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Title: A QUADRATIC PROGRAMMING ALGORITHM FOR DERIVING EFFICIENT FARM PLANS IN A RISK SETTING

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The main focus was on developing an algorithm and supporting computer programs for use by extension personel to counsel farm managers on problems of enterprise choice.

Investigation was initiated from the complete certainty viewpoint of linear programming. Upon introducing uncertainty, ramifications of changing expected income, variance and the correlation coefficient between enterprises were explored. This was extended to develop a quadratic programming algorithm which resulted in complete algebraic specification of the efficiency frontier through integration of the Lagrangian multipliers.

The Von Neuman-Morgenstern utility analysis framework was posed for selecting the best alternative but dismissed as being cumbersome for practical application. A probability of loss function which places confidence intervals about the income level of each

alternative was used since it is more amenable for application by extension workers.

Data requirements were found to be no more difficult to satisfy in the quadratic programming model than in the presently used linear programming models. The triangular probability distribution was used in obtaining subjective estimates for the mean and variance of prices and yields. Subjective methods for deriving covariances between incomes from farm enterprises were discarded as being difficult to administer and subject to inconsistencies. A regional correlation matrix was used from which specific covariance estimates for individual decision problems were computed.

Seven cases were studied as a test of the computer programs and the algorithm. Four of these cases were submitted from actual farm situations by an extension agent. Output from the computer provided each farmer with a report containing the composition of every efficient plan, the pattern of resource use, the shadow prices of limiting resources and confidence statements about achieving certain levels of gross margin. The report was presented in tabular form, in graphic form and as a set of algebraic equations. Although no extensive test of acceptance by farm decision makers was made, results with the four cases studied appeared encouraging.

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A Quadratic Programming Algorithm for Deriving Efficient Farm Plans in a Risk Setting

by

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Advising Under Uncertainity - A Gap

Applied Farm Management by extension personnel has traditionally been of a prescriptive nature. Risk and uncertainty largely have been ignored.[1] Input and product prices and technical coefficients have been assumed to occur with certainty. In general these coefficients have either been projections of historical data or expected values (a long run implication) of random variables. Partial budgets and linear programming have been the principal planning tools used in this problem-solving framework.

Extension workers sometimes are perplexed to find that clients do not implement recommendations based on that combination of activities which will achieve a maximum expected net income. Often the

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[1] Often the term "risk" is reserved for describing future events which can be predicted in an actuarial sense and "uncertainty" is used to describe future events about which such empirical predictions can not be made. In this thesis no such distinction between the two terms will be made. Risk and uncertainty will be used interchangeably to mean that the occurance of a future event is not known with certainty but the decision-maker has, on the basis of historical information or a subjective feeling, some notion about the probability distribution of the event.

client has chosen some modification that results in an income level less than the optimum perceived by the extension worker.

This raises a question about the applicability and completeness of extension advice. Might it be that the extension worker perceives the decision maker's goals and objectives differently from what they in fact are? Might this not be further magnified in an environment of uncertainty where the decision maker stands the chance of economic disaster? It is not so much a lack of theory that inhibits the solution as it is in operational tools.

Evolution of Theory and Operational Planning Tools

During this century there has been rapid development of theory and tools to solve management problems. Although there were some writings (46) prior to the 1920's, it was not until J. D. Black wrote his now classic book Introduction to Production Economics (3) that there emerged a systematic treatment of economics which focused on the use of marginal analysis criteria in agricultural decision making. In his book, Black incorporated the ideas of: (a) statistical methods applied to production relationships by Spillman (40); (b) statistical analyses using individual farm survey data by Tolley, Black and Ezekiel (42) and; (c) neo-classical theory of the firm. This marked the birth of experimentalist philosophy in agricultural economics, a blend of the empiricist[2] and rationalist[3] schools (27). The experimentalist philosophy began to grow in the 1930's nurtured by developments in the field of general economics including the contributions of J. R. Hicks (22) who applied basic concepts of mathematics to the theory of the firm. Developments in agricultural economics followed with Heady's (19) integrative work in the late 1940's, which was continued into the 1950's and 60's by his disciples. Once the concepts of marginal analysis were refined and adopted for use, interest of several agricultural economists, including Johnson (28) and Halter (16) focused on the management processes of farmers.

While developments described above were taking place, a new field called operations research, conceived by engineers, mathematicians and statisticians was taking form. A major contributor to operations research was Dantzig (9) who in 1947 devised the simplex method for optimizing linear functions subject to linear constraints. This tool became known as linear programming. It was soon adopted for use in agricultural economics because of its operational depth and simplicity in solving production problems. In 1958 Dorfman, Samuelson and Solow (11) provided an economic interpretation to linear programming.

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[2] The empiricist philosophy is predicted on collecting "facts", unhampered and unbiased by considerations of theory.

[3] The rationalist philosophy contends that questions of theory must be answered before facts are worthy of consideration.

In that same year Heady and Candler (20) published their widely used text book on applications of linear programming to solving economic problems in agriculture.

Also during the 1940's, a most productive era for economics, Von Neuman and Morgenstern (44) revived the concept of cardinal utility[4] and introduced the theory of games. This rekindled an interest in problems of risk and uncertainty which had been discussed in the 1920's by Knight (30) but had lacked a practical mechanism for application. A theorem concerning probabilities, proven nearly two centuries ago by Thomas Bayes, an English mathematician and clergyman, was brought to bear on decision problems. Since the 1950's, increased emphasis has been placed upon theory. The names of Wald (45), Hurwicz, as cited by Luce and Raiffa (32, p. 492), and Friedman and Savage (15) stand as important contributors to the theory. Halter and Dean (17) give an excellent treatment of the present state of decision theory and its application to agriculture.

Computer technology development became an important precursor of another new approach--simulation and systems analysis. Forrester's (13) Industrial Dynamics is a notable contribution in this area. The computer age made it feasible to perform the vast number of

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[4]Neo-classical economists in the 1930's substituted ordinal utility analysis using indifference curves for the cardinal measure of pleasure and pain envisioned by the classicists. Von Neuman and Morgenstern's concept of cardinal utility was something different. It involved a preference ranking of risky alternatives.

calculations, thus permitting widespread adoptions of the new techniques.

Philosophy and Mechanism for Giving Planning Advice

Concurrent with advances in economic theory and methodology, institutional structures emerged which fostered the dissemination of knowledge. Passing of the Smith-Lever Act in 1914 established the Co-operative Extension Service which had as an objective "---to aid in the diffusing among the people of the United States useful and practical information on subjects relating to agriculture and home economics, and to encourage application of the same---" (43, p. 343).

The extension worker serves as a resource upon which the decision maker can draw to perform his function of management. Bradford and Johnson (5, p. 3) define management as a set of steps in the process of thought and action.

> "Management is the intangible part of production which develops within the lives of men. It is first a mental process, a concentration of desires, a will power. Management functions when a farmer is (1) observing and conceiving ideas; (2) analyzing with further observation; (3) making decisions on the basis of the analysis; (4) taking action; and (5) accepting responsibilities. Management can be seen only through observing the decision making process and its results."

It is generally accepted by agricultural economists that the place of the extension worker is in the steps of observation and analysis. His function is to provide information and present alternatives. He aids in problem definition and raises relevent questions; but making the decision is clearly outside his domain. In practice there is not always a sharp line between presenting alternatives and choosing a course of action from among them. However, the distinction between the domain of the decision maker and that of the advisor is clear in the fifth step of accepting responsibility. The decision maker must live with the consequences of his decision whether the result be success or failure. While the traditional theory postulates economic man as one whose objectives are to maximize profit within a static dimension, the possibility of financial ruin may cause a real world man to behave in a much different manner.

Problem and Purpose - Narrowing The Gap

Despite advancements in decision theory, there has been only minor implementation of planning techniques that account for uncertainty (41). Most planning techniques presently in use assume static, certainty conditions. The objective of the decision maker is taken to be maximum profit, usually measured as net income, or return to labor and management. Solutions are generally given as a single best plan, i.e. the one which results in maximum profit. Although an aura of certainty surrounds the advice, the farmer may be given an estimate of income variability associated with the plan. Furthermore advice is often concluded with the statement, "This plan is only a guide and you should apply your own judgment about how to use it."

The farmer, if unversed in the particular analytical technique used must either follow the advice blindly or be confused as to how he should apply his judgment.

Farm management text books generally give a superficial treatment to the topic of farm planning in the face of uncertainty. They leave off with the notion that it is unwise to "put all of your eggs in one basket." Very little is said in a positive way about how one might determine the proper number of baskets, or how to select the eggs to be placed in them.

Objectives of the Study

A gap exists between theoretical developments in problem solving under uncertainty and methodology for application of this theory in a practical setting. This study will attempt to narrow that gap. The prime objective is to develop a planning technique which actively[5] accounts for uncertainty. Focus will be on the enterprise selection problem with the basic method coming from Markowitz's (34) portfolio selection criteria designed for use by investment consultants. This problem in security analysis has much in common with the agricultural problem of choosing the "correct" combination of enterprises. The

[5] The term "active" distinguishes this approach from the term "passive" which refers to giving a single plan and including a statement about its income variability.

similarity has been recognized by Freund (14), Carter and Dean (7), How and Hazell (26), Boussard (4) and others. For methodology to be operational from the decision makers point of view it should possess several characteristics including (a) the problem it is designed to solve must exist in the real world and answers must be worth at least as much as the cost of getting them, (b) the decision maker for whom the program is designed must recognize that he has the problem and must be able to provide data for its solution, and (c) the answer to the problem must be presented in such a form that the decision maker can understand the various suggested actions. The development of operational tools which focus on enterprise selection under uncertainty remains to be solved and it is to this end that the thesis is directed.

Plan of the Thesis

Chapter II initiates the inquiry with a review of ecnomic theory under the assumption of certainty which is later relaxed to account for crucial issues of uncertainty. The problem is first formulated in a linear programming framework. Then as the concepts of uncertainty are introduced, "deterministic" assumptions of the linear model are relaxed. This reformulation results in a quadratic programming model. A two enterprise example is used to illustrate the transition from traditional non-stochastic linear programming to a more realistic model of quadratic programming.

Chapter III focuses on operational aspects for implementing the quadratic model. An algorithm, with supporting computer program is first developed. This is followed by problems of parameters estimation. Requirements of accuracy, efficiency and simplicity in result interpretation are borne in mind as the development proceeds.

Empirical testing is undertaken in the fourth chapter. This test is restricted primarily to the computational accuracy and efficiency of the algorithm. General conclusions and suggestions for further investigation are the topic of the fifth and final chapter.

II. THE ENTERPRISE SELECTION PROBLEM - METHODOLOGY FOR SOLUTION

The enterprise selection problem is one of several issues which economic theory seeks to answer. This is the question of what and how much to produce. Initially, this chapter will examine the traditional certainty case employing the theory of production and marginal analysis. These restrictive assumptions will be relaxed so that a solution, first in the certainty case and finally in the uncertainty case, will become operationally possible.

The Traditional Certainty Case

The Theory - Static Certainty

The theoretical framework within which the short-run enterprise selection problem is solved comes directly from the theory of production in a purely competitive market. Here the decision maker is assumed to have perfect knowledge about factor and product prices but does not have sufficient control in the markets to exert a pricing influence. Further, it is assumed that this perfect knowledge extends to the technical relationships between factor inputs and resulting products. These relationships are expressed mathematically in a production function (21, p. 72-75). The decision maker is left to choose that combination of input and corresponding output levels which maximizes his profit. Mathematically he is required to solve the following maximization problem:

$$\text{Max:}^{6} \quad \sum_{i=1}^{n} p_i y_i - \sum_{j=1}^{m} r_j x_j = Y$$

$$\text{S. T:}^{7} \quad F(y_1, \cdots, y_n, x_1, \cdots, x_m) = 0 \qquad (2.1)$$

$$y_i \geq 0 \qquad i = 1, \cdots, n$$

$$x_j \geq 0 \qquad j = 1, \cdots, m$$

where Y is profit

y_i is the output of the ith product and p_i its price

x_j is the input level of the jth productive factor and r_j its cost

F is the production function stated in implicit form and chosen so that the non-negativity restrictions always held.

This set of simultaneous equations is usually solved through the application of Lagrangian multipliers. The Lagrangian function (2. 2) is formed and then partially differentiated with respect to its arguments.

$$R(y, x, \lambda) = \sum_{i=1}^{n} p_i y_i - \sum_{j=1}^{m} r_j x_j - \lambda [F(y_1, \cdots, y_n, x_1, \cdots, x_m)] \qquad (2.2)$$

where λ is the Lagrangian multiplier.

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[6] The abreviation "Max:" denotes maximize.

[7] The abreviation "S. T:" denotes subject to.

This establishes the first order condition for an extremum as shown in (2. 3). The sufficient condition for the extreme value of Y to be a maximum is that the matrix of second order cross partial derivatives is negative definite when evaluated at the optimizing levels of y and x. It is assumed that the production function is of such a nature that the second order condition holds.

$$\begin{aligned}
\frac{\partial R}{\partial y_i} &= p_i - \lambda \frac{\partial F}{\partial y_i} = 0 \qquad i = 1, \cdots, n \\
\frac{\partial R}{\partial x_i} &= r_j - \lambda \frac{\partial F}{\partial x_j} = 0 \qquad j = 1, \cdots, m \qquad (2.3) \\
\frac{\partial R}{\partial \lambda} &= F(y_1, \cdots, y_n, x_1, \cdots, x_m) = 0
\end{aligned}$$

Solution of the system of Equations (2. 3) demonstrates a fundamental concept of economics--namely the principle of equimarginal returns. The principle states that in order for profit to be maximum:

(a) the rate of transformation between any two products must equal the ratio of their respective prices. Mathematically this is:

$$-\frac{\partial y_i}{\partial y_k} = \frac{p_k}{p_i} \qquad (2.4)$$

(b) the rate of technical substitution between any two factors of production must equal the ratio of their respective costs. Mathematically this is:

$$- \frac{\partial x_j}{\partial x_s} = \frac{r_s}{r_j} \tag{2.5}$$

(c) the marginal factor cost of any factor of production must equal its marginal value product. Mathematically this is:

$$r_j = p_i \frac{\partial y_i}{\partial x_j} \tag{2.6}$$

Although all of the Equations (2.4), (2.5) and (2.6) must hold simultaneously, the relationship expressed in Equation (2.4) directly answers the question of what and how much to produce, the central issue of this thesis.

Empirical Tools

The Econometric Production Function

The theory of production is rich in explanatory hypotheses about economic phenomena and provides a rigorous framework within which to "think through" economic problems. However, as an operational tool it departs substantially from reality for providing specific answers to a particular firm on questions of input and output levels. As Dillon (2,p.103) points out, the estimation of response surfaces is beset by difficulties, not the least of which are statistical problems of design and measurement. Variability in response over time and space further complicates the issue. These contribute to discrepancies that exist between results obtained under controlled investigation and an actual farm situation. Most response surface experimentation has been conducted on a multiple input, single output basis. Data are generally analyzed using a multiple regression routine with a single equation model. This virtually eliminates investigation of joint product relationships which form the very heart of the enterprise selection problem. Intent of these remarks is not to discredit inter-disciplinary work done on investigating production processes. Such work has produced many insights into agricultural production problems. However, important as these functions may be for providing some of the data useful in farm planning, they alone are not sufficiently powerful to cope with the high level of complexity surrounding many farm units.

The Partial Budget

In the early stages of empirical tool development many operational difficulties were assumed away by describing the production process in terms of straight line segments. The process was called partial budgeting. It provides the simplest form of a linear production function and is probably the most widely used empirical tool even though it is not always presented in a formal written manner. The main philosophy underlying the partial budget revolves around three equations:

(a) ADDED PROFIT = ADDED RETURNS - REDUCED RETURNS

(b) ADDED PROFIT = REDUCED COSTS - ADDED COSTS

(c) ADDED PROFIT = ADDED RETURNS - ADDED COSTS

Although there are no optimizing criteria built into the partial budget as such, it is of interest to note that these equations do have a firm basis in the fundamentals of profit maximization; see Equations (2.4), (2.5) and (2.6). The usual method is to construct a number of partial budgets and then compare the projected outcomes, i. e. added profits, from each. The highest paying alternative, after due consideration is given to other important factors not explicitally included in the budget, can then be chosen.

Introduction of high-speed computers and diligent efforts by Danzig (9) and others added, an optimizing technique to the rather simple notion of partial budgets thereby producing the now well known technique of linear programming.

Linear Programming

Linear programming is a mathematical concept defined as the optimization (maximization or minimization) of a linear function in several variables subject to a set of linear inequality constraints (11, p. 8).

Assumptions of Linear Programming

Since there is an abundance of writing on the subject of linear programming both with respect to theory and application, a detailed review will not be pursued here. Naylor (35) gives a particularly clear and concise treatment of the relation between traditional theory of the firm and linear programming. Certain assumptions about the relation between inputs and outputs are basic to linear programming. It will suit the purpose here to reproduce only its essential features. The list is adopted from Hillier and Lieberman (23). The basic assumptions are:

<u>Proportionality</u>: If one unit of the ith activity requires one unit of the jth resource, then two units of the ith activity will require two units of the jth resource. In terms of the calculus this means that the marginal physical productivity of the jth resource in the ith activity is constant over the interval of concern. At first this appears to be a rather serious limitation of the model, especially in view of the so-called principle of diminishing returns. However, it is possible to preserve the essential nonlinear features in many cases through specification of several activities over an appropriate size range.

<u>Additivity</u>: Engaging in one activity will in no way affect the per unit profit of any other activity, nor will it affect the per unit

resource requirement of any other activity. In the Carlson (6, p. 79) sense there is technical and economic independence between every pair of activities, between every pair of resources and between all resources and activities.

Divisibility: Resources and activities must be perfectly divisible. The implication of this assumption is optimum output levels and their corresponding levels of resource use need not be in whole numbers. For instance the solution may require that there be 10-1/2 sows rather than 10 or 11. Unfortunately there are no good techniques to know, in general, whether to round up to 11 or down to 10 so as to minimize departure from the optimal combination.[8]

Deterministic: The linear programming model treats all of the coefficients as though they were constants occuring with certainty. In dealing with reality, it is seldom, if ever, that such a degree of certainty exists. In actuality, the coefficients are expected values of some random distribution but treated as though they were non-stochastic.[9]

[8] To resolve this difficulty one must go to the more elaborate integer programming methods which are not yet highly developed.

[9] It is usual to use the expected value of the random variable, although in some cases it may make sense to use the most frequently occurring or modal value.

It is unlikely that there exist any situations that completely satisfy the assumptions of linear programming. However, there is a broad set of management problems that come sufficiently close such that the linear model gives reasonably satisfactory results.

The Enterprise Selection Problem in a Linear Programming Setting

The enterprise selection problem can be stated formally as the linear program:

$$\text{Max:} \quad \sum_{i=1}^{n} \mu_i y_i = Y$$

$$\text{S. T:} \quad \sum_{i=1}^{n} a_{ij} y_i \leq G_j \qquad j = 1, \cdots, m \qquad (2.7)$$

$$y_i \geq 0 \qquad i = 1, \cdots, n$$

where Y is total net income

y_i is the level of the ith activity

μ_i is the net income per unit of the ith activity[10]

G_j is the amount of the jth resource available

a_{ij} is the amount of the jth resource used in producing one unit of the ith activity.

To examine some implications of linear programming in the enterprise selection problem a numerical example has been chosen.

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[10]Net income is defined as the return above variable cost.

A farmer has the opportunity to grow any combination of two crops as long as he does not use more than a total of four acres of land or six hours of labor. After deducting variable costs, crop one (y_1) will return one dollar per acre. Crop two (y_2) returns two dollars per acre. It takes one hour of labor to grow an acre of the crop one and three hours for an acre of crop two. This information is known with certainty. The farmer wishes to get maximum return above variable cost. The problem stated in linear programming terms is:

$$\begin{aligned} \text{Max:}\ & y_1 + 2y_2 = Y \\ \text{S.T:}\ & y_1 + 3y_2 \leq 6 \\ & y_1 + y_2 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned} \qquad (2.8)$$

The graphic solution to this problem is found in Figure 2.1. Any point in the area obb', or on its boundary represents a possible choice as far as land is concerned. Likewise any point in the area oaa', or on its boundary represents a possible choice as far as labor is concerned. Any point in the areas adb or b'da', or on their upper boundaries are infeasible, because such a combination would exceed the quantity of labor or land available. Any point lying on or within oadb' represents a feasible choice. The line cc' indicates

[11] This simple problem will be made more elaborate in succeeding sections as the concepts of risk are introduced. It is the intent to provide the reader with a smooth transition to less familiar ground.

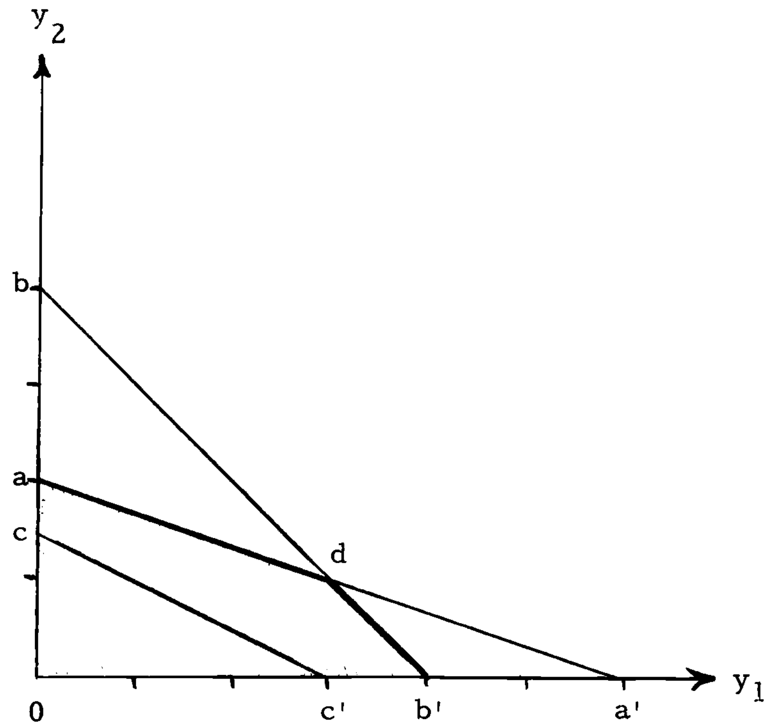

Figure 2. 1. The linear programming problem.

combinations of y_1 and y_2 yielding the same total return above variable cost, in this case three dollars. Any line drawn parallel to cc' and further from the origin represents a higher income. The highest income attainable is found on that line running parallel to cc' and passing through point d. At this point income is five dollars. The amount of land in crop one (y_1) is three acres and in crop two (y_2) one acre. For those more elaborate linear programming problems which contain more than two activities, a graphic solution becomes impossible. In such a case an algorithm called the simplex method is employed to obtain the income maximizing combination of activities. Several good references are available which present the simplex method in detail. Hillier and Lieberman's book (23) is elementary but thorough. However, knowledge of linear programming, beyond what has been discussed here is not essential for the reader to proceed.

Specification Problems in Linear Programming

The objective function in the numerical example of linear programming used here was taken to be maximum profit. This is the usual case in farm planning. Such an objective function may be an inadequate specification of the decision maker's goals. It may be that the farmer has a "dislike" for some enterprises, even though they appear to be generally profitable with farmers in the area. For instance, he may simply "not want a pig on the place." This is easily handled by excluding "pigs" as an activity or enterprise in the model.

Another specification error might arise as a result of the so-called work-leisure concept. For a given production function, additional income can result only if additional labor is applied. As more work is done, less time is available for leisure. This results in a distinction between labor and managerial effort as production resources and leisure, which forms the compliment of labor but is an ingredient of consumption. This topic is pursued by Skitovsky (38, p. 142-147) although not in the linear programming context. In a very real sense, a farmer will wish to put in additional time only if the income derived from it adds more to satisfaction than is lost from the leisure time given up. In formulation of the numerical example of Equation (2. 8), value of additional leisure was assumed implicitly to be zero. This specification problem, when it exits, can be overcome by incorporating an amount reflecting the salvage value of labor (28).

Decision making tools must of necessity be forward looking.[12] Consequently a third possible source of faulty specification results from the deterministic assumption. In real life it is unlikely that all of the information needed for decision making can be known with certainty. Even though payoffs and resource requirements of each activity are stated as parameters, they in fact are estimates--which by

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[12] Of course analytic use of linear programming is also made in a posteriori sense.

their very nature are found only in an environment of uncertainty. Thus the linear programming solution to the enterprise selection problem in reality becomes that combination of activities which results in maximum expected return.[13]

If decision makers were maximizers of expected return, it would not be necessary to focus attention on the randomness of coefficients in the model. However, in reality farmers do concern themselves with questions of failure and bankrupcy. Therefore it becomes necessary to set the stage for examining conditions under which a decision maker is a maximizer of expected profit and the conditions under which he is not.

The Uncertainty Case

Theoretical Considerations

Utility Theory - The Preference for and Aversion to Risk

In 1943 Von Neuman and Morgenstern (44) reintroduced the concept of cardinal utility. Their concept was quite different from the cardinal utility of the early demand theory. In the early theory, cardinal utility was taken to be an absolute measure of pleasure and pain

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[13] It may of course be that the estimate is the most frequently occuring level of per unit profit, in which case the objective function is to maximize most likely profit rather than expected profit.

(2, p. 523). The more recent concept was, instead, a preference ranking of risky alternatives.

The Von Neuman-Mogenstern notion of the utility function proceeds from a set of basic assumptions which are quoted directly from Chernoff and Moses (8, p. 82).

"Assumption 1. With sufficient calculation an individual faced with two prospects P_1 and P_2 will be able to decide whether he prefers prospect P_1 to P_2, whether he likes each equally well, or whether he prefers P_2 to P_1.

Assumption 2. If P_1 is regarded at least as well as P_2 and P_2 at least as well as P_3, then P_1 is regarded at least as well as P_3.

Assumption 3. If P_1 is prefered to P_2 which is prefered to P_3 then there is a mixture of P_1 and P_3 which is prefered to P_2, and there is a mixture of P_1 and P_3 over which P_2 is prefered.

Assumption 4. Suppose the individual prefers P_1 to P_2 and P_3 is another prospect, Then we assume that the individual will prefer a mixture of P_1 and P_3 to the same mixture of P_2 and P_3."

If an individual satisfies the basic assumptions, then for every prospect P there exists a corresponding utility number u(P). If the prospects represent different levels of income Y then the result is a utility function for income. It has the following properties (17, p. 62).

Property 1. If Y_1 is prefered to Y_2 then $u(Y_1) > u(Y_2)$.

Property 2. If Y_1 occurs with probability p and Y_2 with probability 1-p, then $U = E(u(Y)) = p\,u(Y_1) + (1-p)\,u(Y_2)$, where Y is a random variable and $U = E(u(Y))$ is its expected utility.

Property 3. The utility function is bounded, i. e. the utility number to be assigned lies between positive and negative infinity.

Property 4. The utility function is monotone increasing.

From the monotonic property it is known that higher certain incomes result in greater utility than do lower certain incomes. While the first derivative is positive throughout, the second derivative may be positive, negative or zero and accordingly the marginal utility of income will be increasing, decreasing or constant. The three possible shapes of the utility function are shown in Figures 2. 2, 2. 3 and 2. 4. If a wide enough range in income is allowed, then the individual's utility function will include each of the three stages (15).

To permit the utility function to be used for analysis, it can be expressed as a Taylor series expansion about the fixed point of expected income E(Y) (17, p. 100).

$$u(Y) = u(E(Y)) + [Y-E(Y)]\frac{du(E(Y))}{dY} + \frac{[Y-E(Y)]^2}{2}\frac{d^2u(E(Y))}{dY^2} + \sum_{n=3}^{\infty}\frac{1}{n!}[Y-E(Y)]^n\frac{d^nu(E(Y))}{dY^n} \tag{2. 9}$$

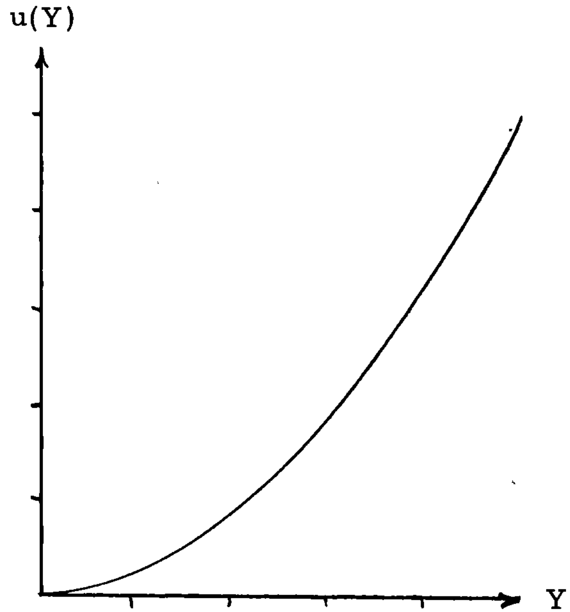

Figure 2. 2. Utility function for an individual who prefers risk.

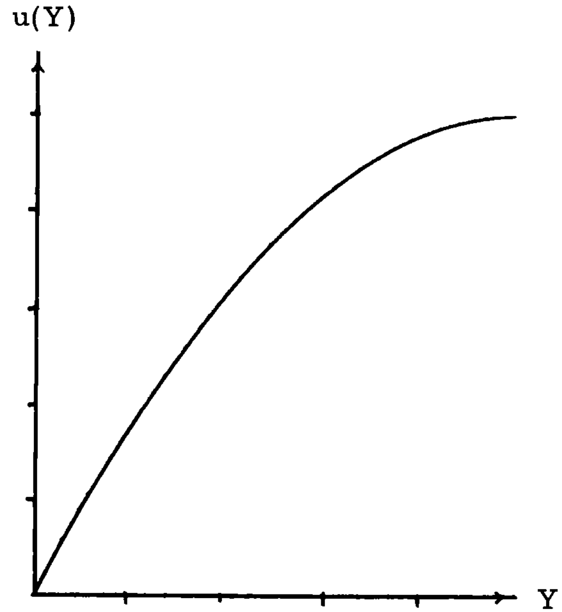

Figure 2. 3. Utility function for an individual who is a risk averter.

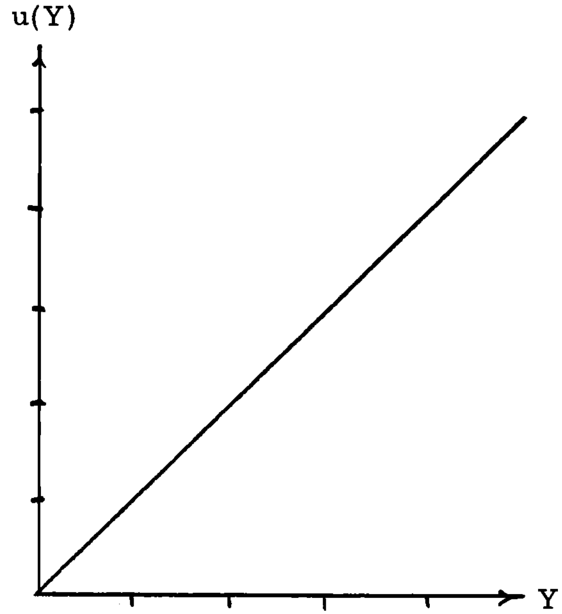

Figure 2. 4. Utility function for an individual who is risk neutral.

Taking the mathematical expectation of Equation (2.9) results in

$$U = E[u(Y)] = E[u(E(Y))] + E[Y-E(Y)]\frac{du(E(Y))}{dY} + E[(Y-E(Y))^2]\frac{d^2u(E(Y))}{dY^2} + \sum_{n=3}^{\infty}\frac{1}{n!}E[(Y-E(Y))^n]\frac{d^nu(E(Y))}{dY^n} \qquad (2.10)$$

where U is expected utility.

The terms of the expansion are made up of the derivatives of the utility function and the moments of the random variable, i.e. income. The first term $E[u(E(Y))]$ reduces to $u(E)$ which is the utility of expected income, the second term $E[Y-E(Y)]$ is zero, and the third term

$$E[(Y-E(Y))^2]\frac{d^2u(E(Y))}{dY^2}$$

is the product of the variance of income and the second derivative of the utility function evaluated at the level of expected income $E(Y)$. If the random variable has no moments higher than the second or the utility function has no derivatives of higher order than the second or if both conditions hold then the remainder term of the Taylor series summed from three to infinity is zero. To permit analysis in the variance expected income space it will be assumed that either or both of these conditions hold. Then expected utility becomes a function of expected income and variance as shown in Equation (2.11).

$$U = u(E) + \frac{1}{2} V \frac{d^2 u(E)}{dY^2} \tag{2.11}$$

where Y is the income variable

E is the expected income i.e. $E = E(Y)$

u(E) is the utility of expected income

V is variance of income i.e. $V = V(Y)$

Equation (2.11) can be rearranged such that variance becomes a function of expected utility and expected income as shown by Equation (2.12).

$$V = 2[U - u(E)] / \frac{d^2 u(E)}{dY^2} \tag{2.12}$$

For fixed levels of expected utility, say U°, variance as a function of expected income produces an indifference curve. Changing the level of U° results in a family of indifference curves. These curves are presented graphically as U_1°, U_2° and U_3° on Figures 2.5, 2.6 and 2.7. The shape of the indifference curves depends upon whether the individual has increasing, decreasing or constant utility for income.

The family of indifference curves has the following characteristics.

1. For any two alternatives, each with the same variance, the one with the higher expected income will yield the greater

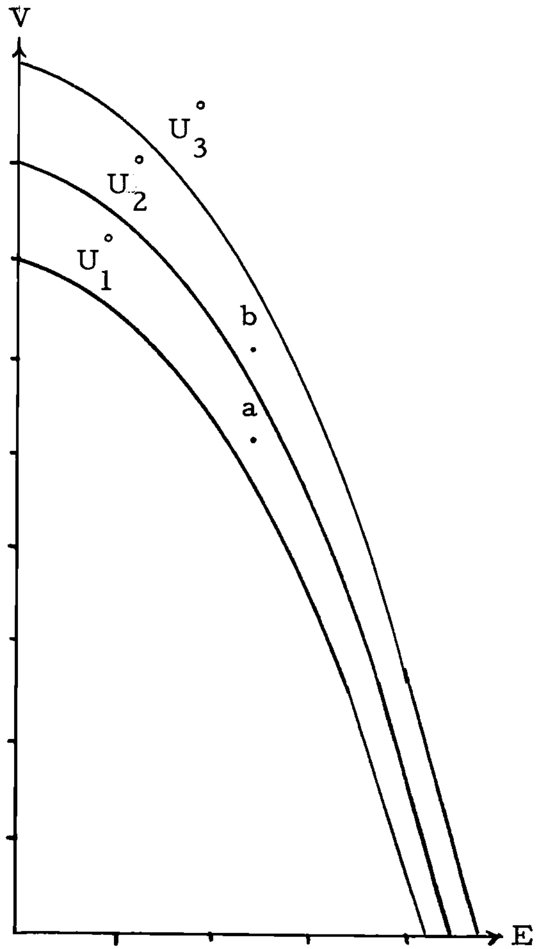

Figure 2. 5. Indifference curves for an individual who prefers risk (increasing marginal utility for money).

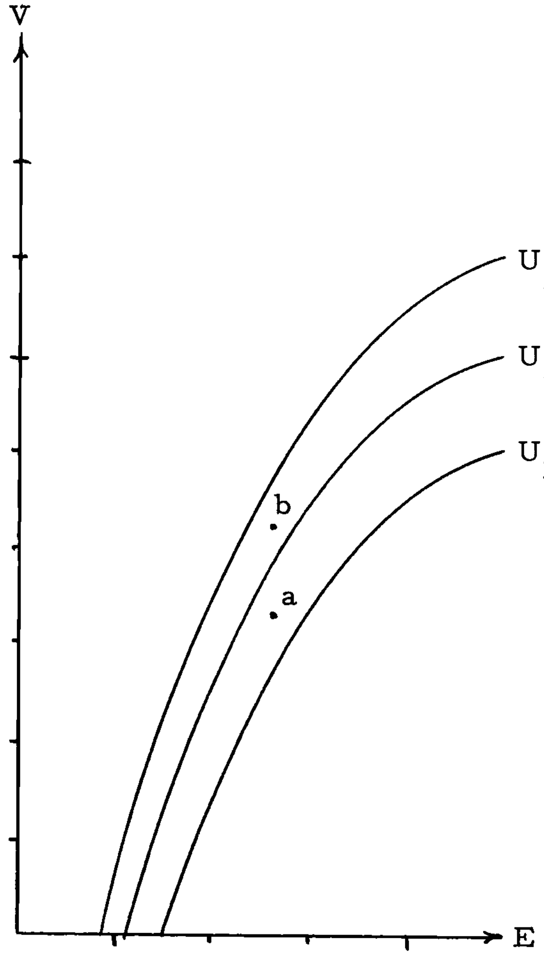

Figure 2. 6. Indifference curves for an individual who is a risk averter (decreasing marginal utility for money).

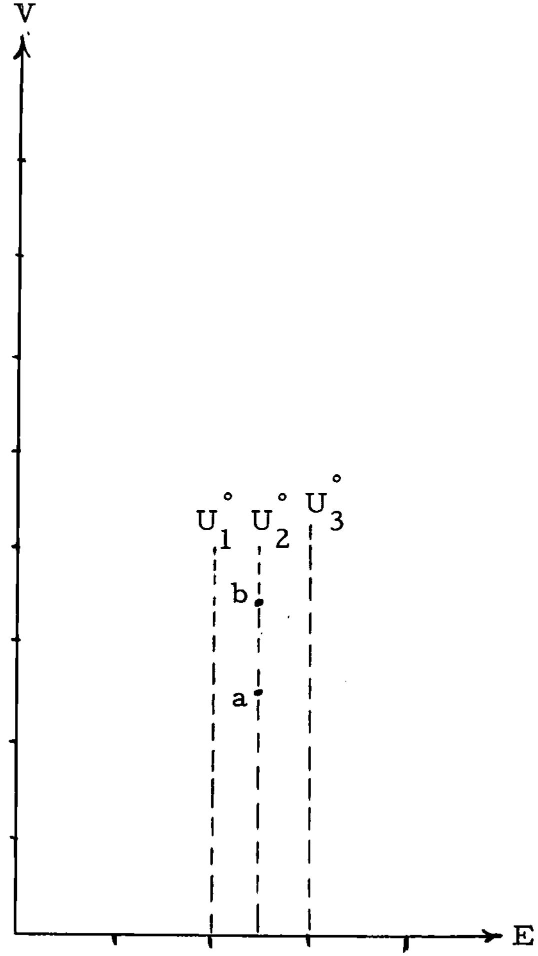

Figure 2. 7. Indifference curves for an individual who is risk neutral (constraint marginal utility for money).

expected utility.

2. For any two alternatives, a and b, each having the same expected income:

   (a) where the marginal utility of incomes is increasing the alternative with the greater variance will yield the greater expected utility as shown in Figure 2. 5.

   (b) where the marginal utility of income is decreasing the alternative with the lower variance will yield the higher expected utility as shown in Figure 2. 6 .

   (c) where the marginal utility of income is constant both alternatives will have the same expected utility as shown in Figure 2. 7.

These characteristics of the indifference curves are derived from Equation (2. 11) and the monotonic property of the utility function. It is possible for an indifference surface to exhibit all three forms of indifference curves.

The theoretical framework for evaluating risky alternatives is now complete and attention can be directed toward specifying enterprise alternatives in terms of their expected incomes and variances.

Feasible Enterprise Choices

Suppose that the income from a particular activity is a random variable. The profitability of that activity is measured by expected income, and its riskiness by variance.[14] No higher moments than the mean and variance are assumed. The expected income of a combination of activities is expressed as:

$$E = E(Y) = \sum_{i=1}^{n} \mu_i y_i \qquad (2.13)$$

where μ_i is the expected income per unit of y_i.

The variance of income of a combination of activities is expressed as:

$$V = V(Y) = \sum_{i=1} \sigma_i^2 y_i^2 + 2\sum_{i=1}^{n} \sum_{j<i}^{n} r_{ij}\sigma_i\sigma_j y_i y_j \qquad (2.14)$$

where σ_i^2 is the variance of income per unit of y_i

r_{ij} is the correlation coefficient between the incomes of y_i and y_j.[15]

These combinations of activities or enterprises can be viewed as alternatives or plans. There is an infinite number of alternatives, each having the same expected income but different variances. Likewise there is an infinite number of alternatives, each having the same variance but different expected incomes. This raises the question

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[14] The coefficient of variation, the ratio of the standard deviation to the mean, is a better measure of riskiness. This notion will be pursued later.

[15] The correlation coefficient r_{ij} measures the degree of statisical interdependence between the incomes of the ith and jth activities.

"Is there some rationale whereby this infinite number can be reduced to a single superior alternative? Its answer is found in the Von Neuman-Morgenstern utility theory.

Efficient Enterprise Choices

It has been shown that if a decision maker satisfies the basic postulates of utility theory and is also a maximizer of expected utility, he will choose from among alternatives having the same variance, the alternative having the highest expected income. This problem is solved mathematically by maximizing expected income subject to some fixed level of variance.

$$\text{Max:} \quad \sum_{i=1}^{n} \mu_i y_i = E$$

$$\text{S. T:} \quad \sum_{i=1}^{n} \sigma_i^2 y_i^2 + \sum_{i=1}^{n} \sum_{j<i}^{n} r_{ij} \sigma_i \sigma_j y_i y_j = V^{\circ} \qquad (2.15)$$

$$y_i \geq 0 \qquad i = 1, \cdots, n$$

The problem expressed in Equation (2.15) can also be stated as minimizing variance subject to some fixed level of expected income.

$$\text{Min:}^{16} \quad \sum_{i=1}^{n} \sigma_i^2 y_i^2 + \sum_{i=1}^{n} \sum_{j<i}^{n} r_{ij} \sigma_i \sigma_j y_i y_j = V$$

$$\text{S. T:} \quad \sum_{i=1}^{n} \mu_i y_i = E^{\circ} \tag{2.16}$$

$$y_i \geq 0 \qquad i = 1, \cdots, n$$

The form used in Equation (2.16) will be required because of computational necessity, however, it is proper to view the problem in terms of Equation (2.15) because it allows for the three basic shapes of the utility function.

For graphic interpretation, the number of activities initially will be restricted to two. A more general model will be introduced later. To proceed it will be helpful to examine the mathematical form of the expected income and variance functions. In the two activity case the expected income and variance equations are:

$$E = \mu_1 y_1 + \mu_2 y_2 \tag{2.17}$$

and

$$V = \sigma_1^2 y_1^2 + 2r\sigma_1 \sigma_2 y_1 y_2 + \sigma_2^2 y_2^2 \tag{2.18}$$

The expected income function is linear. It is shown graphically as line segment cc' on Figure 2.8 with expected income fixed at level E°.

[16] The abreviation "Min:" denotes minimize.

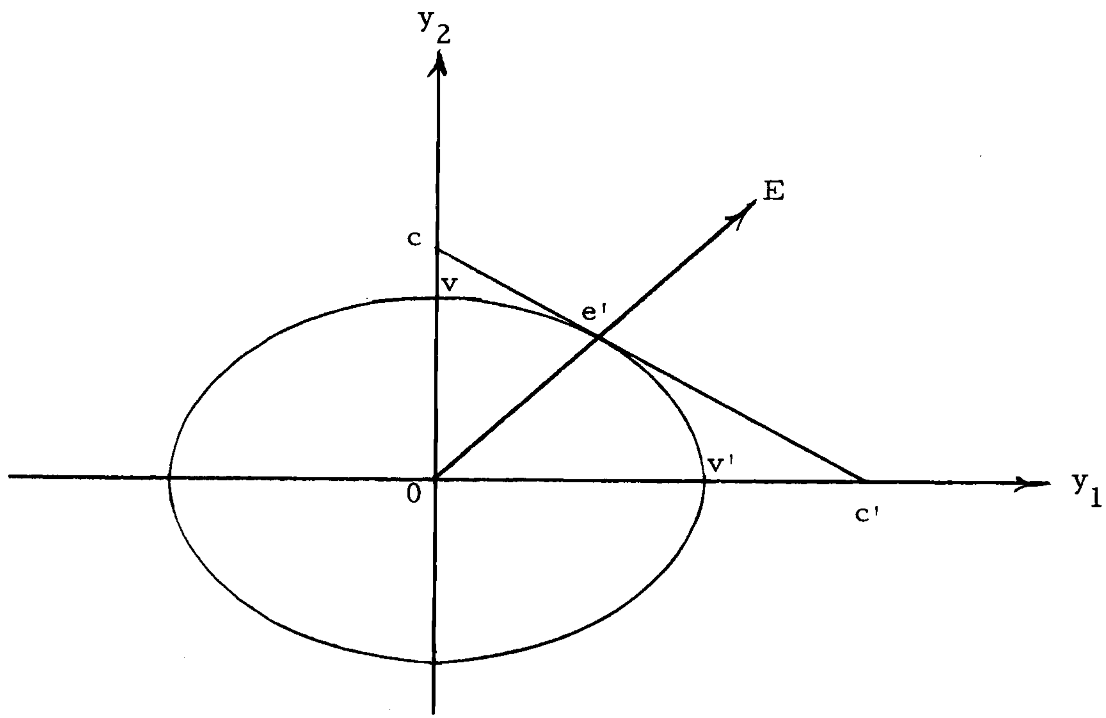

Figure 2.8. Iso-expected income and iso-variance in two dimensions.

This is an iso-expected income line since any combination of y_1 and y_2 that lies on cc' has the same expected income. Varying the level of income produces a family of parallel iso-expected income lines. A fixed expected income level E° is presented in the three dimensional graph of Figure 2.9 as the plane cc'f'f.

