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Title: Time Domain Simulation of Multiconductor Lossy Transmission Lines by the Method of Characteristics

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A numerical technique to compute the time domain response of multiconductor lossy uniform and nonuniform lines terminated in general nonlinear elements is presented. The technique is based on the generalized method of characteristics. The method transforms the original system of transmission line equations into a system of ordinary differential equations. The differentials are then approximated by finite differences and solved numerically. The technique is used to study signal delay, distortion and crosstalk in interconnections in integrated circuits and chip carriers. Typical examples illustrating the time domain response of uniform and nonuniform multiconductor lines representing integrated circuit interconnections on chips, chip carriers and packaging are included.
Time Domain Simulation of Multiconductor Lossy Transmission Lines by the Method of Characteristics

by

Neven Orhanovic

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Head of department of Electrical and Computer Engineering

Redacted for Privacy

Dean of Graduate School

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# Table of Contents

1. Introduction ......................................................... 1

2. Numerical Solution of Hyperbolic Partial Differential Equations
   by the Method of Characteristics
   2-1 Definitions and Derivation of the Method .................. 5
   2-2 Finite Difference Approximations .......................... 7
   2-3 Specified Time (y-direction) Intervals .................... 9
   2-4 Boundary Points ............................................ 11
   2-5 Extrapolation Procedures ................................. 13

3. Extension of the Method of Characteristics to a System
   of 2n Partial Differential Equations
   3-1 Derivation of the Method .................................. 14
   3-2 Specified Time (y-direction) Intervals .................... 17
   3-3 Boundary Points ............................................ 20

4. Application of the Method of Characteristics to Transmission Line
   4-1 Problem Formulation ......................................... 22
   4-2 Numerical Solution ......................................... 23
   4-3 Nonuniform Lines ........................................... 28
   4-4 Examples .................................................... 33

Conclusion ............................................................ 39

Bibliography .......................................................... 40

Appendixes

Appendix A
   A-1 Quadratic Interpolation ................................. 41
   A-2 Derivation of (3-30) and (3-31) ......................... 43
   A-3 Iterative Solution of (4-50) and (4-51) ................ 44
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>Examples of single and multilevel interconnections</td>
<td>2</td>
</tr>
<tr>
<td>1-2</td>
<td>Generic example of multiconductor lines</td>
<td>3</td>
</tr>
<tr>
<td>2-1</td>
<td>Computation of ( u ) and ( v ) at an arbitrary point</td>
<td>8</td>
</tr>
<tr>
<td>2-2</td>
<td>Specified time (y-direction) intervals</td>
<td>10</td>
</tr>
<tr>
<td>2-3</td>
<td>Computation of ( u ) and ( v ) at the boundary</td>
<td>12</td>
</tr>
<tr>
<td>3-1</td>
<td>Computation of ( u ) and ( v ) at a point on the line ( y = Y + \Delta y )</td>
<td>18</td>
</tr>
<tr>
<td>3-2</td>
<td>Computation of ( u ) and ( v ) at the boundary</td>
<td>20</td>
</tr>
<tr>
<td>4-1</td>
<td>Computation of currents and voltages on the line</td>
<td>25</td>
</tr>
<tr>
<td>4-2</td>
<td>The geometry of Example 1</td>
<td>33</td>
</tr>
<tr>
<td>4-3</td>
<td>The results of Example 1</td>
<td>34</td>
</tr>
<tr>
<td>4-4</td>
<td>The geometry of Example 2</td>
<td>36</td>
</tr>
<tr>
<td>4-5</td>
<td>The results of Example 2</td>
<td>36</td>
</tr>
<tr>
<td>4-6</td>
<td>The geometry of Example 3</td>
<td>37</td>
</tr>
<tr>
<td>4-7</td>
<td>The results of Example 3</td>
<td>38</td>
</tr>
<tr>
<td>A-1</td>
<td>Quadratic interpolation</td>
<td>41</td>
</tr>
<tr>
<td>A-3</td>
<td>Iterative computation of ( e ) and ( i ) at the boundary</td>
<td>44</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

The recent advances in electronic technology have reduced the switching time of single devices in integrated circuits to the order of picoseconds or less. At these switching speeds, the time taken by a signal to propagate across on-chip or inter-chip connections is often significantly longer than the device switching time and the delay and distortion characteristics of the connections can no longer be neglected. In addition, because of the increase of the level of integration, the connecting lines are closely spaced, resulting in electromagnetic coupling. Sudden voltage changes in excited lines can generate transients in neighboring lines which can trigger logic gates and other devices at the terminations, resulting in false switching and corrupted data transmission. It is anticipated that in the near future performance in terms of time delay and speed will be determined primarily by interconnections, rather than by device limitations.

These on-chip and inter-chip connections can be modeled as coupled distributed parameter circuits or multiconductor transmission lines. Figures 1-1 and 1-2 show examples of single and multilevel interconnections. The propagation of voltages and currents on such lines is described by the transmission line equations together with the boundary and initial conditions.
Figure 1-1: Examples of single and multilevel interconnections
Various methods have been developed to solve the single and multiconductor transmission line equations, both in frequency and time domain. Some of these methods lead to equivalent circuit representations of the line (e.g., [1]–[3]). This is convenient, because such equivalent circuits can be readily implemented into the existing tools for computer-aided analysis and design of integrated circuits. For the lossy line—which is a more realistic model—things become more complex and the methods developed over the years have been limited to special cases of a homogeneous medium [4], analysis based on frequency domain solutions [5] or cases where the coupling between non-adjacent lines is neglected (e.g., [6]). Some attempts have been made to model the skin effect losses by equivalent RL network representation, primarily for single lines or interconnections ([7]–[9]).

The method of characteristics is a standard mathematical method for solving partial differential equations of the hyperbolic type. It transforms the original system of partial differential equations into a system of ordinary differential equations which are valid along
a family of *characteristic curves*, often abbreviated to *characteristics*. For a lossless single line, the ordinary differential equations obtained can be directly integrated leading to an equivalent circuit model of the line [10]. The method can also be extended to a multiconductor lossless line leading to an equivalent circuit representation. For the lossy line case, the ordinary differential equations that are obtained cannot be integrated directly. They must be solved numerically.

In this thesis, the method of characteristics is generalized and applied to lossy uniform and nonuniform multiconductor lines.

Chapter 2 describes the method of characteristics in general. Some simplifications are made regarding the shape of the boundaries but with small extensions the method could be applied to any physical problem that can be described by similar equations.

Chapter 3 extends the method to a system of $2n$ coupled hyperbolic partial differential equations. The transmission line equations for a lossy multiconductor line can be transformed into this form.

Chapter 4 shows how the transmission line equations for a lossy uniform multiconductor line can be transformed into the system of equations that were solved in Chapter 3. The procedure developed in Chapter 3 is then used to solve these equations. Nonuniform lines are discussed in a similar manner. First they are transformed to equations of Chapter 3, then they are solved using the same procedure. Typical examples illustrating the time domain response of uniform and nonuniform multiconductor lines terminated in linear and nonlinear loads are included.
Chapter 2

Numerical Solution of Hyperbolic Partial Differential Equations by the Method of Characteristics

This thesis deals with lossy transmission line equations which are special cases of quasilinear partial differential equations of the hyperbolic type. This chapter gives the technique used to find the numerical solution to such equations. The technique is given in [10] and is reviewed here for completeness.

2-1 Definitions and Derivation of the Method

The general form of a quasilinear system of equations with two independent variables \(x, y\) and two dependent variables \(u, v\) has the form:

\[
L_1 = A_1 u_x + B_1 u_y + C_1 v_x + D_1 v_y + E_1 = 0,
(2-1)
\]

\[
L_2 = A_2 u_x + B_2 u_y + C_2 v_x + D_2 v_y + E_2 = 0,
(2-2)
\]

where \(A_1, B_1, ..., E_2\) are known functions of \(x, t, u, v\) and \(u_x = \partial u/\partial x, u_y = \partial u/\partial y, v_x = \partial v/\partial x, v_y = \partial v/\partial y\). It is assumed that all the functions introduced above are continuous, possess as many continuous derivatives as may be required and for no value of \(x, y\) is the following condition satisfied:

\[
A_1/A_2 = B_1/B_2 = C_1/C_2 = D_1/D_2.
(2-3)
\]

Let \(y = y(x)\) define a curve in the \(x\)-\(y\) plane, then \(dy/dx\) is the slope of the tangent at any point of this curve and, if \(u = u(x, y)\) and \(v = v(x, y)\) are solutions to (2-1) and (2-2) then

\[\frac{du}{dx} = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad \frac{dv}{dx} = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.\]  

Consider a linear combination of \(L_1\) and \(L_2\):
\[ L = \lambda_1 L_1 + \lambda_2 L_2 = (\lambda_1 A_1 + \lambda_2 A_2)u_x + (\lambda_1 B_1 + \lambda_2 B_2)u_y + \\
+ (\lambda_1 C_1 + \lambda_2 C_2)v_x + (\lambda_1 D_1 + \lambda_2 D_2)v_y + \lambda_1 E_1 + \lambda_2 E_2. \]  

(2-5)

If the constants \( \lambda_1 \) and \( \lambda_2 \) are chosen so that

\[
\frac{dx}{dy} = \frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2} = \frac{\lambda_1 C_1 + \lambda_2 C_2}{\lambda_1 D_1 + \lambda_2 D_2},
\]

(2-6)

(2-5) can be written in the form

\[ dxL = (\lambda_1 A_1 + \lambda_2 A_2) \, du + (\lambda_1 C_1 + \lambda_2 C_2) \, dv + (\lambda_1 E_1 + \lambda_2 E_2) \, dx. \]  

(2-7)

In the above expression the derivatives of \( u \) and \( v \) are combined so that they are taken in the same direction, namely, \( dy/dx \). This direction is called a \textit{characteristic direction}.

