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The sampling theorem states that any frequency bandlimited signal can be exactly reconstructed from its sampled values. Different forms of this theorem have been reviewed since 1841 up to the present time.

Kramer's generalized sampling theorem has been extended to two dimensions, and the form for higher dimensions can be derived by the same method.

Upper bounds for the variations of sampled signals were investigated, and it was shown that tighter bounds can be established.

Although the sampling theorem for sequency bandlimited signals was proved by the aid of Kramer's generalized sampling theorem we showed that this theorem can be proved directly or by the method of Shannon's sampling theorem in Fourier analysis. This is in agreement with the result of Campbell, that for some cases Kramer's generalized theorem does not enlarge the class of functions to which sampling theorems can be applied.

Because of the recent interest in applications of sampling theo-
rems in discrete and finite Walsh-Fourier analysis we introduced new forms of these theorems and proved them by simple techniques. These results were applied easily for the development of periodic and nonperiodic sampling theorems for two dimensions.

Finally, a definition in Haar-Fourier analysis is created, which is analogous to the definition of M-sequency bandlimited signals. Based on this definition several useful theorems are established which show similarities between sampling theorems in Walsh and Haar-Fourier analysis.
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<table>
<thead>
<tr>
<th>Chapter</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Review of the Sampling Theorem</td>
</tr>
<tr>
<td>1.1 The Sampling Theorem in the Time Domain</td>
</tr>
<tr>
<td>1.2 The Cardinal Series and Shannon’s Sampling Theorem</td>
</tr>
<tr>
<td>1.3 On Cauchy’s Interpolation Formula and Sampling Theorems</td>
</tr>
<tr>
<td>1.4 A Physical Interpretation of the Sampling Theorem</td>
</tr>
<tr>
<td>1.5 Different Forms of the Sampling Theorems</td>
</tr>
<tr>
<td>1.6 Sampling Theorem in n Dimensional Space</td>
</tr>
<tr>
<td>1.7 Sampling Theorems for Stationary and Nonstationary Random Processes</td>
</tr>
<tr>
<td>II. A Generalization of Sampling Theorem</td>
</tr>
<tr>
<td>2.1 Kramer’s Generalized Sampling Theorem</td>
</tr>
<tr>
<td>2.2 Sampling and Differential Equations</td>
</tr>
<tr>
<td>2.3 Generalized Sampling Theorem in Two-Dimensional Space</td>
</tr>
<tr>
<td>2.4 Sampling Theorem for Bandlimited Hankel Transform</td>
</tr>
<tr>
<td>2.5 A Comparison of Sampling Theorems of Kramer and Shannon-Whittaker</td>
</tr>
<tr>
<td>III. Limits on Bandlimited Signals</td>
</tr>
<tr>
<td>3.1 Bounds on Output of a Linear Time Invariant System with Bandlimited Input</td>
</tr>
<tr>
<td>3.2 Truncated One-Dimensional Sampling Expansion</td>
</tr>
<tr>
<td>3.3 Truncated Two-Dimensional Sampling Expansion</td>
</tr>
<tr>
<td>3.4 Upper Bounds for the Variation of Sampled Signals</td>
</tr>
<tr>
<td>IV. Walsh Functions and Sampling Expansions</td>
</tr>
<tr>
<td>4.1 A Closed Set of Orthogonal Functions</td>
</tr>
<tr>
<td>4.2 Discrete Walsh Functions</td>
</tr>
<tr>
<td>4.3 Non-Denumerable Walsh Functions</td>
</tr>
<tr>
<td>4.4 Sampling Theorem in Walsh-Fourier Analysis</td>
</tr>
<tr>
<td>4.5 A Comparison of the Sampling Theorems of Kak and Shannon</td>
</tr>
<tr>
<td>4.6 Sampling Expansions in Discrete and Finite Walsh-Fourier Analysis</td>
</tr>
<tr>
<td>4.7 A New Form of Sampling Theorem in Discrete and Finite Walsh-Fourier Analysis</td>
</tr>
<tr>
<td>4.8 Two-Dimensional Sampling Expansions in Discrete and Finite Walsh-Fourier Analysis</td>
</tr>
</tbody>
</table>
V. Haar Functions and Sampling Expansions

5.1 A Complete Orthonormal Sequence of Haar Functions 69
5.2 Sampling Expansions in Discrete and Finite Haar-Fourier Analysis 72
5.3 Two-Dimensional Sampling Expansions in Discrete and Finite Haar-Fourier Analysis 74
5.4 Discrete Representation of Continuous Time Signals 76

Bibliography 79

Appendix A 82

A.1 Walsh Functions Form a Complete Orthonormal Set in $L^2[0,1]$ 82
A.2 Haar Functions Form a Complete Orthonormal Set in $L^2[0,1]$ 84
ON THE RECONSTRUCTION OF BANDLIMITED SIGNALS FROM SAMPLED VALUES

Chapter I

Review of the sampling theorem

In this chapter we shall review some of the different forms of sampling theorems for bandlimited signals, both deterministic and random. Also, we shall attempt to give a brief historical background of the problem, particularly with reference to Cauchy (1841), Whittaker (1915), Ferrar (1925), Shannon (1949), Balakrishnan (1957), Reza (1961), Papoulis (1968), Wang (1970), and Gardner (1972).

1.1. The sampling theorem in the time domain.

Consider a deterministic signal in the form of a continuous function, which has passed through a transmission system having a finite bandwidth. For signals of this type we shall prove the following theorem, due to Shannon (1949). First we need the following definition:

Definition 1.1.1. We say that \( f(t) \) is bandlimited if its Fourier transform,

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt,
\]

is zero in value outside the interval \((-\omega_0, \omega_0)\). That is,

\[
F(\omega) = 0, \quad |\omega| \geq \omega_0.
\]

Shannon's sampling theorem 1.1.1. "If a function \( f(t) \) contains no frequencies higher than \( W \) cps it is completely determined by giving its ordinates at a series of points spaced \( 1/2W \) seconds apart".

Proof: Let \( F(\omega) \) be the Fourier transform of \( f(t) \). Then
\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega) e^{i\omega t} d\omega, \quad (1.1.2)
\]
since \( F(\omega) \) is assumed to have value zero outside the band. If we let
\[
t = \frac{n}{2W}, \quad (1.1.3)
\]
where \( n = 0, \pm 1, \pm 2, \ldots \), we obtain
\[
f(\frac{n}{2W}) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega) e^{i\omega \frac{n}{2W}} d\omega. \quad (1.1.4)
\]
Let us expand \( F(\omega) \) in a Fourier series in the indicated interval. Thus
\[
F(\omega) = \sum_{-\infty}^{\infty} c_n e^{-i\omega \frac{n}{2W}}, \quad (1.1.5)
\]
where
\[
c_n = \frac{1}{4\pi W} \int_{-2\pi W}^{2\pi W} F(\omega) e^{i\omega \frac{n}{2W}} d\omega = \frac{1}{2W} f(\frac{n}{2W}), \quad (1.1.6)
\]
because of Eq. (1.1.4). Therefore,
\[
F(\omega) = \frac{1}{2W} \sum_{-\infty}^{\infty} f(\frac{n}{2W}) e^{-i\omega \frac{n}{2W}}. \quad (1.1.7)
\]
Consequently, from Eqs. (1.1.2) and (1.1.7),
\[
f(t) = \frac{1}{4\pi W} \int_{-2\pi W}^{2\pi W} \sum_{-\infty}^{\infty} f(\frac{n}{2W}) e^{-i\omega \frac{n}{2W} + i\omega t} d\omega
\]
\[
= \frac{1}{4\pi W} \sum_{-\infty}^{\infty} f(\frac{n}{2W}) \int_{-2\pi W}^{2\pi W} e^{i\omega(t - \frac{n}{2W})} d\omega.
\]
Finally,
Therefore, the function can be simply reconstructed from the sample by using a pulse of the type \( \sin \frac{2\pi Wt}{2\pi Wt} \), which is zero at all sample points \( (t_n = n/2W) \) except at \( t = 0 \), where the value is unity, as shown in Fig. (1.1).

At each sample point a pulse of this type is placed whose amplitude is adjusted to equal that of the sample. The sum of these pulses is the required function. The process of reconstructing a continuous function from a set of sample values is commonly referred to as interpolation. Fig. (1.2) illustrates interpolation.
1.2. The cardinal series and Shannon's sampling theorem.

Shannon's sampling expansion has been given previously by E. T. Whittaker (1915) in the form of the cardinal series. Whittaker showed that the series

\[ \frac{\omega}{\pi} \sin \frac{\pi}{\omega} (t-a) \sum_{-\infty}^{\infty} \frac{(-1)^{n} f(a+n\omega)}{t-a-n\omega} \quad (1.2.1) \]

represents a function which is cotabular with the given function \( f(t) \), but which has no periodic constituent of period less than \( 2\omega \). In the above summation the \( a+n\omega \) form a set of equidistant values of the argument. If in Eq. (1.2.1) we let \( a = 0 \), \( \omega = 1/2W \), we have

\[ \frac{\omega}{\pi} \sin \frac{\pi}{\omega} (t-a) \sum_{-\infty}^{\infty} \frac{(-1)^{n} f(a+n\omega)}{t-a-n\omega} = \frac{1}{2\pi W} \sum_{-\infty}^{\infty} \cos n\pi \frac{f(\frac{n}{2W})}{t - \frac{n}{2W}} \]

\[ = \sum_{-\infty}^{\infty} \frac{f(n)}{2W} \sin 2\pi W (t - \frac{n}{2W}) \frac{1}{2\pi W (t - \frac{n}{2W})} \quad (1.2.2) \]

which is Shannon's sampling expansion of Eq. (1.1.8).

1.3. On the Cauchy interpolation formula and sampling theorem.

Although use of the sampling theorem in communication theory is due to Shannon (1949) we point out here that Cauchy (1841) some seventy years before Whittaker (1915) arrived at such a theorem, as mentioned by Wang (1970). Ferrar (1925) considered the following Lagrangian interpolation formula for the polynomial \( F(t) \) of degree \( n \):

\[ F(t) = \sum_{i=0}^{n} A_i F(t_i) \quad (1.3.1) \]
where \( t_0, t_1, t_2, \ldots, t_n \) are \( n+1 \) particular values of \( t \) for which \( F(t) \) is known, and

\[
A_1 = \frac{(t-t_0)(t-t_1)(t-t_{i-1})(t-t_{i+1}) \ldots (t-t_n)}{(t_i-t_0)(t_i-t_{i-1})(t_i-t_{i+1}) \ldots (t_i-t_n)}. \tag{1.3.2}
\]

Suppose that \( t_0 = 0 \) and that a further set of values, \( t_{-1} = -t_i \), is introduced. Then the limiting form of the \( A_1 \) in Eq. (1.3.2) is \( (1 \neq 0) \)

\[
\lim_{n \to \infty} \frac{t(t+t_1)(t-t_1)(t-t_{1-1})(t-t_{1+1}) \ldots (t-t_{2-1})(t-t_{2+1})}{t_1(2t_1)(t_1-t_1)(t_1-t_{1-1})(t_1-t_{1+1}) \ldots (t_1-t_{2-1})(t_1-t_{2+1})}.
\]

Let \( t_1 = i\omega \), then \( A_1 = \lim_{n \to \infty} \frac{N}{D} \), where

\[
N = \frac{t(1 - \frac{t^2}{1^2 \omega}) (1 - \frac{t^2}{2^2 \omega}) \ldots (1 - \frac{t^2}{n^2 \omega})}{1 - \frac{t}{i \omega}}.
\]

provided that \( t \neq i\omega \), and

\[
D = 2i\omega(1 - \frac{1}{1^2})(1 - \frac{1}{2^2}) \ldots (1 - \frac{1}{n^2}) \cdot \frac{1}{(1-1)^2} \frac{1}{(i+1)^2} \ldots \frac{1}{n^2}.
\]

Using the well-known infinite product for \( \sin t \),

\[
\sin t = t(1 - \frac{t^2}{n^2})(1 - \frac{t^2}{(2n)^2}) \ldots, \quad |t| < 1,
\]

we have

\[
\lim_{n \to \infty} \frac{i \omega^2}{n(i \omega - t)} \sin \frac{n t}{\omega}.
\]
and
\[ \lim_{n \to \infty} D = 2i\omega(-1)^{i-1} \frac{(i-1)!}{(i+1)(i+2) \ldots \ldots (2i-1)} \cdot \frac{1}{[(1-\frac{1}{i+1})e -(1-\frac{1}{i+2})e \ldots \ldots ] \cdot [(1+\frac{1}{i+1})e -(1+\frac{1}{i+2})e \ldots \ldots ]}. \]

A well-known identity is
\[ \int_{n=1}^{\infty} \frac{x}{(1 - \frac{x}{z+n})e^n} = e^{-\gamma} \frac{\Gamma(z+1)}{\Gamma(z-x+1)}, \]
where \( \gamma \) is the Euler constant and \( \Gamma(x) \) is the Gamma function. If we let \( x = z = i \) we arrive at
\[ \int_{n=1}^{\infty} (1 - \frac{i}{i+n}) = e^{-\gamma} \frac{\Gamma(i+1)}{\Gamma(i-i+1)} = e^{-\gamma} i! \cdot \]
and, of course, at
\[ \int_{n=1}^{\infty} (1 + \frac{i}{i+n}) = e^{-\gamma} \frac{\Gamma(i+1)}{(i+1)!} = e^{-\gamma} \frac{i!}{(i+1)(i+2) \ldots \ldots 2i}. \]

Therefore,
\[ \lim_{n \to \infty} D = 2i\omega(-1)^{i-1} \frac{(i+1)(i+2) \ldots \ldots (2i-1) i!}{(i-1)!(i+1)(i+2) \ldots \ldots 2i} = i\omega(-1)^{i-1}. \]

Thus, when \( i \neq 0 \), the limiting form of \( A_i \) is \( \frac{\omega \sin \frac{\pi t}{\omega} (-1)^i}{\omega} \).

It is easy to show that the limiting form of \( A_o \) is \( \frac{\omega \sin \frac{\pi t}{\omega}}{\omega} \), so the cardinal function
\[ \frac{\omega}{\pi} \sin \frac{\pi t}{\omega} \sum_{n=-\infty}^{\infty} \frac{(-1)^n f(n\omega)}{t-n\omega} \]
is the limiting form of Lagrange's interpolation formula, which in
turn is Shannon's sampling series by (1.2).

1.4. A physical interpretation of the sampling theorem.

Reza (1961, p. 305) gave the following interpretation of the sampling theorem:

Suppose that a continuous, bandlimited voltage signal \( f(t) \) is given. We could quantize \( f(t) \) at instants

\[
\ldots - \frac{2}{2W}, - \frac{1}{2W}, 0, \frac{1}{2W}, \frac{2}{2W}, \frac{3}{2W}, \ldots
\]

The quantized voltages are successively applied to an ideal lowpass filter as impulses of appropriate magnitude at the specified instants of time. It is well-known that the impulse response \( h(t) \) of an ideal lowpass filter can be given by \( \sin \frac{2\pi W t}{2W} \), where \( 2\pi W \) is the cutoff frequency.

![Fig. (1.3). Physical interpretation of the sampling theorem.](image-url)
Thus the total output of the filter will be the original time function,

\[ f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin 2\pi W(t - \frac{n}{2W})}{2\pi W(t - \frac{n}{2W})}. \quad (1.4.1) \]

1.5. Different forms of the sampling theorems.

Papoulis (1968, pp. 119-132) considered the following forms of the sampling theorem:

1. If \( f(t) \) is bandlimited by \( w_1 \) and \( T \leq \frac{\pi}{w_1}, w_2 = \frac{\pi}{T}, \)
then

\[ f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin w_0(t-nT)}{w_2(t-nT)}. \quad (1.5.1) \]

2. The sampling expansion can be written in the form

\[ f(t-a) = \sum_{n=-\infty}^{\infty} f(nT-a) \frac{\sin w_0(t-nT)}{w_2(t-nT)}, \quad (1.5.2) \]

where \( a \) is any constant.

