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The Stokes Problem of Fluid Mechanics, Riesz Transform, and the Helmholtz-Hodge Decomposition: Probabilistic Methods and their Representations

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The flow of incompressible, viscous fluids in $\mathbb{R}^3$ is governed by the non-linear Navier-Stokes equations. Two common linearizations of the Navier-Stokes equations, the Stokes equations and the Oseen equations, are studied in this thesis using probabilistic methods. The incompressibility condition presents new challenges for the well known theory relating partial differential equations and stochastic processes. In this thesis we construct probabilistic solutions to the incompressible Stokes equations in the absence of boundaries, in the case of the half space, and we make some observations for general domains. Also we give probabilistic representations for the iterated Riesz transforms, the Helmholtz-Hodge decomposition on domains with smooth boundaries as well as the free space, and the solutions to the Neumann problems on exterior domains and the half space.

The Stokes Problem of Fluid Mechanics, Riesz Transform, and the Helmholtz-Hodge Decomposition: Probabilistic Methods and their Representations

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

HoeWoon Kim, Author

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Academic

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1 INTRODUCTION

The 3-dimensional Navier-Stokes equations governing viscous, incompressible fluid velocity $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ for $x \in \mathbb{R}^3, t \geq 0$, with initial data $u(x,0) = u_0(x)$, are mathematical description of the second law of Newton and mass conservation:

$$\frac{\partial u}{\partial t} + (u \cdot \triangledown)u = \nu \Delta u - \triangledown p + f, \quad \triangledown \cdot u = 0, \quad u(x,0) = u_0(x)$$

where $\triangledown = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $u \cdot \triangledown = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$, and $\Delta = \triangledown \cdot \triangledown = \sum_{j=1}^{3} \frac{\partial^2}{\partial {x_j}^2}$ (these operators $u \cdot \triangledown$ and Δ are applied component-wise to the velocities $u(x,t)$). The term $p(x,t)$ is the (scalar) pressure, $f(x,t)$ represents external forcing, and $\nu > 0$ is the kinematic viscosity. The condition $\triangledown \cdot u = 0$ is referred to as the incompressibility condition. See [11] for details. If we assume that the flow is so slow(i.e., large viscosity) that $(u \cdot \triangledown)u$ can be neglected, then the Navier-Stokes system reduces to the linear system, which is called the Stokes equations,

$$\frac{\partial u}{\partial t} = \nu \Delta u - \triangledown p + f, \quad \triangledown \cdot u = 0$$

In 1910 Oseen had suggested that the Stokes equations can be replaced and he formulated what is now known as the Oseen equations as follows:

$$\frac{\partial u}{\partial t} + (U \cdot \triangledown)u = \nu \Delta u - \triangledown p + f, \quad \triangledown \cdot u = 0$$

This is also a linearization of the Navier-Stokes equations about U, whereas the Stokes equations may be viewed as a linearization about 0. We thus would conjecture that Oseen equations are a good approximation to the Navier-Stokes when the flow is close to the velocity U far away from boundaries(the free stream velocity) and Stokes equations are good where the velocity is 0(near boundaries). In particular, Stokes/Oseen equations are important to understanding the motion of an object moving very slowly or steadily in the fluid such as the swimming of microorganisms and the sedimentation, under the force of gravity, of small particles and organisms, in water. See [16] for detail. It can also be expected that open problems on the Navier-Stokes equations such as global existence in time of strong solutions, uniqueness of weak solutions and asymptotic behavior of solutions are closely related to the properties of solutions to the Stokes/Oseen equations.

It is well known that there is a deep relationship between the theory of partial differential equations and probability theory which allows us to derive new results of random processes and properties of the solutions of partial differential equations. It is worth making an attempt to represent the solutions of the Stokes system in terms of stochastic processes. In this dissertation we construct probabilistic solutions to Stokes problems in the absence of boundaries, in the case of the half space, and we make some observations for general domains. Also we explore probabilistic versions of the Helmholtz-Hodge decomposition on domains with smooth boundary as well as the free space. In addition, we obtain probabilistic representations of iterated Riesz transforms. Lastly, we investigate the Neumann problem on the exterior domains as well as bounded domains.

1.1 Motivations and Main Results

In this section we are going to describe motivations and summarize briefly results for each topic.

1.1.1 Iterated Riesz Transforms

The Riesz transform on $\mathbb{R}^n$, one of the most basic examples of the theory of singular integrals, is defined by the principal value of the singular integral

$$R_j f(x) = p.v.c_n \int_{\mathbb{R}^n} f(x-y)\frac{y_j}{|y|^{n+1}}\,dy, \quad j = 1, 2, \cdots, n, \tag{1.1}$$

where $f \in L^p(\mathbb{R}^n), p = 1, 2,$ c_n is a constant that is chosen so that $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, and $\hat{f}$ is the Fourier transform of f, see [37]. For a significant applications of the Riesz transforms let us consider the Navier-Stokes equations of incompressible flows. See, for example, [28]. Because there is no time derivative of the pressure, one might try to eliminate the pressure. The so-called Leray projection operator $\mathcal{P}$ is defined by the orthogonal projection of the Hilbert space $L^2(\mathbb{R}^3)^3$ into the closed subspace $\{u \in L^2(\mathbb{R}^3)^3 : \nabla \cdot u = 0\}$ to project the Navier-Stokes equation on the space of divergence-free vector fields to remove the pressure term. The orthogonal projection property leads us to have $\mathcal{P} = I + \mathcal{R}$ where I is the 3×3 identity matrix and $\mathcal{R} = (R_i R_j), 1 \leq i, j \leq 3$, is the matrix of iterated Riesz transforms. Applying the projection operator to the Navier-Stokes equation to get the projected equation which can be thought of as a non-linear heat equation, we have the solution of the projected equation in the form of the integral equation involving the unknown term. Solutions to that integral equation with $\nabla \cdot u = 0$ are called mild solutions. This is the Kato's approach in which he proved existence of mild solutions for initial data using Picard iterations. The pressure can be obtained from the velocity by solving the Poisson equation. See, for example, [28]. So it is important to study Riesz transforms in the context of incompressible fluid flow and we are, in particular, interested in probabilistic representations of them. There are at least two known probabilistic approaches to define Riesz transforms. One is Gundy-Varopoulos-Silverstein's approach by using the background radiation process and conditional expectation, and the other is Bass' approach by the Doob's h-path transform of Brownian motion and generalized Cauchy-Riemann equations, see [21],[22], and [2]. The properties of the background radiation $\Theta = \{\Theta_t \in [0,\infty) \times \mathbb{R}^3 : -\infty < t < 0\}$ give the following formula for the iterated Riesz transforms of the functions f in Schwartz space

$$R_j R_k f(x) = (-1/2)\mathbb{E}(\int_{-\infty}^{0} A_{jk} \nabla u \cdot d\Theta_t | \Theta_t = (0,x)) \tag{1.2}$$

where the matrix $A_{ij} = e_i \otimes e_j$, i.e. having 1 in location (i,j) and 0's elsewhere, defines the martingale transform in (1.2). The formula (1.2) gives the iterated Riesz transforms as a martingale transform of the background radiation process Θ. The background radiation process Θ must be defined on a path space having infinite measure, and therefore not normalizable to a probability measure. Now, it is natural to ask the following questions.

Problem 1.1. *Is there an explicit probabilistic representation of the iterated Riesz transform different from G-V-S formula (1.2), i.e. the representation in terms of standard Brownian motion defined on a probability space (of measure one) in place of the background radiation process?*

We will see a new probabilistic representation of iterated Riesz transform in terms of standard Brownian motion $\{Z_t : t \geq 0\}$ extending an approach introduced by Bass in [2] as follows.

Theorem 1.1. *Suppose $f \in C_0^\infty$. Then there exists c independent of f s.t.*

$$R_i R_j f(x) = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial^2 u}{\partial x_i \partial x_j} dY_r = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau (e_{d+1} \otimes e_i H e_j \otimes e_{d+1} \cdot Z_r) dZ_r \tag{1.3}$$

where H is the Hessian of the harmonic extension u of f and C_0^∞ is the collection of smooth functions with compact support.

Comparing this formula (1.3) to the Gundy-Varapoulos-Silverstein's formula (1.2) we should note that the process Z_t in Theorem 1.1 is a Brownian motion in the half space while the process Θ in the Gundy-Varapoulos-Silverstein's formula (1.2) is not itself Brownian motion, but it belongs to the class of approximate Markov processes defined, as a collection of measurable functions, on a σ-finite measure space $(\Omega, \mathcal{F}, P)$. Also, as we mentioned in Problem 1.1, the representation in Theorem 1.1 is no longer related to conditional expectation (beyond the initial state) which is required in the Gundy-Varapoulos-Silverstein's representation of iterated Riesz transforms. In addition, the formula in Theorem 1.1 contains the second order partial derivatives of the harmonic extension u of f rather than the first order partial derivatives in the Gundy-Varapoulos-Silverstein's representation. Lastly, the matrix A_{jk} in the Gundy-Varapoulos-Silverstein's formula defines the indicated martingale transform while the matrix $e_{d+1} \otimes e_i H e_j \otimes e_{d+1}$ in the second formula in Theorem 1.1 defines a new transform.

1.1.2 Neumann Problems

There is a different way to define Riesz transforms, which is related to a Neumann problem. Let u be the solution to the Neumann problem in the upper half space $\mathbb{R}^{n+1}_+$;

$$\triangle_{\mathbb{R}^{n+1}_+} u = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \quad \frac{\partial u}{\partial \eta} = f \quad \text{in} \quad \mathbb{R}^n,$$

where f is a Schwartz function and $\frac{\partial}{\partial \eta}$ is the outward normal derivative to $\mathbb{R}^{n+1}_+$ at the boundary $\mathbb{R}^n$, i.e., $\frac{\partial u}{\partial x_{n+1}}|_{x_{n+1}=0} = -f$. Then we have

$$R_j f = \frac{\partial}{\partial x_j}(u|_{\mathbb{R}^n})$$

or, formally,

$$R_j f = \frac{\partial}{\partial x_j}(\frac{\partial}{\partial \eta})^{-1} f,$$

where $(\frac{\partial}{\partial \eta})^{-1} f$ means the restriction to $\mathbb{R}^n$ of the solution u. It is very interesting to investigate the Neumann problem on, more generally, unbounded domains as well as the half space for the Riesz transforms because if we have stochastic representation of the Neumann problem then we might have a chance to obtain a probabilistic representation of the Riesz transform on unbounded domains as well as the Riesz transform on the Euclidean space $\mathbb{R}^n$. In the case of the half space $\mathbb{R}^{n+1}_+$, Ramasubramanian in [35] considered the

Neumann problem with compatibility condition and several restrictions on the boundary data f such as f has finite second moment and $\int_{\mathbb{R}^n} |x|^r |f(x)|\, dx < \infty, r = 0, 1, 2$. If the domain is an unbounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, with compact boundary ∂D, Chen, Williams and Zhao in [14] solved the Neumann problem and gave the probabilistic representation of solution to the problem by constructing the Green-tight class for the boundary ∂D which is the subset of the Kato class with Green-tightness. Now we ready to pose questions.

Problem 1.2. *What if we have the Neumann problem on an exterior domain which is the complement of a compact set in $\mathbb{R}^n$? Under what conditions on the boundary can we solve the Neumann problem? i.e. Can we have weaker conditions on the boundary data than the above two cases?*

The probabilistic representation and properties of the Neumann function for the Neumann problem on exterior domains enable us to present the same formula as in [35] and [14] in more straightforward way. The following are the new probabilistic representations of solutions to the Neumann problems on an exterior domain and the half space $\mathbb{R}^{n+1}_+$, respectively, that we will present in Chapter 3 in this thesis.

Proposition 1.1. *Let $u(x) = \int_{\partial D} N(x,y) f(y)\, dS(y)$ on the exterior domain D and $f \in L^1(\partial D)$. Assume that $\frac{\partial u}{\partial \eta}|_{\partial D} = f$. Then the solution to the Neumann problem on the exterior domain D is given by*

$$u(x) = -\lim_{t \to \infty} \mathbb{E}_x \int_0^t f(B_s^r)\, dLs \tag{1.4}$$

where L_t is the local time process of reflecting Brownian motion B_t^r on $\bar{D}$ and $N(x,y)$ is the Neumann function with respect to B_t^r.

Proposition 1.2. *If $u(x) = \int_{\mathbb{R}^{n-1}} N(x', y', x_n) f(y')\, dy'$, and the boundary data $f \in L^1(\mathbb{R}^{n-1})$, then*

$$\triangle u = 0, \quad \frac{\partial u}{\partial \eta}|_{x_n = 0} = f$$

and

$$u(x) = -\lim_{t\to\infty} \mathbb{E}_{(x_0,y_0)} \int_0^t f(B_s^r)\, dLs \tag{1.5}$$

where L_t is the local time process of reflecting Brownian motion B_t^r on $\mathbb{R}_+^n$ and $N(x,y)$ is the Neumann function on the half space $\mathbb{R}_+^{n+1}$.

1.1.3 Helmholtz-Hodge Decomposition

According to the Helmholtz-Hodge decomposition, any smooth vector field F in $\mathbb{R}^n$ which decays sufficiently fast at infinity may be uniquely represented as a superposition of a gradient and a curl, i.e. $F = G + \nabla\phi$ where $G = \nabla \times \psi$ for a scalar potential ϕ and vector potential ψ obtained by solving a Poisson equation $\triangle\phi = \nabla \cdot F$, $\triangle\psi = -\triangle \times F$, $\nabla \cdot \psi = 0$. See [30], [18]. With this idea the Leray projection operator $\mathcal{P}$ suggests a more general concept of decomposing vector fields into the divergence-free part and the gradient part. Once we have a probabilistic representation of the iterated Riesz transform on the free space, we might expect to obtain the probabilistic version of the Helmholtz-Hodge decomposition on the free space. We will present the probabilistic representation of the gradient part of a smooth vector field in $\mathbb{R}^n$ as the application of the iterated Riesz transforms. Then we ask the following questions.

Problem 1.3. *Is it possible to decompose any smooth vector field on general domains such as exterior domains rather that $\mathbb{R}^n$? What is the probabilistic representation of the Helmholtz-Hodge decomposition on general domains?*

In Chapter 2 and 3 we find new probabilistic representations of the scalar potential as follows.

Proposition 1.3. *Let $F = (F_1, F_2, F_3)$ be a smooth vector field in $\mathbb{R}^n$ which decays sufficiently fast at infinity and $F = G + \nabla\phi$ where $G = \nabla \times \psi$ for a scalar potential ϕ and vector potential ψ. Then if $v_\rho(x,y) = \sum_{i=0}^{3} \frac{\partial u_{F_i}}{\partial x_i}$, where u_{F_i} is the harmonic extension of*

$F_i, i = 1, 2, 3,$ and $\rho := \nabla \cdot F$, then for some constant c

$$\frac{\partial \phi(x)}{\partial x_i} = c \sum_{j=1}^{3} \lim_{s \to \infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial u_{F_j}}{\partial x_i \partial x_j} \, dY_r \tag{1.6}$$

or

$$\frac{\partial \phi(x)}{\partial x_i} = c \lim_{s \to \infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i} \, dY_r \tag{1.7}$$

Also, we can define the Riesz transform of f on a domain D in terms of reflecting Brownian motion from Chapter 3 in this thesis by

$$R_j f(x) = \frac{\partial}{\partial x_j} (- \lim_{t \to \infty} \mathbb{E}^x \int_0^t f(B_s^r) \, dL_s)|_{\partial D}. \tag{1.8}$$

Thus through the Leray projection operator $\mathcal{P}$ we can decompose a smooth vector field into the curl-free vector fields in terms of reflecting Brownian motion. This result, together with the probabilistic representation of the iterated Riesz transform in Theorem 1.1, give us one way to study the Helmholtz-Hodge decomposition on vector fields probabilistically.

1.1.4 Stokes problems on the Free space

We consider Stokes problem on $\mathbb{R}^3$:

$$\frac{\partial u}{\partial t} = \nu \Delta u - \nabla p, \quad \nabla \cdot u = 0 \tag{1.9}$$

Then it follows from the relation between the Leray projection $\mathcal{P}$ and the Riesz transforms in Chapter 2 that the fundamental solution of the Stokes equations can be written as

$$\Gamma(x, y, t) = -\triangle_y \psi(x, y, t) I + Hess\psi(x, y, t), \tag{1.10}$$

where for each $x \in \mathbb{R}^3, t > 0$, ψ satisfies $\triangle_y \psi(x, y, t) = -k(x, y, t)$ with the heat kernel in $\mathbb{R}^3$, $k(x, y, t) = \frac{1}{(4\pi\nu t)^{3/2}} exp(-\frac{|x-y|^2}{4\pi\nu t})$, and $Hess\psi$ denotes the matrix of the second order partial derivatives with respect to the y variable, and I denotes the 3×3 identity matrix. Oseen in [19] used the fundamental solution tensor for the steady problems in this form in $\mathbb{R}^2$. In 3-dimensional case, Solonnikov in [38] had a similar expression in his analysis of the time dependent problem in $\mathbb{R}^3$. More recently Guenther and Thomann in [23] obtained an explicit formula for the fundamental solution in terms of Kummer functions. Now, we recall that for $f \in \mathcal{S}(\mathbb{R}^3)$, the Schwartz class,

$$R_i R_j f = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\triangle^{-1} f.$$

See [37], pg243. Then we have $\Gamma = (I + R)k$ where R is the matrix of iterated Riesz transform $R_i R_j$ and so if an initial data $u(x, 0) = u_0(x)$ is given, then we get

$$u(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t) u_0(y)\, dy = \int_{\mathbb{R}^3} [I+R] K(x,y,t) u_0(y)\, dy, \tag{1.11}$$

where $K = kI$. The probabilistic representation of the iterated Riesz transform (2.12) in Chapter 2 allows us to have a new probabilistic representation of $u = (u_1, u_2, u_3)$ on the free space as follows.

Proposition 1.4. *Let $u = (u_1, u_2, u_3)$ be the solution to the Stokes problem with an initial data $u_0 = (u_0^1, u_0^2, u_0^3)$ on the free space. Then if v is the harmonic extension of u_0 and $v_\rho = \nabla \cdot v$ then for some constant c*

$$u_i(x,t) = \mathbb{E}_x[u_0^i(B_t)] + c\mathbb{E}_x[\lim_{s\to\infty} \mathbb{E}_{h_{B_t}}^{(0,s)}[\int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i}\, dY_r]] \tag{1.12}$$

Then a more interesting and challenging question arises.

What is the probabilistic representation of the solution to the Stokes problem on a bounded or exterior domain with smooth boundary?

1.1.5 Stokes Problems on Domains with Smooth Boundary

We consider the Stokes problem on a domain D with smooth boundary ∂D;

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \nabla p, \tag{1.13}$$

$$\nabla \cdot u = 0, \tag{1.14}$$

$$u(x,0) = u_0(x), \quad u|_{\partial D} = a(x,t) \tag{1.15}$$

The incompressibility condition $\nabla \cdot u = 0$ leads us to the fact that the pressure p and ∇p are harmonic. Hence ∇p is determined by its boundary data. We also note that the equation (1.13) can be thought of as a nonhomogeneous heat equation if the boundary data of ∇p is given. Then we can say that u is the solution to the heat equation with forcing ∇p. The following is the probabilistic representation of the solution u to the heat equation on a domain D with smooth boundary ∂D when ∇p is considered to be known on the boundary.

Proposition 1.5. *Assume that u and p are the solution to the Stokes problem (1.13), (1.14), and (1.15). Let $H = \nabla p|_{\partial D}$ be given. Then then*

$$u(x,t) = \mathbb{E}_x[u_0(B_t)\mathbf{1}_{[t<\tau]}] - \mathbb{E}_x[\int_0^{t\wedge\tau} \mathbb{E}_{B_s}[H(B_\tau, t-s)]\, ds] + \mathbb{E}_x[a(B_\tau, t-\tau)\mathbf{1}_{[\tau<t]}] \quad (1.16)$$

where τ is the first hitting time of B_t on the boundary ∂D and B_t is Brownian motion starting at x.

We will show that the boundary data H of ∇p can be obtained in terms of the initial and boundary data of u in Proposition 1.5 so that u satisfies the incompressibility condition $\nabla \cdot u = 0$. We notice that $\nabla \cdot Q = 0$ and $\nabla \times Q = 0$ on D, where $Q = \nabla p$ since p is harmonic. In particular, on the half space $D = \mathbb{R}^3_+$, the conditions $\nabla \cdot Q = 0$ and $\nabla \times Q = 0$ are the generalized Cauchy-Riemann equations. Hence $h_1 = -R_1 h_3$ and $h_2 = -R_2 h_3$, where $R_i f$ is the Riesz transform of f, $i = 1, 2$, because $\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial x_3}|_{x_3=0} = -h_3$ where $\frac{\partial}{\partial \eta}$ is the outward normal derivative to the boundary $\mathbb{R}^2$. See [37].

For a general domain D in $\mathbb{R}^3$ such as a manifold with smooth boundary we define the gradient of a differentiable function f on ∂D by a differentiable map

$$\widetilde{\nabla} f : \partial D \longrightarrow \mathbb{R}^3$$

which assigns to each point $x \in \partial D$ a vector $\widetilde{\nabla} f(x) \in T_x(\partial D) \subset \mathbb{R}^3$ such that $< \widetilde{\nabla} f(x), v >_x = df_x(v)$ for all $v \in T_x(\partial D)$, where $T_x(\partial D)$ is the tangent plane at x, $<,>_x$

is the inner product in $T_x(\partial D)$, and $df_x(v)$ is the differential of f at x. See [17]. Let H be a vector field on the boundary ∂D. Then

$$H = \Pi_{T_x(\partial D)}(H) + (H \cdot \eta)\eta$$

where $\Pi_{T_x(\partial D)}$ is the projection onto the tangent plane $T_x(\partial D)$.

If we have $\triangle p = 0$ on D and $\frac{\partial p}{\partial \eta} = H \cdot \eta = h$, then

$$H = \Pi_{T_x(\partial D)}(H) + (H \cdot \eta)\eta = \Pi_{T_x(\partial D)}(H) + h\eta.$$

Thus we have

$$\widetilde{\nabla} p|_{\partial D} = \Pi_{T_x(\partial D)}(H). \tag{1.17}$$

In 1997 Arcozzi and Xinwei in [1] described three possible definitions of Riesz transform on the sphere in $\mathbb{R}^n$. One of them is that the Riesz transforms of f can be defined by solving the Neumann problem with boundary data f, restricting the solution to the boundary, and taking the tangential gradient of the restricted solution. With this definition if $\frac{\partial p}{\partial \eta} = H \cdot \eta = h$, then the tangential part of H is the tangential gradient of pressure restricted to the boundary from (1.17). Thus by the definition related to the Neumann problem the tangential part of H is the vector of Riesz transforms of h. We can say that the normal part of H determines the tangential part of H through the Riesz transformation. Thus three unknowns of H go down to only one unknown. Therefore we have an integral equation involving $h = \frac{\partial p}{\partial \eta}$ from $\nabla \cdot u = 0$. The existence of h in this integral equation allows us to solve the Stokes problem.

In the case of the half space $D = \mathbb{R}^3_+$, h can be explicitly determined in terms of the initial and boundary data of u.

Proposition 1.6. *Let u be the solution to the Stokes problem on $\mathbb{R}^3_+$:*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u - \nabla p, \qquad \nabla \cdot u = 0$$

$$u(x,0) = u_0(x), \quad u|_{x_3=0} = 0$$

Assume that $u_0(x) \in \mathcal{C}^1(\mathbb{R}^3_+)$ *satisfies the compatibility conditions* $\nabla \cdot u_0 = 0$ *and* $u_0|_{x_3=0} = 0$*. Then* $h = -\frac{\partial p}{\partial x_3}|_{x_3=0}$ *is given by*

$$h(x_1, x_2, t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t)[\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)]\, dy' dy_3 \tag{1.18}$$

where $u_0 = (u_0^1, u_0^2, u_0^3)$*,* $\nabla' = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})$*,* $R' f = (R_1 f, R_2 f)$ *with the Riesz transform of* f $R_i f$*,* $i = 1, 2$*, and*

$$\widetilde{K}(x', x_3, t) = k^{(2)}(x', t) \frac{1}{\sqrt{2\pi t}} (2)(\frac{x_3}{t}) \exp(-\frac{x_3^2}{2t}).$$

Moreover, a probabilistic representation of h *in terms of reflecting Brownian motion is given as*

$$h(x', t) = \mathbb{E}_{(x',0)}[\frac{\partial g}{\partial y_3}(B_t^r)]$$

where $g(y', y_3) = \nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)$ *and* B_t^r *is reflecting Brownian motion.*

Therefore a new probabilistic representation of the solution u to the Stokes problem can be obtained using the stochastic formula for the Riesz transform $R_j, j = 1, 2$ of h, the formula (2.11) in Chapter 2 to get H since H can be determined by Riesz transforms of h: $H = (-R_1 h, -R_2 h, h)$ and

$$R_j f(x) = c \lim_{s \to \infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau A \nabla u(Z_r) \cdot dZ_r$$

where $A = e_{n+1} \otimes e_k$, i.e., having 1 in location $(n+1, k)$ and 0's elsewhere.