The ratio \( \lambda_1/\lambda_2 \) can be obtained from (2-6):

\[
-\frac{\lambda_1}{\lambda_2} = \frac{A_2 dy - B_2 dx}{A_1 dy - B_1 dx} = \frac{C_2 dy - D_2 dx}{C_1 dy - D_1 dx}.
\]

(2-8)

Hence

\[ a(dy)^2 - b \, dx \, dy + c(dx)^2 = 0, \]

(2-9)

where

\[ a = A_1 C_2 - A_2 C_1, \quad b = A_1 D_2 - A_2 D_1, \quad c = B_1 D_2 - B_2 D_1. \]

(2-10)

For hyperbolic partial differential equations \( b^2 - 4ac > 0 \) and it is assumed that \( a \neq 0 \). Consequently, \( dx \neq 0 \) for a characteristic direction, and a slope \( \zeta = dy/dx \) can be introduced. \( \zeta \) satisfies the equation

\[ a\zeta^2 - b\zeta + c = 0, \]

(2-11)

which has two different solutions \( \zeta_+ \) and \( \zeta_- \).

Thus, at the point \((x, y)\), the two different characteristic directions are given by:

\[ \frac{dy}{dx} = \zeta_+(x, y, u, v), \quad \frac{dy}{dx} = \zeta_-(x, y, u, v), \]

(2-12)

which are functions of \( x, y, u, v \), since \( a, b, \) and \( c \) are functions of \( x, y, u, v \).

Once a solution \( u(x,y), v(x,y) \) of (2-1) and (2-2) has been obtained, equations (2-12) become two separate ordinary differential equations of the first order. These define two one-parameter families of \textit{characteristic curves} (often abbreviated to characteristics), \( C_+ \).
and \( C_+ \) in the \((x,y)\) plane, belonging to this solution \( u(x,y), v(x,y) \). These two families form a curvilinear coordinate net.

If \( \zeta_+ \) and \( \zeta_- \) are functions of \( x \) and \( y \) only i.e.,

\[
\frac{dy}{dx} = \zeta_+(x,y), \quad \frac{dy}{dx} = \zeta_-(x,y),
\]

the problem is simplified and it is not necessary to find the solution of (2-1) and (2-2) in order to find the equations of the characteristics.

Substituting the solutions (2-12) into the expression for \( \lambda_1/\lambda_2 \) given in (2-8) yields

\[
\frac{\lambda_1}{\lambda_2} = -\frac{A_2\zeta_+ - B_2}{A_1\zeta_+ - B_1}, \quad \frac{\lambda_1}{\lambda_2} = -\frac{A_2\zeta_- - B_2}{A_1\zeta_- - B_1}.
\]

Finally, combining (2-1), (2-2), (2-7) and (2-14) gives

\[
\begin{align*}
F \, du + (a\zeta_+ - G) \, dv + (K\zeta_+ - H) \, dx &= 0, \\
F \, du + (a\zeta_- - G) \, dv + (K\zeta_- - H) \, dx &= 0,
\end{align*}
\]

where

\[
\begin{align*}
F &= A_1B_2 - A_2B_1, \quad G = B_1C_2 - B_2C_1, \\
K &= A_1E_2 - A_2E_1, \quad H = B_1E_2 - B_2E_1.
\end{align*}
\]

Thus, the following four characteristic equations have been obtained:

\[
\begin{align*}
dy - \zeta_+ \, dx &= 0, \\
F \, du + (a\zeta_+ - G) \, dv + (K\zeta_+ - H) \, dx &= 0, \\
dy - \zeta_- \, dx &= 0, \\
F \, du + (a\zeta_- - G) \, dv + (K\zeta_- - H) \, dx &= 0,
\end{align*}
\]

where the first two equations hold along \( C_+ \) and the second two along \( C_- \).

The above equations have a simpler form than the original system (2-1)–(2-2), each of them containing only total derivatives of the variables.

According to the derivation, every solution of the original system (2-1) and (2-2) also satisfies the system (2-18)–(2-21). It can be shown that the converse is also true.

### 2.2 Finite Difference Approximations

First- and second-order approximation methods for solving (2-18)–(2-21) are described here. The first finite difference approximation is given by the expression:
\[
\int_{x_0}^{x_1} f(x)dx = f(x_0)(x_1 - x_0),
\]

(2-22)

and will be called a first-order or linear approximation.

The second finite difference approximation is expressed by the trapezoidal rule formula:

\[
\int_{x_0}^{x_1} f(x)dx = \frac{1}{2} [f(x_0) + f(x_1)](x_1 - x_0),
\]

(2-23)

and will be called a second-order approximation.

Let P be the point of intersection of the \(C_+\) characteristic through A and the \(C_-\) characteristic through B, as shown in Fig. 2-1. It is assumed that \(x_A, y_A, u_A, v_A, x_B, y_B, u_B, \) and \(v_B\) are known, and that \(x_p, y_p, u_p, v_p\) are to be found. \(x_A\) means value of \(x\) at \(A\), etc. Using (2-22), the linear approximation to (2-18) to (2-21) is

\[
y_p - y_A - (\zeta_+)_A(x_p - x_A) = 0, \quad (2-24)
\]
\[
F_A(u_p - u_A) + (a\zeta_+ - G)_A(v_p - v_A) + (K\zeta_+ - H)_A(x_p - x_A) = 0, \quad (2-25)
\]
\[
y_p - y_B - (\zeta_-)_B(x_p - x_B) = 0, \quad (2-26)
\]
\[
F_B(u_p - u_B) + (a\zeta_- - G)_B(v_p - v_B) + (K\zeta_- - H)_B(x_p - x_B) = 0. \quad (2-27)
\]

\(\zeta_+, \zeta_-\), \(F, G, K, H\), and \(a\) are functions of \(x, y, u, \) and \(v\). They are evaluated at points A and B and are known. Therefore, the unknowns \(x_p\) and \(y_p\) can be obtained by solving simultaneously (2-24) and (2-26). The solutions can then be substituted into (2-25) and (2-27). These can then be solved simultaneously to give the values for \(u_p\) and \(v_p\).
The second order approximation to (2-18)–(2-21) is

\[
\begin{align*}
yp - y_{A} - \frac{1}{2} \left[ (\zeta_{+})_{A} + (\zeta_{+})_{p} \right] (x_{p} - x_{A}) &= 0, \quad (2-28) \\
\frac{1}{2} \left( F_{A} + F_{p} \right) (u_{p} - u_{A}) + \frac{1}{2} \left[ (a_{\zeta_{+}} - G)_{A} + (a_{\zeta_{+}} - G)_{p} \right] (v_{p} - v_{A}) + \\
&\quad + \frac{1}{2} \left[ (K_{\zeta_{+}} - H)_{A} + (K_{\zeta_{+}} - H)_{p} \right] (x_{p} - x_{A}) = 0, \quad (2-29) \\
y_{p} - y_{B} - \frac{1}{2} \left[ (\zeta_{-})_{B} + (\zeta_{-})_{p} \right] (x_{p} - x_{B}) &= 0, \quad (2-30) \\
\frac{1}{2} \left( F_{B} + F_{p} \right) (u_{p} - u_{B}) + \frac{1}{2} \left[ (a_{\zeta_{-}} - G)_{B} + (a_{\zeta_{-}} - G)_{p} \right] (v_{p} - v_{B}) + \\
&\quad + \frac{1}{2} \left[ (K_{\zeta_{-}} - H)_{B} + (K_{\zeta_{-}} - H)_{p} \right] (x_{p} - x_{B}) = 0. \quad (2-31)
\end{align*}
\]

Unless \( \zeta_{+}, \zeta_{-}, F, G, K, H, \) and \( a \) are independent of \( x, y, u, \) and \( v, \) the above four equations are no longer linear in the unknowns \( x_{p}, y_{p}, u_{p}, \) and \( v_{p}, \) and some iterative method has to be used to find the solution.

The set of equations (2-24)–(2-27) or the set (2-28)–(2-31) can be solved numerically by using a number of techniques. A straightforward numerical procedure is given below.

2-3 Specified Time (y-direction) Intervals

In this process it is assumed that \( u \) and \( v \) are known on the line \( y = Y \) and are to be found on the line \( y = Y + \Delta y. \) Let \( P \) be a typical point on the line \( y = Y + \Delta y \) and \( A, B, C \) be three adjacent points on the line \( y = Y, \) as shown in Fig. 2-2. Let the \( C_{+} \) characteristic at \( P \) intersect \( ACB \) at \( R \) and the \( C_{-} \) characteristic at \( P \) intersect \( ACB \) at \( S. \)

Since \( x_{p} \) and \( y_{p} \) are known, \( u_{p} \) and \( v_{p} \) are to be found. The computation proceeds as follows:

1. The equations

\[
\begin{align*}
y_{p} - y_{R} &= (\zeta_{+})_{C} (x_{p} - x_{R}), \quad (2-32) \\
y_{p} - y_{S} &= (\zeta_{-})_{C} (x_{p} - x_{S}), \quad (2-33)
\end{align*}
\]

give the \( x \)-coordinates of \( R \) and \( S, \) respectively.
2. Using a formula for linear interpolation $u_R$, $v_R$, $u_S$, $v_S$ can be found from:

$$u_R = u_C[1 - (\zeta_+)_C^{-1}\theta] + u_A(\zeta_+)_C^{-1}\theta, \quad (2-34)$$

$$v_R = v_C[1 - (\zeta_+)_C^{-1}\theta] + v_A(\zeta_+)_C^{-1}\theta, \quad (2-35)$$

$$u_S = u_C[1 - (\zeta_-)_C^{-1}\theta] + u_B(\zeta_-)_C^{-1}\theta, \quad (2-36)$$

$$v_S = v_C[1 - (\zeta_-)_C^{-1}\theta] + v_B(\zeta_-)_C^{-1}\theta, \quad (2-37)$$

where $\theta = \Delta y/\Delta x$.

