The crankshaft of an internal combustion engine experiences torsional vibrations which become critical at certain running speeds and can cause overstressing in the crankshaft, leading to failure in some cases.

The object of this study was to design, construct and test an instrument to measure these oscillations at one point on the crankshaft. This information along with a knowledge of the natural frequencies and mode shapes of the crankshaft can be used to determine the maximum amplitude of vibration.

The output sensitivity of the instrument built compared favorably with that predicted from its analytical model except for an increase in sensitivity with frequencies above approximately 40 hz. This was thought to be due to a characteristic of the calibrator system itself.

Torsional vibrations at the front of an engine crankshaft for different speeds and loading conditions were
measured with the instrument. The absence of natural frequency and mode shape data for the crankshaft prevented any determination of the maximum amplitudes of vibration.

The instrument fulfilled the original criteria and performed satisfactorily in the several hours of testing. When used in conjunction with a storage oscilloscope, it provided a means of monitoring or permanently recording a torsional vibration waveform.
Design, Construction and Testing of Torsiograph

by

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LIST OF SYMBOLS

$A_k$ $\frac{1}{2}$ Coefficient in the series expansion of $T_j$

$B_k$ $\frac{1}{2}$ Coefficient in the series expansion of $T_j$

$C$ Spring constant of torsiograph leaf spring

$C_T$ Torsional spring constant of idealized model of torsiograph

$C^{(j)}_k$ Coefficient in the series expansion of $P_i$ (See equation (12))

$D$ Diameter of crankshaft journal

$D^{(j)}_k$ Coefficient in the series expansion of $P_i$ (See equation (13))

$E$ Young's Modulus of Elasticity

$E_{in}$ Input voltage of strain gage bridge

$E_{out}$ Output voltage of strain gage bridge

$E^{(i)}_i$ Coefficient in the series expansion of $P_i$ (See equation (15))

$F^{(i)}_i$ Coefficient in the series expansion of $P_i$ (See equation (16))

$G$ Shear modulus of elasticity

$G.F.$ Gage factor of strain gage

$G^{(i)}_m$ Coefficient in the series expansion of $P_i$ (See equation (18))

$I$ Mass moment of inertia

$I_i$ Equivalent mass moment of inertia of idealized model of internal combustion engine installation in principal coordinates (See equation (10))

$I_x$ Second moment of inertia of leaf spring
\( J_j \) Equivalent mass moment of inertia of idealized model of internal combustion engine installation in system coordinates

\( k_{j} \) Equivalent spring constant of crankshaft between equivalent inertias \( j \) and \( j + 1 \)

\( L \) Length of crankshaft journal

\( M \) Moment on leaf spring

\( P \) Load on leaf spring

\( P_i \) The forcing term of the \( i^{th} \) differential equation of motion in principal coordinates (See equation (9))

\( T_m \) Average torque on a cylinder

\( T_{n-1} \) Applied torque acting on the inertia of the flywheel

\( T_n \) Applied torque acting on the inertia of the load

\( T_j \) Applied torque acting on the equivalent inertia of cylinder \( j \) (See equation (1))

\( Z \) Constant in Hooke's joint relation (See equation (31))

\( c \) \( h/2 \)

\( d \) Distance from point of force to strain gage location on leaf spring

\( f \) Frequency in hz

\( f_n \) Natural frequency in hz

\( h \) Thickness of leaf spring

\( l \) Length of leaf spring

\( P_i \) The \( i^{th} \) principal coordinate

\( r(i)^j \) The \( i^{th} \) principal mode shape of the \( j^{th} \) system coordinate in equation (4)
\( t \)  Time from top dead center position of cylinder No. 1

\( y \)  Displacement of leaf spring

\( \alpha \)  Angle between input and output shafts of Hooke's joint

\( \alpha_j \)  Phase angle between cylinder No. 1 and \( j \)

\( \beta \)  Angular rotation of output shaft of Hooke's joint

\( \beta_m^{(i)} \)  Phase angle in series expansion of \( P_i \)
(See equation (19))

\( \gamma \)  Angular displacement of input of idealized model of torsiograph

\( \delta \)  Angular displacement of seismic mass of idealized model of torsiograph

\( \varepsilon \)  Strain in leaf spring at strain gage location

\( \theta \)  Amplitude of torsional vibration of output shaft of Hooke's joint

\( \theta_j \)  Angular displacement of equivalent inertia of \( j^{th} \) cylinder

\( \mu \)  Mass per unit length of leaf spring

\( \sigma \)  Stress in leaf spring at strain gage location

\( \sigma_s \)  Shear stress in crankshaft journal

\( \phi \)  Relative angular displacement between input shaft and seismic mass of idealized model of torsiograph

\( \psi \)  Angular rotation of input shaft of Hooke's joint

\( \omega \)  Running frequency of engine

\( \omega_i \)  Natural frequency of crankshaft in \( i^{th} \) mode
(See equation (7))

\( \omega_n \)  Natural frequency of idealized model of torsiograph
(See equation (26))
DESIGN, CONSTRUCTION AND TESTING OF TORSIOGRAPH

Introduction

The work presented in this thesis is a study of the torsional crankshaft vibrations that occur in an internal combustion engine and the design and testing of an instrument to measure these vibrations at a point on the crankshaft. With the experimental data obtained with the instrument and an analysis of the natural frequencies and modes of vibration of the crankshaft, the maximum angular deflections can be calculated for a certain running speed. From the angular deflections the corresponding stresses can then be found and this information is used to determine whether or not crankshaft failure may be expected to occur.
Analysis of Forced Torsional Vibration

Idealized Model

Consider an idealized model of the rotating system of an internal combustion engine installation with $n$ degrees of freedom.

\[ \begin{align*}
\theta_1 & \quad J_1 \quad T_1 \\
\theta_2 & \quad J_2 \quad T_2 \\
\theta_3 & \quad J_3 \quad T_3 \\
\theta_{n-2} & \quad J_{n-2} \quad T_{n-2} \\
\theta_{n-1} & \quad J_{n-1} \quad T_{n-1} \\
\theta_n & \quad J_n \quad T_n
\end{align*} \]

Figure 1. Idealized model of the rotating system of an internal combustion engine installation.

Applied Torque

The typical applied torque on each cylinder of a four-cycle internal combustion engine is shown in Figure 2. The torque-displacement curve is assumed to be the same for each cylinder, differing only in phase shift from the T.D.C. position of cylinder No. 1.
Figure 2. Typical applied torque of a four-cycle internal combustion engine cylinder.

This applied torque, acting on the equivalent inertia of a \( j \)th cylinder, can be expressed as a Fourier series:

\[
T_j = T_m + A_1 \sin \frac{1}{2}(\omega t - \alpha_j) + A_1 \sin (\omega t - \alpha_j) \\
+ A_3 \sin \frac{3}{2}(\omega t - \alpha_j) + \ldots + A_N \sin \frac{N}{2}(\omega t - \alpha_j) \\
+ B_1 \cos \frac{1}{2}(\omega t - \alpha_j) + B_1 \cos (\omega t - \alpha_j) \\
+ B_3 \cos \frac{3}{2}(\omega t - \alpha_j) + \ldots + B_N \cos \frac{N}{2}(\omega t - \alpha_j)
\]
or more compactly as:

\[
T_j = T_m + \sum_{k=1}^{N} A_k \sin \frac{k}{2}(\omega t - \alpha_j) + B_k \cos \frac{k}{2}(\omega t - \alpha_j)
\] (1)

The applied torque \(T_{n-1}\) acting on the inertia \(J_{n-1}\) of the flywheel is assumed to be zero.