The variance function is an elliptic paraboloid(37,p.329). This is so because the correlation coefficient r lies between positive and negative unity making the term $\sigma_1^2\sigma_2^2(1-r^2)$ always positive (24, p. 67).

For a fixed level of variance, say V° the equation can be shown in two dimensions as the iso-variance ellipse in Figure 2.8. Varying the level of variance produces a family of iso-variance ellipses. Such a family forms the elliptic paraboloid in Figure 2.9. The correlation coefficient serves to rotate the ellipses in the y_1, y_2 activity plane. If $r = 0$, then the degree of rotation is zero and if $\sigma_1 < \sigma_2$ the y_1 axis becomes the major axis. To maintain perspective in later graphic analyses the activity with the higher variance will be denoted y_2.

Incorporating Equations (2.17) and (2.18) into the Lagranian form results in:

$$R(y_1, y_2, \lambda) = \sigma_1^2 y_1^2 + 2r\sigma_1\sigma_2 y_1 y_2 + \sigma_2^2 y_2^2 - \lambda[E^\circ - \mu_1 y_1 - \mu_2 y_2] \tag{2.19}$$

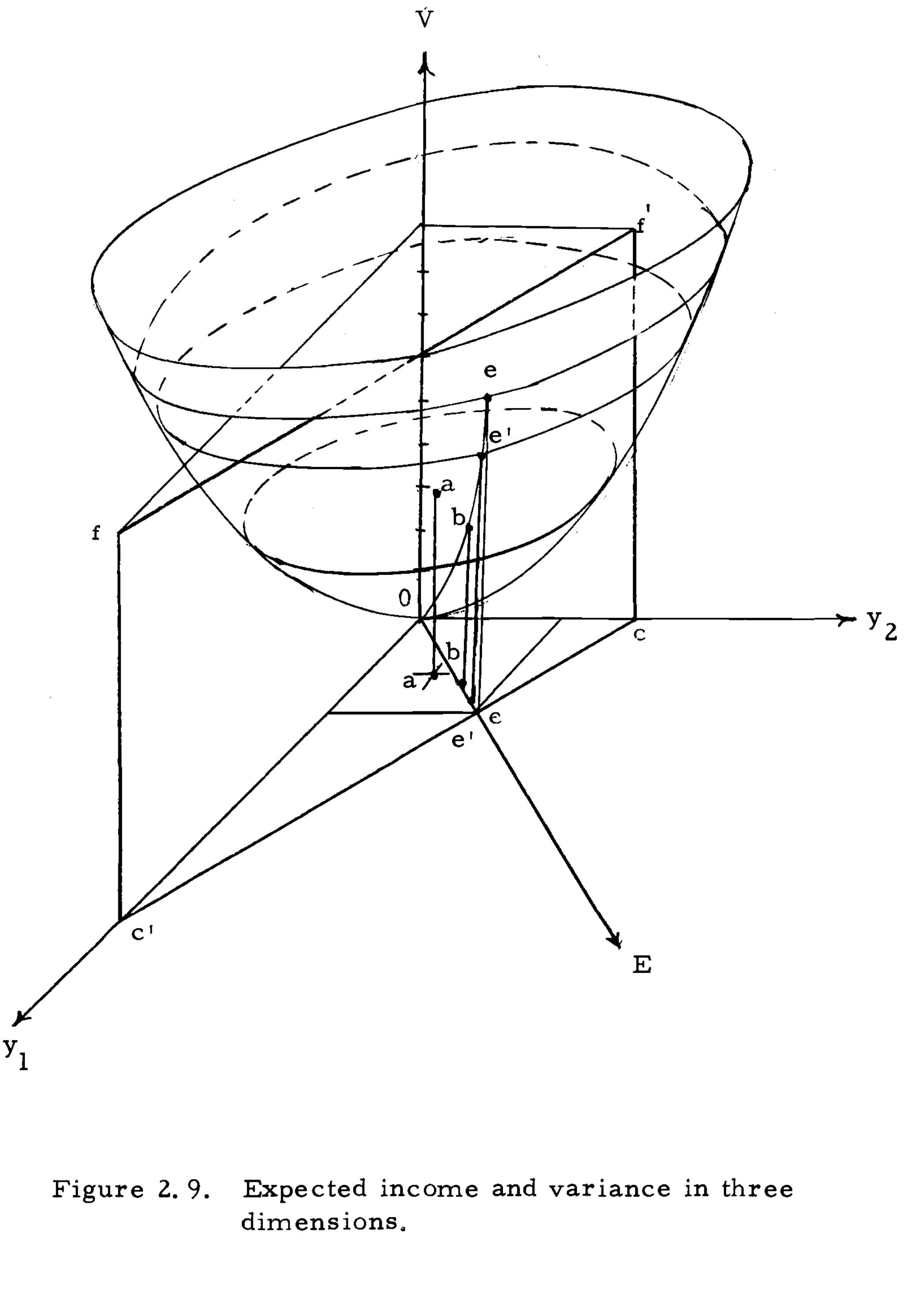

Figure 2. 9. Expected income and variance in three dimensions.

Partially differentiating Equation (2.19) with respect to its arguments and setting the results equal to zero yields the first order equations for a minimum.[17] It is momentarily assumed that r is such that the non-negativity restrictions are fulfilled.

$$\frac{\partial R}{\partial y_1} = 2\sigma_1^2 y_1 + 2r\sigma_1\sigma_2 y_2 + \lambda\mu_1 = 0$$

$$\frac{\partial R}{\partial y_2} = 2r\sigma_1\sigma_2 y_1 + 2\sigma_2^2 y_2 + \lambda\mu_2 = 0 \qquad (2.20)$$

$$\frac{\partial R}{\partial \lambda} = \mu_1 y_1 + \mu_2 y_2 - E^{\circ} = 0$$

Solving this set of simultaneous linear equations for y_1, y_2 and λ results in a number of relationships which have a familiar counterpart in production theory. These include the expansion path, the activity equations and the efficiency frontier.

The expansion path. In Figure 2.8 line cc' is the infinite number of alternatives having the same expected income but different variance. The contour vv' represents the infinite number of alternatives having the same variance but different expected incomes. The tangency of vv' to cc' at the point e' is the combination of y_1 and y_2 at where, for the given level of expected income, variance is as small as possible. This is the solution to Equation (2.19). Varying the level of expected income results in a locus of tangency points tracing out the minimum variance expansion path.

[17] Since variance is a positive definite quadratic form, the sufficient conditon for a minimum is also satisfied.

This forms the line segment oe' in Figure 2. 8. In the two activity case the equation for the expansion path derived from Equation (2. 20) is given by:

$$y_2 = \left(\frac{\sigma_1}{\sigma_2}\right)\left(\frac{\mu_2\sigma_1 - r\mu_1\sigma_2}{\mu_1\sigma_2 - r\mu_2\sigma_1}\right)y_1 \tag{2. 21}$$

The Activity Equations. Each of the activity variables are derived from the set of Equations (2. 20) as linear functions of expected income. In the two activity case the equations are:

$$y_1 = \left[\frac{\sigma_2(\sigma_2\mu_1 - r\sigma_1\mu_2)}{\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2}\right]E$$

$$y_2 = \left[\frac{\sigma_1(\sigma_1\mu_2 - r\sigma_2\mu_1)}{\mu_1^2\sigma_2^2 - r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2}\right]E \tag{2. 22}$$

Graphic presentation of the equations is found in Figure 2. 10. These equations show the level of the activity (decision) variables for each level of expected income such that minimum variance is attained. These equations are analogous to supply functions in production theory.

The Lagrangian form in Equation (2. 19) requires that E be held fixed at some level E°. However, since any $E \geq 0$ will satisfy the Lagrangian function, E will be looked upon as a non-negative continuous variable in the activity equations. This permits specification of y_1 and y_2 for all possible levels of E.

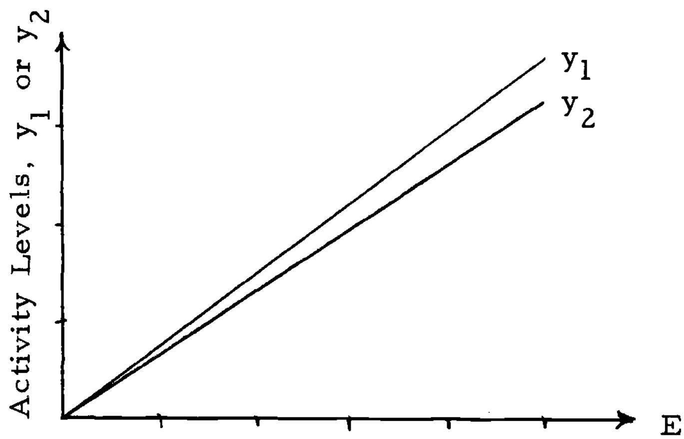

Figure 2.10. Activity level equations.

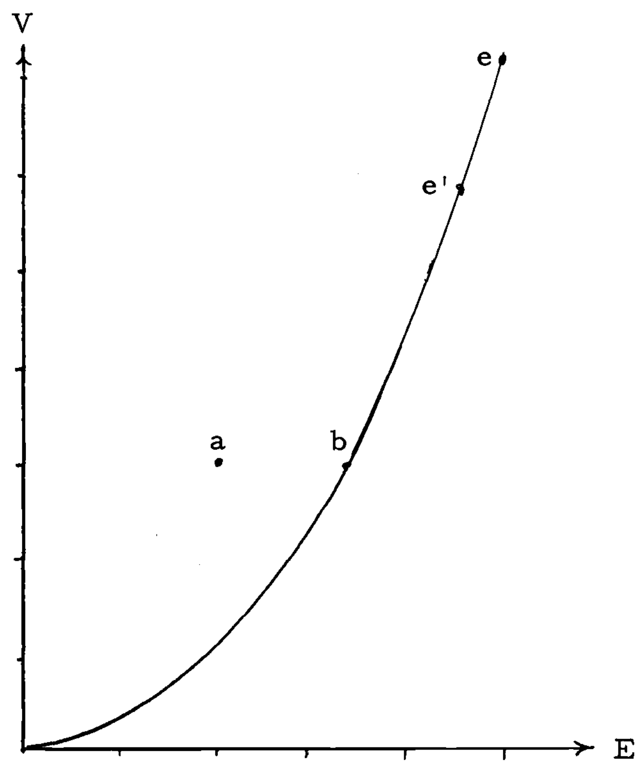

Figure 2.11. The efficiency frontier.

<u>The Efficiency Frontier</u>. There exists a functional relationship between expected income and variance which can be specified exactly in algebraic form by making use of an important but much overlooked feature of the Lagrangian multiplier. This relationship will be referred to as the efficiency frontier. The Lagrangian multiplier is the rate of change in the objective function with respect to a change in the level of the constraint.[18] In the present problem, the Lagrangian multiplier[19] is the increase in variance, attributable to an increment in expected income. Its algebraic form is:

$$-\lambda = \frac{dV}{dE} = \left[\frac{2\sigma_1^2\sigma_2^2(1-r^2)}{\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2}\right]E \qquad (2.23)$$

Like the activity equations, the Lagrangian multiplier is a continuous function of expected income. Since the Lagrangian multiplier is the first derivative of the efficiency frontier, its antiderivative or integral[20] will be the algebraic equation of the efficiency frontier.

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[18] A more detailed interpretation is to be found in the appendix.

[19] Because of the formulation it is actually the negative of the Lagrangian multiplier that represents the rate of change.

[20] Because of the variance form is centered at zero the constant term in the integral is zero.

Hence

$$V = \int dV = \left[\frac{2\sigma_1^2\sigma_2^2(1-r^2)}{\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2}\right]\int E\,dE$$

$$V = \left[\frac{\sigma_1^2\sigma_2^2(1-r^2)}{\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2}\right]E^2 \qquad (2.24)$$

The curve oe of Figure 2.11 is the efficiency frontier. Every alternative whose expected income and variance is given by a point interior to oe is dominated by an alternative which has the same variance but a higher expected income. For example point a is dominated by point b. The efficiency frontier is the locus of expected income-variance points of dominant alternatives. These dominant alternatives are the efficient plans from the total listing of the feasible enterprise choices.

The efficiency frontier is similar to the total variable cost curve in production theory with variance being analogous to cost and expected income analogous to output.

The parameters of the variance and expected income equations have a direct bearing upon the composition of efficient plans and upon the shape and position of the efficiency frontier. Results of varying the parameters in the two activity model are stated as assertions.

<u>Assertion 1</u>. As the correlation coefficient r is increased from 0 to 1, the variance ellipse elongates and its major axis

rotates in a clockwise direction from an angle $\theta = 0^{\circ}$ to at most $\theta = -45^{\circ}$.[21] As r decreases from 0 to -1, the variance ellipse again elongates but the major axis rotates in a counter clockwise direction from an agle $\theta = 0^{\circ}$ to at most $\theta = +45^{\circ}$. Figure 2.12 displays these results.

Assertion 2. As r increases to a number larger than the ratio of the coefficients of variation of the least risky activity to the most risky activity, σ_i/μ_i, the equation of the most risky activity and the expansion path take on negative first derivatives. Let this critical value of r where the derivative becomes negative be denoted r^*.

Assertion 3. As r increases from -1 to r^* the least risky activity replaces the most risky one. At values of r greater than or equal to r^* complete specialization in the least risky activity will take place. This is shown in Figure 2.13.

Assertion 4. An increase in r from -1 to r^* causes the efficiency frontier to rise more steeply with the consequence that, for any level of expected income the variance is increased. This is shown in Figure 2.14.

Assertion 5. An increase in the expected income of an activity

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[21] The major axis of the variance ellipse, when $r = 0$, is the axis of the activity y_i having the smallest variance. All statements concerning the angle of rotation are made from this perspective.

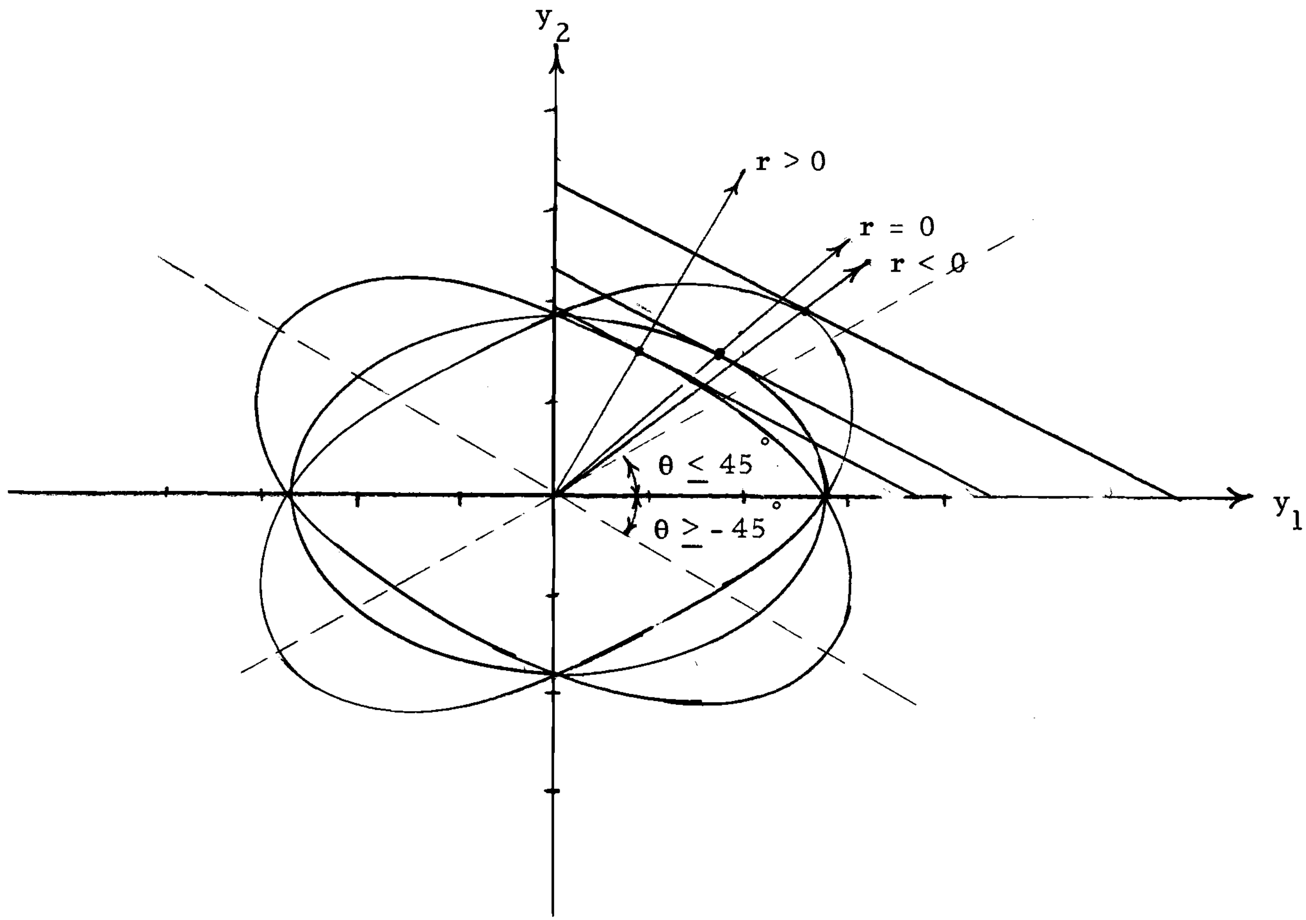

Figure 2.12. Behavior of the variance ellipse and expansion path with changes in the correlation coefficient.

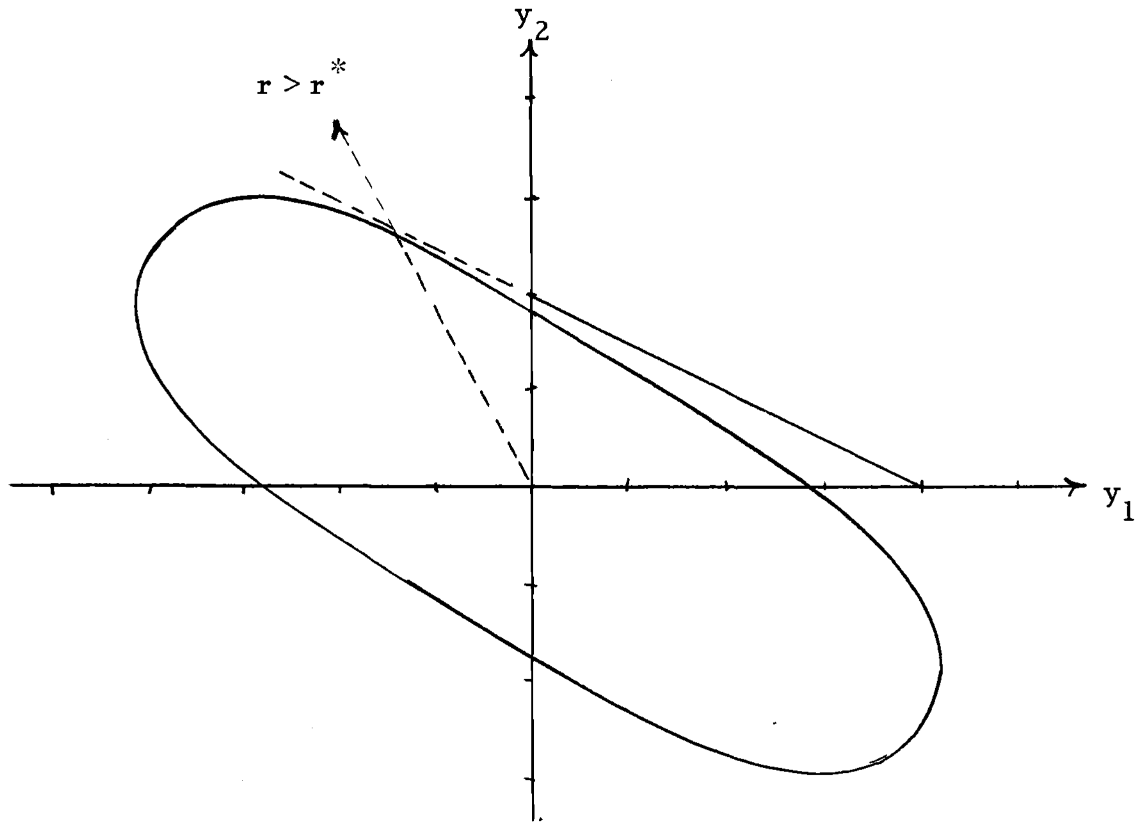

Figure 2.13. The variance ellipse and expansion path in the highly positive correlation case.

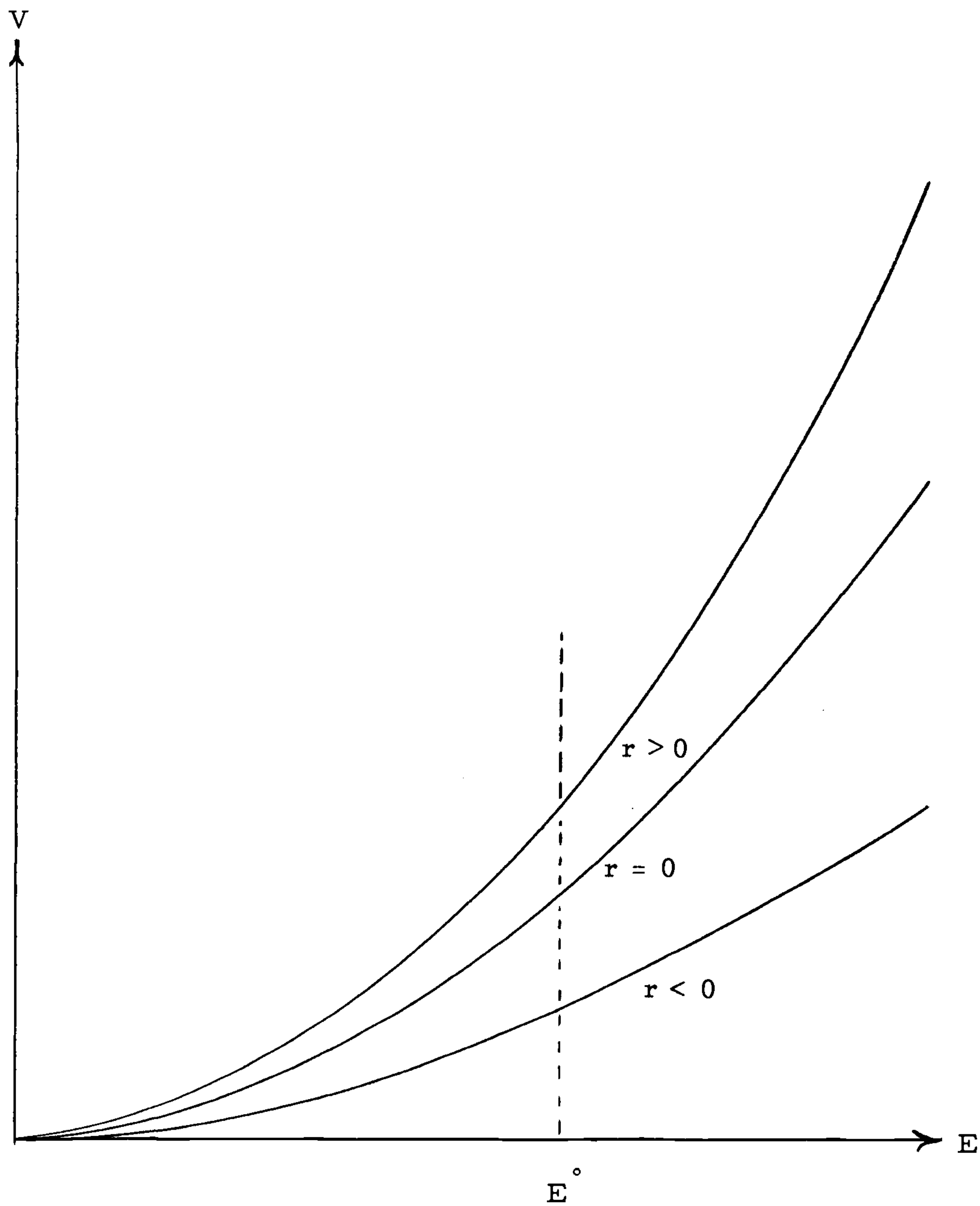

Figure 2. 14. Behavior of efficiency frontier with changes in the correlation coefficient.

will cause that activity to become relatively less risky with the consequence that it will replace the other activity. This is shown in Figure 2.15. With further increase in the activity's expected income r^* will become equal to r. At that point complete specialization occurs in this now least risky activity.

Assertion 6. An increase in the expected income of an activity will cause the efficiency frontier to rise less steeply with the consequence that for any level of expected income, variance is decreased.

Assertion 7. An increase in the variance of an activity will cause that activity to become relatively more risky with the consequence that it will be replaced by the other activity. With further increases in the activity's variance r^* will become equal to r. At that point complete specialization occurs in the other activity which is now least risky.

Assertion 8. An increase in the variance of an activity will cause the efficiency frontier to rise more steeply with the consequence that for any level of expected income the variance is increased.

Proof of these assertions is found in the appendix.

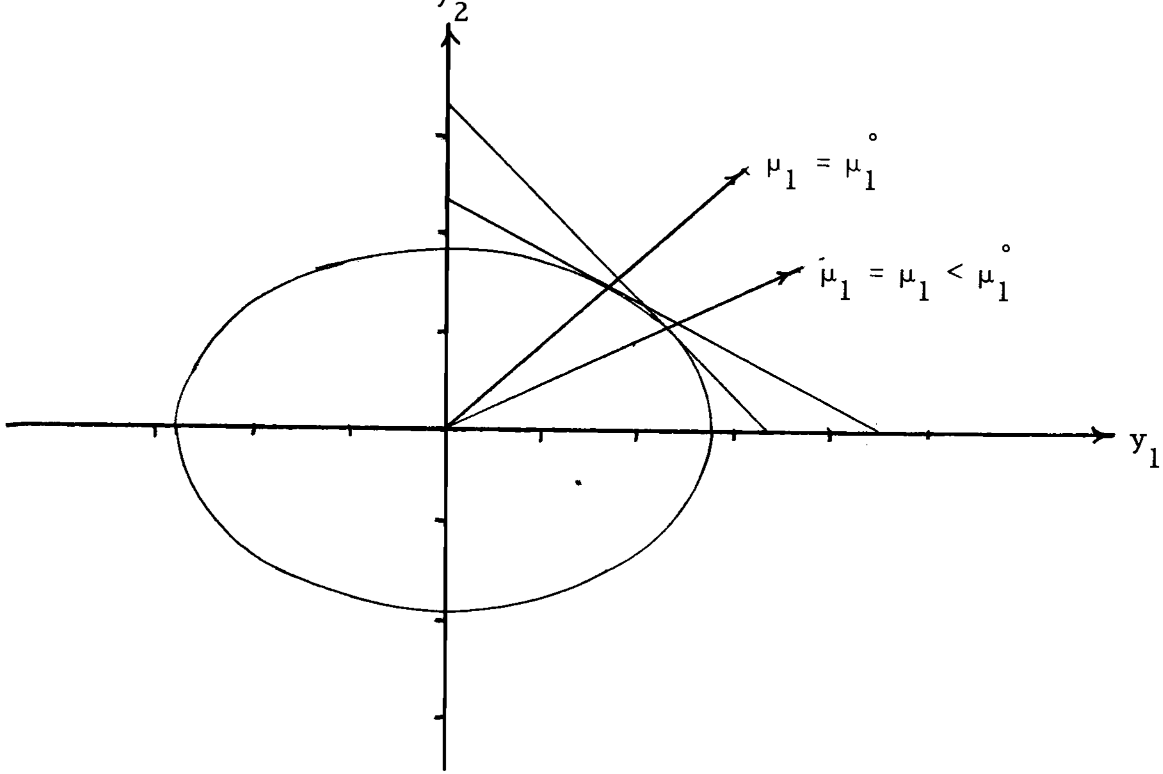

Figure 2.15. Behavior of the iso-expected income line and the expansion path with changes in the expected income of y_1.

A Mathematical Technique for Deriving the Efficient Enterprise Choices

A Numerical Example

To tie the linear model and the risk minimization together it is well to return to the numerical example of the static certainty problem summarized in Equation (2. 8) and to modify it by accounting for risk. The profit per unit of activity figures will now be random variables with expected value of one dollar and standard deviation of two dollars for enterprise crop one (y_1) and expected value of two dollars and standard deviation of three dollars for crop two (y_2). The correlation coefficient between the incomes of the crops is zero. Crop one requires one hour per acre and crop two requires three hours. The farmer is limited to six hours of labor and four acres of land. Production constraints and variability of income must be considered simultaneously in formulating efficient combinations of the two crops. The objective of this problem becomes one of finding that combination of crops which will minimize variance for each level of expected income subject to specified resource constraints. The problem is expressed algebraically as Equation (2. 25).

$$\begin{aligned} \text{Min:}\quad & 4y_1^2 + 9y_2^2 = V \\ \text{S. T:}\quad & y_1 + 2y_2 = E \\ & y_1 + 3y_2 \leq G_1 = 6 \end{aligned} \tag{2. 25}$$

$$y_1 + y_2 \leq G_2 = 4$$

$$y_1, y_2 \geq 0 \tag{2.25}$$

To illustrate the problem graphically in two dimensions, the variance ellipse of Figure 2. 8 is superimposed on the production constraints of Figure 2. 1 with the resulting Figure 2. 16. In the three dimensional case the reader is asked to visualize the elliptic paraboloid of Figure 2. 9 superimposed in the constraint set of Figure 2. 17.

Because the Lagrangian multiplier technique does not permit inequality constraints, disposal or slack activities are introduced to change each inequality to an equality. The transformed set is Equation (2. 26):

$$\begin{array}{llcl}
\text{Min:} & 4y^2 + 9y_2^2 & = & V \\
\text{S. T:} & y_1 + 2y_2 & = & E \\
 & y_1 + 3y_2 + y_3 & = & G_1 = 6 \\
 & y_1 + y_2 + y_4 & = & G_2 = 4 \\
 & y_1 - y_5 & = & G_3 = 0 \\
 & y^2 - y_6 & = & G_4 = 0
\end{array} \tag{2.26}$$

$$y_3, y_4, y_5, y_6 \geq 0$$

where y_3 represents unused labor

y_4 represents unused land

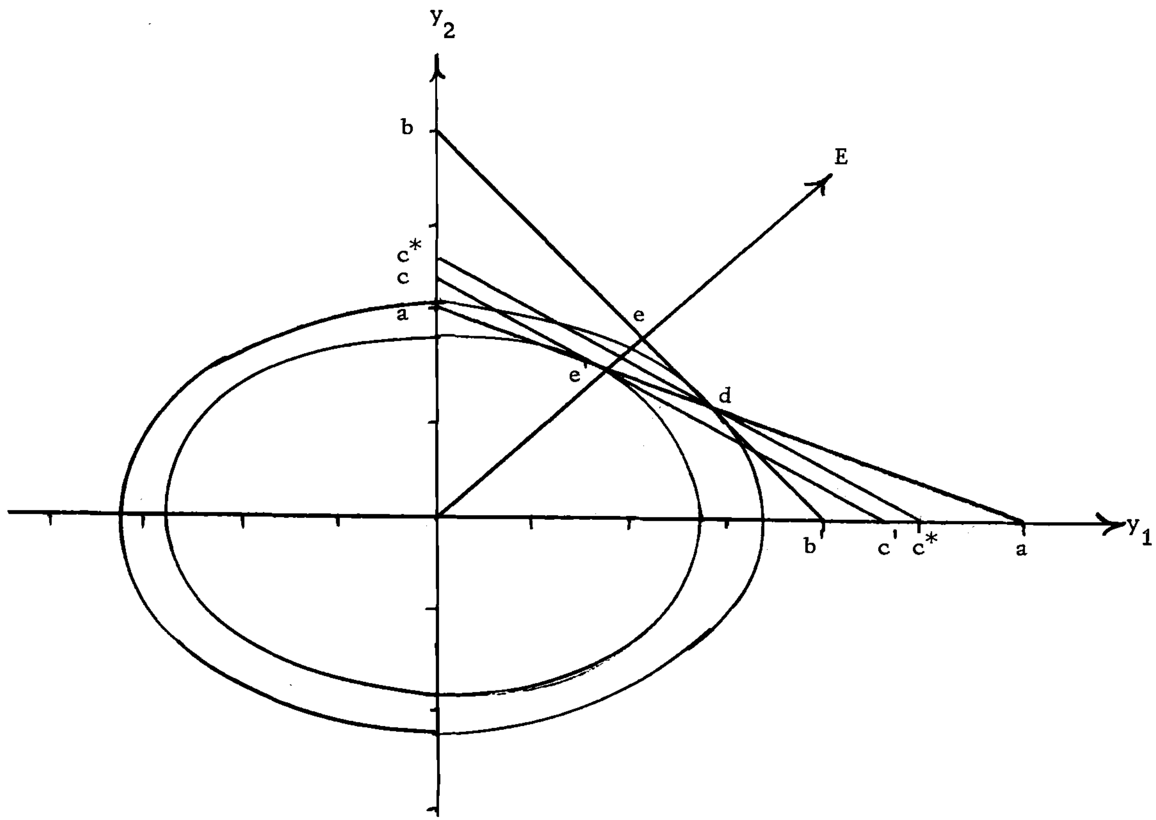

Figure 2.16. Quadratic programming problem in two dimensions.

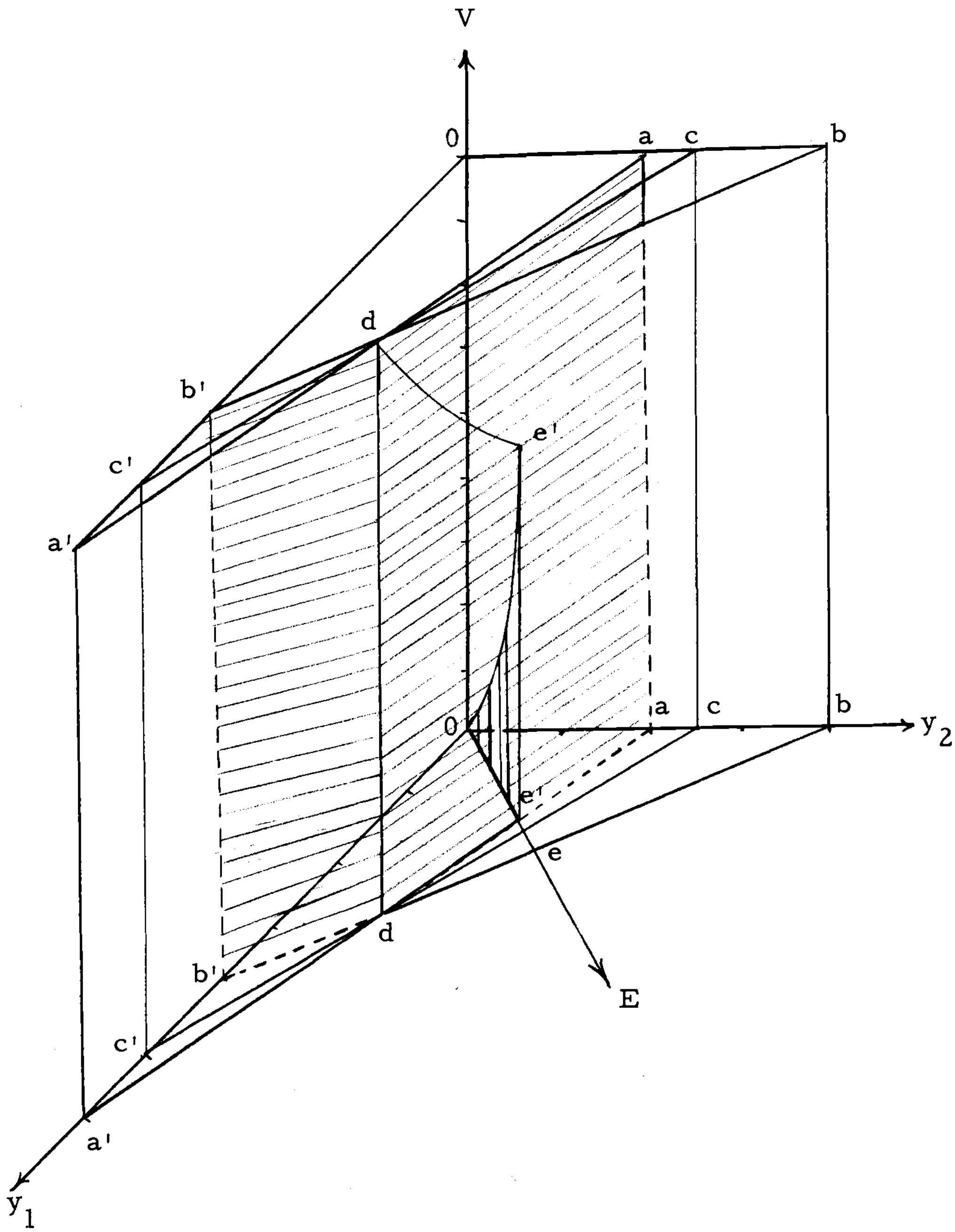

Figure 2.17. Constraint set of the quadratic programming problem in three dimensions.

y_5 and y_6 are required by the Lagrangian technique to insure that the real activities y_1 and y_2 will not become negative.

Equation (2. 26) can now be expressed as the Lagrangian function (2. 27).

$$R(y_1 \cdots y_6, \lambda_0, \lambda_1, \cdots \lambda_4) = 4y_1^2 + 9y_2^2 - \lambda_0[E - y_1 - 2y_2]$$
$$-\lambda_1[G_1 - y_1 - 3y_2 - y_3] - \lambda_2[G_2 - y_1 - y_2 - y_4]$$
$$-\lambda_3[-G_3 + y_1 - y_5] - \lambda_4[-G_4 + y_2 - y_6]$$

(2. 27)

where λ_0 is the Lagrangian multiplier of the expected income constraint.

$\lambda_1 \cdots \lambda_4$ are Lagrangian multipliers of the resource constraints.

The non-negativity requirements for the slack variables (y_3, y_4, y_5 and y_6) cause this traditional Lagrangian procedure to break down because non-feasible solutions occur. This procedural difficulty is overcome by employing the Kuhn-Tucker conditions (31). These optimality conditions require that if a Lagrangian multiplier is positive the slack variable must be zero and if the Lagrangian multiplier is zero the slack variable must be greater than or equal to zero. If the objective function is a positive definite quadratic form and if the constraints are linear then the optimum is also a minimum.

The solution to the constrained variance minimization problem is obtained by partially differentiating Equation (2. 27) with respect to its arguments and setting the results equal to zero. The resulting first order conditions are shown in Equation (2. 28). The matrix form is shown in Equation (2. 29). In the matrix it should be noted that $\partial R/\partial \lambda_0$ has been moved to the position immediately following $\partial R/\partial y_2$. This row and column transposition will prove useful for solving the system. The solution is obtained by inverting the matrix and appears as Equation (2. 30). Equations for the activity levels, expansion path and efficiency frontier are obtained by carrying out the multiplication of the inverted system. These are specified in Equation set (2. 31).

$$
\begin{aligned}
\frac{\partial R}{\partial y_1} &= 8y_1 + \lambda_0 + \lambda_1 + \lambda_2 - \lambda_3 &&= 0 \\
\frac{\partial R}{\partial y_2} &= 18y_2 + 2\lambda_0 + 3\lambda_1 + \lambda_2 - \lambda_4 &&= 0 \\
\frac{\partial R}{\partial y_3} &= \lambda_1 &&= 0 \\
\frac{\partial R}{\partial y_4} &= \lambda_2 &&= 0 \\
\frac{\partial R}{\partial y_5} &= \lambda_3 &&= 0 \\
\frac{\partial R}{y6} &= \lambda_4 &&= 0 \\
\frac{\partial R}{\partial \lambda_0} &= y_1 + 2y_2 - E &&= 0 \\
\frac{\partial R}{\partial \lambda_1} &= y_1 + 3y_2 + y_3 - G_1 &&= 0 \\
\frac{\partial R}{\partial \lambda_2} &= y_1 + y_2 + y_4 - G_2 &&= 0
\end{aligned}
\qquad (2.\ 28)
$$

$$\frac{\partial R}{\partial \lambda_3} = -y_1 + y_5 \qquad + G_3 \qquad = 0$$

$$\frac{\partial R}{\partial \lambda_4} = -y_2 + y_6 \qquad + G_4 \qquad = 0 \qquad \text{(2. 28) cont.}$$

Inspection of Equation set (2. 31) reveals some interesting information about the problem. Real activities y_1 and y_2 are linear functions of expected income. The slope of the efficiency frontier is represented by $-\lambda_0$. It is this equation which can be integrated to obtain the equation for the efficiency frontier. Slack activities y_3, y_4, y_5 and y_6 are represented by linear equations also. The equations must be restricted by the value E takes on so that they remain non-negative.

A level of expected income exceeding 50/11 requires more than the six hours of labor available. This violates the non-negativity restriction on y_3. A level of expected income exceeding 100/17 requires more than four acres of land hence violating the restriction on y_4. Since labor becomes limiting at a lower level of expected income, the upper limit on E is 50/11. A level of expected income less than zero would require y_1 and y_2 to be negative, a violation of the conditions of the problem. This is reflected by y_5 and y_6 being forced negative if E were allowed to take on values less than zero. If E is restricted to the interval $0 \leq E \leq 50/11$ the Kuhn-Tucker conditions are satisfied and variance minimizing combinations of crop one and crop two are assured. In Figure 2. 16 the point e' corresponds to $E = 50/11$ and point e corresponds to $E = 100/17$. But is 50/11

$$
\begin{bmatrix}
8 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\
0 & 18 & 2 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & -1 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ \lambda_0 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ G_1 \\ G_2 \\ -G_3 \\ -G_4
\end{bmatrix}
\qquad (2.29)
$$

$$\begin{bmatrix} NR^{22} & NR & 9/25 & NR & NR & NR & NR & 0 & 0 & 0 & 0 \\ NR & NR & 8/25 & NR & NR & NR & NR & 0 & 0 & 0 & 0 \\ 9/25 & 8/25 & -72/25 & -33/25 & -17/25 & 9/25 & 8/25 & 0 & 0 & 0 & 0 \\ NR & NR & -33/25 & NR & NR & NR & NR & 1 & 0 & 0 & 0 \\ NR & NR & -17/25 & NR & NR & NR & NR & 0 & 1 & 0 & 0 \\ NR & NR & 9/25 & NR & NR & NR & NR & 0 & 0 & 1 & 0 \\ NR & NR & 8/25 & NR & NR & NR & NR & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ G_1 \\ G_2 \\ -G_3 \\ -G_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \lambda_0 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \tag{2.30}$$

[22] Note NR denotes not relevant.

$$
\begin{aligned}
y_1 &= \frac{9}{25} E \\
y_2 &= \frac{8}{25} E \\
-\lambda_0 &= \frac{72}{25} E \\
y_3 &= -\frac{33}{25} E + G_1, \quad E \leq \frac{50}{11}, \quad G_1 = 6 \\
y_4 &= -\frac{17}{25} E + G_2, \quad E \leq \frac{100}{17}, \quad G_2 = 4 \\
y_5 &= \frac{9}{25} E - G_3 \quad , \quad E \geq 0, \quad G_3 = 0 \\
y_6 &= \frac{8}{25} E - G_4 \quad , \quad E \geq 0, \quad G_4 = 0 \\
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
\qquad (2.31)
$$

the maximum expected income that can be produced on this farm? It is not, for the linear programming problem presented earlier showed that E could be increased to a maximum of five dollars. The question of how to increase E while fulfilling the minimum variance requirement must now be answered.

Even though all of the available labor supply is utilized at the level of $E = 50/11$ only $34/11$ acres of land are used leaving a surplus of $10/11$ acres. Is it not possible that the composition of the plans could be changed so that additional expected income may be obtained through greater use of the surplus land resource? The answer is yes. It can not be achieved by movement from e' to e since this would violate the labor constraint but it can be achieved by movement along the labor constraint boundary from e' to d. This allows a further increase of expected income without violating any conditions of the problem. Mathematically this is accomplished by setting y_3, the slack activity for labor equal to zero in the Lagrangian function of Equation (2. 27). The amended Lagrangian form appears as Equation (2. 32) with the assurance that the Kuhn-Tucker conditions will be fulfilled.

$$R(y_1, y_2, y_4, y_5, y_6, \lambda_0, \lambda_1 \cdots \lambda_4)$$

$$= 4y_1^2 + 9y_2^2 - \lambda_0[E-y_1-2y_2] - \lambda_1[G_1-y_1-3y_2] - \lambda_2[G_2-y_1-y_2-y_4]$$

$$- \lambda_3[-G_3+y_1-y_5] - \lambda_4[-G_4+y_2-y_6] \tag{2. 32}$$

The first order conditions are displayed in matrix Equation (2. 33), inverted to produce Equation (2. 34) yielding solution Equation (2. 35). Note that $\partial R/\partial\lambda_0$ and $\partial R/\partial\lambda_1$ have been moved into position immediately following $\partial R/\partial y_2$.

$$
\begin{bmatrix}
8 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 18 & 2 & 3 & 0 & 0 & 0 & 1 & 0 & -1 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ \lambda_0 \\ \lambda_1 \\ y_4 \\ y_5 \\ y_6 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ E \\ G_1 \\ 0 \\ 0 \\ 0 \\ G_2 \\ -G_3 \\ -G_4
\end{bmatrix}
\qquad (2.33)
$$

$$
\begin{bmatrix}
NR & NR & 3 & -2 & NR & NR & NR & 0 & 0 & 0 \\
NR & NR & -1 & 1 & NR & NR & NR & 0 & 0 & 0 \\
3 & -1 & -90 & 66 & -2 & 3 & -1 & 0 & 0 & 0 \\
-2 & 1 & 66 & -50 & 1 & -2 & 1 & 0 & 0 & 0 \\
NR & NR & -2 & 1 & NR & NR & NR & 1 & 0 & 0 \\
NR & NR & 3 & -2 & NR & NR & NR & 0 & 1 & 0 \\
NR & NR & -1 & 1 & NR & NR & NR & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\ 0 \\ E \\ G_1 \\ 0 \\ 0 \\ 0 \\ G_2 \\ -G_3 \\ -G_4
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ y_2 \\ \lambda_0 \\ \lambda_1 \\ y_4 \\ y_5 \\ y_6 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4
\end{bmatrix}
\qquad (2\ 34)
$$

$$
\begin{aligned}
y_1 &= 3E - 2G_1 \\
y_2 &= -E + G_1 \\
-\lambda_0 &= 66E - 50G_1 \quad , \; E \geq 22/5 \; , \; G_1 = 6 \\
\lambda_1 &= 66E - 50G_1 \quad , \; E \geq 50/11 \; , \; G_1 = 6 \\
y_3 &= 0 \\
y_4 &= -2E + G_1 + G_2, \quad E \leq 5 \; , \; G_1 = 6, \; G_2 = 4 \\
y_5 &= 3E - 2G_1 - G_3 \; , \quad E > 4 \; , \; G_1 = 6, \; G_3 = 0 \\
y_6 &= -E + G_1 - G_4 \; , \quad E \leq 6 \; , \; G_1 = 6, \; G_4 = 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
\qquad (2.35)
$$

The Equations (2.35) are linear functions of E. Values of E greater than five would require more than the four acres of available land thus causing the slack variable y_4 to become negative. Values of E less than $50/11$ would result in the Lagrangian multiplier λ_1 becoming negative and violating the minimum variance requirement. The valid range of E is established as $50/11 \leq E \leq 5$. The absolute maximum level of expected income consistent with the land and labor constraints is five dollars as determined by linear programming. The expansion path, activity equations and Lagrangian multiplier equations resulting from the two step solution to the variance minimization problem are shown in Equations (2.36), (2.37) and (2.38).