3. $u_P$ and $v_P$ can now be obtained by solving simultaneously the following two equations:

$$F_C(u_p - u_R) + (a\xi_+ - G)_C(v_p - v_R) + (K\xi_+ - H)_C(x_p - x_R) = 0, \quad (2-38)$$

$$F_C(u_p - u_S) + (a\xi_- - G)_C(v_p - v_S) + (K\xi_- - H)_C(x_p - x_S) = 0. \quad (2-39)$$

Equations (2-32)–(2-39) form a process with first-order accuracy. If a higher degree of accuracy is required, the values of $u_P$ and $v_P$ obtained from steps 1, 2, and 3 can be used as initial estimates for the second-order process, which is as follows:

4. The equations

$$y_P - y_R - \frac{1}{2} [((\xi_+_R)^{(k)} + (\xi_+_R)^{(k)})](x_p - x_R^{(k+1)}) = 0, \quad (2-40)$$

$$y_P - y_S - \frac{1}{2} [((\xi_-S)^{(k)} + (\xi_-S)^{(k)})](x_p - x_S^{(k+1)}) = 0, \quad (2-41)$$

give the $x$-coordinates of $R$ and $S$ at the $(k+1)$st iteration.
5. Using a formula for quadratic interpolation (Appendix A-1), \( u_R^{(k+1)} \), \( v_R^{(k+1)} \), \( u_S^{(k+1)} \), \( v_S^{(k+1)} \) can be calculated as

\[
\begin{align*}
  u_R^{(k+1)} &= u_C + \frac{u_B - u_A}{2\Delta x} (x_R^{(k+1)} - x_C) + \frac{u_A + u_B - 2u_C}{2(\Delta x)^2} (x_R^{(k+1)} - x_C)^2, \\
  v_R^{(k+1)} &= v_C + \frac{v_B - v_A}{2\Delta x} (x_R^{(k+1)} - x_C) + \frac{v_A + v_B - 2v_C}{2(\Delta x)^2} (x_R^{(k+1)} - x_C)^2, \\
  u_S^{(k+1)} &= u_C + \frac{u_B - u_A}{2\Delta x} (x_S^{(k+1)} - x_C) + \frac{u_A + u_B - 2u_C}{2(\Delta x)^2} (x_S^{(k+1)} - x_C)^2, \\
  v_S^{(k+1)} &= v_C + \frac{v_B - v_A}{2\Delta x} (x_S^{(k+1)} - x_C) + \frac{v_A + v_B - 2v_C}{2(\Delta x)^2} (x_S^{(k+1)} - x_C)^2.
\end{align*}
\] (2-42) (2-43) (2-44) (2-45)

6. \( u_p^{(k+1)} \) and \( v_p^{(k+1)} \) can then be obtained by solving simultaneously the equations:

\[
\begin{align*}
  \frac{1}{2} (F_R^{(k+1)} + F_p^{(k)}) (u_p^{(k+1)} - u_R^{(k+1)}) + \frac{1}{2} [(a_\zeta_+ - G)_R^{(k+1)} + (a_\zeta_+ - G)_p^{(k)}] \\
  (v_p^{(k+1)} - v_R^{(k+1)}) + \frac{1}{2} [(K\zeta_+ - H)_R^{(k+1)} + (K\zeta_+ - H)_p^{(k)}] (x_p - x_R^{(k+1)}) &= 0, \\
  \frac{1}{2} (F_S^{(k+1)} + F_p^{(k)}) (u_p^{(k+1)} - u_S^{(k+1)}) + \frac{1}{2} [(a_\zeta_- - G)_S^{(k+1)} + (a_\zeta_- - G)_p^{(k)}] \\
  (v_p^{(k+1)} - v_S^{(k+1)}) + \frac{1}{2} [(K\zeta_- - H)_S^{(k+1)} + (K\zeta_- - H)_p^{(k)}] (x_p - x_S^{(k+1)}) &= 0.
\end{align*}
\] (2-46) (2-47)

2-4 Boundary Points

A special, useful case of the boundary is discussed here. It is assumed that the equation of the boundary is given by \( x = \text{constant} \) and that \( u \) is related to \( v \) on the boundary by

\[
  v = f(u),
\] (2-48)

where \( f \) is a known function.

The values of \( u \) and \( v \) at A and B in Fig. 2-3 are assumed known.

1. \( x_S \) can be obtained from the relation

\[
x_S = x_A - (\zeta_-)_A^{-1} (y_A - y_S).
\]

2. \( u_S \) and \( v_S \) are obtained using linear interpolation.
3. $u_M$ and $v_M$ are calculated by solving

$$F_A(u_M - u_S) + (a\zeta - G)_A(v_M - v_S) + (K\zeta - H)_A(x_M - x_S) = 0,$$

(2-51)

together with

$$v_M = f(u_M).$$

(2-52)

The formulas for the second-order process are as follows:

5. $x_S^{(k+1)}$ can be calculated from the equation

$$y_M - y_S = \frac{1}{2} [((\zeta)_M^{(k)} + (\zeta)_S^{(k)})(x_M - x_S^{(k+1)});$$

(2-53)

6. $u_S^{(k+1)}$ and $v_S^{(k+1)}$ can be found using

$$u_S^{(k+1)} = u_B + \frac{u_C - u_A}{2\Delta x} (x_S^{(k+1)} - x_B) + \frac{u_C + u_A - 2u_B}{2(\Delta x)^2} (x_S^{(k+1)} - x_B)^2,$$

(2-54)

$$v_S^{(k+1)} = v_B + \frac{v_C - v_A}{2\Delta x} (x_S^{(k+1)} - x_B) + \frac{v_C + v_A - 2v_B}{2(\Delta x)^2} (x_S^{(k+1)} - x_B)^2.$$

(2-55)
7. \( u_M^{(k+1)} \) and \( v_M^{(k+1)} \) can be calculated from

\[
\frac{1}{2} (F_{S}^{(k+1)} + F_{M}^{(k)}) (u_M^{(k+1)} - u_S^{(k+1)}) + \frac{1}{2} [(a\zeta_+ - G)_S^{(k+1)} + (a\zeta_+ - G)_M^{(k)}] \\
(v_M^{(k+1)} - v_S^{(k+1)}) + \frac{1}{2} [(K\zeta_+ - H)_S^{(k+1)} + (K\zeta_+ - H)_M^{(k)}] (x_M^{(k+1)} - x_S^{(k+1)}) = 0 ,
\]

(2-56)

together with

\[ v_M^{(k+1)} = f(u_M^{(k+1)}). \]

(2-57)

### 2.5 Extrapolation Procedures

When specified time intervals are used, it is possible to employ extrapolation procedures to increase the accuracy of the computation.

Consider a function \( f(x,y) \) which is to be determined at \( y = 2n \Delta y \), in terms of its known value at \( y = 0 \). This may be accomplished in \( n \) steps of \( 2\Delta y \) by repeating a linear process at a constant value of \( x \). Let this value of \( f(x,y) \) be denoted by \( f_2(2n \Delta y) \). Alternatively, \( 2n \) steps of length \( \Delta y \) can be used. Let \( f_1(2n \Delta y) \) denote this value of \( f(x,y) \).

Then, if \( f(2n \Delta y) \) is computed, where

\[
\overline{f}(2n \Delta y) = 2 f_1(2n \Delta y) - f_2(2n \Delta y), \quad (2-58)
\]

then \( \overline{f}(2n \Delta y) \) and the true value \( f(2n \Delta y) \) agree when terms of order \( (\Delta y)^2 \) are neglected. Further, it can be shown that after \( n \) steps, if \( n \Delta y = o(1) \), then the error is of \( o(\Delta y)^2 \).

If it is desired to eliminate higher order errors, then one of the following methods can be used:

(i) Compute \( f(2n \Delta y) \) for three steps, \( 2\Delta y, \Delta y, 0.5\Delta y \). If \( f^{1/2}(2n \Delta y) \) denotes the value of \( f(2n \Delta y) \) computed by the linear process in \( 4n \) steps of length \( 0.5\Delta y \), then \( \overline{f}(2n \Delta y) \) given by

\[
\overline{f}(2n \Delta y) = \frac{8}{3} f^{1/2}(2n \Delta y) - 2 f_1(2n \Delta y) + \frac{1}{3} f^2(2n \Delta y) \quad (2-59)
\]

and the true value of \( f(2n \Delta y) \) are related by

\[
\overline{f}(2n \Delta y) = f(2n \Delta y) + o(\Delta y)^3 ; \quad (2-60)
\]

(ii) Compute \( f(2n \Delta y) \) for two steps, \( 2\Delta y \) and \( \Delta y \), using the second-order process given by (2-32)–(2-47). Then, \( \overline{f}(2n \Delta y) \) computed from the relation

\[
\overline{f}(2n \Delta y) = \frac{1}{3} [4 f_1(2n \Delta y) - f^2(2n \Delta y)] \quad (2-61)
\]

is again related to \( f(2n \Delta y) \) by (2-60).
Chapter 3

Extension of the Method of Characteristics to a System of 2n Partial Differential Equations

In this chapter, the method introduced in the previous chapter is extended to a system of matrix equations. In the next chapter it will be shown that the equations for the voltage and current vectors on n coupled lossy transmission lines can be transformed to a special case of these equations. The arguments closely follow those of the previous chapter.

3-1 Derivation of the Method

Consider the following system of matrix equations:

\[
\begin{align*}
L_1 &= A_1 u_x + B_1 u_y + C_1 v_x + D_1 v_y + E_1 = 0, \\
L_2 &= A_2 u_x + B_2 u_y + C_2 v_x + D_2 v_y + E_2 = 0.
\end{align*}
\]

(3-1)  
(3-2)

Here \( x \) and \( y \) are the independent variables, \( u \) and \( v \) are n-dimensional column matrices (vectors) of the dependent variables \( u_1(x,y), u_2(x,y), \ldots, u_n(x,y), v_1(x,y), v_2(x,y), \ldots, v_n(x,y) \); \( u_x = \frac{\partial u}{\partial x}, \ u_y = \frac{\partial u}{\partial y}, \ v_x = \frac{\partial v}{\partial x}, \ v_y = \frac{\partial v}{\partial y} \) and \( A_1, B_1, \ldots, E_2 \) are \( n \times n \) real matrices whose elements may be functions of \( x, y, u, v \). It is also assumed that the matrices \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 \) are diagonal.

