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The Picard group is investigated for a certain class of non-singular surfaces: two sheeted branched coverings of CP^2. While I confine the study to double coverings, the results apply to coverings of any degree. The approach is to describe the Picard group in terms of properties of the branch curve.

The Picard group of a higher dimensional covering of complex projective space has a simple description. Let $\pi: W \to CP^n$ denote a covering, branched over a nonsingular hypersurface, and let $[H]$ denote the line bundle of a hyperplane H in CP^n. If $n \geq 3$, the Picard group of W is infinite cyclic, and generated by $\pi^*[H]$. This is a consequence of the topology of W, and therefore is independent of the branch locus.

Let $\pi: W \to CP^2$ denote a double covering, branched over a nonsingular curve in CP^2. The line bundle $\pi^*[H]$ can always be taken as a generator of the Picard group. Additional generators arise when an irreducible curve in CP^2 has reducible inverse

image in the double cover. Some of these curves are in one-to-one correspondence with divisors on the branch curve possessing special properties. Necessary and sufficient conditions are given for the existence of such divisors. More information is obtained by observing that these divisors correspond to square roots on the Jacobian Variety of the branch curve. Relations between these square roots on the Jacobian Variety provide interesting linear equivalence relations between curves on the double cover.

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TABLE OF CONTENTS

ON THE PICARD GROUP OF BRANCHED COVERINGS OF CP^2

INTRODUCTION

In this paper, we study several properties of a certain class of complex algebraic varieties culminating in an investigation of their Picard group. These varieties are double coverings of complex projective n-space CP^n branched over a nonsingular hypersurface. Already, this setting is sufficiently complicated, however, the techniques involved can be expanded to study cyclic, branched coverings of more general varieties.

The overall theme of this investigation is to exploit the heavy dependence a covering has to its branch locus. We relate properties of the covering to properties of the branch locus which are often simpler or known.

An extra effort is made to maintain a self contained exposition. Often well known facts are discussed or stated. This also serves to fix notation.

The first four sections are devoted to computing various topological and numerical invariants of these varieties. In keeping with the theme, they are stated in terms of properties of the branch locus. It is not surprising that a large set of invariants are similar to that of both CP^n and the branch locus.

These invariants become useful when the investigation of the Picard group is undertaken in Section 5. For example, if $\pi : W \to CP^n$ is a double covering of CP^n, it is shown, as a result of the topology of W, for $n \geq 3$ the Picard group of W is infinite cyclic.

If $n = 2$ the situation is complicated. Consequently, most of Section 5 is devoted to the case when W is a surface. The problem of describing the Picard group of a particular covering divides naturally into two problems--generators and relations.

Generators arise when irreducible curves in CP^2 have reducible inverse image in W. Necessary and sufficient conditions are given for their existence. There are difficulties relating all of these curves to the branch locus. This forces one to further divide the problem by considering two types of curves. The two types are determined by singling out those for which their only singularities are locally irreducible. The existence of the first type of curves, that is, irreducible curves with reducible inverse image and which have only locally irreducible singularities, can be completely characterized by certain divisors on the branch locus. The remaining curves evade such a characterization.

The problem of describing the linear equivalence relations among the inverse image of these curves suffers the same fate. It is shown that the linear equivalence relations between the curves of the

first type can be reduced to linear equivalence relations between divisors on the branch locus.

There remains a very interesting question as to whether all the generators and relations can be described in terms of the branch locus. This problem is discussed throughout Section 5.

The results here, go a long way toward describing the Picard group and in a few special cases, describe it completely. The techniques developed also provide a useful way of producing coverings with 'large' Picard group, and have some interesting applications to the classification of surfaces.

1.   DIVISORS AND LINE BUNDLES

We begin by reviewing some definitions and well known facts about divisors and line bundles.

Let   W   be a nonsingular complex algebraic variety.   A Weil divisor of   W   is an element of the group

> Div W = free abelian group generated by the
>
>   irreducible subvarieties of codimen-
>
>   sion one.