The expansion path[23]

$$
\begin{aligned}
y_2 &= \frac{8}{9}y_1, \quad 0 \leq y_1 < \frac{18}{11} \\
y_2 &= 2 - \frac{1}{3}y_1, \quad \frac{18}{11} \leq y_1 \leq 3
\end{aligned}
\qquad (2.36)
$$

The activity equations

$$
\begin{aligned}
y_1 &= \frac{9}{25}E, \quad 0 \leq E < \frac{50}{11} \\
y_1 &= 3E - 12, \quad \frac{50}{11} < E \leq 5 \\
y_2 &= \frac{8}{25}E, \quad 0 \leq E \leq \frac{50}{11} \\
y_2 &= -E + 6, \quad \frac{50}{11} \leq E \leq 5
\end{aligned}
\qquad (2.37)
$$

The Lagrangian multiplier equations

$$
\begin{aligned}
-\lambda_0 &= \frac{72}{25}E, \quad 0 \leq E < \frac{50}{11} \\
-\lambda_0 &= 90E - 396, \quad \frac{50}{11} \leq E \leq 5
\end{aligned}
\qquad (2.38)
$$

[23] The expansion path equation does not appear directly in the solution to the system of equations. It is obtained indirectly by eliminating E from the activity equations and expressing y_2 as a function of y_1.

$$\lambda_1 = 0, \quad 0 \leq E < \frac{50}{11}$$

$$\lambda_1 = 66E - 300, \quad \frac{50}{11} \leq E \leq 5 \qquad (2.38) \text{ cont.}$$

The algebraic form of the efficiency frontier is derived by solving the differential Equations (2.39) which are formed by the Lagrangian multiplier.

$$dV = \frac{72}{25} E dE, \quad 0 \leq E < \frac{50}{11}$$

$$dV = (90 - 66G_1)dE + (-66E + 50G_1)dG_1, \quad \frac{50}{11} \leq E \leq 5 \qquad (2.39)$$

The anti-derivative or integral of Equation (2.39) results in the algebraic specification of the efficiency frontier as Equation (2.40)

<u>The efficiency frontier</u>

$$V = \frac{72}{50} E^2, \quad 0 \leq E < \frac{50}{11}$$

$$V = 45E^2 - 66EG_1 + 25G_1^2, \quad \frac{50}{11} \leq E \leq 5, \quad G_1 = 6 \qquad (2.40)$$

Figure 2.18 displays the efficiency frontier graphically as two parabolas with d'e'd being nested in oe'e. The curve oe'd is the efficiency frontier. The segment e'e is a series of points that can not be attained because of the labor constraint. The segment d'e' is a series of inferior points dominated by points on the segment oe'd and not part of the efficiency frontier.

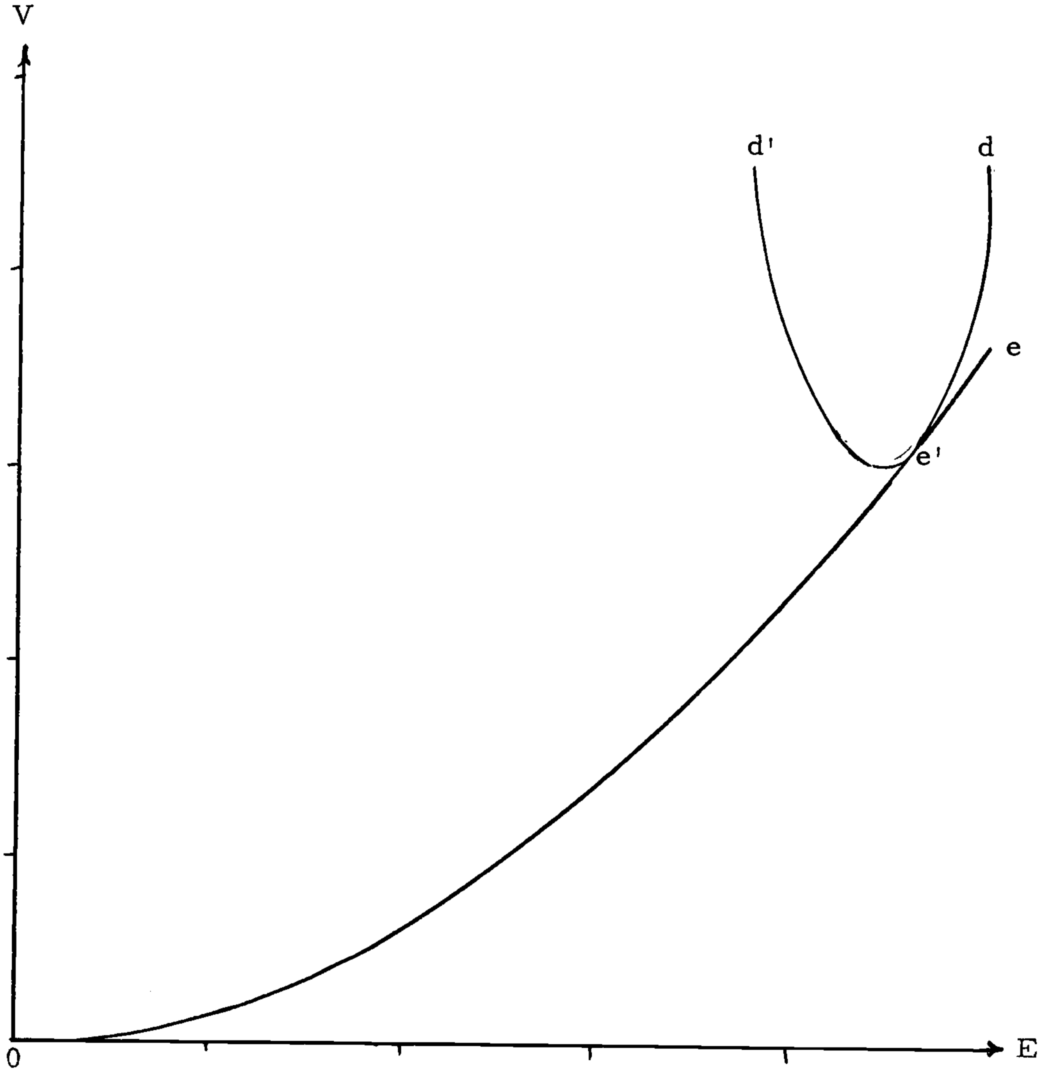

Figure 2.18. The complete efficiency frontier as a result of adding a constraint.

The problem stated as Equation (2. 25) is now solved. A simplified problem was used to facilitate understanding and clarity. Even in the simple two activity model procedural complications can arise and these become the next order of business.

Methodological Complications and Their Resolution

One major difficulty is that the initial basis may be elusive. In assertion two it is noted that high positive values of the correlation coefficient r caused the expansion path to have a negative slope in the y_1, y_2 plane. In this example $r^* = 3/4$. Suppose $r = 7/8$ rather than zero as has been assumed in the example. Then the expansion path becomes the negatively sloped line segment oe'' in Figure 2. 19. This results in a revision of the original example with the minimum variance objective function becoming

$$\text{Min:} \quad 4y_1^2 + \frac{21}{2} y_1 y_2 + 9y_2^2 = V \tag{2. 41}$$

The supply of land and labor are not affected by this change hence the constraint remains the same as before. The Lagrangian function is set up, its first order conditions derived, and the system is solved with results appearing in Equation (2. 42).

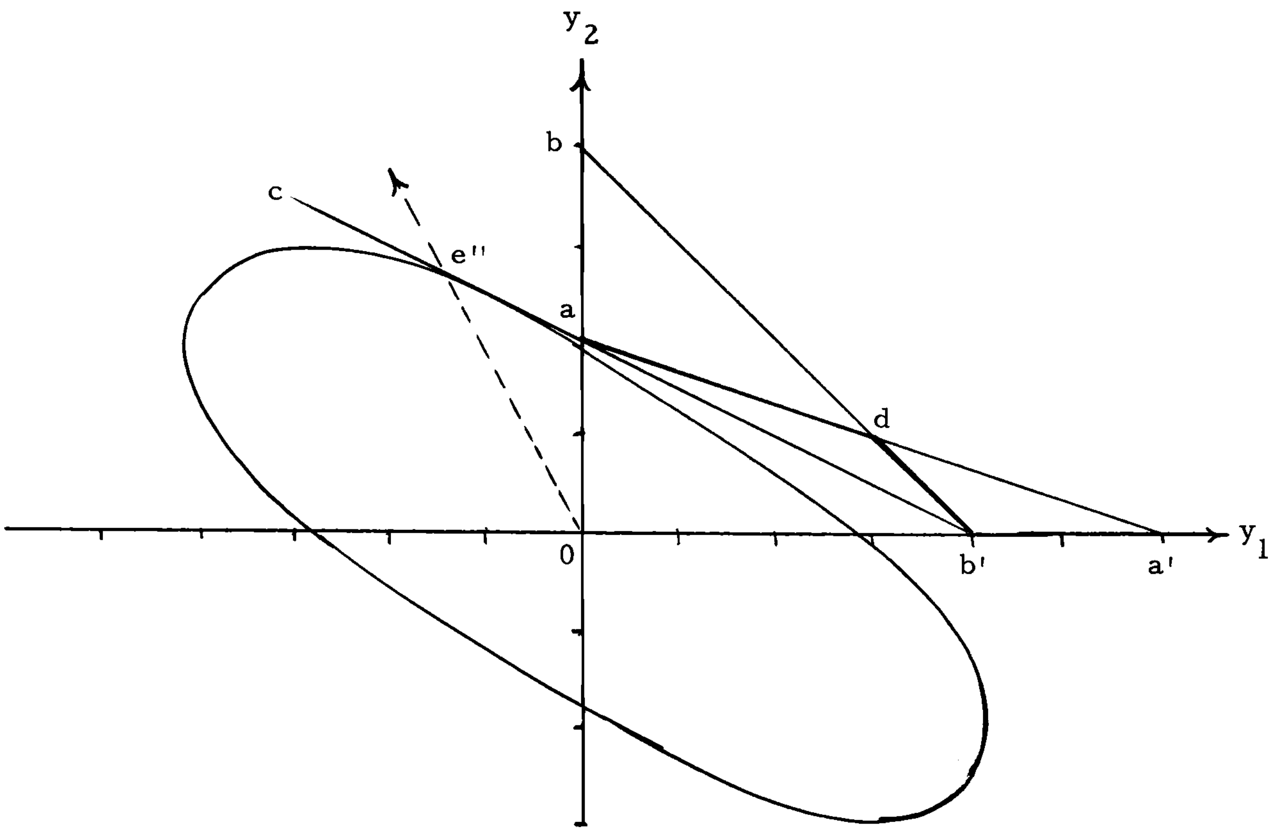

Figure 2. 19. Quadratic programming model - high positive correlation.

$$
\begin{aligned}
y_1 &= -\frac{3}{8}E \\
y_2 &= \frac{11}{16}E \\
-\lambda_0 &= \frac{135}{32}E \qquad , \quad E \geq 0 \\
y_3 &= -\frac{27}{16}E + G_1, \quad E \leq \frac{32}{9}, \quad G_1 = 6 \\
y_4 &= -\frac{5}{16}E + G_2, \quad E \leq \frac{64}{5}, \quad G_2 = 4 \\
y_5 &= -\frac{3}{8}E - G_3 \;, \quad E \leq 0, \quad G_3 = 0 \\
y_6 &= \frac{11}{16}E - G_4 \;, \quad E \geq 0, \quad G_4 = 0 \\
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
\tag{2.42}
$$

For a plan to be feasible it is required that real activities y_1 and y_2 be greater than or equal to zero. Since slack activities, y_5 and y_6, were introduced to represent y_1 and y_2 in the Lagrangian formulation, the range of E must be restricted so that y_5 and y_6 remain non-negative. In Equation (2.42) it is noted that a positive value of E forces y_5 to be negative and a negative value of E forces y_6 to be positive. Thus the Kuhn-Tucker conditions hold only at the point $E = 0$. To resolve the difficulty the same procedure as was followed in the previous section where labor became limiting can be applied. This requires setting $y_5 = 0$ and moving along the y_2

axis in Figure 2.19 resulting in complete specialization in the least risky activity in accordance with assertion three. Mathematically it is required that the Lagrangian function is set up with $y_5 = 0$, the first order conditions derived and the system solved with the results appearing in Equation (2.43).

$$
\begin{aligned}
y_1 &= 0 \\
y_2 &= \frac{1}{2}E - \frac{1}{2}G_3 \\
-\lambda_0 &= \frac{9}{2}E - \frac{3}{4}G_3 &&, \; E \geq 0,\; G_3 = 0 \\
\lambda_3 &= \frac{3}{4}E + 2G_3 &&, \; E \geq 0,\; G_3 = 0 \\
y_3 &= -\frac{3}{2}E + \frac{1}{2}G_3 + G_1, && E \geq 0,\; G_1 = 6,\; G_3 = 0 \\
y_4 &= -\frac{1}{2}E - \frac{1}{2}G_3 + G_2, && E \leq 4,\; G_2 = 4,\; G_3 = 0 \\
y_5 &= 0 \\
y_6 &= \frac{1}{2}E - \frac{1}{2}G_3 - G_4, && E \geq 0,\; G_3 = 0,\; G_4 = 0 \\
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
\qquad (2.43)
$$

Since y_5 was set equal to zero, y_1 is automatically set to zero. The basis is valid only on the interval $0 \leq E \leq 4$. This establishes the expansion path as the segment oa falling on the y_2 axis in Figure 2.19.

At point a the labor supply is exhausted and $E = 4$. The only way to increase E further is to move along the labor boundary from point a to d in Figure 2.19. This requires y_3, the slack activity representing surplus labor to be set equal to zero. Movement from a to d can not occur unless the real activity y_1 is allowed to be positive which requires that y_5 be reintroduced into the system[24]. This results in an amended Lagrangian function where y_3 is set equal to zero and y_5 is replaced. The system is solved as before with results shown in Equation (2.44).

$$
\begin{aligned}
y_1 &= 3E - 2G_1 \\
y_2 &= -E + G_1 \\
-\lambda_0 &= 27E - \frac{27}{2}G_1 \quad , \; E \geq 3, \; G_1 = 6 \\
\lambda_1 &= \frac{27}{2}E - 8G_1 \quad , \; E \geq \frac{32}{9}, \; G_1 = 6 \\
y_3 &= 0 \\
y_4 &= -2E + G_1 + G_2, \; E \leq 5, \; G_1 = 6, \; G_2 = 4 \\
y_5 &= 3E - 2G_1 - G_3, \; E \geq 4, \; G_1 = 6, \; G_3 = 0 \\
y_6 &= -E + G_1 - G_4, \; E \leq 6, \; G_1 = 6, \; G_4 = 0
\end{aligned}
\tag{2.44}
$$

[24] In terms of the matrices, having both y_3 and y_5 set equal to zero would produce a singular system. In this problem there can not be more than n-1 effective production constraints, where n is the number of real activities. In linear programming there can be as many constraints as real activities, however, here the income constraint uses up one row and column of the matrix.

$$\lambda_2 = 0$$

$$\lambda_3 = 0 \qquad (2.44)\ \text{cont.}$$

$$\lambda_4 = 0$$

Note that now the lower limit of E is four dollars and the upper limit is five dollars. The lower limit occurs at point a in Figure 2.19 where y_1 was introduced at a positive level. The upper limit occurs at point d where the land supply is exhausted as indicated by its slack variable y_4 becoming zero. The level of expected income of five dollars has been reached. From both the graphs and a solution identical to that obtained with linear programming, it can be observed that the maximum attainable E has been reached. But what assurance is there that the maximum attainable E has been attained? This can be checked mathematically by noting from Equations (2.44) that the only possible way for expected income to increase is for land to be fully utilized. For land to be fully utilized requires that its slack activity y_4 be set to zero. But this can not be done in the two activity model because there must not be more than one resource fully utilized at one time. There is one possible way to proceed and that is to allow y_3, the slack activity of labor to become positive. The Lagrangian function is amended to exclude y_4 and include y_3 at positive levels. The solution of the system is shown in Equation (2.45).

$$
\begin{aligned}
y_1 &= -\frac{1}{2}E + G_2 \\
y_2 &= \frac{1}{2}E - \frac{1}{2}G_2 \\
-\lambda_0 &= \frac{5}{2}E - \frac{5}{4}G_2 \qquad , \; G \geq 2, \; G_2 = 4 \\
\lambda_2 &= \frac{5}{4}E - 4G_2 \qquad , \; E \geq \frac{64}{5}, \; G_2 = 4 \\
y_3 &= -E + \frac{1}{2}G_2 + G_1, \; E \leq 8, \; G_1 = 6, \; G_2 = 4 \\
y_4 &= 0 \qquad\qquad (2.45) \\
y_5 &= -\frac{1}{2}E + G_2 - G_3, \; E \leq 8, \; G_2 = 4, \; G_3 = 0 \\
y_6 &= \frac{1}{2}E - \frac{1}{2}G_2 - G_4, \; E \geq 4, \; G_2 = 4, \; G_4 = 0 \\
\lambda_1 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
$$

Checking the equations it is found that for the Kuhn-Tucker conditions to hold E must be greater than or equal to $64/5$. At the same time E must not exceed eight. It is impossible that these restrictions hold simultaneously. Thus it is established that trading the labor constraint for the land constraint is not permissable. Since no other trades are possible there is no way expected income can be increased. This assures that the level of E attained in the previous valid basis is infact the maximum possible. From a graphic standpoint movement along the land constraint boundary from point d toward b' reduces E. Conversely movement d toward b violates the labor constraint.

The stepwise procedure just completed produces the equations for the expansion path, the activity levels, the Lagrangian multipliers and the efficiency frontier.

The expansion path

$$
\begin{aligned}
y_1 &= 0, \quad 0 \le y_2 < 2 \\
y_2 &= 2 - \frac{1}{2} y_1, \quad 0 \le y_1 \le 3
\end{aligned}
\tag{2.46}
$$

The activity equations

$$
\begin{aligned}
y_1 &= 0, \quad 0 \le E < 4 \\
y_1 &= 3E - 12, \quad 4 \le E \le 5 \\
y_2 &= \frac{1}{2} E, \quad 0 \le E < 4 \\
y_2 &= -E + 6, \quad 4 \le E \le 5
\end{aligned}
\tag{2.47}
$$

The Lagrangian multiplier equations

$$
\begin{aligned}
-\lambda_0 &= \frac{9}{2} E, \quad 0 \le E < 4 \\
-\lambda_0 &= \frac{27}{2} E - 81, \quad 4 \le E < 5 \\
\lambda_1 &= 0, \quad 0 \le E < 4 \\
\lambda_1 &= \frac{27}{2} E - 48, \quad 4 \le E \le 5 \\
\lambda_3 &= \frac{3}{4} E, \quad 0 \le E < 4 \\
\lambda_3 &= 0, \quad 4 \le E \le 5
\end{aligned}
\tag{2.48}
$$

<u>The efficiency frontier</u>

$$V = \frac{9}{4}E^2, \quad 0 \leq E < 4$$
$$V = \frac{27}{2}E^2 - 81E + 144, \quad 4 \leq E \leq 5 \qquad (2.49)$$

The efficiency frontier can be graphed in the expected income variance coordinate system. This is done in Figure 2. 20. Points on line segments d'f and ff' are infeasible since they violate the land and labor constraints. The line segment ofd is the efficiency frontier. Comparison of Figures 2. 18 and 2. 20 reveals an important difference. In both cases variance is described in terms of parabolas. In the case of Figure 2. 18 where a constraint was simply added to form the second basis there is a smooth transition from the curve oe'e to the curve d'e'd. In the case of Figure 2. 20 where it was necessary to trade constraints there is a sharp corner at point f where the basis change occurs. In both cases the efficiency frontier is completely defined on the interval $0 \leq E \leq 5$.

Shadow Prices - Implications of Changes in Constraint Levels

Thus far the problem perspective has been mainly in the activity space. Similar to the dual of linear programming, the problem also can be specified in the constraint space. In the context of variance minimization this is in the expected income - production resource

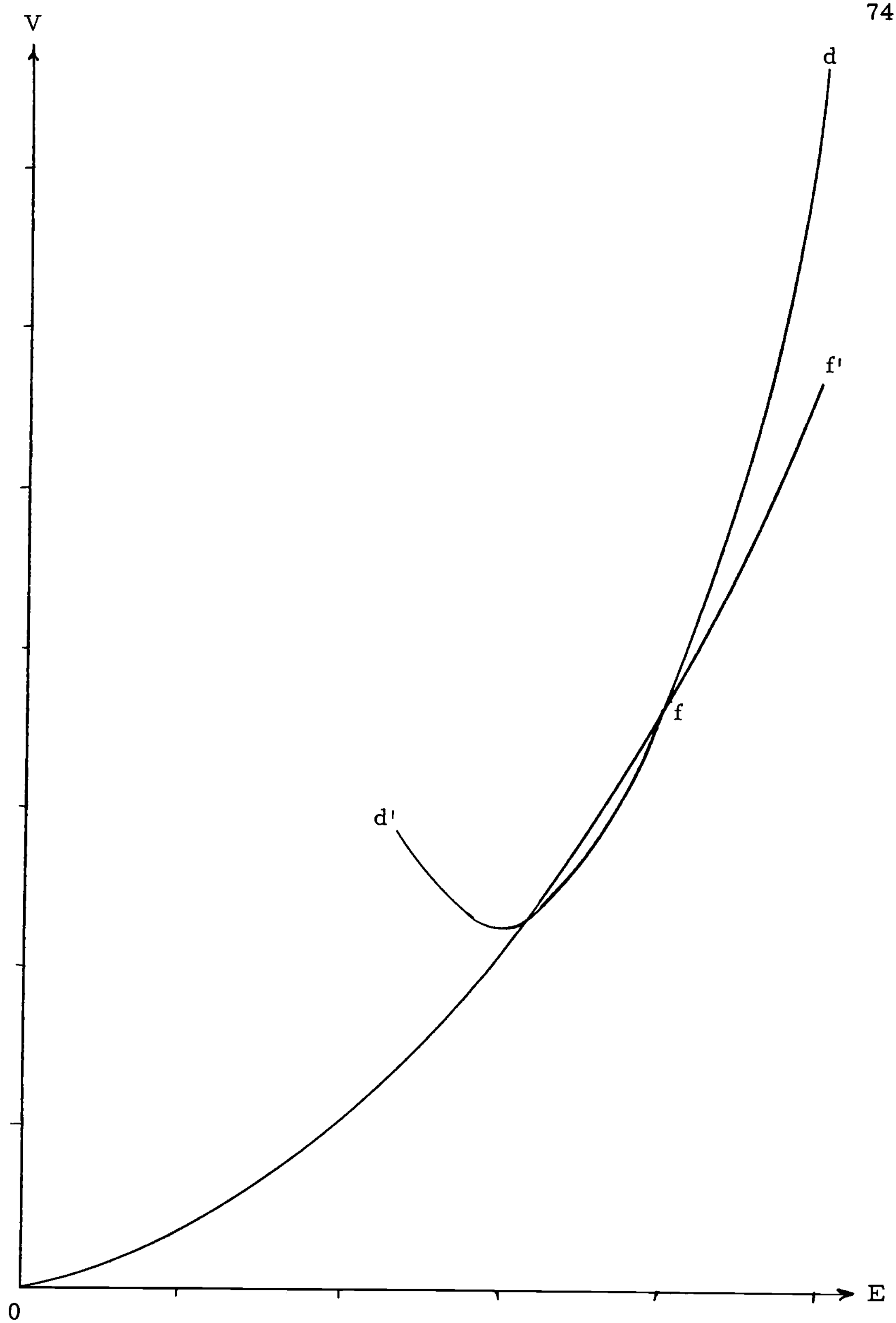

Figure 2. 20. The efficiency frontier as a result of trading constraints.

coordinate system. Although not all of the ramifications of the dual problem will be pursued, the matter of shadow prices deserves special attention.

In Equation (2. 24) the method for algebraically specifying the efficiency frontier was given in the absence of production constraints. A numerical derivation was presented in Equation (2. 40) which included resource constraints. The generalized form of Equation (2. 40) is Equation (2. 50).

$$V = \frac{a}{2} E^2 - bEG_k + \frac{c}{2} G_k^2 \tag{2. 50}$$

where a, b and c are elements taken from the inverse matrix. For example, see Equation (2. 34) where $a = 90$, $b = 66$ and $c = 50$.

The total differential of the variance function is

$$dV = (aE - bG_k)dE + (-bE + cG_k)dG_k \tag{2. 51}$$

where $aE - bG_k = \frac{\partial V}{\partial E} = -\lambda_0$

$-bE + cG_k = \frac{\partial V}{\partial G_k} = -\lambda_k$

The partial derivatives are the negatives of the Lagrangian multipliers. Because of the solution procedure and the nature of the variance function, the Lagrangian multiplier associated with the expected income constraint is never positive. Hence the partial derivative $\frac{\partial V}{\partial E} = -\lambda_0$ is never negative. This indicates that an increase in expected income, holding the level of the production constraint G_k constant results in higher variance. The graphic interpretation of $-\lambda_0$ is given in Figure 2.21 as the slope of curve d'd'' at the point d. The Lagrangian multiplier associated with the production constraint is required never to be negative. Accordingly, the partial derivative $\frac{\partial V}{\partial G_k} = -\lambda_k$ is never positive. If expected income is held constant, an increase in the level of the kth resource will reduce variance since this allows the decision maker to expand in the direction of a less risky activity. The graphic interpretation of $-\lambda_k$ is shown in Figure 2.22 as the slope of curve g''g' at point d.

One additional ramification bears investigation. What will be the effect upon expected income if variance is held fixed and the constraint level is increased. This is shown by the derivative

$$\frac{dE}{dG_k} = -\frac{\partial V/\partial G_k}{\partial V/\partial E} = \frac{bE - cG_k}{aE - bG_k} \qquad (2.52)$$

Extending the arguments used earlier to verify the algebraic sign of $\frac{\partial V}{\partial E}$ and $\frac{\partial V}{\partial G_k}$ it follows that $\frac{dE}{dG_k}$ is non-negative and an increase in the level of the production constraint, holding variance constant, will increase expected income. This is shown as the slope of the variance ellipse at point d in Figure 2.23. The magnitude of the derivative is the value of an additional unit of the resource G_k and the interpretation is similar to the shadow price of linear programming. However a major difference exists. In linear programming the shadow price is

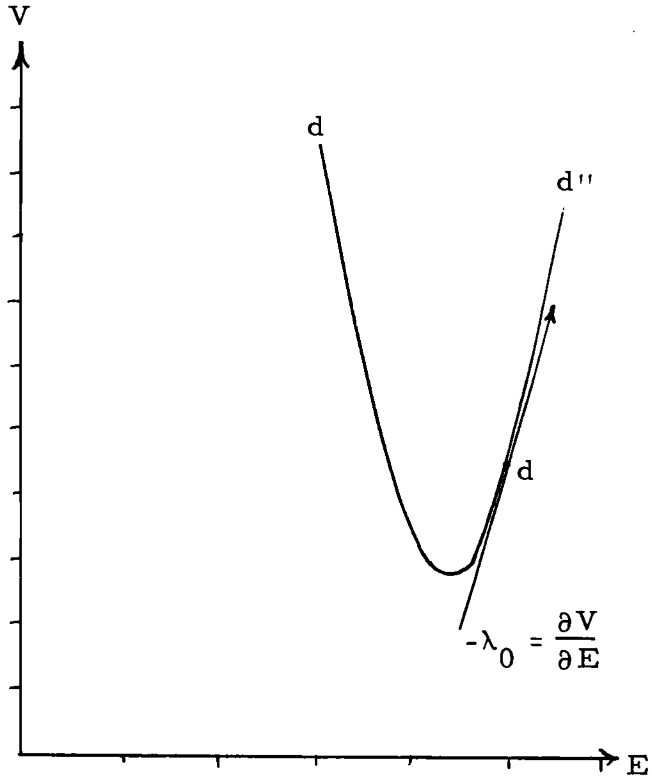

Figure 2. 21. Response of variance to changes in expected income.

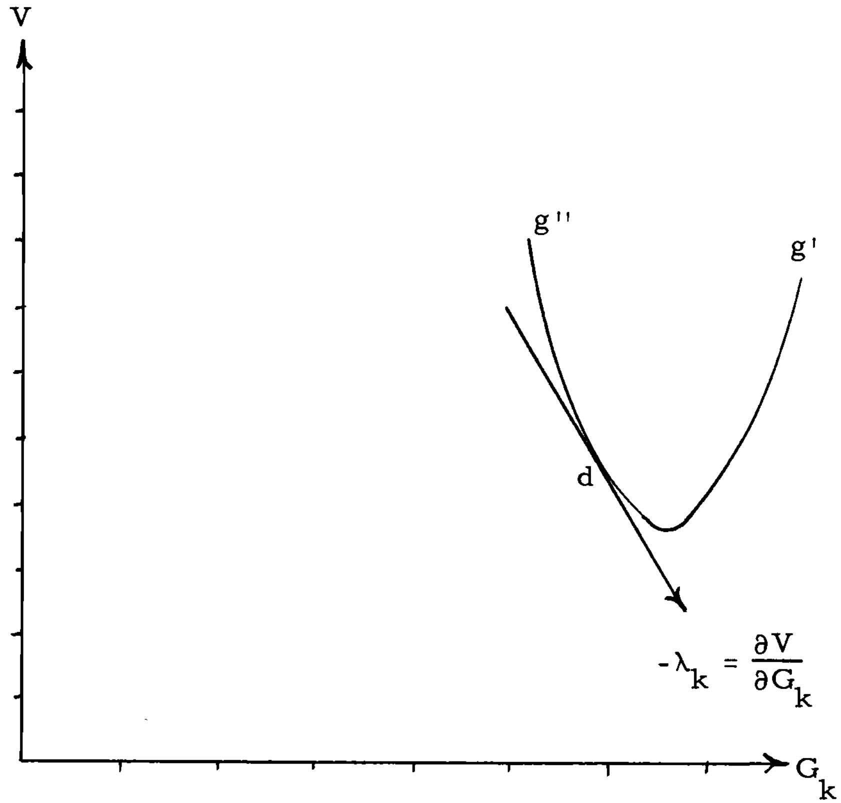

Figure 2. 22. Response of variance to changes in constraint levels.

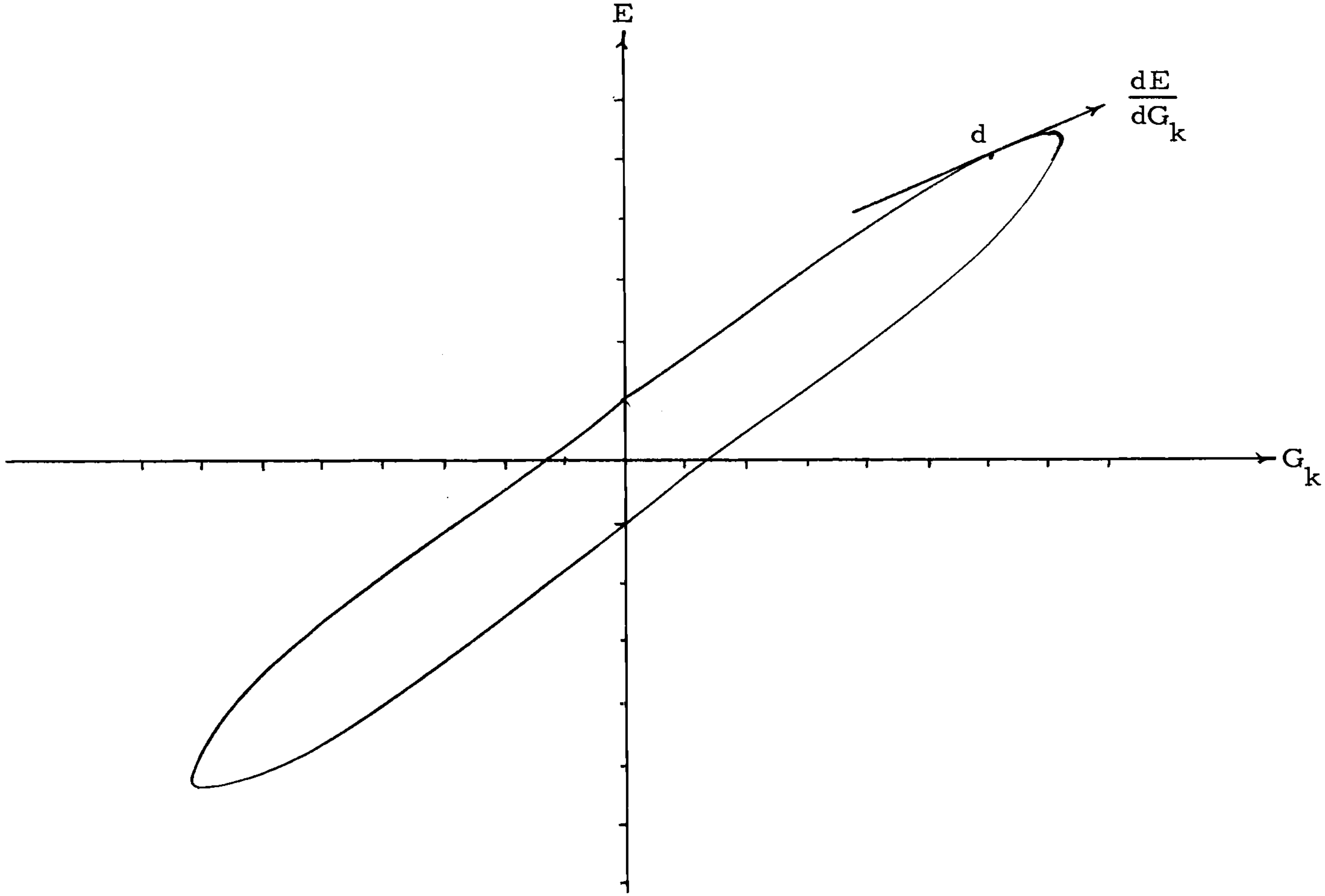

Figure 2. 23. Shadow prices - the response in expected income to increased resource levels.

a constant, valid over the range of the basis. But here the shadow price Equation (2. 52), although valid over the range of the basis, is a non-linear function of expected income and resource level. In the numerical example, the shadow price for labor was 5/9. Although approximately indicating the increase in expected income resulting from an addition of one hour of labor, the addition of another 100 hours certainly would not add 500/9 to expected income. It can be seen from Figure 2. 23 and confirmed by the second derivative of the iso-variance curve, Equation (2. 53),

$$\frac{d^2E}{dG_k^2} = \frac{(b^2 - ac)E}{(bE - cG_k)^2} < 0 \qquad (2.53)$$

that the shadow price of the resource becomes progressively less as the level of the resource is increased. Thus greater caution must be exercized in interpreting shadow prices from the quadratic model than with the linear programming model.

The following assertions review the implications of changing constraint levels in the variance minimization problem.

<u>Assertion 9</u>. For a specified level of production constraints, any increase in expected income occurs only by greater risk as measured by an increase in variance. This results from the positive slope of the efficiency frontier.

<u>Assertion 10</u>. For a specified level of expected income, any increase in the level of a limiting production constraint, holding all other production constraints fixed, will reduce risk as measured by decreased variance.

<u>Assertion 11</u>. For a specified level of variance, an increase in the level of a limiting production constraint, holding all other production constraints fixed, will increase expected income.

Most Risky Alternatives

The discussion thus far has centered on the lower boundary of the feasible set consisting of the least risky enterprise choices. Attention should also be focused on another set of enterprise choices, those which are most risky. This establishes the upper boundary and completely defines the feasible set of alternatives. The upper boundary is the maximum variance frontier and results from movement along the segment ob' in Figure 2. 16, the axis of the most risky activity y_1, and then along the land constraint from b' to d. This traces the locus of variance maximizing points and can be expressed algebraically in the expected income - variance coordinate system as Equation (2. 54).

$$\begin{aligned} V &= 4E^2, \quad 0 \leq E < 4 \\ V &= 13E^2 - 136E + 400, \quad 4 \leq E \leq 5 \end{aligned} \qquad (2.54)$$

The entire set of feasible alternatives appears in Figure 2.24 as the area oedh including its boundary.[25] The lower boundary oed is the expected income - variance locus of least risky alternatives. The upper boundary ohd is the locus of most risky alternatives.

Selecting the "Best" Plan

The Von Neumann Morgenstern Utility Function

All possible enterprise choices from the least to the most risky have been specified. It is from this infinite set that the "best one" is to be chosen. But how is this choice made? The appropriate choice is the one which best meets the objectives of the decision maker. These objectives are specified in the utility function of Equation (2.12).

There are three possible shapes of the utility function. Consider three decision makers. Each is faced with the same set of enterprise choices but one has a preference for risk, the second has an aversion for risk and the third is risk neutral.

Decision maker one prefers risk and has the utility function.

$$u_1(Y) = Y^2, \quad 0 \leq Y \leq 10 \tag{2.55}$$

[25] In the literature the feasible set of alternatives is frequently described as a "cigar shaped" convex set. It is true as stated by Stoval (41) that the maximum variance need not occur at the maximum attainable expected income. However, since the upper boundary results from specialization in the most risky activity and since variance is a homogeneous function of second degree it follows that the maximum variance frontier must increase at an increasing rate contrary to the convex set in Stovall's diagram.

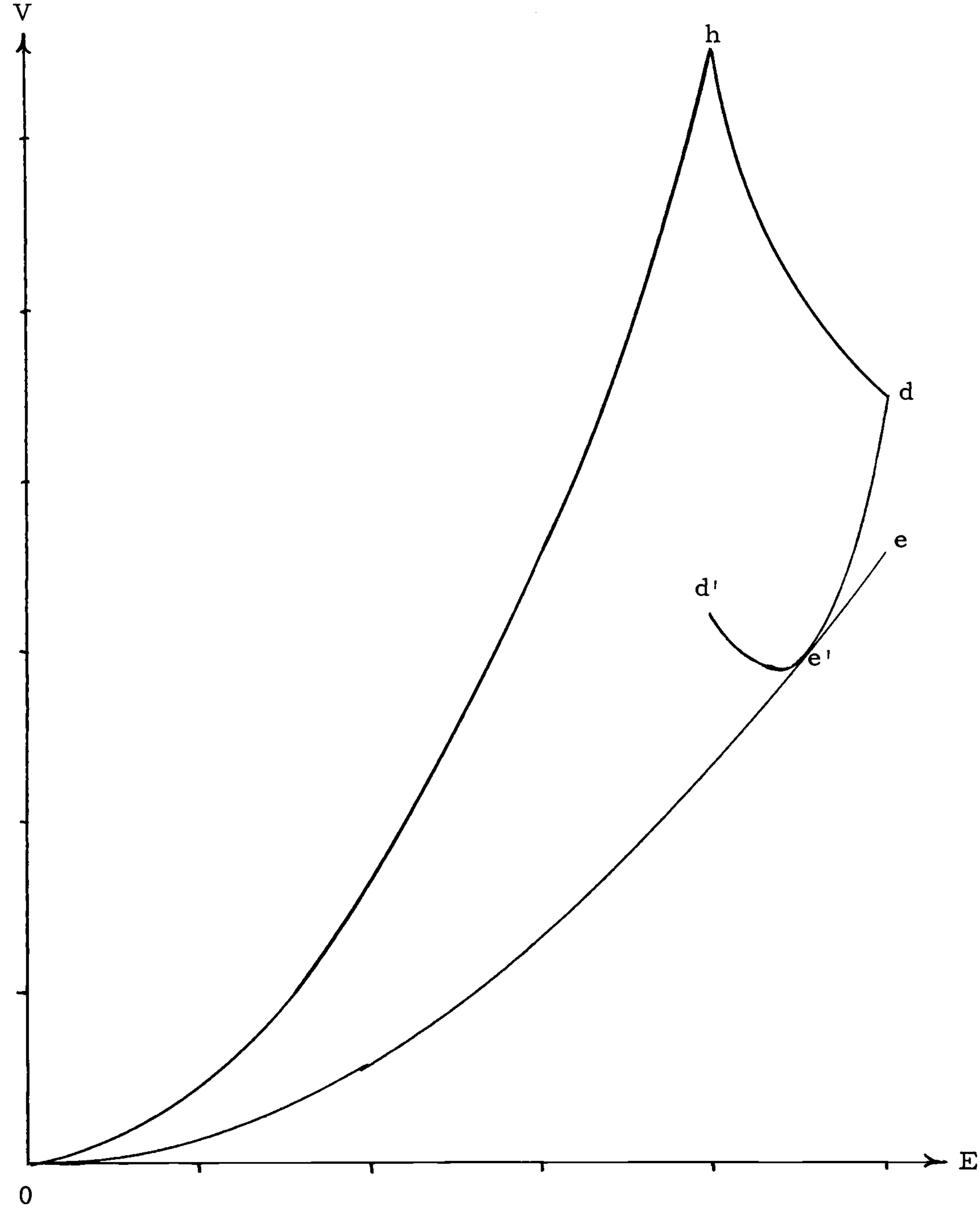

Figure 2. 24. The complete set of feasible alternatives.

Decision maker two is a risk averter and has the utility function

$$u_2(Y) = 20Y - Y^2, \quad 0 \leq Y \leq 10 \tag{2.56}$$

Decision maker three is risk neutral and has the utility function

$$u_3(Y) = 10Y, \quad 0 \leq Y \leq 10 \tag{2.57}$$

Decision maker one, acting rationally to maximize his expected utility chooses the combination of enterprises represented by point h on Figure 2.25 where expected income is four and variance is 64. The highest indifference curve attainable by decision maker one is U_1° passing through point h. An indifference curve passing through any other point representing a feasible combination would result in lower expected utility and any indifference curve representing greater expected utility can not be achieved. If, however, decision-maker one had a utility function like the one represented by U_1^{*} indicating a more cautious gambler the expected utility maximizing point would be point d which is also the maximum expected income combination.

<u>Assertion 12.</u> Decision makers who have a preference for risk will choose either that combination of enterprise representing maximum attainable expected income or a combination lying on the upper boundary of the feasible choices depending upon the intensity of the gambling spirit as reflected by the marginal utility of income.

Decision maker two, acting rationally to maximize his expected utility selects the enterprise combination represented by point g on Figure 2. 26. The highest indifference curve which will be in or on the feasible set is $\overset{\circ}{U}_2$ which is tangent at point g. Mathematically, point g can be derived by substituting the variance Equation (2. 40) into the expected utility Equation (2. 11) to establish Equation (2. 58) where expected utility is a function of expected income.

$$U_2 = 20E - E^2 - \frac{72}{50}E^2 \quad 0 \leq E < \frac{50}{11} \tag{2. 58}$$

$$U_2 = 20E - E^2 - (45E^2 - 39E + 900), \quad \frac{50}{11} \leq E \leq 5$$

Differentiating (2. 58) with respect to E and setting the result equal to zero establishes the expected utility maximizing value of expected income to be 4. 0984 with variance 24. 1875.[26] The activity levels are $y_1 = 1.4745$ and $y_2 = 1.3115$.

<u>Assertion 13</u>. Decision makers who are risk averters will choose a combination of activities which results in a level of expected income and variance lying on the lower boundary of the feasible set. The choice will lie farther from the maximum attainable

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[26] The second derivative of the expected utility function (2. 58) is always negative thus assuring that maximum expected utility is achieved. If the expected utility function for decision maker one had been set up in the same way it would be found that setting the derivative equal to zero does not achieve a maximum because of the shape of his utility function. It becomes necessary to evaluate his expected utility function at the extreme points d and h on Figure 2. 24 to determine which yields the greater expected utility.

expected income point (the linear programming solution) as the feeling of aversion to risk, measured by the marginal utility for income becomes more intense.

Decision maker three, acting rationally to maximize his expected utility, selects the enterprise combination represented by point d on Figure 2. 27. Being risk neutral, variance is not an argument in the utility function. The choice which maximizes his expected utility is the one which maximizes his expected income and is identical to the optimum solution derived in linear programming.