Finally a solution of the Stokes problem can be given as

Theorem 1.2. *Assume that* $u_0(x) \in \mathcal{C}^1(\mathbb{R}^3_+)$ *and* $a \in \mathcal{C}^{1,1}(\mathbb{R}^2 \times (0, \infty))$ *satisfy the compatibility conditions* $\nabla \cdot u_0 = 0$*,* $u_0|_{x_3=0} = a|_{t=0}$*, and* $a \cdot \eta = 0$*. Let*

$$h(x_1, x_2, t) := \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t)[\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)]\, dy' dy_3$$

$$+ \triangle_{x'}(R_1 a_1 + R_2 a_2) + 4 \int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial s} - \frac{1}{2} \triangle') \frac{\exp(-\frac{|x'-y'|^2}{2s})}{s^{3/2}} (\nabla' \cdot a')(y', t - s)\, dy' ds,$$

$$u(x,t) := \{-\int_0^t \int_{\mathbb{R}^2} [\frac{\partial}{\partial z_3}|_{z_3=0} \int_0^\infty E(x,z,t'+s)\, dt'] H(z,t-s)\, dz ds + \int_{\mathbb{R}^3_+} E(x-y,t) u_0(y)\, dy$$
$$+ \int_0^t \int_{\mathbb{R}^2} \frac{\partial}{\partial y_3}|_{y_3=0} E(x,y,t) a(y',t-s)\, dy' ds$$

and let p be the solution to the Neumann problem with boundary data h on the half space $\mathbb{R}^3_+$. Then u and p satisfy the Stokes problem with the initial data u_0 and the boundary data a on the half space $\mathbb{R}^3_+$.

In 2010 Bikri, Guenther, and Thomann in [5] utilized repeatedly the Fourier transforms in $\mathbb{R}^2$ together with the Laplace transforms with respect to x_3 of a function defined in $\mathbb{R}^3_+$ to obtain some results on the DtN map for Laplace and heat operators in $\mathbb{R}^3_+$. We use one of results in [5] to find the explicit formula for the pressure on the boundary in the Stokes problem as the following corollary shows.

Corollary 1.1. *Let $h = -\frac{\partial p}{\partial x_3}|_{x_3=0}$ in the Proposition (1.6). Then*

$$p(x_1,x_2,t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x'-y', y_3, t)(u_0^3 - R_1 u_0^1 - R_2 u_0^2)\, dy' dy_3$$
$$+\nabla' \cdot a' + 4\int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial s} - \frac{1}{2}\triangle') \frac{\exp -(\frac{|x'-y'|^2}{2s})}{s^{3/2}} (R_1 a_1 + R_2 a_2)$$

where $a' = (a_1, a_2)$.

In 1987, Ukai in [39] gave the explicit solution to the Stokes problem in terms of Riesz operators, the heat operator, and the Laplace operator in the half space $\mathbb{R}^n_+$. If we just follow the operators in [39] and compute the pressure, then we can see that the integral representation of Ukai's formula for the pressure is exactly the same as the formula in corollary 1.1. However, Ukai's approach is different; he used a differential equation to remove the term involving the pressure in the Stokes equations. Then he found the velocity to the Stokes problem in terms of the initial and boundary data and finally obtained the pressure from the velocity. However, we focus on the pressure at the first place.

More interestingly, pressure on the boundary $\mathbb{R}^2$ can be expressed in terms of a special function, Kummer's function(see [32]), when the pressure p is the solution to the Neumann problem on $\mathbb{R}^3_+$

$$\nabla p = 0, \quad \frac{\partial p}{\partial z_3}|_{z_3=0} = h(z') \tag{1.19}$$

where $h = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x'-y', y_3, t) g(y', y_3)\, dy' dy_3$ with $g(y', 0) = 0$. Recall that

$$ {}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1}(1-t)^{c-a-1}\, dt,$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$.

Proposition 1.7. *Let u and p be the solution to the Stokes problem in the half space $\mathbb{R}^3_+$ with $u|_{t=0} = u_0$, $u|_{x_3=0} = 0$ and satisfy compatibility conditions $\nabla \cdot u_0 = 0$, $u_0|_{x_3=0} = 0$. Then one has the restriction to the boundary of the pressure given by*

$$p(x', t) = c \int_0^\infty \int_{\mathbb{R}^2} \frac{\exp(\frac{-|y'-x'|^2}{2t})}{\sqrt{t}} {}_1F_1(\frac{1}{2}, 1, \frac{|y'-x'|^2}{2t}) \frac{\exp -\frac{y_3^2}{2t}}{\sqrt{t}} \frac{\partial g}{\partial y_3}|_{(y', y_3)}\, dy' dy_3. \tag{1.20}$$

To summarize, looking at the probabilistic representation of the solution u to the Stokes problem on a bounded or exterior domain, we conclude that the velocity u and the pressure p can be determined by only information of the gradient of pressure on the boundary of domain.

1.2 Organization of Thesis

This thesis is organized as follows. In chapter 2 we recall two different approaches to define Riesz transforms and present a new probabilistic representation of iterated Riesz transforms. Also we prove the L^p-boundednes of iterated Riesz transforms. In chapter 3 we obtain probabilistic representations of solutions to the Neumann problems on bounded, unbounded domains such as an exterior domain and the half space. In chapter 2 and 3 we present probabilistic representation of the Helmholtz-Hodge decomposition theorem on the free space and a domain with smooth boundary, respectively. In chapter 4 we give the probabilistic representation of the Stokes problem and study the existence of solutions to the Stokes problem in terms of the boundary data of the gradient of pressure. We also obtain the explicit formula of the boundary data of the gradient of pressure on the half space in terms of the initial and boundary conditions of the solution to the Stokes problem. In chapter 5 we summarize briefly new results in this thesis and some remarks for future work.

2 PROBABILISTIC REPRESENTATIONS OF ITERATED RIESZ TRANSFORMS IN $\mathbb{R}^N$

In this chapter, we explain the probabilistic representations of the Riesz transforms on the Euclidean space $\mathbb{R}^n$ in the context of the incompressible Navier-Stokes equations for a significant application of the Riesz transforms and present the probabilistic representations of the Riesz transforms with two approaches, Gundy-Varopoulos-Silverstein in [22] and Bass in [2]. As the main result in this chapter, we will give a new probabilistic representation of the iterated Riesz transforms. As an application we will see the Helmholtz-Hodge decomposition. Before going into sections, let us recall some known results on Riesz transforms on $\mathbb{R}^n$.

Riesz in [34] proved that the Hilbert transform on the real line $\mathbb{R}$, defined by the principal value of the singular integral

$$Hf(x) = p.v.\frac{1}{2\pi}\int \frac{f(y)}{x-y}\,dy,$$

is bounded in $L^p(\mathbb{R})$ for all $1 < p < \infty$. In 1952, Calderón and Zygmund in [15] extended it from $\mathbb{R}$ to $\mathbb{R}^n$ and developed the theory of singular integrals. In particular, Riesz transforms on $\mathbb{R}^n$ are the most fundamental example of singular integrals and are defined by

$$R_j f(x) = p.v.c_n \int_{\mathbb{R}^n} f(x-y)\frac{y_j}{|y|^{n+1}}\,dy, \quad j = 1, 2, \cdots, n,$$

where c_n is a constant that is chosen so that $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, and $\hat{f}$ is the Fourier transform of f. It is well known that the Riesz transforms R_j are bounded in $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, see [37].

It is also well known that the relation between the theory of partial differential equations and probability theory is deep and enables us to have new results of random processes and properties of the solutions of partial differential equations. For the incompressible Navier-Stokes system, in particular, the iterated Riesz transforms play an important role and so it is natural to make an attempt to construct a probabilistic representation of this iterated Riesz transform.

2.1 Riesz Transforms on $\mathbb{R}^n$ in the context of Incompressible Navier-Stokes Equations

Recall the 3-dimensional Navier-Stokes equations governing viscous, incompressible fluid velocity $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ for $x \in \mathbb{R}^3, t \geq 0$, with initial data $u(x,0) = u_0(x)$, are mathematical description of the second law of Newton and mass conservation:

$$\frac{\partial u}{\partial t} + (u \cdot \triangledown)u = \nu \Delta u - \triangledown p + f, \quad \triangledown \cdot u = 0, \quad u(x,0) = u_0(x) \tag{2.1}$$

where $\triangledown = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}), u \cdot \triangledown = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$, and $\Delta = \triangledown \cdot \triangledown = \sum_{j=1}^{3} \frac{\partial^2}{\partial {x_j}^2}$ (two operators $u \cdot \triangledown$ and Δ are applied component-wise to the velocities $u(x,t)$). The term $p(x,t)$ is the (scalar) pressure, $f(x,t)$ represents external forcing, and $\nu > 0$ is the kinematic viscosity. The condition $\triangledown \cdot u = 0$ is referred to as the incompressibility condition. This incompressibility condition and the Green formula give the following well-known lemma, see [30] for proof.

Lemma 2.1. *Let v be a smooth vector field in $\mathbb{R}^n$ with $\triangledown \cdot v = 0$ and let q be a smooth real valued function such that $|v(x)||q(x)| = O(|x|^{1-n})$ as $|x| \to \infty$.*
Then v and $\triangledown q$ are orthogonal:

$$\int_{\mathbb{R}^n} v \cdot \triangledown q = 0. \tag{2.2}$$

Therefore for the velocities $u(x,t)$ and pressure $p(x,t)$ in Navier-Stokes equations the L^2 orthogonality

$$\int_{\mathbb{R}^n} u \cdot \triangledown p = 0 \tag{2.3}$$

is obtained from (2.2). This orthogonality leads us to eliminate the pressure term by projection on divergence-free vector fields as follows.

We define an operator $\mathcal{P}$ by the orthogonal projection of the Hilbert space $L^2(\mathbb{R}^3)^3$ onto divergence-free vector fields $\{u \in L^2(\mathbb{R}^3)^3 : \nabla \cdot u = 0\}$. The operator $\mathcal{P}$ is called the Leray projection operator.

Lemma 2.2. *Let $\hat{f}(\xi)$ be the Fourier transform of $f \in L^2(\mathbb{R}^3)$. Then for $u = (u_1, u_2, u_3) \in L^2(\mathbb{R}^3)^3$*

$$\widehat{\mathcal{P}u}(\xi) = (I - \frac{\xi \otimes \xi}{|\xi|^2})\hat{u}, \quad \xi \in \mathbb{R}^3 \tag{2.4}$$

where $\mathcal{P}$ is the Leray projection operator, I is the identity matrix and the tensor product $\otimes$ of two vectors $a, b \in \mathbb{R}^3$ is defined by the matrix $a \otimes b = (a_i b_j)_{1 \leq i,j \leq 3}$ with $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

Proof. See [40]. □

Applying the projection operator to Navier-Stokes equations with the incompressibility condition and L^2-orthogonality of u and ∇p, we obtain for initial data $u_0 \in L^2(\mathbb{R}^3)^3$, the projected equation

$$\frac{\partial u}{\partial t} + \mathcal{P}((u \cdot \nabla)u) = \nu \Delta u + \mathcal{P}(f). \tag{2.5}$$

Viewing the projected equation (2.5) as a heat equation, i.e.

$$\frac{\partial u}{\partial t} = \nu \Delta u + \mathcal{P}(f - (u \cdot \nabla)u) \tag{2.6}$$

we have

$$u(x,t) = \int_{\mathbb{R}^3} u_0(y) k(x-y, \nu t)\, dy + \int_0^t \int_{\mathbb{R}^3} \mathcal{P}(f - (u \cdot \nabla)u)(y,s) k(x-y, \nu(t-s))\, dyds \tag{2.7}$$

where $k(x,t) = \frac{1}{(2\pi)^{3/2}} \exp -\frac{|x|^2}{4\pi t}$ is the fundamental solution of heat equation. Solutions to (2.6) with $\nabla \cdot u = 0$ are called mild solutions. Using the equation (2.6) Kato proved existence of mild solutions for initial data $u_0 \in L^3(\mathbb{R}^3)^3$ in [27].

From the projection onto the divergence-free vector field we recover the pressure as follows: The elements $-\frac{\xi_i \xi_j}{|\xi|^2}$ in (2.4) can be thought of as iterated Riesz transforms $R_i R_j$ in Fourier symbols if we define the Riesz transform on $L^2(\mathbb{R}^3)$ by $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\widehat{f}(\xi)$. Thus $\mathcal{P} = I + \mathcal{R}$ where $\mathcal{R}(R_i R_j)_{1\leq i,j\leq 3}$ is the matrix of iterated Riesz transforms. Taking the divergence in Navier-Stokes equations, and assuming no external forcing $f \equiv 0$ we have

$$\nabla \cdot (u \cdot \nabla)u = -\nabla \cdot \nabla p = -\Delta p \tag{2.8}$$

because of incompressibility condition $\nabla \cdot u = 0$. Taking Fourier transforms and using incompressibility we obtain

$$p = \sum_{1\leq i,j\leq 3} R_i R_j (u_i u_j). \tag{2.9}$$

2.2 Probabilistic Representations of Riesz Transforms on $\mathbb{R}^n$

In this section we introduce two kinds of probabilistic representations of Riesz transforms on $\mathbb{R}^n$ even if it is well known that there are several equivalent methods to define Riesz transform on $\mathbb{R}^n$. One is obtained by Gundy-Varopoulos-Silverstein's Background radiation and the other is Bass' probabilistic representation through the Doob's h-path transform. Also we present a new probabilistic representation of the iterated Riesz transform with Bass' approach and show the boundedness of the iterated Riesz transform with that probabilistic representation.

2.2.1 Gundy-Varopoulos-Silverstein's Background radiation on $\mathbb{R}^n$

Given a function f in the Schwartz space $\mathcal{S}$ (or the space of rapidly decreasing functions) we let $u(x,t)$ denote its unique bounded harmonic extension to the half space $\mathbb{R}^{n+1}_+ := \{(x,y) : x \in \mathbb{R}^n, \quad y \geq 0\}$. Gundy and Varopoulos in [22] had the probabilistic interpretation of Riesz transforms with Brownian motion from infinity (or background radiation) by applying a martingale transform, and then taking the conditional expectation with respect to the terminal position. We define first background radiation as follows.

Definition 2.1. *The background radiation process* $\Theta = \{\Theta_t = (X_t, Y_t) : -\infty < t < 0\}$ *is a continuous path process taking values in the half space* $\mathbb{R}^{n+1}_+$ *and having the following properties:*

1. $Y_{-\infty} = \lim_{t \to -\infty} Y_t = \infty$.

2. $Y_0 = \lim_{t \to 0} Y_t = 0$ *and* $\Theta_0 = \lim_{t \to 0} \Theta_t$ *exists as a point on the boundary* $\mathbb{R}^n$.

3. *If, for* $a > 0$, *we let* $T^a = \inf\{t : Y_t = a\}$, *then* $\Theta^a_t = \Theta_{T^a + t}$, $0 \le t \le -T^a$ *is a copy of standard Browian motion on* $\mathbb{R}^{n+1}_+$ *with initial distribution being Lebesgue measure on the level* $\{y = a\}$ *and which terminates upon hitting the boundary* $\mathbb{R}^n$.

From the third property 3 in the definition of background radiation Θ we can see that the underlying sample space has infinite measure and hence Θ is not normaizable to a probability measure. Also it is not hard from the first property 1 of Θ to see that Θ is not itself Browian motion since a 1-dimensional standard Brownian motion is recurrent, i.e., in one dimension, the sample path of Brownian motion sweeps back and forth in such a way that

$$\underline{\lim}_{t \to \infty} = -\infty, \quad \overline{\lim}_{t \to \infty} = +\infty,$$

and therefore, by continuity of sample path of Brownian motion, must visit every point infinitely often.

The properties of the background radiation process were used to prove the following formula for iterated Riesz transforms of a function $f \in \mathcal{S}$, the Schwartz class,

$$R_j R_k f(x) = (-1/2)\mathbb{E}\left(\int_{-\infty}^{0} A_{jk} \nabla u \cdot d\Theta_t \middle| \Theta_t = (0, x)\right) \tag{2.10}$$

where the matrix $A_{ij} = e_i \otimes e_j$, i.e. having 1 in location (i, j) and 0's elsewhere. See [40]. An expression like $\int_{-\infty}^{0} A_{jk} \nabla u \cdot d\Theta_t$ is called the martingale transform of $\int_{-\infty}^{0} \nabla u \cdot d\Theta_t$ by the matrix A. Hence the formula (2.10) gives the iterated Riesz transforms as a martingale transform of the background radiation process Θ.

2.2.2 Bass Probabilistic Representation of Riesz Transforms on $\mathbb{R}^n$

On the other hand there is another probabilistic representation of Riesz transforms from Bass [2]. In what follows we will denote elements of $\mathbb{R}^{n+1}_+$ by $z = (x, y)$ with $x \in \mathbb{R}^n$, $y \in [0, \infty)$. Let $X = \{X_t : t \geq 0\}$ be a n-dimensional Browian motion, $Y = \{Y_t : t \geq 0\}$ a one-dimensional Browian motion independent of X, and $Z = \{Z_t = (X_t, Y_t) : t \geq 0\}$. We will write $\partial_j u$ for $\frac{\partial u}{\partial x_j}$ and $\partial_y u = \frac{\partial u}{\partial y}$. Given a function $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$ we define the harmonic extension u of f by

$$u(z) = \mathbb{E}^z f(X_\tau)$$

where $\tau = \tau_{\mathbb{R}^{n+1}_+} = \inf\{t > 0 : X_t \notin \mathbb{R}^{n+1}_+\}$ is the first time that X_t leaves the half space $\mathbb{R}^{n+1}_+$. We can also represent the harmonic extension u by the Poisson kernel for the half space $\mathbb{R}^{n+1}_+$:

$$u(z) = u(x, y) = \int_{\mathbb{R}^n} f(w) P_y(x - w)\, dw$$

where $P_y(x) = c_n \frac{y}{(|x|^2+y^2)^{(n+1)/2}}$, $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$.

Let us introduce Doob's h-path transforms which is an example of the use of change of measure. The motivation of the Doob's h-path transforms is that one would like to consider a diffusion process X to exit from a domain D at a point z only, i.e., Brownian motion conditioned to exit the domain at a point. Intuitively, if h is a positive harmonic function that is 0 everywhere on the boundary of a domain D except at one point z, for example we can take the Poisson kernel for the half space $\mathbb{R}^{n+1}_+$ for h, then we have by the Markov property at time t

$$\mathbb{P}^x(X_t \in dy | X_{\tau_D} = z) = \frac{\mathbb{P}^x(X_t \in dy, X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D} = z)} = \frac{\mathbb{P}^x(X_t \in dy)\mathbb{P}^y(X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D} = z)}$$

and so we might expect the probability for Brownian motion conditioned to exit D at z should be $\frac{h(y)}{h(x)} p(t, x, dy)$ if $p(t, x, dy)$ is the probability that Brownian motion started at

x and killed on leaving D is in dy at time t. More precisely, let D be a domain and $\widehat{X_t}$ Brownian motion killed on exiting the domain D. Let h be a positive harmonic function that is 0 everywhere on the boundary of D except at a point z. Since h is harmonic, i.e. $\triangle h = 0$ in D, $h(\widehat{X}_{t\wedge\tau_D})$ is a martingale by the Itô formula where $\tau_D = \inf\{t > 0 : \widehat{X_t} \notin D\}$. Setting $M_t := h(\widehat{X}_{t\wedge\tau_D})/h(\widehat{X_0})$, M_t is a positive continuous martingale with $M_0 = 1$, a.s. So we define the Doob's h-path transform of Browian motion by

$$\mathbb{P}_h^x(A) = \mathbb{E}^x[M_t; A] = \int_A M_t(w)\mathbb{P}(dw), \quad A \in \mathcal{F}_t.$$

Before going further to a probabilistic interpretation of Riesz transforms we need to have three lemmas, two of which are proved in [2].

Lemma 2.3. *Suppose that* $F \geq 0$, $\int_{\mathbb{R}^n}\int_0^\infty yF(x,y)\,dydx < \infty$, *and there exists* c_1 *and* $\beta > 0$ *s.t.*

$$\sup_{(x,y)\in B((0,s),s/2)} F(x,y) \leq c_1 s^{-n-2-\beta}, \quad s \geq 1.$$

Then there exists c_2 *not depending on* F *s.t.*

$$\lim_{s\to\infty} s^n \mathbb{E}^{(0,s)} \int_0^\tau F(X_t, Y_t)\,dt = c_2 \int_{\mathbb{R}^n}\int_0^\infty yF(x,y)\,dydx.$$

Proof. See [2], p 244. □

Lemma 2.4. *Suppose either that* $f \in L^1$ *and* u *is its harmonic extension or* u *is the Poisson kernel for some point* $w \in \mathbb{R}^n$. *Then*

$$\sup_{z\in B((0,s),s/2)} |\nabla u(z)|^2 \leq cs^{-2n-2}, \qquad \sup_{z\in B((0,s),s/2)} |\frac{\partial^2 u}{\partial x_i \partial x_j}| \leq cs^{-n-2}$$

Proof. See [2], p 247. □

Lemma 2.5. *Let* $h_x(w,y) := P_y(w-x)$ *be the positive harmonic function with pole at* x *that is* 0 *on the boundary* $\mathbb{R}^n$ *of* $\mathbb{R}^{n+1}_+$ *except for* 0, u *a harmonic extension of a function*

$f \in L^1(\mathbb{R}^n)$ with $|\partial_y u(x,y)| \le cy^{-1-n}$ for some constant c, and $S_y := \inf\{t : Y_t \le y\}$. Then

$$\lim_{y\to 0} \mathbb{E}^{(0,s)} \int_0^{S_y} \partial_y h_x(Z_r)\partial_y u(Z_r)\, dr = \mathbb{E}^{(0,s)} \int_0^{\tau} \partial_y h_x(Z_r)\partial_y u(Z_r)\, dr$$

Proof. We recall that the Green function for $D = \mathbb{R}^{n+1}_+$, $n \ge 3$, is given by

$$g_D((x_1,y_1),(x_2,y_2)) = c\{[(x_1-x_2)^2+(y_1-y_2)^2]^{-(n-1)/2} - [(x_1-x_2)^2+(y_1+y_2)^2]^{-(n-1)/2}\}$$

We also note that $g_{D_{y_0}}((0,s),(w,y)) \le g_D((0,s),(w,y))$ where $D_{y_0} = \{(x,y) : x \in \mathbb{R}^n, y \ge y_0 > 0\}$ since $g_{D_{y_0}}((0,s),(w,y)) = c\{[w^2+(y-s)^2]^{-(n-1)/2} - [w^2+(s-2y_0+y)^2]^{-(n-1)/2}\}$ and distance between $(w,-y)$ and $(0,s)$ is bigger than distance between $(w, 2y_0 - y)$ and $(0,s)$. Then we have

$$|\mathbb{E}^{(0,s)} \int_0^{S_{y_0}} \partial_y h_x(Z_r)\partial_y u(Z_r)\, dr| \le \int_{\mathbb{R}^n}\int_0^{\infty} \partial_y h_x(w,y)|\partial_y u(w,y)|g_{D_{y_0}}((0,s),(w,y))|\, dydw$$

$$\le \int_{\mathbb{R}^n}\int_0^{\infty} \partial_y h_x(w,y)|\partial_y u(w,y)|g_D((0,s),(x,y))\, dydx$$

By dominated convergence, the proof is completed. □

Now we give a probabilistic representation of Riesz transformations as a corollary of the following Proposition in [2].

Proposition 2.1. *If $\widehat{f} \in L^1(\mathbb{R}^n)$ and there exists c s.t. $|\partial_y u(x,y)| \le cy^{-1-n}$, then*

$$\lim_{s\to\infty} \mathbb{E}^{(0,s)}_{h_x} \int_0^{\tau} \partial_y u(Z_r)\, dY_r = cf(x),$$

where $\mathbb{E}^{(0,s)}_{h_x}$ is the expectation w.r.t the Doob's h-path transform of Brownian motion with the Poisson kernel h_x for the half space $\mathbb{R}^{n+1}_+$.

Proof. See [2], p 249. □

Before looking at the corollary let us introduce the generalized Cauchy-Riemann equations. It is well know that the Riesz transforms and the theory of harmonic functions are tied together closely as follows. See [37] for detail.

Theorem 2.1. *Let f and $f_1, \cdots, f_n$ all belong to $L^2(\mathbb{R}^n)$, and let their respective harmonic extensions be*

$$u_0(x,y) = P_y * f, u_1(x,y) = P_y * h_1, \cdots u_n(x,y) = P_y * f_n$$

where $P_y(x) = \frac{c_n y}{(|x|^2+y^2)^{\frac{n+1}{2}}}, c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$ is the Poisson kernel. Then a necessary and sufficient condition that

$$f_j = R_j(f), \quad j = 1, \cdots, n,$$

is that the following generalized Cauchy-Riemann equations hold:

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0,$$
$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad j \neq k, \quad x_0 = y.$$

The generalized Cauchy-Riemann equations allows us to have the following corollary. See [2] for the proof.