If

\[ x = x_1(y), x = x_2(y), \ldots, x = x_n(y) \]

(3-3)

are the equations of \( n \) curves in the \((x,y)\) plane, then \( dx_i/dy \) (\( i = 1, 2, \ldots, n \)) are the slopes of the tangents to these curves and along each curve of (3-3)

\[
\begin{align*}
&du_i = \frac{\partial u_i}{\partial x} dx_i + \frac{\partial u_i}{\partial y} dy, \\
&dv_i = \frac{\partial v_i}{\partial x} dx_i + \frac{\partial v_i}{\partial y} dy, \quad (i = 1, 2, \ldots, n).
\end{align*}
\]

(3-4)
The above equations can be written in matrix form as

\[
\begin{align*}
\frac{du}{dx} = & \ dx \ \frac{\partial u}{\partial x} + dy \ \frac{\partial u}{\partial y}, \\
\frac{dv}{dx} = & \ dx \ \frac{\partial v}{\partial x} + dy \ \frac{\partial v}{\partial y},
\end{align*}
\]  

(3-5)

where

\[
\begin{bmatrix}
dx_1 & 0 & \ldots & 0 \\
0 & dx_2 & 0 & \ldots & 0 \\
\vdots & & \ddots & & \vdots \\
0 & \ldots & 0 & dx_n
\end{bmatrix}
\]

\[
dx = \text{diag}(dx_1, \ldots, dx_n)
\]  

(3-6)

Let \( L \) be a combination of \( L_1 \) and \( L_2 \):

\[
L = \lambda_1 L_1 + \lambda_2 L_2 = (\lambda_1 A_1 + \lambda_2 A_2)u_x + (\lambda_1 B_1 + \lambda_2 B_2)u_y + \\
(\lambda_1 C_1 + \lambda_2 C_2)v_x + (\lambda_1 D_1 + \lambda_2 D_2)v_y + \lambda_1 E_1 + \lambda_2 E_2,
\]  

(3-7)

where

\[
\lambda_1 = \text{diag}(\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(n)}),
\]

\[
\lambda_2 = \text{diag}(\lambda_2^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_2^{(n)}),
\]

\[
\lambda_1^{(j)}, \lambda_2^{(j)} \in \mathbb{R}, \ (j = 1, 2, \ldots, n).
\]  

(3-8)

(3-9)

Using (3-5), (3-7) can be written as:

\[
dx L = (\lambda_1 A_1 + \lambda_2 A_2)(du - dy \ uy) + (\lambda_1 B_1 + \lambda_2 B_2)dx \ uy + \\
(\lambda_1 C_1 + \lambda_2 C_2)(dv - dy \ vy) + (\lambda_1 D_1 + \lambda_2 D_2)dx \ vy + dx(\lambda_1 E_1 + \lambda_2 E_2). 
\]  

(3-10)

Now, if

\[
(\lambda_1 A_1 + \lambda_2 A_2) \ dy = (\lambda_1 B_1 + \lambda_2 B_2) \ dx,
\]

(3-11)

\[
(\lambda_1 C_1 + \lambda_2 C_2) \ dy = (\lambda_1 D_1 + \lambda_2 D_2) \ dx,
\]

(3-12)

or

\[
\frac{dx}{dy} = (\lambda_1 B_1 + \lambda_2 B_2)^{-1}(\lambda_1 A_1 + \lambda_2 A_2),
\]

(3-13)

\[
\frac{dx}{dy} = (\lambda_1 D_1 + \lambda_2 D_2)^{-1}(\lambda_1 C_1 + \lambda_2 C_2),
\]

(3-14)

where

\[
\frac{dx}{dy} = \text{diag}(\frac{dx_1}{dy}, \frac{dx_2}{dy}, \ldots, \frac{dx_n}{dy}),
\]

(3-15)

then (3-10) becomes
\[ dx \mathbf{L} = (\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2)dx + (\lambda_1 \mathbf{C}_1 + \lambda_2 \mathbf{C}_2)dy + dx(\lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2). \] (3-16)

The conditions (3-13), (3-14) give \( \lambda_1 \lambda_2^{-1} \) as

\[ \lambda_1 \lambda_2^{-1} = - (A_2 dy - B_2 dx)(A_1 dy - B_1 dx)^{-1} = \]
\[- (C_2 dy - D_2 dx)(C_1 dy - D_1 dx)^{-1}. \] (3-17)

The above expression gives:

\[ a (dy)^2 - b dx \ dy + c (dx)^2 = 0, \] (3-18)

where

\[ a = A_1 C_2 - A_2 C_1, \ b = A_1 D_2 - A_2 D_1, \ c = B_1 D_2 - B_2 D_1. \] (3-19)

If a matrix \( \xi = \frac{dx}{dy} \) of slopes to the curves (3-3) is introduced, then \( \xi \) satisfies the equation

\[ c \xi^2 - b \xi + a = 0. \] (3-20)

The above equation has two different solutions \( \xi_+ \neq \xi_- \), where

\[ \xi_+ = \text{diag} (\xi_+^{(1)}, \xi_+^{(2)}, ..., \xi_+^{(n)}), \]
\[ \xi_- = \text{diag} (\xi_-^{(1)}, \xi_-^{(2)}, ..., \xi_-^{(n)}), \quad \xi_+^{(i)}, \xi_-^{(i)} \in \mathbb{R}, \ (i = 1, 2, ..., n). \] (3-21)

Thus, \( \xi_+ \) and \( \xi_- \) give the characteristic slopes at the point \((x, y)\)

\[ \frac{dx}{dy} = \xi_+, \quad \frac{dx}{dy} = \xi_- \] (3-23)

and since \( a, b, c \) are in general functions of \( x, y, u, v \), \( \xi_+ \) and \( \xi_- \) will also be functions of these quantities:

\[ \frac{dx_i}{dy} = \xi_+^{(i)}(x, y, u_i, v_i), \quad \frac{dx_i}{dy} = \xi_-^{(i)}(x, y, u_i, v_i), \ (i = 1, 2, ..., n). \] (3-24)

Equations (3-24) define \( 2n \) families of characteristic curves, \( C_+^{(1)}, C_+^{(2)}, ..., C_+^{(n)} \), \( C_-^{(1)}, C_-^{(2)}, ..., C_-^{(n)} \).

Substituting the solutions (3-23) into the expression for \( \lambda_1 \lambda_2^{-1} \) given in (3-17) yields

\[ \lambda_1 \lambda_2^{-1} = - (A_2 - B_2 \xi_+)(A_1 - B_1 \xi_+)^{-1}, \] (3-25)
\[ \lambda_1 \lambda_2^{-1} = - (A_2 - B_2 \xi_-)(A_1 - B_1 \xi_-)^{-1}. \] (3-26)

If two new matrixes \( \xi_+, \xi_- \) are introduced, where
\( \zeta_+ = \zeta_+^{-1}, \zeta_- = \zeta_-^{-1}, \) \hspace{1cm} (3-27)

(3-25) and (3-26) can be written as

\[
\begin{align*}
\lambda_1 \lambda_2^{-1} &= -(A_2 \zeta_+ - B_2)(A_1 \zeta_+ - B_1)^{-1}, \hspace{1cm} (3-28) \\
\lambda_1 \lambda_2^{-1} &= -(A_2 \zeta_- - B_2)(A_1 \zeta_- - B_1)^{-1}. \hspace{1cm} (3-29)
\end{align*}
\]

Finally, combining (3-28), (3-29) and (3-16) gives (Appendix A-2)

\[
\begin{align*}
F \, du + (\zeta_+ a - G) \, dv + dx (\zeta_+ K - H) &= 0, \hspace{1cm} (3-30) \\
F \, du + (\zeta_- a - G) \, dv + dx (\zeta_- K - H) &= 0, \hspace{1cm} (3-31)
\end{align*}
\]

where

\[
\begin{align*}
F &= A_1 B_2 - A_2 B_1, \hspace{0.5cm} G = B_1 C_2 - B_2 C_1, \\
K &= A_1 E_2 - A_2 E_1, \hspace{0.5cm} H = B_1 E_2 - B_2 E_1. \hspace{1cm} (3-32)
\end{align*}
\]

Thus, the following four characteristic equations have been obtained:

\[
\begin{align*}
I \, dy - \zeta_+ \, dx &= 0, \hspace{1cm} (3-33) \\
F \, du + (\zeta_+ a - G) \, dv + dx (\zeta_+ K - H) &= 0, \hspace{1cm} (3-34) \\
I \, dy - \zeta_- \, dx &= 0, \hspace{1cm} (3-35) \\
F \, du + (\zeta_- a - G) \, dv + dx (\zeta_- K - H) &= 0, \hspace{1cm} (3-36)
\end{align*}
\]

where the first two equations hold along \((C_+^{(1)}, C_+^{(2)}, \ldots, C_+^{(n)})\) and the second two along \((C_-^{(1)}, C_-^{(2)}, \ldots, C_-^{(n)})\). Here, I is the identity matrix.

### 3-2 Specified Time (y-direction) Intervals

It is assumed that \(u_i, v_i\) \((i = 1, 2, \ldots, n)\) are known on the line \(y = Y\), either as given initial conditions or as the results of a previous stage of calculations. Let \(P\) be a typical point on the line \(y = Y + \Delta y\) and \(A, B, C\) be three adjacent points on the line \(y = Y\). Let the \(C_+^{(i)}\) characteristic through \(P\) intersect \(ACB\) at \(R_i\) and the \(C_-^{(i)}\) characteristic through \(P\) intersect \(ACB\) at \(S_i\) \((i = 1, 2, \ldots, n)\). Fig. 3-1 shows this for the first characteristic \((i = 1)\).

Since \(x_p\) and \(y_p\) are known, \(u_p\) and \(v_p\) are to be found. The computation proceeds as follows:

Using (3-22) for each row of a matrix equation, the linear approximation to (3-33)-(3-36) becomes

\[
I(y_p - y_{R_i}) - (\zeta_+\rangle_{R_i}(x_p - x_{R_i}) = 0, \hspace{1cm} (3-37)
\]
\[
\begin{align*}
F_{R_i}(u_p - u_{R_i}) + (\zeta_+ a - G)_{R_i}(v_p - v_{R_i}) + (x_p - x_{R_i})(\zeta_+ K - H)_{R_i} &= 0, \\
I(y_p - y_{S_i}) - (\zeta_-)_{S_i}(x_p - x_{S_i}) &= 0, \\
F_{S_i}(u_p - u_{S_i}) + (\zeta_- a - G)_{S_i}(v_p - v_{S_i}) + (x_p - x_{S_i})(\zeta_- K - H)_{S_i} &= 0,
\end{align*}
\]

where \( x = \text{diag}(x_1, x_2, ..., x_n) \) and \( A_{P_i} \) means evaluating row \( i \) of matrix \( A \) at point \( P_i \) \((i = 1, 2, ..., n)\).