And finally, the applied torque \(T_n\) acting on the inertia \(J_n\) of the load is assumed to be a function of the running speed, as would be the case if the load in question were a generator. Hence:

\[
T_n = T_L(\omega)
\] (2)

Note that this load torque will act in the opposite direction of the cylinder torques.

Equations of Motion

The differential equation of motion for the idealized model can be written as:

\[
\begin{bmatrix}
J_1 \ddot{\theta}_1 \\
J_2 \ddot{\theta}_2 \\
\vdots \\
J_n \ddot{\theta}_n
\end{bmatrix} + [K] \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_n
\end{bmatrix} = \begin{bmatrix}
T_1 \\
T_2 \\
\vdots \\
T_n
\end{bmatrix}
\] (3)
The relation between the system coordinates $\theta_j$ and the principal coordinates $p_i$ can be expressed as:

$$
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_n
\end{bmatrix}
= p_1 \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ \vdots \\ r_n^{(1)} \end{bmatrix}
+ p_2 \begin{bmatrix} r_1^{(2)} \\ r_2^{(2)} \\ \vdots \\ r_n^{(2)} \end{bmatrix}
+ \cdots
+ p_n \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \\ \vdots \\ r_n^{(n)} \end{bmatrix}
$$

(4)

where $\{r^{(i)}\}$ is the $i^{th}$ mode shape of the idealized model in free vibration. It can also be expressed more compactly as:

$$
\{\theta\} = [r]\{p\}
$$

Substituting the above into the differential equation (3) gives:

$$
[J][r]\{\ddot{p}\} + [k][r]\{p\} = \{T\}
$$

(5)

where:

$$
[J] = \begin{bmatrix}
J_1 \\
\vdots \\
J_n
\end{bmatrix}
$$

and:

$$
\{T\} = \{T_j\}.
$$

Multiplying equation (5) by $[r]^T$, the result is:

$$
[r]^T[J][r]\{\ddot{p}\} + [r]^T[k][r]\{p\} = [r]^T\{T\}
$$
and rearranging:

\[
\begin{bmatrix}
\ddot{\mathbf{p}} \\
\mathbf{p}
\end{bmatrix} + \left[ \begin{bmatrix} \mathbf{r}^T \mathbf{k} \mathbf{r} \end{bmatrix} \right] \begin{bmatrix} \mathbf{p} \end{bmatrix} = \left[ \begin{bmatrix} \mathbf{r}^T \mathbf{J} \mathbf{P} \end{bmatrix} \right] \begin{bmatrix} \mathbf{T} \end{bmatrix}
\]

Finally the differential equation of motion in principal coordinates can be expressed as:

\[
\begin{bmatrix}
\ddot{\mathbf{p}} \\
\mathbf{p}
\end{bmatrix} + \left[ \begin{bmatrix} \omega^2 \end{bmatrix} \right] \begin{bmatrix} \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{P} \end{bmatrix} \tag{6}
\]

where \( \left[ \begin{bmatrix} \omega^2 \end{bmatrix} \right] \) is a diagonal matrix expressed as:

\[
\left[ \begin{bmatrix} \omega^2 \end{bmatrix} \right] = \left[ \begin{bmatrix} \omega_1^2 & \cdots & \omega_n^2 \end{bmatrix} \right] \tag{7}
\]

and:

\[
\begin{bmatrix} \mathbf{p} \end{bmatrix} = \left[ \begin{bmatrix} \mathbf{r}^T \\
\mathbf{r}^T \mathbf{J} \mathbf{r} \end{bmatrix} \right] \begin{bmatrix} \mathbf{T} \end{bmatrix}
\]

**Steady State Solution**

Consider the \( i^{th} \) equation from the equations of motion (equation (6)):

\[
\ddot{p}_i + \omega_i^2 p_i = P_i \tag{8}
\]

where:

\[
P_i = \frac{\mathbf{r}(i)}{I_i} \begin{bmatrix} \mathbf{T} \end{bmatrix} \tag{9}
\]

and:

\[
I_i = \mathbf{r}(i)^T \mathbf{J} \mathbf{r}(i) \tag{10}
\]

Expanding equation (9):
\[ P_i = \frac{r^{(i)}_1}{I_i} T_1 + \frac{r^{(i)}_2}{I_i} T_2 + \ldots + \frac{r^{(i)}_n}{I_i} T_n \]

and substituting equations (1) and (2) into the above, the result is:

\[
P_i = \frac{r^{(i)}}{I_i} \left[ T_m + \sum_{k=1}^{N} A^*_k \sin \frac{k}{2} (\omega t - \alpha_1) + B^*_k \cos \frac{k}{2} (\omega t - \alpha_1) \right]
+ \ldots + \frac{r^{(i)}}{I_i} \left[ T_m + \sum_{k=1}^{N} A^*_k \sin \frac{k}{2} (\omega t - \alpha_{n-2}) + B^*_k \cos \frac{k}{2} (\omega t - \alpha_{n-2}) \right]
- \frac{r^{(i)}}{I_i} T_L \omega \]

The above may be rewritten as:

\[
P_i = \frac{r^{(i)}}{I_i} \left[ T_m + \sum_{k=1}^{N} (A_k \cos \frac{k\alpha_1}{2} + B_k \sin \frac{k\alpha_1}{2}) \sin \frac{k}{2} \omega t + (B_k \cos \frac{k\alpha_1}{2} - A_k \sin \frac{k\alpha_1}{2}) \cos \frac{k}{2} \omega t \right] + \ldots
+ \frac{r^{(i)}}{I_i} \left[ T_m + \sum_{k=1}^{N} (A_k \cos \frac{k\alpha_{n-2}}{2} + B_k \sin \frac{k\alpha_{n-2}}{2}) \sin \frac{k}{2} \omega t + (B_k \cos \frac{k\alpha_{n-2}}{2} - A_k \sin \frac{k\alpha_{n-2}}{2}) \cos \frac{k}{2} \omega t \right] - \frac{r^{(i)}}{I_i} T_L \omega \]

(11)
and by letting:

\[ C_k^{(j)} = A_k \cos \frac{k\alpha}{2} j + B_k \sin \frac{k\alpha}{2} j \]  \hspace{1cm} (12)

and:

\[ D_k^{(j)} = B_k \cos \frac{k\alpha}{2} j - A_k \sin \frac{k\alpha}{2} j \]  \hspace{1cm} (13)

\((j = 1, 2, \ldots, n-2)\)