3. Replacing \( t-a \) by \( t \) in Eq. (1.5.2) we obtain

\[ f(t) = \sum_{n=-\infty}^{\infty} f(nT-a) \frac{\sin w_0(t-a-nT)}{w_2(t+a-nT)}. \quad (1.5.3) \]

4. If we expand the function \( F(\omega) \) into a Fourier series in the interval \((-w_2, w_2)\) we have

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt = T \sum_{n=-\infty}^{\infty} f(nT) e^{-jnw}, \quad |\omega| \leq \frac{\pi}{T}, \quad (1.5.4) \]

hence
\[ F(0) = \int_{-\infty}^{\infty} f(t) \, dt = T \sum_{-\infty}^{\infty} f(nT), \quad T \leq \frac{\pi}{w_1}. \] (1.5.5)

We also note that
\[
\frac{1}{2\pi} \int_{-\frac{\omega_1}{2}}^{\frac{\omega_1}{2}} |F(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} f^2(t) \, dt = T \sum_{-\infty}^{\infty} f^2(nT), \quad T \leq \frac{\pi}{w_1}. \] (1.5.6)

5. With \( f(t) = \frac{\sin \omega_1 t}{t} \), \( a = t_1 \), \( t+a = t_2 \), we obtain from Eq. (1.5.3) the identity
\[
\sum_{-\infty}^{\infty} \frac{\sin \omega_1 (t_1-nT) \sin \omega_0 (t_2-nT)}{t_1-nT} \frac{\sin \omega_0 (t_2-nT)}{t_2-nT} = \frac{\sin \omega_1 (t_1-t_2)}{t_1-t_2}. \] (1.5.7)

6. If \( f(t) \) is bandlimited by \( \omega_1 \) then \( f^2(t) \) is bandlimited by \( 2\omega_1 \), because from the frequency convolution theorem we have
\[
f^2(t) \leftrightarrow \frac{1}{2\pi} F(\omega)^* F(\omega),
\]
since \( F(\omega) = 0 \) for \( |\omega| \geq \omega_1 \) the function \( F(\omega)^* F(\omega) \) is zero for \( |\omega| \geq 2\omega_1 \). Thus \( f^2(t) \) is bandlimited by \( 2\omega_1 \). Applying Eq. (1.5.1) to this function we have
\[
f^2(t) = \sum_{-\infty}^{\infty} f^2(nT) \frac{\sin \omega_0 (t-nT)}{\omega_2(t-nT)}, \quad \omega_2 = \frac{\pi}{T}, \] (1.5.8)
provided that \( 2\omega_1 \leq \omega_0 \leq 2\omega_2 - 2\omega_1 \).

7. If the function \( g(t) = f'(t) \) is the output of a linear system with input \( f(t) \) and system function \( H(\omega) = i\omega \), for \( |\omega| < \omega_0 \) we have
\[
f'(t) = \omega_0 \sum_{-\infty}^{\infty} \frac{(-1)^n}{(n\pi - \frac{T}{2})^2} f(t+nT - \frac{T}{2}), \quad T = \frac{\pi}{\omega_0}. \] (1.5.9)
1.6. Sampling theorems in n-dimensional space.

Definition 1.6.1. We say that \( f(t_1, t_2, \ldots, t_n) \) is bandlimited if its Fourier transform \( F(w_1, w_2, \ldots, w_n) \) has zero value outside a region \( A \). Reza (1961, p. 453) gave the following sampling theorem for \( n \)-dimensional space.

Theorem 1.6.1. Let \( f(t_1, t_2, \ldots, t_n) \) be a function of \( n \) real variables, whose \( n \)-dimensional Fourier transform exists and is identically zero outside an \( n \)-dimensional rectangle symmetric about the origin. That is, if

\[
F(w_1, w_2, \ldots, w_n) = 0, \quad |w_k| > 2\pi W_k, \quad k = 1, 2, \ldots, n,
\]

then

\[
f(t_1, t_2, \ldots, t_n) = \sum_{n_1 = -\infty}^{\infty} \ldots \sum_{n_n = -\infty}^{\infty} f\left(\frac{n_1}{2W_1}, \ldots, \frac{n_n}{2W_n}\right) \frac{\sin 2\pi W_1(t - \frac{n_1}{2W_1}) \sin 2\pi W_n(t - \frac{n_n}{2W_n})}{2\pi W_1 t - \frac{n_1}{2W_1}} \ldots \frac{\sin 2\pi W_n(t - \frac{n_n}{2W_n})}{2\pi W_n t - \frac{n_n}{2W_n}}.
\]

(1.6.1)

The following theorem is due to Papoulis (1968, pp. 126-128).

Theorem 1.6.2. If \( f(t, s) \) is bandlimited in the open region \( A \) contained within a rectangle \( R \) with sides of length \( 2a \) and \( 2b \), then

\[
f(t, s) = \frac{\pi}{ab} \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} f\left(\frac{n_1}{a}, \frac{n_2}{b}\right) K_B(t - \frac{n_1}{a}, s - \frac{n_2}{b}),
\]

(1.6.2)

where \( B \) is a region in \( R \) that contains \( A \) and

\[
K_B(t, s) = \frac{1}{4\pi^2} \int \int_B e^{i(ut + vs)} \, du \, dv.
\]

(1.6.3)
Proof: We expand the function $F(u,v)$ into a Fourier series in the rectangle $R$,

$$
F(u,v) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} D_{n,m} e^{-i \frac{nm u}{a} + \frac{mn v}{b}}, \quad |u| \leq a, \quad |v| \leq b, \quad (1.6.4)
$$

where

$$
D_{n,m} = \frac{1}{4ab} \int_{-b}^{b} \int_{-a}^{a} F(u,v) e^{i \frac{nm u}{a} + \frac{mn v}{b}} dudv, \quad (1.6.5)
$$

since $F(u,v) = 0$ for $(u,v) \notin A$. Then since

$$
f(t,s) = \frac{1}{4\pi} \int_{A} \int_{A} F(u,v) e^{i(ut + vs)} dudv, \quad (1.6.6)
$$

generally the present case goes as

$$
f\left(\frac{nm}{a}, \frac{nm}{b}\right) = \frac{1}{4\pi^2} \int_{A} \int_{A} F(u,v) e^{i \frac{nm u}{a} + \frac{mn v}{b}} dudv, \quad (1.6.7)
$$

From Eqs. (1.6.5) and (1.6.7) we have

$$
D_{n,m} = \frac{n^2}{4ab} f\left(\frac{nm}{a}, \frac{nm}{b}\right), \quad (1.6.8)
$$
and the double Fourier series has the form

\[ F(u,v) = \frac{\pi}{ab} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f\left(\frac{nm}{a}, \frac{mn}{b}\right) e^{-i\left(\frac{nm}{a} + \frac{mn}{b}\right)} P_B(u,v), \quad (1.6.9) \]

where

\[ P_B(u,v) = \begin{cases} 1, & (u,v) \in B, \\ 0, & (u,v) \notin B. \end{cases} \]

Taking the inverse transformation of both sides of Eq. (1.6.9) we have

\[ f(t,s) = \frac{\pi}{ab} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f\left(\frac{nm}{a}, \frac{mn}{b}\right) K_B(t - \frac{nm}{a}, s - \frac{mn}{b}). \quad (1.6.10) \]

Depending on the choice of \( B \) we can have various expansions for the signal \( f(t,s) \):

(a) If \( B = \mathbb{R} \) we get the familiar Shannon's sampling expansion for two dimensional signals,

\[ f(t,s) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f\left(\frac{nm}{a}, \frac{mn}{b}\right) \frac{\sin a(t - \frac{nm}{a})}{a(t - \frac{nm}{a})} \frac{\sin b(s - \frac{mn}{b})}{b(s - \frac{mn}{b})}. \quad (1.6.11) \]

(b) If \( B \) is a circular disc of radius \( r_0 \),

\[ K_B(t,s) = \frac{r_0}{2\pi \sqrt{t^2 + s^2}} J_1\left(\frac{r_0}{2\pi} \sqrt{t^2 + s^2}\right). \quad (1.6.12) \]

With \( R \) taken as a square, as in Fig. (1.5), we have \( a = b = r_0 \),

\[ f(t,s) = \frac{\pi}{2} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f\left(\frac{nm}{r_0}, \frac{mn}{r_0}\right) \frac{J_1\left(\sqrt{(r_0 t - mn)^2 + (r_0 s - mn)^2}\right)}{\sqrt{(r_0 t - mn)^2 + (r_0 s - mn)^2}}. \quad (1.6.13) \]
1.7. Sampling theorems for stationary and nonstationary random processes.

The following theorem for a stationary random process is due to Balakrishnan (1957).

**Theorem 1.7.1.** Let \( \{X(t): t \in (-\infty, \infty) \} \) be a wide sense stationary process with power spectral density \( S_X(\omega) \) bandlimited to the interval \((-2\pi W, 2\pi W)\), then

\[
\lim_{N \to \infty} \mathbb{E} \left\{ \left[ X(t) - \sum_{n=-N}^{N} X(n) \frac{\sin 2\pi W (t - \frac{n}{2W})}{2\pi W (t - \frac{n}{2W})} \right]^2 \right\} = 0, \quad \forall t \in (-\infty, \infty). \tag{1.7.1}
\]

Recently Gardner (1972) generalized the above theorem for nonstationary random processes as follows:

**Theorem 1.7.2.** Let \( X \) be a random process with autocorrelation function \( R_X(t,s) \). If the Fourier transform \( S_X(\omega, \nu) \) of \( R_X(t,s) \) is bandlimited to \(-2\pi W < \omega < 2\pi W, -2\pi W < \nu < 2\pi W\), then

\[
\lim_{N \to \infty} \mathbb{E} \left\{ \left[ X(t) - \sum_{n=-N}^{N} X(n) \frac{\sin 2\pi W (t - \frac{n}{2W})}{2\pi W (t - \frac{n}{2W})} \right]^2 \right\} = 0, \quad \forall t \in (-\infty, \infty). \tag{1.7.2}
\]
Chapter II

A generalization of sampling theorems

2.1. Kramer's generalized sampling theorems.

Kramer (1959) has established a generalized sampling theorem which includes the theorems of Shannon and Whittaker as particular classes.

Theorem 2.1.1. Let $I$ be an interval and $L_2(I)$ the class of functions $\varphi(x)$ for which $\int_I |\varphi(x)|^2 \, dx < \infty$. Suppose that for each real $t$, $f(t) = \int_I k(t,x) \varphi(x) \, dx$,

where $g \in L_2(I)$. Suppose, furthermore, that for each real $t$, $k(t,x) \in L_2(I) \quad \text{and that there exists a countable set} \quad \{t_n\} \quad \text{such that} \quad \{k(t_n,x)\}$ is a complete orthogonal set on $L_2(I)$. Then

$$f(t) = \lim_{N \to \infty} \sum_{|n| \leq N} f(t_n) S_n(t),$$

where

$$S_n(t) = \frac{\int_I k(t,x) \overline{k(t_n,x)} \, dx}{\int_I |k(t_n,x)|^2 \, dx}.$$  \hspace{1cm} (2.1.3)

**Proof:** Let $f_n(t) = \sum_{|n| \leq N} f(t_n) S_n(t)$,

then, from Eqs. (2.1.1) and (2.1.4) we have

$$f(t) - f_n(t) = \int_I k(t,x) \varphi(x) \, dx - \sum_{|n| \leq N} \int_I k(t_n,x) \varphi(x) S_n(t) \, dx$$  \hspace{1cm} (2.1.5)
\[
= \int [k(t,x) - \sum_{|n| \leq N} k(t_n,x) S_n(t)] g(x) \, dx.
\]

By Schwartz's inequality Eq. (2.1.5) can be written as

\[
|f(t) - f_N(t)| \leq \int \left| \sum_{|n| \leq N} k(t_n,x) S_n(t) \right|^2 \, dx \int |g(x)|^2 \, dx^{1/2}.
\]

(2.1.6)

Since \( k(t_n,x) \) is a complete orthogonal set we have

\[
\lim_{N \to \infty} \int \left| \sum_{|n| \leq N} k(t_n,x) S_n(t) \right|^2 \, dx = 0.
\]

(2.1.7)

From Eqs. (2.1.6) and (2.1.7) we have

\[
\lim_{N \to \infty} |f(t) - f_N(t)| = 0
\]

(2.1.8)

or

\[
f(t) = \sum_{-\infty}^{\infty} f(t_n) S_n(t).
\]

(2.1.9)

2.2. Sampling and differential equations.

In the above theorem \( k(t_n,x) \) must form a complete orthogonal set of functions for the sample points \( t_n \). This condition is satisfied by the solution of some ordinary differential equations with eigenvalue parameter \( t \). Weiss (1957) has established sampling theorems associated with Sturm-Liouville systems, which are particular cases of Kramer's work. Briefly, let an nth order operator be defined by

\[
L(v) = \sum_{k=0}^{n} p_{n-k} \frac{d^k v}{dx^k},
\]

(2.2.1)

where the \( p_j \) are complex-valued functions of class \( C^{n-j} \) on the
closed interval $a \leq x \leq b$, and $p_0(x) \neq 0$ on that interval. Let $L^*(v)$ be adjoint to $L(v)$; that is,

$$L^*(v) = \sum_{k=0}^{n} (-1)^k \frac{d^k(p_{n-k}v)}{dx^k}.$$  \hspace{1cm} (2.2.2)

We assume $L = L^*$ to insure reality of the sampling points $t_n$ (Courant and Hilbert, 1955, Ch. 7). Let $B_1, B_2, \ldots, B_n$ be n linearly independent boundary conditions on $(a, b)$, such that the problem $Lv = tv, B_1(v) = B_2(v) = \cdots = B_n(v) = 0$ is self adjoint. That is, for every pair of functions $f$ and $g$ of class $C^n$, each of which satisfies the boundary conditions, we have

$$\int_{a}^{b} [f L(g) - L(f) g] \, dx = 0. \hspace{1cm} (2.2.3)$$

Suppose that $v(t, x)$ is a solution of the equation $L(v) = tv$ and the set $E_i$ of zeros of the $B_i(v(t))$ is independent of $i$. That is, $E_i = E$ for all $i$, then the set $\{v(t_n, x)\}$, $t_n \in E$, is a complete orthonormal system for $L_2(I)$, since, for every $t$, $v(t, x)$ is continuous as a function of $x$, $v(t, x) \in L_2(I)$. Thus we have the following theorem due to Kramer (1959).

**Theorem 2.2.1:** Let $L(v) = tv, B_1(v) = B_2(v) = \cdots = B_n(v) = 0$, be a self adjoint boundary value problem for an nth order differential expression $L$ on the finite interval $(a, b)$. Suppose that there exists a solution $v(t, x)$ of the differential equation such that the set of zeros $E_i$ of $B_i(v(t, x))$ is independent of $i$. Then, for all functions having the representation
\[ f(t) = \int_a^b v(t,x) g(x) \, dx, \quad (2.2.4) \]

with \( g(x) \in L^2(a,b) \), there is the representation

\[ f(t) = \lim_{N \to \infty} \sum_{|k| \leq N} f(t_k) S_k(t) \quad (2.2.5) \]

where

\[ S_k(t) = \frac{\int_I [v(t,x) v(t_k,x)] \, dx}{\int_I |v(t_k,x)|^2 \, dx}. \quad (2.2.6) \]

Using the Lagrange-Green formula (Coddington and Levinson, 1955, p. 86) we have

\[ \int_a^b [v(t,x) L(v(t_k,x)) - L(v(t,x)v(t_k,x))] \, dx \]

\[ = [v(t,x), v(t_k,x)](b) - [v(t,x), v(t_k,x)](a), \quad (2.2.7) \]

where

\[ [u,v](x) = \sum_{m=1}^n \sum_{j+k=m-1} (-1)^j v^{(j)}(x)(p_{m-j} v(x))^{(k)} \quad (2.2.8) \]

or

\[ \int_I [v L(v_k) - L(v) v_k] \, dx = (t_k - t) \int_I v(t,x) v(t_k,x) \, dx \]

\[ = (t_k - t) S_k(t) \int_I |v(t_k,x)|^2 \, dx. \quad (2.2.9) \]

Therefore,

\[ S_k(t) = \frac{[v,v_k](b) - [v,v_k](a)}{(t_k - t) \int_I |v(t_k,x)|^2 \, dx}. \quad (2.2.10) \]
The following two examples were given by Kramer to illustrate his theorem.