Corollary 2.1. *Let $f \in C_0^\infty$, a smooth function with compact support. Then there exists c independent of f s.t.*

$$R_j f(x) = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau A\nabla u(Z_r) \cdot dZ_r \tag{2.11}$$

where $A = e_{n+1} \otimes e_k$, i.e., having 1 in location $(n+1,k)$ and 0's elsewhere.

We notice that $\int_0^\tau A\nabla u(Z_r) \cdot dZ_r$ is the martingale transform of the martingale $\int_0^\tau \nabla u(Z_r) \cdot dZ_r$ by a matrix A.

The main result of this chapter is the following.

Theorem 2.2. *Suppose $f \in C_0^\infty$. Then there exists c independent of f s.t.*

$$R_i R_j f(x) = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial^2 u}{\partial x_i \partial x_j} dY_r = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau (e_{n+1} \otimes e_i H e_j \otimes e_{n+1} Z_r) \cdot dZ_r \tag{2.12}$$

where H is the Hessian of the harmonic extension u of f to $\mathbb{R}_+^{n+1}$.

Proof. First we note that $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u_{ij}}{\partial y^2}$ where u_{ij} is the harmonic extension of $R_i R_j f$ since $\widehat{u_{ij}} = \widehat{P_y}\widehat{R_i R_j f} = -e^{-|\xi|y}\frac{\xi_i \xi_j}{|\xi|^2}\widehat{f}$ and $\frac{\partial^2 \widehat{u_{ij}}}{\partial y^2} = -\xi_i\xi_j e^{-|\xi|y}\widehat{f} = \widehat{\frac{\partial^2 u}{\partial x_i \partial x_j}}$. Let $h_x(w,y) = P_y(w-x)$, the positive harmonic function with pole at x that is 0 on $\mathbb{R}^n - \{x\}$. Set $S_y = \inf\{t : Y_t \le y\}$. Then

$$\mathbb{E}_{h_x}^{(0,s)} \int_0^{S_y} Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dY_r = \mathbb{E}^{(0,s)}\{h_x(Z_{S_y}) \int_0^{S_y} Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dY_r\}/h_x(0,s)$$

By the Itô formula

$$h_x(Z_{S_y}) = h_x(Z_0) + \int_0^{S_y} \nabla h_x(Z_r) \cdot dZ_r.$$

Sine the stochastic integral is a martingale with mean 0,

$$\mathbb{E}^{(0,s)}\{h_x(Z_0) \int_0^{S_y} Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dY_r\} = 0.$$

Since X_t is independent of Y_t,

$$\mathbb{E}^{(0,s)}\{\int_0^{S_y} \nabla h_x(Z_r) \cdot dZ_r \int_0^{S_y} Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dY_r = \mathbb{E}^{(0,s)}\{\int_0^{S_y} \partial_y h_x(Z_r) Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dr\}$$

Thus

$$\mathbb{E}_{h_x}^{(0,s)} \int_0^{S_y} Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dY_r = c[h_x(0,s)]^{-1}\mathbb{E}^{(0,s)}\{\int_0^{S_y} \partial_y h_x(Z_r) Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dr\}.$$

Using the argument to justify the passage to the limit in the lemma 2.5 and letting $y \to 0$, we have

$$\mathbb{E}_{h_x}^{(0,s)} \int_0^{\tau} Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dY_r = cs^n \mathbb{E}^{(0,s)}\{\int_0^{\tau} \partial_y h_x(Z_r) Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dr\}.$$

By two lemmas 2.3 and 2.4 we mentioned earlier and the Plancherel identity

$$\begin{aligned}
&\lim_{s\to\infty} s^n \mathbb{E}^{(0,s)}\{\int_0^\tau \partial_y h_x(Z_r) Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}(Z_r)\, dr\} \\
&= c_n \int_{\mathbb{R}^n} \int_0^\infty y^2 \partial_y h_x(w,y) \frac{\partial^2 u}{\partial x_i \partial x_j}(w,y)\, dydw \\
&= c_n \int_0^\infty \int_{\mathbb{R}^n} y^2 \widehat{\partial_y h_x}(\xi,y) \widehat{\frac{\partial^2 u}{\partial x_i \partial x_j}}(\xi,y)\, d\xi dy \\
&= c_n \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \underbrace{(\int_0^\infty y^2 |\xi|^3 e^{-2y|\xi|}\, dy)}_{constant} \widehat{R_i R_j f}(\xi)\, dy d\xi \\
&= c_n \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \widehat{R_i R_j f}(\xi)\, d\xi \\
&= c_n R_i R_j f(x).
\end{aligned}$$

$\square$

Remark: There are some remarks on this result we should note:

1. The process Z in Theorem 2.2 is a Brownian motion in the half space while the process Θ in the Gundy-Varapoulos-Silverstein formula is not itself Brownian motion, but it belongs to the class of approximate Markov processes defined, as a collection of measurable functions, on a σ-finite measure space $(\Omega, \mathcal{F}, P)$.

2. The representation in Theorem 2.2 is no longer related to conditional expectation which is required in the Gundy-Varapoulos-Silverstein representation of iterated Riesz transforms.

3. The formula in Theorem 2.2 contains the second order partial derivatives of the harmonic extension u of f rather than the first order partial derivatives in the Gundy-Varapoulos-Silverstein representation.

4. The matrix A_{jk} in the Gundy-Varapoulos-Silverstein formula defines the indicated martingale transform while the matrix $e_{d+1} \otimes e_i H e_j \otimes e_{d+1}$ in the second formula in Theorem 2.2 defines a new transform.

In addition, we also notice that the boundedness of iterated Riesz transforms can be proved by the result of Theorem 2.2 with the polarization method and Burkholder-Davis-Gundy inequality. We are going to see the proof of the boundedness of iterated Riesz transforms. In order to get to the result we need the following lemma.

Lemma 2.6. *Let $R_if, R_jg \in C_0^1$, C^1 functions with compact support, where R_if, R_jg are the Riesz transforms of f and g, respectively. Then $\int(R_if)(R_jg)\,dx = \int(R_iR_jf)(g)\,dx$.*

Proof. By the polarization on Lemma 3.4 in [2] we have for a constant c

$$\int (R_if)(R_jg)\,dx = \lim_{s\to\infty} s^d \mathbb{E}^{(0,s)} \int_0^\tau \partial_y u_i(Z_r)\partial_y v_j(Z_r)\,dr, \tag{2.13}$$

where u_i and v_j are the harmonic extension of R_if and R_jg, respectively. The generalized Cauchy-Riemann equations, Lemma 3.2 in [2], and the Plancherel identity allow us to have the result as follows. From the equation (2.13)

$$\begin{aligned}
\lim_{s\to\infty} s^d \mathbb{E}^{(0,s)} \int_0^\tau \partial_y u_i(Z_r)\partial v_j(Z_r)\,dr &= c \lim_{s\to\infty} s^d \mathbb{E}^{(0,s)} \int_0^\tau \frac{\partial u}{\partial x_i}(Z_r)\frac{\partial v}{\partial x_j}(Z_r)\,dr \\
&= c\int\int_0^\infty y\frac{\partial u}{\partial x_i}(x,y)\frac{\partial v}{\partial x_j}(x,y)\,dydx \\
&= c\int\int_0^\infty y\widehat{\frac{\partial u}{\partial x_i}}(\xi,y)\widehat{\frac{\partial v}{\partial x_j}}(\xi,y)\,dxd\xi \\
&= c\int\int_0^\infty y|\xi|e^{-|\xi|y}i\frac{\xi_i}{|\xi|}\widehat{f}|\xi|e^{-|\xi|y}i\frac{\xi_j}{|\xi|}\widehat{g}\,dyd\xi \\
&= c\int\int_0^\infty y(e^{-|\xi|y}|\xi|\frac{\xi_i\xi_j}{|\xi|^2}\widehat{f})(|\xi|e^{-|\xi|y}\widehat{g})\,dyd\xi \\
&= c\int\int_0^\infty y(\partial_y u_{ij}(x,y))(\partial_y v(x,y))\,dydx \\
&= c\lim_{s\to\infty} s^d \mathbb{E}^{(0,s)} \int_0^\tau \partial_y u_{ij}(Z_r)\partial_u v(Z_r)\,dr \\
&= \int (R_iR_jf)g.
\end{aligned}$$

$\square$

Finally we have the last new result in this section as follows.

Proposition 2.2. *If $f \in C_0^1$ and $1 < p < \infty$, then there exists c, independent of dimension, such that*

$$\|R_iR_jf\|_p \leq c\|f\|_p.$$

Proof. Suppose that $g \in C_0^1$. Then Lemma 2.6 and the proof of Theorem 3.5 in [2] give the following.

$$\begin{aligned}
\int (R_if)(R_jg) &= c \lim_{s\to\infty} s^d \mathbb{E}^{(0,s)} \int_0^\tau \partial_y u_i(Z_r)\partial_y v_j(Z_r)\,dr \\
&= c \lim_{s\to\infty} s^d \mathbb{E}^{(0,s)} \int_0^\tau \partial_i u(Z_r)\partial_j v(Z_r)\,dr \\
&\leq c \lim_{s\to\infty} s^d \mathbb{E}[(\int_0^\tau |\partial_i u(Z_r)|^2\,dr)^{1/2}(\int_0^\tau |\partial_j v(Z_r)|^2\,dr)^{1/2}] \\
&\leq c(\limsup_{s\to\infty} s^d \mathbb{E}^{(0,s)}[s(f)]^p)^{1/p}(\limsup_{s\to\infty} s^d \mathbb{E}^{(0,s)}[s(g)]^q)^{1/q}
\end{aligned}$$

where $s(f) = (\int_0^\tau |\nabla u(Z_r)|^2\,dr)^{1/2}$. Using Doob's inequality and Burkholder-Davis-Gundy inequality, we have

$$s^d \mathbb{E}^{(0,s)}[s(f)^p] \leq cs^d \mathbb{E}^{(0,s)}|f(X_\tau)|^p$$

and so we get

$$\begin{aligned}
\lim_{s\to\infty} s^d \mathbb{E}^{(0,s)}[s(f)^p] &\leq c \lim_{s\to\infty} s^d \mathbb{E}^{(0,s)}|f(X_\tau)|^p \\
= c \lim_{s\to\infty} s^d \int \frac{s}{(|x|^2+s^2)^{(d+1)/2}}|f(x)|^p\,dx &= c||f||_p^p.
\end{aligned}$$

Similarly with g, and we have

$$\int (R_if)(R_jg) \leq c||f(x)||_p||g(x)||_q.$$

Taking the supremum over gs with $||g||_q \leq 1$ and Lemma 2.6 complete the proof. □

Recently, Bañuelos and Méndez-Hernández in [8] presented a probabilistic representation of the iterated Riesz transforms in terms of the heat extension of a function to the half space and the martingale transform of the extension in the weak sense. They used the formula of the iterated Riesz transform to estimate the iterated Riesz operator. The heat extension of the function $\phi \in C_0^\infty(\mathbb{R}^2)$ to the upper half space $\mathbb{R}^3_+$ is defined by

$$U_\phi(z,t) = \int_{\mathbb{R}^2} \phi(w) P_t(z-w)\, dw,$$

where $P_t(z)$ is the Green function for the half space. That is, this function is the solution to the heat equation in the half space with boundary data ϕ on $\mathbb{R}^2$;

$$\frac{\partial U_\phi}{\partial t}(z,t) = \frac{1}{2} \triangle U_\phi(z,t), \quad (z,t) \in \mathbb{R}^3_+$$

$$U_\phi(z,0) = \phi(z), \quad z \in \mathbb{R}^2.$$

Let Z_t be two dimensional Brownian motion with initial distribution the Lebesgue measure m. Fix $T > 0$ and define the space-time Brownian motion $B_t := (Z_t, T-t), t \in [0,T]$. When $t = 0$ this process starts on the hyperplane $\mathbb{R}^2 \times T$ with initial distribution $m \otimes \delta_T$. Let P^T denote the probability density associated with this process and E^T the corresponding expectation. We define, for a 2×2 matrix A, the martingale transform of $U_\phi(B_t)$ by

$$A * U_\phi = \int_0^T [A \nabla_z U_\phi(B_t)] \cdot dZ_t$$

and its projection in $\mathbb{R}^2$ by

$$\mathcal{S}_A^T \phi(x) = E^T[A * U_\phi | B_T = (x,0)].$$

The following is a probabilistic representation of the iterated Riesz transform in the weak sense in [8].

Proposition 2.3. *Let $i,j \in 1,2$ and $A^{i,j} = (a_{r,s}^{i,j})$ be the 2×2 real matrix defined by*

$$a_{i,j}^{i,j} = -1 \quad \textit{and} \quad a_{r,s}^{i,j} = 0 \quad \textit{if} \quad r \neq i \quad \textit{or} \quad s \neq j.$$

Then for all $\phi \in C_0^\infty(\mathbb{R}^2)$

$$\lim_{T \to \infty} \int_{rtwo} g(z) \mathcal{S}_{A^{i,j}}^T \phi(z)\, dz = \int_{\mathbb{R}^2} g(z) R_i R_j \phi(z)\, dz, \tag{2.14}$$

for any $g \in L^q(\mathbb{R}^2)$, $1 < q < \infty$.

They used this result and properties of martingale transforms to show that

$$||R_j R_i f||_p \leq \frac{1}{2}(p^* - 1)||f||_p, j \neq k,$$

where $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ and $p^* = \max p, p/(p-1)$.

Remark: We are closing this section with some remarks on the differences between Bañuelos and Méndez-Mernández representation, Gundy-Varopoulos-Silverstein's background radiation representation, and the formula in (1.3). In the Gundy-Varopoulos-Silverstein's background radiation representation the martingale transform is given in terms of the harmonic extension of ϕ to the half space instead of the heat extension in the Bañuelos and Méndez-Mernández representation. However, the matrix $e_{n+1} \otimes e_i H e_j \otimes e_{n+1}$ in the second formula in (1.3) defines a new transform rather than a martingale transform. Bañuelos and Méndez-Mernández representation contains the space-time Brownian motion and Gundy-Varopoulos-Silverstein's background radiation representation has the background radiation process while we used the standard Brownian motion on the half space to obtain the formula in (1.3). The natural question arises if we can use the standard Brownian motion to improve the estimates on the iterated Riesz operators, or perhaps to have the same estimates. This is one of nice issues for the future work.

2.3 Application: Probabilistic Representation of the Helmholtz-Hodge Decomposition on $\mathbb{R}^n$

In this section we apply the probabilistic formula of iterated Riesz transforms in this chapter to the Helmholtz-Hodge decomposition. As we have already mentioned, the Leray projection operator is the orthogonal projection of the Hilbert space $L^2(\mathbb{R}^3)^3$ onto divergence-free vector fields $\{u \in L^2(\mathbb{R}^3)^3 : \nabla \cdot u = 0\}$. This operator suggests a more concept of decomposing vector fields into divergence-free part and the gradient part, which leads to the Helmholtz-Hodge decomposition. Let us look at another motivation for the

Helmholtz-Hodge decomposition in terms of PDEs theory rather than operators. We consider the Stokes problem, i.e.,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \nabla p, \quad \nabla \cdot u = 0, \quad u(x,0) = u_0(x) \tag{2.15}$$

where u_0 satisfies the compatibility condition $\nabla \cdot u_0 = 0$. We have time derivatives of only three out of the four unknown functions. Since there is no time derivative of the pressure p we might want to eliminate the pressure in (2.15). In order to do that we need to take the curl of (2.15) and set $w = \nabla \times u$ to obtain the heat equation

$$\frac{\partial w}{\partial t} = \frac{1}{2}\Delta w, \quad w(0,x) = \nabla \times u_0(x). \tag{2.16}$$

We have the solution w by solving the heat equation. Next, we need to consider the existence of a solution u which satisfies $\nabla \times u = w, \nabla \cdot u = 0$. So the following fact is very useful for studying incompressible flows. Here is the statement of the Helmholtz-Hodge decomposition on the vector fields, see [18] for the proof.

Proposition 2.4. *Any smooth vector field F in $\mathbb{R}^n$ which decays sufficiently fast at infinity may be uniquely represented as a superposition of a gradient and a curl, i.e. $F = G + \nabla\phi$ where $G = \nabla \times \psi$ for a scalar potential ϕ and vector potential ψ obtained by solving a Poisson equation $\triangle\phi = \nabla \cdot F$, $\triangle\psi = -\triangle \times F$, and $\nabla \cdot \psi = 0$.*

Roughly speaking, we can find the scalar potential ϕ of a smooth vector field F by taking the Leray projection operator of F or by solving the Poisson equation. Then it is also interesting to seek for the probabilistic interpretation of the Helmholtz-Hodge decomposition. The following is a new probabilistic representations of the scalar potential ϕ by using the probabilistic representation of the iterated Riesz transforms obtained in Theorem 2.2 in Chapter 2 as follows.

Proposition 2.5. *Let $F = (F_1, F_2, F_3)$ be a smooth vector field which decays sufficiently fast at infinity and $F = G + \nabla\phi$ where $G = \nabla \times \psi$ for a scalar potential ϕ and vector* *potential* ψ*. Then if* $v_\rho(x,y) = \sum_{i=0}^{3} \frac{\partial u_{F_i}}{\partial x_i}$*, where* u_{F_i} *is the harmonic extension of* $F_i, i = 1,2,3,$ *and* $\rho := \nabla \cdot F$*, then for some constant* c

$$\frac{\partial \phi(x)}{\partial x_i} = c \sum_{j=1}^{3} \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial u_{F_j}}{\partial x_i \partial x_j}\, dY_r \tag{2.17}$$

or

$$\frac{\partial \phi(x)}{\partial x_i} = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i}\, dY_r \tag{2.18}$$

Proof. First, we are going to get the first probabilistic representation (2.17) of the gradient of the scalar potential ϕ through the Riesz transformations. If we take the divergence of $F = G + \nabla\phi$ to obtain $\triangle\phi = \nabla \cdot F$, formally, $\phi = (\triangle^{-1})\nabla \cdot F$. Hence we have

$$\nabla\phi = \nabla(\triangle^{-1})\nabla \cdot F = RF$$

where $R = [RiRj]$ is the matrix of the iterated Riesz transforms. The probabilistic formula for the iterated Riesz transforms in Chapter 2 gives the first representation (2.17). For the second representation (2.18) we use the argument in proof of Theorem 2.2 in the Chapter 2 as follows. By two lemmas 2.3 and 2.4 we mentioned in the Chapter 2 and the Plancherel identity we have

$$\begin{aligned}
c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i}\, dY_r &= \lim_{s\to\infty} s^n \mathbb{E}^{(0,s)} \{\int_0^\tau \partial_y h_x(Z_r) Y_r \frac{\partial v_\rho}{\partial x_i}(Z_r)\, dr\} \\
&= c_n \int_{\mathbb{R}^n} \int_0^\infty y^2 \partial_y h_x(w,y) \frac{\partial v_i}{\partial y}(w,y)\, dydw \\
&= c_n \int_0^\infty \int_{\mathbb{R}^n} y^2 \widehat{\partial_y h_x}(\xi,y) \widehat{\frac{\partial v_i}{\partial y}}(\xi,y)\, d\xi dy \\
&= c_n \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \underbrace{(\int_0^\infty y^2 |\xi|^3 e^{-2y|\xi|}\, dy)}_{constant} \frac{\xi_i}{|\xi|^2} \widehat{\rho}(\xi)\, dyd\xi \\
&= c_n \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \widehat{\frac{\xi_i}{|\xi|^2}} \widehat{\rho}(\xi)\, dyd\xi \\
&= \frac{\partial}{\partial x_i} (\triangle^{-1})\rho = \frac{\partial \phi(x)}{\partial x_i}.
\end{aligned}$$

where v_i is the harmonic extension of $R_i\rho$. □

3 PROBABILISTIC REPRESENTATION OF NEUMANN PROBLEMS ON EXTERIOR DOMAINS - RELATION TO RIESZ TRANSFORMS AND HELMHOLTZ-HODGE DECOMPOSITION

This chapter is concerned with giving probabilistic representations of solutions to Neumann problems. We construct reflecting Brownian motion by introducing local times and discuss the Neumann problems on bounded domains and exterior domains. Throughout we will assume that the given partial differential equation has a solution, the solution is unique, and the solution is sufficiently regular. We recall that the Riesz transform on $\mathbb{R}^n$ is defined by the principal value of the singular integral

$$R_j f(x) = p.v.c_n \int_{\mathbb{R}^n} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy, \quad j = 1, 2, \cdots, n,$$

where $f \in L^p(\mathbb{R}^n), p = 1, 2$, c_n is a constant that is chosen so that $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, and $\hat{f}$ is the Fourier transform of f. It is well known that there are several equivalent methods to define Riesz transforms on $\mathbb{R}^n$. One of them is to use a Neumann problem on the half space as follows. Let u be the solution in the upper half space $\mathbb{R}^{n+1}_+$ of $\triangle_{\mathbb{R}^{n+1}_+} u = 0$ in $\mathbb{R}^{n+1}_+$, $\frac{\partial u}{\partial \eta} = f$ in $\mathbb{R}^n$, where f is a Schwartz function and $\frac{\partial}{\partial \eta}$ is the outward normal derivative to $\mathbb{R}^{n+1}_+$ at the boundary $\mathbb{R}^n$, i.e., $\frac{\partial u}{\partial x_{n+1}}|_{x_{n+1}=0} = -f$. Then $R_j f = \frac{\partial}{\partial x_j}(u|_{\mathbb{R}^n})$ or, formally, $R_j f = \frac{\partial}{\partial x_j}(\frac{\partial}{\partial \eta})^{-1} f$, where $(\frac{\partial}{\partial \eta})^{-1} f$ means the restriction to $\mathbb{R}^n$ of the solution u. Since $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, $R_j = \frac{\partial}{\partial x_j}(-\triangle_{\mathbb{R}^n})^{1/2}$. Let $Rf := (R_1 f, \ldots, R_n f)$ where $R_j f$ is the Riesz transform of f with respect to j-th component. Then we formally have

$$R = \nabla_{\mathbb{R}^n}(-\triangle_{\mathbb{R}^n})^{1/2}. \tag{3.1}$$

To summarize we can define Riesz transforms of f by solving the Neumann problem on the half space with boundary data f, restricting the solution to the Neumann problem to the boundary $\mathbb{R}^n$, and taking the tangential gradient of restricted solution. This idea suggests that we might define Riesz transforms on a general domain rather than the free space $\mathbb{R}^n$. In 1992 Arcozzi and Xinwei in [1] introduced two nonequivalent possible definitions of Riesz transform on the sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, ball type and cylinder type, by replacing $\nabla_{\mathbb{R}^n}$ in (3.1) with the spherical gradient $\nabla_{\mathbb{S}^{n-1}}$. We also have two ways to define Riesz transforms on the free space. On the one hand, Riesz transforms on $\mathbb{R}^n$ can be characterized by invariant properties with respect to dilations and translations. On the other hand, the generalize Cauchy-Riemann equations can define Riesz transformations on the free space. Recall the Hilbert transform, which is the Riesz transform on $\mathbb{R}$,

$$Hf(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} \, dy.$$

We notice that if $\delta > 0$,

$$H[f(\delta x)] = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(\delta x - \delta y)}{y} \, dy$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(\delta x - y)}{y} \, dy = [Hf](\delta x).$$

Hence the Hilbert transform is invariant with respect to dilation. Similarly, we observe the reflection property as follows.

$$H[f(-x)] = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(-x+y)}{y} \, dy$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(-(x-y))}{y} \, dy = -[Hf](-x).$$

Obviously, we see the translation invariance of the Hilbert transform. These consideration of invariant properties characterize the Hilbert transform as follows. See [37] for proof.

Proposition 3.1. *Suppose T is a bounded operator on $L^2(\mathbb{R})$ which is invariant with respect to translations, positive dilations with the reflection property. Then T is a constant multiple of the Hilbert transform.*

This idea extends to $\mathbb{R}^n$ to define Riesz transforms. For definition of Riesz transform through the generalized Cauchy-Riemann equations we consider the Poisson integral of a function f given on $\mathbb{R}^n$. The relation of theory of harmonic functions, in particular, Poisson integral and the Riesz transforms allows us to define Riesz transforms on $\mathbb{R}^n$. Namely, let u be the harmonic extension of a function f on $\mathbb{R}^n$ and suppose that $u_1, \cdots, u_n, u$ are functions in $\mathbb{R}^{n+1}_+$ which satisfy the generalized Cauchy-Riemann equations,

$$\sum_{j=1}^{n+1} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad for \quad j \neq k, \quad j, k = 1, \cdots, n+1$$

where $u = u_{n+1}$, and are such that $u_1(0) = \cdots = u_{n+1}(0) = 0$. Then $u_j|_{\mathbb{R}^n} = R_j(f), j = 1, \cdots, n$. See [37] for details. In this chapter we consider the approach to solve the Neumann problem rather than to use invariant properties or the generalized Cauchy-Riemann equations to define Riesz transforms on general domain with smooth boundary. Hence it is very interesting to investigate the Neumann problems on more general bounded and unbounded domains because if we have stochastic representation of the Neumann problems on general domains then we might have a chance to obtain probabilistic representation of the Riesz transform on the domains as well as the Riesz transform on the Euclidean space $\mathbb{R}^n$. Let us start our discussion with the Neumann problem on bounded domains.