Assertion 14. Decision makers who are risk neutral will choose that combination of activities which results in the maximum expected income plan as derived by linear programming.

The solution procedure for deriving efficient enterprise combination will not provide the decision maker who prefers risk with the information he requires. For the risk neutral decision maker, not all of the information provided is needed and linear programming yields the required solution more efficiently. However, empirical observation on the behavior of farmers indicates that a significant portion, like decision maker two are concerned with the chances of bankruptcy and failure (36) and act accordingly.

Probability of Loss Function

Decision makers probably do not think of utility functions per se. However they are frequently familiar with probability statements such as those associated with weather forecasting. This suggests a possible substitute for the utility function which involves expressing efficient enterprise alternatives in terms of the probability of losses. The probability of loss function is a set of confidence statements about achieving various levels of income. The task of constructing the confidence bands becomes manageable if one assumes that the income from every efficient plan is normally distributed with mean E and variance V. Then one can use Equation (2. 59) to compute, for every level of expected income E, the critical value Y^* such that there is probability α that the actual level of income Y will not be less than Y^* i. e. $P(Y < Y^*) = \alpha$

$$Y^* = E + N_\alpha \sqrt{V} \qquad (2.\ 59)$$

where Y^* is the critical level of income

N_α is the factor from the standard normal density function (24 p. 370) taken at the desired probability level α.

Figure 2. 28 displays the confidence statements about achieving actual levels of income for each of the alternative plans available. For example, suppose the plan represented by a level of expected

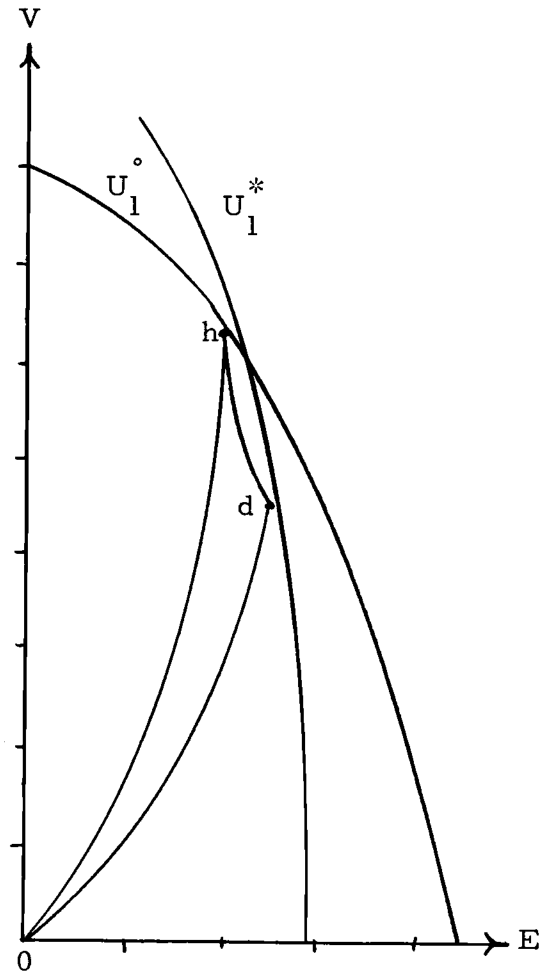

Figure 2.25. "Best" choice for risk preferring individual.

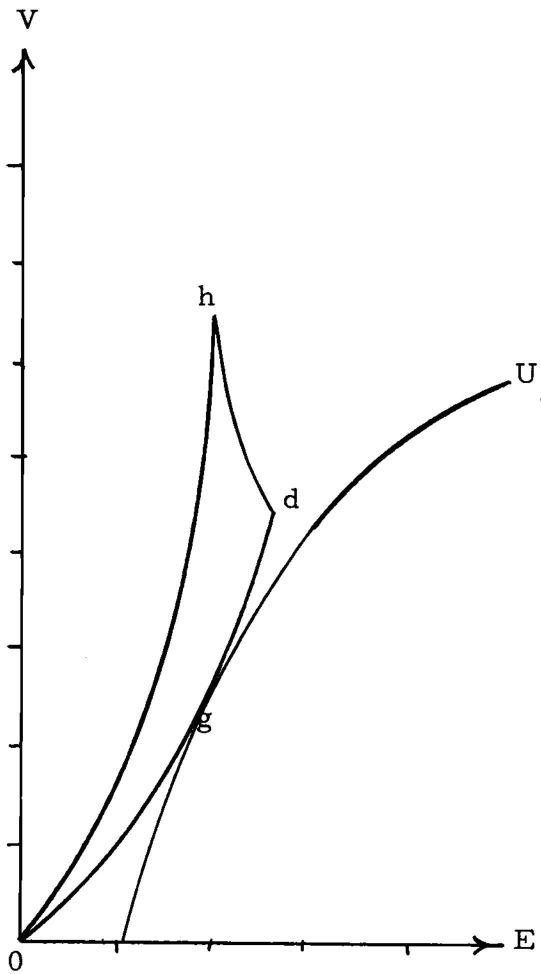

Figure 2.26. "Best" choice for risk averting individuals.

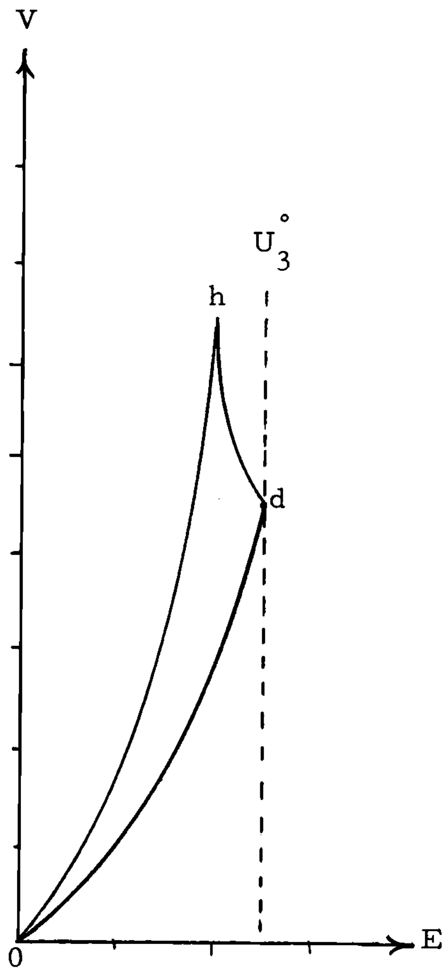

Figure 2.27. "Best" choice for risk neutral individual.

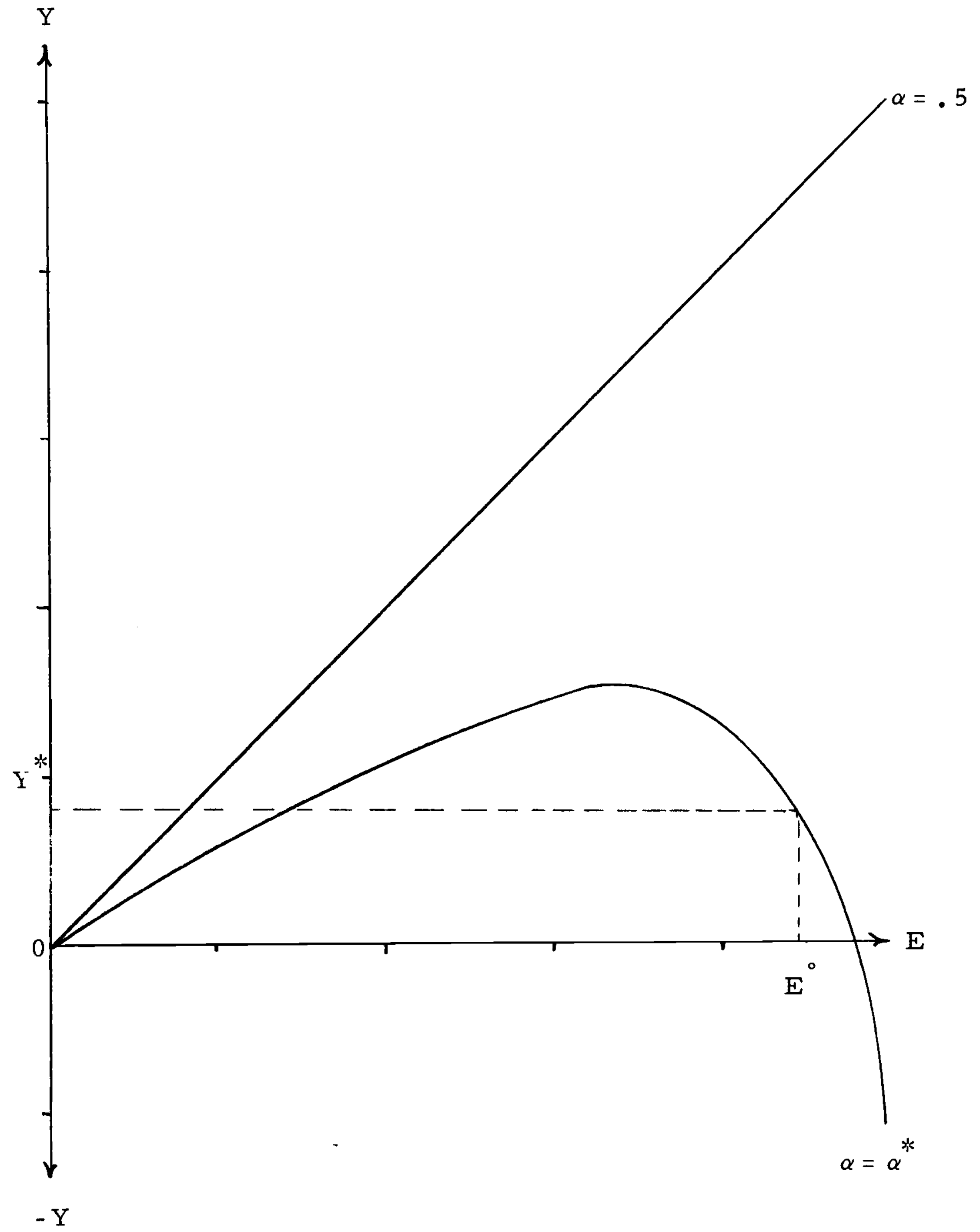

Figure 2. 28. Probability of loss function.

income E° was selected. Then there is probability α^* such that the income in a specific year will be less than Y^*. Because of the symetry properties of the normal density function the confidence band for $\alpha = .5$ is a 45° rayline from the origin. For $\alpha < .5$ the confidence band will have the characteristic shape shown in Figure 2. 28. There is an infinite number of such confidence bands for $0 < \alpha \leq 5$, however presenting bands for a few selected points like $\alpha =$ (.01, .05 .10 .20 .30, .40 and .50) should be ample to allow the decision maker to choose an acceptable level of expected income and hence an acceptable combination of enterprises.

The individuals age, health and propensity to gamble have a bearing on the ultimate choice he makes. He may also wish to guarantee that income for his family to live on, after discharging fixed cash obligations, does not fall below a specified amount. In the case of indebtedness he may not be the sole decision maker; his banker, too may influence the choice especially where potentially high income plans are also highly variable causing an abrupt downturn of the confidence bands.

The factors of age, health, debt position and the gambling spirit are also the same factors which formed the corner stones of the utility function.[27] Estimation of the utility function, although a worthwhile

[27] The probability of loss function approach will not provide the decision maker who has a preference for risk with the required information since it is derived solely from the lower boundary of the feasible set of plans.

endeavor for predicting decision maker behavior, seems less efficient from the extension advising view-point than to present the decision maker with all the relevant choices and let him select the one which is best on the basis of confidence statements surrounding each plan.

The enterprise selection problem formulated in this chapter has now been solved. To keep the problem and its solution understandable, only two activities were considered, however for the model to have practical relevance it must be able to handle problems of greater dimension. The extension of the model to the more general case will be the concern of the next chapter.

III. THE GENERAL MODEL - ENTERPRISE SELECTION UNDER UNCERTAINTY

AN ALGORITHM TO SOLVE FOR THE SET OF EFFICIENT PLANS

Attention was directed in the previous chapter to the mathematical requirements of the variance minimization problem. A numerical example was used to give a preview of the general method to follow. Although two activities were used for simplicity the model must be expanded to include more than two activities if it is to have relevance for farm decision makers. Consequently the two and three dimensional graphs of Chapter II will be inadequate for explaining the solution of the problem. It will still be possible to interpret the efficiency frontier, the activity equations and the probability of loss function graphically.

Description of the Model

The multi-dimensional risk minimization problem stated in matrix form as:

$$\begin{aligned} \text{Min:}\quad & y' X y = V \\ \text{S. T:}\quad & \mu' y = E \\ & ay \leq G \\ & y \geq 0 \end{aligned} \qquad (3.1)$$

where y is an $n \times 1$ vector of the decision variables i. e. activity levels

y' is the transpose of y

X is an $n \times n$ variance-covariance matrix of the incomes per unit of activity

V is the variance of total income

μ is an $n \times 1$ vector of expected incomes per unit of activity and μ' its transpose

E is the total expected income

a is an $m \times n$ matrix of resource requirements per unit of activity

G is an $m \times 1$ vector of available resources.

The full matrix specification of Equation (3. 1) is presented in Equation (3. 2).

Solving the Model

Introduction of Slack Variables

Each inequality of Equation (3. 1) or (3. 2) must be transformed into an equality by introducing disposal or slack activities. The non-negativity constraints on the real activites are also transformed into equations.

Min:

$$[y_1 y_2 \cdots y_n] \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y^n \end{bmatrix} = V$$

S. T:

$$[\mu_1 \mu_2 \cdots \mu_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = E \qquad (3.\ 2)$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \leq \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_m \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Upon transformation, the problem is re-written as:

$$\begin{aligned} \text{Min: } & y'Xy = V \\ \text{S. T: } & \mu' y = E \\ & [a:I] = G \\ & y \geq 0 \end{aligned} \qquad (3.3)$$

where y is now $(2n+m) \times 1$

X is now $(2n+m) \times (2n+m)$

μ is now $(2n+m) \times 1$

$a:I$ is now $(n+m) \times n$

G is now $(m+n) \times 1$

The expanded form appears as Equation (3.4). The $m+n$ additional elements in y are slack activities. The first m of these account for resource non-use and the remaining n of them account for the non-negativity constraints on real activities. The variance-covariance matrix X is expanded in dimension from n to $(2n+m)$ to account for the variances and co-variances of the slack activities which are assumed to be zero. The matrix μ has been increased in length from n to $(2n+m)$ to account for the expected incomes of the slack activities which are also assumed to be zero. The matrix a is first augmented by an $n \times n$ negative identity matrix. These negative coefficients insure that the real activity levels will not fall below their lower limits. The matrix a is again augmented by an $(n+m) \times (n+m)$

$$[y_1 \cdots y_n \, y_{n+1} \cdots y_{n+m}] \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1n} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \sigma_{1n} & \cdots & \sigma_n^2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{n+m} \end{bmatrix} = V$$

$$[\mu_1 \cdots \mu_n \, 0 \cdots 0] \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{n+m} \end{bmatrix} = E \qquad (3.4)$$

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 0 \\ \hline -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1 \end{array}\right] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+m} \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_m \\ -G_{m+1} \\ -G_{m+2} \\ \vdots \\ -G_{m+n} \end{bmatrix}$$

$$\begin{bmatrix} y_{n+1} \\ \vdots \\ y_{n+m} \end{bmatrix} \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

identity matrix to account for the slack variables. The vector G is increased in length from m to m+n with the additional elements explicitly accounting for lower bounds on the real activities which may be zero or positive.[28] With these amendments to the formulation of Equation (3.1), the problem is in proper form for applying the Lagrangian multiplier technique.

For convenience in notation assume that the n equations required for insuring non-negative values for $y_1 \cdots y_n$ are already present in the matrix a and vector G of Equation (3.1). Then the dimensions of the matrices after introducing slack activites are as follows:

y is $(n+m) \times 1$

a is $n \times m$

μ is $(n+m) \times 1$

G is $m \times 1$

X is $(n+m) \times (n+m)$

The Lagrangian Form and the Kuhn-Tucker Conditions

The Lagrangian form is:

$$R(y, \lambda_0, \lambda) = y'Xy - \lambda_0[E-\mu'y) - \lambda'[G-(a:I)y] \qquad (3.5)$$

[28] A positive lower bound on a real activity requires that the corresponding element in the vector G be entered as a negative number. The reader may wish to refer to Equation (2.30) for clarification on this point.

where λ_0 is a scaler representing the Lagrangian multiplier attached to the income constraint

λ is an $m \times 1$ vector of Lagrangian multipliers attached to the production constraints

and all other variables are as previously defined

The presence of non-negativity constraints on slack variables causes the traditional Lagrangian multiplier technique to be ineffective unless the Kuhn-Tucker conditions are observed. The Kuhn-Tucker theorems state that y^* is an optimum solution to the minimization problem of Equation (3.5) if and only if the matrix X is positive definite and the following conditions hold:

if $y_k^* > 0$

$$\text{then } \frac{\partial R}{\partial y_k} = \sum_{i=1}^{n} 2r_{ik}\sigma_i\sigma_k y_k y_i + \lambda_0\mu_k + \sum_{j=1}^{m} \lambda_j a_{kj} = 0\ , \quad k=1, \cdots, n$$

if $y_k^* = 0$

$$\text{then } \frac{\partial R}{\partial y_k} = \sum_{i=1}^{n} 2r_{ik}\sigma_i\sigma_k y_i y_k + \lambda_0\mu_k + \sum_{j=1}^{m} \lambda_j a_{kj} \geq 0\ , \quad k=1, \cdots, n$$

if $\lambda_j > 0$

$$\text{then } \frac{\partial R}{\partial \lambda j} = \sum_{i=1}^{n} a_{ij}y_i - G_j = 0\ , \quad j=1, \cdots, m$$

if $\lambda_j = 0$

then $\frac{\partial R}{\partial \lambda_j} = \sum_{i=1}^{n} a_{ij} y_i - G_j \leq 0, \; j=1, \cdots, m$

$$y_i \geq 0, \; i=1, \cdots, n$$

$$\lambda_j \geq 0, \; j=1, \cdots, m$$

Partially differentiating $R(y, \lambda_0, \lambda)$ with respect to its arguments and setting the derivatives to zero results in the first order conditions as expressed in the matrix of simultaneous linear Equations (3.6). In Equation (3.6), E is a variable and is allowed to take on only those values which satisfy the Kuhn-Tucker condition.

Matrices of the First Order Conditions

Partitions to Facilitate Inversion

Solving the system of equations is routine but formidable even for second generation computers. A modest problem of ten activities and fifty constraints requires inverting a 121 x 121 matrix. However, because of the position of zeros and its symetry,the matrix can be partitioned to reduce the magnitude of the inversion routine.

To facilitate partitioning,the same row operation of Equation (2.29) is performed to move the vector μ' into position n+1. To maintain symetry, a column operation is performed to move the vector

$$
\begin{array}{c}
\begin{matrix} 1 & \cdots & n & n+1 & \cdots & n+k & \cdots & n+m & n+m+1 & n+m+2 & \cdots & n+m+k+1 & \cdots & n+2m+1 \end{matrix} \\
\begin{bmatrix}
2\sigma_1^2 & \cdots & 2\sigma_{1n} & 0 & \cdots & 0 & \cdots & 0 & \mu_1 & a_{11} & \cdots & a_{1k} & \cdots & a_{1m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
2\sigma_{1n} & \cdots & \sigma^2 & 0 & \cdots & 0 & \cdots & 0 & \mu_n & a_{n1} & \cdots & a_{nk} & \cdots & a_{nm} \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\
\mu_1 & \cdots & \mu_n & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
a_{11} & \cdots & a_{n1} & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
a_{1k} & \cdots & a_{nk} & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
a_{1m} & \cdots & a_{nm} & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\end{array}
\begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{n+k} \\ \vdots \\ y_{n+m} \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_k \\ \vdots \\ \lambda_m \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ E \\ G_1 \\ \vdots \\ G_k \\ \vdots \\ G_m \end{bmatrix}
\qquad (3.6)
$$

$$
\begin{array}{ccccccccccccc}
1 & & n & n+1 & n+2 & & n+k+1 & & n+m+1 & n+m+2 & & n+m+k+1 & n+2m+1
\end{array}
$$

$$
\left[\begin{array}{ccc|c|ccccc|cccc}
2\sigma_1^2 & \cdots & 2\sigma_{1n} & \mu_1 & 0 & \cdots & 0 & \cdots & 0 & a_{11} & \cdots & a_{1k} \cdots & a_{1m} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
2\sigma_{1n} & \cdots & 2\sigma_n^2 & \mu_n & 0 & \cdots & 0 & \cdots & 0 & a_{n1} & \cdots & a_{nk} \cdots & a_{nm} \\
\mu_1 & \cdots & \mu_n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \cdots & 1 \\
a_{11} & \cdots & a_{n1} & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
a_{1k} & \cdots & a_{nk} & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
a_{1m} & \cdots & 0 & 0 & 0 & & 0 & \cdots & 1 & 0 & \cdots & 0 \cdots & 0
\end{array}\right]
\left[\begin{array}{c}
y_1 \\ \vdots \\ y_n \\ \lambda_0 \\ y_{n+1} \\ \vdots \\ y_{n+k} \\ \vdots \\ y_{n+m} \\ \lambda_1 \\ \vdots \\ \lambda_k \\ \vdots \\ \lambda_m
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ E \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ G_1 \\ \vdots \\ G_k \\ \vdots \\ G_m
\end{array}\right]
\qquad (3.7)
$$

μ into column position $n+1$. The result appears as matrix Equation (3.7). The resulting matrix is then partitioned according to the dashed lines through the matrix system.

For convenience in manipulation let the matrix of Equation (3.7) be abreviated as

$$A = \left[\begin{array}{c|c} A11 & A12 \\ \hline A21 & A22 \end{array}\right] \tag{3.8}$$

then

$$A^{-1} = B = \left[\begin{array}{c|c} B11 & B12 \\ \hline B21 & B22 \end{array}\right] \tag{3.9}$$

where

$$B11 = [A11 - A12\,A22^{-1}A21]^{-1} \tag{3.10}$$

$$B12 = -B11A12A22^{-1} \tag{3.11}$$

and

$$B22 = A22^{-1} - A22^{-1}B12 \tag{3.12}$$

Referring to Equation (3.7) note that A22 is of the form

$$A22 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{3.13}$$

and

$$A22^{-1} = A22 \tag{3.14}$$

Further note that A12 is of the form

$$A12 = [\,0 : a\,] \tag{3.15}$$

and likewise because of symetry

$$A21 \;=\; A12' \;=\; \begin{bmatrix} 0 \\ \text{---} \\ a' \end{bmatrix} \tag{3.16}$$

Substituting the facts of Equations (3.13), (3.14), (3.15) and (3.16) into Equations (3.10), (3.11) and (3.12) results in:

$$B11 \;=\; A11^{-1} \tag{3.17}$$

$$B12 \;=\; -A11^{-1}[a:0] \;=\; [b:0] \tag{3.18}$$

$$B21 \;=\; B12' \;=\; \begin{bmatrix} b' \\ \text{---} \\ 0 \end{bmatrix} \tag{3.19}$$

$$B22 = \begin{bmatrix} a'A11^{-1}a & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} -a'b & I \\ I & 0 \end{bmatrix} \tag{3.20}$$

and finally

$$A^{-1} = \begin{bmatrix} A11 & 0 & a \\ 0 & 0 & I \\ a' & I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A11^{-1} & b & 0 \\ b' & -a'b & I \\ 0 & I & 0 \end{bmatrix} = B \tag{3.21}$$

Since only $A11^{-1}$ must be found, the matrix to be inverted has been reduced from order $n+2m+1$ to order $n+1$ and is now of manageable size. The full form of the inverted system of Equation (3.7) is expressed as Equation (3.22). Note the strategic location of zero elements in the resultant vector G. Since the inverted matrix in Equation (3.22) is to be postmultiplied by the vector G, every column

$$
\left[\begin{array}{cccc|cccccc}
V_{11} & \cdots & V_{m} & z_{01} & b_{11} & \cdots & b_{1m} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
V_{m} & \cdots & V_{nn} & z_{0n} & b_{m} & \cdots & b_{nm} & 0 & \cdots & 0 \\
z_{01} & & z_{0n} & w_{00} & b_{n+1,1} & \cdots & b_{n+1,m} & 0 & \cdots & 0 \\
\hline
b_{11} & \cdots & b_{nl} & b_{n+1,1} & h_{11} & \cdots & h_{1m} & 1 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
b_{1m} & \cdots & b_{nl} & b_{n+1,m} & h_{1m} & \cdots & h_{mm} & 0 & \cdots & 1 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0
\end{array}\right]
\left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ E \\ 0 \\ \vdots \\ 0 \\ G_{1} \\ \vdots \\ G_{m}
\end{array}\right]
=
\left[\begin{array}{c}
y_{1} \\ \vdots \\ y_{n} \\ \lambda_{0} \\ y_{n+1} \\ \vdots \\ y_{n+m} \\ \lambda_{1} \\ \vdots \\ \lambda_{m}
\end{array}\right]
\qquad (3.22)
$$

$$
\begin{bmatrix}
z_{01} & 0 \cdots 0 \\
\vdots & \vdots \quad \vdots \\
z_{0n} & 0 \cdots 0 \\
w_{00} & 0 \cdots 0 \\
b_{n+1,1} & 1 \cdots 0 \\
\vdots & \vdots \quad \vdots \\
b_{n+1,m} & 0 \cdots 1 \\
0 & 0 \cdots 0 \\
\vdots & \vdots \quad \vdots \\
0 & 0 \cdots 0
\end{bmatrix}
\begin{bmatrix}
E \\
G_1 \\
\vdots \\
G_m
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\
\vdots \\
y_n \\
\lambda_0 \\
y_{n+1} \\
\vdots \\
y_{n+m} \\
\lambda_1 \\
\\
\lambda_m
\end{bmatrix}
\qquad (3.23)
$$

corresponding to a zero element in G can be ignored, thus further simplifying the calculations required. The inverted system with the non-relevent elements removed is displayed in matrix Equation (3. 23). Carrying out the indicated multiplications of Equation (3. 23) yields the linear functions in E for each of the activities and the Lagrangian multipliers of Equation (3. 24).

$$
\begin{aligned}
y_i &= z_{01}E & i=1,\cdots,n \\
-\lambda_0 &= -w_{00}E & \\
y_{n+j} &= b_{n+j}E + G_j & j=1,\cdots,m \\
\lambda_j &= 0 & j=1,\cdots,m
\end{aligned}
\qquad (3.\ 24)
$$

If the first n elements in column n+1 of matrix B11 are positive i. e. $z_{0i} > 0$ for $i = 1,\cdots,n$, then all of the real activity levels will be positive for positive values of E.[29]

Limits on Expected Income

The linear Equations (3. 24) are presented in the graph of Figure 3. 1. The line segments oc and od are representative activity equations and line segments ef and gh represent the levels of slack activities. To insure that the Kuhn-Tucker conditions are not violated one must establish the range over which E is valid. If E exceeds

[29]The first n elements will be positive if there is zero correlation between the incomes of the activities. This will be discussed more fully in a later section.

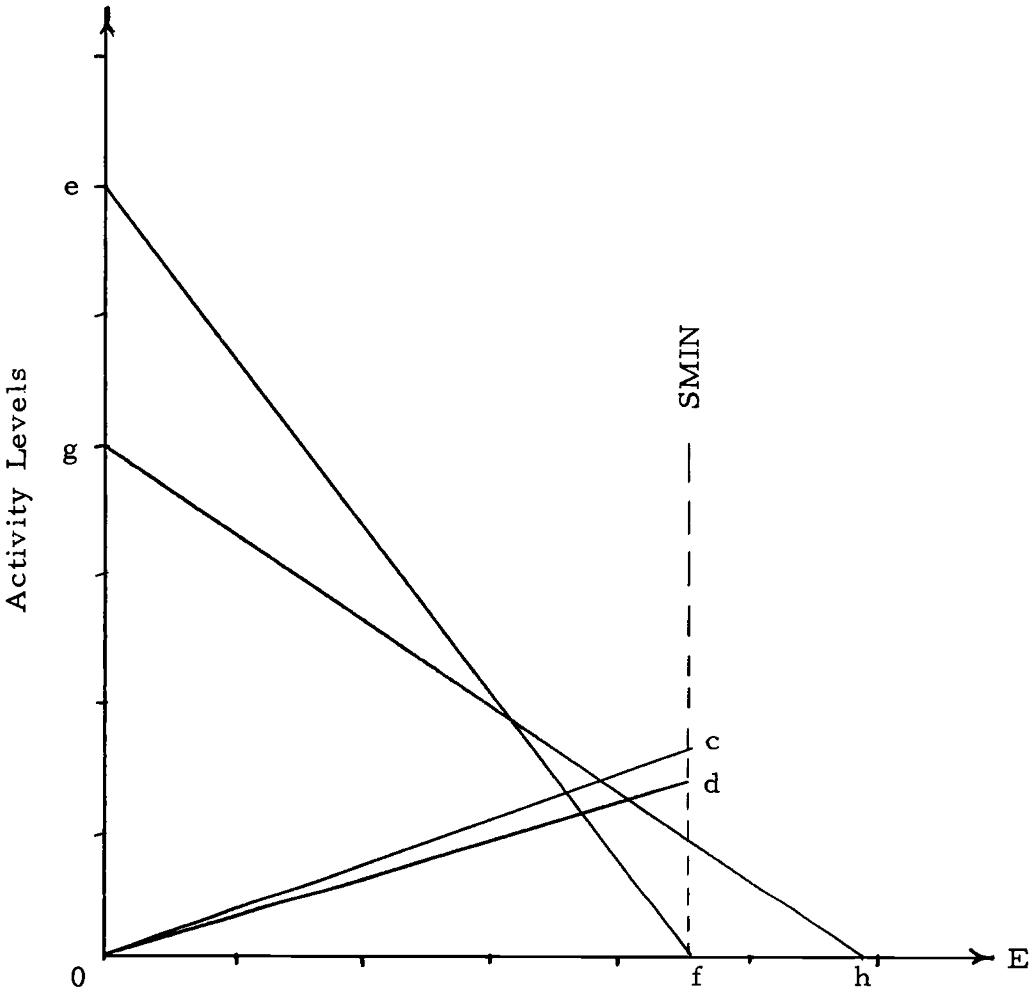

Figure 3.1. The valid range of expected income.

the magnitude of, the slack activity represented by the line ef will be forced negative. This establishes the upper limit on E, denoted SMIN, as being the minimum of the maximum values E can take on. The lower limit on E, denoted SMAX, is established as the maximum of the minimum values E can take on. As E is increased along the expansion path to the point E = SMIN, the level of a real activity increases to the point where a particular resource becomes exhausted. The corresponding slack activity then takes on a level of zero. To proceed into the next basis the level of the slack activity must be maintained at zero to assure complete use of the limiting resource.

Change of Basis

To initiate the next basis let the limiting resource be denoted as the kth resource. The slack activity y_{n+k} representing the kth resource is set at zero. The revised problem is expressed in the Lagrangian form and differentiated to form the matrix of the next basis shown in Equation (3. 25). This matrix differs from Equation (3. 6) only in that the (n+k)th row and the (n+k)th column are removed.

To facilitate solution of the system the vectors μ' and a_k' are moved from position n+m+1 and n+m+k+1 to position n+1 and n+2 respectively. This is done also for vectors μ and a_k to result in matrix Equation (3. 26). The dashed lines show where the partitioning is done for ease of inversion. The sub matrix

$$
\begin{array}{c}
\begin{array}{ccccccccccccc}
1 & & n & n+1 & & n+m & n+m+1 & n+m+2 & & n+m+1+k & & n+2m+1
\end{array}
\\
\left[\begin{array}{ccccccccccccc}
2\sigma_1^2 & \cdots & 2\sigma_{1n} & 0 & \cdots & 0 & \mu & a_{11} & \cdots & a_{1k} & \cdots & a_{1m} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
2\sigma_{1n} & \cdots & 2\sigma_n^2 & 0 & \cdots & 0 & \mu_n & a_{n1} & \cdots & a_{nk} & \cdots & a_{nm} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\
\mu_1 & \cdots & \mu_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
a_{11} & \cdots & a_{n1} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
a_{1k} & \cdots & a_{nk} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
a_{1m} & \cdots & a_{nm} & 0, & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right]
\left[\begin{array}{c}
y_1 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{n+m} \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_k \\ \vdots \\ \lambda_m
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ E \\ G_1 \\ \vdots \\ G_k \\ \vdots \\ G_m
\end{array}\right]
\end{array}
$$

(3. 25)

$$
\begin{array}{c}
\begin{array}{ccccccccccccccccc}
1 & & n & n+1 & n+2 & n+3 & & n+2+k-1 & n+2+k & & n+(m-1)+1 & n+(m-1)+2 & & \begin{array}{c} n+(m-1) \\ +(k-1) \end{array} & n+(m-1)+k & & n+2m
\end{array} \\
\left[\begin{array}{ccccc:cccccccccccc}
2\sigma_1^2 & \cdots & 2\sigma_{1n} & \mu_1 & a_{1k} & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{11} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1m} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
2\sigma_{1n} & \cdots & 2\sigma_n^2 & \mu_n & a_{nk} & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{n1} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{n,m} \\
\mu_1 & \cdots & \mu_n & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{1k} & \cdots & a_{nk} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hdashline
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
a_{11} & \cdots & a_{n1} & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_{1,k-1} & \cdots & a_{n,k-1} & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{1,k+1} & \cdots & a_{n,k+1} & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_{1m} & \cdots & a_{nm} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c}
y_1 \\ \vdots \\ y_n \\ \lambda_0 \\ \lambda_k \\ y_{n+1} \\ \vdots \\ y_{n+k-1} \\ y_{n+k+1} \\ \vdots \\ y_{n+m} \\ \lambda_1 \\ \vdots \\ \lambda_{k-1} \\ \lambda_{k+1} \\ \vdots \\ \lambda_m
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ E \\ G_k \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ G_1 \\ \vdots \\ G_{k-1} \\ G_{k+1} \\ \vdots \\ G_m
\end{array}\right]
\tag{3.26}
$$

A11 is now of order n+2 as opposed to n+1 in Equation (3. 7). It is the variance-covariance matrix multiplied by two and bordered by the vectors μ and a_k. The same procedure of inversion again is followed and those columns which are to be multiplied by zeros in the vector G can be ignored. The relevant part of the inverted system is displayed in Equation (3. 27). The activity equations and the equations for the Lagrangian multipliers which result from performing the indicated multiplication found in Equation (3. 28).

Again the limits of E, SMIN and SMAX, are found by examining each equation in the set (3. 28). The lower limit of E is the upper limit on E from the previous basis. Smaller values of E than the lower limit are not permissible since this would cause the Lagrangian multiplier attached to the kth resource to become negative, violating the Kuhn-Tucker conditions. The upper limit of E represents the point where another constraint becomes limiting. To proceed, the slack associated with the limiting resource must be set to zero and a new basis formed.

After several resource constraints have become limiting it becomes considerably more likely that the upper limit of E may be determined by a Lagrangian multiplier being forced to zero. This means that a resource constraint is no longer binding and the slack variable associated with it must be reintroduced into basis. This requires that the row and column in the sub-matrix A11 which contain

$$
\begin{array}{cccccc}
n+1 & n+2 & n+(m-1)+2 & \begin{array}{c} n+(m-1) \\ +(k-1) \end{array} & n+(m-1)+k & n+2m
\end{array}
$$

$$
\begin{bmatrix}
z_{01} & z_{k1} & 0 \cdots & 0 & 0 \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{0n} & z_{kn} & 0 \cdots & 0 & 0 \cdots & 0 \\
w_{00} & w_{0k} & 0 & 0 & 0 & 0 \\
w_{0k} & w_{kk} & 0 & 0 & 0 & 0 \\
b_{n+1,1} & b_{n+2,1} & 1 \cdots & 0 & 0 \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{n+1,k-1} & b_{n+2,k-1} & 0 \cdots & 1 & 0 \cdots & 0 \\
b_{n+1,k+1} & b_{n+2,k+1} & 0 \cdots & 0 & 1 \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{n+1,m} & b_{n+2,m} & 0 \cdots & 0 & 0 \cdots & 1 \\
0 & 0 & 0 \cdots & 0 & 0 \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \cdots & 0 & 0 \cdots & 0
\end{bmatrix}
\begin{bmatrix}
E \\ G_k \\ G_1 \\ \vdots \\ G_{k-1} \\ G_{k+1} \\ \vdots \\ G_m
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ \vdots \\ y_n \\ \lambda_0 \\ \lambda_k \\ \vdots \\ y_{n+k-1} \\ y_{n+k+1} \\ \vdots \\ y_{n+m} \\ \lambda_1 \\ \vdots \\ \lambda_m
\end{bmatrix}
\qquad (3.27)
$$

$$
\begin{aligned}
y_i &= z_{0i}E + z_{ki}G_k && i = 1, n \\
\lambda_0 &= w_{00}E + w_{0k}G_k \\
\lambda_k &= w_{0k}E + w_{kk}G_k && (3.28) \\
y_{n+j} &= b_{n+1,j}E + b_{n+2,j}G_k + G_j && j = 1, \cdots, k-1, k+1, \cdots, m \\
\lambda_j &= 0 && j = 1, \cdots, k-1, k+1, \cdots, m
\end{aligned}
$$

the coefficients of the limiting resource must be restored to their original places in submatrices A12 and A21. Once this is done the system can be solved.

Identifying the Maximum Attainable Expected Income

The procedure continues until there is one less limitating constraint than there are real activities. Having more effective constraints than this number causes the sub-matrix A11 to be singular. Unfortunately this does not mean that the maximum attainable E has been reached. It may be possible to increase E by trading a presently limiting constraint for the one whose slack activity was forced to zero by E = SMIN in the basis. The entering constraint is identified as the one whose slack has gone to zero but there is no direct method to determine the constraint to be removed. Since there is a relatively small number of effective constraints it is possible by trial and error to find the one, if it exists, which allows E to increase. If there are no constraints that can be released then there is no feasible way that a larger value of E can be attained. At that point the maximum attainable E is reached and the problem is solved.

Complications in Solution of the Model

The Initial Basis

The Zero Correlation Case

In the case of zero correlation between the income of real activities, all real activities will be in the initial basis. The necessary condition for this is that the first n elements of the (n+1)th column of the sub-matrix B11 be positive. That this condition will always be fulfilled when $r_{ij} = 0$ for all $i \neq j$ can be verified by observing that

$$B11_{k,n+1} = (-1)^{2(n+k)+1}\, 2^{n-1}\, \frac{\mu_k}{D} \prod_{i=1}^{n} \sigma_i^2/\sigma_k^2 > 0, \quad k=1, \cdots, n$$

D is the determinant of A11

where

$$D = \sum_{k=1}^{n} [(-1)^{2(n+k)+1}\, 2^{n-1}\, \mu_k^2 \prod_{i=1}^{n} \sigma_i^2/\sigma_k^2] < 0$$

since

$$(-1)^{2(n+k)+1} = -1 \text{ and } \mu_k > 0$$

The Non-Zero Correlation Case

In the more usual case where the correlation coefficients are not all zero the conditions for including all of the activities in the initial basis are not necessarily fulfilled. With the two activity case a negatively sloped expansion path results when the coefficient r is sufficiently large. In the two activity model of Chapter II it was easy to identify the offending real activity as being the most risky one and the problem easily remedied by setting the slack variable representing the lower limit constraint of the real activity to zero. In the more general case, the identification of offending activities is not as straight forward. In the present algorithm, a trial and error procedure is employed to find the initial basis when activities are correlated. The procedure is to set all real activities except the least risky one equal to their lower limits. Since the problem is to minimize risk it seems reasonable that the least risky activity is a most likely candidate for the initial basis. The matrix A11 is inverted and the relevant range for E is determined. If SMIN exceeds SMAX then the initial basis is found and contains only the least risky real activity. It is more likely that the initial basis will include more than one real activity especially if there are several real activities to be considered. If SMAX exceeds SMIN the Kuhn-Tucker conditions are violated because a Lagrangian multiplier attached to the lower limit constraint of a real activity is forced negative. This requires that the slack activity attached to the lower limit constraint must be introduced into the system thereby allowing the real activity to exceed its lower bound. Once this is done the resulting matrix A11 is inverted again and the quantities SMIN and SMAX computed. If SMAX still exceeds SMIN, the source of the conflict must be located and the proper modifications made. It may be a Lagrangian multiplier that is forced negative or it may be a slack activity that was introduced at a positive level that causes the conflict. In the former case, the particular constraint must be made non-effective by introducing the slack activity while in the latter, the particular constraint must be made effective by removing the slack activity. As soon as a situation is encountered where SMIN exceeds SMAX, a starting basis is established and the solution may proceed.[30]

Positive Lower Limits on Real Activities

If there are positive lower limits on some real activities, it is not necessarily true that SMAX computed from the initial basis is the minimum attainable expected income. This can be demonstrated by

[30]This trial and error method has worked satisfactorily during the testing procedure of the algorithm. However, there is a danger of cycling such that the initial basis will not be found. Should such an event occur one could set the level of E at some level greater than the absolute minimum satisfying the production constraints and solve using a standard quadratic programming technique such as the Frank and Wolfe simplex method.

imposing lower limit constraints on Figure 2.16 as in Figure 3.2. The initial basis is not changed from what it is was in the numerical example, however, the valid expansion path in this initial basis is he' rather than oe'. Expected income could be reduced by moving from h to o' along the lower limit constraint of y_1. The entire efficiency frontier in the positive lower limit case is diagrammed as the segment o'he'd in Figure 3.3. To establish the minimum attainable E the same procedure of trading constraints as was done in checking to see if the maximum attainable E had been reached would have to be applied, only in reverse order. Since it is of minor practical relevance to locate the absolute minimum point on the efficiency frontier such procedures will not be pursued further.

The Efficiency Frontier and Activity Equations

Once the various inverses have been computed, the variance function can be expressed in terms of expected income and resource levels by making use of the Lagrangian multiplier equations. If one partitions the sub-matrix B11 further into four sub-matrices and denotes the sub-matrix of order k+1, where k is the number of effective constraints, in the southwest corner as W, then W contains all of the information about the Lagrangian multipliers. The equations representing the Lagrangian multiplier is expressed in matrix form as Equation (3.39). The exact differential dV is expressed as

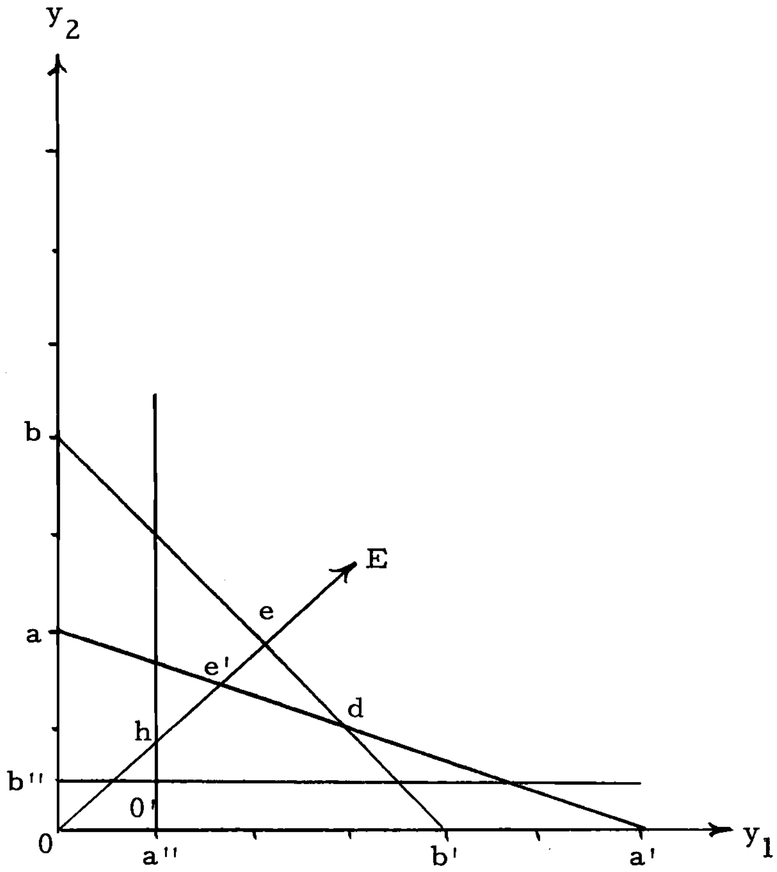

Figure 3. 2. Quadratic model with positive lower limit constraints.