Example 2.2.1. Let \( L(v) = -\frac{1}{2\pi} \frac{dv}{dx} \), \( B_1(v) = v(a) - v(b) \). The general solution of the equation \( L(v) = tv \) is \( v(t,x) = C e^{2\pi i t x} \). The eigenvalues are \( t_n = \frac{n}{b-a} \), \( n = 0, 1, 2, \ldots \), and the corresponding eigenfunctions are

\[
v(t_n, x) = e^{\frac{2\pi i n x}{b-a}}.
\]  

(2.2.11)

Hence

\[
S_n(t) = \frac{2\pi i b (t - \frac{n}{b-a})}{(b-a) 2\pi i (t - \frac{n}{b-a})} - \frac{2\pi i a (t - \frac{n}{b-a})}{(b-a) 2\pi i (t - \frac{n}{b-a})}.
\]  

(2.2.12)

The theorem insures that if

\[
f(t) = \int_a^b 2\pi i t x e^{-\pi i g(x)} dx
\]  

(2.2.13)

then

\[
f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{b-a}\right) S_n(t).
\]  

(2.2.14)

If we let \( a = -W, b = W \) in Eqs. (2.2.14) and (2.2.12) we get Shannon sampling theorem for bandlimited signals,

\[
f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi W(t - \frac{n}{2W})}{\pi W(t - \frac{n}{2W})}.
\]  

(2.2.15)

Example 2.2.2. Let \( L(v) = -\frac{d^2 v}{dx^2} + \frac{n^2}{x^2} v \) so the boundary value problem takes the form
\[- \frac{d^2 v}{dx^2} + \frac{n - \frac{1}{2}}{x^2} v = tv, \quad (2.2.16)\]

with boundary conditions

\[B_1(v) = v(t,0) = 0 \quad (2.2.17a)\]

and

\[B_2(v) = v(t,1) = 0. \quad (2.2.17b)\]

The well-behaved solution of the differential equation involves the Bessel function of the first kind, order \( n \),

\[v(x,t) = \sqrt{x} J_n(\sqrt{tx}), \quad (2.2.18)\]

and Eq. (2.2.17b) leads to the condition

\[v(t,1) = J_n(\sqrt{t}) = 0, \quad (2.2.19)\]

so the set of sample points is the very well tabulated squared set of zeros of the Bessel function of order \( n \). Fig. (2.1) is a crude sketch for the case \( n = 1 \).

![Diagram](Fig. (2.1))

If

\[f(t) = \int_0^1 x J_n(\sqrt{tx}) g(x) \, dx \quad (2.2.20)\]

then

\[f(t) = \sum_{k=1}^{\infty} f(t_k) S_k(t), \quad (2.2.21)\]
By writing the standard Bessel differential equation twice with parameters \(\alpha\) and \(\beta\) it is easily established that
\[
\int_0^1 x J_n'(ax) J_n'(-bx) \, dx = \frac{\alpha J_n'(\alpha) J_n(\beta) - \beta J_n'(\beta) J_n(\alpha)}{\beta^2 - \alpha^2}, \quad \alpha \neq \beta, \ n+1 > 0,
\]
and a simple application of the L'Hopital rule yields
\[
\int_0^1 x J_n''(ax) \, dx = \frac{1}{2} \left[ J_n'(\alpha) + (1 - \frac{n^2}{\alpha^2}) J_n(\alpha) \right]. \quad (2.2.23)
\]
If \(\alpha\) and \(\beta\) are regarded as different members of the set of numbers \(\{\alpha_k\}, \ k = 1, 2, \ldots\), defined as zeros of \(\alpha J_n'(\alpha) + h J_n(\alpha), \ h\) a constant, then the orthogonal property of interest is
\[
\int_0^1 x J_n'(\alpha_k x) J_n'(\alpha_j x) \, dx = \begin{cases} 0, & k \neq j, \\ 1 \left[ J_n'(\alpha_k) + (1 - \frac{n^2}{\alpha_k^2}) J_n(\alpha_k) \right], & k = j. \end{cases} \quad (2.2.25)
\]
Another well-known relationship is
\[
\alpha J_n'(\alpha) = n J_n(\alpha) - \alpha J_{n+1}(\alpha), \quad (2.2.26)
\]
\[ \int_0^1 x J_n(\alpha x) J_n(\beta x) \, dx = \frac{\beta J_n(\alpha) J_{n+1}(\beta) - \alpha J_n(\beta) J_{n+1}(\alpha)}{\frac{1}{2} (\beta^2 - \alpha^2) [J_n^2(\alpha) + (1 - \frac{n^2}{\alpha^2}) J_n(\alpha) - \frac{1}{\alpha^2}]} \]
\[ = \frac{2\alpha \beta J_n(\alpha) J_{n+1}(\beta) - \alpha J_n(\beta) J_{n+1}(\alpha)}{\beta - \alpha [J_n^2(\alpha) + J_{n+1}^2(\beta)] - 2n J_n(\alpha) J_{n+1}(\beta)} \]  

(2.2.27)

If \( \alpha \) is identified with \( \sqrt{t_k} \), \( \beta \) with \( \sqrt{t} \), and the boundary condition of Eq. (2.2.19) (equivalent to letting \( h \to \infty \)) is applied, then Eq. (2.2.22) takes the form

\[ S_k(t) = \frac{2\sqrt{t_k}}{t_k - t} \frac{J_n(\sqrt{t})}{J_{n+1}(\sqrt{t_k})} K \]

\[ k = 1, 2, \cdots, J_n(\sqrt{t_k}) = 0, \]  

and Eq. (2.1.9) goes as

\[ f(t) = 2 J_n(\sqrt{t}) \sum_{k=1}^{\infty} \frac{\sqrt{t_k} f(t_k)}{(t_k - t) J_{n+1}(\sqrt{t_k})}, J_n(\sqrt{t_k}) = 0, \]  

(2.2.29)

2.3. Generalized sampling theorem in two dimensional space.

Let \( R \) be a rectangular region defined by \( a \leq x \leq b, c \leq y \leq d \), and \( L_2(R) \) the class of functions \( \varphi(x,y) \) such that

\[ \int_R \int_R |\varphi(x,y)|^2 \, dx \, dy < \infty. \]

We have the following two dimensional version of Kramer's sampling theorem, which has not appeared in the literature.

Theorem 2.3.1. Suppose that for each real \( t \) and \( s \),
\[ f(t,s) = \int_R \int R \left( k(t,s,x,y) g(x,y) \right) \, dx \, dy, \quad (2.3.1) \]

where \( g(x,y) \in L_2(R) \). Suppose that for each real \( t \) and \( s \)
\( k(t,s,x,y) \in L_2(R) \), and there exists a countable set, \( E = \{ (t_n, s_m) \} \),
such that \( \{ k(t_n, s_m, x, y) \} \) is a complete orthogonal set on \( L_2(R) \).

Then
\[
 f(t,s) = \sum_{n \leq N} \sum_{m \leq M} f(t_n, s_m) S_{n,m}(t,s), \quad (2.3.2)
\]

where
\[
 S_{n,m}(t,s) = \frac{\int_R \int R k(t,s,x,y) k(t_n, s_m, x, y) \, dx \, dy}{\int_R \int R |k(t_n, s_m, x, y)|^2 \, dx \, dy}. \quad (2.3.3)
\]

**Proof:** Let
\[
 f_{N,M}(t,s) = \sum_{n \leq N} \sum_{m \leq M} f(t_n, s_m) S_{n,m}(t,s). \quad (2.3.4)
\]

From Eq. (2.3.1) we have
\[
 f(t, s) - f_{N,M}(t, s) = \int_R \int R k(t,s,x,y) g(x,y) \, dx \, dy. \quad (2.3.5)
\]

From Eqs. (2.3.1) and (2.3.4) we obtain
\[
 f(t, s) - f_{N,M}(t, s) = \int_R \int R k(t,s,x,y) g(x,y) \, dx \, dy
- \sum_{n \leq N} \sum_{m \leq M} f(t_n, s_m) S_{n,m}(t,s). \quad (2.3.6)
\]
Inserting Eq. (2.3.5) into Eq. (2.3.6) we get

\[ f(t,s) - f_{N,M}(t,s) = \int_\mathbb{R} \int k(t,s,x,y) g(x,y) \, dx \, dy \]

\[ - \sum_{|n| \leq N} \sum_{|m| \leq M} \int_\mathbb{R} \int k(t_n,s_m,x,y) g(x,y) \, dx \, dy \]

\[ = \int_\mathbb{R} \int [k(t,s,x,y) - \sum_{|n| \leq N} \sum_{|m| \leq M} k(t_n,s_m,x,y) s_{n,m}(t,s)] g(x,y) \, dx \, dy, \]

where \( s_{n,m}(t,s) \) is defined by Eq. (2.3.3). So,

\[ |f(t,s) - f_{N,M}(t,s)| \leq \int_\mathbb{R} \int |k(t,s,x,y) - \sum_{|n| \leq N} \sum_{|m| \leq M} k(t_n,s_m,x,y) s_{n,m}(t,s)|^2 \, dx \, dy \]

\[ \cdot s_{n,m}(t,s) |g(x,y)| \, dx \, dy. \]

By Schwartz's inequality for two dimensions this becomes

\[ |f(t,s) - f_{N,M}(t,s)| \leq \left[ \int_\mathbb{R} \int |k(t,s,x,y)| \, dx \, dy \right]^{1/2} \left[ \int_\mathbb{R} \int |g(x,y)|^2 \, dx \, dy \right]^{1/2}. \]  

Since \( \{k(t_n,s_m,x,y)\} \) is a complete orthogonal set,

\[ k(t,s,x,y) = \text{l.i.m.} \sum_{N \to \infty} \sum_{M \to \infty} k(t_n,s_m,x,y) s_{n,m}(t,s) \]

\[ \sum_{|n| \leq N} \sum_{|m| \leq M} k(t_n,s_m,x,y) s_{n,m}(t,s) \]

where l.i.m. stands for limit in the mean and \( s_{n,m}(t,s) \) is defined by Eq. (2.3.3). By assumption
\[
\int \int_R |g(x,y)|^2 \, dx \, dy < \infty
\]

so from Eqs. (2.3.9) and (2.3.10) we have

\[
f(t,s) = \lim_{N \to \infty} \lim_{M \to \infty} f(t,s)_{|n| \leq N, |m| \leq M}
\]

or

\[
f(t,s) = \sum_{|n| \leq N} \sum_{|m| \leq M} f(t_n, s_m) S_{n,m}(t,s),
\]

\[
N \to \infty, \ M \to \infty.
\]

Example 2.3.1. Let \( k(t,s,x,y) = e^{-i(tx + sy)} \), \( R \) the rectangle \(-a \leq x \leq a, -b \leq y \leq b, E = \left\{ \frac{nn}{a}, \frac{mn}{b} \right\} \). If

\[
f(t,s) = \int_{-b}^{b} \int_{-a}^{a} e^{-i(tx + sy)} g(x,y) \, dx \, dy
\]

then by theorem (2.3.1) we obtain

\[
f(t,s) = \sum_{|n| \leq N} \sum_{|m| \leq M} f(\frac{nn}{a}, \frac{mn}{b}) S_{n,m}(t,s),
\]

\[
N \to \infty, \ M \to \infty,
\]

where

\[
S_{n,m}(t,s) = \int_{-b}^{b} \int_{-a}^{a} e^{-i(tx + sy)} \sin(\frac{nnx}{a} + \frac{mnx}{b}) \, dx \, dy
\]

\[
= \sin(\pi n x - ta) \sin(\pi m y - sb)
\]

Therefore, from Eq. (2.3.14) we get
\[ f(t, s) = \sum_{|n| \leq N} \sum_{|m| \leq M} f\left( \frac{m_n}{a}, \frac{m_n}{b} \right) \frac{\sin a(t - \frac{m_n}{a}) \sin b(s - \frac{m_n}{b})}{a(t - \frac{m_n}{a})b(s - \frac{m_n}{b})}, \]

\[ N \to \infty, \quad M \to \infty. \]

If we let \( t = t_1, s = t_2, n = m_1, m = m_2, a = 2\pi W_1, b = 2\pi W_2, \) then we recover Eq. (1.6.1) for \( n = 2, \)

\[ f(t_1, t_2) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f\left( \frac{m_1}{2W_1}, \frac{m_2}{2W_2} \right) \frac{\sin 2\pi W_1(t_1 - \frac{m_1}{2W_1}) \sin 2\pi W_2(t_2 - \frac{m_2}{2W_2})}{2\pi W_1(t - \frac{m_1}{2W_1}) 2\pi W_2(t - \frac{m_2}{2W_2})}. \]

(2.3.17)

2.4. Sampling theorem for bandlimited Hankel transforms.

**Definition 2.4.1:** We say that \( f(t) \) is a bandlimited Hankel transform function if \( \tilde{f}(\omega) = 0 \) for \( \omega > a, \) where

\[ \tilde{f}(\omega) = \int_0^\infty t f(t) J_0(\omega t) dt. \]  

(2.4.1)

The following theorem is due to Papoulis (1968, p. 163).

**Theorem 2.4.1:** A function \( f(t) \) with a bandlimited Hankel transform can be recovered from its discrete values \( f\left( \frac{t_n}{a} \right) \), where the \( \left\{ t_n \right\} \) are the zeros of \( J_0(t) \) and \( a \) is the bandwidth.

**Proof.** Papoulis's proof is based on the use of the Fourier-Bessel expansion of \( \tilde{f}(\omega) \). The Kramer theorem approach is equivalent but care is required. That is, according to Jerri (1969a),

\[ f(t) = 2 J_0(at) \sum_{n=1}^\infty \frac{t_n f\left( \frac{t_n}{a} \right)}{(t_n^2 - a^2 t^2) J_1(t)}, \]

where \( J_0(t_n) = 0, \)

(2.4.2)

follows from Eq. (2.4.1) by first setting \( t = t_n/a \) in the inversion.
\[ f(t) = \int_{0}^{\infty} \omega \tilde{f}(\omega) J_{o}(\omega t) \, d\omega \quad (2.4.3) \]

and using the Fourier-Bessel expansion of the bandlimited function
\[ \tilde{f}(\omega) = \sum_{n=1}^{\infty} c_{n} J_{o}(\omega t_{n}), \quad J_{o}(t_{n}) = 0, \quad (2.4.4) \]

where, from Eqs. (2.2.25), (2.4.3) and (2.2.26)
\[ c_{n} = \frac{\int_{0}^{a} \omega \tilde{f}(\omega) J_{o}(\omega t_{n}) \, d\omega}{\int_{0}^{a} \omega J_{o}^{2}(\omega t_{n}) \, d\omega} = \frac{f(t_{n})}{a^{2} 2} , \quad J_{o}(t_{n}) = 0. \quad (2.4.5) \]

So
\[ f(t) = \int_{0}^{a} \omega J_{o}(\omega t) \frac{2}{a^{2}} \sum_{n=1}^{\infty} \frac{f(t_{n})}{a^{2}} J_{o}(\omega t_{n}) \, d\omega \quad (2.4.6) \]
\[ = \frac{2}{a^{2}} \sum_{n=1}^{\infty} \frac{f(t_{n})}{J_{1}^{2}(t_{n})} \frac{a^{2} t_{n} J_{o}(at) J_{1}(t_{n})}{t^{2} - a^{2} t^{2}} = 2 J_{o}(at) \sum_{n=1}^{\infty} \frac{t_{n}}{(t^{2} - a^{2} t^{2}) J_{1}(t_{n})} \]
\[ J_{o}(t_{n}) = 0 , \]

and in this case,
\[ S_{n}(t) = \frac{2 \ t_{n} J_{o}(at)}{(t^{2} - a^{2} t^{2}) J_{1}(t_{n})} , \quad J_{o}(t_{n}) = 0. \quad (2.4.7) \]

It is interesting to show that the spacing between sample points is asymptotically equal to \( n/a \). That is, \( \lim_{n \to \infty} t_{n}/n = 1, \quad J_{o}(t_{n}) = 0. \)
(Campbell, 1964; Courant and Hilbert, 1953, p. 416).
2.5. A comparison of the sampling theorems of Kramer and Shannon-

Whittaker.

Campbell (1964) has considered some special cases of the general-
ized sampling theorem and has shown that in these cases the generalized
theorem does not enlarge the class of functions to which the sampling
theorem can be applied, and it does not alter the average sampling
rate. That is, if a function can be expanded with the aid of the gen-
eralized theorem it can also be expanded with the aid of the older Whitt-
taker theorem. Moreover, the asymptotic spacing between sample points
in the generalized expansion is the same as the spacing between sample
points for the Whittaker case. Campbell (1964) established a sampling
theorem associated with regular first order differential equations and
regular second order equations with separated boundary equations. We
consider his result for the well-known second order Legendre different-
ial equation.