Figure 3-1: Computation of \( u \) and \( v \) at a point on the line \( y = Y + \Delta y \)

1. Equations (3-37) and (3-39) give the x-coordinates of \( R_i \) and \( S_i \) \((i = 1, 2, ..., n)\).

2. Using a formula for linear interpolation, \( u_{R_i}, v_{R_i}, u_{S_i}, v_{S_i} \) can be found from:

\[
\begin{align*}
  u_{R_i} &= [I - (\zeta_+)C^{-1}\theta]u_C + (\zeta_+)C^{-1}\theta u_A, \\
  v_{R_i} &= [I - (\zeta_+)C^{-1}\theta]v_C + (\zeta_+)C^{-1}\theta v_A, \\
  u_{S_i} &= [I - (\zeta_-)C^{-1}\theta]u_C + (\zeta_-)C^{-1}\theta u_B, \\
  v_{S_i} &= [I - (\zeta_-)C^{-1}\theta]v_C + (\zeta_-)C^{-1}\theta v_B,
\end{align*}
\]

where \( \theta = I \Delta y/\Delta x \).

3. \( u_p \) and \( v_p \) can now be obtained by solving simultaneously (3-38) and (3-40).

The above three steps form a process with first-order accuracy. For a higher degree of accuracy, the values of \( u_p \) and \( v_p \) obtained from steps 1, 2, 3 can be used as initial estimates for the second-order process, which is as follows:

A second-order approximation to (3-33)–(3-36) is
\[ I(y_p - y_{R_i}) - \frac{1}{2} \left[ (\zeta_+)_R + (\zeta_+)_p \right] (x_p - x_{R_i}) = 0, \quad (3-45) \]
\[ \frac{1}{2} (F_{R_i} + F_p) (u_p - u_{R_i}) + \frac{1}{2} \left[ (\zeta_- a - G)_{R_i} + (\zeta_- a - G)_p \right] (v_p - v_{R_i}) + \]
\[ + \left( x_p - x_{R_i} \right) \frac{1}{2} \left[ (\zeta_+ K - H)_{R_i} + (\zeta_+ K - H)_p \right] = 0, \quad (3-46) \]
\[ I(y_p - y_{S_i}) - \frac{1}{2} \left[ (\zeta_-)_{S_i} + (\zeta_-)_p \right] (x_p - x_{S_i}) = 0, \quad (3-47) \]
\[ \frac{1}{2} (F_{S_i} + F_p) (u_p - u_{S_i}) + \frac{1}{2} \left[ (\zeta_- a - G)_{S_i} + (\zeta_- a - G)_p \right] (v_p - v_{S_i}) + \]
\[ + \left( x_p - x_{S_i} \right) \frac{1}{2} \left[ (\zeta_- K - H)_{S_i} + (\zeta_- K - H)_p \right] = 0. \quad (3-48) \]

4. Equations (3-45) and (3-47) written as
\[ I(y_p - y_{R_i}) - \frac{1}{2} \left[ (\zeta_+)^{(k)}_{R_i} + (\zeta_+)^{(k)}_p \right] (x_p - x_{R_i}^{(k+1)}) = 0, \quad (3-49) \]
\[ I(y_p - y_{S_i}) - \frac{1}{2} \left[ (\zeta_-)^{(k)}_{S_i} + (\zeta_-)^{(k)}_p \right] (x_p - x_{S_i}^{(k+1)}) = 0, \quad (3-50) \]
give the x-coordinates of \( R_i \) and \( S_i \) at the \( (k + 1) \)st iteration.

5. Using a formula for quadratic interpolation \( u_{R_i}^{(k+1)}, v_{R_i}^{(k+1)}, u_{S_i}^{(k+1)}, v_{S_i}^{(k+1)} \) can be calculated as
\[ u_{R_i}^{(k+1)} = u_C + \frac{1}{2\Delta x} (x_{R_i}^{(k+1)} - x_C) (u_B - u_A) + \]
\[ + \frac{1}{2(\Delta x)^2} (x_{R_i}^{(k+1)} - x_C)^2 (u_A + u_B - 2u_C), \quad (3-51) \]
\[ v_{R_i}^{(k+1)} = v_C + \frac{1}{2\Delta x} (x_{R_i}^{(k+1)} - x_C) (v_B - v_A) + \]
\[ + \frac{1}{2(\Delta x)^2} (x_{R_i}^{(k+1)} - x_C)^2 (v_A + v_B - 2v_C), \quad (3-52) \]
\[ u_{S_i}^{(k+1)} = u_C + \frac{1}{2\Delta x} (x_{S_i}^{(k+1)} - x_C) (u_B - u_A) + \]
\[ + \frac{1}{2(\Delta x)^2} (x_{S_i}^{(k+1)} - x_C)^2 (u_A + u_B - 2u_C), \quad (3-53) \]
\[ v_{S_i}^{(k+1)} = v_C + \frac{1}{2\Delta x} (x_{S_i}^{(k+1)} - x_C) (v_B - v_A) + \]
\[ + \frac{1}{2(\Delta x)^2} (x_{S_i}^{(k+1)} - x_C)^2 (v_A + v_B - 2v_C). \quad (3-54) \]
6. $u_p^{(k+1)}$ and $v_p^{(k+1)}$ can then be obtained by solving simultaneously the equations:

$$\frac{1}{2} (F_{R_i}^{(k+1)} + F_{P}^{(k)}) (u_{R_i}^{(k+1)} - u_{R_i}^{(k+1)}) + \frac{1}{2} \left[ (\zeta_+ a - G)^{R_i}_{(k+1)} + (\zeta_+ a - G)^{P}_{(k)} \right]$$

$$\left( v_{p}^{(k+1)} - v_{R_i}^{(k+1)} \right) + (x_p - x_{R_i}^{(k+1)}) \frac{1}{2} \left[ (\zeta_+ K - H)^{R_i}_{(k+1)} + (\zeta_+ K - H)^{P}_{(k)} \right] = 0, \quad (3-55)$$

$$\frac{1}{2} (F_{S_i}^{(k+1)} + F_{P}^{(k)}) (u_{P}^{(k+1)} - u_{S_i}^{(k+1)}) + \frac{1}{2} \left[ (\zeta_- a - G)^{S_i}_{(k+1)} + (\zeta_- a - G)^{P}_{(k)} \right]$$

$$\left( v_{p}^{(k+1)} - v_{S_i}^{(k+1)} \right) + (x_p - x_{S_i}^{(k+1)}) \frac{1}{2} \left[ (\zeta_- K - H)^{S_i}_{(k+1)} + (\zeta_- K - H)^{P}_{(k)} \right] = 0. \quad (3-56)$$

### 3-3 Boundary Points

It is assumed that the equation of the boundary is given by $x = \text{constant}$ and that $u$ is related to $v$ on the boundary by

$$v = f(u), \quad (3-57)$$

where $f$ is a known vector function.

![Figure 3-2: Computation of $u$ and $v$ at the boundary](image)

The values of $u$ and $v$ at A and B in Fig. 3-2 are assumed known.

1. $x_{S_i}$ can be obtained from the relation
\[ x_{S_i} = x_A - (\zeta_-)_A^{-1} I \( y_A - y_{S_i} \). \] (3-58)

2. \( u_{S_i} \) and \( v_{S_i} \) are obtained using linear interpolation

\[ u_{S_i} = u_A + \frac{1}{\Delta x} \( x_{S_i} - x_A \)(u_B - u_A), \] (3-59)

\[ v_{S_i} = v_A + \frac{1}{\Delta x} \( x_{S_i} - x_A \)(v_B - v_A). \] (3-60)

3. \( u_M \) and \( v_M \) are calculated by solving

\[ F_A(u_M - u_{S_i}) + (\zeta_- a - G)_A(v_M - v_{S_i}) + (x_{S_i} - x_A)(\zeta_- K - H)_A = 0, \] (3-61)

together with

\[ v_M = f(u_M). \] (3-62)

The formulas for the second-order process are as follows:

5. \( x_{S_i}^{(k+1)} \) can be calculated from the equation

\[ I(y_M - y_{S_i}) = \frac{1}{2} \( x_M - x_{S_i}^{(k+1)} \)[(\zeta_-)^{(k)}_M + (\zeta_-)^{(k)}_{S_i}] \]. \] (3-63)

6. \( u_{S_i}^{(k+1)} \) and \( v_{S_i}^{(k+1)} \) can be found using

\[ u_{S_i}^{(k+1)} = u_B + \frac{1}{2\Delta x} \( x_{S_i}^{(k+1)} - x_B \)(u_C - u_A) + \]
\[ + \frac{1}{2(\Delta x)^2} \( x_{S_i}^{(k+1)} - x_B \)^2(u_C + u_A - 2u_B), \] (3-64)

\[ v_{S_i}^{(k+1)} = v_B + \frac{1}{2\Delta x} \( x_{S_i}^{(k+1)} - x_B \)(v_C - v_A) + \]
\[ + \frac{1}{2(\Delta x)^2} \( x_{S_i}^{(k+1)} - x_B \)^2(v_C + v_A - 2v_B). \] (3-65)

7. \( u_M^{(k+1)} \) and \( v_M^{(k+1)} \) can be calculated from

\[ \frac{1}{2} \( F_{S_i}^{(k+1)} + F_M^{(k)} \)(u_M^{(k+1)} - u_{S_i}^{(k+1)}) + \frac{1}{2} [(\zeta_- a - G)^{(k+1)}_{S_i} + (\zeta_- a - G)^{(k)}_M] \]
\[ (v_M^{(k+1)} - v_{S_i}^{(k+1)}) + \frac{1}{2} [(\zeta_- K - H)^{(k+1)}_{S_i} + (\zeta_- K - H)^{(k)}_M](x_M - x_{S_i}^{(k+1)}) = 0, \] (3-66)

together with

\[ v_M^{(k+1)} = f(u_M^{(k+1)}). \] (3-67)
Chapter 4

Application of the Method of Characteristics to Transmission Line Equations

In this chapter, the method developed in Chapter 3 is used to solve for the voltages and currents on a general lossy multiconductor transmission line. Both uniform and nonuniform lines are considered. The technique is exemplified by computing the time domain response of typical systems terminated in linear as well as nonlinear elements.