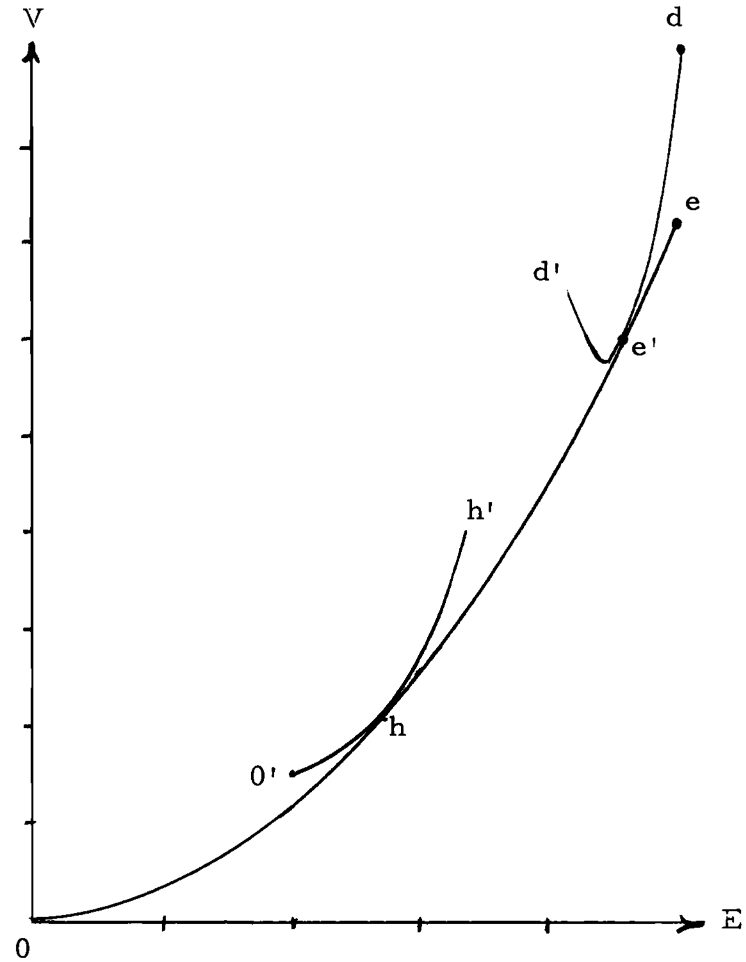

Figure 3. 3. Efficiency frontier with positive lower limit constraints.

$$\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0k} \\ w_{01} & w_{11} & \cdots & w_{1k} \\ \vdots & \vdots & & \vdots \\ w_{0k} & w_{1k} & \cdots & w_{kk} \end{bmatrix} \begin{bmatrix} E \\ G_1^* \\ \vdots \\ G_k^* \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1^* \\ \vdots \\ \lambda_k^* \end{bmatrix} \tag{3.39}$$

$$[dE\ dG_1^* \cdots dG_k^*] \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} w_{oo} & w_{01} & \cdots & w_{0k} \\ w_{01} & w_{11} & \cdots & w_{1k} \\ \vdots & \vdots & & \vdots \\ w_{0k} & w_{1k} & \cdots & w_{kk} \end{bmatrix} \begin{bmatrix} E \\ G_1^* \\ \vdots \\ G_k^* \end{bmatrix} = dV \tag{3.40}$$

$$[E\ G_1^* \cdots G_k^*] \begin{bmatrix} -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0k} \\ w_{01} & w_{11} & \cdots & w_{1k} \\ \vdots & \vdots & & \vdots \\ w_{0k} & w_{1k} & \cdots & w_{kk} \end{bmatrix} \begin{bmatrix} E \\ G_1^* \\ \vdots \\ G_k^* \end{bmatrix} = V \tag{3.41}$$

---

[31] The notation G_1^* means that G_1^* is an effective production constraint. They are listed in the same order as constraint vector G except that the non-limiting ones are removed. λ_1^* is the Lagrangian multiplier attached to G_1^*.

Equation (3.40) and its solution is given by Equation (3.41).[32] Equation (3.42) for the efficiency frontier in the variance-expected income plane is determined by substituting the actual numerical values for the production constraint levels into Equation (3.41) and observing the proper limits on E:

$$V = \alpha_1 E^2 + \alpha_2 E + \alpha_3 \tag{3.42}$$

where α_1, α_2 and α_3 are constants similar to $\frac{a}{2}$, bG_k and $\frac{cG_k}{2}$ in Equation (2.50).

The complete frontier is described by a series of parabolas all having the general form of Equation (3.42). The parabolas from later bases will be nested in the parabolas of earlier bases or intersect with them depending upon whether the later basis was constructed by addition or deletion of a constraint or whether it was formed by trading one constraint for another.

The level of the ith real activity is expressed as

$$y_i = \beta_{1i} E + \beta_{2i} \quad i = 1, \cdots, n \tag{3.43}$$

where β_{2i} is a constant resulting from holding all constraint levels fixed

β_{1i} is the slope of the activity equation.

The magnitude of β_{1i} indicates the stability of the solution at a particular point in E. For instance, if β_{1i} differs greatly from zero,

[32] A more formal interpretation of Lagrangian Multipliers and the solution of the differential equation is given in the appendix.

then small changes in E bring about large changes in y_i. As the solution nears the maximum attainable E, high paying, high risk activities begin to dominate the solution precipitating major changes in the efficient plans.

Slack activity levels are represented by

$$y_{n+j} = \beta_{1,n+j}E + \beta_{2,n+j} \qquad j = 1, \cdots, m \tag{3.44}$$

in the case where the jth resource is not an effective constraint. In the case where the jth resource is an effective constraint the Lagrangian multiplier equation is

$$\lambda_j = \beta_{1,n+j}E + \beta_{2,n+j} \qquad j = 1, \cdots, m \tag{3.45}$$

Since slack activities represent unused resources, and because of drastic changes in the composition of the plans as the maximum attainable E is approached there may be major changes in the resource use pattern.

A Summary of the Algorithm

At this point it appears useful to summarize, briefly, the steps involved in solving the variance minimization problem. These steps correspond to the computer program which was developed as part of this research project. The computer program appears in the appendix.

STEP I:

(a) Set up the matrices of the problem as in Equation (3. 4).

(b) Move the vector of means into the position n+1 in the matrix A11. See Equation (3. 7).

(c) Identify the real activity having the lowest coefficient of variation.

(d) Make all of the lower limit production constraints on the real activities limiting[33] except the one identified in Step I(c).

(e) Solve the system.[34]

(f) Compare SMIN and SMAX. If SMIN is greater than SMAX go to Step II. If SMIN is less than or equal to SMAX go to Step I(g).

(g) There is a conflict among the activities. If the conflict is due to a Lagrangian multiplier being forced to zero go to I(i).

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[33]Making the constraint limiting: Suppose the kth production constraint has become exhausted as indicated by a slack variable being forced to zero, then the following row and column operations must be performed to make it a limiting constraint.

(a) strike out the row and column representing the slack activity and its coefficient.

(b) Move the row vector and the column vector containing the production coefficients of the limiting resource from its original position in A21 and A12 to its proper position in A11, as specified in the discussion immediately following Equation (3. 25).

[34]Solving the system: This refers to finding the inverse matrix B which is postmultiplied by the vector G to find the parameters of the activity and the Lagrangian multiplier equations and the limits SMIN and SMAX.

(h) Make the indicated constraint a limiting constraint. Go to Step I(e).

(i) Make the indicated constraint non-limiting.[35] Go to Step I(e).

STEP II:

(a) Record the number of the basis and the parameters of the activity equations, Lagrangian multiplier equations and the variance equation.

(b) Identify the constraint of concern at SMIN. If a slack variable has been forced to zero, go to Step II(c). If a Lagrangian multiplier has been forced to zero, make the constraint non-limiting and go to Step I(e).

(c) If there are n-2 or fewer limiting constraints make the constraint identified in Step II(b) limiting and go to Step I(e). If there are already n-1 limiting constraints in the basis go to Step III.

STEP III:

(a) Make the constraint identified in Step II(b) limiting.

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[35] Making a constraint non-limiting: Suppose the kth resource is no longer limiting as indicated by a previously positive Lagrangian multiplier being forced to zero. Then the following row and column operations must be performed to make it a non-limiting constraint.

(a) Move the row vector and the column vector containing the production coefficients of the resource from the position in A11 to its original position in A12 and A21.

(b) Replace the row and column representing the slack variable and its coefficient.

(b) Make the constraint having the smallest Lagrangian multiplier as evaluated at SMIN in Step II(b) non-limiting.

(c) Solve the system.

(d) Compare SMIN and SMAX. If SMIN is greater than SMAX go to Step II. If SMIN is less than or equal to SMAX go to Step III(e).

(e) If all of the n-1 limiting constraints in the basis of Step II(c) have been made non-limiting one by one, and there has been no increase in E go to Step III(g). If there are still some constraints which have not been tried, go to Step III(f).

(f) Retain the constraint made limiting just prior to Step III(c). Make the constraint, having the next largest Lagrangian multiplier to the one just attempted, non-limiting. Go to Step III(c).

(g) The absolute maximum E has been reached and the problem is solved.

Parameter Estimation

Error in decisions[36] can result from two sources. No matter how accurate the information about a particular situation, erroneous conclusions can result from faulty reasoning. It has been the purpose of Chapter II and the first part of Chapter III to develop a methodological framework such that this type of error is minimized. However, no matter how accurate, precise or elegant the reasoning framework may be, a second source of error can result from misinformation or faulty data. It is this second source of error upon which the remainder of the chapter is focused.

The confidence that can be placed ultimately on the efficient plans depends in no small way upon the reliability of the estimates of the parameters. Thus it becomes necessary to examine ways by which these numerical values can be found so that they communicate the impressions of the decision-maker about the future prices and yields in an accurate and simple manner.

Resource requirements and resource limits continue to be considered non-stochastic. These are the elements of the matrix a and

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[36] Error in this context refers to whether the choice was consistent with the goals and aspirations of the decision maker not whether the desired result was obtained. Suppose an individual having certain fixed debt commitments chooses a plan where the probability of bankruptcy is but 1%. Yet a catastrophe strikes and he loses his farm. This is not an error in decision making but rather the consequence of the random disturbance that has caused his failure.

the vector G of Equation (3.1). Since these elements are identical to those encountered in linear programming, the problems pertaining to their estimation, are not discussed here. The means, variances and covariances of real activities do present new problems and merit the attention of this thesis. The quest for the elements of the matrix X and vector μ begins with a definition of gross margin.

Gross Margin - Definitions and Assumptions

Gross margin is defined as gross income less variable costs, where gross income refers to the physical yield multiplied by the market price. Variable costs, assumed non-stochastic, are direct production costs and do not include overhead or fixed costs. Gross margin used here is synonymous with the term "net price" used by Heady and Candler (20, p. 112). The contribution of the ith activity or enterprise to the total gross margin of the farm is expressed as:

$$Y_i = y_i q_i p_i - y_i c_i = y_i(q_i p_i - c_i) \tag{3.46}$$

where Y_i is the gross margin contributed by the ith activity,

y_i is the level of the ith activity,

q_i is the per unit yield of the ith activity,

p_i is the price per unit of yield of the ith activity,

and c_i is the variable cost per unit of the ith activity.

The quantities q_i and p_i are random variables and shall be assumed stochastically independent. Such an assumption is not inconsistent with that made in perfect competition where the actions of an individual do not affect the market in the aggregate.[37] Letting:

$$Z_i = q_i p_i \tag{3.47}$$

where Z_i is gross income

and applying the appropriate statistical theorems (24, p. 148) it follows that:

$$E(Z_i) = E(q_i p_i) = E(q_i)E(p_i) \tag{3.48}$$

where $E(Z_i)$ is expected gross income per unit of activity,

$E(q_i)$ is expected yield per unit of activity,

and $E(p_i)$ is expected price per unit of yield.

Furthermore:

$$V(Z_i) = V(q_i)V(p_i) + V(q_i)[E(p_i)]^2 + V(p_i)[E(q_i)]^2 \tag{3.49}$$

where $V(Z_i)$ is variance of gross income per unit of activity,

$V(q_i)$ is variance of the yield per unit of activity,

$V(p_i)$ is variance of the price per unit of yield.

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[37] It is recognized that this may lead to some difficulty in the case where yield is highly dependent upon some variables such as weather and the total supply of the commodity in question comes from a small geographic area. Such a case might indicate a high correlation between an individuals yield and the price he receives. This, however, is thought to be the exception rather than the rule.

Letting:

$$X_i = Z_i - c_i \tag{3.50}$$

then X_i is a random variable representing the gross margin contributed by one unit of the ith enterprise. From this relationship one can define:

$$\mu_i = E(X_i) = E(q_i)E(p_i) - E(c_i) \tag{3.51}$$

and

$$\sigma_i^2 = V(X_i) = V(Z_i) \tag{3.52}$$

where μ_i is expected gross margin contributed by the ith activity and σ_i^2 its variance.

Extending these relationships to include the entire farming operation results in equations

$$E = E(Y) = \sum_{i=1}^{n} \mu_i y_i \tag{3.53}$$

and

$$V = V(Y) = \sum_{i=1}^{n} y_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j<1}^{n} y_i y_j \sigma_i \sigma_{ij} r_{ij} \tag{3.54}$$

All of the results obtained thus far in this section are completely general and do not depend upon the parent distribution of prices, yields or gross margin.

Estimated Expected Value and Variance of Gross Margin

One possible source for estimating gross margin parameters is aggregated time series data of prices and yields. While such series have their place in predicting response in the aggregate, they give a downward bias to variance estimates for farm planning studies because the aggregation process "averages out" variability, (12). Also they carry with them the implied assumption that history will repeat itself. For estimates to be relevant, the data source should be closer to the individual farm situation. Another possible source is historical data recorded by the farmer himself. Unfortunately farmers do not as a rule keep such detailed listings of yields and prices and they may wish to consider engaging in new enterprises about which they could not possibly have recorded the information. They do, however, often have strong subjective notions about the profitability and riskiness of various enterprises. Since the prime purpose is to organize the decision maker's information so that the efficient enterprise combinations can be derived, it is necessary only to have him quantify his impressions about future prospects of each enterprise.

Engineers, (23, p. 229) under similar circumstances of forward planning in critical path analysis are concerned in completing a project in optimum time. To do so requires coordinated scheduling of interrelated sub-activities. Decision makers are asked to provide three estimates of the completion time of the sub-activities: (a) the most optimistic; (b) the most likely; and (c) the most pessimistic completion time. These estimates specify a "beta" distribution of the completion time. MacCrimmon and Ryavec (33) in their review of the assumptions underlying critical path analysis suggest that the triangular distribution results in about the same degree of error[38] as does the beta distribution but has a much simpler mathematical form. It is not necessary, as required of the beta distribution, to solve for the roots of a cubic equation to obtain the parameters.

The probability density function of the triangular distribtuion is:

$$\begin{aligned} f(x) &= \frac{2(x-a)}{(m-a)(b-a)}, \quad a \leq x < m \\ &= \frac{2(b-x)}{(b-m)(b-a)}, \quad m \leq x \leq b \qquad (3.55) \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

where x is the random variable

a and b are the end points

m is the most frequently occuring value.

The triangular density function is graphed in Figure 3.4. The triangular cumulative frequency distribution is:

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[38] There are two kinds of errors involved. First the random variable of concern may not be from either a beta or a triangular distribution. Secondly errors may result in estimating the parameters. It is the errors in estimation that are of concern here.

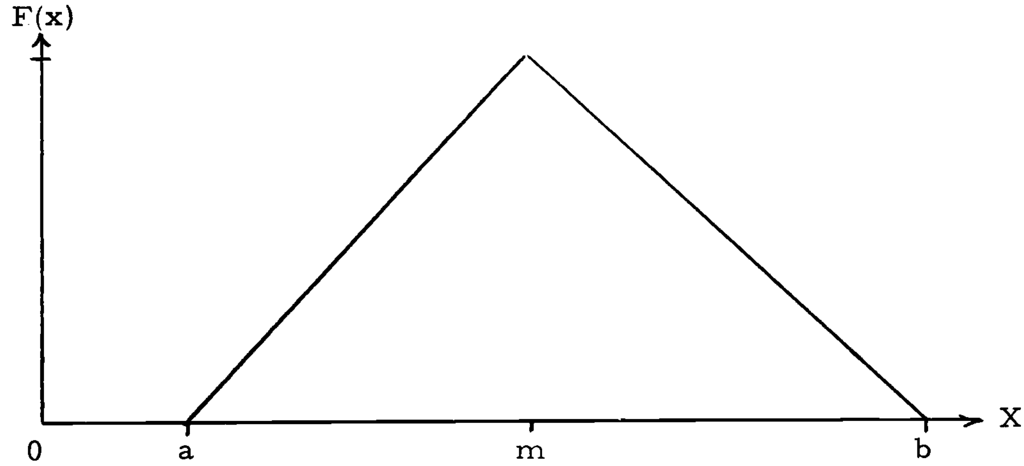

Figure 3.4. The triangular probability distribution function.

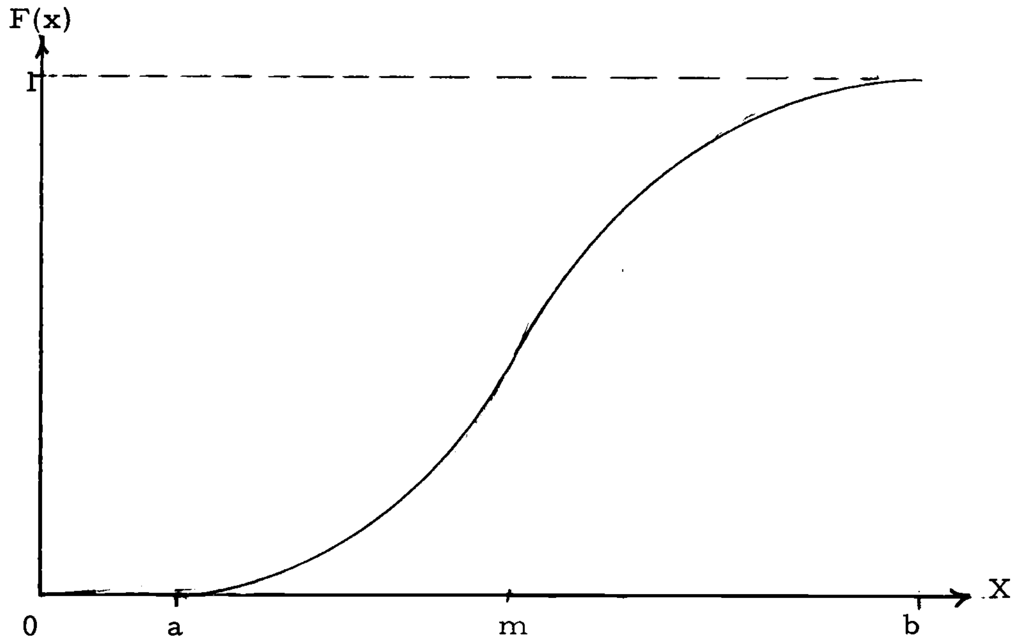

Figure 3.5. The triangular cumulative distribution function.

$$F(x) = \begin{cases} 0 & , \ x < a \\ \dfrac{(x-a)^2}{(m-a)(b-a)} & , \ a \leq x < m \\ 1 - \dfrac{(b-x)^2}{(b-m)(b-a)} & , \ m \leq x \leq b \\ 1 & , \ b < x \end{cases} \tag{3.56}$$

The cumulative distribution function is graphed in Figure 3.5.

The mean of the triangular distribution is:

$$\mu = \frac{1}{3}(a + m + b) \tag{3.57}$$

From the partial derivatives

$$\frac{\partial \mu}{\partial a}, \frac{\partial \mu}{\partial m}, \frac{\partial \mu}{\partial b} > 0 \tag{3.58}$$

it can be noted that increases in the estimates of a, m, or b cause increases in the mean.

The variance of the triangular distribution is

$$\sigma^2 = \frac{1}{18}[(b-a)^2 - (m-a)(b-m)] \tag{3.59}$$

From the partial derivatives

$$\frac{\partial \sigma^2}{\partial a} < 0, \ \frac{\partial \sigma^2}{\partial b} > 0$$

and

$$\frac{\partial \sigma^2}{\partial m} > 0 \quad \text{for} \quad m > \frac{a+b}{2} \tag{3.60}$$

$$\frac{\partial \sigma^2}{\partial m} < 0 \quad \text{for} \quad m < \frac{a+b}{2}$$

it can be noted that increases in a reduce variance, increases in b increase variance and increases in m will either increase or decrease variance depending whether m lies to the right or the left of the midpoint between a and b.

If a, m, and b are respectively the most pessimistic, most likely and most optimistic estimates for price or yield, then the triangular distribution quantifies, the decision makers impressions about profitability and risk of the enterprises he is considering. The decision maker could be asked to give the three estimates for gross income. However, it is felt that he will, in most cases, give clearer thought to the problem if asked for the price and yield components separately. Once the price and yield estimates are obtained, their corresponding means and variances come directly from Equations (3.57) and (3.59). After an estimate for variable costs has been made, the mean and variance of gross margin follow directly from Equations (3.51) and (3.52).

It is important that the questions concerning the three points of the distribution be asked in the proper time context. For instance, one would expect different answers, probably leading to a lower variance, if the planning horizon were for next year as opposed to a longer run of say 10 years. This approach will allow the decision maker to subjectively account for factors that exert an influence on the future behavior of gross margin.

The expected values of gross margin establish the elements of the vector μ in Equation (3. 1). The variance estimates of gross margin establish the elements on the main diagonal of the matrix X. The estimation of covariances, i. e. the off-diagonal elements of X poses a more difficult problem.

Estimating Covariances

Empirical evidence indicates substantial degrees of correlation between certain farm enterprises. To account for this interdependency, an estimate of the covariance must be made.

Ideally, one should construct a subjective joint probability density function involving gross margins of all enterprises to be considered. Through integration, the mean, variance and covariance would be derived. The covariance term is given by:

$$\sigma_{ij} = r_{ih}\sigma_i\sigma_j \qquad (3.61)$$

where σ_{ij} is the covariance between the ith and jth activities,

r_{ij} is the correlation coefficient between the ith and jth activities,

σ_i is the standard deviation of the ith activity,

σ_j is the standard deviation of the jth activity,

The expression in Equation (3.61) is general and does not depend upon any specific density function. Since the values σ_i and σ_j have been established by the triangular distribution one need be concerned only with estimation of the correlation coefficients.[39]

Farmers often think in terms of worst, best and most likely outcomes hence do not have difficulty in estimating the triangular distribution. However they find it virtually impossible to answer questions concerning enterprise interdependency.

If there are n enterprises, then n(n-1)/2 correlation coefficients must be specified. Not only must the correlation coefficients lie between negative and positive unity, they must also form a positive definite matrix. While this matrix could be established through an interview in the simple case of two or even three enterprises, the task becomes impossible for the decision maker as more activities are added.

An alternate method is suggested by Markowitz (34, p. 100). To

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[39] The estimates a, m, and b can be thought of as describing a marginal triangular distribution. If one assumes stochastic independence, then the joint distribution is the product of the marginal distributions. Such an assumption implies that the enterprises are uncorrelated.

find the covariance between two securities, s_i and s_j, the simple linear regression coefficient of each of the security on some common element such as an index of business activity is used resulting in:

$$\sigma_{ij} = b_i b_j V(I) \tag{3.62}$$

where σ_{ij} is the covariance between s_i and s_j,

I is the common element index,

b_i and b_j are the simple linear regression coefficients on the index I,

$V(I)$ is the variance of index I.

The diversity of farming enterprises makes it difficult to establish a common element index to be used in estimating the covariance σ_{ij}. For example, should weather be chosen as the common element one notes that the introduction of irrigation might make a crop uncorrelated with rainfall. It does not necessarily follow that the irrigated crop is then uncorrelated with dryland crops.

A third alternative is to use historical price and yield data. If an individual has such a series for the enterprises he wishes to consider, then it is advisable to use his data. In most cases individual data is non-existant and one is required to resort to aggregate time series. Added to the variance bias discussed earlier, there is the possibility of time trends in the data. These trends may be the result of technological advances, long run weather patterns, business cycles and other causes. The longer the series the more likely the presence of trends. Due to the short run nature of the problem, interest here is only in the random elements and it may be necessary to remove the influence of time. This can be done in a number of ways. One method is to determine the regression equation of time on the gross income of each activity by the least squares technique. The deviations of the observed gross incomes from those predicted by the regression equation can be computed. The resulting deviations are interpreted as the random disturbances and the correlation coefficients are computed according to the following formulation:

$$r_{ij} = \frac{\sum_{t=1}^{T} d_{it} d_{jt}}{\sqrt{\sum_{t=1}^{T} d_{it}^2 \sum_{t=1}^{T} d_{jt}^2}} \tag{3.63}$$

where r_{ij} is the coefficient of correlation between the ith and jth enterprise gross incomes,

d_{it} is the deviation of the ith enterprise in the tth year from the regression line of the gross income,

d_{jt} is the deviation of the jth enterprise in the tth year from the regression line of the gross income.

Computation of the correlation matrix is described in matrix notation as:

$$R = QD'DQ \tag{3.64}$$

where R is the NxN correlation matrix.

D is the TxN matrix of deviations and D' is its transpose,

Ω is an NxN diagonal matrix containing on its main diagonal the elements

$$\frac{1}{\sqrt{\sum_{t=1}^{T} d_{it}^{2}}} \qquad i = 1, \cdots, N$$

N is the number of enterprises considered,

T is the number of observations on each enterprise.

The matrix R, can be constructed for the region in which the decision maker resides. The correlation matrix for the decision maker, denoted by R^* is constructed by transfering the relevant rows and colums representing the enterprises of interest from the regional matrix R to the individual's matrix R^*. The matrix X is obtained by premultiplying and postmultiplying R^* by a diagonal matrix composed of the standard deviations of each enterprise.

For clarity the matrix in full is:

$$X = \begin{bmatrix} \sigma_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \sigma_n \end{bmatrix} \begin{bmatrix} 1 & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{1n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \sigma_n \end{bmatrix} \qquad (3.65)$$

where X is the variance-covariance matrix of Equation (3.1),

σ_i is the standard deviation of the ith activity, and n is the number of activities.

The matrix R which results from the product D'D is positive semi-definite (1, p. 141) and has rank not exeeding the minimum dimension of D. Since the $n x n$ matrix X is required to be positive definite there can be no more enterprises considered by an individual than there are observations in the time series used to construct the matrix D.

Usually there are more enterprises in a given region than there are years of data about their gross margins. Advances in technology bring about changes in farming practices from one time period to another thereby shortening the time period for which a complete set of data can be obtained. For example, bush beans were unheard of prior to the introduction of mechanical harvestors about 10 years ago. They are now steadily replacing the pole-type varieties which required hand labor. This will limit the entire set of observations to 10 years.

In summary, data requirements of the enterprise selection problem can be met by using the triangular distribution as a subjective measure of the mean and variance of prices and yields. The mean estimates establish the vector μ. The variance estimate are combined with the appropriate rows and column of a regional correlation matrix derived from time series data to construct the matrix X.

IV. EMPIRICAL EXAMPLES AND RESULTS

Algorithm Development--Accuracy and Efficiency Comparisons

The computational procedures discussed earlier were incorporated into a sequence of three computer programs.[40] The first program, called INPUT, prepares the data for use of the second program, called PROCESS, which solves the problem.[41] The third program, called OUTPUT, prepares the report in graphs and tables for use by the decision maker. Numerous hypothetical examples were used in the early development stage with the chief rôle played by a problem, (see Case 1, Table 4.1), adapted from the Oregon Farm Management Game (39). To verify the accuracy of results obtained by the algorithm under development, a comparison was made to a standard quadratic programming routine. The composition of plans obtained at the selected points on the efficiency frontier were identical for both methods. Later in the development a problem reported by Carter and Dean (7), (see Case 2 in Table 4.1), was used as a further check on accuracy and to obtain a comparison on efficiency. The solution values were identical.

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[40] The programs were written in Fortran IV and run on a Control Data Corporation 3300 computer under OS3, a time sharing Executive System at Oregon State University.

[41] The program PROCESS will accommodate up to 10 real activities and 99 production constraints.

Table 4.1. Problem dimensions and computer costs.

	Test Problems			Willamette Valley Farms			
	Case One	Case Two	Case Three	Case Four	Case Five	Case Six	Case Seven
No. of Activities	7	7	8	9	10	9	4
Total Constraints	23	10	15	48	46	43	37
land	15	7	11	20	22	19	9
labor	8	0	4	12	12	12	12
irrigation water	0	0	0	4	0	0	4
capital	0	0	0	12	12	12	12
Total Inversions	21	---	26	39	22	26	12
valid bases	12	12	11	19	9	16	7
inversions for initial basis	0	---	8	9	4	2	2
Total Computer time (seconds)	94.804	96.672	157.963	235.576	187.300	219.475	79.21
input	---	---	6.662	6.035	10.121	9.739	7.67
process	---	---	62.541	176.985	108.217	117.394	23.62
output	---	---	88.800	52.556	68.962	92.342	48.92
Total Computer Cost (dollars)	10.66	11.65	21.51	24.75	20.40	23.64	12.78
cost of report only	---	---	11.62	8.82	9.98	12.42	8.29

Carter and Dean obtained only a number of points on the frontier in just under three minutes of computing time. The algorithm developed here accomplished the task in about half the time (95 seconds). Furthermore, the exact algebraic equation of the entire frontier was obtained. Consequently if one wishes to do more extensive utility analysis requiring the entire frontier, it is not necessary to use some regression. tehcnique as an approximation (17, p. 200). A final check on accuracy was made against results obtained by How and Hazell (26), (see Case 3 in Table 4.1). The algorithm they used also specified only a finite number of points on the frontier and seemed to violate a number of the production constraints.

In each of the three cases tested, the results obtained by the algorithm under development were identical or superior to those obtained by the other methods. This made it possible to attempt the solution of real world management problems submitted by farmers in the Willamette Valley of Oregon.

Tests of Applicability - Four Case Studies

Problem Specifications and Data Collection

Four farm operators submitted crop enterprise selection problems for solution.

Case 4 was submitted by a Yamhill County, Oregon partnership interested in determining the advisability of renting additional land and deciding upon the optimal combination of crops should the renting prove advantageous.

Case 5 was submitted by a Polk County, Oregon farmer interested in the optimum combination of irrigated and dryland crops.

Case 6 was submitted by a Polk County, Oregon farmer interested in the optimum combination of dryland crops.

Case 7 was submitted by the Agricultural Representative of a bank on behalf of a Marion County, Oregon farmer having similar interest to those expressed in Cases 5 and 6.

Since these were crop farms, located in the Willamette Valley using similar cultural practices, the production coefficients were also similar. Cereal grains, grass seeds, legume seeds and more intensive crops like beans and strawberries were considered. The basic constraints were identical for all farms, and included four categories; land, labor, irrigation water and operating capital. The land constraint consisted of two classes; irrigated and dry land. In addition there was a maximum and a minimum acreage limit on each crop. Labor coefficients and constraints were specified by month. Irrigation water requirements and constraints were established for the critical season beginning with May and ending with September. Total annual operating costs per acre for each crop were obtained separately for machinery and equipment operation, fertilizer, spray and dust, seed, supplies and miscellaneous cash costs. These costs were then allocated to the month in which they normally occur to establish the operating capital requirements. The percentage of the revenue to be received in each month was recorded to establish a cumulative cash flow statement per acre for each crop. As an example of this procedure, suppose a particular crop required an expenditure of five dollars in January, \$15.00 in February, \$10.00 in March, \$25.00 in April and \$35.00 in October, and the produce was sold in November for \$150.00. The resulting cumulative cash flow statement for this example appears as Table 4.2. The cumulative cash flow concept is incorporated into the model by addition of a column vector in the matrix a. A maximum limit on cumulative operating capital permitted for the farm throughout the operating year was imposed.

While production constraints in either the quadratic or linear programming models are the same, there is a difference in formulating the objective function. The objective function in this quadratic programming was to minimize variance. Hence, one must also obtain variance and covariance estimates in addition to the normal linear programming requirements. Farmers frequently think in terms of an interval rather than a point estimate (47) when asked about prices and yields. If one interprets this interval to be the interval ab in the triangular distribution of Figure 3.4, and asks the additional question about the most likely yield or price, then estimates for mean and

Table 4. 2. Monthly cash flow statement.

Month	Monthly Cash Flow[42]	Cumulative Cash Flow[43]
January	5.00	5.00
February	15.00	20.00
March	10.00	30.00
April	25.00	55.00
May	0.00	55.00
June	0.00	55.00
July	0.00	55.00
August	0.00	55.00
September	0.00	55.00
October	35.00	90.00
November	-150.00	-60.00
December	0.00	-60.00

[42] Positive numbers indicate an outflow of cash while negative numbers indicate an inflow.

[43] Positive numbers indicate that there has been a cumulative net outflow while negative numbers indicate a cumulative net inflow.

variance are established. Since farmers usually go through such a thought process anyway, the only additional requirement is to record their pessimistic, optimistic and most likely estimates of price and yield. Thus data collection is no more difficult for the quadratic model than for linear programming.

Production coefficients, price and yield data and resource constraint levels were obtained for each of the farms using the forms appearing in the appendix. A regional correlation matrix for the Willamette Valley was prepared from a 10 year aggregate time series on 46 different crop enterprises.[44]

Report and Interpretation of Results

The program OUTPUT was designed to provide a report which could be interpreted by farm decision makers. The report for Case 4 follows. Although this represents a real farm, the names Smith and Jones are ficticious.

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[44] The data was obtained from the files of D. L. Rassmisson and H. G. Ottaway, County Extension Agents, Marion County, Salem, Oregon. The regional correlation matrix was computed with the program CORRELATE, a copy of which appears in the Appendix.

MR. SMITH AND JONES
SOMEWHERE ORE.

DEAR MR. SMITH AND JONES

THE FOLLOWING REPORT GIVES A DETAILED DESCRIPTION OF EFFICIENT PLANS FOR YOUR FARM BUSINESS. THE PLANS ARE ARRANGED IN ORDER OF INCREASING PROFITABILITY. PROFITABILITY IS MEASURED BY EXPECTED GROSS MARGIN. GROSS MARGIN IS THE DOLLAR VALUE OF PRODUCTION AFTER THE VARIABLE COSTS SUCH AS FUEL,FERTILIZER, REPAIRS, ETC. HAVE BEEN DEDUCTED. THE TERM ¬EXPECTED¬ VALUE IS USED TO INDICATE THAT WE ARE DEALING WITH THE ¬AVERAGE¬ YEAR. NOTHING IS SAID ABOUT THE GROSS MARGIN FOR A SPECIFIC YEAR. AS THE EXPECTED GROSS MARGIN OR PAYOFF OF A PLAN INCREASES, SO DOES ITS RISKINESS. HOWEVER, THE PLANS ARE SO CONSTRUCTED THAT AT A SPECIFIC LEVEL OF EXPECTED GROSS MARGIN, THE RISK IS AS SMALL AS IT CAN BE. THIS IS WHY THE PLANS ARE SAID TO BE EFFICIENT. AS EXPECTED GROSS MARGIN IS INCREASED, THE GENERAL NATURE OF THE PLAN MUST CHANGE. FOR EXAMPLE, THE LOWER PAYING, LESS RISKY CROPS BECOME REPLACED BY HIGHER PAYING, BUT MORE RISKY ONES. AS THE COMPOSITION OF THE PLAN CHANGES, SOME RESOURCES, FOR EXAMPLE LABOR OR OPERATING CAPITAL, MAY BECOME LIMITING. WHEN THIS HAPPENS A NEW PLAN IS MADE. NEW PLANS ARE CONSTRUCTED UNTIL THERE IS NO WAY IN WHICH EXPECTED GROSS MARGIN CAN BE INCREASED FURTHER. IN ORDER TO DETERMINE ALL EFFICIENT PLANS A STEP BY STEP PROCEDURE IS FOLLOWED. THE COMPOSITION OF THE PLAN AND ITS PAYOFF AND RISKINESS IS GIVEN AT THE END OF EACH STEP. WHILE EACH OF THE PLANS IS EFFICIENT IT REMAINS FOR YOU TO DECIDE WHICH ONE IS BEST. SELECT THAT PLAN WHICH FOR YOU HAS THE MOST ACCEPTABLE COMBINATION OF EXPECTED GROSS MARGIN AND RISK.

THE REPORT IS DIVIDED INTO A NUMBER OF PARTS.

PART ONE IS A SUMMARY OF ALL THE STEPS. THE PLAN AS IT EXISTS AT THE END OF EACH STEP IS GIVEN. THIS SUMMARY SHOWS THE EXPECTED GROSS MARGIN, THE RISKINESS AND THE NUMBER OF ACRES IN EACH CROP. YOU WILL ALSO FIND A STATEMENT SHOWING THE AMOUNT OF RESOURCES USED AND THE VALUE OF ONE MORE UNIT OF THE RESOURCE.

PART TWO IS DESIGNED TO HELP YOU CHOOSE YOUR BEST PLAN. REMEMBER THAT EXPECTED GROSS MARGIN IS A LONG_RUN_AVERAGE CONCEPT, AND SAYS NOTHING DIRECTLY ABOUT NEXT YEAR. IN THIS SECTION YOU ARE GIVEN THE PROBABILITIES OR CHANCES THAT NEXT YEAR, GROSS MARGIN WILL EXCEED A SPECIFIED AMOUNT. THIS STATEMENT IS MADE FOR EACH OF THE PLANS OF PART ONE.

PART THREE GIVES THE COMPLETE SPECIFICATION FROM WHICH ANY POSSIBLE PLAN CAN BE CALCULATED. THE PLANS, AND THEIR RESPECTIVE PAYOFFS AND RISKINESS ARE GIVEN AS EQUATIONS. TO DETERMINE ANY PLAN YOU NEED ONLY PLUG THE PROPER VALUES INTO THE EQUATIONS.

PART FOUR PRESENTS THE ENTIRE SET OF EFFICIENT PLANS IN GRAPHIC FORM. IT ALLOWS YOU TO KNOW, AT A GLANCE, THE CHARACTERISTICS OF EACH POSSIBLE EFFICIENT PLAN.

IT IS HOPED THAT THE FOLLOWING INFORMATION WILL BE OF VALUE TO YOU AS YOU PLAN YOUR FUTURE FARMING ACTIVITIES.

YOURS TRULY,

PART ONE

SUMMARY OF EFFICIENT FARM PLANS

A STATEMENT OF THE LEVELS OF ACTIVITIES AND THE EXPECTED PAYOFF

NAME OF CROP	UNIT	PLAN 1	PLAN 2	PLAN 3	PLAN 4	PLAN 5	PLAN 6	PLAN 7	PLAN 8	PLAN 9
WHEAT	AC	29.00	29.00	29.00	29.00	29.00	29.00	29.00	29.00	29.00
RED CLOVER	AC	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
ALFALFA IRG	AC	37.65	50.00	50.00	44.23	40.10	30.36	27.44	0	.00
ALFALFA DRY	AC	15.56	23.23	27.04	40.46	50.00	50.00	50.00	50.00	50.00
CORN SILAGE	AC	-0.00	.00	-0.00	5.77	9.90	19.64	22.56	50.00	50.00
[illegible]	AC	50.00	50.00	50.00	50.00	52.25	101.71	141.11	143.92	144.00
ORCH GRASS	AC	-0.00	-0.00	-0.00	-0.00	-0.00	.00	5.45	3.51	4.47
HAIRY VETCH	AC	3.17	7.90	8.86	11.69	13.44	9.56	4.44	3.57	2.52
PINTO BEANS	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00
EXP GR MARG	$$	18718.35	20553.81	20833.07	22367.28	23582.66	27318.83	30124.56	32856.55	32884.38
STD DEV	$$	1041.04	1130.69	1146.52	1238.88	1315.69	1592.03	1829.00	2110.08	2113.58

PART ONE CONTINUED

A STATEMENT OF THE AMOUNT OF EACH RESOURCE USED AND THE EXPECTED PAYOFF

NAME OF RESOURCE	UNIT	PLAN 1	PLAN 2	PLAN 3	PLAN 4	PLAN 5	PLAN 6	PLAN 7	PLAN 8	PLAN 9
JAN LAB	HR	0	0	0	2.02	3.46	6.88	13.34	21.01	21.97
FEB LAB	HR	0	0	0	2.02	3.46	6.88	7.90	17.50	17.50
MARCH LAB	HR	13.22	13.69	13.79	16.09	18.16	31.07	40.01	49.89	49.90
APRIL LAB	HR	20.00	20.00	20.00	20.00	20.90	40.69	56.44	57.57	57.60
MAY LAB	HR	188.22	208.23	212.04	226.04	236.22	242.14	246.37	249.39	249.40
JUNE LAB	HR	141.80	160.79	160.89	162.32	163.32	164.88	164.96	170.36	170.25
JULY LAB	HR	184.75	226.48	230.50	238.28	243.54	229.91	232.78	200.59	201.47
AUG LAB	HR	226.24	259.24	261.53	264.96	268.96	295.78	323.75	302.80	303.34
SEPT LAB	HR	99.49	109.74	110.32	117.20	121.97	128.41	127.97	152.14	151.51
OCT LAB	HR	24.62	29.46	30.42	34.90	37.94	38.04	36.25	42.39	41.86
NOV LAB	HR	57.40	57.40	57.40	61.73	65.73	92.82	110.76	132.47	132.50
DEC LAB	HR	2.90	2.90	2.90	4.92	6.36	9.78	10.80	20.40	20.40
MAY WATER	AI	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
JUNE WATER	AI	350.61	400.00	400.00	400.00	400.00	400.00	400.00	400.00	400.00
JULY WATER	AI	350.61	400.00	400.00	400.00	400.00	400.00	400.00	400.00	400.00
AUG WATER	AI	350.61	400.00	400.00	400.00	400.00	400.00	400.00	400.00	400.00
JAN CAP	$$	0	0	0	-285.06	-488.94	-970.39	-1432.58	-2674.94	-2731.27
FEB CAP	$$	0	0	0	-570.12	-977.88	-1940.78	-2546.98	-5144.94	-5201.27
MARCH CAP	$$	1029.09	1077.87	1087.83	261.80	-303.44	-1167.15	-1331.75	-5310.26	-5350.87
APRIL CAP	$$	1474.09	1522.87	1532.83	706.80	161.63	-261.90	-75.89	-4029.37	-4069.26
MAY CAP	$$	1378.98	548.06	393.93	-504.58	-1098.46	-654.18	-201.22	-1752.19	-1792.06
JUNE CAP	$$	1610.54	827.19	677.36	-206.27	-790.69	-359.80	71.58	-1472.31	-1516.82
JULY CAP	$$	-309.29	-1821.54	-2076.90	-3076.21	-3742.33	-2885.70	-2315.05	-2646.29	-2689.43
AUG CAP	$$	-4416.39	-6399.38	-6745.99	-7945.88	-8835.45	-9652.38	-10420.68	-10360.12	-10367.98
SEPT CAP	$$	-20053.57	-22164.05	-22496.51	-23583.29	-24478.24	-27165.53	-29389.90	-29211.14	-29186.14
OCT CAP	$$	-19117.82	-21052.16	-21350.68	-22360.67	-23202.68	-25979.28	-28252.64	-28266.07	-28247.07
NOV CAP	$$	-18965.34	-20899.68	-21198.20	-22486.90	-23525.77	-26718.69	-29026.99	-30384.99	-30354.83
DEC CAP	$$	-18693.47	-20527.74	-20807.19	-22342.71	-23558.40	-27281.50	-30076.86	-32810.77	-32838.75
DRY LAND	AC	97.73	110.13	114.90	131.15	144.69	190.27	230.00	230.00	230.00
IRG LAND	AC	87.65	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
MAX WHT	AC	29.00	29.00	29.00	29.00	29.00	29.00	29.00	29.00	29.00
MAX RD CLOV	AC	-0.00	0	-0.00	-0.00	0	0	-0.00	-0.00	-0.00
MAX ALF IRG	AC	37.65	50.00	50.00	44.23	40.10	30.36	27.44	.00	.00
MAX ALF DRY	AC	15.56	23.23	27.04	40.46	50.00	50.00	50.00	50.00	50.00
MAX CORN	AC	0	0	0	5.77	9.90	19.64	22.56	50.00	50.00
MAX BLY	AC	50.00	50.00	50.00	50.00	52.25	101.71	141.11	143.92	144.00
MAX OR GR	AC	0	0	-0.00	0	0	0	5.45	3.51	4.47
MAX VETCH	AC	3.17	7.90	8.86	11.69	13.44	9.56	4.44	3.57	2.52
MAX BEANS	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00
MIN WHT	AC	-29.00	-29.00	-29.00	-29.00	-29.00	-29.00	-29.00	-29.00	-29.00
MIN RD CLOV	AC	0	0	0	0	0	0	0	0	0
MIN ALF IRG	AC	37.65	50.00	50.00	44.23	40.10	30.36	27.44	.00	0
MIN ALF DRY	AC	15.56	23.23	27.04	40.46	50.00	50.00	50.00	50.00	50.00
MIN CORN	AC	0	0	0	5.77	9.90	19.64	22.56	50.00	50.00
MIN BLY	AC	-50.00	-50.00	-50.00	-50.00	-47.75	1.71	41.11	43.92	44.00
MIN OR GR	AC	0	0	0	0	0	0	5.45	3.51	4.47
MIN VETCH	AC	3.17	7.90	8.86	11.69	13.44	9.56	4.44	3.57	2.52
MIN BEANS	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00
EXP GR MARG	$$	18718.35	20553.81	20833.07	22367.28	23582.66	27318.83	30124.56	32856.55	32884.38
STD DEV	$$	1041.04	1130.69	1146.52	1238.88	1315.69	1592.03	1829.00	2110.08	2113.58