Example 2.5.1. The Legendre differential equation goes as

$$\frac{d}{dx}[(1-x^2)\frac{du}{dx}] + (t^2 - \frac{1}{4})u = 0 \quad (2.5.1)$$

We choose the boundary conditions in such a way that solutions will be
bounded at $x = -1$ and at $x = 1$. The eigenfunctions are the Legendre
polynomials and the eigenvalues are $t_n^2 = (n + \frac{1}{2})^2$, $n = 0, 1, 2, \ldots$.

A solution of Eq. (2.5.1) is $P_{t-\frac{1}{2}}(x)$, which reduces to the polynom-
ials $P_n(s)$ for $t = n + \frac{1}{2}$. If we let

$$f(t) = \int_{-1}^{1} P_{t-\frac{1}{2}}(x) g(x) \, dx \quad (2.5.2)$$
then, by Kramer's generalized sampling theorem,

\[ f(t) = \sum_{n=0}^{\infty} f(n+\frac{1}{2}) S_n(t), \quad (2.5.3) \]

where

\[ S_n(t) = -\frac{\int_{-1}^{1} P_{t-\frac{1}{2}}(x) P_n(x) \, dx}{\int_{-1}^{1} P_n^2(x) \, dx}. \quad (2.5.4) \]

It is well-known that [Erdelyi, et al., 1953, Eqs. 3.12 (7) and 3.12 (10)]

\[ \int_{-1}^{1} P_{\alpha}(x) P_{\beta}(x) \, dx = \frac{2}{n^2}[\gamma(\beta-\alpha)(\alpha+\beta+1)] \]

\[ = 2 \sin(\pi \alpha) \sin(\pi \beta)[\gamma(\alpha+1) - \gamma(\beta+1)] + \pi \sin(\pi(\beta - \alpha)), \quad \alpha + \beta + 1 \neq 0, \]

where \( \gamma(x) \) is the logarithmic derivative of the Gamma function, and

\[ \int_{-1}^{1} P_n^2(x) \, dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \ldots. \quad (2.5.6) \]

From the three equations just above

\[ S_n(t) = \frac{(2n+1) \sin \left(t - n - \frac{1}{2}\right)}{\pi \left(t - n - \frac{1}{2}\right) \left(t + n + \frac{1}{2}\right)}. \quad (2.5.7) \]

Since \( P_t(x) = P_{-t-1}(x) \) we have \( P_{t-\frac{1}{2}}(x) = P_{-t-\frac{1}{2}}(x) \), and we conclude from Eq. (2.5.2) that \( f(t) \) is an even function of \( t \). Because of
Eqs. (2.5.3) and (2.5.7) and the evenness of \( f(t) \), and if we set

\[ F(t) = f(t + \frac{1}{2}) \]

we can write

\[
F(t) = \sum_{-\infty}^{\infty} F(n) \frac{\sin \pi(t-n)}{\pi(t-n)} .
\]  

(2.5.8)

Thus if \( f(t) \) is given by Eq. (2.5.2) the generalized sampling theorem is just Whittaker's or Shannon's sampling theorem for \( F(t) = f(t + \frac{1}{2}) \).

Since

\[
P(t) = \frac{\sqrt{2}}{\pi} \frac{\cos \theta}{\sqrt{\cos \alpha - \cos \theta}}
\]

from Eq. (2.5.2) we get

\[
f(t) = \int_{-\pi}^{\pi} \frac{\sin \theta}{\pi} H(\alpha) \, d\alpha
\]  

(2.5.9)

where

\[
H(\alpha) = \frac{1}{n^{1/2}} \int_{-\pi}^{\pi} \frac{g(\cos \theta) \sin \theta \, d\theta}{|\alpha| \sqrt{\cos \alpha - \cos \theta}}.
\]  

(2.5.10)

Thus \( f(t) \) can be expanded by Shannon's theorem with a sampling interval equal to unity.

We shall show in Chapter IV, section five, that the sampling theorem in Walsh-Fourier analysis, which was proved by Kak (1970) with the aid of Kramer's generalized sampling theorem, can be proved directly or by the aid of the Shannon-Whittaker theorem.
Chapter III
Limits on Bandlimited Signals

3.1. Bounds on output of a linear time invariant system with bandlimited input.

Most of the results of this chapter are due to Papoulis (1967). Consider a linear, time invariant system with input and output related by the convolution integral

\[ g(t) = f(t)^* h(t) = \int_{-\infty}^{\infty} f(u) h(t-u) \, du, \tag{3.1.1} \]

where \( f(t) \) is input, \( g(t) \) is output, \( h(t) \) is impulse response of the linear, time invariant system. Suppose \( F(\omega), G(\omega) \) and \( H(\omega) \) are the Fourier transforms of the above functions. We consider two classes of functions, deterministic and random.

Case 1. Deterministic signals. Suppose \( f(t) \) is bandlimited by \( w_0 \) with finite energy

\[ E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt. \tag{3.1.2} \]

From Eq. (3.1.1) it is well-known that

\[ G(\omega) = F(\omega) H(\omega), \tag{3.1.3} \]

and the inversion goes as

\[ g(t) = \frac{1}{2\pi} \int_{-w_0}^{w_0} F(\omega) H(\omega) e^{i\omega t} \, d\omega. \tag{3.1.4} \]

Application of Schwartz's inequality yields

\[ |g(t)|^2 \leq \frac{1}{4\pi^2 w_0} \int_{-w_0}^{w_0} |F(\omega)|^2 \, d\omega \int_{-w_0}^{w_0} |H(\omega)| e^{i\omega t} \, d\omega. \tag{3.1.5} \]
By Parseval's formula,

\[ E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} |F(\omega)|^2 \, d\omega \]  
(3.1.6)

so, from Eqs. (3.1.5) and (3.1.6),

\[ |g(t)|^2 \leq \frac{E}{2\pi} \int_{-\omega_0}^{\omega_0} |H(\omega)|^2 \, d\omega , \]  
(3.1.7)

for any bandlimited input of energy \( E \).

**Case 2. Random signals.** Consider a random signal \( f(t) \) with finite average power,

\[ P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt , \]  
(3.1.8)

and autocorrelation,

\[ R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t + \tau) \overline{f(t)} \, dt \]  
(3.1.9)

Suppose \( f(t) \) is bandlimited in the sense that \( S(\omega) = 0 \) for \( |\omega| \geq \omega_b \), where \( S(\omega) \) is the power spectrum associated with \( f(t) \). If we consider our random signals to be stationary, stochastic processes with autocorrelation defined as

\[ R(\tau) = E[f(t+\tau)\overline{f(t)}] \]  
(3.1.10)

then

\[ P = E[|f(t)|^2] = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} S(\omega) \, d\omega = R(0) . \]  
(3.1.11)

If the input \( f(t) \) is a random signal with power spectrum \( S(\omega) \), then the output power spectrum goes as
\[ S_g(\omega) = S(\omega) |H(\omega)|^2. \quad (3.1.12) \]

From Eq. (3.1.11) we get
\[ E[|g(t)|^2] = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} S(\omega) |H(\omega)|^2 \, d\omega. \quad (3.1.13) \]

Denote by \( \omega_m \) the value of \( \omega \) for which \( |H(\omega)| \) takes on maximum value in the interval \((-\omega_0, \omega_0)\),
\[ |H(\omega)| \leq |H(\omega_m)|, \quad |\omega| \leq \omega_0. \quad (3.1.14) \]

Then from Eqs. (3.1.11), (3.1.13) and (3.1.14) we get
\[ E[|g(t)|^2] \leq P |H(\omega_m)|^2. \quad (3.1.15) \]

3.2. **Truncated one-dimensional sampling expansion.**

If \( f(t) \) is a deterministic signal, bandlimited by \( \omega_0 \), it is easy to show that \( f(t+\tau) \) is also bandlimited by \( \omega_0 \), and the sampling theorem can be applied to \( f(t+\tau) \). That is,
\[ f(t+\tau) = \sum_{-\infty}^{\infty} f(t+n\tau) \frac{\sin \omega_0(\tau-n\tau)}{\omega_0(\tau-n\tau)}, \quad \tau = \frac{\pi}{\omega_0}. \quad (3.2.1) \]

**Definition 3.2.1.** Let
\[ e_N(t+\tau) = f(t+\tau) - \sum_{-N}^{N} f(t+n\tau) \frac{\sin \omega_0(\tau-n\tau)}{\omega_0(\tau-n\tau)}, \quad \tau = \frac{\pi}{\omega_0}. \quad (3.2.2) \]

Let \( g(t) = e_N(t+\tau) \) be the output of a system with input \( f(t) \) and system function
\[ H(\omega) = e^{-j\omega T} - \sum_{-N}^{N} e^{jnt\omega} \frac{\sin \omega_0(\tau-n\tau)}{\omega_0(\tau-n\tau)}. \quad (3.2.3) \]
From Eq. (3.1.7) we get

\[ |e_N(t+\tau)|^2 \leq \frac{E}{2\pi} \int_{-w_0}^{w_0} \left| e^{i\omega \tau} - \sum_{n=-N}^{N} \sin \frac{\omega_0 (\tau - nT)}{\omega_0 (\tau - nT)} \right|^2 d\omega. \quad (3.2.4) \]

The Fourier series expansion of \( e^{i\omega \tau} \) in the interval \((-\omega_0, \omega_0)\) is

\[ e^{i\omega \tau} = \sum_{-\infty}^{\infty} c_n e^{i\omega \tau}. \quad (3.2.5) \]

where

\[ c_n = \frac{1}{2w_0-w_0} \int_{-\infty}^{\infty} e^{-i\omega \tau} e^{i\omega \tau} d\omega = \frac{\sin \omega_0 (\tau - nT)}{\omega_0 (\tau - nT)}. \quad (3.2.6) \]

So Eq. (3.2.5) has the form

\[ e^{i\omega \tau} = \sum_{-\infty}^{\infty} \frac{\sin \omega_0 (\tau - nT)}{\omega_0 (\tau - nT)} e^{i\omega \tau}. \quad (3.2.7) \]

In view of the above formula and Eq. (3.2.4) we conclude that

\[ |e_N(t+\tau)|^2 \leq \frac{E\omega_0}{\pi} \sum_{|n| > N} \frac{\sin^2 \omega_0 (\tau - nT)}{\omega_0^2 (\tau - nT)^2}. \quad (3.2.8) \]

In the random signal case we get from Eq. (3.1.15)

\[ \mathbb{E}[|e_N(t+\tau)|^2] \leq P \max_{|\omega| \leq \omega_0} |H(\omega)|^2. \quad (3.2.9) \]

Therefore, if \( f(t) \) is a random signal the mean square value of \( e_N(t) \) is bounded by the maximum error of the approximation of \( e^{i\omega \tau} \) by a truncated Fourier series (Papoulis, 1967).

3.3. Truncated two-dimensional sampling expansions.
For a bandlimited signal \( f(t,s) \) with the Fourier transform \( F(u,v) \) the energy \( E \) is defined as
\[
E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t,s)|^2 \, dt \, ds = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |F(u,v)|^2 \, du \, dv. \quad (3.3.1)
\]
where \( \mathbb{R} \) is the region such that \( F(u,v) = 0 \) for \((u,v) \notin \mathbb{R}\). Consider now a two-dimensional linear shift invariant system with impulse response \( h(t,s) \) and system function \( H(u,v) \). If \( f(t,s) \) is taken as input to this system the output will be
\[
g(t,s) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} F(u,v) H(u,v) e^{i(ux + vy)} \, du \, dv. \quad (3.3.2)
\]
Applying the two-dimensional form of the Schwartz inequality to this equation and using the one above that we get
\[
|g(t,s)|^2 \leq \frac{E}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |H(u,v)|^2 \, du \, dv. \quad (3.3.3)
\]
Recall Shannon's sampling theorem in two-dimensional space as given by Eq. (1.6.11). That is,
\[
f(t,s) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f \left( \frac{n}{a} \frac{m}{b} \right) \frac{\sin a(t - \frac{nm}{a}) \sin b(s - \frac{nm}{b})}{a(t - \frac{nm}{a}) b(s - \frac{nm}{b})}. \quad (3.3.4)
\]
Define
\[
e_N(t+c,s+v) = f(t+c,s+v)
- \sum_{-N}^{N} \sum_{-N}^{N} f \left( t + \frac{nm}{a}, s + \frac{nm}{b} \right) \frac{\sin a(t - \frac{nm}{a}) \sin b(v - \frac{nm}{b})}{a(t - \frac{nm}{a}) b(v - \frac{nm}{b})}. \quad (3.3.5)
\]
If \( e_N(t_1, s_1) \) is the output of a two-dimensional, shift invariant system with input \( f(t, s) \), then it is easy to see that the system function goes as

\[
H(u, v) = e^{i(u\tau + v\nu)} \quad (3.3.6)
\]

\[
- \sum_{-N}^{N} \sum_{-N}^{N} e^{i \left( \frac{nu}{a} + \frac{mv}{b} \right)} \sin a \left( \frac{nu}{a} \right) \sin b \left( \frac{mv}{b} \right) \frac{1}{a \left( \frac{nu}{a} \right) b \left( \frac{mv}{b} \right)}.
\]

From Eq. (3.3.3) we get

\[
|e_N(t_1, s_1)|^2 \leq \frac{E}{4n^2} \int \int_R |e^{i(u\tau + v\nu)}|
\]

\[
- \sum_{-N}^{N} \sum_{-N}^{N} e^{i \left( \frac{nu}{a} + \frac{mv}{b} \right)} \sin a \left( \frac{nu}{a} \right) \sin b \left( \frac{mv}{b} \right) \frac{1}{a \left( \frac{nu}{a} \right) b \left( \frac{mv}{b} \right)}|dudv.
\]

If we expand \( e^{i(u\tau + v\nu)} \) into a Fourier series in the rectangle

\(-a \leq u \leq a, -b \leq v \leq b \) we arrive at

\[
e^{i(u\tau + v\nu)} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} D_{n,m} e^{i \left( \frac{nu}{a} + \frac{mv}{b} \right)} \quad (3.3.8)
\]

where

\[
D_{n,m} = \frac{1}{4ab} \int_{-b}^{b} \int_{-a}^{a} e^{i(u\tau + v\nu)} - e^{i \left( \frac{nu}{a} + \frac{mv}{b} \right)}|dudv
\]

\[
= \frac{\sin a \left( \frac{nu}{a} \right) \sin b \left( \frac{mv}{b} \right)}{a \left( \frac{nu}{a} \right) b \left( \frac{mv}{b} \right)}.
\]

From the three equations just above we have
\[ |e_N(t+T_s+\nu)|^2 \leq \frac{E}{\pi^2} \sum_{|n| > N} \sum_{|m| > N} \frac{\sin^2 a(T - \frac{nm}{a}) \sin^2 b(\nu - \frac{mn}{b})}{a^2 (T - \frac{nm}{a})^2 b^2 (\nu - \frac{mn}{b})^2}. \]

That is,

\[ \lim_{N \to \infty} e_N(t+T_s+\nu) = 0, \]

which proves Shannon's sampling theorem in two-dimensional space. We point out here that the above analysis can be applied to n-dimensional space, but two-dimensional signals have lots of applications in different areas, especially in modern optics. An optical arrangement can be considered as a linear system with input the amplitude of an object and response \( g(t,s) \) of its image. The Fourier transform \( F(u,v) \) of \( f(t,s) \) is proportional to the far field of its object.

3.4. Upper bounds for the variation of sampled signals.

If a bandlimited signal \( f(t) \) is sampled at the Nyquist rate it is interesting to determine upper bounds for the variation of \( f(t) \) between two successive sampling points. In certain applications the sampled values \( f\left(\frac{n}{2W}\right) \) are stored on magnetic tape or other devices for displaying or recording increments, therefore it is necessary to know this type of variation before recording.

