) of f_{jl} on l results from the singular S-L equation (4.3) having a unique (normalized) bounded solution, see [8, 1, 2]. □

The proof of the following lemma is included in the appendix, section 6.1.

Lemma 4.2. *For a radial acoustic profile* $c(r)$ *the coefficient function (4.6) with* $n(r) = c^{-2}(r)$ *is bounded on the interval* $[0, C]$, $C = \eta(1)$.

We will use $X_m(\eta)$ to denote a solution of equation (4.5). The boundedness of f_j at $r = 0$ imposes a boundary condition on the $X_m(\eta)$ at $\eta = 0$, namely

$$\lim_{\eta\to 0} \eta^{-(m+1)} X_m(\eta) < \infty, \tag{4.7}$$

since $\frac{\eta}{r}$ has a finite positive limit at $r = 0$.

Following [8, 1, 2], (4.5) has a fundamental set of solutions X_{m1}, X_{m2} satisfying

$$\lim_{\eta\to 0} \eta^{-(m+1)} X_{m1}(\eta, k) = 1, \ \lim_{\eta\to 0} \eta^{m} X_{m2}(\eta, k) = 1. \tag{4.8}$$

From (4.7) we see that $X_m(\eta, k)$ must be a constant multiple of X_{m1}.

We now consider two radially symmetric acoustic profiles. If k is a transmission eigenvalue for the pair on B_1, then equality of the Cauchy data of the corresponding Helmholtz equations implies equality of each spherical harmonic component of the Cauchy data, and non-triviality of the solution implies that some term in the spherical harmonic expansion is non-trivial. Combined with the respective Liouville transformations, this proves most of the following result.

Lemma 4.3. *For two radially symmetric acoustic profiles $c(r)$ and $b(r)$ on $B_1(0) \subset \mathbb{R}^d$ the transmission spectrum is equal to the set of all $k \in \mathbb{R}_+$ such that, for some $m = j + \frac{1}{2}(d-3)$, there exists non-trivial solutions $(X_m(\eta, k), Z_m(\xi, k)) \in C^2((0, C]) \times C^2((0, B])$ satisfying*

$$\begin{aligned} \ddot{X}_m(\eta, k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta, k) &= p_{1m}(\eta) X_m(\eta, k),\ \eta \in [0, C] \\ \ddot{Z}_m(\xi, k) + \left(k^2 - \frac{m(m+1)}{\xi^2}\right) Z_m(\xi, k) &= p_{2m}(\xi) Z_m(\xi, k),\ \xi \in [0, B] \\ \lim_{\eta \to 0} \eta^{-(m+1)} X_m(\eta, k) < \infty,\ &\lim_{\xi \to 0} \xi^{-(m+1)} Z_m(\xi, k) < \infty \\ X_m(C, k) = Z_m(B, k),\ \dot{X}_m(C, k) &= \dot{Z}_m(B, k). \end{aligned} \tag{4.9}$$

In the statement of the lemma, recall that $n_c(r) = c^{-2}(r)$, $n_b(r) = b^{-2}(r)$. The Liouville transform (4.4) then yields the coefficients $p_{1m}(\eta)$ and $p_{2m}(\xi)$ defined in (4.6). Here $\xi = \xi(r)$ is the new independent variable introduced by (4.4) using $n_b(r)$. The right endpoints are defined by $C = \int_0^1 \sqrt{n_c(s)}\, ds$ and $B = \int_0^1 \sqrt{n_b(s)}\, ds$.

Proof. The only matter left to check is the equality of X_m, Z_m and their derivative at the endpoints of their respective intervals. This just requires tracing through their definition by the Liouville transform. □

Remark 4.4. The condition above depends on $m = j + \frac{1}{2}(d-3)$. Thus the transmission spectrum in odd dimensions greater than three is a subset of the transmission spectrum in dimension three. (However, the multiplicity of each transmission eigenvalue is greater, since the dimension of the space of spherical harmonics grows with dimension.) Any result which implies sparseness in dimension three implies sparseness in higher odd dimensions.

Let $X_{m1}(\eta, k)$ and $Z_{m1}(\xi, k)$ be the solutions of (4.9) corresponding to $c(r)$ and $b(r)$, respectively, satisfying

$$\lim_{\eta \to 0} \eta^{-(m+1)} X_{m1}(\eta, k) = 1 \text{ and } \lim_{\xi \to 0} \xi^{-(m+1)} Z_{m1}(\xi, k) = 1.$$

We denote the Wronskian of two fundamental solutions evaluated at different endpoints by

$$W(Z_{m1}(B, k), X_{m1}(C, k)) = Z_{m1}(B, k) \dot{X}_{m1}(C, k) - \dot{Z}_{m1}(B, k) X_{m1}(C, k).$$

Corollary 4.5. *Let $d \geq 3$ be odd. For two radially symmetric acoustic speeds $c(r)$ and $b(r)$ on $B_1(0) \subset \mathbb{R}^d$, $k \in \mathbb{R}_+$ is a transmission eigenvalue if and only if*

$$W(Z_{m1}(B, k), X_{m1}(C, k)) = 0 \tag{4.10}$$

for some integer $m \geq \frac{1}{2}(d-3)$.