\((k = 1, 2, \ldots, N)\)

which when substituted into equation (11), one has then:

\[
P_i = \frac{r^{(i)}}{I_i} \left[ T_m + \sum_{k=1}^{N} \left( C_k^{(1)} \sin \frac{k\omega t}{2} + D_k^{(1)} \cos \frac{k\omega t}{2} \right) \right] + \ldots \\
+ \frac{r^{(i)}}{I_i} \left[ T_m + \sum_{k=1}^{N} \left( C_k^{(n-2)} \sin \frac{k\omega t}{2} + D_k^{(n-2)} \cos \frac{k\omega t}{2} \right) \right] \\
- \frac{r^{(i)}}{I_i} \left[ T_L \omega \right]
\]

Rewriting the above:

\[
P_i = \frac{T_m}{I_i} \left[ r^{(i)}_1 + r^{(i)}_2 + \ldots + r^{(i)}_{n-2} \right] - \frac{T_L \omega}{I_i} \left[ r^{(i)}_n \right] \\
+ \frac{1}{I_i} \left[ r^{(i)}_1 C_1^{(1)} + r^{(i)}_2 C_1^{(2)} + \ldots + r^{(i)}_{n-2} C_1^{(n-2)} \right] \sin \frac{1}{2} \omega t \\
+ \frac{1}{I_i} \left[ r^{(i)}_1 D_1^{(1)} + r^{(i)}_2 D_1^{(2)} + \ldots + r^{(i)}_{n-2} D_1^{(n-2)} \right] \cos \frac{1}{2} \omega t \\
+ \ldots + \frac{1}{I_i} \left[ r^{(i)}_1 C_N^{(1)} + r^{(i)}_2 C_N^{(2)} + \ldots + r^{(i)}_{n-2} C_N^{(n-2)} \right] \sin \frac{N}{2} \omega t \\
+ \frac{1}{I_i} \left[ r^{(i)}_1 D_N^{(1)} + r^{(i)}_2 D_N^{(2)} + \ldots + r^{(i)}_{n-2} D_N^{(n-2)} \right] \cos \frac{N}{2} \omega t
\]  \hspace{1cm} (14)
and letting:

\[ E_{i}^{(i)} = r_{1}^{(i)} C_{i}^{(1)} + r_{2}^{(i)} C_{i}^{(2)} + \ldots + r_{n-2}^{(i)} C_{i}^{(n-2)} \]  \hspace{1cm} (15)

and:

\[ F_{i}^{(i)} = r_{1}^{(i)} D_{i}^{(1)} + r_{2}^{(i)} D_{i}^{(2)} + \ldots + r_{n-2}^{(i)} D_{i}^{(n-2)} \]  \hspace{1cm} (16)

\((i = 1,2,\ldots,N)\)

Upon substitution into equation (14), the result is:

\[ P_{i}^{(i)} = \frac{T_{m}}{I_{i}} \left[ r_{1}^{(i)} + r_{2}^{(i)} + \ldots + r_{n-2}^{(i)} \right] - \frac{L}{I_{i}} (\omega) \left[ r_{n}^{(i)} \right] \]

\[ + \frac{1}{I_{i}} \left[ E_{1}^{(i)} \sin \frac{1}{2}\omega t + F_{1}^{(i)} \cos \frac{1}{2}\omega t \right] + \ldots \]

\[ + \frac{1}{I_{i}} \left[ E_{N}^{(i)} \sin \frac{N}{2}\omega t + F_{N}^{(i)} \cos \frac{N}{2}\omega t \right] \]  \hspace{1cm} (17)

Now letting:

\[ G_{m}^{(i)} = \sqrt{(E_{m}^{(i)})^{2} + (F_{m}^{(i)})^{2}} \]  \hspace{1cm} (18)

and:

\[ \beta_{m}^{(i)} = \tan^{-1} \frac{F_{m}^{(i)}}{E_{m}^{(i)}} \]  \hspace{1cm} (19)

\((m = 1,2,\ldots,N)\)

And substituting the above into equation (17) the result is:
\[ P_i = \frac{T_m}{I_i} \left[ r_1(i) + r_2(i) + \ldots + r_{n-2}(i) \right] - \frac{T_L}{I_i(\omega)} \left[ r_n(i) \right] \\
+ \frac{1}{I_i} \left[ G_1^{(i)} \sin \left( \frac{1}{2} \omega t + \beta_1^{(i)} \right) \right] + \ldots \\
+ \frac{1}{I_i} \left[ G_N^{(i)} \sin \left( \frac{N}{2} \omega t + \beta_N^{(i)} \right) \right] \] (20)

And finally, substituting equation (20) into equation (8) the resulting differential equation is:

\[ \ddot{P}_i + \omega^2_{1} P_i = \frac{T_m}{I_i} \left[ r_1(i) + r_2(i) + \ldots + r_{n-2}(i) \right] - \frac{T_L}{I_i(\omega)} \left[ r_n(i) \right] \\
+ \frac{1}{I_i} \left[ G_1^{(i)} \sin \left( \frac{1}{2} \omega t + \beta_1^{(i)} \right) \right] + \ldots \\
+ \frac{1}{I_i} \left[ G_N^{(i)} \sin \left( \frac{N}{2} \omega t + \beta_N^{(i)} \right) \right] \] (21)

Special attention should be given the rigid body mode, where:

\[ \omega^2_1 = 0 \]

and:

\[ \left\{ r_1 \right\} = \left\{ 1 \right\} \]

Equation (21) for \( i = 1 \) is then:

\[ \ddot{P}_1 = \frac{T_m}{I_1(n-2)} - \frac{T_L}{I_1(\omega)} + \frac{1}{I_1} \sum_{k=1}^{N} G_k^{(1)} \sin \left( \frac{k}{2} \omega t + \beta_k^{(1)} \right) \]

which has a steady state solution of the form:

\[ P_1 = \left[ T_m(n-2) - T_L(\omega) \right] \frac{\omega^2}{2I_1} - \frac{1}{I_1} \sum_{k=1}^{N} \left[ G_k^{(1)} \omega^2 \sin \left( \frac{k}{2} \omega t + \beta_k^{(1)} \right) \right] \]
Hence the condition for constant speed operation of the engine is:

\[ T_m(n-2) = T_L \omega \]

or the total average cylinder torque of the engine must equal the load torque. Since in this case the magnitude of the torque was constant and the load was a function of engine speed there existed only one equilibrium or constant speed point. In an actual case the torque is a function of speed and other parameters giving such systems a variable operating speed.

The general solution to equation (21) is:

\[
P_i = \frac{T_m}{I_i \omega_i} 2 \left[ \sum_{i=1}^{n-2} r(i) \right] - \frac{T_L}{I_i \omega_i} \left[ \omega_n(i) \right] + \frac{1}{I_i} \sum_{k=1}^{N} \frac{G_k(i)}{\omega_i^2 - \left( \frac{k(\omega)}{2} \right)^2} \sin \left( \frac{k}{2} \omega t + \beta_k(i) \right)
\]  \hspace{1cm} \text{(22)}

The first two terms in the above equation represent a constant deflection in the torsional system, and the last term represents the oscillatory motion.