Let \( \Delta f \) be defined as

\[ \Delta f = |f\left(\frac{n+1}{2W}\right) - f\left(\frac{n}{2W}\right)|. \]

From the above we get
\[ \max \triangle f = \max_n |f(\frac{n+1}{2W}) - f(\frac{n}{2W})| \leq \max_t |f(t + \frac{1}{2W}) - f(t)| \]

\[ = \max_t |\frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega) (e^{2\pi W} - 1) e^{i\omega t} d\omega|, \]

where \( F(\omega) \) is the spectrum of \( f(t) \). Lepscy and Todero (1969) showed different upper bounds for the right side of Eq. (3.4.2), as follows:

\[ \max |\frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega) (e^{2\pi W} - 1) e^{i\omega t} d\omega| \leq \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} |F(\omega)| |e^{2\pi W} - 1| d\omega \]

\[ = \frac{1}{\pi} \int_{-2\pi W}^{2\pi W} |F(\omega)| |\sin \frac{\omega}{4W}| d\omega. \]

If we let \( M = \max |F(\omega)| \), then by Schwartz's inequality Eq. (3.4.2) can be written as

\[ (\max \triangle f)^2 \leq \frac{1}{4\pi^2} \int_{-2\pi W}^{2\pi W} |F(\omega)|^2 d\omega \int_{-2\pi W}^{2\pi W} [4 \sin^2 \frac{\omega}{4W}] d\omega = 8W M^2. \]

Using the above formulas and the following inequalities,

\[ |\sin \frac{\omega}{4W}| \leq 1, \quad \text{(3.4.5)} \]

\[ |\sin \frac{\omega}{4W}| \leq |\frac{\omega}{4W}|, \quad \text{(3.4.6)} \]

\[ |\sin \frac{\omega}{4W}| \leq \frac{3}{4} - \frac{1}{4} \cos \frac{\omega}{2W}, \quad \text{(3.4.7)} \]

we get different upper bounds for \( \max \triangle f \), as shown in Table 3.1.
<table>
<thead>
<tr>
<th>max $f \leq$</th>
<th>deduced from</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4W$M</td>
<td>Eqs. (3.4.3) and (3.4.5)</td>
</tr>
<tr>
<td>$3W$M</td>
<td>Eqs. (3.4.3) and (3.4.7)</td>
</tr>
<tr>
<td>$\pi W$M</td>
<td>Eqs. (3.4.3) and (3.4.6)</td>
</tr>
<tr>
<td>$2\sqrt{2}$W$M$</td>
<td>Eqs. (3.4.3) and (3.4.4)</td>
</tr>
</tbody>
</table>

Table 3.1.

Remarks: By direct integration of Eq. (3.4.3) (recall that $M = \max |F(\omega)|$), we get

$$\frac{1}{\pi} \int_{-2\pi W}^{2\pi W} |F(\omega)| \sin \frac{\omega}{4W} \, d\omega \leq \frac{M}{\pi} \int_{-2\pi W}^{2\pi W} |\sin \frac{\omega}{4W}| \, d\omega = \frac{8}{\pi} WM ,$$

which is tighter than those bounds in Table 3.1 due to Lapschy and Todero (1969). Also, using the inequality $|\sin \frac{\omega}{4W}| \leq \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cos \frac{\omega}{2W}$ and Eq. (3.4.3), we have

$$\max \Delta f \leq \frac{M}{\pi} \int_{-2\pi W}^{2\pi W} [\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cos \frac{\omega}{2W}] \, d\omega = 2\sqrt{2} WM ,$$

which is the same as the last row of table 3.1 and proved with the aid of Schwartz's inequality by Lapschy and Todero (1969). These authors presented more interesting upper bounds by considering absolute moments of $F(\omega)$ and the energy of the signal $E$, namely

$$\mu_0 = \int_{-2\pi W}^{2\pi W} |F(\omega)| \, d\omega \quad (3.4.8)$$

and

$$\mu_1 = \int_{-2\pi W}^{2\pi W} \omega |F(\omega)| \, d\omega \quad (3.4.9)$$
\[ E = \int_{-\infty}^{\infty} f^2(t) \, dt = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} |F(\omega)|^2 \, d\omega. \]  
(3.4.10)

The most useful spectrum is a trapezoid with ratio \( k \) between the larger and the smaller bases. The case \( k = 1 \) corresponds to the rectangular spectrum, and \( k = 0 \) corresponds to a triangular spectrum.

![Fig. (3.1)](image)

For any trapezium, \( \max \Delta f \leq CWM \), where \( C \) is a function of \( k \) as shown in Table 3.2 and Fig. 3.2.

<table>
<thead>
<tr>
<th>max ( f \leq )</th>
<th>deduced from</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\pi}</td>
<td>V_0</td>
<td>)</td>
</tr>
<tr>
<td>( \frac{1}{\pi}</td>
<td>V_0</td>
<td>)</td>
</tr>
<tr>
<td>( \frac{1}{4\pi W}</td>
<td>V_1</td>
<td>)</td>
</tr>
<tr>
<td>( 2\sqrt{E}N )</td>
<td>Eqs. (3.4.2) and (3.4.10)</td>
<td>( 2\sqrt{\frac{2}{3}(1+2k)} )</td>
</tr>
</tbody>
</table>

Table 3.2.
Fig. (3.2)
Chapter IV
Walsh Functions and Sampling Expansions

4.1. A closed set of orthonormal functions.

Walsh (1923) introduced a complete set of orthonormal functions (see Appendix A.1). The definition is based on Rademacher or Haar functions. Harmuth (1969, p. 19) discussed a difference equation for Walsh functions as follows:

\[
\text{Wal}(0,t) = \begin{cases} 
1 & \text{for } -\frac{1}{2} \leq t < \frac{1}{2} , \\
0 & \text{for } t < -\frac{1}{2} , t \geq \frac{1}{2} ,
\end{cases}
\]  

(4.1.1)

and

\[
\text{Wal}(2^j + p,t) = (-1)^{[\frac{j}{2}]+p} \left\{ \text{Wal}[j,2(t+\frac{1}{4})] + (-1)^{j+p} \text{Wal}[j,2(t-\frac{1}{4})] \right\}, \\
p = 0 \text{ or } 1, j = 0, 1, 2, \ldots .
\]  

(4.1.2)

Harmuth (1969, pp. 49-51) introduced the term sequency for the Walsh functions, analogous to frequency for the two sinusoidal functions. In terms of sign changes or zero crossings sequency is defined as

\[
\text{sequency} = \begin{cases} 
\frac{\alpha + 1}{2} & \text{if } \alpha \text{ is odd}, \\
\frac{\alpha}{2} & \text{if } \alpha \text{ is even},
\end{cases}
\]  

(4.1.3)

where \( \alpha \) is the number of sign changes. Walsh functions can be classified as

\[
\text{Wal}(2i,t) = \text{cal}(i,t), \text{ sequency } = i = 0, 1, 2, \ldots .
\]  

(4.1.4)

and

\[
\text{Wal}(2i-1,t) = \text{sal}(i,t), \text{ sequency } = i = 1, 2, \ldots .
\]  

(4.1.5)
where the calc and sal notation was chosen from cosine and sine to indicate that connection and the "al" the connection with Walsh.

In Fig. (4.1) the first set of eight Walsh functions, with corresponding cosine and sine functions, is shown in the half open interval, \(-1/2 \leq t < 1/2\):

\[
\begin{array}{ccc}
\text{Sequence} & \text{Frequency} \\
\hline
+1 & 0 & \cos(0t) \\
\text{Cal}(0,t) & 0 & \cos(0t) \\
+1 & 1 & \sin(2\pi t) \\
\text{Sal}(1,t) & 1 & \sin(2\pi t) \\
-1 & 1 & \cos(2\pi t) \\
\text{Cal}(1,t) & 1 & \cos(2\pi t) \\
2 & 2 & \sin(4\pi t) \\
\text{Sal}(2,t) & 2 & \sin(4\pi t) \\
2 & 2 & \cos(4\pi t) \\
\text{Cal}(2,t) & 2 & \cos(4\pi t) \\
3 & 3 & \sin(6\pi t) \\
\text{Sal}(3,t) & 3 & \sin(6\pi t) \\
3 & 3 & \cos(6\pi t) \\
\text{Cal}(3,t) & 3 & \cos(6\pi t) \\
4 & 4 & \sin(8\pi t) \\
\text{Sal}(4,t) & 4 & \sin(8\pi t)
\end{array}
\]

Fig. (4.1)
Walsh functions can be represented by the signum of a product expansion of sine and cosine functions. The general expansion for Wal\(j,t\) is based on the binary representation of \(j\), i.e.,

\[
j = j_n 2^n + j_{n-1} 2^{n-1} + \ldots + j_1 2^1 + j_0 2^0. \tag{4.1.6}
\]

Ross and Kelly (1972) established the following formula for \(-1/2 \leq t < 1/2\),

\[
 Wal(j,t) = \text{sign}[\sin 2\pi t)(\cos 2\pi nt) \sum_{k=1}^{j_0^n} j_k]. \tag{4.1.7}
\]

**Example 4.1.1.** Construct Wal(5,t) from Eq. (4.1.7).

Solution: In binary form 5 = 101, therefore Wal(5,t) = sign[\sin 2\pi t \cos 4\pi t] (see Fig. (4.2)).

![Fig. (4.2)](image_url)

**4.2. Discrete Walsh Functions.**

The discrete Walsh functions are sampled values of the continuous set defined on \([0,1]\) by Walsh (1923) (see Fig. 4.3).
The $N$-length discrete Walsh function may be defined for $N = 2^n$, where $n$ is a positive integer, by a Hadamard matrix whose rows are ordered so that the number of sign changes in each row increases by one.

**Example 4.2.1.**

For $N = 2$:

- $j = 0, 1$
- $k = 0$
- Number of sign changes
- Sequency

$$Wal(j,k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

For $N = 4$:

- $j = 0, 1, 2, 3$
- $k = 0$
- Number of sign changes
- Sequency

$$Wal(j,k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

It is well-known that if the matrix $H$ is a Hadamard matrix of order $N$ then

$$G = \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$
is a Hadamard matrix of order \(2N\). Using this representation of discrete Walsh functions we have the following properties (Kenenth, 1970)

\[
\sum_{j=0}^{N-1} \text{Wal}(j,k) \text{Wal}(j,m) = \begin{cases} N, & k = m, \\ 0, & k \neq m. \end{cases}
\]

(4.2.2)

Walsh functions are symmetric,

\[
\text{Wal}(j,k) = \text{Wal}(k,j),
\]

(4.2.3)

and there is an addition formula

\[
\text{Wal}(m,j) \text{Wal}(k,j) = \text{Wal}(m \oplus k,j),
\]

(4.2.4)

where \(\oplus\) indicates addition modulo 2, i.e., bit by bit addition under the rules 0 + 1 = 1 + 0 = 1, 1 + 1 = 0 (no carry).

For an \(N\)-length sequence \(f(j)\) the finite Walsh transform is defined as

\[
F(k) = \frac{1}{N} \sum_{j=0}^{N-1} f(j) \text{Wal}(k,j), \quad k = 0, 1, \ldots, N-1,
\]

(4.2.5)

and the inverse finite Walsh transform of \(F(k)\) is

\[
f(j) = \sum_{k=0}^{N-1} F(k) \text{Wal}(j,k), \quad j = 0, 1, \ldots, N-1.
\]

(4.2.6)

4.3. Non-denumerable Walsh functions.

The system \(\{\sin 2\pi it, \cos 2\pi it\}\) is orthogonal in the interval \(-\frac{1}{2} \leq t < \frac{1}{2}\), where \(i\) is an integer; replacing \(i\) by a real number \(x\) we get an orthogonal system \(\{\sin 2\pi xt, \cos 2\pi xt\}\), for \(-\infty < t < \infty\).

The system of Walsh functions, orthogonal and complete in the whole interval \(-\infty < t < \infty\), is defined by \(\{\text{sal}(x,t), \text{cal}(x,t)\}\).

Pichler (1967) defined \(\text{sal}(x,t)\) and \(\text{cal}(x,t)\) by periodic
continuation of \( \text{sal}(1,t) \) and \( \text{cal}(1,t) \) as follows:

\[
\begin{align*}
\text{cal}(2^k, t) &= \text{cal}(1, 2^k t), \quad k = \pm 1, \pm 2, \pm 3, \cdots, \quad (4.3.1a) \\
\text{sal}(2^k, t) &= \text{sal}(1, 2^k t), \quad -\infty < t < \infty, \quad (4.3.1b)
\end{align*}
\]

Let \( x \) have the binary representation

\[
x = \sum_{-\infty}^{\infty} x_i 2^{-i}, \quad x_i \text{ is either} \ 0 \text{ or} \ 1, \quad (4.3.2)
\]

then

\[
\text{cal}(x, t) = \bigcap_{-\infty}^{\infty} \text{cal}(x_i 2^{-i}, t), \quad -\infty < t < \infty, \quad (4.3.3)
\]

and

\[
\text{sal}(x, t) = \begin{cases} 
-\text{cal}(x, t), & -\infty < t < 0, \ x \text{ dyadic irrational,} \\
\text{cal}(x, t), & 0 < t < \infty. 
\end{cases} \quad (4.3.4)
\]

Also

\[
\text{sal}(x, t) = \text{cal}(g 2^{-m}, t) \text{sal}(2^{-m}, t), \quad -\infty < t < \infty, \quad (4.3.5)
\]

where \( g \) is an even number and \( x = \frac{g+1}{2^m} \) is dyadic rational.

The following additional formulas are very important for computation with Walsh functions:

\[
\begin{align*}
\text{Wal}(x, t) &= \text{Wal}(0, t), \quad 0 \leq x < 1, \quad (4.3.6a) \\
\text{cal}(x, t) &= \text{cal}(i, t), \quad i \leq x < i+1, \quad -\frac{1}{2} \leq t < \frac{1}{2}, \quad (4.3.6b) \\
\text{sal}(x, t) &= \text{sal}(i, t), \quad i-1 < x \leq i, \quad (4.3.6c)
\end{align*}
\]

In terms of \( x \) and \( t \) real numbers,

\[
\begin{align*}
\text{cal}(2^k x, t) &= \text{cal}(x, 2^k t), \quad (4.3.7a) \\
\text{sal}(2^k x, t) &= \text{sal}(x, 2^k t), \quad (4.3.7b)
\end{align*}
\]
and as with the familiar sinusoidal functions,

\[ \text{cal}(x, -t) = \text{cal}(x, t), \quad (4.3.8a) \]
\[ \text{cal}(-x, t) = \text{cal}(x, t), \quad (4.3.8b) \]
\[ \text{sal}(x, -t) = -\text{sal}(x, t), \quad (4.3.8c) \]
\[ \text{sal}(-x, t) = -\text{sal}(x, t), \quad (4.3.8d) \]

For real \( x \) and \( t \) there is symmetry,

\[ \text{cal}(x, t) = \text{cal}(t, x), \quad (4.3.9a) \]
\[ \text{sal}(x, t) = \text{sal}(t, x). \quad (4.3.9b) \]

Pichler (1967) established two figures for the calculation of \( \text{cal}(x, t) \) and \( \text{sal}(x, t) \) in the rectangle \( 0 \leq x < 4, -3 < t < 3 \). Because of the symmetric and skew symmetric properties of the \( \text{cal} \) and \( \text{sal} \) functions Harmuth (1969, p. 24) obtained the following interesting figures for computation of \( \text{cal}(x, t) \) and \( \text{sal}(x, t) \) in the rectangle \( -4 < x < 4, -3 < t < 3 \). Black areas mean 1 and white areas indicate -1.

By drawing a line parallel to the \( t \)-axis we obtain \( \text{cal}(x, t) \) or \( \text{sal}(x, t) \) as a function of \( t \) for a particular value of \( x \), and vice versa. (See Fig. 4.4).
4.4. Sampling theorem in Walsh-Fourier analysis.

Definition 4.4.1. We say that \( f(t) \) is sequency limited if its Walsh-Fourier transform \( F(x) \), defined as

\[
F(x) = \int_{-\infty}^{\infty} f(t) \text{Wal}(x,t) \, dt,
\]

is zero outside the interval \((-Z,Z)\), or \( F(x) = 0 \) for \( |x| > Z \). The following theorem is due to Kak (1970).