3.1 Neumann Problems on bounded Domains

First of all, we need to introduce local times. Credit for the discovery of local time should go to P. Levý. Intuitively, the local time is a method for measuring the density of the time spent by the 1-dimensional Brownian motion around a point $x \in \mathbb{R}$. P.Levý introduced the following stochastic process

$$L_t(x) = \lim_{\epsilon \to 0} \frac{1}{4\epsilon} |\{0 \leq s \leq t : |B_s - x| \leq \epsilon\}|, t \geq 0, x \in \mathbb{R},$$

where $|\cdot|$ is the Lebesgue measure, and showed that this limit exists and is finite, but not identically zero, see, for example, [26] . This concept also extends the Itô formula from C^2-functions to convex functions, see [33]. Specifically, the Itô formula showed that if $f : \mathbf{R} \to \mathbf{R}$ is C^2 and X is a semimartingale, then $f(X)$ is also semimartingale. This property extends to convex functions, and leads the important notion of local times. We are about to see local times in the extension of the Itô formula from C^2 to convex functions as follows.

Theorem 3.1. *Let X be a continuous semimartingale and let f be convex. Then there exists a continuous increasing process A^f such that*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\, dX_s + (1/2)A_t^f \tag{3.2}$$

where f'_- is the left-hand derivative of f.

Proof. See, for example, [36] □

For x, y real numbers, let $x^+ = max(x, 0)$ and $x^- = -min(x, 0)$.

Corollary 3.1. *Let X be a semimartingale. Then $|X|$, X^+, and X_- are all semimartingales.*

Let $B = \{B_t : t \geq 0\}$ be one-dimensional Brownian motion starting at 0. By the above corollary it can be written as a martingale plus an increasing process. Since B is itself a martingale, the increasing process grows only at times when the Brownian motion is at 0. This increasing process is known as local time at 0. We give the explicit decomposition of B and then define reflecting Brownian motion which is useful in studying the Neumann problems. For each $\epsilon > 0$, define a g_ϵ on $\mathbb{R}$ by

$$g_\epsilon = \begin{cases} |x|, & \text{if } |x| \geq \epsilon \\ (1/2)(\epsilon + \frac{x^2}{\epsilon}) & \text{if } |x| < \epsilon. \end{cases} \tag{3.3}$$

We note that the function $g_\epsilon(x)$ is not C^2-function since the second derivative $g''\epsilon(x)$ is discontinuous at $x = 0$. However, we can still apply Itô formula to the function g_ϵ due to the following known result.

Proposition 3.2. *Assume that $g : \mathbf{R} \to \mathbf{R}$ is C^1 everywhere and C^2 outside finitely many points $z_1, \dots, z_N$ with $|g''(x)| \le M$ for $x \in \{z_1, \dots, z_n\}$. Then*

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)\, dB_s + \frac{1}{2}\int_0^t g''(B_s)\, ds. \tag{3.4}$$

For the proof of Proposition 3.2 see, for example, [31]. Therefore, we get the equality

$$\begin{aligned} g_\epsilon(B_t) &= g_\epsilon(B_0) + \int_0^t g'_\epsilon(B_s)\, dB_s + \frac{1}{2}\int_0^t g''_\epsilon(B_s)\, ds & (3.5)\\ &= g_\epsilon(B_0) + \int_0^t g'_\epsilon(B_s)\, dB_s + \frac{1}{2\epsilon}|\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}| & (3.6)\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure.

Moreover, applying the Itô isometry to $\mathbb{E}\int_0^t g'_\epsilon(B_s)\mathbf{1}_{B_s\in(-\epsilon,\epsilon)}\, dB_s$, we have $\int_0^t g'_\epsilon(B_s)\mathbf{1}_{B_s\in(-\epsilon,\epsilon)}\, dB_s = \int_0^t \frac{B_s}{\epsilon}\mathbf{1}_{B_s\in(-\epsilon,\epsilon)\, dB_s} \to 0$ in L^2 as $\epsilon \to 0$. Thus letting $\epsilon \to 0$ we get the equality

$$|B_t| = |B_0| + \int_0^t sgn(B_s)\, dB_s + \lim_{\epsilon\to 0}\frac{1}{2\epsilon}|\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}| \tag{3.7}$$

Note that the equality implies implicitly that the limit in L^2 exists.

Definition 3.1. *The local time of a Brownian motion B_t at 0 is defined to be the random variable*

$$L_t = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}|\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}| \quad \text{in } L^2 \tag{3.8}$$

From the equation (3.8) we have the Tanaka's formula as follows.

Theorem 3.2. *Let B_t be a Brownian motion on the real line starting at 0. Then for any $a \in \mathbf{R}$,*

$$|B_t| = |B_0| + \int_0^t sgn(B_s)\, dB_s + L_a(t), \tag{3.9}$$

where $L_a(t) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}|\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}|$.

We note that by the conditional Jensen's inequality, the stochastic process $|B_t|$ is a submartingale. Thus the Tanaka's formula gives the Doob-Meyer decomposition of $|B_t|$.

Moveover, the stochastic process

$$M_t := \int_0^t sgn(B_s)\, dB_s$$

is a Brownian motion by the Levy's characterization theorem, i.e. $\mathbb{P}(M_0 = 0) = 1$, M_t is a continuous martingale with respect to the filtration $\mathcal{F} = \sigma\{B_s : s \le t\}$ and the quadratic variation of M_t is given by

$$< M >_t = \int_0^t |sgn(B_s)|^2\, ds = \int_0^t 1\, ds = t.$$

Therefore we conclude that if B_s is a Brownian motion on the real line, then the Tanaka's formula says that

$$|B_t| = W_t + L_t, \tag{3.10}$$

where W_t is another Brownian motion and L_t is a continuous non-decreasing process that increases only when $|B_t|$ is at 0.

We call $X_t =: |B_t|$ reflecting Brownian motion and L_t the local time (at 0) of X_t as the previous definition of local time. Let's look at an interesting example of the reflecting Brownian motion on the half space.

Example 3.1. *(A diffusion in $\mathbb{R}^n$ with reflection, $n \ge 2$)*
Let D be the upper-half space, let $Y_t = (Y_t^1, \ldots, Y_t^n)$ be standard n-dimensional Brownian motion, and let L_t be the local time of $|Y_t^n|$. Then $X_t = (Y_t^1, \ldots, |Y_t^n|)$ is reflecting Brownian motion with normal reflection in D. If $|Y_t^n| = \widetilde{W_t} + L_t$, then X_t solves the SDE

$$dX_t = dW_t + \eta(X_t)dL_t, \quad X_t \in \bar{D}, \tag{3.11}$$

where $W_t = (Y_t^1, \ldots, Y_t^{n-1}, \widetilde{W_t})$ is a n-dimensional Brownian motion, $\eta(x) = (0, \ldots, 0, 1)$ is the inward pointing unit normal vector, and L_t is continuous nondecreasing process that increases only when X_t is on the boundary of D.

The equation (3.11) is an example of what is known as the Skorokhod equation. Let D be a domain, σ be a matrix, b a vector, W_t a standard d-dimensional Brownian motion, and $v(x)$ defined on ∂D such that $v(x) \cdot \eta(x) > 0$ for all $x \in \partial D$. Here $\eta(x)$ is the inward pointing unit normal vector at x. Then the Skorokhod equation is the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt + v(X_t)dL_t, \quad X_0 = x_0 \tag{3.12}$$

where $X_t \in \bar{D}$ for all $t, x_0 \in \bar{D}$, and L_t is a continuous nondecreasing process that increases only when $X_t \in \partial D$. In 1984 P. L. Lions and A. S. Sznitman in [29] proved the uniqueness and existence of the solution to the Skorokhod equation as follows. See [29] for detail.

Theorem 3.3. *Let D be a bounded smooth domain and let σ and b be uniformly bounded real-valued functions on $\mathbb{R}^d$ satisfying a uniform Lipschitz condition: there exists $K > 0$ such that*

$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq K|x-y|, \quad |b_i(x) - b_j(x)| \leq K|x-y|, \quad i,j = 1, 2, \cdots, d.$$

Also let $v(x) \in C_b^2(\mathbb{R}^d)$ be satisfy $v(x) \cdot \eta(x) > 0$ for all $x \in \partial D$. Then there exists a solution to the Skorokhod equation (3.12). If X_t and X_t' are two solutions to (3.12), then $X_t = X_t'$ a.s. for all t.

Now we are ready to look at the Neumann problem. Suppose D is a bounded smooth domain. The Neumann problem for D is the following: *Find $u \in C(\bar{D})$ such that u is C^2 on D and $\triangle u = 0$ in D, $\frac{\partial u}{\partial \eta} = f$ on ∂D, where $\eta(x)$ denotes the inward pointing unit normal vector at $x \in \partial D$.* In order for the Neumann problem to have a solution we need side conditions such as compatibility conditions. For example, by the Green's identity, we have $0 = \int_D 1 \triangle u + \int_D \nabla 1 \cdot \nabla u = \int_{\partial D} 1 \frac{\partial u}{\partial \eta}\, d\sigma = \int_{\partial D} f\, d\sigma$ where $d\sigma$ is surface measure on ∂D.

In this section, we want to derive a representation for the solution to the Neumann problem in terms of reflecting Brownian motion on smooth bounded domains. Such a probabilistic representation was dicussed in [20], [6] and [3]. Following Bass' approach in

[3] let us get the probabilistic representation in terms of reflecting Brownian motion. To avoid dealing with side conditions for the Neumann problem, let us introduce a smooth compact subset K of D and find $u \in C(\bar{D})$ such that $u \in C^2$ on D and

$$\triangle u = 0 \quad \text{in} \quad D - K, \quad \frac{\partial u}{\partial \eta} = f \quad \text{on} \quad \partial D, \quad u = 0 \quad \text{on} \quad K. \tag{3.13}$$

Suppose X_t satisfies

$$dX_t = dW_t + \eta(X_t) dL_t \tag{3.14}$$

where W_t is d-dimensional Brownian motion ($d \geq 3$), L_t is a nondecreasing continuous process that increases only when $X_t \in \partial D$, D is a bounded C^2 domain. By the Itô formula we get the probabilistic representation of solution to the Neumann problem on a bounded domain D as follows.

Theorem 3.4. *Suppose τ_K, the hitting time to K, is finite a.s. and $\mathbb{E}^x L_{\tau_K} < \infty$ for all x. The solution to the problem (3.13) is given by*

$$u(x) = -\mathbb{E}^x \int_0^{\tau_K} f(X_s)\, dL_s. \tag{3.15}$$

Proof. By the Itô formula,

$$\begin{aligned} u(X_{t\wedge\tau_K}) = u(X_0) + \int_0^{t\wedge\tau_K} \nabla u \cdot dB_s + \int_0^{t\wedge\tau_K} \triangle u(X_s)\, ds \\ + \int_0^{t\wedge\tau_K} (\nabla u \cdot \eta)(X_s)\, dL_s. \end{aligned}$$

We take expectation and then let $t \to \infty$. Since $u = 0$ on K and $\nabla u = 0$ in D, we obtain

$$u(x) + \mathbb{E} \int_0^{\tau_K} f(X_s)\, dL_s = 0.$$

□

3.2 Neumann Problems on Unbounded Domains

In this section we investigate the Neumann problems on an exterior domain $D \in \mathbb{R}^d$, which is the complement of a compact domain in $\mathbb{R}^d$, and the half space $\mathbb{R}^d_+$, i.e., we consider $u \in C(\bar{D})$ s.t. u is C^2 on D and $\triangle u = 0$ in D, $\frac{\partial u}{\partial \eta} = f$ on ∂D where $\eta(x)$ denotes the outward pointing unit normal vector at $x \in \partial D$. In the case of the upper half space, Ramasubrammnian in [35] considered the Neumann problem on the half space with compatibility condition and several restrictions on the boundary data f such as the boundedness of f, finite second moment and $\int_{\mathbb{R}^n} |x|^r |f(x)|\, dx < \infty$, $r = 0, 1, 2$. He extended the probabilistic representation of solutions to the Neumann problem in bounded domains, which was investigated by ergodic theory, to the case of the half space. If a domain D is an unbounded domain in $\mathbb{R}^d (d \geq 3)$ with compact Lipschitz boundary ∂D, Chen, Williams and Zhao in [14] solved the Neumann problem on that domain and gave the probabilistic representation of solution to the problem by constructing the Green-tight class for the boundary ∂D which is the subset of the Kato class with Green-tightness and a function space Γ that we will explain just below. The following is the precise definitions of Green-tightness and of the class Γ.

Definition 3.2. *A function w is Green-tight on D if w is a real-valued Borel measurable function defined on D such that the family of functions $\{w(\cdot)/|x - \cdot|^{d-2}, x \in D\}$ defined on D is uniformly integrable in the sense that w satisfies*

$$\lim_{m(A)\to 0, A\subset D} \{\sup_{x\in D} \int_A \frac{|w(y)|}{|x-y|^{d-2}}\, dy\} = 0,$$

and

$$\lim_{M\to\infty} \{\sup_{x\in D} \int_{|y|>M, y\in D} \frac{|w(y)|}{|x-y|^{d-2}}\, dy\} = 0,$$

where m denotes Lebesgue measure on $\mathbb{R}^d$.

It is known [13] that a Borel measurable function w is Green-tight on D if and only if $\mathbf{1}_D w \in \mathbf{K}_d^\infty$, where

$$\mathbf{K}_d^\infty = \{v \in \mathbf{K}_d : \lim_{M\to\infty} [\sup_{x\in\mathbb{R}^d} \int_{|y|>M} \frac{|v(y)|}{|x-y|^{d-2}}] = 0\}.$$

Here $\mathbf{K}_d$ denotes the Kato class for $\mathbb{R}^d$ which consists of all real-valued Borel measurable functions v defined on $\mathbb{R}^d$ such that

$$\lim_{r\to 0}[\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le r}\frac{|v(y)|}{|x-y|^{d-2}}]=0$$

For the Neumann boundary data they introduced the class Γ which is an analogue of the Green-tight class but for the boundayr ∂D in place of D.

Definition 3.3. *A function w is in the class $\Gamma = \Gamma(\partial D)$ if w is a real-valued Borel measurable function defined on ∂D such that the family of functions $\{w(\cdot)/|x-\cdot|^{d-2}, x\in \partial D\}$ is uniformly integrable with respect to the surface measure σ on ∂D, i.e.,*

$$\lim_{\sigma(A)\to 0, A\subset\partial D}\{\sup_{x\in\partial D}\int_A \frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\}=0, \tag{3.16}$$

.

We notice that if w is in the class Γ then w satisfies:

$$\lim_{r\to 0}\{\sup_{x\in\partial D}\int_{\partial D\cap B(x,r)}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\}=0.$$

and so w satisfies:

$$\lim_{\sigma(A)\to 0, A\subset\partial D}\{\sup_{x\in\bar{D}}\int_A \frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\}=0.$$

where $B(x,r)$ denotes the open ball in $\mathbb{R}^d$ centered at x with radius r. Hence $x\in\partial D$ in (3.3) can be replaced by $x\in\bar{D}$. Finally we can say that if $w\in\Gamma$,

$$\sup_{x\in\bar{D}}\int_{\partial D}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)<\infty.$$

Since ∂D is compact, we have

$$\int_{\partial D}|w(y)|\,\sigma(dy)=\int_{\partial D}\frac{|w(y)|}{|x-y|^{d-2}}(|x-y|^{d-2})\,\sigma(dy)$$
$$\le c\int_{\partial D}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)\le c\sup_{x\in\bar{D}}\int_{\partial D}\frac{|w(y)|}{|x-y|^{d-2}}\,\sigma(dy)<\infty$$

and so $\Gamma \subset L^1(\partial D, \sigma)$. In [14] it is shown in Proposition 2.2 that Γ contains all functions in $L^p(\partial D, \sigma)$ for $p > d-1$. To summarize they solved weakly the Neumann problem with the boundary data in Γ which is the subset of the Kato class with Green-tightness.

The aim in this section is to get a probabilistic representation of the solution to the Neumann problem in terms of reflecting Brownian motion with more straightforward method when we assume the existence of the solution. Before our new main theorem, we need the following result which was proved in [13].

Proposition 3.3. *The transition density function* $(t, x, y) \to p(t, x, y)$ *of reflecting Brownian motion* B_t^r *exists as a continuous function on* $(0, \infty) \times \bar{D} \times \bar{D}$*, where* D *is an unbounded Lipschitz domain in* $\mathbb{R}^n (n \geq 3)$ *with compact boundary* ∂D*. Furthermore, there exist constants* $c_1 = c_1(D) > 0$ *and* $c_2 = c_2(D)$ *such that*

$$p(t, x, y) \leq \frac{c_1}{t^{n/2}} \exp(-\frac{|x-y|^2}{c_2 t}) \tag{3.17}$$

for all $t > 0$*,* $x, y \in \bar{D}$*.*
Let $N(x, y) = \int_0^\infty p(t, x, y)\, dt$*, the Neumann function with respect to* B_t^r*. Then* $N(x, y)$ *is finite and continuous on* $\bar{D} \times \bar{D}$*, except on the diagonal. Furthermore, there exists a constant* $c = c(D)$ *such that* $N(x, y) \leq \frac{c}{|x-y|^{n-2}}$ *for all* $x, y \in \bar{D}$*.*

With the above Proposition 3.3 we are going to show the following lemma which is necessary to prove the main result. Recall that $N(x, y) = \int_0^\infty p(t, x, y)\, dt$ is the Neumann function with respect to reflecting Brownian motion B_t^r.

Lemma 3.1. *Let* $u(x) = \int_{\partial D} N(x, y) f(y)\, dS(y)$ *on the exterior domain* D *and* $f \in L^1(\partial D)$*. Then* $\lim_{t\to\infty} \mathbb{E}_x[u(B_t^r)] = 0$*.*

Proof. If $p(t, x, y)$ is the probability density of reflecting Browian motion B_t^r on D, then the Fubini-Tonelli theorem, semi-group property of transition density and change of variables give the boundedness of $\mathbb{E}_x[u(B_t^r)]$ as follows. First, we note that

$$\mathbb{E}_x[u(B_t^r)] = \int_D u(x')p(t,x,x')\,dx' = \int_D [\int_{\partial D} N(x',y)f(y)\,dS(y)]p(t,x,x')\,dx'$$
$$= \int_{\partial D}\int_0^\infty [\int_D p(s,x',y)p(t,x,x')\,dx']\,ds f(y)\,dS(y)$$
$$= \int_{\partial D}[\int_0^\infty p(t+s,x,y)\,ds]f(y)\,dS(y) = \int_{\partial D}[\int_t^\infty p(s,x,y)\,ds]f(y)\,dS(y).$$

Hence we have

$$|\mathbb{E}_x[u(B_t^r)]| \le \int_{\partial D} N(x,y)|f(y)|\,dS(y)$$

Since $f \in L^1(\partial D)$ and the Neumann function $N(x,y)$ is finite, by the Dominated Convergence Theorem, we have $\lim_{t\to\infty}\mathbb{E}_x[u(B_t^r)] = 0$. □

Now we ready to present and prove the main result.

Proposition 3.4. *Under the same assumption in Lemma 3.1 and assuming that* $\frac{\partial u}{\partial \eta}|_{\partial D} = f$ *the solution* u *to the Neumann problem on the exterior domain* D *is given by*

$$u(x) = -\lim_{t\to\infty}\mathbb{E}_x\int_0^t f(B_s^r)\,dLs \tag{3.18}$$

where L_t *is the local time process of reflecting Brownian motion* B_t^r *on* $\bar{D}$.

Proof. By the Itô formula with $dB_t^r = dB_t + \eta(B_t^r)dL_t$ or $(dB_t^r)^i = (dB_t)^i + (\eta(B_t^r))^i dL_t, i = 1, 2, \cdots n$, we have

$$u(B_t^r) = u(B_0^r) + \int_0^t \sum_{i=0}^n \partial_i u(B_s^r)\,(dB_s^r)^i + \frac{1}{2}\int_0^t \sum_{i,j=1}^n \partial_{ij}u(B_s^r)\,d<(B^r)^i,(B^r)^j>_s$$

$$= u(x) + \int_0^t \sum_{i=0}^n \partial_i u(B_s^r)\,(dB_s)^i + \int_0^t \sum_{i=1}^t \partial_i u(B_s^r)(\eta(B_s^r)^i)\,dL_s + \frac{1}{2}\int_0^t \triangle u(B_s^r)\,ds$$
$$= u(x) + \int_0^t \nabla u\cdot dB_s + \int_0^t \nabla u\cdot\eta(B_s^r)\,dL_s + \frac{1}{2}\int_0^t \triangle u(B_s^r)\,ds.$$

We take expectation on both sides and then let $t\to\infty$. By the Lemma 3.1 we obtain our result. □

Let us make a remark on the decayness of the solution to the Neumann problem.

Remark: For the decayness of the solution u to the Neumann problem on unbounded domains let us look at the solution u to the Neumann problem on the half space $\mathbb{R}^3_+$ with boundary data $f \in L^1(\mathbb{R}^2)$:

$$u(x) = u(x', x_3) = c \int_{\mathbb{R}^2} \frac{1}{(|x' - y'|^2 + x_3^2)^{\frac{1}{2}}} f(y')\, dy' \tag{3.19}$$

If $x_3 \neq 0$, then the Neumann function $\frac{1}{(|x'-y'|^2+x_3^2)^{\frac{1}{2}}}$ is bounded and so L^1 boundary data gives us the boundedness of u. Also, we notice that the normal derivative of the Neumann function is the Poison kernel and hence by the approximation of identity $-\frac{\partial u}{\partial_3}|_{x_3=0} = f(x')$. Thus u in (3.19) is well defined for each $x \in \mathbb{R}^3_+$. However, we can not guarantee that u vanishes uniformly at infinity as the following counterexample shows.

Example 3.2. *Let $f(y') = \sum_{k\neq 0} a_k \chi_{B(k,r_k)}(y')$, where k is an non zero integer and $B(k, r_k)$ is the ball centered at k with radius r_k. If $a_k = k^2$ and $r_k = \frac{1}{k^2}$, then we have*

$$\int_{\mathbb{R}^2} |f(y')|\, dy' = \pi \sum_{k\neq 0} a_k (\frac{1}{k^2})^2 = \pi \sum_{k\neq 0} \frac{1}{k^2} < \infty.$$

Let $x' = m$. Then for $x_3 = 0$ in (3.19)

$$1 = a_m r_m = \frac{1}{2\pi} a_m \int_{|y'|<r_m} \frac{1}{|y'|}\, dy' \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|y'|} f(x' - y')\, dy'.$$

Therefore u does not vanish at infinity.

Even if we take $f \in L^p(\mathbb{R}^2), p < 3/2$, in the above Example, u is still greater than 2π at infinity. However, if f is in the Green-tight class Γ on the boundary, then u decays uniformly as x goes to ∞ because

$$|u(x)| = |u(x', x_3)| = c \int_{\mathbb{R}^2} \frac{1}{(|x' - y'|^2 + x_3^2)^{\frac{1}{2}}} |f(y')|\, dy' \leq c \int_{\mathbb{R}^2} \frac{1}{|x' - y'|} |f(y')|\, dy'$$

$$\leq \int_{|y'|\leq M} \frac{1}{|y'|} |f(x' - y')|\, dy' + \int_{|y'|>M} \frac{1}{|y'|} |f(x' - y')|\, dy'$$

and $\int_{|y'|>M} \frac{1}{|y'|}|f(x'-y')|\,dy' \leq \frac{1}{M}||f||_{L^1(\mathbb{R}^2)}$. That means the Green-tight condition on the boundary is a sufficient condition for us to have the solution to the Neumann problem vanishing uniformly at infinity. Natural questions arise such as what sufficient and necessary condition on the boundary data f is for uniform decayness of u as well as the existence of u or at least what are the weaker conditions for that rather than the Green-tightness. These questions are not easy to answer right away.

The key idea of the main proposition (3.4) on the exterior domain can be applied to the Neumann problem on the upper half space $\mathbb{R}^n_+, n \geq 3$, which is not an exterior domain itself. Let us obtain a probability representation of the solution to the Neumann problem on the half space $\mathbb{R}^n_+$. For the next Lemma and Proposition we introduce the Neumann function

$$N(x', y', x_n) = (\frac{2}{n(n-2)\alpha(n)})\frac{1}{(|x'-y'|^2 + x_n^2)^{\frac{n-2}{2}}}, \quad x' \in \mathbb{R}^{n-1}, \quad x_n > 0$$

for Laplace equation on the half space $\mathbb{R}^n_+$

Lemma 3.2. *If* $u(x) = \int_{\mathbb{R}^{n-1}} N(x', y', x_n) f(y')\,dy'$, *and the boundary data* $f \in L^1(\mathbb{R}^{n-1})$, *then*

$$\lim_{t\to\infty} \mathbb{E}_{(x_0,y_0)}[u(B^r_t)] = 0.$$

where L_t *is the local time process of reflecting Brownian motion* B^r_t *on* $\mathbb{R}^n_+$.