Considering only the oscillatory motion of the crankshaft, equation (4) can finally be written as:
\[
\begin{align*}
&\{\theta_1\} \\
&\{\theta_2\} \\
&\{\vdots\} \\
&\{\theta_n\} = \sum_{j=1}^{n} \frac{1}{r_j} \\
&\{r(j)\} \\
&\{\vdots\} \\
&\{r_n\} \\
&\sum_{k=1}^{N} G_k(j) \sin\left(\frac{k\omega t}{2} + \beta_k(j)\right) \\
&\omega_2 - \left(\frac{k\omega}{2}\right)^2
\end{align*}
\]

A typical torque-time curve can usually be described adequately by equation (1) with terms up to about the ninth order.\(^1\) The largest values of the coefficients \(A_k\) and \(B_k\) occur around the first order, which is the running frequency of the engine. Hence the expansions of the coefficients \(G_k^{(i)}\) which are related to \(A_k\) and \(B_k\) from the preceding analysis follow in the same suit, with the most influential terms occurring at low values of \(k\).

Noting the terms on the right-hand side of equation (23) it can be seen that as the driving frequency or some multiple of it (i.e., \(\omega/2\), \(\omega\), \(3/2\omega\), \(2\omega\), etc.) approaches one of the natural frequencies of the torsional system \(\omega_i\), the terms in which this occurs will approach infinity along with the displacements related to that particular mode. Therefore, in practice it is important to insure that none of the torque harmonics

\(^1\) This is covered in detail in: Den Hartog, (1956), p. 197-200.
coincide with any of the natural frequencies of the torsional system for at least any length of time.

In practice, the test data from the front of the crankshaft ($\Theta_1$ location) is also composed of many harmonics. This data is harmonically analyzed to reduce it to a form which shows the contributions of each harmonic.\(^2\) The engine tests are normally run at a speed which will excite one of the natural frequencies of the system and, as in the analytical case, the experimental $\Theta_1$ is expected to be dominated by the term corresponding to that natural frequency. It is possible too that other natural frequencies of the actual system could be excited by multiples of the running speed with their influences also showing up in $\Theta_1$.

**Shear Stress**

Once the displacements are known, the shear stress can then be computed.\(^3\) Referring to Figure 3, the relation for the shear stress in the crankshaft journal between cylinders $j$ and $j+1$ is expressed as:

$$\sigma_s = \frac{\left(\Theta_{j+1} - \Theta_j\right) DG}{2L} \tag{24}$$

\(^2\) This relation would have a form similar to the $\Theta_1$ portion of equation (23).

\(^3\) An example of how the displacements are determined from test data is given in the following chapter.
The value of the shear stress is then used to determine whether or not crankshaft failure will occur for the particular set of conditions corresponding to the displacements.
Procedure for Determining Torsional Vibrations in a Crankshaft

Introduction

The procedure for determining the torsional vibrations in a crankshaft may be divided into two parts:

1. Analytical
2. Experimental

Analytical

The analytical portion of the procedure is concerned with the determination of the natural frequencies of the crankshaft and the respective modes of vibration. This analytical procedure is clearly outlined in several textbooks on vibrations, including References 1 and 4.

Generally this information is calculated by the engine manufacturer. Table 1 and Figure 4 are an example of this type of information.
Table 1. Natural Frequencies and Mode Shape Values of System Shown in Figure 4.

<table>
<thead>
<tr>
<th>Sta.</th>
<th>Natural Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5.54 Hz</td>
</tr>
<tr>
<td>1</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.9997</td>
</tr>
<tr>
<td>3</td>
<td>0.9991</td>
</tr>
<tr>
<td>4</td>
<td>0.9982</td>
</tr>
<tr>
<td>5</td>
<td>0.9970</td>
</tr>
<tr>
<td>6</td>
<td>-0.0892</td>
</tr>
</tbody>
</table>

Figure 4. Mode shapes of system composed of four-cylinder engine, driveshaft and rear wheels.
Experimental

The experimental portion of the procedure, as mentioned in the previous chapter, consists of measuring the torsional vibration at the front of the crankshaft (Sta. 1, Figure 4). The engine installation is tested at a frequency or speed in the operating range that will excite one of the natural frequencies of the system, since operation at a natural frequency is considered to be the worst possible vibration condition. Also, the mode of vibration at a natural frequency is known and it can be used with the one experimentally determined displacement ($\theta_1$) to find the others ($\theta_2, \theta_3, \ldots, \theta_n$). From these displacements the shear stresses in the crankshaft are calculated.

The speeds that excite the natural frequencies of the system are called critical speeds. They occur when one of the orders of the torque relation coincides with a natural frequency. Each critical engine speed is expressed as:

\[
\text{Critical engine speed} = \frac{\text{Natural frequency}}{\text{Order number}}
\]

The critical speeds for the system of Figure 4 are shown in Table 2. Although they are numerous, fortunately not all of them are of practical significance. The most important critical speeds lie in the range of engine operation and are of lower orders, since, for example, a first order torque harmonic can have a magnitude that is several
Table 2. Critical Speeds for System Shown in Figure 4.

<table>
<thead>
<tr>
<th>Torque Harmonic</th>
<th>Critical Engine Speeds Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st Mode $f_n = 5.54$ hz</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>11.08</td>
</tr>
<tr>
<td>1</td>
<td>5.54</td>
</tr>
<tr>
<td>$1\frac{1}{2}$</td>
<td>3.69</td>
</tr>
<tr>
<td>2</td>
<td>2.77</td>
</tr>
<tr>
<td>$2\frac{1}{2}$</td>
<td>2.215</td>
</tr>
<tr>
<td>3</td>
<td>1.845</td>
</tr>
<tr>
<td>$3\frac{1}{2}$</td>
<td>1.583</td>
</tr>
<tr>
<td>4</td>
<td>1.383</td>
</tr>
<tr>
<td>$4\frac{1}{2}$</td>
<td>1.230</td>
</tr>
<tr>
<td>5</td>
<td>1.108</td>
</tr>
<tr>
<td>$5\frac{1}{2}$</td>
<td>1.006</td>
</tr>
<tr>
<td>6</td>
<td>0.923</td>
</tr>
<tr>
<td>$6\frac{1}{2}$</td>
<td>0.852</td>
</tr>
<tr>
<td>7</td>
<td>0.792</td>
</tr>
<tr>
<td>$7\frac{1}{2}$</td>
<td>0.739</td>
</tr>
<tr>
<td>8</td>
<td>0.6925</td>
</tr>
<tr>
<td>$8\frac{1}{2}$</td>
<td>0.652</td>
</tr>
<tr>
<td>9</td>
<td>0.615</td>
</tr>
<tr>
<td>$9\frac{1}{2}$</td>
<td>0.583</td>
</tr>
<tr>
<td>10</td>
<td>0.554</td>
</tr>
</tbody>
</table>
times greater than a fourth order.

The data obtained from one of these tests at a critical speed is harmonically analyzed. The resulting relation is generally composed of several harmonics with the amplitude of the excited natural frequency predominating. This amplitude is then used with the corresponding mode shape to find the deflections and stresses in the crankshaft for that particular condition of operation.