PART ONE CONTINUED

A STATEMENT OF THE VALUE OF AN ADDITIONAL UNIT OF RESOURCE

NAME OF	UNIT	PLAN 1	PLAN 2	PLAN 3	PLAN 4	PLAN 5	PLAN 6	PLAN 7	PLAN 8	PLAN 9
JAN LAB	HR	0	0	0	0	0	0	0	0	0
FEB LAB	HR	0	0	0	0	0	0	0	0	0
MARCH LAB	HR	0	0	0	0	0	0	0	0	0
APRIL LAB	HR	0	0	0	0	0	0	0	0	0
MAY LAB	HR	0	0	0	0	0	0	0	0	0
JUNE LAB	HR	0	0	0	0	0	0	0	0	0
JULY LAB	HR	0	0	0	0	0	0	0	0	0
AUG LAB	HR	0	0	0	0	0	0	0	0	0
SEPT LAB	HR	0	0	0	0	0	0	0	0	0
OCT LAB	HR	0	0	0	0	0	0	0	0	0
NOV LAB	HR	0	0	0	0	0	0	0	0	0
DEC LAB	HR	0	0	0	0	0	0	0	0	0
MAY WATER	AI	0	0	0	0	0	0	0	0	0
JUNE WATER	AI	0	0	0	0	0	0	0	0	0
JULY WATER	AI	0	0	0	0	0	0	0	0	0
AUG WATER	AI	0	0	0	0	0	0	0	0	0
JAN CAP	$$	0	0	0	0	0	0	0	0	0
FEB CAP	$$	0	0	0	0	0	0	0	0	0
MARCH CAP	$$	0	0	0	0	0	0	0	0	0
APRIL CAP	$$	0	0	0	0	0	0	0	0	0
MAY CAP	$$	0	0	0	0	0	0	0	0	0
JUNE CAP	$$	0	0	0	0	0	0	0	0	0
JULY CAP	$$	0	0	0	0	0	0	0	0	0
AUG CAP	$$	0	0	0	0	0	0	0	0	0
SEPT CAP	$$	0	0	0	0	0	0	0	0	0
OCT CAP	$$	0	0	0	0	0	0	0	0	0
NOV CAP	$$	0	0	0	0	0	0	0	0	0
DEC CAP	$$	0	0	0	0	0	0	0	0	0
DRY LAND	AC	0	0	0	0	0	0	0	17.45	23.35
IRG LAND	AC	0	0	3.63	8.54	10.31	18.95	21.91	47.86	66.27
MAX WHT	AC	0	0	0	0	0	0	0	0	0
MAX RD CLOV	AC	0	0	0	0	0	0	0	0	0
MAX ALF IRG	AC	0	0	0	0	0	0	0	0	0
MAX ALF DRY	AC	0	0	0	0	0	21.29	25.83	28.26	24.03
MAX CORN	AC	0	0	0	0	0	0	0	0	0
MAX BLY	AC	0	0	0	0	0	0	0	0	0
MAX OR GR	AC	0	0	0	0	0	0	0	0	0
MAX VETCH	AC	0	0	0	0	0	0	0	0	0
MAX BEANS	AC	0	80.44	85.95	94.68	102.81	127.50	131.90	97.05	78.59
MIN WHT	AC	198.72	130.45	118.66	110.40	104.95	49.89	36.55	8.66	0
MIN RD CLOV	AC	797.75	471.33	400.36	366.01	338.67	174.73	136.28	116.97	98.45
MIN ALF IRG	AC	0	0	0	0	0	0	0	0	12.58
MIN ALF DRY	AC	0	0	0	0	0	0	0	0	0
MIN CORN	AC	76.33	11.23	-0.00	0	0	0	0	0	0
MIN BLY	AC	5.90	3.92	1.34	0	0	0	0	0	0
MIN OR GR	AC	167.03	73.42	64.18	57.65	51.49	0	0	[illegible]	0
MIN VETCH	AC	0	0	0	0	0	[illegible]			0
MIN BEANS	AC	0	0	0	0	0	0	0	0	0
EXP GR MARG	$$	16718.35	20553.81	20833.07	22367.28	23582.66	27318.83	30124.56	32856.55	32884.38
STD DEV	$$	1041.04	1130.69	1146.52	1238.88	1315.69	1592.03	1829.00	2110.08	2113.58

PART ONE CONTINUED

A STATEMENT OF THE LEVELS OF ACTIVITIES AND THE EXPECTED PAYOFF

NAME OF CROP	UNIT	PLAN 10	PLAN 11	PLAN 12	PLAN 13	PLAN 14	PLAN 15	PLAN 16	PLAN 17	PLAN 18	PLAN 19
WHEAT	AC	54.61	62.03	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00
RED CLOVER	AC	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	28.72	50.00	50.00	100.00
ALFALFA IRG	AC	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
ALFALFA DRY	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	13.00	13.00
CORN SILAGE	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	0
BARLEY	AC	124.24	117.97	113.00	105.65	105.65	82.49	50.00	50.00	50.00	50.00
ORCH GRASS	AC	1.15	.00	-0.00	7.35	7.35	30.51	63.00	63.00	100.00	100.00
HAIRY VETCH	AC	.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
PINTO BEANS	AC	50.00	50.00	50.00	50.00	50.00	50.00	21.28	.00	.00	.00
EXP GR MARG	$$	34466.43	34913.34	35225.18	35344.90	35344.90	35722.39	37457.37	38350.68	38895.65	38974.54
STD DEV	$$	2357.07	2435.77	2493.01	2526.17	2526.17	2740.11	4490.47	5576.16	6341.80	9449.16

PART ONE CONTINUED

A STATEMENT OF THE AMOUNT OF EACH RESOURCE USED AND THE EXPECTED PAYOFF

NAME OF RESOURCE	UNIT	PLAN 10	PLAN 11	PLAN 12	PLAN 13	PLAN 14	PLAN 15	PLAN 16	PLAN 17	PLAN 18	PLAN 19
JAN LAB	HR	18.65	17.50	17.50	24.85	24.85	48.01	80.50	80.50	117.50	100.00
FEB LAB	HR	17.50	17.50	17.50	17.50	17.50	17.50	17.50	17.50	17.50	0
MARCH LAB	HR	47.92	47.30	46.80	46.07	46.07	43.75	40.50	40.50	44.20	26.70
APRIL LAB	HR	49.70	47.19	45.20	42.26	42.26	33.00	20.00	20.00	20.00	20.00
MAY LAB	HR	247.42	246.80	246.30	245.57	245.57	243.25	194.05	160.00	123.00	118.00
JUNE LAB	HR	170.00	170.00	170.00	170.00	170.00	170.00	164.26	160.00	160.00	150.00
JULY LAB	HR	192.30	190.00	190.00	204.69	204.69	251.02	272.92	241.00	285.40	210.40
AUG LAB	HR	305.77	306.00	306.00	304.53	304.53	299.90	250.32	218.40	214.70	139.70
SEPT LAB	HR	150.00	150.00	150.00	150.00	150.00	150.00	150.00	150.00	150.00	150.00
OCT LAB	HR	55.38	59.72	62.70	63.43	63.43	65.75	71.87	74.00	74.00	61.50
NOV LAB	HR	139.96	141.91	142.90	139.96	139.96	130.70	106.21	97.70	97.70	60.20
DEC LAB	HR	22.96	23.70	24.20	24.20	24.20	24.20	24.20	24.20	24.20	6.70
MAY WATER	AI	200.00	200.00	200.00	200.00	200.00	200.00	85.12	.00	.00	.00
JUNE WATER	AI	400.00	400.00	400.00	400.00	400.00	400.00	400.00	400.00	400.00	400.00
JULY WATER	AI	400.00	400.00	400.00	400.00	400.00	400.00	285.12	200.00	200.00	.00
AUG WATER	AI	400.00	400.00	400.00	400.00	400.00	400.00	285.12	200.00	200.00	.00
JAN CAP	$$	-2537.22	-2470.00	-2470.00	-2899.08	-2899.08	-4251.90	-9388.75	-11789.20	-13950.00	-17120.00
FEB CAP	$$	-5007.22	-4940.00	-4940.00	-5369.08	-5369.08	-6721.90	-11858.75	-14259.20	-16420.00	-17120.00
MARCH CAP	$$	-5192.96	-5140.45	-5139.61	-5466.71	-5466.71	-6498.00	-10968.52	-13209.37	-14392.26	-12247.26
APRIL CAP	$$	-4087.22	-4090.53	-4133.91	-4526.40	-4526.40	-5763.86	-10523.52	-12764.37	-13947.26	-11802.26
MAY CAP	$$	-1815.95	-1821.13	-1866.01	-2260.70	-2260.71	-3505.11	-9482.46	-12618.37	-12208.78	-12080.78
JUNE CAP	$$	-1551.95	-1557.13	-1602.01	-1996.71	-1996.71	-3241.11	-9167.62	-12265.87	-11856.28	-11630.78
JULY CAP	$$	-2742.18	-2751.63	-2796.51	-3163.95	-3163.95	-4322.41	-10100.82	-13178.64	-11588.01	-11318.51
AUG CAP	$$	-10684.60	-10775.21	-10842.80	-10908.63	-10908.63	-11116.19	-15622.14	-18745.28	-16569.68	-16407.69
SEPT CAP	$$	-31388.24	-32027.76	-32430.55	-32187.65	-32187.65	-31421.82	-28549.94	-27217.77	-25504.67	-25274.68
OCT CAP	$$	-30002.84	-30508.74	-30821.87	-30577.43	-30577.43	-29806.74	-26767.77	-25316.86	-23869.79	-23390.80
NOV CAP	$$	-32141.80	-32659.49	-32972.53	-32651.82	-32651.82	-31640.68	-28169.14	-26647.58	-24775.75	-21650.76
DEC CAP	$$	-34420.27	-34867.09	-35179.13	-35301.39	-35301.39	-35686.85	-37413.56	-38292.45	-38836.35	-38881.35
DRY LAND	AC	230.00	230.00	230.00	230.00	230.00	230.00	230.00	230.00	230.00	230.00
IRG LAND	AC	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
MAX WHT	AC	54.61	62.03	67.00	67.00	67.00	67.00	67.00	67.00	67.00	67.00
MAX RD CLOV	AC	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	28.72	50.00	50.00	100.00
MAX ALF IRG	AC	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
MAX ALF DRY	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	13.00	13.00
MAX CORN	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	-0.00
MAX BLY	AC	124.24	117.97	113.00	105.65	105.65	82.49	50.00	50.00	50.00	50.00
MAX OR GR	AC	1.15	.00	-0.00	7.35	7.35	30.51	63.00	63.00	100.00	100.00
MAX VETCH	AC	0	0	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
MAX BEANS	AC	50.00	50.00	50.00	50.00	50.00	50.00	21.28	.00	.00	.00
MIN WHT	AC	-3.39	4.03	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00
MIN RD CLOV	AC	0	0	0	0	0	0	28.72	50.00	50.00	100.00
MIN ALF IRG	AC	0	0	0	0	0	0	0	0	0	0
MIN ALF DRY	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	13.00	13.00
MIN CORN	AC	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	-0.00
MIN BLY	AC	24.24	17.97	13.00	5.65	5.65	-17.51	-50.00	-50.00	-50.00	-50.00
MIN OR GR	AC	1.15	.00	0	7.35	7.35	30.51	63.00	63.00	100.00	100.00
MIN VETCH	AC	0	0	0	0	0	0	0	0	0	0
MIN BEANS	AC	50.00	50.00	50.00	50.00	50.00	50.00	21.28	.00	0	0
EXP GR MARG	$$	34466.43	34913.34	35225.18	35344.90	35344.90	35722.39	37457.37	38350.68	38895.65	38974.54
STD DEV	$$	2357.07	2435.77	2493.01	2526.17	2526.17	2740.11	4490.47	5576.16	6341.80	9449.16

PART ONE CONTINUED

A STATEMENT OF THE VALUE OF AN ADDITIONAL UNIT OF RESOURCE

E VALUE

NAME OF	UNIT	PLAN 10	PLAN 11	PLAN 12	PLAN 13	PLAN 14	PLAN 15	PLAN 16	PLAN 17	PLAN 18	PLAN 19
JAN LAB	HR	0	0	0	0	0	0	0	0	0	0
FEB LAB	HR	0	0	0	0	0	0	0	0	0	0
MARCH LAB	HR	0	0	0	0	0	0	0	0	0	0
APRIL LAB	HR	0	0	0	0	0	0	0	0	0	0
MAY LAB	HR	0	0	0	0	0	0	0	0	0	0
JUNE LAB	HR	0	0	0	0	0	0	0	0	0	0
JULY LAB	HR	0	0	0	0	0	0	0	0	0	0
AUG LAB	HR	0	0	0	0	0	0	0	0	0	0
SEPT LAB	HR	0	0	0	0	0	0	0	0	0	0
OCT LAB	HR	0	0	0	0	0	0	0	0	0	0
NOV LAB	HR	0	0	0	0	0	0	0	0	0	0
DEC LAB	HR	0	0	0	0	0	0	0	0	0	0
MAY WATER	AI	0	0	0	0	0	0	0	0	0	0
JUNE WATER	AI	0	0	0	0	0	0	0	0	0	0
JULY WATER	AI	0	0	0	0	0	0	0	0	0	0
AUG WATER	AI	0	0	0	0	0	0	0	0	0	0
JAN CAP	$$	0	0	0	0	0	0	0	0	0	0
FEB CAP	$$	0	0	0	0	0	0	0	0	0	0
MARCH CAP	$$	0	0	0	0	0	0	0	0	0	0
APRIL CAP	$$	0	0	0	0	0	0	0	0	0	0
MAY CAP	$$	0	0	0	0	0	0	0	0	0	0
JUNE CAP	$$	0	0	0	0	0	0	0	0	0	0
JULY CAP	$$	0	0	0	0	0	0	0	0	0	0
AUG CAP	$$	0	0	0	0	0	0	0	0	0	0
SEPT CAP	$$	0	0	0	0	0	0	0	0	0	0
OCT CAP	$$	0	0	0	0	0	0	0	0	0	0
NOV CAP	$$	0	0	0	0	0	0	0	0	0	0
DEC CAP	$$	0	0	0	0	0	0	0	0	0	0
DRY LAND	AC	33.99	35.86	37.23	48.55	48.55	55.84	58.55	60.65	61.61	61.13
IRG LAND	AC	96.43	102.28	106.34	148.18	148.18	146.81	143.53	143.40	152.14	192.60
MAX WHT	AC	0	0	0	28.10	28.10	45.80	53.71	53.72	54.39	61.01
MAX RD CLOV	AC	0	0	0	0	0	0	0	0	0	0
MAX ALF IRG	AC	0	0	0	0	0	0	0	0	0	0
MAX ALF DRY	AC	12.90	11.14	10.00	5.67	5.67	2.88	2.19	.35	0	0
MAX CORN	AC	0	0	0	0	0	28.10	40.98	42.37	35.53	0
MAX BLY	AC	0	0	0	0	0	0	0	0	0	0
MAX OR GR	AC	0	0	0	0	0	0	0	0	0	14.22
MAX VETCH	AC	0	0	0	0	0	0	0	0	0	0
MAX BEANS	AC	52.56	47.57	43.46	0	0	0	0	0	0	0
MIN WHT	AC	0	0	0	0	0	0	0	0	0	0
MIN RD CLOV	AC	60.43	53.79	51.66	36.53	36.53	0	0	0	0	0
MIN ALF IRG	AC	34.46	38.57	41.10	68.28	68.28	57.36	49.71	49.25	57.51	100.12
MIN ALF DRY	AC	0	0	0	0	0	0	0	0	0	0
MIN CORN	AC	0	0	0	0	0	0	0	0	0	0
MIN BLY	AC	0	0	0	0	0	0	0	1.86	2.49	1.59
MIN OR GR	AC	0	0	1.32	0	0	0	0	0	0	0
MIN VETCH	AC	0	.91	1.85	8.24	8.24	12.53	15.28	17.25	18.40	13.35
MIN BEANS	AC	0	0	0	0	0	0	0	0	7.69	40.63
EXP GR MARG	$$	34466.43	34913.34	35225.18	35344.90	35344.90	35722.39	37457.37	38350.68	38895.65	38974.54
STD DEV	$$	2357.07	2435.77	2493.01	2526.17	2526.17	2740.11	4490.47	5576.16	6341.80	9449.16

PART TWO

PROBABILITY STATEMENTS ABOUT ATTAINING SPECIFIED LEVELS OF ACTUAL GROSS MARGIN FOR A GIVEN LEVEL OF EXPECTED GROSS MARGIN

PLAN	EXP GR MAR	PROBABILITY LEVEL 1%	5%	10%	20%	30%	40%	50%
1	18718.35	16296.17	17005.84	17384.05	17842.00	18165.56	18457.05	18718.35
2	20553.81	17923.04	18693.83	19104.61	19601.99	19953.41	20270.00	20553.81
3	20833.07	18165.46	18947.05	19363.58	19867.93	20224.27	20545.30	20833.07
4	22367.28	19484.77	20329.32	20779.41	21324.39	21709.43	22056.32	22367.28
5	23582.66	20521.44	21418.35	21896.34	22475.11	22884.03	23252.43	23582.66
6	27318.83	23614.65	24699.94	25278.32	25978.66	26473.46	26919.23	27318.83
7	30124.56	25869.04	27115.87	27780.34	28584.92	29153.37	29665.49	30124.56
8	32856.55	27947.02	29385.46	30152.06	31080.28	31736.09	32326.92	32856.55
9	32884.38	27966.73	29407.55	30175.41	31105.18	31762.07	32353.88	32884.38
10	34466.43	28982.23	30589.05	31445.37	32482.25	33214.83	33874.81	34466.43
11	34913.34	29246.03	30906.49	31791.41	32862.91	33619.94	34301.96	34913.34
12	35225.18	29424.68	31124.17	32029.88	33126.56	33901.39	34599.43	35225.18
13	35344.90	29467.26	31189.35	32107.11	33218.37	34003.51	34710.83	35344.90
14	35344.90	29467.26	31189.35	32107.11	33218.37	34003.51	34710.83	35344.90
15	35722.39	29346.97	31214.90	32210.39	33415.76	34267.39	35034.62	35722.39
16	37457.37	27009.40	30070.55	31701.94	33677.30	35072.93	36330.26	37457.37
17	38350.68	25376.64	29177.90	31203.72	33656.67	35389.74	36951.07	38350.68
18	38895.65	24140.18	28463.38	30767.36	33557.12	35528.16	37303.86	38895.65
19	38974.54	16989.17	23430.67	26863.55	31020.23	33957.03	36602.80	38974.54

PART THREE CONTINUED

DETAILED DESCRIPTION OF EFFICIENT PLANS IN EQUATION FORM CONTINUED

THIS PLAN WAS GENERATED DURING STEP 16
IT IS VALID FOR VALUES OF EXP GR MARG FROM 35722.39TO 37457.37

ALL EQUATIONS PERTAINING TO THIS PLAN ARE EVALUATED AT EXP GR MARG = 37457.37

THE VARIANCE EQUATION

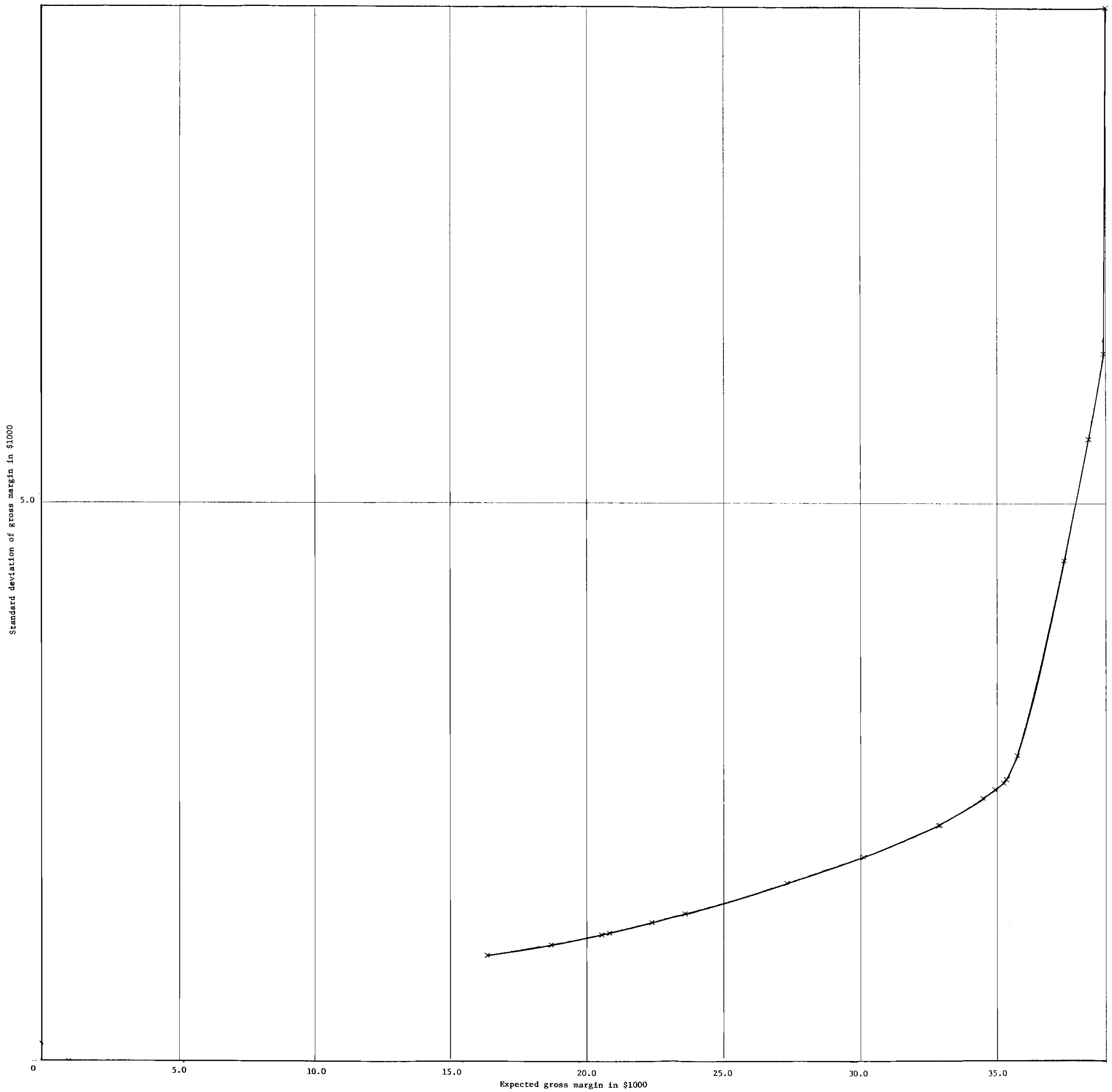
ALPHA1	ALPHA2	ALPHA3	VARIANCE	STD DEV
1.786378	-123432.099236	2137219775.875000	20164284.50	4490.47

THE ACTIVITY EQUATIONS

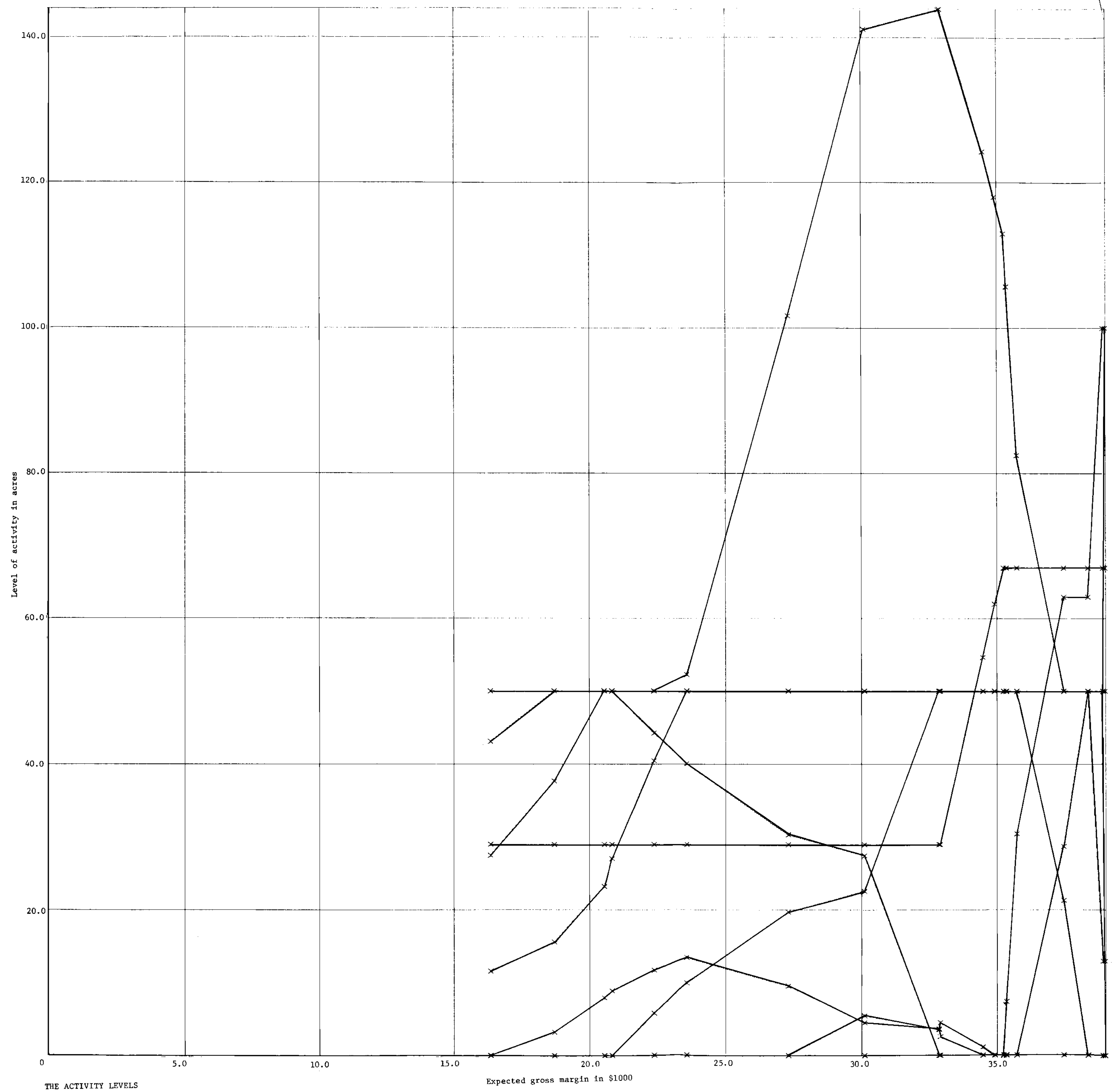
NO OF ACTIVITY	NAME OF ACTIVITY	UNIT	BETA1	BETA2	LEVEL OF ACTIVITY
1	WHEAT	AC	-0.000000	67.000001	67.00
2	RED CLOVER	AC	.016553	-591.315859	28.72
3	ALFALFA IRG	AC	.000000	-0.000004	.00
4	ALFALFA DRY	AC	-0.000000	50.000001	50.00
5	CORN SILAGE	AC	.000000	49.999999	50.00
6	BARLEY	AC	-0.018725	751.398047	50.00
7	ORCH GRASS	AC	.018725	-638.398048	63.00
8	HAIRY VETCH	AC	-0.000000	.000001	-0.00
9	PINTO BEANS	AC	-0.016553	641.315866	21.28

THE RESOURCE EQUATIONS

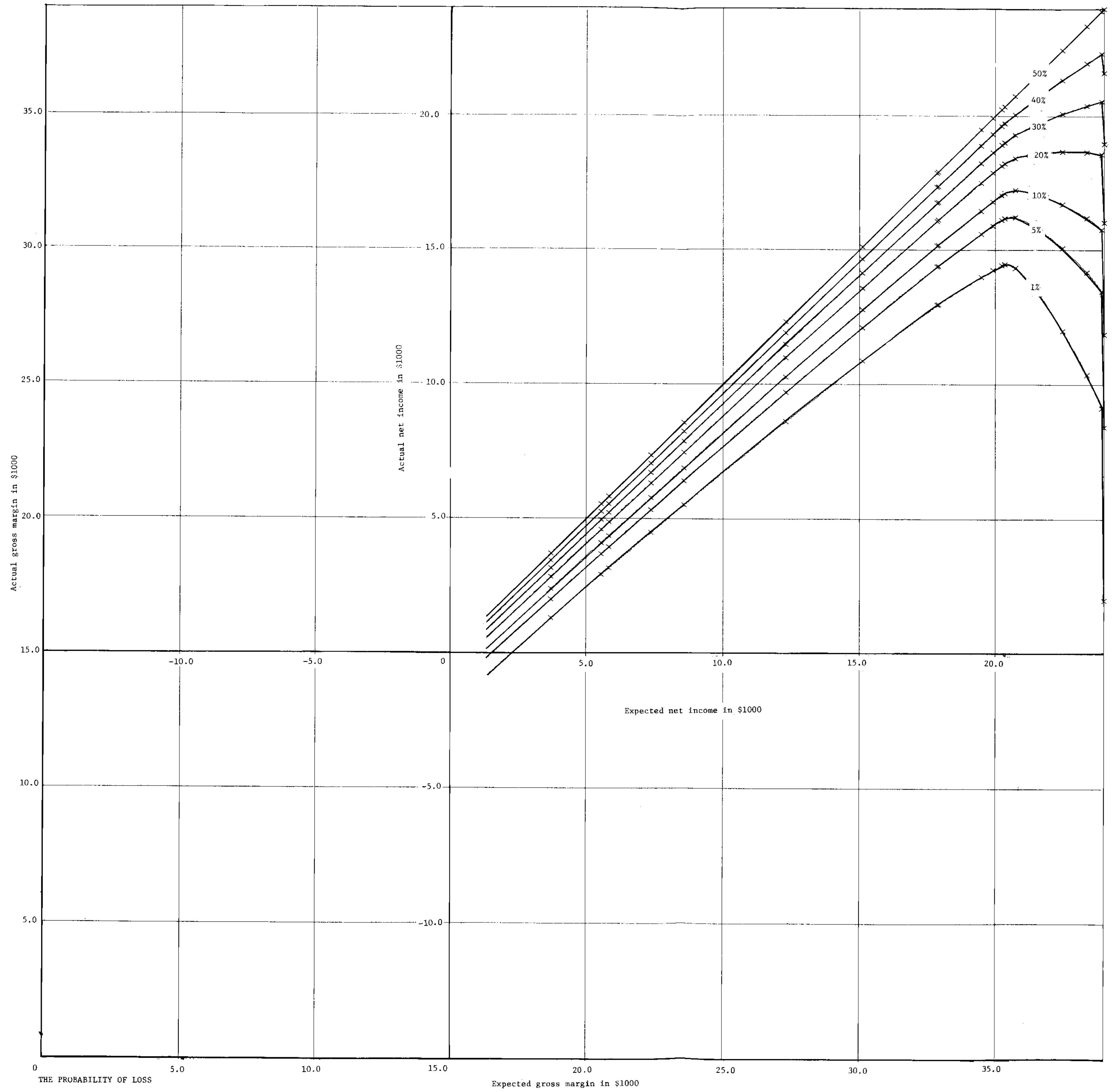
NO OF CONSTRAINT	NAME OF CONSTRAINT	UNIT	BETA1	BETA2	LEVEL OF CONSTRAINT	VALUE OF LAGRANGIAN
0	EXP GR MARG	$$	-3.572757	123432.099236		-10393.97
1	JAN LAB	HR	-0.018725	1620.898049	919.50	0
2	FEB LAB	HR	-0.000000	982.500000	982.50	0
3	MARCH LAB	HR	.001873	889.360196	959.50	0
4	APRIL LAB	HR	.007490	699.440781	980.00	0
5	MAY LAB	HR	.028357	-256.245193	805.95	0
6	JUNE LAB	HR	.003311	711.736824	835.74	0
7	JULY LAB	HR	-0.012621	1199.822308	727.08	0
8	AUG LAB	HR	.028575	-320.653398	749.68	0
9	SEPT LAB	HR	.000000	849.999993	850.00	0
10	OCT LAB	HR	-0.003528	1060.271391	928.13	0
11	NOV LAB	HR	.014111	365.214435	893.79	0
12	DEC LAB	HR	-0.000000	975.800000	975.80	0
13	MAY WATER	AI	.066212	-1365.263464	1114.88	0
14	JUNE WATER	AI	.000000	799.999993	800.00	0
15	JULY WATER	AI	.066212	-1765.263444	714.88	0
16	AUG WATER	AI	.066212	-1765.263444	714.88	0
17	JAN CAP	$$	2.960742	-76512.874992	34388.75	0
18	FEB CAP	$$	2.960742	-74042.875032	36858.75	0
19	MARCH CAP	$$	2.576688	-60547.431203	35968.52	0
20	APRIL CAP	$$	2.743342	-67234.873819	35523.52	0
21	MAY CAP	$$	3.445183	-94565.038691	34482.46	0
22	JUNE CAP	$$	3.415884	-93782.409632	34167.62	0
23	JULY CAP	$$	3.330522	-89651.789818	35100.82	0
24	AUG CAP	$$	2.597110	-56658.762796	40622.14	0
25	SEPT CAP	$$	-1.655280	115552.362879	53549.94	0
26	OCT CAP	$$	-1.751578	117377.288973	51767.77	0
27	NOV CAP	$$	-2.000902	128117.689325	53169.14	0
28	DEC CAP	$$	.995231	25134.842040	62413.56	0
29	DRY LAND	AC	215.749898	-7472818.849731	0	608605.12
30	IRG LAND	AC	504.878677	-17419551.780762	0	1491876.07
31	MAX WHT	AC	211.006047	-7345514.749756	0	558217.01
32	MAX RD CLOV	AC	-0.016553	691.315859	71.28	0
33	MAX ALF IRG	AC	-0.000000	100.000004	100.00	0
34	MAX ALF DRY	AC	6.157530	-207860.043182	0	22784.84
35	MAX CORN	AC	177.529466	-6223870.957031	0	425916.11
36	MAX BLY	AC	.018725	-521.398047	180.00	0
37	MAX OR GR	AC	-0.018725	738.398049	37.00	0
38	MAX VETCH	AC	.000000	49.999999	50.00	0
39	MAX BEANS	AC	.016553	-591.315866	28.72	0
40	MIN WHT	AC	-0.000000	38.000001	38.00	0
41	MIN RD CLOV	AC	.016553	-591.315859	28.72	0
42	MIN ALF IRG	AC	159.119868	-5443504.453491	0	516707.45
43	MIN ALF DRY	AC	-0.000000	50.000001	50.00	0
44	MIN CORN	AC	.000000	49.999999	50.00	0
45	MIN BLY	AC	-0.018725	701.398047	-0.00	0
46	MIN OR GR	AC	.018725	-638.398049	63.00	0
47	MIN VETCH	AC	61.240406	-2135093.122314	0	158811.47
48	MIN BEANS	AC	-0.016553	641.315866	21.28	0



THE EFFICIENCY FRONTIER



THE ACTIVITY LEVELS


50%
40%
30%
20%
10%
5%
1%
Actual gross margin in $1000
Actual net income in $1000
Expected net income in $1000
THE PROBABILITY OF LOSS
Expected gross margin in $1000
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0
20.0
15.0
10.0
5.0
-5.0
-10.0
-10.0
-5.0
0
5.0
10.0
15.0
20.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0

The report begins with a letter which outlines the results to be presented, defines the terminology used and describes the main concepts the farmer will encounter. As the letter indicates, the report is divided into four parts. The reader is now asked to put himself in the farmers position as he reads the approximate discussion during interpretation of the report to Mr. Smith and Mr. Jones.

"Part one deals with the composition and attributes of the efficient plans. Here you are given the number of acres planted to each crop and the gross margin you can expect as a consequence. You are also given the standard deviation of gross margin which indicates the riskiness of the plan. In your report, 19 plans are presented. Plan one has an expected gross margin of $18, 718. 35 and standard deviation of $1041. 04. The plans are arranged in order of increasing expected gross margin. As expected gross margin increases, standard deviation increases at an increasing rate. The absolute maximum expected gross margin and the maximum standard deviation occur at plan 19. For example in plan 19 gross margin is $38, 974. 54 and standard deviation is $9449. 16. This indicates that about two thirds of the time you will find gross margin lying within one standard deviation of its expected level i. e., in the range from $29, 525. 38 to $48, 423. 70. Note the rapid increase in standard deviation from plan 18 to plan 19. This is because 50 acres was transferred from corn silage, a high paying low risk crop, to red clover, a slightly higher paying crop than corn silage but a considerably more risky one. The net gain in expected gross margin was \$78.89 while standard deviation has increased \$3107.36. Since the composition of the plans changes as expected gross margin increases so does the amount of each resource used. Those resources which are completely used up have a shadow price attached to them. The shadow price indicates the value of one more unit of limiting resource. Note at plan 17 the value of an additional unit of irrigated land is \$143.40 indicating the approximate amount by which expected gross margin would increase if one acre were added. The shadow prices must be interpreted with caution because they are valid only over a small range.

Part two is prepared as an aid in helping you select the "best" plan. Since you are the decision maker, and you must live with the outcome of your actions the choice of the "best" plan can be made only by you. The probability statements in part two can, however, help you make the choice by pointing out the chances of failure. For example if you choose plan 19 your gross margin will be \$38,974.54 on the average, however in any given year you stand one chance in 100 that your gross margin will be less than \$16,989.17. On the other hand, if you were to choose plan 12 your expected gross margin is only $35, 225.18. However it is much less risky since there is one chance in 100 that gross margin will fall below $29, 424.68. Probability statements are also made for the 5, 10, 20, 30, 40 and 50% levels. You will notice that expected gross margin is $38, 895.65 which is only $78.89 less than the maximum possible expected gross margin. However, the variability of gross margin is much less under plan 18 than plan 19 as reflected by the fact that there is a 1% chance of gross margin falling below $24, 140.18. Your own personal circumstances and your willingness to take chances are the factors important in deciding upon the proper plan. However, any of the 19 plans carries with it the assurance that there is no less risky way in which you can produce that level of expected income.

Part three describes the plans in equation form. If, for example, you wish to choose a plan having an expected gross margin somewhere between that given for plan 15 and plan 16 you can determine the acres in each crop and the amount of unused resources according to the formula:

ACRES = (BETA1)x(EXP GR MAR) + (BETA 2)

If you wish to know the variability of the plan use the formula:

VARIANCE = (ALPHA 1)x(EXP GR MAR)x(EXP GR MAR)

+ (ALPHA 2)x(EXP GR MAR) + (ALPHA 3)

For example, if you evaluated the equations at an expected gross margin of $36,500, about midway between plan 15 and plan 16 you would find the result as shown in Table 4.3 under the heading of plan 15a.

Part four is composed of three graphs. The first graph shows the degree of riskiness for every level of expected gross margin. Note that as expected gross margin becomes higher the riskiness as measured by standard deviation increases more rapidly. The second graph shows the composition of the plans for every level of expected gross margin. You can read the number of acres in each crop directly from the graph. If you wish to determine the composition of plan 15a you need only draw a vertical line at the expected gross margin of $36,500 and read the number of acres in each crop directly on the vertical axis of the graph. It is also interesting to note the drastic changes in the composition of plans as the maximum expected gross margin is approached. The third graph displays the probability statements tabulated in part two. If you pick a specific level of expected gross margin on the horizontal axis you can read the levels on the vertical axis, below which actual gross

Table 4. 3. Composition of an intermediate plan.

Name of Crop	Units	Plan 15a
wheat	ac	67. 00
red clover	ac	12. 87
alfalfa irg	ac	0. 00
alfalfa dry	ac	50. 00
corn silage	ac	50. 00
barley	ac	67. 94
orch. grass	ac	45. 06
hairy vetch	ac	0. 00
pinto beans	ac	37. 13
EXP GR MARG	$	36, 500
Std. Dev.	$	3442. 95

margin will fall at the 1, 5, 10, 20, 30, 40 and 50% probability levels. For example, suppose you wish to determine the level below which gross margin will fall five times in 100 for plan 15a. First find $36,500 on the horizontal axis then draw a vertical line up to the five percent probability curve and then across to the vertical axis where you can read $30,836.35. Thus if you choose plan 15a there is a five percent chance that your gross margin in a specific year will fall below $30,836.35. Usually farmers have fixed cash commitments such as debt payments and family living costs. In such a case it may be more appropriate to deduct these costs from the gross margin figures before examining probability of loss graph. The second set of axis on the graph are with respect to net income. In your case there is a $10,000 rental payment and $5,000 repayment on a loan for irrigation equipment. Hence, if you choose plan 15a there is a five percent chance of having less than $15,836.35 of net income. This figure is read from the net income axis."

After some deliberation, the partners chose plan 17 as "best" in their circumstances. They were in agreement that the added expected gross margin that would accrue in choosing plan 18 or plan 19 over plan 17 was not sufficient, in their opinion, to compensate for the increase in standard deviation. Their choice of plan 17 was reinforced by examination of the probability of loss graph with the knowledge that there would be a $15,000 dollar fixed cash commitment.

Operational Costs

Once an algorithm is operational, it is the human time involved in setting up the problem, collecting the data and preparing it for computer processing that tends to be the most expensive item.[45] This is true regardless of whether quadratic or linear programming is used since they take about the same set up time. Approximately seven hours were required for each of the four cases studied. This included three hours for data collection, two hours for computer input preparation and three hours for discussion and interpretation of results with the farmer. The computer cost alone is likely to be in the range of $20.00 - $30.00 depending upon the dimensions of the problem. About one-half of the computer cost represents printing the report and drawing the graphs. Since the equations for each step are of limited use to the farmer, the program OUTPUT contains the facility to suppress printing this part of the report. Further computer cost could be eliminated by not plotting the activity level graph since the large amount of information tends to be confusing to the farmer.

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[45] These costs are exclusive of the overhead cost in developing the algorithm.

V. SUMMARY AND CONCLUSIONS

The main objective of this research was to develop an operational tool for solving the enterprise selection problem under conditions of uncertainty. The central purpose was to develop an algorithm amenable to use by extension workers and/or farm management consultants as they counsel farmers on problems of enterprise choice. To accomplish this, the problem was formulated as the minimization of variance subject to a level of expected income and a set of production constraints. It was found that by making use of some important properties of Lagrangian multipliers, properly constrained by the Kuhn-Tucker conditions, one could compute the entire array of efficient choices.

This permitted presentation of all relevant alternatives to the farm decision maker rather than the single expected income maximizing plan of linear programming which is not infrequently sub-optimal when evaluated in light of the decision makers risk preference.

The framework of analysis used here is comparable to Markowitz's (34) portfolio selection method designed for use by investment consultants. Houthakker's (25) capacity method of solving quadratic programs provided many insights into procedures that were ultimately built into the program. The algorithm developed in this research is problem specific and deal only with minimizing positive definite quadratic forms containing no linear components.[46] Previous existing quadratic programming algorithms provided only a finite number of solution points on the efficiency frontier (7, 26). The algorithm developed here provides exact algebraic specification for the frontier.

In the theory portion of the thesis, a two dimensional model was developed and used to provide a transition from the traditional certainty framework to the more realistic uncertainty environment in which decision makers find themselves. Variations in the model parameters σ, μ and r demonstrated the sensitivity to change in the efficient plans and emphasized the error that is introduced by ignoring uncertainty. Capital restriction, debt payments, family living requirements and other fixed cash commitments become important considerations in the decision problem. The adage "fixed costs have no bearing upon short run decisions" is simply not true if the decision maker is confronted with variations in income.

To test its applicability, the algorithm was used to solve enterprise selection problems submitted by four farmers. The results appeared encouraging. The data requirements, although substantial, were no more difficult to satisfy than for the linear programming model where uncertainty is assumed non-existent. Crop enterprise selection problems lend themselves particularly well to the method used. Livestock enterprise choice problems could be handled in the

[46] The algorithm will not maximize a quadratic form.

same way, although difficulties could arise because the algorithm cannot accommodate transfer equations which may be needed to account for activities like home grown feed.

The results, although appearing more difficult to interpret because of the presence of probability statements can be given in a more realistic setting, and were no more difficult for the farmers to comprehend than the non-stochastic linear programming case. Suggestions made by the farmers have been incorporated into the report with the result that it is more understandable and meaningful to the decision maker. Results of this study indicate additional areas for research.

The algorithm is deficient in at least two areas; (a) the initial basis is found by a trial and error approach which could result in cycling; and (b) it is not possible to include transfer equations in the model. These two unanswered questions could prove to be interesting and fruitful avenues of exploration.

Additional computational efficiencies could undoubtedly result from revisions in the three computer programs.[47] Clerical time needed for organizing data and key punch time could certainly be reduced by streamlining the input routine. The report form which has benefited from comments of farmers and colleagues could stand further

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[47] The writer does not claim more than a rudimentary knowledge of computer programming, and although the programs have benefited immeasurably by others more gifted in the field, some inefficiencies no doubt remain.

improvement.

An empirical question surrounds the triangular distribution and its ability to transmit the farmer's impressions about the future performance of price and yield variables. The data needs of the triangular distribution are small compared to more elaborate methods of establishing subjective probability density functions, but no direct check has been made on the reliability of the estimates. Additional work in this area is warranted. Extending the subjective probability concept to the joint distribution case poses a difficult but interesting question. The subjective establishment of correlation coefficients was dismissed because of the burden placed upon the respondent and because of the high chance for inconsistencies. Perhaps the dismissal was premature and additional investigation could result in practical methods for accomplishing the task.

Questions of practical relevance and acceptability also remain. It is in this area that additional research efforts need be expended to evaluate whether or not the research in this thesis has narrowed the gap between theoretical developments and practical application by providing an operationally feasible quadratic programming algorithm.

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APPENDIX

APPENDIX A

LAGRANGIAN MULTIPLIERS AND TRANSFORMATIONS

Lagrangian multipliers are used frequently in the main body of the thesis. A general statement about their behavior and interpretation may be of value to the reader who wishes to pursue the topic further.