Proof. By Theorem 4.3, k is transmission eigenvalue if and only if there exists a non-trivial pair X_m, Z_m as in the theorem. These must be multiples of X_{m1}, Z_{m1} by the condition at 0. The equalities at the right endpoints imply $\alpha X_{m1}(C) = \beta Z_{m1}(B)$ and similarly for the derivatives, which is a linear system for α, β which has a non-trivial solution if and only if the Wronskian condition holds. □

We will often use the shorter notation

$$d_m(k) = Z_{m1}(B)\dot{X}_{m1}(C) - \dot{Z}_{m1}(B)X_{m1}(C)$$

As in the case of the usual transmission eigenvalue problem, the determinant is the restriction to the positive real axis of an entire function of exponential type. We note the following

Proposition 4.6. *The transmission spectrum associated to radial acoustic profiles $c(r), b(r)$ is uncountable if and only if for some m, $W(Z_m(B), X_m(C))$ is identically zero as a function of k.*

Proof. If $d_m(k)$ is identically zero, then every k is a transmission eigenvalue, since X_m and Z_m are non-trivial. Conversely, if the transmission spectrum is uncountable, then the transmission spectrum is uncountable for some m, and so $d_m(k)$ determinant vanishes on an uncountable set. Since d_m is analytic, it vanishes identically. □

Let us now assume that the transmission spectrum associated to radial sound speeds $b(r), c(r)$ is uncountable. Our main result of this section is the following theorem.

Theorem 4.7. *Let $b(r), c(r)$ be radial sound speeds with n_b, n_c the associated index of refraction. If the transmission spectrum for the pair n_b, n_c is uncountable, then $n_c = n_b$.*

Proof. The proof has two steps. The first is to establish that $B = C$; that is, that the interval $[0, 1]$ has the same length in the respective slowness metrics. In the case of the standard transmission eigenvalue problem when one of the speeds is unity, there is an asymptotic expression for the determinant ([3], eq. (8.38)) which implies discreteness of the zeros when $C \neq 1$ (taking $b = 1$). The proof in the general situation is slightly more complicated, but follows the same line making use of the asymptotics of the solutions resulting from their expression as analytic functions of the potentials. We omit the details here: they can be found in [9].

By proposition 4.6, we may now assume that for some fixed m, the determinant d_m is identically zero. We will select some particular values of k. Let k_1^2 be a Dirichlet eigenvalue for (4.5) for $p = p_{1m}$ on $[0, C]$. Since the space of solutions satisfying the boundary condition at $\eta = 0$ is one dimensional, X_m must be a Dirichlet eigenfunction. Since X_m has the same Cauchy data at $\eta = C$ as Z_m, then Z_m must also be a Dirichlet eigenfunction and so k_1^2 is a Dirichlet eigenvalue for (4.5) for $q = p_{2m}$ on $[0, C]$. Reversing the roles shows that potentials p and q have the same Dirichlet spectrum. A similar argument shows that if k^2 is chosen from the Neumann spectrum of p ($X_m'(C) = 0$ with the boundedness condition at $\eta = 0$) then k^2 must also belong to the Neumann spectrum for q, and conversely. However, it is known ([2], Theorem 1.3) that equality of spectra for the Bessel type operator for two independent boundary conditions implies equality of the potentials. Thus $p_{1m} = p_{2m}$ on $[0, C]$. The proof is finished by applying the following proposition. □

Proposition 4.8. *Let p_{1m}, p_{2m} be the coefficients defined through (4.4)-(4.6) on $[0, C]$. Also, assume that $[0, 1]$ has the same length with respect to the two metrics defined by n_c and n_b, that is assume $B = C$. If $p_{1m} = p_{2m}$ then $n_c = n_b$.*