For example, if the engine system in Figure 4 was run at 5.54 hz and the harmonically analyzed data yielded an amplitude of $\theta_1 = \pm 2.5^\circ$ for the first natural frequency (5.54 hz), then the corresponding mode of vibration would be multiplied by a factor of 2.5 to give the maximum angular deflections (see Figure 5).
Figure 5. Amplitude of first mode of vibration for system of Figure 4 with $\theta_1 = 2.5^\circ$. 
Current Devices Used to Measure Torsional Oscillations

The following are some of the more widely used devices for present day measurement of torsional oscillations:
General Motors Mechanical Torsiograph

The General Motors Mechanical Torsiograph is a refinement of the earlier Geiger and Sommers instruments. It is a displacement type pickup consisting of an inertial mass elastically coupled to the shaft under test by four springs. The relative displacement between the seismic mass and the shaft actuates an indicating stylus finger through a mechanical linkage. The stylus then records the motion on a sensitized paper attached to a hand held member which is momentarily pressed against it. Hence a polar diagram of the torsional oscillation is obtained.

TECHNICAL DATA

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency Range</td>
<td>16.7 to 334 hz</td>
</tr>
<tr>
<td>Amplitude Range</td>
<td>±0.05° to 2.5°</td>
</tr>
<tr>
<td>Recording Method</td>
<td>Stylus on Sensitized Paper</td>
</tr>
<tr>
<td>Natural Frequency</td>
<td>8 hz</td>
</tr>
<tr>
<td>Power Requirement</td>
<td>None</td>
</tr>
<tr>
<td>Attachment Method</td>
<td>1&quot; Collet</td>
</tr>
<tr>
<td>Dimensions</td>
<td>6 1/4&quot; dia. X 10&quot;</td>
</tr>
<tr>
<td>Weight</td>
<td>9 1/2 lbs.</td>
</tr>
</tbody>
</table>
Figure 6. General Motors Mechanical Torsiograph.
C. E. C. Torsional Vibration Pickup

The Consolidated Electrodynamics Corp. torsional vibration pickup is also a seismic type of instrument. The seismic mass in this case is a magnet mounted on ball bearings around an armature which is attached to the crankshaft by means of a collet. The mass is elastically coupled to the armature by its strong magnetic field. An induction coil is attached to the armature and its turns are cut by the magnetic lines of flux between the seismic mass and the armature. The relative motion between the mass and the induction coil generates a voltage which is proportional to the angular velocity between the two. This voltage signal is picked off the rotating instrument by means of silver slip rings. The voltage can be used directly for indication of vibratory angular velocity or integrated to find angular displacement oscillations.
<table>
<thead>
<tr>
<th><strong>TECHNICAL DATA</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Frequency Range</strong></td>
<td>10 - 10,000 hz</td>
</tr>
<tr>
<td><strong>Amplitude Range</strong></td>
<td>± 0.05° to 2°</td>
</tr>
<tr>
<td><strong>Recording Method</strong></td>
<td>Varies</td>
</tr>
<tr>
<td><strong>Natural Frequency</strong></td>
<td>3 hz</td>
</tr>
<tr>
<td><strong>Power Requirement</strong></td>
<td>Self Generating</td>
</tr>
<tr>
<td><strong>Sensitivity</strong></td>
<td>9 Millivolts/degree/second</td>
</tr>
<tr>
<td><strong>Attachment Method</strong></td>
<td>1 1/4&quot; Collet</td>
</tr>
<tr>
<td><strong>Dimensions</strong></td>
<td>3 3/4&quot; dia. X 5 3/4&quot;</td>
</tr>
<tr>
<td><strong>Weight</strong></td>
<td>9.5 lbs.</td>
</tr>
</tbody>
</table>
Figure 7. C. E. C. Torsional Vibration Pickup.
General Motors Phase Shift Torsiograph

This device utilizes a gear placed on the end of the crankshaft and two magnetic pickups placed in series. During operation the teeth of the gear move past the pickups generating a relatively high frequency A.C. voltage. A saw tooth wave is then synchronized to the average frequency of the pickup signal and it is analogous to the seismic mass of the instruments previously discussed. This saw tooth wave is applied to the vertical channel of an oscilloscope. The positive peaks of the pickup signal are then converted into sharp pulses; these pulses are then used to modulate the intensity of the saw tooth waveform on the oscilloscope screen. Hence along each slope of the saw tooth wave there will appear a bright dot. If the pickup signal has a constant frequency the dots will appear at the same spot on each slope of the saw tooth, but if the pickup signal has a variable frequency due to a torsional vibration the dots will rise and fall on the saw tooth slopes because of the change in phase (see Figure 8).

When the sweep speed of the oscilloscope is adjusted to the crankshaft speed, the saw tooth wave will appear as a rectangular band of light with the dots appearing as a broken horizontal line if the pickup signal has a constant frequency, or as a dotted waveform depicting the
torsional vibration if the pickup signal has a variable frequency. The amplitude of the torsional vibration can be referenced to the height of the saw tooth wave which is the angular displacement between the gear teeth.

a) 

b) 

c) 

d) 

Figure 8.  
a) Signal from magnetic pickup.  
b) Saw tooth wave synchronized to the average frequency of the pickup signal.  
c) Pickup signal broken into series of sharp pulses.  
d) Saw tooth wave with intensity modulated by pulses from pickup signal. Dots trace out the torsional vibration.  

Technical data on General Motors Phase Shift Torsiolongraph was not available.
Figure 9. General Motors Phase Shift Torsiograph.
Design of Instrument

Introduction

A seismic type of torsiograph utilizing a strain gage displacement transducer to measure the relative motion between the seismic mass and the input shaft was decided upon for a number of reasons:

1. This would be a direct displacement readout device.
2. Static as well as dynamic calibrations could be performed on the instrument.
3. Recording equipment compatible to strain gage transducers was readily available.

Viscous metal (mercury) slip rings provided the means of transferring the electrical signal from the rotating instrument to the recording device. This type of slip ring, with its low noise and almost constant resistance throughout the speed range, solved one of the primary problems involved with the instrumentation of rotating systems.

Design Criteria

The design criteria for the instrument were as follows:
1. Amplitude range: 0.1° to 10° of peak to peak angular displacement
2. Bandwidth: 20 to 400 hz
3. Natural frequency: Less than 10 hz
4. Full scale output: At least 2 millivolts/volt
5. Mounting: Self centering collet for 1" shaft
6. Dimensions: Less than 7" dia. and 6" long
7. Weight: Less than 5 lbs.

Instrument Theory

Consider an idealized model of a torsiograph as shown in Figure 10, with the input $\gamma$ composed of a constant velocity displacement and an oscillatory motion expressed as:

$$\gamma = \omega_0 t + \gamma(t)$$

The differential equation of motion of the idealized model can be written as:

$$I \ddot{\phi} + C_T \dot{\phi} = 0$$

or

$$\ddot{\phi} + \frac{\omega_n^2}{I} \phi = 0$$  \hspace{1cm} (25)

where the natural frequency is:

$$\omega_n^2 = \frac{C_T}{I} \text{ (rad/sec)}^2$$  \hspace{1cm} (26)
Figure 10. Idealized model of torsiograph.