Consider the general problem:

$$\text{Min or Max: } G(X_1, X_2 \cdots X_n) = G$$

$$\text{S. T: } K_j - F(X_1, X_2 \cdots X_n) = 0 \quad j = 1, \; m \leq n$$

The Lagrangian form is:

$$R(X, \lambda) = G(X_1 \cdots X_n) + \sum_{j=1}^{m} \lambda_j [K_j - F(X_1 \cdots X_n]$$

and the first order condition becomes:

$$\frac{\partial R}{\partial X_i} = \frac{\partial G}{\partial X_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial X_i} = 0 \qquad i = 1, n$$

$$\frac{\partial R}{\partial \lambda_j} = K_j - F_j(X_1 \cdots X_n) = 0 \qquad j = 1, m$$

From the objective function it follows that the exact differential of G is:

$$dG = \sum_{i=1}^{n} \frac{\partial G}{\partial X_i} dX_i$$

and from the constraints:

$$dK_j = \sum_{i=1}^{n} \frac{\partial F_j}{\partial X_i} dX_i \qquad j = 1, m$$

In the first order conditions it was established that:

$$\frac{\partial G}{\partial X_i} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial X_i} \qquad i = 1, n$$

Substituting this information into the differential of G establishes that

$$dG = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial X_i} dX_i$$

Changing the order of summation results in:

$$dG = \sum_{j=1}^{m} \lambda_j \sum_{j=1}^{n} \frac{\partial F_j}{\partial X_i} dX_i$$

which upon simplification yields:

$$dG = \sum_{j=1}^{m} \lambda_j dK_j$$

If G is a positive definite quadratic form in X and F is a set of linear equations in X, then the first order conditions resulting from minimizing G subject to F can be expressed as:

$$\left[\begin{array}{ccc|ccc}
\frac{\partial^2 G}{\partial X_1^2} & \cdots & \frac{\partial^2 G}{\partial X_1 \partial X_n} & \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_m}{\partial X_1} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 G}{\partial X_n X_1} & \cdots & \frac{\partial^2 G}{\partial X_n^2} & \frac{\partial F_1}{\partial X_n} & \cdots & \frac{\partial F_m}{\partial X_n} \\
\hline
\frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial F_m}{\partial X_1} & \cdots & \frac{\partial F_m}{\partial X_n} & 0 & \cdots & 0
\end{array}\right]
\begin{bmatrix} X_1 \\ \vdots \\ X_n \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ K_1 \\ \vdots \\ K_m \end{bmatrix}$$

Note that $\frac{\partial^2 G}{\partial X_r \partial X_s}$ and $\frac{\partial F_j}{\partial X_s}$ are constants. Further more the matrix is symetric and non-singular if $n \geq m$ and there are no linear dependencies in F.

This system has a solution for $X_1, \cdots, X_n$ and $\lambda_1, \cdots, \lambda_m$ which can be obtained from the inverted system:

$$\left[\begin{array}{ccc|ccc}
C_{11} & \cdots & C_{11} & a_{11} & \cdots & a_{1m} \\
\vdots & & \vdots & \vdots & & \vdots \\
C_{1n} & \cdots & C_{nn} & a_{n1} & \cdots & a_{nm} \\
\hline
a_{11} & \cdots & a_{n1} & b_{11} & \cdots & b_{1m} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{1m} & \cdots & a_{nm} & b_{m1} & \cdots & b_{nm}
\end{array}\right]
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ K_1 \\ \vdots \\ K_m \end{bmatrix}
=
\begin{bmatrix} X_1 \\ \vdots \\ X_0 \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}$$

In the case where $m = 3$, $n \geq 3$

$$dG = \lambda_1 dK_1 + \lambda_2 dK_2 + \lambda_3 dK_3$$

where
$$\lambda_1 = b_{11}K_1 + b_{12}K_2 + b_{13}K_3 = \frac{\partial G}{\partial K_1}$$

$$\lambda_2 = b_{12}K_1 + b_{22}K_2 + b_{23}K_3 = \frac{\partial G}{\partial K_2}$$

$$\lambda_3 = b_{13}K_1 + b_{23}K_2 + b_{33}K_3 = \frac{\partial G}{\partial K_3}$$

then

$$G(K_1K_2K_3) = \int (b_{11}K_1 + b_{12}K_2 + B_{13}K_3)dK_1 + g_1(K_2K_3)$$

$$= \frac{b_{11}K_1^2}{2} + b_{12}K_1K_2 + b_{13}K_1K_3 + g_1(K_2K_3)$$

and

$$\frac{\partial G(K_1K_2K_3)}{\partial K_2} = b_{12}K_1 + \frac{\partial g_1(K_2K_3)}{\partial K_2} = b_{12}K_1 + b_{22}K_2 + b_{33}K_3$$

hence

$$\frac{\partial g_1(K_2, K_3)}{\partial K_2} = b_{22}K_2 + b_{23}K_3$$

and

$$g_1(K_2, K_3) = \frac{b_{22}K_2^2}{2} + b_{23}K_2K_3 + g_2(K_3)$$

thus

$$G(K_1K_2K_3) = \frac{b_{11}}{2}K_1^2 + b_{12}K_1K_2 + b_{13}K_1K_3 + \frac{b_{23}}{2}K_2^2 + b_{23}K_2K_3$$

$$+ g_2(K_3)$$

and

$$\frac{\partial G(K_1 K_2 K_3)}{\partial K_3} = b_{13}K_1 + b_{23}K_2 + \frac{\partial g_2(K_3)}{\partial K_3} = b_{13}K_1 + b_{23}K_2 + b_{23}K_3$$

hence

$$\frac{\partial g_2(K_3)}{\partial K_3} = b_{33}K_3 \;\; => \;\; g_2(K_3) = \frac{b_{33}}{2}K_3^2 + K_0$$

Finally

$$G(K_1 K_2 K_3) = \frac{b_{11}}{2}K_1^2 + \frac{b_{22}}{2}K_3^2 + b_{12}K_1K_2 + b_{13}K_1K_3 + b_{23}K_2K_3 + K_0$$

For $m > 3$ the same step by step procedure can be followed to transform $G(X)$ to $G(K)$ with the general results:

$$G(K_1, \cdots, K_m) = \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} b_{jk}K_jK_k$$

APPENDIX B

PROOF OF ASSERTIONS

Proof of Assertion 1: The direction of rotation is found directly from the derivative of the angle θ with respect to r.

The rotation equation is:

$$\tan 2\theta = \frac{2r\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}, \quad \text{where } -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$\text{and} \quad \sigma_1^2 - \sigma_2^2 < 0$$

then

$$\frac{\partial\theta}{\partial r} = \frac{\sigma_1\sigma_2\cos^2 2\theta}{\sigma_1^2 - \sigma_2^2} < 0$$

hence the direction of rotation is clockwise as r increases.

The properties of elongation are found by examining the ellipse in the rotated coordinate system. Let

$$V = Ay'^2_1 + By'_1y'_2 + Cy'^2_2$$

where $A = \sigma^2\cos^2\theta + 2r\sigma_1\sigma_2\sin\theta\cos\theta + \sigma_2^2\sin^2\theta$

$B = 0$ since the angle θ is so chosen

$C = \sigma_1^2\sin^2\theta - 2r\sigma_1\sigma_2\sin\theta\cos\theta + \sigma_2^2\cos^2\theta$

$V =$ the variance

$y'_1y'_2 =$ the activity levels in the rotated coordinate system.

$$A < C \text{ since } -\frac{\pi}{4} < \theta < \frac{\pi}{4} \text{ and } \sigma_1^2 < \sigma_2^2$$

Then the vertices of the ellipse are at $(\pm\sqrt{V/A}\,,\ 0)$ in the (y_1', y_2') coordinate system. Let

$$K = \sqrt{V/A}$$ [48]

then

$$\frac{dK}{dr} = \Phi\left[r\sigma_1\sigma_2\cos^2 2\theta\left(\frac{\cos^2\theta - \sin^2\theta}{\sigma_2^2 - \sigma_1^2}\right) - \sin\theta\cos\theta\sin^2 2\theta\right]$$

where

$$\Phi = [\sigma_1^2\sigma_2^2 V]^{1/2}[\sigma_1^2\cos^2\theta + 2r\sigma_1\sigma_2\sin\theta\cos\theta + \sigma_2^2\sin^2\theta]^{-3/2}$$

The derivative $\frac{dK}{dr}$ must be evaluated under two cases:

Case 1: where r is positive

$$r > 0 \Rightarrow -\frac{\pi}{4} < \theta < 0 \Rightarrow \sin\theta < 0,\ \cos\theta > 0$$

and

$$\cos^2\theta > \sin^2\theta$$

then

$$r\sigma_1\sigma_2\cos^2 2\theta\left(\frac{\cos^2\theta - \sin^2\theta}{\sigma_2^2 - \sigma_1^2}\right) > \sin\theta\cos\theta\sin^2 2\theta$$

hence $\frac{dK}{dr} > 0$ and the conclusion that the ellipse elongates as r increases from 0 to 1 holds.

Case 2: where r is negative

$$r < 0 \Rightarrow 0 < \theta < \frac{\pi}{4} \Rightarrow \sin\theta > 0,\ \cos\theta > 0,\ \cos^2\theta > \sin^2\theta$$

---

[48] Since only the positive quadrant is of concern the negative root need not be considered.

then

$$r\sigma_1\sigma_2\cos^2 2\theta\left(\frac{\cos^2\theta - \sin^2\theta}{\sigma_2^2 - \sigma_1^2}\right) < \sin\theta\cos\theta\sin^2 2\theta$$

hence $\frac{dK}{dr} < 0$ and the conclusion that the ellipse elongates as r decreases from 0 to -1 holds.

Proof of Assertion 2: It is required to determine the limits on the correlation coefficient so that the expansion path and the activity equations will not have a negative slope.

There are two cases to be evaluated:

Case 1: for

$$\frac{\partial y_1}{\partial E} = \frac{\sigma_2(\sigma_2\mu_1 - r\sigma_1\mu_2)}{(\mu_1^2\sigma_2^2 - 2r\mu_1\mu_2\sigma_1\sigma_2 + \mu_2^2\sigma_1^2)} > 0$$

it must be that $r < \left(\frac{\sigma_2}{\mu_2}\right)/\left(\frac{\sigma_1}{\mu_1}\right) = k_1$

Case 2: for

$$\frac{\partial y_2}{\partial E} = \frac{\sigma_1(\sigma_1\mu_2 - r\sigma_2\mu_1)}{(\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2)} > 0$$

it must be that $r < \left(\frac{\sigma_1}{\mu_1}\right)/\left(\frac{\sigma_2}{\mu_2}\right) = k_2$

Note also that $k_1k_2 = 1$.

Now let r^* be the smaller of k_1 and k_2, then r^* is the ratio of the coefficients of variation of the least risky activity to the most risky activity Only if $r < r^*$ will $y_1, y_2 > 0$ and if $y_1, y_2 > 0$ then the expansion path has a positive slope.

Proof of Assertion 3: The direction of substitution due to variations in the correlation coefficient can be known by taking the derivative of the expansion path with respect to r.

$$y_2 = y_1\left(\frac{\sigma_1}{\sigma_2}\right)\left(\frac{\mu_2\sigma_1 - r\mu_1\sigma_2}{\mu_1\sigma_2 - r\mu_2\sigma_1}\right)$$

$$\frac{\partial y_2}{\partial r} = y_1\left(\frac{\sigma_1}{\sigma_2}\right)\left\{\frac{(\mu_2\sigma_1 - \mu_1\sigma_2)(\mu_2\sigma_1 + \mu_1\sigma_2)}{(\mu_1\sigma_2 - r\mu_2\sigma_1)^2}\right\}$$

If $\left(\frac{\sigma_1}{\mu_1}\right) > \left(\frac{\sigma_2}{\mu_2}\right)$ then $\frac{\partial y_2}{\partial r} > 0$ and similarly $\frac{\partial y_1}{\partial r} < 0$. If $\left(\frac{\sigma_2}{\mu_2}\right) > \left(\frac{\sigma_1}{\mu_1}\right)$ then $\frac{\partial y_2}{\partial r} < 0$ and similarly $\frac{\partial y_1}{\partial r} > 0$. Thus increases in r cause increases in the least risky activity.

Proof of Assertion 4: The shift of the efficiency frontier can be deduced from the change in the slope of frontier with respect to variations in r. This is done by examing the derivative of $-\lambda_0 = \frac{dV}{dE}$.

$$\frac{dV}{dE} = \frac{2\sigma_1^2\sigma_2^2(1-r^2)E}{\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2}$$

$$\frac{\partial(\frac{dV}{dE})}{\partial r} = \Phi[r^2 - r(\frac{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2}{\sigma_1\sigma_2\mu_1\mu_2}) + 1]$$

where

$$\Phi = \frac{4\sigma_1^2\sigma_2^2 E}{(\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2)^2} > 0$$

recalling that

$$k_1 = (\frac{\sigma_2}{\mu_2})/(\frac{\sigma_1}{\mu_1}) \quad \text{and} \quad k_2 = (\frac{\sigma_1}{\mu_1})/(\frac{\sigma_2}{\mu_2})$$

then arranging the terms accordingly

$$\frac{\partial\frac{dV}{dE}}{\partial r} = \Phi\, \sigma_1\sigma_2\mu_1\mu_2[r^2 - rk_1 - rk_2 + 1]$$

Now define r^* as the minimum of k_1 and k_2 and note that $0 < r^* < 1$ since $k_1k_2 = 1$ and $k_1, k_2 > 0$. Suppose $r^* = k_1$, then $k_2 = \frac{1}{r^*}$. Thus

$$\frac{\partial(\frac{dV}{dE})}{\partial r} = \frac{\Phi\,\sigma_1\sigma_2\mu_1\mu_2}{r^*}\,[r^2r^* - rr^{*2} - r + r^*]$$

$$\partial(\frac{dV}{dE}) = \Phi\,\frac{\sigma_1\sigma_2\mu_1\mu_2}{r^*}\,(1 - rr^*)(r^* - r)$$

hence if $-1 < r < r^*$ then $\frac{\partial(\frac{dV}{dE})}{\partial r} > 0$. Thus, increases in r cause the efficiency frontier to rise more steeply throughout.

Proof of Assertion 5: This assertion is established from the derivative of the expansion path. First consider that

$$\frac{\partial y_2}{\partial u_2} = \frac{\sigma_1^2 y_1 \mu_1 (1-r^2)}{(\mu_1 \sigma_2 - r\mu_2 \sigma_1)^2} > 0$$

Thus, increases in μ_2 cause increases in y_2. Next consider that

$$\frac{\partial y_2}{\partial u_1} = \frac{\sigma_1^2 y_1 \mu_2 (r^2 - 1)}{(\mu_1 \sigma_2 - r\mu_2 \sigma_1)^2} < 0$$

Thus, increases in μ_1 cause decreases in y_2.

Proof of Assertion 6: This assertion is established from the derivative of $\frac{dV}{dE}$ with respect to μ_1.

$$\frac{\partial (\frac{dV}{dE})}{\partial \mu_1} = \frac{-4\sigma_1^3 \sigma_2^3 (1-r^2) E \mu_2 (r^* - r)}{(\mu_1^2 \sigma_2^2 - 2r\sigma_1 \sigma_2 \mu_1 \mu_2 + \mu_2^2 \sigma_1^2)^2}$$

If $r < r^*$ then $\frac{\partial (\frac{dV}{dE})}{\partial \mu_1} < 0$. Thus, increases in μ_1 (or μ_2) cause the efficiency frontier to rise less steeply throughout.

Proof of Assertion 7: This assertion is established from the derivatives of the expansion path. First consider

$$\frac{\partial y_2}{\partial \sigma_2} = \frac{-y_1 \sigma_1}{\sigma_2^2 (\mu_1 \sigma_2 - r\mu_2 \sigma_1)^2} [r\mu_1 \sigma_2 (\mu_1 \sigma_2 - r\mu_2 \sigma_1) + (2\mu_1 \sigma_2 - r\mu_2 \sigma_1)(\mu_2 \sigma_1 - r\mu_1 \sigma_2)]$$

letting $r^* = (\frac{\sigma_2}{\mu_2})/(\frac{\sigma_1}{\mu_1})$ and noting that $0 < r^* < 1$ and $-1 < r < r^*$ then

$$\frac{\partial y_2}{\partial \sigma_2} = \frac{-y_1\sigma_1}{\sigma_2^2(\mu_1\sigma_2 - r\mu_2\sigma_1)^2}\left[\mu_2^2\sigma_1^2(r^* - r) + \mu_1^2\sigma_2^2\left(\frac{1}{r^*} - r\right)\right] < 0$$

Thus, increases in σ_2 cause decreased in y_2. Second consider

$$\frac{\partial y_2}{\partial \sigma_1} = \frac{y_1}{\sigma_2(\mu_1\sigma_2 - r\mu_2\sigma_1)^2}\left[(2\mu_2\sigma_1 - r\mu_1\sigma_2)(\mu_1\sigma_2 - r\mu_2\sigma_1) + r\mu_2\sigma_1(\mu_2\sigma_1 - r\mu_1\sigma_2)\right]$$

letting $r^* = \left(\frac{\sigma_2}{\mu_2}\right)/\left(\frac{\sigma_1}{\mu_1}\right)$ and noting that $0 < r^* < 1$ and $-1 < r < r^*$ then

$$\frac{\partial y_2}{\partial \sigma_1^2} = \frac{y_1}{\sigma_2(\mu_1\sigma_2 - r\mu_2\sigma_1)^2}\left[\mu_2\sigma_1^2(r^* - r) + \mu_1^2\sigma_2^2\left(\frac{1}{r^*} - r\right)\right] > 0$$

Thus, increases in σ_1 cause increases in y_2.

<u>Proof of Assertion 8</u>: The proof of the assertion follows from the derivative of the slope of the efficiency frontier.

$$\frac{\partial\left(\frac{dV}{dE}\right)}{\partial \sigma_1} = \frac{4\sigma_1\sigma_2^2(1 - r^2)E}{(\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2)^2}\left[\mu_1^2\sigma_2^2 - r\sigma_1\sigma_2\mu_1\mu_2\right]$$

letting $r^* = \left(\frac{\sigma_2}{\mu_2}\right)/\left(\frac{\sigma_1}{\mu_1}\right)$ and noting $0 < r^* < 1$ and $-1 < r < r^*$ then

$$\frac{\partial\left(\frac{dV}{dE}\right)}{\partial \sigma_1} = \frac{4\sigma_1^2\sigma_2^3\mu_1\mu_2(1-r^2)E(r^*-r)}{(\mu_1^2\sigma_2^2 - 2r\sigma_1\sigma_2\mu_1\mu_2 + \mu_2^2\sigma_1^2)^2} > 0$$

Thus, increases in σ_1 cause the slope of the efficiency frontier to be steeper throughout.

APPENDIX C

FORMS FOR OBTAINING COST AND INCOME DATA

NAME ______________________________

ADDRESS ____________________________

REMARKS: DATE ______________________

I. AVAILABLE RESOURCES

A. Land Available for Crops (acres)

	Class I	Class II	Class III	Total
Owned				
Rented				
Total				

B. Labor Available for Crops (hours per month)

Month	Operator	Family	Hired	Total
January				
February				
March				
April				
May				
June				
July				
August				
September				
October				
November				
December				

C. Irrigation Water (acre inches per month)

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Amount												

D. Operating Captial ($$)

MAXIMUM EXPOSURE

II. CROP INCOME INFORMATION

Crop Name	Price Estimate			Yield Estimate			Land Restrictions		
	Most Pessimistic	Most Likely	Most Optimistic	Most Pessimistic	Most Likely	Most Optimistic	Land Class	Max. Acres	Min. Acres

III. EXPENSE INFORMATION FOR ____________ Name of Crop

A. Labor Required (Hours per acre per month)

Month	Jan	Feb	March	April	May	June	July	Aug	Sept	Oct	Nov	Dec
Hours per Acre												

B. Irrigation Water Requested (Acre inches per acre per month)

Month	Jan	Feb	March	April	May	June	July	Aug	Sept	Oct	Nov	Dec
Acre inches per Acre												

C. Operating Capital Required ($$ per acre and % per month)

Item		Amount	Month of Revenue or Expense in %											
			Jan	Feb	March	April	May	June	July	Aug	Sept	Oct	Nov	Dec
revenue	crop sales													
	Total													
expense	mach. equip													
	fertilizer													
	spray, dust													
	seed													
	supplies													
	Total													
Gross Margin														