Proof. We note that

$$\frac{1}{(|x-x'|^2+y^2)^{(n-2)/2}} = c_n \int_0^\infty \frac{1}{s^{n/2}} \exp{(-\frac{|x-x'|^2+y^2}{2s})}\,ds.$$

Using semi-group property of the transition probability density and change of variables we have

$$\mathbb{E}_{(x_0,y_0)}[u(B_t^r)] = c_n \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{1}{s^{(n-1)/2}} \exp(-\frac{|x-x'|^2}{2s}) \frac{1}{\sqrt{s}} \exp(-\frac{y^2}{2s})$$

$$\times f(x') \frac{1}{t^{(n-1)/2}} \exp(-\frac{|x-x_0|^2}{2t})$$

$$\times [\frac{1}{\sqrt{t}} \exp(-\frac{(y-y_0)^2}{2t}) + \frac{1}{\sqrt{t}} \exp(-\frac{(y+y_0)^2}{2t})] \, ds dx' dx dy$$

$$= c_n \int_{\mathbb{R}^{n-1}} f(x') \int_0^\infty \frac{1}{(t+s)^{(n-1)/2}} \exp(-\frac{|x'-x_0|^2}{2(t+s)})$$

$$\times \int_0^\infty \frac{\exp(-y^2/2s)}{\sqrt{s}\sqrt{t}} [\exp(-\frac{|y-y_0|^2}{2t}) + \exp(-\frac{|y+y_0|^2}{2t})] \, dy ds dx'$$

$$= c_n \int_{\mathbb{R}^{n-1}} f(x') \int_0^\infty \frac{1}{(t+s)^{(n-1)/2}} \exp(-\frac{|x'-x_0|^2}{2(t+s)})$$

$$\times \frac{1}{\sqrt{t+s}} \exp(-\frac{y_0^2}{2(t+s)}) \, ds dx'$$

$$= c_n \int_{\mathbb{R}^{n-1}} f(x') \int_t^\infty \frac{1}{s^{n/2}} \exp(-\frac{|x'-x_0|^2 + y_0^2}{2s}) \, ds dx'$$

Therefore we have

$$\mathbb{E}_{(x_0,y_0)}[u(B_t^r)] = c_n \int_{\mathbb{R}^{n-1}} f(x') [\frac{1}{(|x-x'|^2+y^2)^{(n-2)/2}} - \int_0^t \frac{1}{s^{n/2}} \exp(-\frac{|x'-x_0|^2+y_0^2}{2s}) \, ds] \, dx'.$$

Letting $t \to \infty$, we have the result.

□

Proposition 3.5. *If* $u(x) = \int_{\mathbb{R}^{n-1}} N(x', y', x_n) f(y') \, dy'$, *and the boundary data* $f \in L^1(\mathbb{R}^{n-1})$, *then*

$$\triangle u = 0, \quad \frac{\partial u}{\partial \eta}|_{x_n=0} = f$$

and

$$u(x) = -\lim_{t\to\infty} \mathbb{E}_{(x_0,y_0)} \int_0^t f(B_s^r) \, dLs \tag{3.20}$$

where L_t *is the local time process of reflecting Brownian motion* B_t^r *on* $\mathbb{R}_+^n$.

Proof. By the Itô formula we have

$$u(B_t^r) = u(B_0^r) + \int_0^t \nabla' u \cdot dB_s + \int_0^t \frac{\partial u}{\partial y} \, dB_s^r + \frac{1}{2} \int_0^t \triangle u \, ds$$

$$= u(x) + \int_0^t \nabla' u \cdot \, dB_s + \int_0^t \frac{\partial u}{\partial y} sgn B_s \, dB_s + \int_0^t f \, dL_s$$

We take expectation on both sides and then let $t \to \infty$. By the Lemma 3.2 we obtain our result. $\square$

3.3 Application: Probabilistic Representation of the Helmholtz-Hodge Decomposition on General Domains

In this section we apply the probabilistic representation of solution to the Neumann problem on general domains such as exterior domains in this chapter to the Helmholtz-Hodge decomposition. We have studied probabilistic representation of the Helmholtz-Hodge decomposition on the free space in Chapter 2. In this section we are going to have another probabilistic representation of Helmholtz-Hodge decomposition on a domain D with smooth boundary ∂D. Let us recall that the Helmholtz-Hodge decomposition on a domain D with smooth boundary ∂D, which states that any vector field u on D can be uniquely decomposed in the form

$$u = G + \nabla \varphi \tag{3.21}$$

where G has divergence zero and is parallel to ∂D, i.e., $G \cdot \eta = 0$ on ∂D. From the fact that G is parallel to ∂D, we obtain $\frac{\partial \varphi}{\partial \eta} = \nabla \varphi \cdot \eta = u \cdot \eta$ and if we are going to take the divergence of (3.21), then we have indeed the Neumann problem

$$\Delta \varphi = \nabla \cdot u \quad in D, \quad \frac{\partial \varphi}{\partial \eta} = u \cdot \eta \tag{3.22}$$

We already had the probabilistic representation of solution to the Neumann problem (3.22) in this chapter in terms of reflecting Brownian motions. This gives us another new probabilistic representation of the Helmholtz-Hodge decomposition. We summerize this discussion as

Proposition 3.6. *Let u be a smooth vector field on a domain D with smooth boundary* *such as a bounded domain or an exterior domain with smooth boundary. If a domain D is an exterior domain, then we need to have the property that u decays sufficiently fast at infinity. Suppose $u = G + \triangle\varphi$ where G has zero divergence and is parallel to ∂D. Then if $\nabla \cdot u = 0$ then we have*

$$\varphi(x) = -\lim_{t\to\infty} \mathbb{E}_x \int_0^t (u \cdot n)(B_s^r)\, dLs \tag{3.23}$$

where L_t is the local time process of reflecting Brownian motion B_t^r on $\bar{D}$.

we recall again that if $f \in L^1(\mathbb{R}^n)$, then

$$R_j f(x) = p.v.c_n \int_{\mathbb{R}^n} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy, \quad j = 1, 2, \cdots, n, \tag{3.24}$$

where $f \in L^p(\mathbb{R}^n), p = 1, 2$, c_n is a constant that is chosen so that $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, and $\hat{f}$ is the Fourier transform of f. There is a different way to define Riesz transforms, which is related to a Neumann problem. Let u be the solution in the upper half space $\mathbb{R}_+^{n+1}$ of $\triangle_{\mathbb{R}_+^{n+1}} u = 0$ in $\mathbb{R}_+^{n+1}$, $\frac{\partial u}{\partial eta} = f$ in $\mathbb{R}^n$, where $f \in L(\mathbb{R}^n)$ and $\frac{\partial}{\partial \eta}$ is the outward normal derivative to $\mathbb{R}_+^{n+1}$ at the boundary $\mathbb{R}^n$. Then $R_j f = \frac{\partial}{\partial x_j}(u|_{\mathbb{R}^n})$ or, formally, $R_j f = \frac{\partial}{\partial x_j}(\frac{\partial}{\partial \eta})^{-1} f$, where $(\frac{\partial}{\partial \eta})^{-1} f$ means the restriction to $\mathbb{R}^n$ of the solution u. Since $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, $R_j = \frac{\partial}{\partial x_j}(-\triangle_{\mathbb{R}^n})^{1/2}$.

Therefore we can define the Riesz transform of f in terms of reflecting Brownian motion from Chapter 3 by

$$R_j f = \frac{\partial}{\partial x_j}(-\lim_{t\to\infty} \mathbb{E} \int_0^t f(B_s^r)\, dL_s)|_{\mathbb{R}^n}. \tag{3.25}$$

Through the Leray projection operator $\mathcal{P}$ we can decompose a smooth vector field into the curl-free vector fields in terms of reflecting Brownian motion.

This result, together with the probabilistic representation of the iterated Riesz transform in Chapter 2, give us a way to study of the Helmholtz-Hodge decomposition on vector field probabilistically.

4 OSEEN AND STOKES PROBLEMS - LINEARIZATION OF NAVIER-STOKES EQUATIONS

In this chapter we consider the 3-dimensional linearized Navier-Stokes equations, Stokes and Oseen equations, on bounded and exterior domains with smooth boundaries in $\mathbb{R}^3$. The aim is to study existence of solutions to Stokes and Oseen problems and to give probabilistic representations of the solutions in terms of stochastic process. This representation uses absorbed Brownian motion and boundary data on $\triangledown$p, which can be determined by the initial and boundary data of the solutions. We recall that the 3-dimensional Navier-Stokes equations on $\mathbb{R}^3$ are given by

$$\frac{\partial u}{\partial t} + (u \cdot \triangledown)u = \nu \Delta u - \triangledown p + f, \quad \triangledown \cdot u = 0, \quad u(x,0) = u_0(x) \tag{4.1}$$

where $\triangledown = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, and $u \cdot \triangledown = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$, $\Delta = \triangledown \cdot \triangledown = \sum_{j=1}^{3} \frac{\partial^2}{\partial {x_j}^2}$ are applied component-wise. The term $p(x,t)$ is the (scalar) pressure, $f(x,t)$ represents external forcing, and $\nu > 0$ is the kinematic viscosity. In general, there are two common linearizations of the Navier-Stokes equations. One is the Stokes equation and the other is the Oseen equation. Suppose that the Reynolds number is very small. Then the very small Reynolds number implies slow velocity, large viscosity, or small bodies enough to ignore the convective term $(u \cdot \triangledown)u$ in (4.1), i.e.,

$$\frac{\partial u}{\partial t} = \nu \Delta u - \triangledown p + f, \quad \triangledown \cdot u = 0, \tag{4.2}$$

which are the Stokes equations for incompressible flow. In 1910 Oseen suggested that Stokes equations can be expressed by

$$\frac{\partial v}{\partial t} + (U \cdot \triangledown)v = \nu \Delta v - \triangledown p + f, \quad \triangledown \cdot v = 0, \quad v(x,0) = v_0(x) \tag{4.3}$$

where U is a constant vector. This is the linearizaiton of the Navier-Stokes equations about U, while Stokes equations may be viewed as a linearization about 0.

Let us get started by considering Stokes problem on $\mathbb{R}^3$.

$$\frac{\partial u}{\partial t} = \nu \Delta u - \nabla p + f, \quad \nabla \cdot u = 0, \quad u(x,0) = u_0 \tag{4.4}$$

Then it follows from the relation between the Leray projection $\mathcal{P}$ and the Riesz transforms in Chapter 2 that the fundamental solution of the Stokes equations can be written as

$$\Gamma(x,y,t) = -\triangle_y \psi(x,y,t) I + Hess\psi(x,y,t), \tag{4.5}$$

where for each $x \in \mathbb{R}^3, t > 0$, ψ satisfies $\triangle_y \psi(x,y,t) = -k(x,y,t)$ with the heat kernel in $\mathbb{R}^3$, $k(x,y,t) = \frac{1}{(4\pi\nu t)^{3/2}} exp(-\frac{|x-y|^2}{4\pi\nu t})$, and $Hess\psi$ denotes the matrix of the second order partial derivatives with respect to the y variable, and I denotes the 3×3 identity matrix. Oseen in [19] used the fundamental solution tensor for the steady problems in this form in $\mathbb{R}^2$. In 3-dimensional case, Solonnikov in [38] had a similar expression in his analysis of the time dependent problem in $\mathbb{R}^3$. More recently Guenther and Thomann in [23] obtained an explicit formula for the fundamental solution in terms of Kummer functions. Now, recall that for $f \in \mathcal{S}(\mathbb{R}^3)$, the Schawrtz class,

$$R_i R_j f = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \triangle^{-1} f.$$

See [37], pg243. Hence we have $\Gamma = (I + R)k$ where R is the matrix of iterated Riesz transform $R_i R_j$ and so if $f = 0$ and $u(x,0) = u_0(x)$ in the Stokes problem, then we get

$$u(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t) u_0(y)\, dy = \int_{\mathbb{R}^3} [I + R] K(x,y,t) u_0(y)\, dy, \tag{4.6}$$

where $K = kI$. If we want to give a probabilistic representation to the solution u, then we need to find an appropriate stochastic process to explain the behavior of particles in slow-flowing fluid. If we, in particular, use the standard Brownian motion and the iterated Riesz transform formula (2.12) in Chapter 2, then we have probabilistic representation of $u = (u_1, u_2, u_3)$ as follows.

Proposition 4.1. *Let $u = (u_1, u_2, u_3)$ be the solution to the Stokes problem with an initial data $u_0 = (u_0^1, u_0^2, u_0^3)$ on the free space. Then if v is the harmonic extension of u_0 and $v_\rho = \nabla \cdot v$ then for some constant c*

$$u_i(x,t) = \mathbb{E}_x[u_0^i(B_t)] + c\mathbb{E}_x[\lim_{s\to\infty} \mathbb{E}_{h_{B_t}}^{(0,s)}[\int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i}\, dY_r]] \tag{4.7}$$

Proof. From (4.6) properties of Fourier transforms and Riesz transforms allow us to compute

$$\begin{aligned} u_i(x,t) &= \int k(x,y,t)u_0^i(y)\, dy + \int \sum_{j=1}^3 (R_iR_jk)u_0^j\, dy \\ &= \int k(x,y,t)u_0^i(y)\, dy + \int \sum_{j=1}^3 (R_iR_ju_0^j)k\, dy \end{aligned}$$

Letting $w_i := \sum_{j=1}^3 (R_iR_ju_0^j)$, the probabilistic representation of the iterated Riesz transforms of (2.12) in chapter 2

$$R_iR_jf(x) = c\lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}\, dY_r = c\lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau (e_{d+1} \otimes e_iHe_j \otimes e_{d+1} \cdot Z_r) dZ_r, \tag{4.8}$$

where H is the Hessian of the harmonic extension u of f, gives

$$\begin{aligned} u_i(x,t) &= \mathbb{E}_x[u_0^j(B_t)] + \mathbb{E}_x[w_i(B_t)] \\ &= \mathbb{E}_x[u_0^j(B_t)] + \mathbb{E}_x[\sum_{j=1}^3 \lim_{s\to\infty} c\mathbb{E}_{h_{B_t}}^{(0,s)}[\int_0^\tau Y_r \frac{\partial^2 v_j}{\partial x_i \partial x_j}\, dY_r]] \end{aligned}$$

where v_j is the harmonic extension of u_0^j. Letting $v_\rho = \nabla \cdot v$ completes the proof □

Then a more interesting and challenging problem is to consider probabilistic representation of the solution u on a bounded or exterior domain with smooth boundary. We will investigate the initial-boundary value problems of Stokes equation on a domain with smooth boundary as follows.

4.1 A Priori

We recall the Stokes problem on a domain D with smooth boundary ∂D;

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \nabla p, \tag{4.9}$$

$$\nabla \cdot u = 0, \tag{4.10}$$

$$u(x,0) = u_0(x), \quad u|_{\partial D} = a(x,t) \tag{4.11}$$

The incompressibility condition $\nabla \cdot u = 0$ leads us to the fact that the pressure p and ∇p are harmonic. In particular, ∇p is determined by its boundary data, say, $H = (h_1, h_2, h_3)$, i.e. $\frac{\partial p}{\partial x_i}|_{\partial D} = h_i$, $i = 1, 2, 3$, and so we can obtain ∇p by solving the following Dirichlet problem:

$$\triangle q_i = 0, \quad q_i|_{\partial D} = h_i, \quad i = 1, 2, 3,$$

where $Q = (q_1, q_2, q_3) := (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}) = \nabla p$. We note that the equation (4.9) can be thought of as a nonhomogeneous heat equation if ∇p is given, more precisely, the boundary data H. Once we have H in terms of initial and boundary data of the solution u in the Stokes problem, u is just the solution to the heat equation with forcing ∇p. In the next section we will show that H is determined by the initial and boundary data of the solution to the Stokes problem. The following is the probabilistic representation of the solution u to the heat equation on a domain D with smooth boundary ∂D when ∇p is considered to be known.

Proposition 4.2. *Assume that u and p are the solution to the Stokes problem (4.9), (4.10), and (4.11). Let $H = \nabla p|_{\partial D}$. Then*

$$u(x,t) = \mathbb{E}_x[u_0(B_t)\mathbf{1}_{[t<\tau]}] - \mathbb{E}_x[\int_0^{t\wedge\tau} \mathbb{E}_{B_s}[H(B_\tau, t-s)]\, ds] + \mathbb{E}_x[a(B_\tau, t-\tau)\mathbf{1}_{[\tau<t]}] \tag{4.12}$$

where τ is the first hitting time of B_t on the boundary ∂D and B_t is absorbed Brownian motion starting at x.

Proof. First of all, we solve the Dirichlet problem: given the boundary data of H of ∇p

$$\triangle q_i = 0, \quad q_i|_{\partial D} = h_i, \quad i = 1, 2, 3.$$

By the Itô formula for the Brownian motion started at x we have

$$q_i(x) = \mathbb{E}_x[h_i(B_\tau)]$$

if $\tau < \infty$ a.s. Hence we have the probabilistic representation of the gradient of pressure

$$\nabla p(x, t) = \mathbb{E}_x[H(B_\tau, t)].$$

We fix $t_0 > 0$ and let $v(x, t) := u^{(j)}(x, t_0 - t), 0 \leq t \leq t_0$, where $u = (u^{(1)}, u^{(2)}, u^{(3)})$. Then

$$\frac{\partial v}{\partial t} + \frac{1}{2}\triangle v = -\frac{\partial u^{(j)}}{\partial t}(x, t_0 - t) + \frac{1}{2}\triangle u = \frac{\partial p}{\partial x_j}(x, t_0 - t)$$

By Itô formula and optional stopping time theorem, for all $t < t_0$

$$\begin{aligned} v(B_{t\wedge\tau}, t \wedge \tau) &= v(B_0, 0) + \int_0^{t\wedge\tau} \frac{\partial v}{\partial t}(B_s, s)\, ds + \int_0^{t\wedge\tau} \nabla v\, dB_s + \frac{1}{2}\int_0^{t\wedge\tau} \triangle v(B_s, s)\, ds \\ &= v(B_0, 0) + \int_0^{t\wedge\tau} \nabla v\, dB_s + \int_0^{t\wedge\tau} \frac{\partial p}{\partial x_j}(B_s, t_0 - s)\, ds. \end{aligned}$$

Taking the expected value gives us the following

$$\mathbb{E}_x[v(B_{t\wedge\tau}, t \wedge \tau)] = \mathbb{E}_x[v(B_0, 0)] + \mathbb{E}_x[\int_0^{t\wedge\tau} \frac{\partial p}{\partial x_j}(B_s, t_0 - s)\, ds]$$

Since

$$\mathbb{E}_x[v(B_{t\wedge\tau, t\wedge\tau})] = \mathbb{E}_x[v(B_{t_0}, t_0)\mathbf{1}_{[t_0<\tau]} + v(B_\tau, \mathbf{1}_{[\tau<t_0]})]$$

we have

$$\begin{aligned} u^{(j)}(x, t_0) &= \mathbb{E}_x[v(B_{t_0}, t_0)\mathbf{1}_{[t_0<\tau]}] + \mathbb{E}_x[v(B_\tau, \tau)\mathbf{1}_{[\tau<t_0]}] - \mathbb{E}_x[\int_0^{t_0\wedge\tau} \frac{\partial p}{\partial x_j}(B_s, t_0 - s)\, ds] \\ &= \mathbb{E}_x[u(B_{t_0}, 0)\mathbf{1}_{[t_0<\tau]}] + \mathbb{E}_x[u(B_\tau, t_0 - \tau)\mathbf{1}_{[\tau<t_0]}] - \mathbb{E}_x[\int_0^{t_0\wedge\tau} \frac{\partial p}{\partial x_j}(B_s, t_0 - s)\, ds]. \end{aligned}$$

Substitution of initial data u_0 and boundary data a completes the proof. $\square$

Remark: For a general domain D in $\mathbb{R}^3$ such as an exterior domain with smooth boundary we define the gradient of a differentiable function f on ∂D by a differentiable map $\widetilde{\nabla} f$: $\partial D \longrightarrow \mathbb{R}^3$ which assigns to each point $x \in \partial D$ a vector $\widetilde{\nabla} f(x) \in T_x(\partial D) \subset \mathbb{R}^3$ such that $< \widetilde{\nabla} f(x), v >_x = df_x(v)$ for all $v \in T_x(\partial D)$, where $T_x(\partial D)$ is the tangent plane at x, $<,>_x$ is the inner product in $T_x(\partial D)$, and $df_p(v)$ is the differential of f at x. See [17]. Let H be a vector field on the boundary ∂D. Then $H = \Pi_{T_x(\partial D)}(H) + (H \cdot \eta)\eta$ where $\Pi_{T_x(\partial D)}$ is the projection onto the tangent plane $T_x(\partial D)$. If we have $\triangle p = 0$ on D and $\frac{\partial p}{\partial \eta} = H \cdot \eta = h$, then $H = \Pi_{T_x(\partial D)}(H) + (H \cdot \eta)\eta = \Pi_{T_x(\partial D)}(H) + h\eta$. Thus we have

$$\widetilde{\nabla} p|_{\partial D} = \Pi_{T_x(\partial D)}(H). \tag{4.13}$$

4.2 A Posterior

In this section we show that H can be obtained in terms of the initial and boundary data so that u given by Proposition 4.2 satisfies the incompressibility condition $\nabla \cdot u = 0$.

We notice that $\nabla \cdot Q = 0$ and $\nabla \times Q = 0$ on D, where $Q = \nabla p$ since p is harmonic. In particular, on the half space $D = \mathbb{R}^3_+$, the conditions $\nabla \cdot Q = 0$ and $\nabla \times Q = 0$ are the generalized Cauchy-Riemann equations. We recall the equivalent relation of the generalized Cauchy-Riemann equations and Riesz transforms; let u be the harmonic extension to $\mathbb{R}^{n+1}_+$ of a function f defined on $\mathbb{R}^n$ and suppose that $u_1, \cdots, u_n, u$ are functions in $\mathbb{R}^{n+1}_+$ which satisfy the generalized Cauchy-Riemann equations,

$$\sum_{j=1}^{n+1} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad for \quad j \neq k, \quad j, k = 1, \cdots, n+1$$

where $u = u_{n+1}$, and are such that $u_1(0) = \cdots = u_n(0) = 0$. Then $u_j|_{\mathbb{R}^n} = R_j(f), j = 1, \cdots, n$. Hence $h_1 = -R_1 h_3$ and $h_2 = -R_2 h_3$, where $R_i f$ is the Riesz transform of f, $i = 1, 2$, because $\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial x_3}|_{x_3=0} = -h_3$ where $\frac{\partial}{\partial \eta}$ is the outward normal derivative to the boundary $\mathbb{R}^2$. For a general domain D in $\mathbb{R}^3$ such as an exterior domain with smooth boundary, we recall the identity (4.13) in the previous section, i.e.,

$$\widetilde{\nabla} p|_{\partial D} = \Pi_{T_x(\partial D)}(H).$$

If we also recall the definition of Riesz transforms through the Neumann problem in Chapter 3, i.e., the Riesz transform of a function f is the tangential derivative of restriction to the boundary of the solution to the Neumann problem with boundary data f, then the tangential part of H is the Riesz transforms of h. Thus three unknowns of H go down to only one unknown. We will try to find the one unknown h of the boundary data H so that the solution to the Stokes problem in Proposition 4.2 satisfies the incompressibility condition. Therefore we have an integral equation involving h from $\nabla \cdot u = 0$. The existence of h in this integral equation allows us to solve the Stokes problem.