From Figure 10:
\[ \delta = \phi + \gamma \]
and thus:
\[ \ddot{\delta} = \ddot{\phi} + \ddot{\gamma} \]

Upon substitution of the above into equation (25), the resulting expression is:
\[ \ddot{\phi} + \omega_n^2 \phi = -\ddot{\gamma} \tag{27} \]

Assuming that the oscillatory motion \(\dot{\gamma}\) of the input function \(\gamma\) is sinusoidal, then:
\[ \dot{\gamma} = \gamma_0 e^{i\omega t} \]
and it follows that:
\[ \ddot{\gamma} = -\omega^2 \gamma_0 e^{i\omega t} \]
Substituting the above into equation (27), the result is:

\[ \ddot{\phi} + \omega_n^2 \phi = \omega^2 \gamma_o e^{i\omega t} \]

The solution to the above differential equation of motion is of the form:

\[ \phi = \phi_o e^{i\omega t} \]

which, upon substitution, gives:

\[ -\omega^2 \phi_o + \omega_n^2 \phi_o = \omega^2 \gamma_o \]

Finally, the above can be rewritten to express the frequency response:

\[ \frac{\phi_o}{\gamma_o} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \]

or:

\[ \frac{\phi_o}{\gamma_o} = \frac{\left(\frac{f}{f_n}\right)^2}{1 - \left(\frac{f}{f_n}\right)^2} \] \hfill (28)

where:

\[ f = \frac{\omega}{2\pi} \text{(Hz)} \] \hfill (29)

Figure 11 shows the frequency response of the idealized model of the torsiograph with a natural frequency of 5 Hz.
Figure 11. Frequency response curve of idealized torsiograph with $f_n = 5$ hz.
Instrument

The instrument design is shown in Figure 12. Essentially it consisted of an aluminum mass which revolved about a hollow shaft on an oilite bearing. The shaft could be slipped over a one-inch diameter shaft and secured by means of a collet. The mass was coupled to the shaft by two leaf springs, mounted as cantilever beams, one end being fixed to the mass and the other end being displaced by the shaft. The full scale deflection of the instrument was limited to ±5° by rubber stops. The configuration of the springs was the result of an attempt to keep the deflection and corresponding stresses in the leaf springs below a certain value, and the lowest natural frequency of the springs at a maximum. Strain gages were mounted on both sides of each spring, thus forming a displacement transducer with a full bridge. The two input and the two output leads to the bridge were fed through the shaft to a four-pin connector which was coupled to its four-pin counterpart on the slip ring assembly.

The inertia of the mass alone was calculated to be 0.01398 lb-in-sec². The torsional spring constant was found as follows:
Figure 12. Torsiograph assembly.
Figure 13. Strain gage location and circuit.

Figure 14. Torsiograph.
Consider the mounting arrangement of the leaf spring:

\[ y = 0.813 \phi \enspace (\phi \text{ in radians}) \]

and the spring constant of the leaf spring is:

\[ C = \frac{3EI}{y^3} \]

Hence the torsional spring constant due to both leaf springs is:

\[ C_T = (2) \frac{Cy}{\phi} \]

Finally, for a steel leaf spring 0.019" thick and 1/2" wide the torsional spring constant is then:

\[ C_T = 27.4 \text{ lb-in/rad} \]
and the theoretical natural frequency of the instrument is found to be:

$$f_n = \frac{1}{2\pi} \sqrt{\frac{C_T}{I}}$$

$$= 7.03 \text{ Hz}$$

A lower experimental value for the natural frequency was expected since the inertia of the seismic mass would increase with the addition of the springs and mounting hardware.

To determine the natural frequency of the leaf spring, its mounting condition was assumed to be fixed-hinged. The frequency corresponding to the first mode of vibration is: (Harris and Crede, 1961, Chap. 1, p. 14)

$$f = \frac{1}{2\pi} (15.4) \sqrt{\frac{EI}{\mu l^4}}$$

\[\text{FIXED} \quad \text{HINGED}\]

Figure 16. Assumed first mode of vibration of leaf spring.

From the above relation the natural frequency of the leaf spring was found to be 1635 Hz. This value was large
enough to alleviate any fears of the displacement transducer approaching its first resonance point during the operation of this instrument in its intended range.

Theoretical Electro Mechanical Response

The strain gages were located on the leaf springs as shown in Figure 17.

Figure 17. Strain gage installation.

A relation between the stress at the gages and the displacement of the spring can be derived as follows:

Figure 18. Cantilever leaf spring with the properties $E$, $I_x$ and thickness $h$. 
The displacement of the spring can be written:

\[ y = \frac{p l^3}{3EI_x} \]

or by rewriting:

\[ p = \frac{3EI_x y}{l^3} \]

\[ = \frac{Mc}{I_x} = \frac{pdh}{I_x} \]

The strain at the gage location is expressed as:

\[ \varepsilon = \frac{3}{2} \frac{dhy}{l^3} \]

and from the previous section:

\[ y = 0.813 \phi \]

Upon substituting, the result is:

\[ \varepsilon = (\frac{3}{2})(0.813) \frac{dh}{l^3} \phi \]

Rewriting and substituting for the leaf spring, one then has the following relation for the strain at the gage location (strain at each gage):

\[ \frac{\varepsilon}{\phi} = 13,200 \text{ microstrains/radian} \]

or

\[ \frac{\varepsilon}{\phi} = 230.0 \text{ microstrains/degree} \]

The full scale output can be found from the relation:

\[ \frac{F_{\text{out}}}{F_{\text{in}}} = \left( \frac{G \cdot F_{\cdot}}{4} \right) \text{ (total strain)} \]
The gages used in the instrument have a gage factor of 1.83, and the total strain for a deflection of 5° would be 4600 microstrains. The full scale output is then:

\[ \frac{E}{E_0} = 2.1 \text{ millivolts/volt} \]

To summarize, the idealized model of the instrument has the following properties:

\[ f_n = 7.03 \text{ Hz} \]

Sensitivity = 920 microstrains/degree

These properties were determined to give an idea of what to expect from the instrument, and to be used for comparison with the actual properties in order to check the assumptions made about linearity.
Calibration of Instrument

Static Calibration

The instrument was both statically and dynamically calibrated. The static calibration was accomplished by displacing the instrument a known amount and reading the output in microstrains on a C.R.T.

The same readout system was used for both the calibrations and the engine test. Input and output of the full bridge of the instrument were connected to a Tektronix 3C66 carrier amplifier and Type 564 storage oscilloscope through mercury slip rings.

The carrier amplifier was calibrated for 120 ohm strain gages with a gage factor of 2.00. The strain gages used in the instrument were 120 ohm with a gage factor of 1.83. Hence the readings from the calibration setup were adjusted by a factor of \( \frac{2.00}{1.83} \) to give the actual strain. The average sensitivity of the instrument was found to be 1025 microstrains per degree, or in terms of full scale output (5°) the sensitivity was 2.55 millivolts per volt.