Theorem 4.4.1: "A signal limited in sequency \( Z \) zeros/sec can be reconstructed completely from samples spaced \( 1/2^{k+1} \) seconds apart, where \( k \) is an integer such that \( 2^k > Z \). In particular,

\[
f(t) = f(-\frac{n}{2^{k+1}}), \quad \frac{n}{2^{k+1}} < t < \frac{n+1}{2^{k+1}}.
\]

Proof. Kak assumed \( f(t) \) to be an even function. Similar results can be obtained for \( f(t) \) an odd function and, in general, for an arbitrary function since \( f(t) = f_0(t) + f_e(t) \), where \( f_0(t) = \frac{1}{2} [f(t) - f(-t)] \) and \( f_e(t) = \frac{1}{2} [f(t) + f(-t)] \). If \( f(t) \) is sequency limited to \( Z \) zeros/sec then

\[
f(t) = \frac{1}{2} \int_{-Z}^{Z} F(x) \text{cal}(x,t) \, dx.
\]

Here the form of Kramer's generalized sampling theorem is as follows: over \((-Z,Z)\) the set \( \{\text{cal}(\frac{x}{2^Z},n), \text{sal}(\frac{x}{2^Z},n)\} \) is orthogonal and complete, so

\[
f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n}{2^Z}) S_n(t),
\]

where
Putting $x = 2Zy$ Eq. (4.4.4) can be written as

$$S_n(t) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} \text{cal}(2Zy, t) \text{cal}(y, n) \ dy.$$  

(4.4.5)

Whenever $Z$ is of the form $2^k$, $k$ an integer, we have, because of Eq.(4.3.7a),

$$\text{cal}(2Zy, t) = \text{cal}(y, 2Zt), \ -\frac{1}{2} \leq y < \frac{1}{2}.$$  

(4.4.6)

From Eqs. (4.4.5) and (4.4.6),

$$S_n(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} \text{cal}(y, 2Zt) \text{cal}(y, n) \ dy.$$  

(4.4.7)

or

$$S_n(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} \text{cal}(y, 2Zt \oplus n) \ dy = \begin{cases} 1, & 2Zt = n, \\ 0, & \text{otherwise}. \end{cases}$$  

(4.4.8)

Therefore $S_n(t) = 1$ whenever $t = n/2Z$. Because of Eq.(4.3.6b) we obtain

$$\text{cal}(y, 2Zt) = \text{cal}(y, n), \ n \leq 2Zt < n+1.$$  

(4.4.9)

Therefore $\text{cal}(y, 2Zt) \text{cal}(y, n) = \text{cal}(y, n) \text{cal}(y, n) = \text{cal}(y, n \oplus n)$

$= \text{cal}(y, 0) = 1$ or

$$\text{cal}(y, 2Zt \oplus n) = \text{cal}(y, 0) = 1.$$  

(4.4.10)
So \( S_n(t) = 1 \) for \( \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \) and

\[
f(t) = f\left(-\frac{n}{2^{k+1}}\right), \quad \frac{n}{2^{k+1}} \leq t < \frac{n+1}{2^{k+1}}. \quad (4.4.11)
\]

If \( Z \), the sequency bandwidth of the signal, is not of the form \( 2^k \), \( k \) an integer, we can consider a number of the latter form which is larger than the given \( Z \). We have not been able to get a reconstruction procedure for arbitrary \( Z \) because Eq. (4.4.6) is not valid for all \( Z \).

4.5. A comparison of the sampling theorems of Kak and Shannon.

Kak proved his theorem (4.4.1) with the aid of Kramer's generalized sampling theorem (2.1.1). Here we will give a simple proof of Kak's theorem without the use of the Kramer theorem. Since \( f(t) \) is sequency limited to \( Z \) zeros/sec

\[
f(t) = \frac{1}{2} \int_{-Z}^{Z} F(x) \text{cal}(x,t) \, dx. \quad (4.5.1)
\]

For \( x = 2Zy \) this becomes

\[
f(t) = 2Z \int_{-\frac{1}{2}}^{\frac{1}{2}} F(2Zy) \text{cal}(2Zy,t) \, dy. \quad (4.5.2)
\]

If \( Z = 2^k \), \( k \) an integer, then because of Eq. (4.3.7a) we obtain

\[
\text{cal}(2Zy,t) = \text{cal}(y,2Zt). \quad (4.5.3)
\]

Using Eqs. (4.3.6b) and (4.3.9a) we get

\[
\text{cal}(y,2Zt) = \text{cal}(y,n), \quad n \leq 2Zt < n+1. \quad (4.5.4)
\]

From Eqs. (4.5.2) and (4.5.4),
\[ f(t) = Z \int_{-1/2}^{1/2} F(2y) \text{cal}(y,n) \, dy, \quad n \leq 2Zt < n+1. \quad (4.5.5) \]

Therefore, in the interval \( \frac{n}{2Z} \leq t < \frac{n+1}{2Z} \) the above integral is independent of \( t \). Hence,

\[ f(t) = f\left(\frac{n}{2^{k+1}}\right), \quad \frac{n}{2^{k+1}} \leq t < \frac{n+1}{2^{k+1}}, \quad (4.5.6) \]

which is Kak's theorem.

From the above discussion we make the following statements:

**Statement 4.5.1.** If a signal \( f(t) \) is sequency limited to \( Z \) zeros per second then it is a step signal with jumps that occur at instants \( t = \frac{n}{2^{k+1}} \), where \( n \) is an integer and \( k \) is the smallest positive integer such that \( 2^k \geq Z \).

We give the following example to illustrate the above statement.

**Example 4.5.1.** Let

\[ F(x) = \begin{cases} 1, & 6 \leq |x| < 7, \\ 0, & \text{otherwise,} \end{cases} \]

then

\[ f(t) = \frac{1}{2} \int_{-7}^{7} F(x) \text{cal}(x,t) \, dx, \]

and \( Z = 7, 2^3 > 7 \), so \( k = 3 \) and \( t = n/16 \). It is easy to see that \( f(t) = \text{cal}(6,t) \). (See Fig. 4.5).

![Fig. (4.5)](image-url)
Statement 4.5.2. In Fourier analysis the Nyquist rate $2W$ is sufficient for the reconstruction of a signal which contains no frequencies higher that $W$ cps (Shannon's sampling theorem 1.1.1.). But in the Walsh-Fourier analysis a rate $2Z$ is not sufficient for the reconstruction of a signal limited in sequency to $Z$ zeros/sec unless $Z = 2^k$.

Statement 4.5.3. In Fourier analysis a signal $f(t)$ cannot be both time and frequency limited, but in Walsh-Fourier analysis a signal can be both time and sequency limited (see Example 4.5.1).

Statement 4.5.4. Kak's theorem can be proved by Walsh-Fourier analysis analogous to the proof of Shannon's sampling theorem by Fourier analysis. Our proof is based on the expansion of the Walsh spectrum $F(x)$ into the Walsh-Fourier series as follows. Since $F(x) = 0$ for $|x| \geq Z$,

$$f(t) = \frac{1}{2} \int_{-Z}^{Z} F(x) \text{cal}(x,t) \, dx, \quad (4.5.7)$$

Let $z = 2Zy$ so

$$f(t) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(2Zy) \text{cal}(2Zy,t) \, dy. \quad (4.5.8)$$

Let

$$G(y) = F(2Zy); \quad (4.5.9)$$

expand $G(y)$ into a cal-Fourier series of the form

$$G(y) = \sum_{i=0}^{\infty} a_i \text{cal}(y,i), \quad (4.5.10)$$

where

$$a_i = \frac{1}{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} G(y) \text{cal}(y,i) \, dy. \quad (4.5.11)$$
Substituting Eq. (4.5.10) into Eq. (4.5.8),

\[ f(t) = Z \int_{-\frac{1}{2}}^{\frac{1}{2}} a_i \cal{a}(y,i) \cal{a}(2Zy,t) \, dy \]

\[ = Z \sum_{i=0}^{\infty} a_i \int_{-\frac{1}{2}}^{\frac{1}{2}} \cal{a}(y,i) \cal{a}(2Zy,t) \, dy. \]  

(4.5.12)

If \( n \leq 2Zt < n+1 \), and \( Z = 2^k \), then because of Eqs. (4.3.7a) and (4.3.6b) we have \( \cal{a}(2Zy,t) = \cal{a}(y,2Zt) = \cal{a}(y,n) \), and Eq. (4.5.12) becomes

\[ f(t) = Z \sum_{i=0}^{\infty} a_i \int_{-\frac{1}{2}}^{\frac{1}{2}} \cal{a}(y,i \oplus n) \, dy. \]

(4.5.13)

But \( \int_{-\frac{1}{2}}^{\frac{1}{2}} \cal{a}(y,i \oplus n) \, dy = 0 \) except for \( i = n \), for which unity, hence

\[ f(t) = a_n Z, \quad \frac{n}{2Z} \leq t < \frac{n+1}{2Z}. \]

(4.5.14)

From Eq. (4.5.8) we get

\[ f(\frac{n}{2Z}) = Z \int_{-\frac{1}{2}}^{\frac{1}{2}} G(y) \cal{a}(2Zy,\frac{n}{2Z}) \, dy = Z \int_{-\frac{1}{2}}^{\frac{1}{2}} G(y) \cal{a}(y,n) \, dy \]

\[ = a_n Z. \]

(4.5.15)

These last two equations constitute Kak's theorem, namely

\[ f(t) = f(\frac{n}{2^{k+1}}), \quad \frac{n}{2^{k+1}} \leq t < \frac{n}{2^{k+1}}. \]

(4.5.16)

So, we have shown the equivalence of Kramer's and Shannon's sampling theorems for the case of Kak's theorem. Equivalence of Kramer's and Shannon's sampling theorems for other functions of mathematics and physics has been shown by Campbell (1964) and Jerri (1969b).
Statement 4.5.5. Upper bounds for the variation of sampled, sequency limited signals can be established as follows: If \( f(t) \) is sequency limited to \( Z \) zeros/sec then it is useful to find an upper bound for the variation of \( f(t) \) between two successive sampling points. Without loss of generality we can assume that \( f(t) \) is even. Similar results can be obtained for odd functions, and, in general, for any function. Let

\[
\Delta f = |f(\frac{n+1}{2Z}) - f(\frac{n}{2Z})|, \tag{4.5.17}
\]

then

\[
\max \Delta f = \max_n |f(\frac{n+1}{2Z}) - f(\frac{n}{2Z})|. \tag{4.5.18}
\]

Since \( F(t) \) is sequency limited to \( Z \) zeros/sec.,

\[
f(t) = \frac{1}{2} \int_{-Z}^{Z} F(x) \text{cal}(x,t) \, dx, \tag{4.5.19}
\]

where \( F(x) \) is the Walsh spectrum of \( f(t) \). From this integral we have

\[
|f(\frac{n+1}{2Z}) - f(\frac{n}{2Z})| = \frac{1}{2} \left| \int_{-Z}^{Z} F(x)[\text{cal}(x,\frac{n+1}{2Z}) - \text{cal}(x,\frac{n}{2Z})] \, dx \right| \tag{4.5.20}
\]

\[
\leq \frac{1}{2} \int_{-Z}^{Z} |F(x)| \, |\text{cal}(x,\frac{n+1}{2Z}) - \text{cal}(x,\frac{n}{2Z})| \, dx.
\]

Putting \( x = 2Zy \) the above can be written as

\[
|f(\frac{n+1}{2Z}) - f(\frac{n}{2Z})| \leq \frac{1}{Z} \int_{-\frac{1}{2}}^{\frac{1}{2}} |F(2Zy)| \, |\text{cal}(2Zy,\frac{n+1}{2Z}) - \text{cal}(2Zy,\frac{n}{2Z})| \, dy. \tag{4.5.21}
\]

If \( Z = 2^k \), then \( \text{cal}(2Zy,\frac{n+1}{2Z}) = \text{cal}(y,n+1) \) and \( \text{cal}(2Zy,\frac{n}{2Z}) = \text{cal}(y,n) \).
Let

\[ M = \max_{y} |F(2\pi y)|, \quad -\frac{1}{2} \leq y < \frac{1}{2}, \quad (4.5.22) \]

and Eq. (4.5.18) can be written as

\[ \max \Delta f \leq ZM \int_{-\frac{1}{2}}^{\frac{1}{2}} |\text{cal}(y, n+1) - \text{cal}(y, n)| \, dy. \quad (4.5.23) \]

But

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} |\text{cal}(y, n+1) - \text{cal}(y, n)| \, dy \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |\text{cal}(y, n+1)| \, dy + \int_{-\frac{1}{2}}^{\frac{1}{2}} |\text{cal}(y, n)| \, dy = 2. \quad (4.5.24) \]

Therefore

\[ \max \Delta f \leq 2ZM. \quad (4.5.25) \]

Comparing this result with those of Table I, Chapter III, Section 4, we notice that we have here a tighter bound. This is one of the advantages of working in the Walsh rather than the Fourier domain. The sine and cosine functions have been used in communication theory for a long time, but recently the system of orthogonal Walsh functions has come to the fore in theoretical investigation and for equipment design.

4.6. Sampling expansions in discrete and finite Walsh-Fourier analysis

The problem of sampling expansion in the case of continuous functions was discussed in sections 4 and 5 of this chapter. But digital transmission, high speed digital computer, etc., deal with large sets of finite and discrete signals instead of classical, continuous and infinite functions. Therefore, it is desirable to review the above problem in the case of discrete and finite functions.

Notations 4.6.1. The functions to be considered in this section are
defined on a domain \( B_n \), the set of non-negative integers less than \( N = 2^n \), where \( n \) is a positive integer. That is, \( B_n = \{0,1,\ldots,2^n-1\} \). Each element \( j \) of \( B_n \) has a unique expansion, \( j = \sum_{i=0}^{n-1} j_i 2^i \), where \( j_i = 0 \) or 1.

**Truncated Walsh finite series.** Let us define the partial sum by

\[
f_p(j) = \sum_{k=0}^{p-1} F(k) \text{Wal}(k,j), \tag{4.6.1}
\]

where

\[
F(k) = \frac{1}{N} \sum_{j=0}^{N-1} f(j) \text{Wal}(k,j). \tag{4.6.2}
\]

Inserting Eq. (4.6.2) into Eq. (4.6.1) we get

\[
f_p(j) = \frac{1}{N} \sum_{k=0}^{p-1} \sum_{i=0}^{N-1} f(i) \text{Wal}(k,i) \text{Wal}(k,j)
= \frac{1}{N} \sum_{k=0}^{p-1} \sum_{i=0}^{N-1} f(i) \text{Wal}(k,i \oplus j). \tag{4.6.3}
\]

Let the Walsh-Fourier (WF) kernel be

\[
d_p(j) = \sum_{k=0}^{p-1} \text{Wal}(k,j). \tag{4.6.4}
\]

From Eqs. (4.6.3) and (4.6.4) we have

\[
f_p(j) = \frac{1}{N} \sum_{i=0}^{N-1} f(i) d_p(i \oplus j) = f(j) \odot d_p(j), \tag{4.6.5}
\]

where \( \odot \) denotes the logical convolution.

Consider now the case \( p = 2^m = M \). It is easy to verify that
Theorem 4.6.2.

\[ d_p(j) = M \sum_{i=0}^{R-1} \delta(j \oplus i), \quad (4.6.6) \]

where \( R = N/M \). From Eqs. (4.6.5) and (4.6.6) we have

\[ f_M(j) = \frac{1}{R} \sum_{i=0}^{R-1} f(j \oplus i). \quad (4.6.7) \]

Definition 4.6.1. A function \( f(j) \) is said to be an \( M \) sequency band-limited signal (MBL) if its WF transform \( F(k) \) is zero for \( k > M \).

Various forms of sampling expansion for MBL signals due to Dinh, Le and Goulet (1972). If \( f(j) \) is an MBL signal then it can be reconstructed completely from its \( M \) values.

Theorem 4.6.1.

\[ f(j) = \frac{1}{M} \sum_{i=0}^{M-1} f(iR) d_M(j \oplus iR). \quad (4.6.8) \]

Theorem 4.6.2.

\[ f(j) = \frac{1}{M} \sum_{i=0}^{M-1} f(iR \oplus a) d_M(j \oplus iR \oplus a), \quad (4.6.9) \]

Theorem 4.6.3.

\[ f(j) = \frac{1}{M} \sum_{i=0}^{M-1} f(2iR) d_M(j \oplus 2iR) \]

\[ + \frac{1}{M} \sum_{i=0}^{M-1} f(2iR \oplus (2R-1)) d_M(j \oplus (2R-1) \oplus 2iR). \quad (4.6.10) \]

The sampling points are illustrated in Fig. 4.6 with \( N = 16 \) and \( M = 4 \), \( R = 16/4 = 4 \).
Proof of theorem 4.6.1., due to Dinh, Le and Goulet (1972). The following proof uses the convolution technique. Let us consider the impulse train

\[ h(j) = NR \sum_{i=0}^{M-1} \delta(j \oplus iR); \]  

(4.6.11)

the WF transform is

\[ H(k) = R \sum_{i=0}^{M-1} \text{Wal}(iR, k). \]  

(4.6.12)

Writing the Walsh function in terms of delta functions we have

\[ H(k) = N \sum_{i=0}^{2R-1} \delta(k \oplus 2iM) + N \sum_{i=0}^{2R-1} \delta(k \oplus (2M-1) \oplus 2iM). \]  

(4.6.13)

Let \( F(k) \) be the WF transform of an arbitrary MBL signal \( f(j) \).