4-1 Problem Formulation

The voltages and currents on a set of \( n \) coupled transmission lines are given by

\[
\frac{\partial \mathbf{e}}{\partial x} + L \frac{\partial \mathbf{i}}{\partial t} + \mathbf{R} \mathbf{i} = 0, \tag{4-1}
\]

\[
\frac{i \partial \mathbf{i}}{\partial x} + C \frac{\partial \mathbf{e}}{\partial t} + \mathbf{G} \mathbf{e} = 0, \tag{4-2}
\]

together with the boundary conditions at \( x = 0 \) and \( x = d \). Here, \( \mathbf{e} = \mathbf{e}(x,t) \) and \( \mathbf{i} = \mathbf{i}(x,t) \) are \( n \)-dimensional voltage and current vectors, respectively, and \( \mathbf{R}, \mathbf{L}, \mathbf{G}, \mathbf{C} \) are \( n \times n \) real symmetric matrixes.

Let the boundary conditions (terminations) be given by

\[
\mathbf{i}(0, t) = f_{\text{in}}(\mathbf{v}_g(t), \mathbf{e}(0, t), t), \tag{4-3}
\]

\[
\mathbf{i}(d, t) = f_{\text{out}}(\mathbf{v}_s(t), \mathbf{e}(d, t), t), \tag{4-4}
\]

where \( \mathbf{v}_g(t) \) and \( \mathbf{v}_s(t) \) are the vectors of generator voltages at the input (near) and output (far) ends of the line and \( f_{\text{in}} \) and \( f_{\text{out}} \) are known, reasonably behaved vector functions.

Given \( \mathbf{e}(x, 0) \) and \( \mathbf{i}(x, 0) \) \( \forall x : x \in [0, d] \), the problem is to find \( \mathbf{e}(x, t) \) and \( \mathbf{i}(x, t) \) for particular \( x \) and \( t \).
4-2 Numerical Solution

In order to use the method developed in chapters 1 and 2 to solve the system (4-1)–(4-2), it first has to be transformed into a "decoupled" system whose L and C matrixes are diagonal. This can be done as follows:

Let \( e_T \) and \( i_T \) be new, transformed voltage and current vectors related to \( e, i \) by

\[
e = E e_T, \quad i = H i_T,
\]

(4-5)

where \( E \) and \( H \) are \( n \times n \) real matrixes.

Substituting (4-5) into (4-1)–(4-2) gives

\[
\begin{align*}
E \frac{\partial e_T}{\partial x} + LH \frac{\partial i_T}{\partial t} + RH i_T &= 0, \\
H \frac{\partial i_T}{\partial x} + CE \frac{\partial e_T}{\partial t} + GE e_T &= 0,
\end{align*}
\]

(4-6) (4-7)

or

\[
\begin{align*}
\frac{\partial e_T}{\partial x} + LT \frac{\partial i_T}{\partial t} + RT i_T &= 0, \\
\frac{\partial i_T}{\partial x} + CT \frac{\partial e_T}{\partial t} + GT e_T &= 0,
\end{align*}
\]

(4-8) (4-9)

where

\[
L_T = E^{-1} L H, \\
R_T = E^{-1} R H, \\
C_T = H^{-1} C E, \\
G_T = H^{-1} G E.
\]

(4-10) (4-11) (4-12) (4-13)

The boundary conditions (4-3), (4-4) transform to

\[
i_T(0, t) = H^{-1} f_{in}(v_s(t), E e_T(0, t), t), \]

(4-14)

\[
i_T(d, t) = H^{-1} f_{out}(v_s(t), E e_T(d, t), t).
\]

(4-15)

The matrixes \( E \) and \( H \) can be determined from the requirement that \( L_T \) and \( C_T \) have to be diagonal. Using (4-10) and (4-12):

\[
(LC) \; E = E (L_T C_T). \\
(CL) \; H = H (C_T L_T).
\]

(4-16) (4-17)
It is seen from (4-16) and (4-17) that $E$ is the eigenvector matrix of $LC$ and $H$ is the eigenvector matrix of $CL$. Since $L$ and $C$ are symmetrical matrices, $H$ is related to $E$ by

$$H = (E^T)^{-1},$$

(4-18)

where $E^T$ denotes the transpose of $E$. The above expression can be obtained by transposing (4-17) and using $(CL)^T = LC$.

Equations (4-8), (4-9) have the same form as (3-1) and (3-2) with

$$x = x, \quad y = t, \quad u = e_T, \quad v = i_T,$$

$$A_1 = I, \quad B_1 = C_1 = 0, \quad D_1 = L_T, \quad E_1 = R_T i_T,$$

$$A_2 = D_2 = 0, \quad B_2 = C_T, \quad C_2 = I, \quad E_2 = G_T e_T.$$

(4-19)

Therefore, the method given in Chapter 2 may be directly applied.

Equation (4-20) for the characteristic directions becomes

$$-(C_T L_T) \xi^2 + I = 0,$$

(4-20)

and the two solutions are

$$\xi_+ = (C_T L_T)^{-1/2}, \quad \xi_- = -(C_T L_T)^{-1/2},$$

(4-21)

giving

$$\zeta_+ = (C_T L_T)^{1/2}, \quad \zeta_- = -(C_T L_T)^{1/2}.$$  

(4-22)

The system (3-33)–(3-36) becomes

$$I dt - \zeta_+ dx = 0,$$

(4-23)

$$C_T de_T + \zeta_+ di_T + dx (\zeta_+ G_T e_T + C_T R_T i_T) = 0,$$

(4-24)

$$I dt - \zeta_- dx = 0,$$

(4-25)

$$C_T de_T + \zeta_- di_T + dx (\zeta_- G_T e_T + C_T R_T i_T) = 0$$

(4-26)

or

$$I dt - (C_T L_T)^{1/2} dx = 0,$$

(4-27)

$$de_T + Z_0 di_T + dx (Z_0 G_T e_T + R_T i_T) = 0,$$

(4-28)

$$I dt + (C_T L_T)^{1/2} dx = 0,$$

(4-29)

$$de_T - Z_0 di_T + dx (-Z_0 G_T e_T + R_T i_T) = 0,$$

(4-30)

where
\[ Z_0 = C_T^{-1} \zeta = \text{diag}(\sqrt{\frac{L_T^{(1)}}{C_T^{(1)}}}, \sqrt{\frac{L_T^{(2)}}{C_T^{(2)}}}, \ldots, \sqrt{\frac{L_T^{(n)}}{C_T^{(n)}}}). \]  

Here, \( L_T^{(k)}, C_T^{(k)} \) (\( k = 1, 2, \ldots, n \)) are the elements of \( L_T \) and \( C_T \).

If the first order approximation is used for the differentials in (4-27)–(4-30) the procedure from Chapter 2 becomes (Fig 4-1):

**Figure 4-1: Computation of voltages and currents on the line**

I Choose the number of points where the voltages will be calculated (\( m+1 \)). This specifies \( \Delta x \) in Fig. 4-1 as \( \Delta x = d/m \).

II Choose \( \Delta t \) as

\[ \Delta t = \min_{i \in \{1 \ldots n\}} (\Delta x \sqrt{\frac{L_T^{(i)}}{C_T^{(i)}}}). \]  

(4-32)

This increases the accuracy of the method.

III *At points along the line*

1. \( x(R_i) \) and \( x(S_i) \) can be calculated from

\[ I \Delta t - (C_T L_T)^{1/2} (x(P) - x(R_i)) = 0 , \]  

(4-33)

\[ I \Delta t + (C_T L_T)^{1/2} (x(P) - x(S_i)) = 0 , \]  

(4-34)
where \( A(P_i) \) means evaluating row \( i \) of matrix \( A \) at point \( P_i \) (\( i = 1, 2, ..., n \)).

2. \( e_T(R_i), i_T(R_i), e_T(S_i), i_T(S_i) \) can be calculated from known \( e_T \) and \( i_T \) at points C, D, E at time \( t \) using

\[
\begin{align*}
e_T(R_i) &= e_T(D) + \frac{1}{\Delta x} (x(D) - x(R_i))(e_T(C) - e_T(D)), \\
i_T(R_i) &= i_T(D) + \frac{1}{\Delta x} (x(D) - x(R_i))(i_T(C) - i_T(D)), \\
e_T(S_i) &= e_T(D) + \frac{1}{\Delta x} (x(D) - x(S_i))(e_T(D) - e_T(E)), \\
i_T(S_i) &= i_T(D) + \frac{1}{\Delta x} (x(D) - x(S_i))(i_T(D) - i_T(E)).
\end{align*}
\]

3. Approximate (4-28) and (4-30) as

\[
\begin{align*}
e_T(P) - e_T(R_i) + Z_0 \left[ i_T(P) - i_T(R_i) \right] + \\
&\quad + (x(P) - x(R_i)) \left( R_T i_T(R_i) + Z_0 G_T e_T(R_i) \right) = 0, \\
e_T(P) - e_T(S_i) - Z_0 \left[ i_T(P) - i_T(S_i) \right] + \\
&\quad + (x(P) - x(S_i)) \left( R_T i_T(S_i) - Z_0 G_T e_T(S_i) \right) = 0,
\end{align*}
\]

and solve for \( e_T(P) \) and \( i_T(P) \).