Every meromorphic function   f   on   W   defines a divisor   (f)   by

$$(f) = (f)_0 - (f)_\infty$$

where $(f)_0$ denotes the zero set of   f   and $(f)_\infty$ denotes the polar set of   f.   The subgroup

> Div_ℓ W = divisors of meromorphic functions on W,

is called the group of divisors linearly equivalent to zero.   The quotient

$$\text{Pic } W = \frac{\text{Div } W}{\text{Div}_\ell W}$$

is called the Picard Group of   W.

There is a sheaf theoretic formulation of the above groups, and is equivalent when W is nonsingular. Define

$\mathcal{O}$ = sheaf of germs of local holomorphic functions on W.

$\mathcal{O}^*$ = sheaf of germs of local nowhere zero holomorphic functions on W.

$\mathcal{m}^*$ = sheaf of germs of local meromorphic functions on W that are not identically zero on any component of W.

A Cartier divisor is a section of the sheaf $\mathcal{D} = \dfrac{\mathcal{m}^*}{\mathcal{O}^*}$, that is, for some covering $\mathcal{U} = (U_i)$ of W, D can be represented by a collection $\{f_i\}$ of meromorphic functions f_i on U_i such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. When W is nonsingular there is an isomorphism

$$\Gamma(W, \mathcal{D}) \to \text{Div } W$$

given by

$$f \mapsto \sum_{X \text{ codim } 1} \text{ord}_X(f) \cdot X$$

providing a natural equivalence between Weil and Cartier divisors.

There is a short exact sequence of sheaves on W

(1.1)
$$0 \to \mathcal{O}^* \to \mathcal{m}^* \to \mathcal{D} \to 0.$$

From the short exact sequence (1.1) we obtain a long exact sequence of cohomology groups

$$0 \to \Gamma(W, \mathcal{O}*) \to \Gamma(W, \mathcal{M}*) \to \Gamma(W, \mathcal{D}) \xrightarrow{\delta} H^1(W, \mathcal{O}*) \to 0.$$

The last zero is because $H^1(W, \mathcal{M}*) = 0$ when W is algebraic. Thus each divisor D defines a $\mathcal{O}*$-bundle δD, or alternatively a complex line bundle we shall denote by $[D]$. The line bundle $[D]$ is the trivial line bundle if and only if D is a divisor of a meromorphic function. Also, every line bundle is the line bundle of a divisor. In other words

$$\text{Pic } W = \frac{\text{Div } W}{\text{Div}_\ell W} \simeq \frac{\Gamma(W, \mathcal{D})}{\text{im}\Gamma(W, \mathcal{M}*)} \cong H^1(W, \mathcal{O}*).$$

There are advantages to both formulations. The varieties studied here will be nonsingular, so we need not distinguish between Weil and Cartier divisors.

Let V be another complex algebraic variety and $\pi : V \to W$ holomorphic. These groups are natural in the sense, that if we define similar sheaves on V then the diagram

$$
\begin{array}{ccccccccc}
0 \to & \Gamma(V, \mathcal{O}*) & \to & \Gamma(V, \mathcal{M}*) & \to & \Gamma(V, \mathcal{D}) & \to & H^1(V, \mathcal{O}*) & \to 0 \\
& \uparrow \pi* & & \uparrow \pi* & & \uparrow \pi* & & \uparrow \pi* & \\
0 \to & \Gamma(W, \mathcal{O}*) & \to & \Gamma(W, \mathcal{M}*) & \to & \Gamma(W, \mathcal{D}) & \to & H^1(W, \mathcal{O}*) & \to 0
\end{array}
$$

commutes. Quite often the symbol $\pi*$ has multiple uses, e.g., pull back of divisors $\pi* : \Gamma(W, \tilde{\mathcal{O}}) \to \Gamma(V, \tilde{\mathcal{O}}')$, pull back of line bundles $\pi* : H^1(W, \mathcal{O}*) \to H^1(V, \mathcal{O}*)$, etc. The correct context will always be clear, for example, we distinguish between a divisor D and its corresponding line bundle $[D]$, so there is no ambiguity in writing $\pi*D$ or $\pi*[D]$.