Remark: Let us look at the integral equation of the solution containing h on a domain D with smooth boundary ∂D. We recall the equations in Proposition 4.2.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u - \nabla p,$$

$$u(x,0) = u_0(x), \qquad u|_{\partial D} = a(x,t)$$

and

$$\triangle p = 0, \quad \nabla p|_{\partial D} = H$$

where H is the boundary data of the gradient of pressure, and D is a bounded domain or an exterior domain with smooth boundary ∂D. Then we have

$$\begin{aligned} u(x,t) = &-\int_0^t \int_D K_D(x-y,s)\nabla p(y,t-s)\,dyds + \int_D K_D(x-y,t)u_0(y)\,dy \\ &+ \int_0^t \int_{\partial D} \frac{\partial}{\partial \eta}|_{\partial D}[K_D(x'-y',s)]a(y',t-s)\,dy'ds, \end{aligned}$$

where K_D is the Green function of the heat equation on a domain D. Solving the Dirichlet problem

$$\triangle q_j = 0 \quad q_j|_{\partial D} = h_j,$$

where $q_j = \frac{\partial p}{\partial x_j}$, $j = 1, 2, 3$, we have

$$q_j(y,t) = -\int_{\partial D} \frac{\partial}{\partial \eta(z)} E_D(y,z) h_j(z,t)\, dz.$$

where E_D is the Green function of the Laplace equation on a domain D. Since $E_D(x,z) = \int_0^\infty K_D(x,z,s)\, ds$, the semi-group property of heat kernels gives us the following.

$$\begin{aligned}\int_D K_D(x,y,s) \frac{\partial}{\partial \eta(z)} E_D(y,z)\, dy &= \frac{\partial}{\partial \eta(z)} \int_0^\infty \int_D K_D(x,y,t) K_D(y,z,t')\, dy dt' \\ &= \frac{\partial}{\partial \eta(z)} \int_0^\infty K_D(x,z,s+t')\, dt'\end{aligned}$$

Therefore we have the solution u involving H as follow.

$$\begin{aligned}u(x,t) = &\int_0^t \int_{\partial D} [\frac{\partial}{\partial \eta(z)} \int_0^\infty K_D(x,z,t'+s)\, dt'] H(z,t-s)\, dz ds + \int_D K_D(x-y,t) u_0(y)\, dy \\ &+ \int_0^t \int_{\partial D} \frac{\partial}{\partial \eta}|_{\partial D} [K_D(x'-y',s)] a(y',t-s)\, dy' ds. \qquad (4.14)\end{aligned}$$

If we apply the incompressibility condition $\nabla \cdot u = 0$, then we have the integral equation for H;

$$\begin{aligned}\nabla \cdot \{ &\int_0^t \int_{\partial D} [\frac{\partial}{\partial \eta(z)} \int_0^\infty K_D(x,z,t'+s)\, dt'] H(z,t-s)\, dz ds + \int_D K_D(x-y,t) u_0(y)\, dy \\ &+ \int_0^t \int_{\partial D} \frac{\partial}{\partial \eta}|_{\partial D} [K_D(x'-y',s)] a(y',t-s)\, dy' ds \} = 0 \qquad (4.15)\end{aligned}$$

In the case of the half space $D = \mathbb{R}^3_+$, h_3 can be explicitly determined in terms of the initial and boundary data of u. We will use h of H rather than h_3.

Before proceeding, we introduce some well-known notations that will be used in the following results. First we define the Laplace transform of a function $f(t)$ with $t > 0$ to be

$$\mathcal{L}(f)(p) = \int_0^\infty e^{-pt} f(t)\, dt.$$

Also, in this section we denote by $\mathcal{F}(f)(\xi)$ the Fourier transform of a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{-i(x' \cdot \xi)} f(x')\, dx'.$$

In addition, we define two functions for the following results. One is the n-dimensional heat kernel

$$k^{(n)}(x,t) = \frac{1}{(2\pi t)^{n/2}} \exp(-\frac{-|x|^2}{2t}),$$

where $n = 1, 2, 3$, and the other is

$$\widetilde{K}(x', x_3, t) = k^{(2)}(x', t)\frac{1}{\sqrt{2\pi t}}(2)(\frac{x_3}{t})\exp(-\frac{x_3}{2t}).$$

We note that $\widetilde{K}$ is inward normal derivative of the Green function K for the heat equation on the half space $\mathbb{R}^3_+$. In this section we always denote the normal derivative by the outward normal derivative. Let us start with a lemma concerning the Laplace and Fourier transform of $\widetilde{K}$.

Lemma 4.1. *With notations we already introduced*

$$\mathcal{LF}(\widetilde{K})(\xi, x_3, p) = \exp(-\sqrt{2p + |\xi|^2}x_3) \tag{4.16}$$

Proof. Consider the heat equation on the half space with zero initial data and boundary data a.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u, \tag{4.17}$$

$$u(x,0) = 0, \quad u|_{x_3=0}(x,t) = a(x',t), \quad x' \in \mathbb{R}^2.$$

Then

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^2} \frac{\partial}{\partial y_3}|_{y_3=0}[K(x,y,s)]a(y', t-s)\, dy'ds \\ &= \int_0^t \int_{\mathbb{R}^2} \widetilde{K}(x,y,s)a(y',t-s)\, dy'ds. \end{aligned}$$

Taking Laplace and Fourier transforms on (4.17) we have

$$2p\mathcal{LF}(u)(\xi, x_3, p) + |\xi|^2\mathcal{LF}(u)(\xi, x_3, p) = \frac{\partial^2}{\partial x_3^2}\mathcal{LF}(u)(\xi, x_3, p) \tag{4.18}$$

Solving the ordinary differential equation (4.18) we have

$$\mathcal{LF}(u) = \mathcal{LF}(\widetilde{K})\mathcal{LF}(a) = \exp(-\sqrt{2p + |\xi|^2}x_3)\mathcal{LF}(a).$$

The above identity completes the proof.

□

With this lemma we will obtain h in terms of the initial and boundary data of the solution to the Stokes problem. For simplicity, the following result is the case of non-zero initial data and zero boundary data.

Proposition 4.3. *Let u be the solution to the Stokes problem on $\mathbb{R}^3_+$:*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u - \nabla p, \qquad \nabla \cdot u = 0$$

$$u(x,0) = u_0(x), \quad u|_{x_3=0} = 0$$

Assume that $u_0(x) \in \mathcal{C}^1(\mathbb{R}^3_+)$ satisfies the compatibility conditions $\nabla \cdot u_0 = 0$ and $u_0|_{x_3=0} = 0$. Then $h = -\frac{\partial p}{\partial x_3}|_{x_3=0}$ is given by

$$h(x_1,x_2,t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x'-y',y_3,t)[\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)](y',y_3)\, dy' dy_3 \quad (4.19)$$

where $u_0 = (u_0^1, u_0^2, u_0^3)$, $\nabla' = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})$, and $R_i f$ is the Riesz transform of f, $i = 1,2$.

Proof. Recall the solution u invoving H on a general domain D from (4.14) when $a = 0$;

$$u(x,t) = \int_0^t \int_{\partial D} [\frac{\partial}{\partial \eta(z)} \int_0^\infty K_D(x,z,t'+s)\, dt']H(z,t-s)\, dz ds + \int_D K_D(x-y,t)u_0(y)\, dy$$

Since we consider $D = \mathbb{R}^3_+$ we have

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^2} [\frac{\partial}{\partial z_3}|_{z_3=0} \int_0^\infty K(x,z,t'+s)\, dt']H(z,t-s)\, dz ds + \int_{\mathbb{R}^3_+} K(x-y,t)u_0(y)\, dy,$$

where K is the Green function of the heat equation on the half space $\mathbb{R}^3_+$. Since $\frac{\partial}{\partial z_3}|_{z_3=0}[K(x,z,t)] = 2(\frac{x_3}{t})k^{(1)}(x_3,t)k^{(2)}(x',z',t)(= \widetilde{K}(x,z',t))$, we have

$$u(x,t) = \int_0^t \int_{\mathbb{R}^2} \int_0^\infty 2(\frac{x_3}{s+t'})k^{(1)}(x_3,s+t')k^{(2)}(x',z',s+t')\, dt']H(z,t-s)\, dz ds \quad (4.20)$$

$$+ \int_{\mathbb{R}^3_+} K(x,y,t)u_0(y)\, dy. \quad (4.21)$$

Noting that $H = (-R_1 h, -R_2 h, h)$ and using $\mathcal{F}(k^{(2)}(x', t)) = \exp(-\frac{|\xi|^2}{2} t)$ and $\mathcal{F}(R_j f) = \frac{i\xi_j}{|\xi|}\mathcal{F}f$ we have the Fourier transform of (4.20) given by

$$\mathcal{F}(4.20) = \int_0^t \int_0^\infty \exp(-\frac{|\xi|^2}{2}(t'+s)) \frac{2}{\sqrt{2\pi}} \frac{x_3}{(t'+s)^{3/2}} \exp(-\frac{x_3^2}{2(t'+s)})\, dt'$$
$$\times(-\frac{i\xi_1}{|\xi|}, -\frac{i\xi_2}{|\xi|}, 1)\mathcal{F}(h)(\xi, t-s)\, ds$$

or, by change of variable of $t' + s$,

$$\mathcal{F}(4.20) = \int_0^t \int_s^\infty \exp(-\frac{|\xi|^2}{2} t') \frac{2}{\sqrt{2\pi}} \frac{x_3}{t'^{3/2}} \exp(-\frac{x_3^2}{2t'})\, dt' (-\frac{i\xi_1}{|\xi|}, -\frac{i\xi_2}{|\xi|}, 1)\mathcal{F}(h)(\xi, t-s)\, ds \tag{4.22}$$

We are going to take Laplace transform of (4.22) with respect to s. First of all, we compute the following.

$$\mathcal{L}[\int_s^\infty \exp(-\frac{1}{2}(|\xi|^2 t' + \frac{x_3^2}{t'})) \frac{1}{t'^{3/2}}\, dt'](p)$$

Using the fact from the definition of Laplace transform that

$$\mathcal{L}[\int_t^\infty f(w)\, dw](p) = \frac{1}{p}(\int_0^\infty f(w)\, dw - \mathcal{L}[f](p))$$

we have

$$\mathcal{L}[\int_s^\infty \exp(-\frac{1}{2}(|\xi|^2 t' + \frac{x_3^2}{t'})) \frac{1}{t'^{3/2}}\, dt'](p)$$
$$= \frac{1}{p}\int_0^\infty \exp(-\frac{1}{2}(|\xi|^2 t' + \frac{x_3^2}{t'})) \frac{1}{t'^{3/2}}\, dt' - \frac{1}{p}\mathcal{L}[\exp(-\frac{1}{2}(|\xi|^2 t' + \frac{x_3^2}{t'})) \frac{1}{t'^{3/2}}](p). \tag{4.23}$$

Recall that the modified Bessel function of order ν is given by

$$K_\nu(z) = \frac{1}{2}\int_0^\infty t^{-(1+\nu)} \exp -(\frac{z}{2}(\frac{1}{t} + t))\, dt$$

and that in particular

$$K_{1/2}(z) = (\frac{\pi}{2z})^{1/2} e^{-z}.$$

See [32] for basic properties of these functions. Let us look at the first term of (4.23). By the change of variables, $t = \frac{|\xi| t'}{|x_3|}$, and the modified Bessel functions we have

$$\frac{1}{p}\int_0^\infty \exp(-\frac{1}{2}(|\xi|^2 t' + \frac{x_3^2}{t'})) \frac{1}{t'^{3/2}}\, dt'$$

$$= \frac{1}{p}\int_0^\infty \exp\,(-\frac{|\xi||x_3|}{2}(\frac{|\xi|t'}{|x_3|}+\frac{|x_3|}{|\xi|t'})\frac{1}{t'^{3/2}})\,dt'$$

$$= \frac{1}{p}\frac{|\xi|^{1/2}}{|x_3|^{1/2}}\int_0^\infty \exp-(\frac{|\xi||x_3|}{2}(t+\frac{1}{t}))t^{-3/2}\,dt$$

$$= \frac{1}{p}\frac{|\xi|^{1/2}}{|x_3|^{1/2}}K_{\frac{1}{2}}(|\xi||x_3|) = \frac{1}{p}(\frac{\sqrt{\pi}}{\sqrt{2}x_3})\exp\,(-|\xi||x_3|).$$

Similarly, with change of variables and the modified Bessel function in (4.23), we have the Laplace and Fourier transform of $\int_s^\infty \exp(-\frac{|\xi|^2}{2}t')\frac{2}{\sqrt{2\pi}}\frac{x_3}{t'^{3/2}}\exp(-\frac{x_3^2}{2t'})\,dt'$;

$$\mathcal{LF}(\int_s^\infty \exp(-\frac{|\xi|^2}{2}t')\frac{2}{\sqrt{2\pi}}\frac{x_3}{t'^{3/2}}\exp(-\frac{x_3^2}{2t'})\,dt') = \frac{1}{p}[\exp(-\sqrt{|\xi|^2+2p}x_3)-\exp(-|\xi|x_3)]$$

Now we are going to use the incompressibility condition $\nabla\cdot u=0$ to get h;

$$\nabla\cdot[-\int_0^t\int_{\mathbb{R}^2}[\frac{\partial}{\partial z_3}|_{z_3=0}\int_0^\infty K(x,z,t'+s)\,dt']H(z,t-s)\,dzds] \tag{4.24}$$

$$= -\nabla\cdot[\int_{\mathbb{R}^3_+}K(x-y,t)u_0(y)\,dy]. \tag{4.25}$$

Since $(i\xi_j)\mathcal{LF}[R_jh] = (i\xi_j)(i\frac{\xi_j}{|\xi|})\mathcal{LF}[h] = -\frac{\xi_j^2}{|\xi|}\mathcal{LF}[h]$, we finally have the Laplace and Fourier transform of (4.24);

$$(i\xi_1,i\xi_2,\frac{\partial}{\partial x_3})\cdot(\frac{1}{p}[\exp(-\sqrt{|\xi|^2+2p}x_3)-\exp(-|\xi|x_3)])(\mathcal{LF}(-R_1h),\mathcal{LF}(-R_2h),\mathcal{LF}(h))$$

$$= |\xi|\frac{1}{p}[\exp(-\sqrt{|\xi|^2+2p}x_3)-\exp(-|\xi|x_3)]\mathcal{LF}(h)$$

$$+\frac{1}{p}[-\sqrt{|\xi|^2+2p}\exp(-\sqrt{|\xi|^2+2p}x_3)+|\xi|\exp(-|\xi|x_3)]\mathcal{LF}(h)$$

$$= \frac{|\xi|-\sqrt{|\xi|^2+2p}}{p}\exp(-\sqrt{|\xi|^2+2p}x_3)\mathcal{LF}[h]. \tag{4.26}$$

For the computation of the right side (4.25) we have

$$\mathcal{F}(4.25) = -(i\xi_1,i\xi_2,\frac{\partial}{\partial x_3})\cdot\int_0^\infty \exp(-\frac{-|\xi|^2t}{2})(\frac{1}{\sqrt{2\pi t}})[\exp(\frac{-|x_3-y_3|^2}{2t})-\exp(\frac{-|x_3+y_3|^2}{2t})]$$

$$\times \mathcal{F}(u_0)(\xi, y_3)\, dy_3 \tag{4.27}$$

By the compatibility condition $\nabla \cdot u_0 = 0$, one has

$$-i\xi_1 \mathcal{F}(u_0^1) - i\xi_2 \mathcal{F}(u_0^2) = \frac{\partial \mathcal{F}(u_0^3)}{\partial y_3}.$$

Using this in (4.27) and integration by parts we have

$$\begin{aligned}
\mathcal{F}(4.25) = &\int_0^\infty \exp(-\frac{-|\xi|^2 t}{2})(\frac{1}{\sqrt{2\pi t}})[\exp(\frac{-|x_3 - y_3|^2}{2t}) - \exp(\frac{-|x_3 + y_3|^2}{2t})] \\
&\times \frac{\partial}{\partial y_3}\mathcal{F}(u_0^3)(\xi, y_3)\, dy_3 \\
&- \int_0^\infty \exp(-\frac{-|\xi|^2 t}{2})(\frac{1}{\sqrt{2\pi t}})\frac{\partial}{\partial x_3}[\exp(\frac{-|x_3 - y_3|^2}{2t}) - \exp(\frac{-|x_3 + y_3|^2}{2t})] \\
&\times \mathcal{F}(u_0^3)(\xi, y_3)\, dy_3 \\
= &-\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty (x_3 + y_3) \exp(-\frac{1}{2}(|\xi|^2 t + \frac{|x_3 + y_3|^2}{t}))\frac{1}{t^{3/2}} \mathcal{F}(u_0^3)(\xi, y_3)\, dy_3
\end{aligned}$$

Using the change of variables and the modified Bessel functions again for Laplace transform we have

$$\mathcal{LF}(4.25) = -\int_0^\infty \exp(-|x_3 + y_3|\sqrt{|\xi|^2 + 2p})\mathcal{F}(u_0^3)(\xi, y_3)\, dy_3. \tag{4.28}$$

From (4.26) and (4.28) we, therefore, have the Laplace and Fourier transform of h

$$\begin{aligned}
\mathcal{LF}(h)(\xi, p) &= \frac{p}{\sqrt{|\xi|^2 + 2p} - |\xi|} \int_0^\infty \exp(-y_3\sqrt{|\xi|^2 + 2p})\mathcal{F}[u_0^3](\xi, y_3)\, dy_3 \\
&= \frac{1}{2}\int_0^\infty (\sqrt{|\xi|^2 + 2p} + |\xi|) \exp(-y_3\sqrt{|\xi|^2 + 2p})\mathcal{F}[u_0^3](\xi, y_3)\, dy_3 \\
&= \frac{1}{2}\int_0^\infty (\sqrt{|\xi|^2 + 2p}) \exp(-y_3\sqrt{|\xi|^2 + 2p})\mathcal{F}[u_0^3](\xi, y_3)\, dy_3 \\
&\quad + \frac{1}{2}\int_0^\infty |\xi| \exp(-y_3\sqrt{|\xi|^2 + 2p})\mathcal{F}[u_0^3](\xi, y_3)\, dy_3
\end{aligned}$$

The compatibility condition $u_0^3|_{x_3=0} = 0$ and integration by parts gives

$$\mathcal{LF}(h)(\xi, p) = \frac{1}{2}\int_0^\infty \exp(-y_3\sqrt{|\xi|^2 + 2p})\frac{\partial}{\partial y_3}\mathcal{F}[u_0^3](\xi, y_3)\, dy_3$$

$$+\frac{1}{2}\int_0^\infty \exp(-y_3\sqrt{|\xi|^2+2p})\frac{|\xi|^2}{\xi}\mathcal{F}[u_0^3](\xi,y_3)\,dy_3$$

$$=\frac{1}{2}\int_0^\infty \exp(-y_3\sqrt{|\xi|^2+2p})(\mathcal{F}[\frac{\partial}{\partial y_3}u_0^3](\xi,y_3))+\mathcal{LF}(\nabla'\cdot R'(u_0^3))\,dy_3$$

Noting that $\mathcal{LF}(\widetilde{K})=\exp-(\sqrt{|\xi|^2+2p}y_3)$ completes the proof. □

Similarly, we have the explicit formula for h in terms of initial and boundary data of u as follows.

Corollary 4.1. *Let u be the solution to the Stokes problem on $\mathbb{R}^3_+$:*

$$\frac{\partial u}{\partial t}=\frac{1}{2}\triangle u-\nabla p,\qquad \nabla\cdot u=0$$

$$u(x,0)=u_0(x),\quad u|_{x_3=0}=a(x',t),\quad x'\in\mathbb{R}^2.$$

Assume that $u_0(x)\in\mathcal{C}^1(\mathbb{R}^3_+)$ and $a\in\mathcal{C}^{1,1}(\mathbb{R}^2\times(0,\infty))$ satisfy the compatibility conditions $\nabla\cdot u_0=0$, $u_0|_{x_3=0}=a|_{t=0}$, and $a\cdot\eta=0$. Then $h=-\frac{\partial p}{\partial x_3}|_{x_3=0}$ is given by

$$h(x_1,x_2,t)=\int_0^\infty\int_{\mathbb{R}^2}\widetilde{K}(x'-y',y_3,t)[\nabla'\cdot(R_1u_0^3-u_0^1,R_2u_0^3-u_0^2)]\,dy'dy_3$$

$$+\triangle_{x'}(R_1a_1+R_2a_2)+4\int_0^t\int_{\mathbb{R}^2}(\frac{\partial}{\partial s}-\frac{1}{2}\triangle')\frac{\exp(-\frac{|x'-y'|^2}{2s})}{s^{3/2}}(\nabla'\cdot a')(y',t-s)\,dy'ds$$

where $u_0=(u_0^1,u_0^2,u_0^3)$, $\nabla'=(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2})$, and R_if is the Riesz transform of f, $i=1,2$.

Proof. From Lemma 4.1, Equation (4.26), and Equation (4.28) in the proof of Proposition 4.3 we have, taking Laplace and Fourier transforms of (4.15),

$$\frac{|\xi|-\sqrt{|\xi|^2+2p}}{p}\exp(-\sqrt{|\xi|^2+2p}x_3)\mathcal{LF}[h]$$

$$=-\exp(-\sqrt{|\xi|^2+2p}x_3)\int_0^\infty\exp(-y_3\sqrt{|\xi|^2+2p})\mathcal{F}[u_0^3](\xi,y_3)\,dy_3$$

$$-\exp(-\sqrt{2p+|\xi|^2}x_3)(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2)).$$

and so the Laplace and Fourier transform of h is

$$\mathcal{LF}[h]=\frac{1}{2}(\sqrt{|\xi|^2+2p}+|\xi|)\int_0^\infty\exp(-y_3\sqrt{|\xi|^2+2p})\mathcal{LF}[u_0^3](\xi,y_3)\,dy_3$$

$$+(\sqrt{|\xi|^2+2p}+|\xi|)(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2)). \tag{4.29}$$

We will compute the terms in (4.29) involving boundary data a, $\sqrt{|\xi|^2+2p}(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2))$ and $|\xi|(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2))$. First, we notice that

$$|\xi|(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2))=|\xi|^2(\frac{i\xi_1}{|\xi|}\mathcal{LF}(a_1)+\frac{i\xi_2}{|\xi|}\mathcal{LF}(a_1))=\mathcal{LF}[\triangle_{x'}(R_1a_1+R_2a_2)].$$

For the other one we see that

$$\sqrt{|\xi|^2+2p}(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2))=[\int_0^\infty(|\xi|^2+2p)\exp(-\sqrt{|\xi|^2+2p}y_3)\,dy_3]\mathcal{LF}[\nabla'\cdot a']$$

$$=\int_0^\infty \mathcal{LF}[(-\triangle+2\frac{\partial}{\partial t})\widetilde{K}(y,t)]\,dy_3\mathcal{LF}[\nabla'\cdot a'].$$

Since $\int_0^\infty \widetilde{K}\,dy_3=\frac{2}{t^{3/2}}\exp(-\frac{|y'|^2}{2t})$, we have

$$\sqrt{|\xi|^2+2p}(i\xi_1\mathcal{LF}(a_1)+i\xi_2\mathcal{LF}(a_2))$$

$$=\mathcal{LF}[4\int_0^t\int_{\mathbb{R}^2}(-\frac{1}{2}\triangle+\frac{\partial}{\partial s})]\frac{\exp(-\frac{|x'-y'|^2}{2s})}{s^{3/2}}(\nabla'\cdot a')(y',t-s)\,dy'ds.$$

This computation completes the proof. $\square$

Finally the above a priori and a posteriori argument allows us to solve the Stokes problem as follows.

Theorem 4.1. *Assume that $u_0(x)\in\mathcal{C}^1(\mathbb{R}^3_+)$ and $a\in\mathcal{C}^{1,1}(\mathbb{R}^2\times(0,\infty))$ satisfy the compatibility conditions $\nabla\cdot u_0=0$, $u_0|_{x_3=0}=a|_{t=0}$, and $a\cdot\eta=0$. Let*

$$h(x_1,x_2,t):=\int_0^\infty\int_{\mathbb{R}^2}\widetilde{K}(x'-y',y_3,t)[\nabla'\cdot(R_1u_0^3-u_0^1,R_2u_0^3-u_0^1)]\,dy'dy_3$$

$$+\triangle_{x'}(R_1a_1+R_2a_2)+4\int_0^t\int_{\mathbb{R}^2}(\frac{\partial}{\partial s}-\frac{1}{2}\triangle')\frac{\exp(-\frac{|x'-y'|^2}{2s})}{s^{3/2}}(\nabla'\cdot a')(y',t-s)\,dy'ds,$$

$$u(x,t):=\{-\int_0^t\int_{\mathbb{R}^2}[\frac{\partial}{\partial z_3}|_{z_3=0}\int_0^\infty E(x,z,t'+s)\,dt']H(z,t-s)\,dzds+\int_{\mathbb{R}^3_+}E(x-y,t)u_0(y)\,dy$$

$$+\int_0^t\int_{\mathbb{R}^2}\frac{\partial}{\partial y_3}|_{y_3=0}E(x,y,t)a(y',t-s)\,dy'ds$$

and let p be the solution to the Neumann problem with boundary data h on the half space $\mathbb{R}^3_+$. Then u and p satisfy the Stokes problem with the initial data u_0 and the boundary data a on the half space $\mathbb{R}^3_+$.

4.3 Pressure on the Boundary

In this section we will represent the pressure on boundary in the Stokes problem. In 2010 Bikri, Guenther, and Thomann in [5] utilized repeatedly the Fourier transforms in $\mathbb{R}^2$ together with the Laplace transforms with respect to x_3 of a function defined in $\mathbb{R}^3_+$ to obtain some results on the DtN map for Laplace and heat operators in $\mathbb{R}^3_+$. The DtN map is a common tool in the analysis of inverse problems in electrical exploration and impedance tomography, see [25] for a modern treatment. Consider the Laplace operator. The following is one of results in [5].