The above two equations give \( e_T(P) \) and \( i_T(P) \) as

\[
\begin{align*}
e_T(P) &= \frac{1}{2} \left[ e_T(R_i) + e_T(S_i) + Z_0 \left[ i_T(R_i) - i_T(S_i) \right] - c - d \right], \\
i_T(P) &= \frac{1}{2} \left[ i_T(R_i) + i_T(S_i) + Z_0^{-1} \left[ e_T(R_i) - e_T(S_i) + d - c \right] \right],
\end{align*}
\]

where

\[
\begin{align*}
c &= (x(P) - x(R_i)) \left( R_T i_T(R_i) + Z_0 G_T e_T(R_i) \right), \\
d &= (x(P) - x(S_i)) \left( R_T i_T(S_i) - Z_0 G_T e_T(S_i) \right).
\end{align*}
\]

4. \( e(P) \) and \( i(P) \) can then be obtained from

\[
e(P) = E e_T(P), \quad i(P) = H i_T(P).
\]

IV At \( x = 0 \)

1. Calculate \( x(U_i) \) from

\[
I \Delta t + (C_T L_T)^{1/2} (x(M) - x(U_i)) = 0.
\]
2. Calculate $e_T(U_j), i_T(U_j)$ from

$$
e_T(U_j) = e_T(A) + \frac{1}{\Delta x} (x(U_j) - x(A))(e_T(B) - e_T(A)), \quad (4-47)$$

$$i_T(U_j) = i_T(A) + \frac{1}{\Delta x} (x(U_j) - x(A))(i_T(B) - i_T(A)). \quad (4-48)$$

3. Solve

$$e_T(M) - e_T(U_i) - Z_0 (i_T(M) - i_T(U_i)) +$$

$$+ (x(M) - x(U_i)) (R_T i_T(U_i) - Z_0 G_T e_T(U_j)) = 0, \quad (4-49)$$

together with (4-14) for $e_T(M), i_T(M)$. This can be done as follows:

Equations (4-49) and (4-14) can be written as

$$e_T(M) = Z_0 i_T(M) + b, \quad (4-50)$$

$$i_T(M) = f(e_T(M)), \quad (4-51)$$

where

$$b = e_T(U_i) - Z_0 i_T(U_i) - (x(M) - x(U_i)) (R_T i_T(U_i) - Z_0 G_T e_T(U_j)) \quad (4-52)$$

and

$$f(e_T) = H^{-1} f_{in}(v_g(t), E e_T(t), \forall e_T \in \mathbb{R}^n). \quad (4-53)$$

Equations (4-50) and (4-51) can then be solved by the following iteration (Appendix A-3):

$$i_T^{(k+1)} = [I - \frac{df(e_T^{(k)})}{de_T} Z_0]^{-1} [f(e_T^{(k)}) + \frac{df(e_T^{(k)})}{de_T} (b - e_T^{(k)})], \quad (4-54)$$

$$e_T^{(k+1)} = f^{-1}(i_T^{(k+1)}), \quad (4-55)$$

where $e_T^{(k+1)}, i_T^{(k+1)}$ are the values of $e_T, i_T$ at point M, at the (k+1)st iteration and $f^{-1}$ is the inverse function of f.

4. Transform back to get $e(M), i(M)$:

$$e(M) = E e_T(M), \quad i(M) = H i_T(M). \quad (4-56)$$

V At $x = d$

1. Calculate $x(V_j)$ from

$$I \Delta t - (C_T L_T)^{1/2} (x(N) - x(V_j)) = 0. \quad (4-57)$$
2. Calculate \( e_T(V_i), i_T(V_i) \) from

\[
e_T(V_i) = e_T(G) + \frac{1}{\Delta x} (x(V_i) - x(G))(e_T(G) - e_T(F)),
\]

(4-58)

\[
i_T(V_i) = i_T(G) + \frac{1}{\Delta x} (x(V_i) - x(G))(i_T(G) - i_T(F)).
\]

(4-59)

3. Solve

\[
e_T(N) - e_T(V_i) + Z_0 (i_T(N) - i_T(V_i)) +
+ (x(N) - x(V_i))(R_T i_T(V_i) + Z_0 G_T e_T(V_i)) = 0,
\]

(4-60)

together with (4-15) for \( e_T(N), i_T(N) \).

4. Transform back to get \( e(N), i(N) \):

\[
e(N) = E e_T(N), \quad i(N) = H i_T(N).
\]

(4-61)

The above procedure is then repeated for each time step.

4-3 Nonuniform Lines

In the above procedure the \( R, L, G, C \) matrixes were constant. For a nonuniform multiconductor line these matrixes will be functions of \( x \) and equations (4-1)–(4-2) become

\[
\frac{d e}{dx} + L(x) \frac{d i}{dt} + R(x) i = 0,
\]

(4-62)

\[
\frac{d i}{dx} + C(x) \frac{d e}{dt} + G(x) e = 0.
\]

(4-63)

The boundary conditions are still given by (4-3)–(4-4).

The above system can be transformed to the form of (3-1)–(3-2) with the transformation

\[
e = E(x) e_T, \quad i = H(x) i_T,
\]

(4-64)

where \( E(x) \) is the eigenvector matrix of \( L(x)C(x) \) and \( H(x) = (E^T(x))^{-1} \).

Substituting (4-64) into (4-62)–(4-63) gives

\[
E \frac{d e_T}{dx} + L H \frac{d i_T}{dt} + R H i_T + \frac{dE}{dx} e_T = 0,
\]

(4-65)
\[
H \frac{\partial i_T}{\partial x} + CE \frac{\partial e_T}{\partial t} + GE e_T + \frac{dH}{dx} i_T = 0, \quad (4-66)
\]

or

\[
\frac{\partial e_T}{\partial x} + LT \frac{\partial i_T}{\partial t} + RT i_T + E^{-1} \frac{dE}{dx} e_T = 0, \quad (4-67)
\]

\[
\frac{\partial i_T}{\partial x} + CT \frac{\partial e_T}{\partial t} + GT e_T + H^{-1} \frac{dH}{dx} i_T = 0, \quad (4-68)
\]

where

\[
L_T = L_T(x) = E^{-1}(x) L(x) H(x), \quad (4-69)
\]

\[
R_T = R_T(x) = E^{-1}(x) R(x) H(x), \quad (4-70)
\]

\[
C_T = C_T(x) = H^{-1}(x) C(x) E(x), \quad (4-71)
\]

\[
G_T = G_T(x) = H^{-1}(x) G(x) E(x). \quad (4-72)
\]

Equations (4-67)-(4-68) have the same form as (3-1)-(3-2) with

\[
x = x, \ y = t, \ u = e_T, \ v = i_T,
\]

\[
A_1 = I, \ B_1 = C_1 = 0, \ D_1 = L_T, \ E_1 = R_T i_T + E^{-1} \frac{dE}{dx} e_T,
\]

\[
B_2 = C_T, \ A_2 = D_2 = 0, \ C_2 = I, \ E_2 = G_T e_T + H^{-1} \frac{dH}{dx} i_T. \quad (4-73)
\]

Using the theory of Chapter 3, the system (4-67)-(4-68) becomes

\[
I \ dt - \left( C_T L_T \right)^{1/2} dx = 0, \quad (4-74)
\]

\[
de_T + Z_0 di_T +
+ \frac{dx}{\left[ Z_0(G_T e_T + H^{-1} \frac{dH}{dx} i_T) + R_T i_T + E^{-1} \frac{dE}{dx} e_T \right]} = 0, \quad (4-75)
\]

\[
I \ dt + \left( C_T L_T \right)^{1/2} dx = 0, \quad (4-76)
\]

\[
de_T - Z_0 di_T +
+ \frac{dx}{\left[ -Z_0(G_T e_T + H^{-1} \frac{dH}{dx} i_T) + R_T i_T + E^{-1} \frac{dE}{dx} e_T \right]} = 0, \quad (4-77)
\]

where \(Z_0 = Z_0(x) = (L_T(x) C_T^{-1}(x))^{1/2}\).

The above system of equations can be numerically solved in a similar manner as the system (4-27)-(4-30). Referring again to Fig. 4-1, a first-order procedure would be:
I Choose the number of points where the voltages will be calculated \((m+1)\). This specifies \(\Delta x\) in Fig. 4-1 as \(\Delta x = d/m\).

II Choose \(\Delta t\) as

\[
\Delta t = \min_{i \in \{1...n\}} \left( \Delta x \sqrt{L_T^{(i)}(x) C_T^{(i)}(x)} \right).
\] (4-78)

III At points along the line

1. \(x(R_i)\) and \(x(S_i)\) can be calculated from

\[
\begin{align*}
I \Delta t - [C_T(D) L_T(D)]^{1/2} (x(P) - x(R_i)) &= 0, \\
I \Delta t + [C_T(D) L_T(D)]^{1/2} (x(P) - x(S_i)) &= 0,
\end{align*}
\] (4-79) (4-80)

where \(A(P_i)\) means evaluating row \(i\) of matrix \(A\) at point \(P_i\) \((i = 1, 2, ..., n)\).

2. \(e_T(R_i), i_T(R_i), e_T(S_i), i_T(S_i)\) can be calculated from known \(e_T\) and \(i_T\) at points C, D, \(E\) at time \(t\) using

\[
\begin{align*}
e_T(R_i) &= e_T(D) + \frac{1}{\Delta x} (x(D) - x(R_i))(e_T(C) - e_T(D)), \\
i_T(R_i) &= i_T(D) + \frac{1}{\Delta x} (x(D) - x(R_i))(i_T(C) - i_T(D)), \\
e_T(S_i) &= e_T(D) + \frac{1}{\Delta x} (x(D) - x(S_i))(e_T(D) - e_T(E)), \\
i_T(S_i) &= i_T(D) + \frac{1}{\Delta x} (x(D) - x(S_i))(i_T(D) - i_T(E)).
\end{align*}
\] (4-81) (4-82) (4-83) (4-84)

3. \(e_T(P)\) and \(i_T(P)\) can be obtained from

\[
\begin{align*}
e_T(P) - e_T(R_i) + Z_0(D) (i_T(P) - i_T(R_i)) + \\
+ (x(P) - x(R_i)) \left[ R_T(D) i_T(R_i) + E^{-1}(D) \frac{E(D) - E(C)}{\Delta x} e_T(R_i) + \\
+ Z_0(D) \left[ G_T(D) e_T(R_i) + H^{-1}(D) \frac{H(D) - H(C)}{\Delta x} i_T(R_i) \right] \right] = 0, \\
e_T(P) - e_T(S_i) - Z_0(D) (i_T(P) - i_T(S_i)) + \\
+ (x(P) - x(S_i)) \left[ R_T(D) i_T(S_i) + E^{-1}(D) \frac{E(D) - E(D)}{\Delta x} e_T(S_i) + \\
- Z_0(D) \left[ G_T(D) e_T(S_i) + H^{-1}(D) \frac{H(E) - H(D)}{\Delta x} i_T(S_i) \right] \right] = 0.
\end{align*}
\] (4-85) (4-86)