The natural frequency of the instrument was experimentally determined by recording the free oscillations of the seismic mass about a fixed input shaft (see Figure 21). The experimental value was 6.25 hz, which was
Figure 19. Block diagram of calibration and test setup.
Table 3. Static Calibration Data.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Actual Strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>+5°</td>
<td>5100 microstrains</td>
</tr>
<tr>
<td>+4°</td>
<td>4100</td>
</tr>
<tr>
<td>+3°</td>
<td>3100</td>
</tr>
<tr>
<td>+2°</td>
<td>2100</td>
</tr>
<tr>
<td>+1°</td>
<td>1050</td>
</tr>
<tr>
<td>0°</td>
<td>0</td>
</tr>
<tr>
<td>-1°</td>
<td>1000</td>
</tr>
<tr>
<td>-2°</td>
<td>2000</td>
</tr>
<tr>
<td>-3°</td>
<td>3050</td>
</tr>
<tr>
<td>-4°</td>
<td>4050</td>
</tr>
<tr>
<td>-5°</td>
<td>5050</td>
</tr>
</tbody>
</table>

Sign convention: + ( ) -
Figure 20. Static calibration curve of torsiograph.
Vertical: 2000 microstrains/division  
Horizontal: 50 milliseconds/division  
Natural Frequency: 6.25 hz  

Figure 21. Free vibration output signal of torsiograph.
lower than the calculated value of 7.03 hz as expected from previous discussion.

Dynamic Calibration

The dynamic response of the instrument was determined, using a Hookes Joint calibrator. This device utilizes the principle that a Hooke's joint with a uniform input velocity, when set at a given angle, will produce at the output a uniform velocity plus a torsional oscillation of amplitude $(\theta)$ and fundamental frequency of twice that of the input.

![Figure 22. Hooke's joint.](image)

The amplitude of the torsional vibration is expressed as: (Wilson, 1941, Vol. II, p. 283)

$$\theta = (\psi - \beta) = z \sin 2\beta + \frac{z^2}{2} \sin 4\beta + \frac{z^3}{3} \sin 6\beta + ...$$

(30)
where:

\[ Z = \frac{1 - \cos \alpha}{1 + \cos \alpha} \quad (31) \]

The components of the second and fourth harmonics of equation (30) for values of \( \alpha \) up to 20° are listed in Table 4. The relation is plotted in Figure 23.

The calibrator was laid out as shown in Figures 24 and 25. The readout system was the same as for the static calibration, as shown in Figure 26. The angle of the Hooke's joint was set at 15° and the corresponding output relation was:

\[ \theta = 0.99307 \sin 2\beta + 0.008606 \sin 4\beta \]

Considering only the second and fourth harmonics as shown above, the maximum amplitude of the output would be approximately the coefficient of the second harmonic.

The procedure consisted of running the calibrator at a certain speed, storing the output trace on the oscilloscope screen and photographing it for later analysis. This procedure was then repeated for another speed. The resulting data was used to plot the frequency response curve shown in Figure 27.

The frequency response curve is the actual strain in the instrument plotted versus the fundamental frequency of the torsional vibration input from the Hooke's joint. The first resonance of this curve occurred at a
Table 4. Second and Fourth Harmonics for Hooke's Joint Relation.

<table>
<thead>
<tr>
<th>α</th>
<th>Z</th>
<th>( \frac{Z^2}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>0°</td>
</tr>
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</tr>
<tr>
<td>20</td>
<td>1.78140</td>
<td>.027693</td>
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</table>
\[ \Theta = (\psi - \beta) = Z \sin 2\beta + \frac{Z^2}{2} \sin 4\beta + \frac{Z^3}{3} \sin 6\beta + \cdots \]

\[ Z = \frac{1 - \cos \alpha}{1 + \cos \alpha} \]

Figure 23. Hooke's joint characteristics.
Figure 24. Calibrator layout.
Figure 25. Calibrator.

Figure 26. Dynamic calibration setup.
Table 5. Dynamic Calibration Data.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Readout</th>
<th>Actual Strain</th>
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<tr>
<td>5.7 hz</td>
<td>1900 microstrains</td>
<td>2055 microstrains</td>
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<tr>
<td>6.06</td>
<td>2500</td>
<td>2730</td>
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<tr>
<td>6.25</td>
<td>2800</td>
<td>3060</td>
</tr>
<tr>
<td>7.15</td>
<td>4200</td>
<td>4590</td>
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<tr>
<td>7.7</td>
<td>2600</td>
<td>2840</td>
</tr>
<tr>
<td>8.45</td>
<td>2300</td>
<td>2510</td>
</tr>
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<td>9.3</td>
<td>1700</td>
<td>1860</td>
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<td>1200</td>
<td>1310</td>
</tr>
<tr>
<td>15.4</td>
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<td>1200</td>
</tr>
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<td>18.2</td>
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</tr>
<tr>
<td>21.7</td>
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<td>1038</td>
</tr>
<tr>
<td>34.5</td>
<td>910</td>
<td>994</td>
</tr>
<tr>
<td>40.7</td>
<td>910</td>
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<td>52.2</td>
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<td>1038</td>
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<td>62.5</td>
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<td>1038</td>
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<td>71.4</td>
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<tr>
<td>104.7</td>
<td>1070</td>
<td>1170</td>
</tr>
<tr>
<td>109.7</td>
<td>1100</td>
<td>1200</td>
</tr>
</tbody>
</table>
Figure 27. Frequency response curve of torsiograph.
frequency of approximately 6.25 hz which agreed with the previously determined natural frequency of the instrument. To the right of the resonance point at about 25 hz the curve became a horizontal line at 994 microstrains. From 40 hz on, the curve assumed a slightly increasing positive slope. The waveforms corresponding to this portion of the curve had sharper peaks as compared with the smoother lower frequency waveforms (see Figure 28). The reason for the differences in the waveforms and the corresponding slope in the response curve was attributed to either one or both of the following:

1. A certain amount of play in the universal joint which, as the frequencies increase and the corresponding inertial forces become larger, could add to the output amplitude.⁴

2. This increasing slope could also be due to the influence of a second resonance point in the response curve which was the first resonance point of the calibration system itself. This first mode of the calibration system was calculated to be 525 hz (see Appendix). The limited

⁴Experimental work in this area is discussed in: S.A.E. War Engineering Board (1945), p. 99-125.
Vertical: 2000 microstrains/division
Horizontal: 50 milliseconds/division
Frequency: 7.7 hz
Amplitude: 2600 microstrains

Figure 28(a). Typical calibration waveform.
Vertical: 1000 microstrains/division
Horizontal: 5 milliseconds/division
Frequency: 109.7 hz
Amplitude: 1100 microstrains

Figure 28(b). Typical calibration waveform.
frequency range of the calibrator prevented any experimental determination of this point.

![Graph showing frequency response curve with first and second resonance points]

Figure 29. Possible extension of frequency response curve.