Consider the logical convolution \( F(k) \oplus H(k) \), with \( H(k) \) given by Eq. (4.6.13).

\[ F(k) \oplus H(k) = \sum_{i=0}^{2R-1} F(k \oplus 2iM) + \sum_{i=0}^{2R-1} F(k \oplus (2M-1) \oplus 2iM). \]  

(4.6.14)

This equation is illustrated by Fig. (4.7b) for \( N = 16 \) and \( M = 4 \). A study of Fig. (4.7b) suggests that to regain \( F(k) \) it is sufficient to
to multiply $F(k) \odot H(k)$ by the function $W(k)$ shown in Fig. (4.7c).

$$F(k) = [F(k) \odot H(k)] W(k). \quad (4.6.15)$$

Fig. (4.7). Proof of theorem 4.6.1.

The inverse transform of Eq. (4.6.15) is

$$f(j) = \frac{1}{N} [f(j) h(j)] \odot w(j). \quad (4.6.16)$$

Substituting Eq. (4.6.11) into Eq. (4.6.16) leads to

$$f(j) = \frac{1}{M} \sum_{i=0}^{M-1} f(iR) w(j \oplus iR). \quad (4.6.17)$$

Since the transform of a WF kernel $d_M(j)$ is

$$D_M(k) = \sum_{j=0}^{M-1} \delta(k \oplus j) \quad (4.6.18)$$

it is clear that $w(j) = d_M(j)$. Hence, from Eqs. (6.4.17) and the pre-
ceding we have
\[ f(j) = \frac{1}{M} \sum_{i=0}^{M-1} f(iR) d_M(j \oplus iR). \]  
(4.6.19)

**Proof of Theorem 4.6.2:** Proof of this theorem is easy by applying Eq. (4.6.8) to \( f(j \oplus a) \), which is a MEL signal, and by replacing \( j \oplus a \) by \( j \) in the sampling expansion of \( f(j \oplus a) \).

**Proof of theorem 4.6.3:** Proof of this theorem is similar to that of theorem 6.4.1, the only difference is to start with the function next below rather than with Eq. (4.6.11).

\[ h(j) = NR \sum_{i=0}^{M-1} [ \delta(j \oplus iR) + \delta(j \oplus (2R-1) \oplus 2iR) ], \]  
(4.6.20)

whose WF transform is
\[ H(k) = N \sum_{i=0}^{R-1} \delta(k \oplus iM). \]  
(4.6.21)

**4.7. A new form of sampling theorem in discrete and finite Walsh-Fourier analysis.**

In the previous section we discussed the theorem of Dinh, Le and Goulet (1972), proved by them by a complicated and lengthy method. In this section we shall attempt to give a new form of that theorem and prove it by a different method. We also give an example to illustrate our proof. Generalization for higher dimensions can be made, and similar results can be obtained for discrete and finite Haar-Fourier analysis. Our notation is the same as in Not. (4.6.1), namely \( N = 2^n \), where \( n \) is a positive integer, \( M = 2^m \), and \( R = N/M \). We have the
following

Theorem 4.7.1. If \( f(j) \) is an MBIL signal then

\[
f(j) = \sum_{i=0}^{M-1} f(iR) \delta \left[ \bigcup_{p=iR}^{(i+1)R-1} (j-p) \right],
\]

where \( \delta(x) = 1 \) if \( x = 0 \), otherwise zero.

Proof: Consider a Walsh-Hadamard matrix of length \( N \). That is,

\[
H = (h_{ij}), \quad i, j = 0, 1, \ldots, N-1,
\]

\[
h_{ij} = 1 \text{ or } -1,
\]

and the number of sign changes in each row increases by one. Since our signal is MBIL the reconstruction procedure needs only the first \( M \) rows of the matrix in Eq. (4.7.2). The \( M \) entries in each row are sampled values of the corresponding Walsh functions (1923). Since we need only \( M \) samples to reconstruct the first \( M \) Walsh functions we conclude that the submatrix, \( H_1 = (h_{ij}), \quad i = 0, 1, \ldots, M-1, \quad j = 0, 1, \ldots, R-1 \), has \( R \) equal columns, and so does \( H_2 = (h_{ij}), \quad i = 0, 1, \ldots, M-1, \quad j = R, R+1, \ldots, 2R-1 \), \( H_m = (h_{ij}), \quad i = 0, 1, \ldots, M-1, \quad j = N-R, \ldots, N-1 \). Because of the inverse transform formulas of Eq. (4.2.6) this means that

\[
f(0) = f(1) = \ldots = f(R-1), \quad f(R) = f(R+1) = \ldots = f(2R-1),
\]

\[
\ldots \ldots \quad f(N-R) = f(N-R+1) = \ldots = f(N-1),
\]

which proves the theorem. The \( \delta \) notation is used to write our signal \( f(t) \) in the form of summation using sample values, \( f(0), f(R), \ldots, f(N-R) \).

Example 4.7.1. Let \( N = 2^3 = 8 \) and \( M = 2^2 = 4 \), then \( R = N/M = 2 \).
Suppose that our signal $f(t)$ is sequency limited by 4, meaning that $F(k) = 0$ for $k \geq 4$. Let $F(k)$ take on arbitrary values for $k = 0, 1, 2, 3$, for instance, $F(0) = 10$, $F(1) = 4$, $F(2) = 5$, $F(3) = 1$. The 8-length Walsh Hadamard matrix is given in Fig. (4.8).

$$
\begin{array}{cccccccc}
j = 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
 1 & 1 & -1 & -1 & -1 & 1 & 1 & 2 \\
 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
 1 & -1 & -1 & 1 & 1 & -1 & -1 & 4 \\
 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
 1 & -1 & 1 & -1 & -1 & 1 & -1 & 6 \\
 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
\end{array}
$$

Fig. (4.8)

This is a Walsh-Hadamard matrix of length 8 with 4 submatrices $4 \times 2$ with the property that each submatrix has equal columns. Corresponding Walsh functions were shown in Fig. (4.3).

By inverse transform Eq. (4.2.6) we have

$$
f(j) = \sum_{k=0}^{3} F(k) \text{Wal}(j,k),
$$

therefore

$$
f(0) = \sum_{k=0}^{3} F(k) \text{Wal}(0,k) = 10 + 4 + 5 + 1 = 20,
$$

$$
f(1) = \sum_{k=0}^{3} F(k) \text{Wal}(1,k) = 10 + 4 + 5 + 1 = 20,
$$

$$
f(2) = \sum_{k=0}^{3} F(k) \text{Wal}(2,k) = 10 + 4 - 5 - 1 = 8,
$$

f(3) = \sum_{k=0}^{3} F(k) \text{Wal}(3,k) = 10 + 4 - 5 - 1 = 8,
\text{f(4) = \sum_{k=0}^{3} F(k) \text{Wal}(4,k) = 10 - 4 - 5 + 1 = 2,}
\text{f(5) = \sum_{k=0}^{3} F(k) \text{Wal}(5,k) = 10 - 4 - 5 + 1 = 2}
\text{f(6) = \sum_{k=0}^{3} F(k) \text{Wal}(6,k) = 10 - 4 + 5 - 1 = 10}
\text{f(7) = \sum_{k=0}^{3} F(k) \text{Wal}(7,k) = 10 - 4 + 5 - 1 = 10.}

If we have only four sampled values, say \text{f(0) = 20, f(2) = 8, f(4) = 2, f(6) = 10}, by theorem (4.7.1),
\text{f(j) = \sum_{i=0}^{3} f(2i) \delta \left[ \prod_{p=0}^{1} (j - 2i - p) \right] \quad (4.7.3)}
\text{or}
\text{f(j) = \sum_{i=0}^{3} f(2i) \delta[(j-2i)(j-2i-1)] \quad (4.7.4)}
\text{or}
\text{f(j) = f(0) \delta[j(j-1)] + f(2) \delta[(j-2)(j-3)] + f(4) \delta[(j-4)(j-5)]}
\text{+ f(6) \delta[(j-6)(j-7)] \quad (4.7.5)}
\text{This gives}
\text{f(0) = f(0) = 20,}
\text{f(1) = f(0) = 20,}
\text{f(2) = f(2) = 8,}
\text{f(3) = f(2) = 8,}
Theorem 4.7.2. If \( f(j) \) is an MBL signal then

\[
f(j) = \sum_{i=0}^{M-1} f(iR \oplus a) \delta \left[ \frac{R(i+1)-1}{R} \right] \left( j \oplus a - p \right),
\]

\( a \in \{0, 1, \ldots, 2^n - 1\} \).

Proof: We can apply theorem 4.7.1 to \( f(j \oplus a) \), which is an MBL signal, and replacing \( (j \oplus a) \) by \( j \) in the sampling expansion. The above theorem is a sampling expansion with periodic sampling points starting from \( a \).

Theorem 4.7.3. If \( f(j) \) is an MBL signal then

\[
f(j) = \sum_{i=0}^{M-1} f(I^R_i) \delta \left[ \frac{R(i+1)-1}{R} \right] \left( j - p \right),
\]

\( I^R_1 \in \{iR, iR+1, \ldots, iR + (R-1)\} \).

This is the sampling theorem with non-periodic sampling points, and it can be proved by the same method as theorem 4.7.1. Actually the above theorem is theorem 4.7.1., if we let \( I^R_1 = iR \). We give the following example to illustrate our theorem.

Example 4.7.2. Suppose \( N = 2^5 = 32, M = 2^3 = 8 \), then \( R = N/M = 4 \), and \( I^L_1 \in \{41, 4i+1, 4i+2, 4i+3\} \). If we choose \( I^L_0 = 2, I^L_1 = 7, I^L_2 = 9, I^L_3 = 14, I^L_4 = 17, I^L_5 = 21, I^L_6 = 27 \) and \( I^L_7 = 28 \), then we have
non-periodic sampling points (see Fig. 4.9) and the values of the signal at those points is enough to reconstruct \( f(j) \) with 32 values by the following formula:

\[
f(j) = \sum_{i=0}^{7} f(i^4) \delta \left( \sum_{p=41}^{41+3} (j-p) \right).
\]

**Fig. (4.9).** Nonperiodic sampling points with the values of signal at those points.

### 4.8. Two dimensional sampling expansions in discrete and finite Walsh-Fourier analysis.

**Notations 4.8.1.** Our domain is the set \( A_n = \{ (i,j) \} \), \( i, j \in B_n \), where \( B_n = \{ 0, 1, \ldots, 2^n - 1 \} \) and \( n \) is a non-negative integer. \( N = 2^n \), \( M = 2^m \), where \( m \in B_n \) and \( R = N/M \).

**Definition 4.8.1.** The Walsh-Fourier transform for a signal \( f(x,y) \) and its inverse transform are respectively defined as

\[
F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \text{Wal}(u,x) \text{Wal}(v,y),
\]

and

\[
f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) \text{Wal}(u,x) \text{Wal}(v,y),
\]

where \( u, v, x, y \) are non-negative integers belonging to \( B_n \).

**Definition 4.8.2.** A signal \( f(x,y) \) is said to be \( M \) sequency band-
Theorem 4.8.1. If \( f(x, y) \) is an \( M_2 \) signal then it may be reconstructed completely from its \( M_2 \) values. That is,

\[
f(x, y) = \sum_{i=0}^{M_1-1} \sum_{j=0}^{M_1-1} f(i, j) \delta \left( \sum_{p=1}^{R(i+1)-1} (x - p) \right) \delta \left( \sum_{q=1}^{R(j+1)-1} (y - q) \right).
\]

Our proof will be similar to the proof of theorem 4.7.1. for the one dimension case. Here we illustrate our theorem by the following example:

Example 4.8.1: Let \( N = 2^3 = 8 \) and \( M = 2^2 = 4 \), then \( R = 2 \). The periodic sampling points are depicted in Fig. (4.10), and the values of the signal at those points are enough to reconstruct the signal. That is,

\[
f(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} f(2i, 2j) \delta \left( \sum_{p=1}^{2i+1} (x - p) \right) \delta \left( \sum_{q=2j}^{2j+1} (y - q) \right).
\]

Fig. (4.10). Periodic sampling points in two dimensions with the (0,0) as the first sample point.

Theorem 4.8.2. If \( f(x, y) \) is an \( M_2 \) signal then
\[ f(x,y) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} f(iR \oplus a, jR \oplus b) \delta \left[ \bigcup_{p=1}^{R(i+1)-1} (x \oplus a - p) \right] \]

\[ \times \left[ \bigcup_{q=1}^{R(j+1)-1} (y \oplus b - q) \right]. \]

where \( a \) and \( b \in \mathbb{B}_n \).

**Example 4.8.2.** If \( N = 2^3 = 8 \), \( M = 2^2 = 4 \), \( a = 1 \), \( b = 3 \), then

\[ f(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} f(2i \oplus 1, 2j \oplus 3) \delta \left[ \bigcup_{p=2i}^{2i+1} (x - p) \right] \delta \left[ \bigcup_{q=2j}^{2j+1} (y - q) \right]. \]

See Fig. (4.11)

![Fig. (4.11). Periodic sampling points in two dimensions with (1,3) as the first sample point.](image)

**Theorem 4.8.3.** If \( f(x,y) \) is \( M \)-BL signal then

\[ f(x,y) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} f(I_i^{R}, I_j^{R}) \delta \left[ \bigcup_{p=1}^{R(i+1)-1} (x - p) \right] \delta \left[ \bigcup_{q=1}^{R(j+1)-1} (y - q) \right], \]

where

\[ I_i^{R} \in \{ iR, iR+1, \ldots, iR + (R-1) \}, \]

and
This is a non-periodic sampling expansion of an $M^{th}$ signal.

**Example 4.8.3.** If $N = 2^3 = 8$, $M = 2^2 = 4$, then $R = 2$, $I_1 \in \{2i, 2i+1\}$, $I_2 \in \{2j, 2j+1\}$, and

$$f(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} f(I_1^2, I_2^2) \delta \left( \frac{2i+1}{2} \right) \delta \left( \frac{2j+1}{2} \right).$$

(See Fig. 4.12).

Fig. (4.12). Nonperiodic sampling points in two dimensions.
5.1. A complete orthonormal sequence of Haar functions.

Because of the recent interest in the applications of Walsh functions several investigators have suggested that other complete, orthogonal systems of functions may be useful. Shore (1973a,b,c) showed that Haar functions, close relatives of the Walsh functions, are useful in various ways. Haar (1910) published the results of his doctoral dissertation, a complete, orthonormal sequence of functions on [0,1] (see appendix, A.2). They are defined as

\[ \{H_n\} = H_0; H_1^1, H_2^2; \ldots; H_k^1, \ldots, H_k^{2^{k-1}} \]

\[ H_0(x) = 1, \quad 0 \leq x \leq 1, \]

\[ H_1^1(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases} \]

\[ H_2^1(x) = \begin{cases} \sqrt{2}, & 0 \leq x < \frac{1}{4} \\ -\sqrt{2}, & \frac{1}{4} < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x \leq 1 \end{cases} \]

\[ \vdots \]

\[ H_n(x) = \begin{cases} \frac{n-1}{2}, & 0 \leq x < \frac{m-1}{2} \\ \frac{n-1}{2^n-1}, & \frac{m-1}{2^n-1} < x < \frac{m}{2^n-1} \\ 0, & 0 \leq x \leq \frac{m-1}{2^n-1} \text{ and } \frac{m}{2^n-1} < x \leq 1. \end{cases} \]
At points of discontinuity the Haar functions are defined to be the average of the limits from the right and from the left. Fig. 5.1 shows the first eight Haar functions.