The above two equations give \(e_T(P)\) and \(i_T(P)\) as
\[ e_T(P) = \frac{1}{2} \{ e_T(R_i) + e_T(S_i) + Z_0(D) [i_T(R_i) - i_T(S_i)] - c - d \}, \]  
(4-87)

\[ i_T(P) = \frac{1}{2} \{ i_T(R_i) + i_T(S_i) + Z_0^{-1}(D) [e_T(R_i) - e_T(S_i) + d - c] \}, \]  
(4-88)

where

\[ c = (x(P) - x(R_i)) \left\{ R_T(D) i_T(R_i) + E^{-1}(D) \frac{E(D) - E(C)}{\Delta x} e_T(R_i) + 
+ Z_0(D) \left[ G_T(D) e_T(R_i) + H^{-1}(D) \frac{H(D) - H(C)}{\Delta x} i_T(R_i) \right] \right\}, \]  
(4-89)

\[ d = (x(P) - x(S_i)) \left\{ R_T(D) i_T(S_i) + E^{-1}(D) \frac{E(E) - E(D)}{\Delta x} e_T(S_i) + 
- Z_0(D) \left[ G_T(D) e_T(S_i) + H^{-1}(D) \frac{H(E) - H(D)}{\Delta x} i_T(S_i) \right] \right\}. \]  
(4-88)

4. \( e(P) \) and \( i(P) \) can then be obtained from

\[ e(P) = E(D) e_T(P), \quad i(P) = H(D) i_T(P). \]  
(4-89)

IV \( \Delta t x = 0 \)

1. Calculate \( x(U_i) \) from

\[ I \Delta t + [C_T(A) L_T(A)]^{1/2} (x(M) - x(U_i)) = 0. \]  
(4-90)

2. Calculate \( e_T(U_i), i_T(U_i) \) from

\[ e_T(U_i) = e_T(A) + \frac{1}{\Delta x} (x(U_i) - x(A)) (e_T(B) - e_T(A)), \]  
(4-91)

\[ i_T(U_i) = i_T(A) + \frac{1}{\Delta x} (x(U_i) - x(A)) (i_T(B) - i_T(A)). \]  
(4-92)

3. Solve

\[ e_T(M) - e_T(U_i) - Z_0(A) (i_T(M) - i_T(U_i)) + 
+ (x(M) - x(U_i)) \left\{ R_T(A) i_T(U_i) + E^{-1}(A) \frac{E(B) - E(A)}{\Delta x} e_T(U_i) + 
- Z_0(A) \left[ G_T(A) e_T(U_i) + H^{-1}(A) \frac{H(B) - H(A)}{\Delta x} i_T(U_i) \right] \right\} = 0, \]  
(4-93)

together with (4-14) for \( e_T(M), i_T(M) \). This can be done as follows:

Equations (4-93) and (4-14) can be written as
\[ e_T(M) = Z_0(A) i_T(M) + b, \] (4-94)
\[ i_T(M) = f(e_T(M)), \] (4-95)

where

\[ b = e_T(U_i) - Z_0(A) i_T(U_i) - (x(M) - x(U_i)) \]
\[ R_T(A) i_T(U_i) + E^{-1}(A) \frac{E(B) - E(A)}{\Delta x} e_T(U_i) + \]
\[ - Z_0(A) \left[ G_T(A) e_T(U_i) + H^{-1}(A) \frac{H(B) - H(A)}{\Delta x} i_T(U_i) \right] \] (4-96)

and

\[ f(e_T) = H^{-1}(A) f_{in}(v_g(t), E(A) e_T, t), \forall e_T \in \mathbb{R}^n. \] (4-97)

Equations (4-94) and (4-95) can then be solved by the following iteration:

\[ i_T^{(k+1)} = \left[ I - \frac{df(e_T^{(k)})}{de_T} Z_0(A) \right]^{-1} \left[ f(e_T^{(k)}) + \frac{df(e_T^{(k)})}{de_T}(b - e_T^{(k)}) \right], \] (4-98)
\[ e_T^{(k+1)} = f^{-1}(i_T^{(k+1)}), \] (4-99)

where \( e_T^{(k+1)}, i_T^{(k+1)} \) are the values of \( e_T, i_T \) at point \( M \), at the \((k+1)\)st iteration and \( f^{-1} \) is the inverse function of \( f \).

4. Transform back to get \( e(M), i(M) \):

\[ e(M) = E(A) e_T(M), \quad i(M) = H(A) i_T(M). \] (4-100)

V At \( x = d \)

1. Calculate \( x(V_i) \) from

\[ I \Delta t - [C_T(G) L_T(G)]^{1/2} (x(N) - x(V_i)) = 0. \] (4-101)

2. Calculate \( e_T(V_i), i_T(V_i) \) from

\[ e_T(V_i) = e_T(G) + \frac{1}{\Delta x} (x(V_i) - x(G))(e_T(G) - e_T(F)), \] (4-102)
\[ i_T(V_i) = i_T(G) + \frac{1}{\Delta x} (x(V_i) - x(G))(i_T(G) - i_T(F)). \] (4-103)

3. Solve

\[ e_T(N) - e_T(V_i) + Z_0(G) (i_T(N) - i_T(V_i)) + \]
\[
+ (x(N) - x(V_i)) \left\{ R(T(G)) i_T(V_i) + E^{-1}(G) \frac{E(G) - E(F)}{\Delta x} e_T(V_i) + \\
+ Z_0(G) \left[ G_T(G) e_T(V_i) + H^{-1}(G) \frac{H(G) - H(F)}{\Delta x} i_T(V_i) \right] \right\} = 0,
\]

together with
\[
i_T(d, t) = H^{-1}(G) f_{out}(v_s(t), E(G) e_T(d, t), t).
\]

for \( e_T(N), i_T(N) \).

4. Transform back to get \( e(N), i(N) \):
\[
e(N) = E(G) e_T(N), \quad i(N) = H(G) i_T(N).
\]

Steps III–V are then repeated for each time point.

4-4 Examples

Example 1

Figure 4-2 shows a lossy three-line microstrip structure. The diode is defined by \( i = I_s(e^{v/V_T} - 1) \), where \( I_s = 1 \text{nA} \) and \( V_T = 25 \text{mV} \). A separate program was used [12] to compute the \( R, L \) and \( C \) matrixes of the structure.

\[
\begin{align*}
2 u(t) \text{V} & \quad 1 \quad 50 \Omega \\
& \quad 2 \quad 50 \Omega \\
& \quad 3 \quad 50 \Omega \\
& \quad 4 \quad 50 \Omega \\
& \quad 5 \quad 50 \Omega \\
& \quad 6 \quad D \\

d = 1 \text{cm} \\
w_1 = w_2 = w_3 = s_1 = s_2 = H = 0.2 \text{mm} \quad \varepsilon_r = 9.9 \quad R_s = 20 \text{m} \Omega /\text{square}
\end{align*}
\]

Figure 4-2: The geometry of Example 1
The values were:

$$L = \begin{bmatrix} 426.32 & 75.184 & 24.923 \\ 75.184 & 424.50 & 75.184 \\ 24.923 & 75.184 & 426.32 \end{bmatrix} \text{ nH/m},$$

Figure 4-3: The results of Example 1
The circuit was analyzed on a personal computer using the procedure described above with the first-order approximation and 100 divisions per line. The results of the analysis are shown in Figure 4-3 in the form of the step response of the multiport.

**Example 2**

Figure 4-4 shows a nonuniform asymmetric coupled microstrip structure. The \( R \), \( L \), and \( C \) matrixes used in the computation were

\[
R = \begin{bmatrix}
    100 & 0 & 0 \\
    0 & 100 & 0 \\
    0 & 0 & 100
\end{bmatrix} \Omega/m,
\]

\[
C = \begin{bmatrix}
    175.17 & -14.234 & -0.64561 \\
    -14.234 & 176.75 & -14.234 \\
    -0.64561 & -14.234 & 175.17
\end{bmatrix} \text{pF/m}.
\]

The circuit was analyzed on a personal computer using first-order interpolation and 100 divisions per line. The results of the analysis are shown in Fig. 4-5.
\[ V_g = u(t) V \]

\[ s(x) = 0.16 + x/d \ (0.32 - 0.16) \ mm \]

\[ w_1 = 0.2 \ mm \quad w_2 = 0.1 \ mm \]

\[ \varepsilon_r = 9.9 \]

\[ H = 0.2 \ mm \quad R_s = 10 \ m\Omega \text{/square} \]

**Figure 4-4:** The geometry of Example 2

**Figure 4-5:** The results of Example 2
Example 3

Figure 4-6 shows a three-line nonuniform microstrip structure. The R, L and C matrixes were computed by using a separate program. The results obtained for the step response computed with 100 divisions per line are shown in Fig. 4-7.

\[
\begin{align*}
R &= 30 \text{ m} \Omega /\text{square} \\
\varepsilon_r &= 9.8 \\
H &= 100 \ \mu \text{m} \\
w_1(x) &= w_2(x) = w_3(x) = \frac{10 + (50 - 10)}{d} \mu \text{m} \\
s_1(x) &= s_2(x) = 10 + (50 - 10) x/d \ \mu \text{m}
\end{align*}
\]

Figure 4-6: The geometry of Example 3
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Conclusion

A numerical technique to compute the time domain response of multiconductor lines terminated in general nonlinear terminations has been presented. The technique is based on the well known method of characteristics and should be useful to formulate CAD tools and study the signal delay, distortion and crosstalk in various multiconductor structures such as interconnections in integrated circuits, including VLSI.

Further work in this area includes the study of general structures encountered in IC packages and chip carriers and inclusion of the skin effect and other dispersion effects in the analysis and modeling of general multiple coupled distributed parameter systems.
Bibliography


Appendixes
Appendix A

A-1

Quadratic Interpolation

The equation of the second order curve that passes through points A, B and C of Fig. A-1 is

\[ y = a \left(x - x_c\right)^2 + b \left(x - x_c\right) + c. \]  

(A-1-1)

The constants \(a\), \(b\) and \(c\) can be determined from the requirement that the points A, B, C with coordinates \((x_c - \Delta x, y_A)\), \((x_c + \Delta x, y_B)\), \((x_c, y_c)\) satisfy (A-1-1):

\[ y_A = a \Delta x^2 + b \Delta x + c, \]  

(A-1-2)

\[ y_B = a \Delta x^2 + b \Delta x + c, \]  

(A-1-3)

\[ y_c = c. \]  

(A-1-4)

From the above three equations:

\[ y_A + y_B - 2y_c = 2a \Delta x^2, \]  

(A-1-5)

\[ y_B - y_A = 2b \Delta x. \]  

(A-1-6)