APPENDIX D

COMPUTER PROGRAMS

```
      PROGRAM INPUT
C     THIS PROGRAM IS THE FIRST OF THREE  PROGRAMS DESIGNED TO SOLVE THE
C     FARM ENTERPRISE SELECTION UNDER UNCERTAINTY PROBLEM. THIS PROGRAM
C     PREPARES THE DATA FOR INPUT INTO THE SECOND PROGRAM. UNDER CERTAIN
C     OPTIONS REGARDING CORRELATION COEFFICIENTS, YOU WILL NEED THE
C     MASTER CORRELATION MATRIX PREPARED BY PROGRAM CORRELATE AND FILED
C     FOR ACCESS BY THIS PROGRAM.
C
C
C     INSTRUCTIONS FOR SETTING UP INPUT FILE
C
C         SET UP OF CONTROL CARD
C
C              COLUMN 1-2, ENTER M, THE NO. OF CONSTRAINTS MAX 99
C              COLUMN 3-4, ENTER N, THE NO. OF ACTIVITIES  MAX 10
C              COLUMN   5, LEAVE BLANK OR ZERO
C              COLUMN   6, ENTER 1 IF YOU WISH TO USE TRIANGLE DISTN FOR
C                                  PRICE AND YIELD DATA
C                          ENTER 2 IF YOU WISH TO USE MEAN AND VARIANCE
C                                  ESTIMATES FOR PRICE AND YIELD DATA
C                          ENTER 3 IF YOU WISH TO USE GROSS MARGIN DATA
C                                  FOR EACH ACTIVITY
C              COLUMN   7, ENTER BLANK OR ZERO
C              COLUMN   8, ENTER 1 IF YOU WISH TO USE MASTER CORRELATION
C                          ENTER 2 IF YOU WISH TO USE ZERO CORRELATION
C                          ENTER 3 IF YOU WISH TO SUPPLY OWN CORRELATION
C              COLUMN   9, ENTER BLANK OR ZERO
C              COLUMN  10, ENTER 0 (ZERO) IF YOU DO NOT WISH EQUATIONS
C                                  FOR EFFICIENCY FRONTIER AND ACTIVITY
C                                  LEVELS.
C                          ENTER 1 IF YOU WISH THESE EQUATIONS.
C
C         SET UP OF LABEL CARDS
C         YOU MUST HAVE EXACTLY M+N+2 LAVEL CARDS. PREPARE LABEL CARDS
C         FIRST FOR ACTIVITIES, THEN FOR CONSTRAINTS, THEN FOR CLIENT
C         IDENTIFICATION AND ADDRESS. IN SUCCEEDING SECTIONS BE SURE TO
C         FOLLOW EXACTLY THE SAME ORDER AS YOU DO IN LABELS.
C
C              COLUMN  1-13, ENTER NAME OF ACTIVITY OR CONSTRAINT
C              COLUMN    14, ENTER BLANK, DO NOT ENTER ZERO
C              COLUMN 15-16, ENTER UNITS  SUCH AS ACRES, HOURS, ETC.
C              COLUMN 17-20, ENTER BLANK
C              COLUMN 21-22, IF YOU ENTERED 2 OR 3 IN COLUMN 8 OF THE
C                            CONTROL CARD LEAVE BLANKS. IF YOU ENTERED
C                            1 IN COLUMN 8 OF THE CONTROL CARD.THEN YOU
C                            MUST ENTER ACTIVITY IDENTIFICATION AS IT
C                            APPEARS IN MASTER CORRELATION MATRIX.
C              PREPARE LABEL CARDS FOR ACTIVITIES FIRST, THEN FOR THE
C              CONSTRAINTS. YOU SHOULD NOW HAVE M+N CARDS. NOW PREPARE A
C              NAME CARD
C
C              COLUMN  1-16, ENTER NAME OF YOUR CLIENT.
C
C              NOW PREPARE AN ADDRESS CARD
C
C              COLUMN  1-16, ENTER ADDRESS OF YOUR CLIENT.
C
C         THIS COMPLETES THE LABEL CARDS. THE BALANCE OF THE DATA MUST
C         BE ENTERED IN FREE FORM. SEPARATE EACH ENTRY  WITH A COMMA (,)
```

```
C       OR BLANKS.
C
C       SET UP OF PRICE AND YIELD DATA.
C
C            IF YOU ENTERED 1 IN COLUMN 6 OF CONTROL CARD YOU MUST
C            SUPPLY PARAMETERS OF PRICE AND YIELD FOR TRIANGULAR
C            DISTRIBUTION AND AN ESTIMATE FOR VARIABLE COST. FOR YOUR
C            CONVENIENCE USE A SEPARATE CARD FOR EACH ACTIVITY. MAKE
C            SURE THAT YOU PUT CARDS IN EXACTLY SAME ORDER AS LABELS.
C            FOR EACH ACTIVITY ENTER THE REQUIRED DATA IN THE
C            FOLLOWING ORDER, SEPARATING EACH ENTRY BY BLANKS OR COMMA
C
C                      MOST PESSIMISTIC PRICE
C                      MOST LIKELY PRICE
C                      MOST OPTIMISTIC PRICE
C                      MOST PESSIMISTIC YIELD
C                      MOST LIKELY YIELD
C                      MOST OPTIMISTIC YIELD
C                      VARIABLE COST PER UNIT OF ACTIVITY
C
C            IF YOU ENTERED 2 IN COLUMN 6 OF CONTROL CARD YOU MUST
C            SUPPLY MEAN AND VARIANCE ESTIMATES FOR PRICE AND YIELD
C            AND AN ESTIMATE FOR VARIABLE COST. FOR YOUR CONVENIENCE
C            USE A SEPARATE CARD FOR EACH ACTIVITY. MAKE SURE THAT YOU
C            PUT CARDS IN EXACTLY SAME ORDER AS LABELS. FOR EACH
C            ACTIVITY ENTER THE REQUIRED DATA IN THE FOLLOWING ORDER
C            SEPARATING EACH ENTRY BY BLANKS OR COMMA.
C
C                      MEAN PRICE
C                      MEAN YIELD
C                      VARIANCE OF PRICE
C                      VARIANCE OF YIELD
C                      VARIABLE COST PER UNIT OF ACTIVITY
C
C            IF YOU ENTERED 3 IN COLUMN 6 OF CONTROL CARD YOU MUST
C            SUPPLY MEAN AND STANDARD DEVIATIONS OF GROSS MARGIN FOR
C            EACH ACTIVITY. FIRST ENTER THE MEAN GROSS MARGIN FOR
C            EACH ACTIVITY IN EXACTLY SAME ORDER AS ACTIVITIES ARE
C            IN LABEL CARDS.SEPARATING EACH ENTRY BY BLANKS OR COMMA.
C            THEN ENTER STANDARD DEVIATIONS OF GROSS MARGIN OF EACH
C            ACTIVITY IN EXACTLY SAME ORDER AS ACTIVITIES ARE ON LABEL
C            CARDS SEPARATING EACH ENTRY BY BLANKS OR COMMA.
C
C       THE PRICE AND YIELD DATA SHOULD NOW BE COMPLETE. ON A NEW CARD
C       ENTER 9999
C
C       SET UP OF CORRELATION MATRIX.
C
C            IF YOU ENTERED 1 OR 3 IN COLUMN 8 OF CONTROL CARD THE
C            CORRELATION MATRIX IS AUTOMATICALLY PREPARED
C
C            IF YOU ENTERED 2 IN COLUMN 8 OF CONTROL CARD, THEN YOU
C            MUST SUPPLY THE UPPER TRIANGLE OF AN N-DIMENSIONAL
C            CORRELATION MATRIX. ENTER THE ELEMENTS BY ROW. WHEN
C            YOU HAVE ENTERED THE REQUIRED ELEMENTS, ENTER 9999 ON A
C            NEW LINE.
C
C       THIS COMPLETES THE CORRELATION DATA.
C       SET UP OF COEFFICIENT MATRIX AA12.
C            ENTER ACTIVITIES AND RESOURCES IN EXACTLY SAME ORDER AS
C            LABEL CARDS, FOR THE COEFFICIENT MATRIX ACTIVITIES ARE
C            ROWS AND RESOURCES ARE COLUMNS. FOR THE FIRST ACTIVITY
```

```
C             ENTER RESOURCE REQUIREMENTS ON HOWEVER MANY CARDS NEEDED
C             SEPARATING EACH ENTRY BY BLANKS OR COMMA. REPEAT UNTIL
C             ALL ACTIVITIES ARE COMPLETE. YOU SHOULD HAVE AN NXM
C             MATRIX WITH THE RIGHT SQUARE PORTION AN NXN NEGATIVE
C             IDENTITY MATRIX. ON A NEW CARD ENTER 9999
C        THIS COMPLETES THE COEFFICIENT MATRIX
C
C        SET UP OF RESOURCE LEVEL VECTOR GG
C             ENTER ALL OF THE RESOURCE LEVELS. MAKE SURE THAT THE
C             LAST N ELEMENTS ARE EITHER ZERO OR NEGATIVE NUMBERS. WHEN
C             YOU HAVE ENTERED EVERY ELEMENT, ENTER 9999 ON A NEW LINE.
C
C        YOU HAVE NOW ENTERED ALL OF THE DATA. AS A FINAL CHECK MAKE
C        SURE ALL DATA LINES CONFORM TO THE ORDER OF THE LABEL CARDS.
C        NOW FILE THE DATA AND GOOD LUCK
C
C
C
C
C     DIRECTORY OF LOGICAL UNIT NUMBERS
C          LUN 1    = DATA FILE
C          LUN 2    = CORRELATION MATRIX FILE
C          LUN 3    = LETTER FILE
C          LUN 4    = FILE    (STORES INFORMATION FOR PROGRAM OUTPUT)
C          LUN 5    = FILE    (STORES INTORMATION FOR INPUT TO PROGRAM
C                              PROCESS)
C          LUN 6    = PLOT    ( PLOTTER)
C          LUN 34   = LP      (LINE PRINTER)
C          LUN 61   = TELETYPE OUTPUT.
C
C
      DIMENSION PARAM(20,7),RLAB(122,2),MD(122),RR(50,50),
     1CORR(20,20),A11(20,20),AA12(20,100),GG(100),VAR(20,4),AMEAN(20,4)
     2,IDSLK(100),IDAA12(100)
      READ(1,1000) M,N,IHAVE1,IHAVE2,IWANT
      N1=N+1
      N2=N+2
      NM=N+M
      NM1=NM+1
      NM2=NM+2
      WRITE(4) M,N,N1,N2,NM,NM1,NM2,IHAVE1,IHAVE2,IWANT
      WRITE(5) M,N,N1,N2,NM,NM1,NM2,IHAVE2
      DO 10 I=1,NM2
10    READ(1,1001)(RLAB(I,J),J=1,2),MD(I)
      WRITE(4) RLAB
      GO TO(20,30,40),IHAVE1
20    DO 21 I=1,N
      DO 21 J=1,7
21    PARAM(I,J)=FFIN(1)
      KCHECK=FFIN(1)
      IF(KCHECK.NE.9999) GO TO 990
      WRITE(4) PARAM
      DO 22 I=1,N
      AMEAN(I,1)=(PARAM(I,1)+PARAM(I,2)+PARAM(I,3))/3.0
      AMEAN(I,2)=(PARAM(I,4)+PARAM(I,5)+PARAM(I,6))/3.0
      VAR(I,1)=((PARAM(I,3)-PARAM(I,1))**2-(PARAM(I,2)-PARAM(I,1))
     1*(PARAM(I,3)-PARAM(I,2)))/18.0
      VAR(I,2)=((PARAM(I,6)-PARAM(I,4))**2-(PARAM(I,5)-PARAM(I,4))
     1*(PARAM(I,6)-PARAM(I,5)))/18.0
      VAR(I,3)=VAR(I,1)*VAR(I,2)+VAR(I,1)*AMEAN(I,2)**2+VAR(I,2)
     1*AMEAN(I,1)**2
      AMEAN(I,3)=AMEAN(I,1)*AMEAN(I,2)
```

```
      AMEAN(I,4)=AMEAN(I,3)-PARAM(I,7)
22    VAR(I,4)=SQRT(VAR(I,3))
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
      GO TO 44
30    DO 31 I=1,N
      DO 31 J=1,3
31    AMEAN(I,J)=FFIN(1)
      DO 32 I=1,N
      DO 32 J=1,2
32    VAR(I,J)=FFIN(1)
      KCHECK=FFIN(1)
      IF(KCHECK.NE.9999) GO TO 991
      DO 33 I=1,N
      AMEAN(I,4)=AMEAN(I,3)
      AMEAN(I,3)=AMEAN(I,1)*AMEAN(I,2)
      AMEAN(I,4)=AMEAN(I,3)-AMEAN(I,4)
      VAR(I,3)=VAR(I,1)*VAR(I,2)+VAR(I,1)*AMEAN(I,2)**2+VAR(I,2)
     1*AMEAN(I,1)**2
33    VAR(I,4)=SQRT(VAR(I,3))
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
      GO TO 44
40    DO 41 I=1,N
41    AMEAN(I,4)=FFIN(1)
      DO 42 I=1,N
42    VAR(I,4)=FFIN(1)
      KCHECK=FFIN(1)
      IF(KCHECK.NE.9999) GO TO 992
      DO 43 I=1,N
      DO 43 J=1,3
      AMEAN(I,J)=0.0
43    VAR(I,J)=0.0
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
44    CONTINUE
      WRITE(34,1005)
      DO 45 I=1,N
45    WRITE(34,1003) I,(AMEAN(I,J),J=1,4),(VAR(I,J),J=1,4)
      WRITE(4) AMEAN
      WRITE(4) VAR
      GO TO (50,60,70),IHAVE2
50    REWIND 2
      READ(2) RR
      DO 51 I=1,N
      II=MD(I)
      DO 51 J=1,N
      JJ=MD(J)
51    CORR(I,J)=RR(II,JJ)
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
      GO TO 72
60    DO 61 I=1,N
      DO 61 J=I,N
      CORR(I,J)=FFIN(1)
61    CORR(J,I)=CORR(I,J)
      KCHECK=FFIN(1)
      IF(KCHECK.NE.9999) GO TO 993
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
      GO TO 72
70    DO 71 I=1,N
      DO 71 J=1,N
      CORR(I,J)=0.0
71    IF(I.EQ.J) CORR(I,J)=1.0
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
72    CONTINUE
```

```
      WRITE(34,1006)
      DO 104 I=1,N
104   WRITE(34,1003) I,(CORR(I,J),J=1,N)
      WRITE (4) CORR
      DO 80 I=1,N
      DO 80 J=1,N
80    CORR(I,J)=CORR(I,J)*VAR(J,4)
      DO 81 J=1,N
      DO 81 I=1,N
81    CORR(I,J)=CORR(I,J)*VAR(I,4)
      DO 82 I=1,N
      DO 82 J=1,N
82    A11(I,J)=2.0*CORR(I,J)
      DO 83 I=1,N
83    A11(I,N1)=AMEAN(I,4)
      DO 84 J=1,N
84    A11(N1,J)=AMEAN(J,4)
      A11(N1,N1)=0.0
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
      WRITE(34,1007)
      DO 105 I=1,N1
105   WRITE(34,1003) I,(A11(I,J),J=1,N1)
      WRITE(4) A11
      WRITE(5) A11
      DO 90 I=1,N
      DO 90 J=1,M
90    AA12(I,J)=FFIN(1)
      KCHECK=FFIN(1)
      IF(KCHECK.NE.9999) GO TO 994
C     IF DESIRED A ERITE STATEMENT CAN GO HERE
      WRITE(34,1002)
      DO 106 J=1,M
106   WRITE(34,1003) J,(AA12(I,J),I=1,N)
      WRITE(4) AA12
      WRITE(5) AA12
      DO 91 I=1,M
91    GG(I)=FFIN(1)
      KCHECK=FFIN(1)
      IF(KCHECK.NE.9999) GO TO 995
C     IF DESIRED A WRITE STATEMENT CAN GO HERE
      WRITE(34,1009)
      DO 107 I=1,M
107   WRITE(34,1003) I,GG(I)
      WRITE(4) GG
      WRITE(5) GG
      K=0
      IF(IHAVE2.EQ.3) GO TO 108
      MN=M-N
      MN1=MN+1
      DO 109 I=1,MN
109   IDAA12(I)=I
      ACV=9999999.
      DO 110 I=1,N
      IF(VAR(I,4)/AMEAN(I,4).GE.ACV) GO TO 110
      ACV=VAR(I,4)/AMEAN(I,4)
      MIN=I
110   CONTINUE
      DO 111 I=1,N
      IF(MIN.NE.I) GO TO 112
      IDAA12(MN1)=I+MN
      GO TO 111
112   K=K+1
```

```
      IDSLK(K)=I+MN
111   CONTINUE
      WRITE(5) K
      WRITE(5)IDAAI2
      WRITE(5) IDSLK
      MK=M-K
      WRITE(61,1017) K
      WRITE(61,1017) (IDAAI2(I),I=1,MK)
      WRITE(61,1017) (IDSLK(I),I=1,K)
1017  FORMAT(/10I4/10I4/10I4/10I4)
108   WRITE(61,1004)(RLAB(NM1,J),J=1,2)
      GO TO 999
990   WRITE(61,1010)
      GO TO 998
991   WRITE(61,1011)
      GO TO 998
992   WRITE(61,1012)
      GO TO 998
993   WRITE(61,1013)
      GO TO 998
994   WRITE(61,1014)
      GO TO 998
995   WRITE(61,1015)
      GO TO 998
998   WRITE(61,1016)
      GO TO 999
1000  FORMAT(5I2)
1001  FORMAT(2A8,4X,I2)
1002  FORMAT(≠1 THE INPUT MATRIX AA12≠)
1003  FORMAT(1X,I2,10F12.2)
1004  FORMAT(1X,2A8≠YOUR INPUT IS PREPARED≠)
1005  FORMAT(≠1THE MEANS AND VARIANCES≠)
1006  FORMAT(≠1THE COVARIANCE MATRIX≠)
1007  FORMAT(≠1THE INPUT MATRIX, AA11 ≠)
1008  FORMAT(≠1 THE ORIGINAL INPUT DATA≠)
1009  FORMAT(≠1 THE INPUT MATRIX GG≠)
1010  FORMAT(≠ ERROR IN THE INPUT OF THE PARAMETERS OF THE≠
     1≠ TRIANGULAR DISTRIBUTION≠)
1011  FORMAT(≠ ERROR IN THE YIELD AND PRICE PARAMETER INPUT≠)
1012  FORMAT(≠ ERROR IN THE GROSS INCOME INPUT≠)
1013  FORMAT(≠ ERROR IN THE CORRELATION COEFFICIENT INPUT≠)
1014  FORMAT(≠ ERROR IN THE PRODUCTION COEFFICIENT INPUT≠)
1015  FORMAT(≠ ERROR IN THE AVAILABLE RESOURCES INPUT≠)
1016  FORMAT(≠ CALCULATION NOT COMPLETED, CHECK THE INDICATED DATA≠)
999   CALL EXIT
      END
```

```
      PROGRAM PROCESS
      COMMON A11,A12,AA12,GG,G,RG,S,IDSLK,IDAA12,K,K1,MK,MK1,
     1B11K,B11KK,
     1N,M,
     1N1,N2,NK1,NK2,NM1,NK,SMAX,SMIN,IMAX,IMIN
      EQUIVALENCE (A11(1,1),B11(1))
     1,(AA12(1,1),C(1,1))
      DIMENSION A11(20,20),L(20),MM(20),B(20),A12(20,100),
     1B11(1),B11K(20,20),C(20,100),B11KK(20,20)
     2,G(100),RG(121),S(121)
     3,IN(100),IDAA12(100),IDSLK(100)
     4,AA12(20,100),GG(100)
     6,OUT1(7),OUT2(21,3),OUT3(100,4)
     5,R(121),ACT(121)
     7,IDSLKB(100),IDAB12(100)
C  READING OF ORIGINAL DATA  **************************
      READ(5) M,N,N1,N2,NM,NM1,NM2,IHAVE2
      READ(5) A11
      READ(5) AA12
      READ(5) GG
      SSMIN=0.0
      ICOUNT=0
      K=0
      IF(IHAVE2.EQ.3) GO TO 6200
      READ(5) K
      READ(5) IDAA12
      READ(5) IDSLK
      GO TO 6300
6200  DO 2 I=1,M
      IDSLK(I)=0
2     IDAA12(I)=I
6300  ISTEP=0
      SSMIN=0.
6000  K1=K+1
      MK=M-K
      MK1=M-K-1
      N1=N+1
      N2=N+2
      NK1=N+K+1
      NK2=N+K+2
      NM1=N+M+1
      NK=N+K
      CALL COMPUT
      CONTINUE
      IF(SMIN-SMAX) 2101,2101,203
204   WRITE(34,1013) ISTEP
C ERROR MESSAGE
1013  FORMAT(≠1 SMAX IS GREATER THAN SMIN DURING STEP≠ I3)
      CALL EXIT
203   ISTEP=ISTEP+1
      OUT1(1)=SMIN
      OUT1(2)=SMAX
      OUT1(3)=-.5*B11K(1,N1)
      OUT1(4)=0.
      DO 205 I=1,K
      II=I+N1
205   OUT1(4)=-B11K(1,II)*G(I)+OUT1(4)
      OUT1(5)=0.
      DO 206 I=1,K
      W=0.
```

```
      DO 207 J=1,K
      JJ=J+N1
207   W=G(J)*B11K(I+1,JJ)+W
      W=W*G(I)
206   OUT1(5)=W+OUT1(5)
      OUT1(5)=-.5*OUT1(5)
      OUT1(6)=OUT1(3)*OUT1(1)**2+OUT1(4)*OUT1(1)+OUT1(5)
      OUT1(7)=SQRT(OUT1(6))
      DO 208 I=1,N1
      OUT2(I,1)=S(I)
      OUT2(I,2)=RG(I)
208   OUT2(I,3)=OUT2(I,1)*OUT1(1)+OUT2(I,2)
      DO 209 I=N2,NK1
      II=I-N1
      JJ=IDSLK(II)
      OUT3(JJ,1)=S(I)
      OUT3(JJ,2)=RG(I)
      OUT3(JJ,3)=0.0
209   OUT3(JJ,4)=OUT3(JJ,1)*OUT1(1)+OUT3(JJ,2)
      DO 210 I=NK2,NM1
      II=I-NK1
      JJ=IDAA12(II)
      OUT3(JJ,1)=S(I)
      OUT3(JJ,2)=RG(I)
      OUT3(JJ,3)=OUT3(JJ,1)*OUT1(1)+OUT3(JJ,2)
210   OUT3(JJ,4)=0.0
      WRITE(4) ISTEP
      WRITE(4) OUT1
      WRITE(4) OUT2
      WRITE(4) OUT3
      WRITE(61,9000) ISTEP,SMIN
9000  FORMAT(≠ STEP≠ I3 ≠F IS≠F20.2)
2101  IF(N.EQ.K1.AND.IMIN.GT.NK1)211,212
212   KNO=0
      CALL SELECT(IMIN,IMIN,KNO)
      GO TO 6000
211   DO 213 I=1,K
      JJ=I+N1
213   ACT(I)=S(JJ)*SMIN+RG(JJ)
      DO 214 I=1,K
      DO 214 J=I,K
      IF(ACT(I).LE.ACT(J))214,215
215   SAVE=ACT(I)
      ACT(I)=ACT(J)
      ACT(J)=SAVE
      SAVE=IDSLK(I)
      IDSLK(I)=IDSLK(J)
      IDSLK(J)=SAVE
214   CONTINUE
      SSMIN=SMIN
      JMINS=IMIN
      DO 2155 J=1,K
2155  IDSLKB(J)=IDSLK(J)
      DO 2152 J=1,MK
2152  IDAB12(J)=IDAA12(J)
      DO 2151 I=1,K
      DO 2153 J=1,K
2153  IDSLK(J)=IDSLKB(J)
      DO 2154 J=1,MK
2154  IDAA12(J)=IDAB12(J)
      KNO=I
      II=I+N1
```

```
      JMIN=JMINS
      CALL SELECT(II,JMIN,KNO)
      CALL COMPUT
      IF (SMIN.GE.SMAX)216,2151
216   IF(SMIN.GE.SSMIN)203,2151
2151  CONTINUE
      ISTEP=9999
      WRITE(4) ISTEP
      REWIND 4
      CALL EXIT
      END

C     SUBROUTINE COMPUTE IS TO BE INSERTED HERE
      SUBROUTINE COMPUT
      COMMON A11,A12,AA12,GG,G,RG,S,IDSLK,IDAA12,K,K1,MK,MK1,
     1B11K,B11KK,
     1N,M,
     1N1, N2,NK1,NK2,NM1,NK,SMAX,SMIN,IMAX,IMIN
      EQUIVALENCE (A11(1,1),B11(1)),(AA12(1,1),C(1,1))
      DIMENSION A11(20,20),L(20),MM(20),B(20),A12(20,100),
     1B11(1),B11K(20,20),C(20,100),B11KK(20,20)
     2,G(100),RG(121),S(121)
     3,IN(100),IDAA12(100),IDSLK(100)
     4,AA12(20,100),GG(100), RLAB(122,2)
     5,R(121),ACT(121)
      REWIND 5
      READ(5) M,N,N1,N2,NM,NM1,NM2,IHAVE2
      READ(5) A11
      READ(5) AA12
      READ(5) GG
C     ADD CONSTRAINTS TO A11
      IDN=N1
      DO 1 J=1,K
      JJ =IDSLK(J)
      DO 2 I=1,N
      A11(I,IDN+1)=AA12(I,JJ)
2     A11(IDN+1,I)=AA12(I,JJ)
1     IDN=IDN+1
      DO 3 I=N1,NK1
      DO 3 J=N1,NK1
3     A11(I,J)=0.
C     SET UP A12
      DO 4 J=1,MK
      JJ=IDAA12(J)
      DO 4 I=1,N
4     A12(I,J)=AA12(I,JJ)
      DO 5 I=N1,NK1
      DO 5 J=1,MK
5     A12(I,J)=0.
C     SET UP G
      DO 6 I=1,K
      II=IDSLK(I)
6     G(I)=GG(II)
      IK=1
      DO 7 I=K1,M
10    CONTINUE
      DO 8 J=1,K
      IF(IDSLK(J)-IK) 8,9,8
9     IK=IK+1
      GO TO 10
8     CONTINUE
      G(I)=GG(IK)
```

```
 7      IK=IK+1
        CALL ARRAY (2,NK1,NK1,20,20,A11,A11)
        CALL ARRAY (2,NK1,MK,20,100,A12,A12)
        CALL MINV (B11,NK1,DET,L,MM)
 C      MESSAGE FOR SINGULAR MATRIX
        IA=1
        DO 11 J=1,NK1
        DO 11 I=N1,NK1
        IR=NK1*(J-1)+I
        B11K(IA)=B11(IR)
 11     IA=IA+1
        C1=-1.0
        CALL SMPY(B11K,C1,B11KK,K1,NK1,0)
        CALL MPRD(B11KK,A12,C,K1,NK1,0,0,MK)
        CALL ARRAY(1,K1,NK1,20,20,B11K,B11K)
        CALL ARRAY(1,K1,MK,20,100,C,C)
        DO 12 J=1,NK1
        RG(J)=0.
        S(J)=B11K(1,J)
        DO 12 I=1,K
 12     RG(J)=RG(J)+B11K(I+1,J)*G(I)
        DO 13 J=NK2,NM1
        JJ=J-N-1
        RG(J)=G(JJ)
        JJ=J-NK1
        S(J)=C(1,JJ)
        DO 13 I=1,K
 13     RG(J)=RG(J)+C(I+1,JJ)*G(I)
        DO 14 I=1,NM1
        IF(S(I)) 15,16,17
 15     R(I)=-RG(I)/S(I)
 C  E MUST BE LESS THAN R(I) !!!!!!!!!!!!!!!!!!!!!!!!!!!!
        IN(I)=1
        GO TO 14
 16     R(I)=9999999.
 C  VALID FOR ALL E**********************************!
        IN(I)=2
        GO TO 14
 17     R(I)=-RG(I)/S(I)
 C  E MUST BE GREATER THAN R(I)******************************!
        IN(I)=3
 14     CONTINUE
        IMAX=0
        IMIN=0
        SMAX=-9999999.
        SMIN=9999999.
        DO 18 I=N2,NM1
        IF(IN(I)-2)19,18,20
 19     IF(R(I)-SMIN)21,18,18
 21     SMIN=R(I)
        IMIN=I
        GO TO 18
 20     IF(R(I)-SMAX)18,18,22
 22     SMAX=R(I)
        IMAX=I
 18     CONTINUE
        WRITE(61,9000) SMIN,SMAX
 9000   FORMAT(≠ FROM COMPUT SMIN ! SMAX ≠2F20.2    )
        RETURN
        END
```

```
C     SUBROUTINE SELECT IS TO BE INSERTED HERE
      SUBROUTINE SELECT(JMIN,IMIN,KNO)
      COMMON A11,A12,AA12,GG,G,RG,S,IDSLK,IDAA12,K,K1,MK,MK1,
     1B11K,B11KK,
     1N,M,
     1N1,N2,NK1,NK2,NM1,NK,SMAX,SMIN,IMAX

      DIMENSION A11(20,20),L(20),MM(20),B(20),A12(20,100),
     1B11(1),B11K(20,20),C(20,100),B11KK(20,20),
     2G(100),RG(121),S(121),
     3IN(100),IDAA12(100),IDSLK(100),
     4AA12(20,100),GG(100),
     5R(121),ACT(121)
      EQUIVALENCE(A11(1,1),B11(1)),(AA12(1,1),C(1,1))
      IF(IMIN.GT.NK1)10,1
10    JJ=IMIN-NK1
      IDSLK(K+1)=IDAA12(JJ)
      DO 11 I=JJ,MK
11    IDAA12(I)=IDAA12(I+1)
      K=K+1
      IF(KNO.EQ.1)7,20
7     IMIN=JMIN
1     JJ=IMIN-N1
      II=MK+1-KNO
      DO 2 I=1,MK
      IF(IDAA12(II-1).GT.IDSLK(JJ))3,4
3     IDAA12(II)=IDAA12(II-1)
      II=II-1
      GO TO 2
4     IDAA12(II)=IDSLK(JJ)
      GO TO 5
2     CONTINUE
5     CONTINUE
      DO 6 I=JJ,K
6     IDSLK(I)=IDSLK(I+1)
      K=K-1
20    CONTINUE
      KM=M-K
C     WRITE(61,9001)(IDAA12(I),I=1,KM)
      WRITE(61,9001)(IDSLK(I),I=1,K)
9001  FORMAT(/10I3/10I3/10I3)
      RETURN
      END
         FINIS
```

```
      PROGRAM OUTPUT
      COMMON IARRY(12),ARRAY(22),LABELS(63),N,KSTEP,MINC3,MAXC1,MAXC2,
     1     EMAX,OUT1,OUT2,OUT3,NINT
      REAL MINC3,MAXC1,MAXC2
      COMMON/DATA/CONST(7)
      DIMENSION OUT1(7),OUT2(21,3),OUT3(100,4),PART1(122,10),PART2(7)
     1,RLETTER(50,10),RLAB(122,2)
     2,PARAM(20,7),AMEAN(20,4),VAR(20,4),CORR(20,20),
     3A11(20,20),AA12(20,100),GG(100)
      EQUIVALENCE (PART1(1,1),PART2(1))
      READ(4) M,N,N1,N2,NM,NM1,NM2,IHAVE1,IHAVE2,IWANT
      READ(4) RLAB
      IF(IHAVE1.GT.1) GO TO 9
      READ(4) PARAM
9     READ(4) AMEAN
      READ(4) VAR
      READ(4) CORR
      READ(4) A11
      READ(4) AA12
      READ(4) GG
      REWIND 3
      IPAGE=1
      READ(3,10005)((RLETTER(I,J),J=1,10),I=1,50)
      WRITE(34,10001)(RLAB(NM1,J),J=1,2),IPAGE
      IPAGE=IPAGE+1
      WRITE(34,10002)(RLAB(NM2,J),J=1,2)
      WRITE(34,10003)(RLAB(NM1,J),J=1,2)
      WRITE(34,10006)
      IF(IWANT.EQ.1) 10,11
10    WRITE(34,10000)((RLETTER(I,J),J=1,10),I=1,48)
      GO TO 12
11    WRITE(34,10000)((RLETTER(I,J),J=1,10),I=1,35)
      WRITE(34,10000)(RLETTER(50,J),J=1,10)
      WRITE(34,10000)((RLETTER(I,J),J=1,10),I=41,48)
12    CONTINUE
C     PREPARATION OF PART ONE
      MINC3=MAXC1=MAXC2=0
      KCC=1
      KC=0
      REWIND 5
100   READ(4) ISTEP
      IF(ISTEP.EQ.9999.AND.I.EQ.9) GO TO 113
      IF(ISTEP.EQ.9999) 104,101
101   I=ISTEP-KC
      KSTEP=ISTEP
      READ(4) OUT1
      READ(4) OUT2
      READ(4) OUT3
      WRITE(5) ISTEP
      WRITE(5) OUT1
      WRITE(5) OUT2
      WRITE(5) OUT3
      MAXC2=OUT1(7)
      EMAX=OUT1(1)
      DO 102 J=1,N
      IF(OUT2(J,3).GT.MAXC1)MAXC1=OUT2(J,3)
102   PART1(J,I)=OUT2(J,3)
      DO 103 J=N1,NM
      JJ=J-N
103   PART1(J,I)=OUT3(JJ,3)
```

```
      PART1(NM1,I)=OUT1(1)
      PART1(NM2,I)=OUT1(7)
      DO 1040 J=N1,NM
      JJ=J-N
      IF(GG(JJ)) 1041,1041,1042
 1041 PART1(J,I)=GG(JJ)+PART1(J,I)
      GO TO 1040
 1042 PART1(J,I)=GG(JJ)-PART1(J,I)
 1040 CONTINUE
      DO 1046 J=1,M
      JJJ=NM2+J
 1046 PART1(JJJ,I)=-OUT3(J,4)/OUT2(N1,3)
      IF(I.EQ.9) 104,100
  104 II=I
      KP=KC+1
      LP=KC+II
      KC=KC+9
      IF(KCC.EQ.1) 105,106
  105 WRITE(34,1009) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,1011)
      KCC=KCC+1
      GO TO 107
  106 WRITE(34,1010) IPAGE
      IPAGE=IPAGE+1
  107 WRITE(34,1012)
      WRITE(34,1001)
      WRITE(34,1003) (IP,IP=KP,LP)
      WRITE(34,1004)
      WRITE(34,1002)
      DO 108 J=1,N
  108 WRITE(34,1006)(RLAB(J,JJ),JJ=1,2),(PART1(J,I),I=1,II)
      WRITE(34,1002)
      WRITE(34,1007)(PART1(NM1,I),I=1,II)
      WRITE(34,1008)(PART1(NM2,I),I=1,II)
      WRITE(34,1000)
      WRITE(34,1010) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,1013)
      WRITE(34,1001)
      WRITE(34,1003)(IP,IP=KP,LP)
      WRITE(34,1005)
      WRITE(34,1002)
      LCC=0
      LC=0
      DO 109 J=N1,NM
      JN=J+N
      LCC=LCC+1
      LC=LC+1
      IF(LCC.EQ.40) 110,111
  110 LCC=0
      LC=0
      WRITE(34,1000)
      WRITE(34,1010) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,1014)
      WRITE(34,1001)
      WRITE(34,1003)(IP,IP=KP,LP)
      WRITE(34,1005)
      WRITE(34,1002)
      GO TO 109
  111 IF(LC.EQ.5) 112,109
```

```
112    LC=0
       WRITE(34,1002)
109    WRITE(34,1006)(RLAB(J,JJ),JJ=1,2),(PART1(J,I),I=1,II)
       WRITE(34,1002)
       WRITE(34,1007)(PART1(NM1,I),I=1,II)
       WRITE(34,1008)(PART1(NM2,I),I=1,II)
       WRITE(34,1000)
       WRITE(34,1010) IPAGE
       IPAGE=IPAGE+1
       WRITE(34,1020)
       WRITE(34,1001)
       WRITE(34,1003)(IP,IP=KP,LP)
       LCC=-1
       LC=-1
       DO 115 J=1,M
       JN=J+N
       JJJ=NM2+J
       LCC=LCC+1
       LC=LC+1
       IF(LCC.LT.40) GO TO 116
       LCC=0
       LC=0
       WRITE(34,1000)
       WRITE(34,1010) IPAGE
       IPAGE=IPAGE+1
       WRITE(34,1021)
       WRITE(34,1001)
       WRITE(34,1003)(IP,IP=KP,LP)
       WRITE(34,1005)
       WRITE(34,1002)
       WRITE(34,1000)
       GO TO 115
116    IF(LC.LT.5) GO TO 115
       LC=0
       WRITE(34,1002)
115    WRITE(34,1006)(RLAB(JN,JJ),JJ=1,2),(PART1(JJJ,I),I=1,II)
       WRITE(34,1002)
       WRITE(34,1007)(PART1(NM1,I),I=1,II)
       WRITE(34,1008)(PART1(NM2,I),I=1,II)
       WRITE(34,1000)
1020   FORMAT(≠0 A STATEMENT OF THE VALUE OF AN ADDITIONAL UNIT OF ≠
      1≠ RESOURCE≠)
1021   FORMAT(≠0 A STATEMENT OF THE VALUE OF AN ADDITIONAL UNIT OF ≠
      1≠ RESOURCE CONTINUED≠)
       IF(ISTEP.EQ.9999) 113,100
113    CONTINUE
       WRITE(61,12000)
C      PREPARATION OF PART TWO
       REWIND 5
       WRITE(34,2000) IPAGE
       IPAGE=IPAGE+1
       WRITE(34,2003)
       WRITE(34,2009)
       WRITE(34,2005)
       WRITE(34,2006)
       WRITE(34,2007)
       WRITE(34,2010)
       WRITE(34,2011)
       LC=0
       LCC=0
       DO 200 I=1,KSTEP
       READ(5) ISTEP
```

```
      READ(5) OUT1
      READ(5) OUT2
      READ(5) OUT3
      PART2(1)=OUT1(1)-2.3267*OUT1(7)
      IF(PART2(1).LT.MINC3) MINC3=PART2(1)
      PART2(2)=OUT1(1)-1.6450*OUT1(7)
      PART2(3)=OUT1(1)-1.2817*OUT1(7)
      PART2(4)=OUT1(1)-0.8418*OUT1(7)
      PART2(5)=OUT1(1)-0.5310*OUT1(7)
      PART2(6)=OUT1(1)-0.2510*OUT1(7)
      PART2(7)=OUT1(1)-0.0000*OUT1(7)
      LCC=LCC+1
      LC=LC+1
      IF(LCC.EQ.40) 201,202
201   LCC=0
      LC=0
      WRITE(34,2001) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,2004)
      WRITE(34,2009)
      WRITE(34,2005)
      WRITE(34,2006)
      WRITE(34,2007)
      WRITE(34,2010)
      WRITE(34,2011)
      GO TO 200
202   IF(LC.EQ.5) 203,200
203   LC=0
      WRITE(34,2011)
200   WRITE(34,2008) I,OUT1(1),(PART2(J),J=1,7)
      WRITE(34,2010)
C     DOES THE CLIENT WANT PART THREE
      IF(IWANT.EQ.1) 300,400
C     PREPARATION OF PART THREE
300   REWIND 5
      WRITE(61,12001)
      DO 301 I=1,KSTEP
      READ(5) ISTEP
      READ(5) OUT1
      READ(5) OUT2
      READ(5) OUT3
      IF(I.EQ.1) 302,303
302   WRITE(34,3000) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,3002)
      GO TO 304
303   WRITE(34,3001) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,3003)
304   WRITE(34,3004) I
      WRITE(34,3005) OUT1(2),OUT1(1)
      WRITE(34,3006) OUT1(1)
      WRITE(34,3007)
      WRITE(34,3011)
      WRITE(34,3012)
      WRITE(34,3013)
      WRITE(34,3014)(OUT1(J),J=3,7)
      WRITE(34,3011)
      WRITE(34,3008)
      WRITE(34,3015)
      WRITE(34,3016)
      WRITE(34,3011)
```

```
      WRITE(34,3019)
      LC=0
      DO 305 J=1,N
      LC=LC+1
      IF(LC.EQ.5) 306,305
306   LC=0
      WRITE(34,3019)
305   WRITE(34,3020)J,(RLAB(J,JJ),JJ=1,2),(OUT2(J,JJ),JJ=1,3)
      WRITE(34,3011)
      WRITE(34,3009) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,3017)
      WRITE(34,3018)
      WRITE(34,3011)
      WRITE(34,3019)
      WRITE(34,3021)(OUT2(N1,JJ),JJ=1,3)
      LC=1
      LCC=0
      DO 307 J=1,M
      JN=J+N
      LC=LC+1
      LCC=LCC+1
      IF(LCC.EQ.40) 308,309
308   LCC=0
      LC=0
      WRITE(34,3010) IPAGE
      IPAGE=IPAGE+1
      WRITE(34,3011)
      WRITE(34,3017)
      WRITE(34,3018)
      WRITE(34,3011)
      WRITE(34,3019)
      GO TO 307
309   IF(LC.EQ.5) 310,307
310   LC=0
      WRITE(34,3019)
307   WRITE(34,3020) J,(RLAB(JN,JJ),JJ=1,2),(OUT3(J,JJ),JJ=1,4)
      WRITE(34,3011)
301   CONTINUE
400   CONTINUE
      WRITE(61,12002)
C     THE PLOTTING ROUTINE FITS HERE
      CALL PLOT
1000  FORMAT(≠ ≠135(≠-≠))
1001  FORMAT(≠0≠135(≠-≠))
1002  FORMAT(≠ I                    I≠9(≠                I≠))
1003  FORMAT(≠ I NAME OF     UNITI≠9(≠   PLAN ≠I3≠   I≠))
1004  FORMAT(≠ I  CROP              I≠9(≠                I≠))
1005  FORMAT(≠ I RESOURCE           I≠9(≠                I≠))
1006  FORMAT(≠ I≠2A8≠I≠9(F11.2≠ I≠))
1007  FORMAT(≠ I EXP GR MARG   $$I≠9(F11.2≠ I≠))
1008  FORMAT(≠ I STD DEV       $$I≠9(F11.2≠ I≠))
1009  FORMAT(≠1PART ONE≠118(≠ ≠)≠PAGE ≠I3)
1010  FORMAT(≠1PART ONE CONTINUED≠108(≠ ≠)≠PAGE ≠I3)
1011  FORMAT(≠0ASUMMARY OF EFFICIENT FARM PLANS≠)
1012  FORMAT(≠0A STATEMENT OF THE LEVELS OF ACTIVITIES AND THE EXPECTED
     1PAYOFF≠)
1013  FORMAT(≠0A STATEMENT OF THE AMOUNT OF EACH RESOURCE USED AND THE
     1EXPECTED PAYOFF≠)
1014  FORMAT(≠0A STATEMENT OF THE AMOUNT OF EACH RESOURCE USED AND THE
     1EXPECTED PAYOFF CONTINUED≠)
2007  FORMAT(≠0IPLANI EXP GR MAR I          1% I          5% I          10%
```

```
     1I         20% I          30% I          40% I          50% I≠)
2009  FORMAT(≠0≠110(≠-≠))
2010  FORMAT(≠ ≠110(≠-≠))
2000  FORMAT(≠1PART TWO≠94(≠ ≠)≠PAGE ≠I3)
2001  FORMAT(≠1PART TWO CONTINUED≠84(≠ ≠)≠PAGE ≠I3)
2008  FORMAT(≠ I ≠I3≠I≠8(F11.2≠ I≠))
2003  FORMAT(≠0PROBABILITY STATEMENTS ABOUT ATTAINING SPECIFIED LEVELS O
     1F ACTUAL GROSS MARGIN FOR A GIVEN LEVEL OF EXPECTED GROSS MARGIN≠)
2004  FORMAT(≠0PROBABILITY STATEMENTS CONTINUED≠)
2005  FORMAT(≠ I     I                  I≠26(≠ ≠)≠PROBABILITY LEVEL≠47(≠ ≠)≠I≠)
2006  FORMAT(≠ I     I                  I≠90(≠-≠)≠I≠)
2011  FORMAT(≠ I     I≠8(≠              I≠))
3000  FORMAT(≠1PART THREE≠117(≠ ≠)≠PAGE ≠I3)
3001  FORMAT(≠1PART THREE CONTINUED≠107(≠ ≠)≠PAGE ≠I3)
3002  FORMAT(≠0DETAILED DESCRIPTION OF EFFICIENT PLANS IN EQUATION FORM
     1≠)
3003  FORMAT(≠0DETAILED DESCRIPTION OF EFFICIENT PLANS IN EQUATION FORM
     1CONTINUED≠)
3004  FORMAT(≠0THIS PLAN WAS GENERATED DURING STEP  ≠I3)
3005  FORMAT(≠ IT IS VALID FOR VALUES OF EXP GR MARG FROM≠F26.2≠TO≠F26.2
     1)
3006  FORMAT(≠0ALL EQUATIONS PERTAINING TO THIS PLAN ARE EVALUATED AT EX
     1P GR MARG =≠F26.2)
3007  FORMAT(≠0THE VARIANCE EQUATION≠)
3008  FORMAT(≠0THE ACTIVITY EQUATIONS≠)
3009  FORMAT(≠1THE RESOURCE EQUATIONS≠100(≠ ≠)≠PAGE ≠I3)
3010  FORMAT(≠1THE RESOURCE EQUATIONS CONTINUED≠90(≠ ≠)≠PAGE ≠I3)
3011  FORMAT(≠0     ≠131(≠-≠))
3012  FORMAT(≠     I≠15(≠ ≠)≠ALPHA1    I≠15(≠ ≠)≠ALPHA2    I≠
     115(≠ ≠)≠ALPHA3    I≠16(≠ ≠)≠VARIANCE I≠17(≠ ≠)≠STD DEV I≠)
3013  FORMAT(≠     I≠5(≠                              I≠))
3014  FORMAT(≠     I≠3(F24.6≠ I≠),2(F24.2≠ I≠))
3015  FORMAT(≠ I NO     OF   I NAME  OF   UNITI≠16(≠ ≠)≠BETA1     I≠
     116(≠ ≠)≠BETA2    I≠16(≠ ≠)≠LEVEL OF I≠)
3016  FORMAT(≠ I ACTIVITY    I ACTIVITY       I≠25(≠ ≠)≠I≠25(≠ ≠)≠I≠
     116(≠ ≠)≠ACTIVITY I≠)
3017  FORMAT(≠ I NO       OF I NAME    OF UNITI≠16(≠ ≠)≠BETA1     I≠
     116(≠ ≠)≠BETA2     I≠14(≠ ≠)≠LEVEL   OF I≠14(≠ ≠)≠VALUE    ≠
     2≠OF I≠)
3018  FORMAT(≠ I CONSTRAINT I CONSTRAINT       I≠25(≠ ≠)≠I≠25(≠ ≠)≠I≠
     114(≠ ≠)≠CONSTRAINT I≠14(≠ ≠)≠LAGRANGIAN I≠)
3019  FORMAT(≠ I≠12(≠ ≠)≠I≠16(≠ ≠)≠I≠25(≠ ≠)≠I≠25(≠ ≠)≠I≠25(≠ ≠)≠I≠
     125(≠ ≠)≠I≠)
3020  FORMAT(≠ I        ≠I3≠   I≠2A8≠I≠2(F24.6≠ I≠),2(F24.2≠ I≠))
3021  FORMAT(≠ I           0    IEXP GR MARG    $$I≠
     1F24.6≠ I≠F24.6≠ I                            I≠F24.2≠ I≠)
10000 FORMAT(≠ ≠10A8)
10001 FORMAT(≠1MR. ≠2A8,50(≠ ≠)≠PAGE ≠I3)
10002 FORMAT(≠    ≠2A8)
10003 FORMAT(≠0DEAR MR. ≠2A8)
10005 FORMAT(10A8)
10006 FORMAT(≠0≠)
12000 FORMAT(≠ YOU ARE NOW GOING INTO PART TWO ≠)
12001 FORMAT(≠ YOU ARE NOW GOING INTO PART THREE ≠)
12002 FORMAT(≠ YOU HAVE NOW COMPLETED PART THREE AND THE REPORT≠)
      CALL EXIT
      END
      SUBROUTINE PLOT
      COMMON IARRY(12),ARRAY(22),LABELS(63),N,KSTEP,MINC3,MAXC1,MAXC2,
     1 EMAX,OUT1(7),OUT2(21,3),OUT3(100,4),NINT
      DIMENSION RLABEL(30)
      EQUIVALENCE (LABELS,RLABEL)
```

```
      REAL MINC3,MAXC1,MAXC2
      COMMON/DATA/CONST(7)
      DATA (CONST=-2.3267,-1.6450,-1.2817,-.8418,-.5310,-.2510,-.0000)
      DIMENSION ESUB(21),STD(21),E(101),A(101,20),P(21,7)
      REWIND 5
      NINT=20
      NINT1=NINT+1
C
C         INITIALIZATION FOR CHART 2
C
      IARRY(1)=IARRY(4)=1
      IARRY(3)=IARRY(9)=0
      IARRY(6)=1
      IARRY(7)=3
      IARRY(2)=1
      IARRY(5)=16
      IARRY(8)=2
      IARRY(10)=6
      IARRY(11)=IARRY(12)=18
      RLABEL(11)=8HEXPECTED
      RLABEL(12)=8H GROSS M
      RLABEL(13)=8HARGIN IN
      RLABEL(14)=6H $1000
      LABELS(62)=30
      RLABEL(21)=8HTHE EFFI
      RLABEL(22)=8HCIENCY F
      RLABEL(23)=7HRONTIER
      LABELS(63)=23
      RLABEL(1)=8HSTANDARD
      RLABEL(2)=8H DEVIATI
      RLABEL(3)=8HON OF GR
      RLABEL(4)=8HOSS MARG
      RLABEL(5)=8HIN IN $1
      LABELS(11)=3H000
      LABELS(61)=43
      IF (EMAX.GT.20.) GO TO 5
      ARRAY(7)=ARRAY(11)=ARRAY(15)=ARRAY(8)=ARRAY(12)=ARRAY(16)=.5
      GO TO 40
5     IF (EMAX.GT.50.) GO TO 7
      ARRAY(7)=ARRAY(11)=ARRAY(15)=ARRAY(8)=ARRAY(12)=ARRAY(16)=1.
      LABELS(62)=26
      RLABEL(14)=6H $
      RLABEL(15)=7HIN IN $
      LABELS(61)=39
      GO TO 40
7     IF (EMAX.GT.100.) GO TO 10
      ARRAY(7)=ARRAY(11)=ARRAY(15)=ARRAY(8)=ARRAY(12)=ARRAY(16)=2.
      LABELS(62)=26
      RLABEL(14)=6H $
      RLABEL(15)=7HIN IN $
      LABELS(61)=39
      GO TO 40
10    IF (EMAX.GT.25000.) GO TO 20
      ARRAY(7)=ARRAY(15)=ARRAY(8)=ARRAY(16)=1000.
      ARRAY(11)=ARRAY(12)=1.
      GO TO 40
20    IF (EMAX.GT.100000.) GO TO 30
      ARRAY(7)=ARRAY(15)=ARRAY(8)=ARRAY(16)=5000.
      ARRAY(11)=ARRAY(12)=5.
      GO TO 40
30    ARRAY(7)=ARRAY(15)=ARRAY(8)=ARRAY(16)=10000.
      ARRAY(11)=ARRAY(12)=10.
```

```
40    ARRAY(1)=EMAX
      ARRAY(3)=ARRAY(4)=ARRAY(5)=ARRAY(6)=ARRAY(9)=ARRAY(10)=ARRAY(13)=
     1    ARRAY(14)=ARRAY(17)=ARRAY(18)=0.
      ARRAY(19)=EMAX
      ARRAY(20)=MAXC2
      ARRAY(21)=ARRAY(22)=1.
      ARRAY(2)=MAXC2
      DO 50 I=11,17
      CALL EQUIP(I,5HFILE )
50    CONTINUE
      READ (5) ISTEP
      READ (5) OUT1
      READ (5) OUT2
      READ (5) OUT3
      STD(1)=SQRT(OUT1(3)*OUT1(2)*OUT1(2)+OUT1(4)*OUT1(2)+OUT1(5))
      CALL MLTIPLT(OUT1(2),STD)
      E(1)=OUT1(2)
      DO 60 I=1,N
60    A(1,I)=OUT2(I,1)*E(1)+OUT2(I,2)
      DO 100 ICT=1,KSTEP
      ICT1=ICT+1
      E(ICT1)=OUT1(1)
      DO 70 I=1,N
70    A(ICT1,I)=OUT2(I,3)
      EINC=(OUT1(1)-OUT1(2))/NINT
      DO 80 J=1,NINT
      ESUB(J)=OUT1(2)+(J-1)*EINC
      STD(J)=SQRT(OUT1(3)*ESUB(J)*ESUB(J)+OUT1(4)*ESUB(J)+OUT1(5))
      DO 80 K=1,7
80    P(J,K)=ESUB(J)+CONST(K)*STD(J)
      IARRY(2)=NINT1
      IARRY(5)=0
      ESUB(NINT1)=OUT1(1)
      STD(NINT1)=OUT1(7)
      DO 85 K=1,7
85    P(NINT1,K)=OUT1(1)+CONST(K)*OUT1(7)
      CALL GRAPH(ESUB,STD)
      IARRY(2)=1
      IARRY(5)=16
      CALL GRAPH(OUT1(1),OUT1(7))
      DO 90 J=1,7
      IJ=J+10
90    WRITE (IJ) (ESUB(K),P(K,J),K=1,NINT1)
      IF (ICT.EQ.KSTEP) GO TO 100
      READ (5) ISTEP
      READ (5) OUT1
      READ (5) OUT2
      READ (5) OUT3
100   CONTINUE
C
C         INITIALIZATION FOR CHART 1
C
      IARRY(1)=N
      IARRY(2)=KSTEP+1
      IARRY(5)=16
      RLABEL(21)=8HTHE ACTI
      RLABEL(22)=8HVITY LEV
      LABELS(45)=3HELS
      LABELS(63)=19
      RLABEL(1)=8HLEVEL OF
      RLABEL(2)=8H ACTIVIT
      RLABEL(3)=8HY IN ACR
```

```
      LABELS(7)=2HES
      LABELS(61)=26
      IF (MAXC1.GT.50.) GO TO 105
      ARRAY(8)=ARRAY(12)=ARRAY(16)=1.
      GO TO 130
105   IF (MAXC1.GT.100.) GO TO 110
      ARRAY(8)=ARRAY(12)=ARRAY(16)=10.
      GO TO 130
110   IF (MAXC1.GT.1000.) GO TO 120
      ARRAY(8)=ARRAY(12)=ARRAY(16)=20.
      GO TO 130
120   ARRAY(8)=ARRAY(16)=100.
      RLABEL(3)=8HY IN 100
      RLABEL(4)=6H ACRES
      LABELS(61)=30
      ARRAY(12)=1.
130   ARRAY(2)=MAXC1
      ARRAY(20)=MAXC1
      CALL MLTIPLT(E,A(1,1))
      DO 140 I=2,N
      CALL GRAPH(E,A(1,I))
140   CONTINUE
C
C          INITIALIZATION FOR CHART 3
C
      IARRY(1)=7
      IARRY(2)=NINT1
      IARRY(5)=0
      IARRY(8)=3
      RLABEL(21)=8HTHE PROB
      RLABEL(22)=8HABILITY
      RLABEL(23)=7HOF LOSS
      LABELS(63)=23
      RLABEL(1)=8HACTUAL G
      RLABEL(2)=8HROSS MAR
      RLABEL(3)=8HGIN IN $
      LABELS(7)=4H1000
      LABELS(61)=28
      AMIN=MIN1(0.,MINC3)
      ALENG=EMAX-AMIN
      IF (ALENG.GT.50.) GO TO 142
      ARRAY(8)=ARRAY(12)=ARRAY(16)=1.
      LABELS(61)=24
      GO TO 149
142   IF (ALENG.GT.100.) GO TO 143
      ARRAY(8)=ARRAY(12)=ARRAY(16)=2.
      LABELS(61)=24
      GO TO 149
143   IF (ALENG.GT.1000.) GO TO 144
      ARRAY(8)=ARRAY(16)=ARRAY(12)=50.
      LABELS(61)=24
      GO TO 149
144   IF (ALENG.GT.25000.) GO TO 146
      ARRAY(8)=ARRAY(16)=1000.
      ARRAY(12)=1.
      GO TO 149
146   IF (ALENG.GT.100000.) GO TO 147
      ARRAY(8)=ARRAY(16)=5000.
      ARRAY(12)=5.
      GO TO 149
147   ARRAY(8)=ARRAY(16)=10000.
      ARRAY(12)=10.
```

```
149   CONTINUE
      ITEMP=AMIN/ARRAY(8)
      ATEMP=ARRAY(8)*ITEMP
      IF (MINC3.LT.ATEMP) ATEMP=ATEMP-1
      ARRAY(10)=ARRAY(14)=ARRAY(4)=ATEMP
      ARRAY(2)=EMAX-ARRAY(4)
      ARRAY(18)=MIN1(0.,MINC3)
      ARRAY(20)=EMAX
      DO 150 I=11,17
150   REWIND I
      READ (11) (ESUB(K),P(K,1),K=1,NINT1)
      CALL MLTIPLT(ESUB,P)
      IARRY(2)=1
      IARRY(5)=16
      CALL GRAPH(ESUB(NINT1),P(NINT1,1))
      DO 160 I=2,KSTEP
      READ (11) (ESUB(K),P(K,1),K=1,NINT1)
      IARRY(2)=NINT1
      IARRY(5)=0
      CALL GRAPH(ESUB,P)
      IARRY(2)=1
      IARRY(5)=16
      CALL GRAPH(ESUB(NINT1),P(NINT1,1))
160   CONTINUE
      DO 170 IGRAPH=2,7
      DO 170 I=1,KSTEP
      IJ=IGRAPH+10
      READ (IJ) (ESUB(K),P(K,1),K=1,NINT1)
      IARRY(2)=NINT1
      IARRY(5)=0
      CALL GRAPH(ESUB,P)
      IARRY(2)=1
      IARRY(5)=16
      CALL GRAPH(ESUB(NINT1),P(NINT1,1))
170   CONTINUE
      IF (AXISXY(0,0,0,0,0,0,0,0,0,0,0,0)) 180,180
180   DO 190 I=11,17
      CALL UNEQUIP(I)
190   CONTINUE
      RETURN
      END
```

```
      PROGRAM CORRELATE
C     THIS PROGRAM IS DESIGNED TO COMPUTE A CORRELATION MATRIX OF THE
C     GROSS MARGINS OF CROPING ACTIVITIES. YOU HAVE THE OPTION OF
C     REMOVING THE INFLUENCE OF TIME BY CORRELATING THE DEVIATIONS FROM
C     A LINEAR TREND REGRESSION EQUATION. THE RESULTING CORRELATION
C     MATRIX BECOMES A SOURCE OF DATA UNDER CERTAIN OPTIONS OF
C     PROGRAM INPUT.
C
C     INSTRUCTIONS FOR SETTING UP INPUT FILE.
C
C         SET UP OF CONTROL CARD
C
C               COLUMN  1- 2, ENTER NCROP, THE NO. OF CROPS, MAX 50
C                       3- 4, ENTER NYEAR, THE NO. OF YEARS, MAX 10
C                       5- 8, ENTER MINYEAR, THE FIRST YEAR IN SERIES
C                       9-12, ENTER MAXYEAR, THE LAST YEAR IN SERIES
C                             MAKE SURE THAT THE DIFFERENCE BETWEEN
C                             MINYEAR AND MAXYEAR IS 9 OR LESS.
C                      13-80, LEAVE BLANK
C         SET UP OF LABEL CARDS
C         YOU MUST HAVE EXACTLY NCROP LABELS. PREPARE LABEL CARD FOR
C         EACH CROP AND MAKE SURE TO USE SAME ORDER FOR SUCCEEDING
C         SECTIONS
C               COLUMN  1-15, ENTER NAME OF CROP
C               COLUMN    16, ENTER BLANK, DO NOT ENTER ZERO
C               COLUMN 17-24, ENTER PRICE UNITS, FOR EXAMPLE #$$/TON#
C               COLUMN 25-32, ENTER YIELD UNITS, FOR EXAMPLE #TON/ACRE#
C               COLUMN 33-80, LEAVE BLANK
C         SET UP OF PRICE MATRIX.
C         YOU MUST HAVE THE SAME ORDER IN THE PRICE MATRIX AS YOU
C         HAVE IN THE LABEL CARDS. THE PRICE MATRIX IS NCROP X NYEAR.
C         FOR EACH ACTIVITY ENTER PRICE FOR EACH YEAR SEPARATING EACH
C         ENTRY BY BLANKS OR A COMMA. WHEN YOU HAVE COMPLETED ALL PRICE
C         DATA ENTER 9999 ON A NEW CARD. THIS COMPLETES PRICE MATRIX.
C
C         SET UP OF YIELD MATRIX.
C         YOU MUST HAVE THE SAME ORDER IN THE YIELD MATRIX AS YOU
C         HAVE IN THE LABEL CARDS. THE YIELD MATRIX IS NCROP X NYEAR.
C         FOR EACH ACTIVITY ENTER YIELD FOR EACH YEAR SEPARATING EACH
C         ENTRY BY BLANKS OR A COMMA. WHEN YOU HAVE COMPLETED ALL YIELD
C         DATA ENTER 9999 ON A NEW CARD. THIS COMPLETES YIELD MATRIX.
C
C         YOU HAVE NOW ENTERED ALL OF THE DATA. AS A FINAL CHECK MAKE
C         SURE ALL DATA LINES CONFORM TO THE ORDER OF THE LABEL CARDS.
C         NOW FILE THE DATA AND GOOD LUCK.
C
C    DIRECTORY OF LOGICAL UNIT NUMBERS
C
C        LUN 1  = DATA FILE
C        LUN 2  = OUTPUT FILE  ( CORRELATION MATRIX)
C        LUN 34 = LP    ( LINE PRINTER)
C        LUN 60 = TELETYPE INPUT
C        LUN 61 = TELETYPE OUTPUT
C
     DIMENSION RNAM(50,4),PRICE(50,10),YIELD(50,10),GROSS(50,10),
    1SUM(50),TSUM(50),XTX(50,50),TGROSS(50,10),XXTXX(50,50),
    2CORR(50,50),STD(50),XBAR(50)
    3,S(50),SS(50),ST(50),TSTAT(50),A(50),B(50)
      EQUIVALENCE(SUM(1),S(1)),(TSUM(1),SS(1)),(ST(1),STD(1))
      EQUIVALENCE (XXTXX(1,1),PRICE(1,1)),(XTX(1,1),YIELD(1,1)),
```

```
      1(CORR(1,1),GROSS(1,1))
       READ(1,1000) NCROP,NYEAR,MINYEAR,MAXYEAR
       DO 10 J=1,NCROP
10     READ(1,1001)(RNAM(J,I),I=1,4)
       DO 20 J=1,NCROP
       DO 20 I=1,NYEAR
20     PRICE(J,I)=FFIN(1)
       KCHECK=FFIN(1)
       IF(KCHECK.NE.9999) GO TO 990
       DO 30 J=1,NCROP
       DO 30 I=1,NYEAR
30     YIELD(J,I)=FFIN(1)
       KCHECK=FFIN(1)
       IF(KCHECK.NE.9999) GO TO 991
       DO 40 J=1,NCROP
       DO 40 I=1,NYEAR
40     GROSS(J,I)=PRICE(J,I)*YIELD(J,I)
       WRITE(34,1003)
       WRITE(34,1004)(II,II=MINYEAR,MAXYEAR)
       WRITE(34,1005)
       LC=0
       DO 50 J=1,NCROP
       LC=LC+1
       IF(LC.EQ.5) 51,50
51     LC=0
       WRITE(34,1005)
50     WRITE(34,1006)(RNAM(J,JJ),JJ=1,3),(PRICE(J,I),I=1,NYEAR)
       DO 80 J=1,NCROP
80     RNAM(J,3)=RNAM(J,4)
       WRITE(34,1007)
       WRITE(34,1004)(II,II=MINYEAR,MAXYEAR)
       WRITE(34,1005)
       LC=0
       DO 60 J=1,NCROP
       LC=LC+1
       IF(LC.EQ.5) 61,60
61     LC=0
       WRITE(34,1005)
60     WRITE(34,1006)(RNAM(J,JJ),JJ=1,3),(YIELD(J,I),I=1,NYEAR)
       WRITE(34,1008)
       WRITE(34,1004)(II,II=MINYEAR,MAXYEAR)
       WRITE(34,1005)
       DO 70 J=1,NCROP
       LC=LC+1
       IF(LC.EQ.5) 71,70
71     LC=0
       WRITE(34,1005)
70     WRITE(34,1009)(RNAM(J,JJ),JJ=1,2),(GROSS(J,I),I=1,NYEAR)
1000   FORMAT(2I2,2I4)
1001   FORMAT(4A8)
1002   FORMAT(≠ YOU CAN NOW CHECK YOUR DATA≠)
1003   FORMAT(≠1                    ANNUAL AVERAGE CROP PRICES≠)
1007   FORMAT(≠1                    ANNUAL AVERAGE CROP YIELDS≠)
1008   FORMAT(≠1               ANNUAL AVERAGE GROSS CROP INCOME≠)
1004   FORMAT(≠0≠24(≠ ≠),10(≠      ≠I4))
1005   FORMAT(≠ ≠)
1006   FORMAT(1X,3A8,10F10.2)
1009   FORMAT(1X,2A8≠ $$/ACRE≠10F10.2)
1010   FORMAT(≠  ≠I2≠      ≠I2≠   ≠3F15.6≠   ≠2A8)
1011   FORMAT(≠  ≠I2≠      ≠I2≠   ≠F15.6,30(≠ ≠)≠   ≠2A8≠ VS ≠2A8)
1012   FORMAT(≠1CROP VS CROP      CORR COEF       STD DEV    ≠
     1≠    MEAN GROSS ≠)
```

```
      WRITE(61,1016)
      READ(60,1000) KILL
      GO TO (2000,3000,4000),KILL
4000  YR=NYEAR
      YR1=YR-1.0
      YR2=YR-2.0
      T=YR*(YR+1.0)/2.0
      TS=YR*(YR+1.0)*(2.0*YR+1.0)/6.0
      TCS=(TS-T**2/YR)/YR1
      TBAR=T/YR
      DO 200 J=1,NCROP
      S(J)=0.0
      SS(J)=0.0
200   ST(J)=0.0
      DO 201 J=1,NCROP
      DO 202 I=1,NYEAR

      S(J)=S(J)+GROSS(J,I)
      SS(J)=SS(J)+GROSS(J,I)**2
202   ST(J)=ST(J)+GROSS(J,I)*I
      XBAR(J)=S(J)/YR
      SS(J)=(SS(J)-S(J)**2/YR)/YR1
      ST(J)=(ST(J)-S(J)*T/YR)/YR1
      B(J)=ST(J)/TCS
      A(J)=XBAR(J)-B(J)*TBAR
      TSTAT(J)=B(J)/SQRT((SS(J)-B(J)*ST(J))/(TCS*YR2))
      DO 203 I=1,NYEAR
203   GROSS(J,I)=GROSS(J,I)-A(J)-B(J)*I
201   CONTINUE
      WRITE(34,1021)
      WRITE(34,1022)
      DO 206 J=1,NCROP
      SS(J)=SQRT(SS(J))
206   WRITE(34,1019)(RNAM(J,JJ),JJ=1,2),XBAR(J),SS(J)
1021  FORMAT(≠1  MEAN AND STANDARD DEVIATION OF GROSS INCOME≠)
1022  FORMAT(≠0NAME OF CROP≠30(≠ ≠)≠MEAN    ≠10(≠ ≠)≠STD DEV≠)
      WRITE(34,1017)
      WRITE(34,1018)
      DO 204 J=1,NCROP
204   WRITE(34,1019)(RNAM(J,JJ),JJ=1,2),A(J),B(J),TSTAT(J)
      WRITE(34,1020)
      WRITE(34,1004)(II,II=MINYEAR,MAXYEAR)
      DO 205 J=1,NCROP
205   WRITE(34,1009)(RNAM(J,JJ),JJ=1,2),(GROSS(J,I),I=1,NYEAR)
1017  FORMAT(≠1REGRESSION ON TIME≠)
1018  FORMAT(≠0NAME OF CROP≠30(≠ ≠)≠ALPHA    ≠
     118(≠ ≠)≠BETA     ≠13(≠ ≠)≠T-STATISTIC≠)
1019  FORMAT(1X,2A8≠$$/ACRE ≠3F26.6)
1020  FORMAT(≠1DEVIATIONS OF ACTUAL GROSS INCOME FROM EXPECTED≠
     1≠ GROSS INCOME≠)
1016  FORMAT(≠ THE GROSS INCOME STATEMENT IS PREPARED ≠/
     1≠  IF YOU WANT TO CHECK DATA TYPE ¬01¬ ≠/
     2≠  IF YOU WANT ORDINARY CORRELATION TYPE ¬02¬ ≠/
     3≠  IF YOU WANT TO REMOVE THE TIME INFLUENCE TYPE ¬03¬ ≠/
     4≠  TYPE THE NUMBER IN AN ¬I2¬ FIELD ≠)
3000  DO 100 J=1,NCROP
100   SUM(J)=0.0
      DO 101 I=1,NYEAR
      DO 101 J=1,NCROP
101   SUM(J)=GROSS(J,I)+SUM(J)
      CALL ARRAY(2,NCROP,NYEAR,50,10,GROSS,GROSS)
      CALL MTRA(GROSS,TGROSS,NCROP,NYEAR,0)
```

```
      CALL MPRD(GROSS,TGROSS,XXTXX,NCROP,NYEAR,0,0,NCROP)
      CALLMTRA(SUM,TSUM,NCROP,1,0)
      CALL MPRD(SUM,TSUM,XTX,NCROP,1,0,0,NCROP)
      YEAR=NYEAR
      YEAR=1./YEAR
      CALL SMPY(XTX,YEAR,XTX,NCROP,NCROP,0)
      CALL MSUB(XXTXX,XTX,CORR,NCROP,NCROP,0,0)
      YEAR=NYEAR
      YEAR=1./(YEAR-1.)
      CALL SMPY(CORR,YEAR,CORR,NCROP,NCROP,0)
      CALL ARRAY(1,NCROP,NCROP,50,50,CORR,CORR)
      DO 102 J=1,NCROP
      I=J
  102 STD(J)=SQRT(CORR(J,I))
      DO 103 J=1,NCROP
      DO 103 I=1,NCROP
  103 CORR(J,I)=CORR(J,I)/STD(I)
      DO 104 I=1,NCROP
      DO 104 J=1,NCROP
  104 CORR(J,I)=CORR(J,I)/STD(J)
      WRITE(2) CORR
      YEAR=NYEAR
      DO 105 J=1,NCROP
  105 XBAR(J)=SUM(J)/YEAR
      LC=0
      LCC=0
      WRITE(34,1012)
      WRITE(34,1005)
      DO 106 I=1,NCROP
      DO 106 J=I,NCROP
      LC=LC+1
      LCC=LCC+1
      IF(LC.EQ.5) 107,108
  107 LC=0
      WRITE(34,1005)
  108 IF(LCC.EQ.45) 109,110
  109 LC=0
      LCC=0
      WRITE(34,1012)
  110 IF(I.EQ.J) 111,112
  111 WRITE(34,1010)I,J,CORR(J,I),STD(J),XBAR(I),
     1(RNAM(I,II),II=1,2)
      GO TO 106
  112 WRITE(34,1011) I,J,CORR(J,I),(RNAM(I,II),II=1,2),
     1(RNAM(J,JJ),JJ=1,2)
  106 CONTINUE
      GO TO 999
  990 WRITE(61,1013)
      GO TO 999
  991 WRITE(61,1014)
      GO TO 999
 2000 WRITE(61,1002)
 1013 FORMAT(≠ THERE IS A CARD ERROR IN THE PRICE INPUT≠)
 1014 FORMAT(≠ THERE IS A CARD ERROR IN THE YIELD INPUT≠)
  999 CALL EXIT
      END
```