Proposition 4.4. *Assume that $\phi(x) \in C^2(\mathbb{R}^3_+)$ and it satisfies $\triangle\phi = g(x)$. Then the Laplace-Fourier transform of ϕ satisfies*

$$\mathcal{L}_{x_3}(\mathcal{F}g(\xi, x_3))|_{|\xi|} = -|\xi|\mathcal{F}(\phi)(\xi, x_3)|_{x_3=0} - \frac{\partial \mathcal{F}(\phi)(\xi, x_3)}{\partial x_3}|_{x_3=0}.$$

Using the Proposition 4.4 we have the pressure on the boundary as follows.

Corollary 4.2. *Under the same assumption in the Proposition (4.3) the boundary of p is given by*

$$p(x_1, x_2, t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t)(u_0^3 - R_1 u_0^1 - R_2 u_0^2)\, dy' dy_3 \tag{4.30}$$
$$+\nabla' \cdot a' + 4 \int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial s} - \frac{1}{2}\triangle')\frac{\exp -(\frac{|x'-y'|^2}{2s})}{s^{3/2}}(R_1 a_1 + R_2 a_2)$$

where $a' = (a_1, a_2)$.

Proof. We note that $\triangle p = 0, \quad -\frac{\partial p}{\partial x_3}|_{x_3=0} = h$. From Proposition 4.4 with $g = 0$, $\phi = p$, we have

$$\mathcal{F}(p)(\xi, x_3, t)|_{x_3=0} = \frac{1}{|\xi|}\mathcal{F}(h)(\xi, t).$$

Recall h from Corollary 4.1;

$$h(x_1, x_2, t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t)[\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^2)]\, dy' dy_3$$

$$+\triangle_{x'}(R_1 a_1 + R_2 a_2) + 4\int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial s} - \frac{1}{2}\triangle')\frac{\exp(-\frac{|x'-y'|^2}{2s})}{s^{3/2}}(\nabla' \cdot a')(y', t-s)\, dy' ds.$$

Thus p on the boundary $\mathbb{R}^2$ in the Fourier side is given by

$$\frac{1}{|\xi|}\mathcal{F}(h) = \int_0^\infty \mathcal{F}(\widetilde{K})(\xi, y_3, t)\frac{i\xi}{|\xi|} \cdot (\frac{i\xi_1}{|\xi|}\mathcal{F}(u_0^3) - \mathcal{F}(u_0^1), \frac{i\xi_2}{|\xi|}\mathcal{F}(u_0^3) - \mathcal{F}(u_2))\, dy_3$$

$$+\frac{|\xi|^2}{|\xi|}(\frac{i\xi_1}{|\xi|}, \frac{i\xi_2}{|\xi|}) \cdot (\widehat{a_1}, \widehat{a_2}) + \frac{1}{|\xi|}4\int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial s} - \frac{1}{2}\triangle')\frac{\exp-(\frac{|x'-y'|^2}{2s})}{s^{3/2}}((i\xi_1, i\xi_2) \cdot (\widehat{a_1}, \widehat{a_2})).$$

This completes the proof. $\square$

In 1987, Ukai in [39] gave the explicit solution to the Stokes problem in terms of Riesz operators, the heat operator, and the Laplace operator in the half space $\mathbb{R}^n_+$. To understand his formula let us introduce several well-known operators. First, we define two kinds of Riesz operators, $R_j, j = 1, \cdots, n$, and $S_j, j = 1, \cdots, n-1$, which are the singular integral operators with the symbols

$$\sigma(R_j) = i\frac{\xi_j}{|xi|}, \quad j = 1, \cdots, n,$$

$$\sigma(S_j) = i\frac{\xi_j}{|\xi'|}, \quad j = 1, \cdots, n-1,$$

where $\xi = (\xi', \xi_n)$ is the dual variable to $x \in \mathbb{R}^n$. Thus R_j is the Riesz transform with respect to $x \in \mathbb{R}^n$ and S_j is the Riesz transform with respect to $x' \in \mathbb{R}^{n-1}$. In our notation, particularly, in the boundary data h of the gradient of pressure ∇p, $S_j = R_j$ because we have Riesz transforms on the boundary in $\mathbb{R}^2$. Set

$$R' = (R_1, R_2, \cdots, R_{n-1}),$$

$$S = (S_1, S_2, \cdots, S_{n-1}),$$

and define the operators V_1 and V_2 by

$$V_1 u_0 = -S \cdot u_0' + u_0^n,$$

$$V_2 u_0 = u_0' + S u_0^n.$$

Further, we define the operator U by

$$Uf = rR' \cdot S(R' \cdot S + R_n)e$$

where r is the restriction operator from $\mathbb{R}^n$ to the half space $\mathbb{R}^n_+$, that is,

$$rf = f|_{\mathbb{R}^n_+}$$

and e is the extension operator from $\mathbb{R}^n_+$ over $\mathbb{R}^n$ with value 0:

$$ef = \begin{cases} f & \text{for } x_n > 0, \\ 0 & \text{for } x_n < 0. \end{cases}$$

Also, we define the operators $E(t)$ and F, respectively, by

$$E(t)f = \int_{\mathbb{R}^n_=} \{k^{(n)}(x-y,t) - k^{(n)}(x'-y', x_n + y_n, t)\} f(y)\, dy,$$

$$Fb = \int_0^t \int_{\mathbb{R}^{n-1}} \partial_n k^{(n)}(x'-y', x_n, t-s) b(s,y')\, ds dy',$$

which are the solution operators to the heat equation in the half space $\mathbb{R}^n$, i.e.,

$$u_t = \triangle u$$

$$u|_{t=0} = u_0, \quad u|_{x_n=0} = b(x', t).$$

Thus $E(t)u_0$ is the solution to the heat equation for the case $b = 0$ while Fb is the solution for $u_0 = 0$. Finally, we define the Poisson operators D and N by

$$Db = \int_{\mathbb{R}^{n-1}} \partial_n G(x'-y', x_n) b(y')\, dy',$$

$$Nb = \int_{\mathbb{R}^{n-1}} G(x'-y', x_n) b(y')\, dy'$$

where

$$G(x) = \begin{cases} c_n |x|^{-(n-2)} & \text{for } n \geq 3, \\ -(2\pi)^{-1} \log |x| & \text{for } n = 2, \end{cases}$$

is the Newton potential with $c_n = 2(n-2)^{-1}\pi^{n/2}\Gamma(\frac{1}{2}n)$. Obviously, D and N are the solution to the Dirichlet and Neumann problem with boundary data b, respectively. With all operators we just defined, let us present his formula as follows.

Theorem 4.2. *Suppose $a = 0$. Then the solution to the Stokes problem can be expressed as*

$$u^n = UE(t)V_1 u_0,$$

$$u' = E(t)V_2 u_0 - SUE(t)V_1 u_0,$$

$$p = -D\gamma\partial_n E(t)V_1 u_0. \tag{4.31}$$

where $\gamma z = z|_{x_3=0}$.

Theorem 4.3. *Suppose $u_0 = 0$. Then the solution to the Stokes problem is*

$$u^n = Da^n + UFV_1 a,$$

$$u' = FV_2 a - S(Da^n + UFV_1 a),$$

$$p = |\nabla'| DV_1 a - D\gamma\partial n F V_1 a - Na^n, \tag{4.32}$$

where $|\nabla'|$ is the pseudo-differential operator having the symbol $|\xi'|$.

Remark: If we just follow the already-defined operators in [39] and compute the pressure, then we can see that the integral representation of the Ukai's formula for the pressure is exactly the same as the formula in corollary 4.2. For simplicity we check the case of $a = 0$

and $n = 3$ in his paper with (4.30) in Corollary 4.2. We use a superscript of U to denote quantities relating to Ukai's formula. For example, R_i^U in Ukak's is different from R_i in this section. Suppose $a = 0$. Then from Ukai's formula

$$p^U = -D^U \gamma \partial_n E^U(t) V_1^U u_0.$$

Noting that

$$V_1^U u_0 = -(S_1^U u_0^1 + S_2^U u_0^2) + u_0^3 = -(R_1 u_0^1 + R_2 u_0^2) + u_0^3,$$

and

$$\gamma \partial_n E^U(t) V_1^U u_0 = \frac{\partial}{\partial x_3} [E^U(t) V_1^U u_0]|_{x_3=0}$$

$$= \int_0^t \int_{\mathbb{R}^2} K^{(2)}(x' - y', t)(2)(\frac{y_3}{t}) K^{(1)}(y_3, t)[-(R_1 u_0^1 + R_0^2) + u_0^3] \, dy' dy_3,$$

we have the pressure p on the boundary in corollary 4.2 is the same as $\gamma \partial_n E(t) V_1 u_0$, i.e.,

$$p = \gamma \partial_n E^U(t) V_1^U u_0.$$

Since D^U is the harmonic extension operator to the half space $\mathbb{R}^3_+$, the pressure p on the boundary in corollary 4.2 is the same as the boundary of p^U. However, Ukai's approach is different. Let us look at his argument briefly as follows. For simplicity, we set $n = 3$. Ukai used the fact that the pressure is harmonic and the Fourier transform with respect to the tangential variable x' to obtain the following ordinary differential equation

$$(\partial n - |\xi'|^2) \mathcal{F}(p) = 0.$$

Assuming that p is bounded, such solution has the form

$$\mathcal{F}(p) = \exp(-|\xi'| x_3) h,$$

where h is the boundary data of the pressure p. Even if h is undetermined, p satisfies

$$(\partial_n + |\xi'|) \mathcal{F}(p) = 0.$$

This is the key in his argument. he used this equation to remove the term involving the pressure in the Stokes equation. And he found the velocity to the Stokes problem in terms of the initial and boundary data and finally obtained the pressure from the velocity. However, we focus on the pressure at the first place. After we have the boundary data h of the gradient of the pressure in terms of the initial and boundary data of the velocity, we find the gradient pressure and solve the nonhomogeneous heat equation to obtain the velocity to the Stokes problem.

More interestingly, pressure on the boundary $\mathbb{R}^2$ can be expressed in terms of special function, Kummer's function, when the pressure p is the solution to the Neumann problem on $\mathbb{R}^3_+$

$$\nabla p = 0, \quad \frac{\partial p}{\partial z_3}|_{z_3=0} = h(z') \tag{4.33}$$

where $h(z') = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(z'-y', y_3, t) g(y', y_3)\, dy' dy_3$ with $g(y', 0) = 0$. Kummer's function ${}_1F_1(a; c; z)$ is defined by the infinite series $\sum \frac{(a)_n}{(c)_n} \frac{z^n}{n!}$, i.e.,

$$\begin{aligned} {}_1F_1(a; c; z) &= \sum_{n=0}^\infty \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, c \neq 0, -1, -2, \cdots \\ &= \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^\infty \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} \end{aligned}$$

where a, c and z are complex numbers. This Kummer's function has the following integral representation, see [32] for detail.

$${}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1}\, dt,$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$. The following is the explicit formula for p on the boundary $\mathbb{R}^2$ in terms of Kummer's function.

Proposition 4.5. *Let p be the solution to the Neumann problem in (4.33). Then*

$$p(z', t) = c \int_0^\infty \int_{\mathbb{R}^2} \frac{\exp(-|y'-z'|^2)}{\sqrt{t}} {}_1F_1\left(\frac{1}{2}, 1, \frac{|y'-z'|^2}{2t}\right) \frac{\exp -\frac{y_3^2}{2t}}{\sqrt{t}} \frac{\partial g}{\partial y_3}|_{(y', y_3)}\, dy' dy_3. \tag{4.34}$$

Proof. First of all, we notice that the solution p to the Neumann problem on $\mathbb{R}^3_+$ is

$$p(z,t) = c\int_{\mathbb{R}^2} \frac{1}{(|x'-z'|^2+z_3^2)^{1/2}} h(x',t)\,dx'.$$

Since

$$\int_0^\infty \frac{\exp -\frac{|x'-z'|^2+z_3^2}{2s}}{(2\pi s)^{3/2}}\,ds = c\frac{1}{(|x'-z'|^2+z_3^2)^{1/2}}$$

and

$$h(x',t) = c\int_0^\infty \int_{\mathbb{R}^2} k^{(2)}(x',y',t)(\frac{y_3}{t})\frac{\exp -\frac{y_3^2}{2t}}{\sqrt{2\pi t}} g(y',y_3)\,dy'dy_3,$$

the semi-group property of probabilistic densities allows us to have

$$p(z',t) = c\int_0^\infty \int_{\mathbb{R}^2} [\int_0^\infty \frac{\exp(-\frac{|y'-z'|^2/2(s+t)}{s+t})}{s+t}(\frac{\exp(-\frac{z_3^2}{2s})}{\sqrt{s}})\,ds](\frac{y_3}{t})\frac{\exp(-\frac{y_3^2}{2t})}{\sqrt{t}} g(y',y_3)\,dy'dy_3.$$

The first change of variables $\lambda = \frac{1}{s+t}$ gives us

$$\int_0^\infty \frac{\exp(-\frac{|y'-z'|^2/2(s+t)}{s+t})}{s+t}(\frac{\exp(-\frac{z_3^2}{2s})}{\sqrt{s}})\,ds = \int_0^{1/t} e^{-\frac{|y'-z'|^2}{2}\lambda}(\frac{e^{-\frac{z_3^2}{2\lambda}(1-\lambda t)}}{\sqrt{1-\lambda t}})\sqrt{t}\frac{d\lambda}{\lambda}.$$

By the second change of variables $\lambda t = u$ we have

$$\int_0^{1/t} e^{-\frac{|y'-z'|^2}{2}\lambda}\frac{e^{-\frac{z_3^2}{2\lambda}(1-\lambda t)}}{\sqrt{1-\lambda t}}\sqrt{t}\frac{d\lambda}{\lambda} = \frac{e^{z_3 t}}{\sqrt{t}}\int_0^1 e^{-\frac{|y'-z'|^2}{2t}u}\frac{e^{-\frac{z_3^2}{u}t}}{\sqrt{1-u}}\frac{du}{\sqrt{u}}.$$

The last change of variables $u = 1-u'$ leads us to have the expression involving Kummer's function as follows.

$$\begin{aligned}\frac{e^{z_3 t}}{\sqrt{t}}\int_0^1 e^{-\frac{|y'-z'|^2}{2t}u}\frac{e^{-\frac{z_3^2}{u}t}}{\sqrt{1-u}}\frac{du}{\sqrt{u}} &= \frac{e^{-\frac{|y'-z'|^2}{2t}}}{\sqrt{t}}\int_0^1 e^{\frac{|y'-z'|^2}{2t}u}u^{-1/2}(1-u)^{-1/2}\,du \\ &= \frac{e^{-\frac{|y'-z'|^2}{2t}}}{\sqrt{t}}\,{}_1F_1(\frac{1}{2};1;\frac{|y'-z'|^2}{2t}).\end{aligned}$$

Letting $z_3 = 0$ we finally have the pressure on the boundary

$$\begin{aligned}p(z',t) &= c\int_0^\infty \int_{\mathbb{R}^2} \frac{e^{-|y'-z'|^2/2t}}{\sqrt{t}}\,{}_1F_1(\frac{1}{2};1;\frac{|y'-z'|^2}{2t})(\frac{y_3}{t})(\frac{e^{-y_3^2/2t}}{\sqrt{t}})g(y',y_3)\,dy'dy_3 \\ &= c\int_{\mathbb{R}^2} \frac{e^{-|y'-z'|^2/2t}}{\sqrt{t}}\,{}_1F_1(\frac{1}{2};1;\frac{|y'-z'|^2}{2t})[\int_0^\infty \frac{\partial}{\partial y_3}(\frac{e^{-y_3^2/2t}}{\sqrt{t}})g(y',y_3)\,dy_3]\,dy'\end{aligned}$$

The integration by parts completes the proof. □

Moreover, in the case of zero boundary condition, $a = 0$, we have a probabilistic representation of h in terms of reflecting Brownian motion as the following proposition shows.

Proposition 4.6. *Let h be given as in Proposition 4.3;*

$$h(x_1, x_2, t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t) g(y', y_3)\, dy' dy_3 \tag{4.35}$$

where $g = [\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)]$ with $g(y', 0) = 0$. Then

$$h(x', t) = \mathbb{E}_{(x',0)}[\frac{\partial g}{\partial y_3}(B_t^r)]$$

where B_t^r is reflecting Brownian motion.

Proof. Consider the heat equation on the half space $\mathbb{R}^3_+$ with initial data g;

$$\frac{\partial u}{\partial t} = \frac{1}{2} \triangle u$$

$$\frac{\partial u}{\partial x_3}|_{x_3=0} = 0, \quad u|_{t=0} = g$$

Then the solution u is given by

$$u(x', x_3, t) = \int_{\mathbb{R}^3_+} K_N(x' - y', x_3 - y_3, t) g(y', y_3)$$

where $K_N(x', y', x_3, y_3, t) = k^{(3)}(x' - y', x_3 - y_3, t) + k^{(3)}(x' - y', x_3 + y_3, t)$. Then we have a probabilistic representation of u in terms of reflecting Brownian motion, see [26], as follows.

$$u(x', x_3, t) = \mathbb{E}_{(x',x_3)}[g(B_t^r)].$$

Reordering integration in (4.35) we have

$$h(x_1, x_2, t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t) g(y', y_3)\, dy' dy_3$$

$$\int_{\mathbb{R}^2} [\int_0^\infty 2 \frac{\partial}{\partial y_3} \frac{\exp -(\frac{y_3^2}{2t})}{\sqrt{2\pi t}} g(y', y_3)\, dy_3] k^{(2)}(x' - y' t). \tag{4.36}$$

Using the condition $g(x',0)=0$ and integration by parts in (4.36) we have

$$\int_{\mathbb{R}^2}[\int_0^\infty 2k^{(2)}(x'-y't)k^{(1)}(y_3,t)\frac{\partial g}{\partial y_3}(y',y_3)\,dy'dy_3$$

Noting that

$$K_N(x',y',x_3,y_3,t)|_{y_3=0}=2k^{(2)}(x'-y',t)k^{(1)}(y_3,t)$$

completes the proof. □

4.4 Remarks

What we have studied for the Stokes problem can be applied to the Oseen problem. Let us consider the Oseen problem on the half space $\mathbb{R}^3_+$.

$$\frac{\partial v}{\partial t}+(U\cdot\nabla)v=\frac{1}{2}\triangle v-\nabla p,\qquad \nabla\cdot v=0$$

$$v(x,0)=v_0(x),\quad v|_{x_3=0}=a(x',t),\quad x'\in\mathbb{R}^2$$

Assuming $U\cdot\eta=0$ with the outward unit normal η and using the change of variables we have the boundary data h^U of ∇p in the Oseen problem in terms of h^0, the boundary of ∇p in the Stokes problem, the Oseen problem with $U=0$.

Proposition 4.7. *Let v be the solution to the Oseen problem with $U=(U_1,U_2,U_3)$:*

$$\frac{\partial v}{\partial t}+(U\cdot\nabla)v=\frac{1}{2}\triangle v-\nabla p,\qquad \nabla\cdot v=0$$

$$v(x,0)=v_0(x),\quad v|_{x_3=0}=a(x',t),\quad x'\in\mathbb{R}^2.$$

Then if $U\cdot\eta=0$, then

$$h^U=h^0(x'-tU,t)$$

where h^0 is the boundary condition of ∇p in the Stokes problem, the Oseen problem with $U=0$.

Proof. Let $w(x,t) := v(x_1 + U_1, x_2 + U_2, x_3, t)$. Then we have the Stokes problem of w

$$\frac{\partial w}{\partial t} = \frac{1}{2}\triangle w - \nabla \widetilde{p}, \qquad \nabla \cdot w = 0$$

$$w(x,0) = v_0(x), \quad w|_{x_3=0} = a(x_1 + U_1, x_2 + U_2, t), \quad x' \in \mathbb{R}^2.$$

where $\widetilde{p}(x,t) = p(x' + tU', x_3, t)$. The change of variables completes the proof as follows.

$$h^0(x,t) = \frac{\partial \widetilde{p}}{\partial x_3}|_{x_3=0}(x,t) = \frac{\partial p^U}{\partial x_3}|_{x_3=0}(x+tU,t) = h^U(x+tU,t).$$

□

To summarize, looking at the probabilistic representation of the solution u to the Stokes/Oseen problem on a bounded or exterior domain, we conclude that the velocity u and the pressure p can be determined by only information of the gradient of pressure on the boundary of domain.

In 2008 Constantin and Iyer in [9] derived a probabilistic representation of the three-dimensional Navier-Stokes equations on the free space $\mathbb{R}^3$ based on stochastic Lagrangian paths. First of all, let us define the particle-trajectory mapping $X_t(\cdot)$ or $X(\cdot,t)$. Given a fluid velocity $v(x,t)$, $X(a,t) : a \in \mathbb{R}^n \to X(a,t) \in \mathbb{R}^n$ is the location at time t of a fluid particle initially placed at the point a at time $t = 0$. The following nonlinear ordinary differential equation defines particle-trajectory mapping:

$$\frac{dX}{dt}(a,t) = v(X(a,t),t), \quad X(a,0) = a. \tag{4.37}$$

The parameter a is called the Lagrangian particle marker. The above differential equation can be thought of the relation between the Lagrangian description of fluid flow and the Eulerian one. Through this Lagrangian formulation we can say that an initial domain $D \in \mathbb{R}^n$ in a fluid evolves in time to $X(D,t) = \{X(a,t) : a \in D\}$, with the vector v tangent to the particle trajectory. We recall the incompressible Navier-Stokes equations with no forcing term

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu\triangle u + \nabla p \tag{4.38}$$

$$\nabla \cdot u = 0 \tag{4.39}$$

describe the evolution of the velocity field u of an incompressible fluid with kinematic viscosity $\nu > 0$ and pressure p. If $\nu = 0$, then (4.46) and (4.47) is known as the Euler equations. This equations describes the evolution of the velocity field of an inviscid incompressible fluid. We can easily notice that the difference between the Euler equations and the Navier-Stokes equations is the term involving the Laplacian and viscosity $\nu > 0$. Their idea of a stochastic representation of the Navier-Stokes equations is a proper expectation of inviscid dynamics from the Euler equations and Browian motion since the Laplacian is the (infinitesimal) generator of a Brownian motion. They started with consideration of the stochastic Lagrangian formulation of the incompressible Euler equations and provided the following a probabilistic representation of the incompressible Navier-Stokes equaition on the free space.

Theorem 4.4. *Let $\nu > 0$, W be an n-dimensional Wiener process, $k \geq 1$, and $u_0 \in C^{k+1,\alpha}$ a given deterministic divergence-free vector field, where $C^{k,\alpha}$ is the Hölder space consisting of functions on $\mathbb{R}^n$ having derivatives up to k and such that the kth partial derivatives are Hölder continuous with exponent α with $0 < \alpha \leq 1$. Let the pair u and X satisfy the stochastic system*

$$dX = udt + \sqrt{2\nu}dW \tag{4.40}$$

$$A = X^{-1} \tag{4.41}$$

$$u = \mathbb{E}\mathbf{P}[(\nabla^T A)(u_0 \circ A))] \tag{4.42}$$

with initial data

$$X(a, 0) = a, \tag{4.43}$$

where $\mathbf{P}$ denotes the Leray-Hodge projection onto divergence free vector fields, the notation ∇^T denotes the transpose of the Jacobian, and for any $t \geq 0$, $A_t = X_t^{-1}$ is the spatial inverse of the map X_t, i.e., $A_t X_t(a) = a$ for all $a in \mathbb{R}^n$ and $X_t(A_t(x)) = x$ for all $x \in \mathbb{R}^n$.

Boundary conditions is that u and $X - I$ are either spatially periodic or decay sufficiently at infinity. Then u satisfies the incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \triangledown)u = \nu \triangle u + \triangledown p \tag{4.44}$$

$$\triangledown \cdot u = 0 \tag{4.45}$$

with initial data u_0.

For example, we consider the Oseen problem with the steady velocity U

$$\frac{\partial u}{\partial t} + (U \cdot \triangledown)u = \nu \triangle u + \triangledown p \tag{4.46}$$

$$\triangledown \cdot u = 0 \tag{4.47}$$

with initial data u_0. Then the stochastic differential equation in (4.40) of the particle trajectory map X_t is $dX = Udt + dW_t$ and so we have $X_t = U + a + W_t$. Let $X_t(a) = x$. Then the spatial inverse of X_t is $A_t(x) = x - Ut - W_t$ since the domain is the free space. Also, the transpose of Jacobian $\triangledown^T$ is the identity matrix. Thus, by the above theorem, the stochastic Lagrangian representation of the solution to the Oseen problem is

$$u = \mathbb{E}\mathbf{P}[A(u_0 \circ A)]$$

where $A_t(x) = x - Ut - W_t$.

However, in case of a domain with boundary, some trajectories of the stochastic Lagrangian flow X will leave the domain and the spatial inverse A_t of X_t may not exist if we use the argument in a free space. Recently, Constantin and Iyer in [10] obtained a stochastic representation of the the 3-dimensional incompressible Navier-Stokes equations on a domain with boundary similar to the probabilistic representation on the free space in [9]. One of main differences between two cases in stochastic Lagrangian formulations in [9] and [10] is that in case of a domain with boundary the vorticity is created on the boundary and has the influence of the fluid velocity. Comparing Constantin-Iyer's representation to our formula for the Stokes problem on a domain with boundary, we focus on the pressure on the boundary rather than the vorticity on the boundary.