It was assumed that the instrument had a flat frequency response curve from 25 hz. The corresponding sensitivity was 1001 microstrains/degree, which varied from the sensitivity arrived at in the static calibration by 2.4%. The instrument could be operated down to a frequency of 10 hz with the use of the frequency response curve.
The theoretical and experimental properties of the instrument are compared as follows:

<table>
<thead>
<tr>
<th></th>
<th>Theoretical</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Frequency</td>
<td>7.03 hz</td>
<td>6.25 hz</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>920 micro-strains/degree</td>
<td>1001 micro-strains/degree</td>
</tr>
</tbody>
</table>

The natural frequency of the actual system was lower than the calculated value which was expected as mentioned previously. The sensitivity of the actual instrument compared favorably with the theoretical value. This indicated that the instrument functioned both mechanically and electrically as predicted from the analytical model, therefore the assumption made about the lineality of the instrument was valid.
Typical Engine Test

The instrument was tested on a Chrysler Corporation six-cylinder engine coupled to a General Electric Dynamometer as shown in Figure 30. It was attached to the crankshaft pulley by means of an adapter. The adapter consisted of a one-inch diameter shaft welded to a three-hole flange which was bolted to the face of the pulley. The concentricity of the adapter with the crankshaft was set with a dial indicator. The instrument readout set up for the test was the same as for the calibrations. A General Radio Strobatach was used to monitor engine speed.

The method of testing consisted of running the engine through a speed range (600 to 2500 RPM) both loaded and unloaded and observing the torsional vibration on the oscilloscope screen. At only one point was the amplitude of the vibration observed to increase considerably over the normal amplitude of 0.7° peak to peak. This was with the engine at approximately 1000 RPM under full load and the amplitude in that case was 2.4° peak to peak.

Two waveforms of the torsional vibration for the same engine speed but different load conditions are shown in Figure 31. The first waveform is typical of
Figure 30(a). Torsiograph test setup.

Figure 30(b). Torsiograph test setup.
Figure 30(c). Torsiograph test setup.

Figure 30(d). Torsiograph test setup.
1000 RPM at no load
Vertical: 1 degree = 1 division
Horizontal: 1 revolution = 1.1 divisions

Figure 31(a). Typical engine test waveform.

Figure 31(b). Same as 31(a) except engine at full load.
what was observed throughout the test. The frequency of this waveform was the same as the frequency of the power strokes of the engine, as would be expected from a system with a forced vibration. The one higher peak of the waveform was periodic with the power stroke of one cylinder and was attributed to a faulty valve which was audible as well as visible on the scope. The second waveform was the only resonance point found in the testing, as mentioned previously. Information about the natural frequencies and modes of vibration of the engine dynamometer system was not available for comparison with test results.
Results and Conclusions

The instrument described in this thesis fulfilled the original criteria except for one question concerning the bandwidth. This question arises from the calibration limit of 100 hz which prevented any means of verifying that the response curve was flat above that frequency. The resolution of the instrument was ten times greater than the criteria when used in conjunction with a Tektronix Type 3C66 Carrier Amplifier. In that case, with the amplifier in the most sensitive position of ten microstrains/division, the output of the instrument was 0.01 degrees/division.

The mechanical portion of the instrument performed well in the several hours of calibration and engine testing. The problems that did occur involved the strain gages, wiring, and the four-pin connector. The only critical aspect of setting up the instrument for testing was aligning the slip ring assembly with the instrument shaft. Any misalignment between the two resulted in strains on the mechanical and electrical connection, with resulting vibrations in the slip ring assembly causing in some instances a loss of mercury.

Further work on this instrument should probably include extending the calibration to a higher frequency.
and a comparison of actual engine test data with that taken with one of the commercially available torsiographs. A further development recommended on the instrument itself is some improvement in the mechanical and electrical connection between the instrument and the slip rings and a means of aligning the two when setting up for a test.

Table 6. Technical Data of Torsiograph.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency Range</td>
<td>10 to 100 hz</td>
</tr>
<tr>
<td>Amplitude Range</td>
<td>± 0.01° to 5°</td>
</tr>
<tr>
<td>Recording Method</td>
<td>Varies</td>
</tr>
<tr>
<td>Natural Frequency</td>
<td>6.25 hz</td>
</tr>
<tr>
<td>Power Requirement</td>
<td>5 volts A.C. or D.C.</td>
</tr>
<tr>
<td>Sensitivity (Full Scale)</td>
<td>2.55 millivolts/volt</td>
</tr>
<tr>
<td>Attachment Method</td>
<td>1&quot; Collet</td>
</tr>
<tr>
<td>Dimensions</td>
<td>5 3/4&quot; dia. X 5 1/2&quot; long</td>
</tr>
<tr>
<td>Weight</td>
<td>3 lbs.</td>
</tr>
</tbody>
</table>

5 When used in conjunction with Tektronix Type 3C66 Carrier Amplifier.

6 Note that torsiograph is connected to recording apparatus by slip ring assembly (Rotocon Type MSD-4).
Bibliography


APPENDIX
Appendix

Natural Frequency of Calibrator System

Assuming that the coupling is rigid, have idealized system:

Figure 32. Calibrator system.

The relation for the natural frequency of the above system is:
\[ \omega_n^2 = \frac{1}{2} \left[ B \pm \sqrt{B^2 - \frac{4k_1k_2}{I_1I_2I_3}(I_1 + I_2 + I_3)} \right] (\text{rad/sec})^2 \]

where:

\[ B = \frac{k_1}{I_1} + \frac{k_2}{I_3} + \frac{k_1+k_2}{I_2} \]

Substituting:

\[ B = \frac{148.7 \times 10^3}{0.123} + \frac{18.4 \times 10^3}{0.00149} + \frac{(148.7+18.4) \times 10^3}{0.0042} \]

\[ = 1210 \times 10^3 + 12350 \times 10^3 + 39750 \times 10^3 \]

\[ = 53610 \times 10^3 = 53.61 \times 10^6 \]

\[ \omega_n^2 = \frac{1}{2} \left[ 53.61 \times 10^6 \pm \sqrt{2870 \times 10^{12} - \frac{(4)(148.7)(18.4) \times 10^6(0.12869)}{(0.123)(0.0042)(0.00149)}} \right] \]

\[ \omega_n^2 = \frac{1}{2} \left[ 53.61 \times 10^6 \pm \sqrt{2870 \times 10^{12} - 1830 \times 10^{12}} \right] \]

\[ = \frac{1}{2} \left[ 21.7 \times 10^6 \right], \quad \frac{1}{2} \left[ 85.8 \times 10^6 \right] \]

\[ \omega_n^2 = 10.9 \times 10^6 \text{ rad/sec, } 42.9 \times 10^6 \text{ rad/sec} \]

\[ \omega_n = \pm \ 3.3 \times 10^3 \text{ rad/sec, } \pm \ 6.5 \times 10^3 \text{ rad/sec} \]

\[ f_n = \frac{\omega_n}{2\pi} = 525 \text{ hz, } 1035 \text{ hz} \]

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