Fig. (5.1). First eight Haar functions.
Haar functions will be referred to by order $m$ and degree $n$, rather than by sequency or frequency, analogous to Legendre functions (Gubbins, Scollar and Wisskirchen, 1971). The discrete Haar functions are sampled values of the continuous set. The $N$-length discrete Haar functions may be defined for $N = 2^n$, $n$ a positive integer, in terms of a Haar matrix:

Example 5.1.1

For $N = 2$, 

$H(k,j) = \begin{bmatrix}
1 & 1 \\
1 & -1 \\
0 & 1
\end{bmatrix}$

For $N = 4$, 

$H(k,j) = \begin{bmatrix}
1 & 1 & -1 & -1 \\
\sqrt{2} & -\sqrt{2} & 0 & 2 \\
0 & 0 & \sqrt{2} & 3
\end{bmatrix}$

For $N = 8$, 

$H(k,j) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -2 & 7
\end{bmatrix}$

Notations 5.1.1. Our domain is the set of non-negative integers less than $N = 2^n$, where $n$ is a positive integer. That is, 

$B_n = \{0, 1, 2, \ldots, 2^n - 1\}$, $M = 2^m$ for $m \in B_n$, and $R = N/M$.

Definition 5.1.2. For an $N$-length real sequence $f(j)$ the finite Haar transform and its inverse are
\[
F(k) = \frac{1}{N} \sum_{j=0}^{N-1} f(j) H(k,j), \quad (5.1.5)
\]
and
\[
f(j) = \sum_{k=0}^{N-1} F(k) H(k,j). \quad (5.1.6)
\]

**Definition 5.1.1.** For an \(N^2\)-length real sequence \(f(i,j)\) the Haar transform and its inverse are:

\[
F(k,m) = \frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} f(i,j) H(k,i) H(m,j), \quad (5.1.7)
\]
and
\[
f(i,j) = \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} f(k,m) H(k,i) H(m,j). \quad (5.1.8)
\]

5.2. **Sampling expansions in discrete and finite Haar-Fourier analysis.**

Fino (1972) showed that for some applications Haar transforms perform as well as, and faster, than Walsh-Hadamard transforms. For this reason we shall attempt to establish sampling expansions associated with the Haar transform which have not as yet appeared in the literature. In this section and the next we shall show that our theorems of Chap. IV, sections seven and eight, shall apply in the Haar domain if we are able to arrive at a definition analogous to the definition 4.6.1 for \(M\)-sequency bandlimited signals. Since we do not have the concept of sequency for Haar functions we can establish our definition based on the order of these functions as follows:

**Definition 5.2.1.** A signal \(f(j)\) is called \(M\)-th order bandlimited if its Haar transform \(F(k)\) is zero for \(k > M\).
For a signal of this type we have the following

**Theorem 5.2.1.** If \( f(j) \) is \( M \)th order bandlimited signal then

\[
f(j) = \sum_{i=0}^{M-1} f(iR) \delta [ \bigcap_{p=iR}^{R(i+1)-1} (j-p) ].
\]

(5.2.1)

**Theorem 5.2.2.** If \( f(j) \) is an \( M \)th order bandlimited signal then

\[
f(j) = \sum_{i=0}^{M-1} f(iR \oplus a) \delta [ \bigcap_{p=iR}^{R(i+1)-1} (j \oplus a - p) ].
\]

(5.2.2)

where \( a \in \{0, 1, 2, \ldots, 2^n - 1\} \).

**Theorem 5.2.3.** If \( f(j) \) is an \( M \)th order bandlimited signal then

\[
f(j) = \sum_{i=0}^{M-1} f(I_1^R) \delta [ \bigcap_{p=iR}^{R(i+1)-1} (j-p) ].
\]

(5.2.3)

where \( I_1^R \subseteq \{iR, iR+1, \ldots, iR + (R-1)\} \).

**Proofs.** The proofs of these three theorems are the same as those for theorems (4.7.1), (4.7.2) and (4.7.3). We give an example to illustrate theorem 5.2.1.

**Example 5.2.1.** Let \( N = 2^3 = 8 \) and \( M = 2^2 = 4 \), then \( R = N/M = 2 \).

Suppose our signal \( f(j) \) is \( 4 \)-th order bandlimited signal, then \( F(k) = 0 \) for \( k \geq 4 \). Let \( F(k) \) take arbitrary values for \( k = 0, 1, 2 \) and \( 3 \), such as \( F(0) = 10, F(1) = 2, F(2) = \sqrt{2}, \) and \( F(3) = 3\sqrt{2} \).

The 8-length Haar matrix is given in Fig. 5.2. By Eq. (5.1.6) the inverse transform goes as

\[
f(j) = \sum_{k=0}^{3} F(k) H(k,j)
\]

or
If we have four sampled values, \( f(0) = 14 \), \( f(2) = 10 \), \( f(4) = 20 \), \( f(6) = 2 \), then by theorem 5.2.1,

\[
f(j) = f(0)H(j,0) + f(1)H(j,1) + f(2)H(j,2) + f(3)H(j,3) = 14,
\]

\[
f(1) = f(0)H(1,0) + f(1)H(1,1) + f(2)H(1,2) + f(3)H(1,3) = 14,
\]

\[
f(2) = f(0)H(2,0) + f(1)H(2,1) + f(2)H(2,2) + f(3)H(2,3) = 10,
\]

\[
f(3) = f(0)H(3,0) + f(1)H(3,1) + f(2)H(3,2) + f(3)H(3,3) = 10,
\]

\[
f(4) = f(0)H(4,0) + f(1)H(4,1) + f(2)H(4,2) + f(3)H(4,3) = 20,
\]

\[
f(5) = f(0)H(5,0) + f(1)H(5,1) + f(2)H(5,2) + f(3)H(5,3) = 20,
\]

\[
f(6) = f(0)H(6,0) + f(1)H(6,1) + f(2)H(6,2) + f(3)H(6,3) = 2,
\]

\[
f(7) = f(0)H(7,0) + f(1)H(7,1) + f(2)H(7,2) + f(3)H(7,3) = 2.
\]

Fig. (5.2). Haar matrix of length 8 with 4 submatrices (4 x 2) with the property that each submatrix has equal columns. The corresponding Haar functions were shown in Fig. 5.1.

If we have four sampled values, \( f(0) = 14 \), \( f(2) = 10 \), \( f(4) = 20 \), \( f(6) = 2 \), then by theorem 5.2.1,

\[
f(j) = \sum_{i=0}^{3} f(2i) \delta \left[ \int_{p=2i}^{2i+1} (j-p) \right],
\]

which gives \( f(0) = f(0) = 14 \), \( f(1) = f(0) = 14 \), \( f(2) = f(2) = 10 \), \( f(3) = f(2) = 10 \), \( f(4) = f(4) = 20 \), \( f(5) = f(4) = 20 \), \( f(6) = f(6) = 2 \), \( f(7) = f(6) = 2 \).

5.3. Two-dimensional sampling expansions in discrete and finite Haar-
Fourier analysis.

In this section we shall generalize our results of previous sections for two-dimensional sampling expansions. The same methods apply for cases of higher dimensions. First we need a definition for two dimensions similar to the definition 5.2.1. for one dimension:

Definition 5.3.1. A signal \( f(x,y) \) is said to be \( M^2 \)th order bandlimited if its Haar transform \( F(u,v) \) is zero for \( u > M, v > M \). Here \( u, v, x, y \) are non-negative integers belonging to \( B_n \), and \( M = 2^m \) with \( m \in B_n \). For a signal of this type we have the following

Theorem 5.3.1. If \( f(x,y) \) is \( M^2 \)th order bandlimited signal then \( f(x,y) \) can be reconstructed completely from its \( M^2 \) values. That is,

\[
f(x,y) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} f(iR,jR) \delta \left( \sum_{p=iR}^{R(i+1)-1} (x-p) \right) \delta \left( \sum_{q=jR}^{R(j+1)-1} (y-q) \right). \tag{5.3.1}
\]

Theorem 5.3.2. If \( f(x,y) \) is \( M^2 \)th order bandlimited signal, then

\[
f(x,y) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} f(iR+a,jR+b) \delta \left( \sum_{p=iR}^{R(i+1)-1} (x+a-p) \right) \delta \left( \sum_{q=jR}^{R(j+1)-1} (y+b-q) \right), \tag{5.3.2}
\]

where \( a \) and \( b \in B_n \).

Theorem 5.3.3. If \( f(x,y) \) is \( M^2 \)th order bandlimited signal, then

\[
f(x,y) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} f(iR_1,jR_2) \delta \left( \sum_{p=iR}^{R(i+1)-1} (x-p) \right) \delta \left( \sum_{q=jR}^{R(j+1)-1} (y-q) \right), \tag{5.3.3}
\]

where \( R_1 \in \{ iR, iR+1, \ldots, iR+(R-1) \} \).
Proofs: The proofs of the above theorems will be similar to the proofs of theorems (4.8.1), (4.8.2) and (4.8.3). We give an example to illustrate the proof of theorem (5.3.1).

Example 5.3.1. Suppose \( f(x,y) \) is second order bandlimited signal. If \( N=4 \) then the following four sampled values, \( f(0,0) = 18, f(0,2) = 4, f(2,0) = 12, f(2,2) = 6 \), are enough to reconstruct \( f(x,y) \), and by theorem 5.3.1.,

\[
f(x,y) = \sum_{i=0}^{1} \sum_{j=0}^{1} f(2i,2j) \delta[(2i+p) \mod 4] \delta[(2j+q) \mod 4].
\]

(See Fig. 5.3).

5.4. Discrete representation of continuous time signals.

In some applications the input and output are continuous time signals but the process is digital. This means that the input must first be represented by a sequence; after this sequence is processed the output sequence is reconverted to the original time signal. When the signal is bandlimited, in Fourier, Walsh or Haar domain, reconstruction procedures are referred to as periodic sampling expansions, as we have
been discussing in this dissertation. The advantage of a discrete representation of a continuous time signal, based on a periodic sampling expansion, is that the signal $f(t)$ is not required to be available for all $t$ in order to obtain the coefficients in the expansion. For example, in Shannon's sampling expansion we need only the sampled values $f(\frac{n}{2W})$ of $f(t)$ to reconstruct the signal. In other words, the periodic sampling is a memoryless transformation. No information of $f(t)$ is required between sample points. The major disadvantage to this representation is that it requires that $f(t)$ be bandlimited. If $f(t)$ is not bandlimited Oppenheim and Johnson (1972) showed that discrete representation of $f(t)$ is possible, but the major disadvantage is that the entire waveform $f(t)$ must be available before any of its values in the discrete sequence can be obtained. We give the following discrete representation of a waveform $f(t)$ which is not bandlimited:

$$f(t) = c_0 + \sum_{i=1}^{\infty} \sum_{m=1}^{2^{i-1}} c_m H_1(t), \tag{5.4.1}$$

where $c_m^i = \int_{0}^{1} f(t) H_1^m(t) \, dt$ and $H_1^m(t)$ is the Haar function of degree $i$ and order $m$. The partial sums

$$S_n(t) = c_0 + \sum_{i=1}^{n} \sum_{m=1}^{2^{i-1}} c_m^i H_1(t), \tag{5.4.2}$$

where $n$ is a positive integer, have the following properties:

(i) $S_n(t)$ contains $2^n$ terms;

(ii) $S_n(t)$ is a step function with $2^n$ equal length steps, and the value of $S_n(t)$ at each step is the mean value of $f(t)$ in that
step.

(iii) \( S_n(t) \) is the best step function approximation of \( f(t) \) in the mean square error sense. We divide the interval \([0,1]\) into \( N = 2^n \) equal parts and denote the average values of \( f(t) \) in these sub-intervals by \( f_1, f_2, \ldots, f_n \). The step function that has the value \( f_k \) in the interval \( \left( \frac{k-1}{2^n}, \frac{k}{2^n} \right) \) is the nth Haar partial sum approximation to \( f(t) \).

Shore (1973a) gave the following physical interpretation of the above. The waveform \( f(t) \) is passed through an integrator that resets to zero every \( \frac{1}{2^n} \) second. Our integrator has been set up in such a way that the integral after a period of \( \frac{1}{2^n} \) seconds is the mean value of \( f(t) \) during that period. The output from the integrator is sampled and held for a period of \( \frac{1}{2^n} \) seconds. Therefore, the output of the integrator, sampler and holder may be recognized as the Haar series partial sum \( S_n(t) \), and the combination of the three devices constitutes a lowpass sequency filter, analogous to the frequency lowpass filter.

Therefore, we conclude that when a waveform \( f(t) \) is not bandlimited it must be available in its entirety before any of the values in the discrete sequence can be obtained.
BIBLIOGRAPHY


APPENDIX A

A.1. Walsh functions form a complete orthonormal set in $L[0,1]$.  

(a) Orthogonality. Lackey and Meltzer (1971) showed that Walsh functions can be written as the product of Rademacher functions, which preserve the order of zero crossings. That is, if $n$ is binary

$$n = b_mb_{m-1} \cdots b_1,$$

then $n$ in the Grey code is

$$n = g_mg_{m-1} \cdots g_1,$$

where

$$g_1 = b_1 \oplus b_2,$$
$$g_2 = b_2 \oplus b_3, \quad \vdots$$
$$g_{m-1} = b_{m-1} \oplus b_m,$$
$$g_m = b_m.$$

With these notations,

$$\text{Wal}(n,t) = \left[R(m,t)\right] \left[R(m-1,t)\right] \cdots \left[R(1,t)\right],$$

where

$$R(0,t) = 1, \quad R(m,t) = \text{sign}(\sin 2\pi t), \quad m = 1, 2, 3, \ldots.$$ 

Alexits (1961, p. 54) showed that

$$\int_0^1 [R(n_1,t) \cdots R(n_p,t)][R(m_1,t) \cdots R(m_q,t)]dt = 0 \quad (A.1.4)$$

This establishes the orthogonality of the Walsh functions and it is
Furthermore, we have
\[ \int_0^1 \left| \text{Wal}(n,t) \right|^2 dt = 1. \]  
(A.1.5)

(b) Completeness: Suppose \( f(t) \in L^2[0,1] \) and is orthogonal to all elements of a Walsh sequence. We shall prove that \( f(t) = 0 \), a.e.

Let
\[ F(x) = \int_0^x f(t) dt. \]  
(A.1.6)

Then
\[ F(1) - F(0) = \int_0^1 \text{Wal}(0,t) f(t) dt = 0, \]  
(A.1.7)

so
\[ F(1) = F(0) = 0. \]  
(A.1.8)

Furthermore, we have
\[ 0 = \int_0^1 \text{Wal}(1,t) f(t) dt = \int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt \]  
(A.1.9)
\[ = F(1/2) - F(0) - [F(1) - F(1/2)] = 2 F(1/2) = 0. \]

Also
\[ 0 = \int_0^1 \text{Wal}(2,t) f(t) dt = \int_0^{1/4} f(t) dt - \int_{1/4}^{3/4} f(t) dt + \int_{3/4}^1 f(t) dt \]  
(A.1.10)
\[ = F(1/4) - [F(3/4) - F(1/4)] + F(1) - F(3/4) = 2[F(1/4) - F(3/4)] = 0, \]

and
\[ 0 = \int_0^1 \text{Wal}(3,t) f(t) dt = \int_0^{1/4} f(t) dt - \int_{1/4}^{1/2} f(t) dt + \int_{1/2}^{3/4} f(t) dt + \int_{3/4}^1 f(t) dt \]  
(A.1.11)
From Eqs. (A.1.10) and (A.1.11) we obtain
\[ F(\frac{1}{4}) = F(\frac{1}{2}) + F(\frac{1}{4}) = 0. \]
By induction on \( n \) we prove that
\[ F(\frac{k}{2^n}) = 0, \quad n = 0, 1, 2, \ldots, \quad k = 0, 1, 2, \ldots, \quad 2^n. \]  
Since \( F(x) \) is continuous and the binary rationals are dense in \([0,1]\), we conclude that \( F(x) \) = 0 a.e.. Therefore \( f(t) = 0 \) a.e., which proves that the Walsh functions form a complete set in \( L^2[0,1] \).

A.2. Haar functions form a complete orthonormal set in \( L^2[0,1] \).

(a) Orthogonality. It is clear that \( H_0 \) and \( H_1 \) are orthogonal to other functions of the set. The product \( H_i H_j \) = 0 for \( i \neq j \), and for \( n > m \) we have
\[ \int_{0}^{1} H_n(x) H_m(x) \, dx = \frac{1}{2} \int_{0}^{1} H_n(x) \, dx = 0. \]  
(b) Completeness. The proof is exactly the same as the proof of completeness for Walsh functions.