5 CONCLUSIONS AND FUTURE WORK

In this thesis we, first of all, solve the Stokes problem in the absence of boundaries, in the case of the half space, and we make some observations for general domains. In particular, we provide the explicit formula for the boundary data of the gradient of pressure in the Stokes problem on the half space in terms of initial and boundary data of the velocity in the Stokes problem. Also, we construct probabilistic solutions to Stokes problems. In addition, we give probabilistic representation of the Helmholtz-Hodge decomposition on domains with smooth boundary as well as the free space. Moreover, we obtain probabilistic representations of iterated Riesz transforms in terms of standard Brownian motion. Lastly, we investigate the Neumann problem on the exterior domains as well as bounded domains probabilistically.

The following are the principal new results in this thesis:

1. The probabilistic representation of the iterated Riesz transform in terms of standard Brownian motion

   Theorem 5.1. *Suppose $f \in C_K^\infty$. Then there exists c independent of f s.t.*

   $$R_iR_jf(x) = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial^2 u}{\partial x_i \partial x_j}\, dY_r = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau (e_{d+1}\otimes e_i H e_j \otimes e_{d+1}\cdot Z_r) dZ_r \tag{5.1}$$

   where H is the Hessian of the harmonic extension u of f.

2. The probabilistic representations of solutions to Neumann problems on unbounded domains.

   Proposition 5.1. *Let $u(x) = \int_{\partial D} N(x,y) f(y)\, d\sigma(y)$ on the exterior domain D and $f \in L^1(\partial D)$. Assume that $\frac{\partial u}{\partial \eta}|_{\partial D} = f$. Then the solution u to the Neumann problem* *on the exterior domain D is given by*

$$u(x) = -\lim_{t\to\infty} \mathbb{E}_x \int_0^t f(B_s^r)\, dLs \tag{5.2}$$

where L_t is the local time process of reflecting Brownian motion B_t^r on $\bar{D}$ and $N(x,y)$ is the Neumann function with respect to B_t^r.

Proposition 5.2. *If $u(x) = \int_{\mathbb{R}^{n-1}} N(x', y', x_n) f(y')\, dy'$, and the boundary data $f \in L^1(\mathbb{R}^{n-1})$, then*

$$\triangle u = 0, \quad \frac{\partial u}{\partial \eta}|_{x_n=0} = f$$

and

$$u(x) = -\lim_{t\to\infty} \mathbb{E}_{(x_0,y_0)} \int_0^t f(B_s^r)\, dLs \tag{5.3}$$

where L_t is the local time process of reflecting Brownian motion B_t^r on $\mathbb{R}_+^n$ and $N(x', y')$ is the Neumann function of the half space $\mathbb{R}_+^{n+1}$.

3. The probabilistic representation of the Helmholtz-Hodge decomposition

Proposition 5.3. *Let $F = (F_1, F_2, F_3)$ be a smooth vector field in $\mathbb{R}^n$ which decays sufficiently fast at infinity and $F = G + \nabla\phi$ where $G = \nabla \times \psi$ for a scalar potential ϕ and vector potential ψ. Then if $v_\rho(x,y) = \sum_{i=0}^3 \frac{\partial u_{F_i}}{\partial x_i}$, where u_{F_i} is the harmonic extension of $F_i, i = 1,2,3$, and $\rho := \nabla \cdot F$, then for some constant c*

$$\frac{\partial \phi(x)}{\partial x_i} = c \sum_{j=1}^3 \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial u_{F_j}}{\partial x_i \partial x_j}\, dY_r \tag{5.4}$$

or

$$\frac{\partial \phi(x)}{\partial x_i} = c \lim_{s\to\infty} \mathbb{E}_{h_x}^{(0,s)} \int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i}\, dY_r \tag{5.5}$$

4. The probabilistic representation of solution to the Stokes problem on the free space

Proposition 5.4. *Let $u = (u_1, u_2, u_3)$ be the solution to the Stokes problem with an initial data $u_0 = (u_0^1, u_0^2, u_0^3)$ on the free space. Then if v is the harmonic extension of u_0 and $v_\rho = \nabla \cdot v$ then for some constant c*

$$u_i(x,t) = \mathbb{E}_x[u_0^i(B_t)] + c\mathbb{E}_x[\lim_{s\to\infty} \mathbb{E}_{h_{B_t}}^{(0,s)}[\int_0^\tau Y_r \frac{\partial v_\rho}{\partial x_i}\, dY_r]] \tag{5.6}$$

5. The Stokes problem on the half space in $\mathbb{R}^3_+$

Proposition 5.5. *Let u be the solution to the Stokes problem on $\mathbb{R}^3_+$:*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u - \nabla p, \qquad \nabla \cdot u = 0$$

$$u(x,0) = u_0(x), \quad u|_{x_3=0} = 0$$

Assume that $u_0(x) \in \mathcal{C}^1(\mathbb{R}^3_+)$ satisfies the compatibility conditions $\nabla \cdot u_0 = 0$ and $u_0|_{x_3=0} = 0$. Then $h = -\frac{\partial p}{\partial x_3}|_{x_3=0}$ is given by

$$h(x_1,x_2,t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x'-y',y_3,t)[\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)]\, dy'dy_3 \qquad (5.7)$$

where $u_0 = (u_0^1, u_0^2, u_0^3)$, $\nabla' = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})$, $R'f = (R_1 f, R_2 f)$ with the Riesz transform of f $R_i f$, $i = 1,2$, and

$$\widetilde{K}(x',x_3,t) = k^{(2)}(x',t)\frac{1}{\sqrt{2\pi t}}(2)(\frac{x_3}{t})\exp(-\frac{x_3^2}{2t}).$$

Moreover, a probabilistic representation of h in terms of reflecting Brownian motion is given as

$$h(x',t) = \mathbb{E}_{(x',0)}[\frac{\partial g}{\partial y_3}(B_t^r)]$$

where $g(y',y_3) = \nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)$ and B_t^r is reflecting Brownian motion.

Theorem 5.2. *Assume that $u_0(x) \in \mathcal{C}^1(\mathbb{R}^3_+)$ and $a \in \mathcal{C}^{1,1}(\mathbb{R}^2 \times (0,\infty))$ satisfy the compatibility conditions $\nabla \cdot u_0 = 0$, $u_0|_{x_3=0} = a|_{t=0}$, and $a \cdot \eta = 0$. Let*

$$h(x_1,x_2,t) := \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x'-y',y_3,t)[\nabla' \cdot (R_1 u_0^3 - u_0^1, R_2 u_0^3 - u_0^1)]\, dy'dy_3$$

$$+\triangle_{x'}(R_1 a_1 + R_2 a_2) + 4\int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial t} - \frac{1}{2}\triangle')\frac{\exp(-\frac{|x'-y'|^2}{2s})}{s^{3/2}}(\nabla' \cdot a')(y',t-s)\, dy'ds,$$

$$u(x,t) := \{-\int_0^t \int_{\mathbb{R}^2} [\frac{\partial}{\partial z_3}|_{z_3=0} \int_0^\infty E(x,z,t'+s)\, dt']H(z,t-s)\, dzds + \int_{\mathbb{R}^3_+} E(x-y,t)u_0(y)\, dy$$

$$+\int_0^t \int_{\mathbb{R}^2} \frac{\partial}{\partial y_3}|_{y_3=0} E(x,y,t)a(y',t-s)\, dy'ds$$

and let p be the solution to the Neumann problem with boundary data h on the half space $\mathbb{R}^3_+$. Then u and p satisfy the Stokes problem with the initial data u_0 and the boundary data a on the half space $\mathbb{R}^3_+$.

6. Pressure on the boundary in the Stokes problem in $\mathbb{R}^3_+$

Corollary 5.1. *Let $h = -\frac{\partial p}{\partial x_3}|_{x_3=0}$ in Theorem 5.2. Then*

$$p(x_1, x_2, t) = \int_0^\infty \int_{\mathbb{R}^2} \widetilde{K}(x' - y', y_3, t)(u_0^3 - R_1 u_0^1 - R_2 u_0^2)\, dy' dy_3$$

$$+\nabla' \cdot a' + 4 \int_0^t \int_{\mathbb{R}^2} (\frac{\partial}{\partial s} - \frac{1}{2}\triangle') \frac{\exp -(\frac{|x'-y'|^2}{2s})}{s^{3/2}} (R_1 a_1 + R_2 a_2)$$

where $a' = (a_1, a_2)$.

For the future work we have the following suggestions:

- In view of the definition of Riesz transform through the Neumann problem on the half space, it is natural question to ask: "What are the properties of that local time process for Riesz transformations?". We might want to get the explicit representation of Riesz transform containing a local time process without the gradient operator.
- Also, it is worth to consider the case $n = 2$ rather than $n \geq 3$.
- As one of applications of the probabilistic representations of Neumann problem, we can investigate the regularity of the Helmholtz-Hodge decompostion using local time process.
- Daniel Strook and S.R.S Varadhan developed the theory of diffusion processes in the form of their martingale problem, making the appropriate family of probability measures on function spaces the centerpiece. So it is also interesting to formulate a martingale problem for the Stokes problem and find the solution to the martingale

problem, which gives us another probabilistic interpretation through a family of probability measures on function spaces.

- When we look at the explicit formula for h in the Stokes problem on the half space we have the conjecture that the h in a general domain D such as an exterior domain might be obtained by two processes: one is Riesz transforms of the initial data and then take tangential derivative of Riesz transformed initial data and the other is Riesz transforms of boundary data and a certain differential operator of boundary data.

BIBLIOGRAPHY

1. Nicola Arcozzi and Xinwei Li, *Riesz Transforms on Spheres*, Mathematical Research Letters 4, 401-412, 1997.
2. Richard F. Bass, *Probabilistic Techniques in Analysis*, Springer-Verlag, 1995.
3. Richard F. Bass, *Diffusions and Elliptic Operators*, Springer-Verlag, 1998.
4. G. A. Brosamler, *A probabilistic solution of the Neumann problem*, Math. Scand., 38, 137-147, 1976.
5. Ihsane Bikri, R. B. Guenther, and E. A. Thomann, *The Dirichlet to Neumann Map-An Application to the Stokes Problem in Half Space*, Discrete and continuous dynamical systems series, vol.3, 2010
6. R.F. Bass and P. Hsu *Some potential theory for reflecting Browian motion in Hölder and Lipschitz domains*, Ann. Probab. 19, 486-508, 1991
7. R. Bhattacharya and E. Waymire *A Basic Course in Probability Theory*, Universitext, Springer, NY, 2007
8. R. Bañuelos and P.J. Méndez-Hernández *Space-time Browian motion and the Beurling-Ahlfors transform*, Indiana Univ. Math.J. 52, 981-990, 2003
9. Peter Constantin and Gautam Iyer *A Stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations*, Comm. Pure Appl. Math. 61, no.3, 330-345, 2010
10. Peter Constantin and Gautam Iyer *A Stochastic-Lagrangian Approach to the Navier-Stokes Equations in Domains with Boundary*, prepint, 2010
11. A.J. Chorin and J.E. Marsden *A Mathematical Introduction to Fluid Mechanics*, 3rd ed., Springer-Verlag, New York, 1993
12. Z. Q. Chen, R. J. Williams, and Z. Zhao *A Sobolev Inequality and Neumann Heat Kernel Estimate for Unbounded Domains*, Mathematical Research Letters 1, 177-184, 1994
13. Z. Q. Chen, R. J. Williams, and Z. Zhao, *On the existence of positive solutions for semilinear elliptic equations with Dirichlet boundary conditions*, Math. Ann. 298, 543-556, 1994
14. Z. Q. Chen, R. J. Williams, and Z. Zhao, *On the existence of positive solutions for semilinear elliptic equations with Neumann boundary conditions*, Probab. Theory Relat. Fields 101, 251-276, 1995

15. A.P. Calderón and Z. Zygmund, *On the existence of certain singular integrals*, Acta. Math. 88, 85-139, 1952

16. David B. Dusenbery, *Living at Micro Scale*, Harvard University Press, Cambridge, Mass, 2009

17. Manfredo Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall; First edition, 1976

18. R. B. Guenther and John W. Lee, *Patial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications, INC., 1996

19. F.K.G. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademische. Math. Z.,32,329-375, 1930

20. G.A. Brosamler, *A probabilistic solution of the Neumann problem*, Math. Scand., 38, 137-147, 1976

21. R.F. Gundy and M.L. Silverstein, *On a probabilistic interpretation for the Riesz transforms*, Functional Analysis in Markov Processes, Proceedings, Katata and Kyoto, Lecture Notes in Mathematics 923 Springer-Verlag, NY, 1981

22. R.F. Gundy and N.Th. Varopoulos, *Les transformations de Riesz et les integrales stochastiques*, C.R. Acad. Sci. Paris Ser. A 289 13-16, 1979

23. R. B. Guenther and E. A. Thomann, *The Fundamental solutions of the Linearized Navier-Stokes Equations for Spinning Bodies in Three Spatial Dimensions-Time Dependent Case*, Journal of Mathematical Fluid Mechanics, Vol 9, 489-505, 2007

24. R. B. Guenther and E. A. Thomann, *Fundamental solutions of Stokes and Oseen problem in Two Spatial Dimensions*, Published electronically Sept 2006 in Journal of Mathematical Fluid Mechanics.

25. V. Isakov, *Inverse Problems for Partial Differential Equations*, Applied Mathematical Sciences, Vol 127. Springer-Verlag, New York, 1998.

26. Ioannis Karatzas and Steven E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag New York Inc., 1988

27. T. Kato, *Strong L^p solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Math. Z. 187, 471-480, 1984

28. P.G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Chapman & Hall/CRC, 2002

29. P. L. Lions and A. S. Sznitman, *Stochastic Differential Equations with Reflecting Boundary Conditions*, Com.on Pure and App. Math., 37, 511-537, 1984

30. Andrew J. Majda and Andrea L. Bertozzi *Vorticity and Incompressible Flow*, Cambridge University Press, 2002

31. Bernt Oksendal, *Stochastic Differential Equations:An Introduction with Applications*, Springer-Verlag Berlin Heidelberg, 2000

32. W. Magnus,F. Oberhettinger and F.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, 1966

33. Philip E. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag Berlin Heidelberg New York, Second Edition, 2004

34. M. Riesz, *Sur les fonctions conjuguées*, Math. Zeit. 27, 218-244, 1927

35. S. Ramasubramanian, *Diffusions and the Neumann Problem in the Half Space*, The indian journal of statistics, 54, Series A, Pt.3, pp 351-378, 1992.

36. Daniel Revuz and Marc Yor *Continous Martingales and Brownian Motion*, Springer-Verlag Berlin Hedelberg, 1991

37. Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No.30. Princeton University Pres, Princeton, N.J., 1970

38. V.A. Solonnikov, *Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov. 70, 213-317, 1964

39. S. Ukai, *A Solution Formula for the Stokes Equation in* $\mathbb{R}^n_+$, CPAM, Vol.XL 611-621, 1987

40. Edward C. Waymire, *Probability & Incompressible Navier-Stokes Equations: An Overview of Some Recent Developments*, Probability Surveys, 2004

41. Z. Zhao, *On the existence of positive solutions of nonlinear elliptic equations - a probablistic potential theory approach*, Duke Math. J. 69, 247-258, 1993

APPENDIX

A APPENDIX Mathematical Background

In this appendix we give some preliminaries on probability theory. We will follow some introductory textbooks, for example, [31], [2] and [7].

Definition 0.1. *If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

- $\emptyset \in \mathcal{F}$
- *If $F \in \mathcal{F}$ then $F^C \in \mathcal{F}$, where $F^C = \Omega \setminus F$ is the complement of F in Ω*
- *If $A_1, A_2, \cdots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$*

The pair $(\Omega, \mathcal{F})$ is called a measurable space. A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that

- $P(\emptyset) = 0, P(\Omega) = 1$
- *if $A_1, A_2, \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint then*

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space. Elements of $\mathcal{F}$ are called events. Measurable functions from Ω to $\mathbb{R}$ are called random variables and are usually denoted X or Y instead of f and g. The integral of X with respect to P is called the expectation of X or the expected value of X, $\mathbb{E}X = \int X(\omega)P(d\omega)$, and $\mathbb{E}[X; A] := \int_A X(\omega)P(d\omega)$.

If an event F occurs with probability one, $P(F) = 1$, we say "almost surely" instead of "almost everywhere" and write a.s. The notation 1_A, the indicator of the set A, is the random variable that is 1 on A and 0 on the complement.

The law or distribution of X is the probability measure P_X on $\mathbb{R}$, given by

$$P_X(A) = P(X \in A).$$

Proposition 0.6. *If $f \geq 0$ or f is bounded,*

$$\mathbb{E}f(X) = \int f(x)P_X(dx).$$

Two events A, B are independent if $P(A \cap B) = P(A)P(B)$. A σ-algebra $\mathcal{F}$ is independent of a σ-algebra $\mathcal{G}$ if each $A \in \mathcal{F}$ is independent of each $B \in \mathcal{G}$. Two random variables are independent if the σ-algebras generated by X, Y are independent. Note that if X and Y are independent and f and g are Borel measurable functions, then $f(X)$ and $g(Y)$ are independent.

Proposition 0.7. *If X, Y, and XY are integrable and X and Y are independent, then $\mathbb{E}XY = \mathbb{E}(X)\mathbb{E}(Y)$.*

Definition 0.2. *If $\mathcal{F} \subseteq \mathcal{G}$ are two σ-fields and X is an integrable $\mathcal{G}$ measurable random variable., the conditional expectation of X given $\mathcal{F}$, written $\mathbb{E}[X|\mathcal{F}]$ is any $\mathcal{F}$ measurable random variable Y such that $\mathbb{E}[Y; A] = \mathbb{E}[X; A]$ for every $A \in \mathcal{F}$.*

If Y_1, Y_2 are two $\mathcal{F}$ measurable random variables with $\mathbb{E}[Y_1; A] = \mathbb{E}[Y_2; A]$ for all $A \in \mathcal{F}$, then $Y_1 = Y_2$, a.s., or conditional expectation is unique up to a.s. equivalence. Note that limit theorems such as monotone convergence theorem and dominated convergence theorem have conditional expectation versions, as do inequalities like Jensen's and Chebyshev's inequalities.

Proposition 0.8. *If X is integrable, then $\mathbb{E}[X|\mathcal{F}]$ exists.*

A stochastic process is a collection of random variables with a parameter t $\{X_t\}_{t \in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$ for $n \geq 1$. The parameter t might be usually in the half line $[0, \infty)$, or an interval $[a, b]$, the non-negative integers and even subsets of $\mathbb{R}^n$ for $n \geq 1$.

Definition 0.3. *A stochastic process X_t is a one-dimensional Brownian motion started at 0 if*

- *$X_0 = 0$ a.s.;*

- *for all $s \le t$, $X_t - X_s$ is a mean zero Gaussian random variable with variance $t - s$;*

- *for all $s < t$, $X_t - X_s$ is independent of $\sigma(X_r; r \le s)$;*

- *with probability 1 the map $t \to X_t(\omega)$ is continuous.*

where $\sigma(X_r; r \le s)$ is the smallest σ-field with respect to which each $X_r, r \le s$, is measurable.

To define d-dimensional Brownian motion, let $X_t^1, \cdots, X_t^d$ be independent one-dimensional Brownian motions. Then

$$X_t = (X_t^1, \cdots, X_t^d)$$

is d-dimensional Brownian motion. If you want to consider Brownian motion started at $x \in \mathbb{R}^d$, we can get this just by looking at $x + X_t$. Let $\mathcal{F} = \sigma(X_r; r < \infty)$. Now define P^x to be the probability measure on $(\Omega, \mathcal{F})$ given by

$$P^x(X_t \in A) = P(x + Z_t \in A), \quad x \in \mathbb{R}^d, A \in \mathcal{F}$$

where Z_t is d-dimensional Brownian motion as defined above.

We next define stopping times. Suppose we have a stochastic process X_t and a filtration $\mathcal{F}_t$, which is an increasing collection of σ-fields. We suppose each $\mathcal{F}_t$ is right continuous (i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$ for each t, where $\mathcal{F}_{t+} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$). We also suppose that X_t is adapted to $\mathcal{F}_t$: for each t, X_t is $\mathcal{F}_t$ measurable.

Definition 0.4. *A random mapping T from Ω to $[0, \infty)$ is called a stopping time if for each t, $(T < t) \in \mathcal{F}_t$.*

Proposition 0.9. *(a) T is a stopping time if and only if $(T \le t) \in \mathcal{F}_t$ for all t.*
(b) Fixed times t are stopping times.

(c) If S and T are stopping times, then so are $S \wedge T$ and $S \vee T$.

(d) If T_n is a nondecreasing sequence of stopping times, then so is $T = \sup_n T_n$.

(e) If T_n is a nonincreasing sequence of stopping times, then so is $T = \inf_n T_n$.

(f) If S is a stopping time, then so is $S + T$.

Stopping times we are interested in are the first times that X_t hits a set A. Let

$$T_A = \inf t > 0 : X_t \in A$$

$$\tau_A = \inf t > 0 : X_t \notin A.$$

Proposition 0.10. *(a) If A is an open set, then T_A is a stopping time.*

(b) If A is a closed set, then T_A is a stopping time.

Proposition 0.11. *$P^x(X_{\tau(B(x,r))} \in dy)$ is normalized surface measure on $\partial B(x,r)$ where $B(x,r)$ is the ball centered at x with radius r.*

If follows that if f is a function defined on the boundary of $B(x,r)$, then

$$\mathbb{E}^x f(X_{\tau(B(x,r))}) = \int_{\partial B(x,r)} f(y)\sigma(dy),$$

where σ is normalized surface measure on the boundary of $B(x,r)$.

We consider martingales. Let $\mathcal{F}_n$ be an increasing sequence of σ-fields. A sequence of random variables M_n is adapted to $\mathcal{F}_n$ if for each n, M_n is $\mathcal{F}_n$ measurable. Similarly a collection of random variables M_t is adapted to $\mathcal{F}_t$ if for each t, M_t is $\mathcal{F}_t$ measurable. We assume that the filtration $\mathcal{F}_t$ is right continuous and complete (i.e., $\mathcal{F}_t$ contains all P-null sets).

Definition 0.5. *M_n is martingale if M_n is adapted to $mathcalF_n$, M_n is integrable for all n, and*

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}, \quad a.s., \quad n = 2, 3, \cdots .$$

Similarly, M_t is martingale if M_t is adapted to $mathcalF_t$, M_t is integrable for all t, and

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s, \quad a.s., \quad ifs \leq t.$$

If M_t is Brownian motion, then it is a martingale using independent increments. By the definition of martingale and induction $\mathbb{E}M_n = \mathbb{E}M_0$. One of important theorems in probability theory is Doob's optional stopping time, which says that the same is true if we replace n by a stopping time N. There are various versions, depending on what conditions one puts on the stopping times. We will give a version of discrete martingales.

Theorem 0.3. *If N is a bounded stopping time with respect to $\mathcal{F}_n$ and M_n a martingale, then $\mathbb{E}M_n = \mathbb{E}M_0$.*

The first interesting consequences of the optional stopping theorems are Doob's inequalities. If M_t and M_n are martingales, denote $M_t^* = \sup_{s \le t} |M_s|$, and similarly M_n^*.

Theorem 0.4.

$$P(M_n^* \ge a) \le \mathbb{E}|M_n|/a$$

Theorem 0.5. *If $p > 1$, there exists c depending only on p such that*

$$\mathbb{E}(M_n^*)^p \le c\mathbb{E}|M_n|^p.$$

The same results hold for M_t if M_t is a martingale or positive submartingale with right continuous paths.

The martingale convergence theorems are another set of important consequences of optional stopping. The main step is the upcrossing lemma. The number of upcrossings of an interval $[a, b]$ is the number of times a process crosses from below a to above b.

Theorem 0.6. *If X_t is a submartingale such that $\sup_n \mathbb{E}X_n^+ < \infty$, then X_t converges a.s. as $n \to \infty$.*

The Doob-Meyer decomposition says that , under mild hypotheses, a supermartingale can be decomposed into a martingale minus an increasing process. If M_t is a continuous square integrable martingale, then M_t^2 is a submartingale and $-M_t^2$ is a supermartingale. By the Doob-Meyer decomposition there exists a continuous increasing process, denoted $< M >_t$, the quadratic variation of M, such that $M_t^2 - < M >_t$ is a martingale. If we have two martingales M, N, we define $< M, N >_t$ by polarization:

$$< M, N >_t = \frac{1}{2}(< M + N >_t - < M >_t - < N >_t).$$

The most important fact about stochastic integration is the change of variables formula or Itôs formula.

Theorem 0.7. *Let X_t ba a semimartingale with continuous paths. Suppose $f \in C^2$. Then with probability one we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d < X >_s, \quad t \geq 0.$$