Proof. The proof is a minor modification of the proof of Theorem 4.3 in [15], which treated the case $m = 1$. Since [15] is not easily accessible (some of its results were summarized in [14]), we present it here. Denote by $r_1(\eta)$ and $r_2(\xi)$ the inverses of $\eta(r)$ and $\xi(r)$ respectively, and replace n_c, n_b by n_1, n_2 respectively. The equality of p_{1m} and p_{2m} is to be interpreted as equality when the first is evaluated at $r = r_1(\eta)$ and the second at $r = r_2(\eta)$. Here both r_1 and r_2 may be evaluated at η since $B = C$ implies r_1 and r_2 are defined over the same domain. Now, for $i = 1, 2$ define $u_i(\eta)$ by

$$u_i(\eta) = \frac{n_i^{1/4}(r_i(\eta))}{r_i(\eta)^m}.$$

Then a calculation shows that u_i satisfies

$$\ddot{u}_i(\eta) - \left(p_{im}(\eta) + \frac{m(m+1)}{\eta^2}\right) u_i = 0, \quad 0 < \eta \le C,$$
$$u_i(C) = C^{-m} \qquad \dot{u}_i(C) = -mC^{-(m+1)},$$

where we have used that $n_i(1) = 1$ and $n_i'(1) = 0$. Applying the uniqueness theorem for ordinary differential equations, it holds that $u_1(\eta) = u_2(\eta)$ on $(0, C]$. Taking reciprocals and squaring gives that $(r_1^{2m+1} - r_2^{2m+1})' = 0$ using that $r_i' = \frac{1}{\sqrt{n_i}}$. Since $r_1(C) = 1 = r_2(C)$, the difference is zero on the interval. Taking $(2m+1)^{th}$ roots gives $r_1 = r_2$ and differentiation finishes the proof. □

Remark 4.9. Theorem 2.1 of [5] has a similar statement to Theorem 4.7, but an examination of the proof shows that the hypothesis is really that d_m is identically zero for all m.

The following theorem and corollary summarizes the implications of the results of this section for the TAT problem.

Theorem 4.10. *Let $c(r)$ and $b(r)$ be non-trapping, radially symmetric, acoustic profiles in the unit ball, $B_1 \subset \mathbb{R}^d$ with $d \ge 3$ odd. If $c(r) \ne b(r)$ then the intersection of the range of the thermoacoustic operators, $\mathcal{L}_{c(r)}$ and $\mathcal{L}_{b(r)}$ reduces to zero.*

Corollary 4.11. *Suppose $d \geq 3$ is odd and that thermoacoustic data $h(x,t)$ on $\partial B_1 \times \mathbb{R}_+$ is generated by a radially symmetric, non-trapping acoustic profile. Then the acoustic profile generating data $h(x,t)$ is uniquely determined among the set of radially symmetric, non-trapping acoustic profiles.*

5 Conclusion

This work details a relation between the unique determination of the acoustic profile of a body from thermoacoustic data and properties of the spectrum of the interior transmission eigenvalue problem. The difference of two acoustic profiles gives a constrast which does not realistically satisfy the constant sign or coercivity hypotheses of previous work on the transmission eigenvalue problem. Radial (non-trapping) sound speeds give the simplest examples of such profiles, and we have succeeded to show that the associated spectrum of pairs within this class is discrete. It would be of great interest to develop a general method to analyze the transmission spectrum for two acoustic profiles without positivity assumptions or radial symmetry assumptions.

Our analysis does not put any *a priori* conditions on the allowable initial impulses $f(x)$ besides $\text{supp}(f) \subset D$. Since we study the uniqueness question in terms of separation of ranges of operators $\mathcal{L}_{c(x)}$, restriction of the domain might lead to new uniqueness results. In particular, focussed initial impulses might offer the possibility of both uniqueness and inversion.

6 Appendix

6.1 Proof of boundedness of $p_m(\eta)$

Here we prove lemma 4.2. First, notice that since $c(r)$ is an acoustic profile it must be smooth, bounded, even in r, and bounded away from zero. Thus, $n(r)$ is smooth, bounded, even in r, and bounded away from zero. Moreover all derivatives of $c(r)$ and $n(r)$ must also be bounded. This implies that the first two terms in $p_m(\eta)$ are bounded on $[0, C]$. It remains to show that the term

$$\frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)}$$

is bounded on $[0, C]$, for which it suffices to prove that it has a finite limit at 0. Writing the difference

$$\frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)} = \frac{1}{n(r)} \left(1 + \sqrt{n}\frac{r}{\eta}\right) \left(\frac{\eta - \sqrt{n}r}{r^2 \eta}\right).$$

The second term on the right has limit equal to 2, while integration by parts in the definition of η gives that $\eta = \sqrt{n}r - \int_0^r s\phi'(s)\, ds$, where $\phi = \sqrt{n} = \frac{1}{c}$. This simplifies the numerator of the last term on the right so that an application of l'Hospital's rule conveniently shows that its limit is $-2\frac{\phi''(0)}{\eta'(0)}$.

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