There is another short exact sequence of sheaves on W

$$0 \to Z \to \mathcal{O} \to \mathcal{O}* \to 0,$$

and a long exact sequence of cohomology groups, a portion of which is

$$\to H^1(W, \mathcal{O}) \to H^1(W, \mathcal{O}*) \xrightarrow{c_1} H^2(W, Z) \to$$

The homomorphism $c_1 : H^1(W, \mathcal{O}*) \to H^2(W, Z)$ assigns a line bundle ξ its first Chern class $c_1(\xi)$, and has the following interpretation. Each divisor D on W determined a homology class. Identifying Čech and Singular cohomology, the Poincaré dual of this homology class is $c_1[D]$. The image of $H^1(W, \mathcal{O}*)$ in $H^2(W, Z)$ is called the Neron-Severi Group of W.

The dimension of $H^1(W, \mathcal{O})$ is sometimes called the irregularity[1] of W and denoted by the symbol q. In case $q = 0$ the Picard Group of W is isomorphic to the Neron-Severi Group of W. This shall be the case for the algebraic varieties studied here.
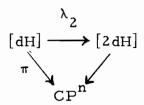
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[1]Actually the irregularity is defined by $q = \dim H^0(W, \Omega^1)$, but by Serre Duality $\dim H^1(W, \mathcal{O}) = \dim H^0(W, \Omega^1)$. See Section 3.

2. SIMPLICIAL HOMOLOGY AND COHOMOLOGY GROUPS

In this section the simplicial homology and cohomology groups are computed for double covers of CP^n branched along a nonsingular hypersurface.

Let B be a nonsingular hypersurface of degree 2d in CP^n. Let [H] denote the line bundle of a hyperplane H in CP^n. The hypersurface B is defined by the zeros of a section s of [2dH], the 2d-fold tensor product of [H], and there is a commutative diagram

$$[dH] \xrightarrow{\lambda_2} [2dH]$$
$$\pi \searrow \qquad \swarrow$$
$$CP^n$$

where λ_2 is the squaring map. Define $W = \lambda_2^{-1} \circ s(CP^n)$. It is clear W is nonsingular and algebraic. The restruction of π to W gives a double covering $\pi : W \to CP^n$ branched along B. More-over, W is endowed with a natural involution $\sigma : W \to W$ that commutes with π. We shall call $B_W = \pi^{-1}(B)$ the ramification locus and B the branch locus of the covering $\pi : W \to CP^n$.

Repeating this construction we have a similar commulative diagram

$$[H] \xrightarrow{\lambda_d} [dH] \xrightarrow{\lambda_2} [2dH]$$

$$\pi_{2d} \searrow \quad \pi \downarrow \quad \swarrow$$

$$CP^n$$

where λ_d is the dth power map. As above, this gives a commutative diagram of nonsingular varieties

(2.1)

$$W'$$
$$\pi_d \swarrow \quad \downarrow \pi_{2d}$$
$$W$$
$$\pi \searrow \quad \downarrow$$
$$CP^n$$

Here $\pi_d : W' \to W$ is a cyclic covering of degree d branched along B_W and $\pi_{2d} : W' \to CP^n$ is a cyclic covering of degree $2d$ branched along B. The advantage of studying W' along with W is that W' can be regarded as a nonsingular hypersurface in CP^{n+1} defined by the zeros of the homogeneous polynomial $z_{n+1}^{2d} - f(z_0, \ldots, z_n)$, where $f = 0$ defines B in CP^n.

Since algebraic varieties admit simplicial triangulations we have the following lemma

Lemma (2.2). There are simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$ with subcomplexes $\mathcal{L}$ and $\mathcal{L}'$ respectively together with simplicial maps $s_\pi : \mathcal{K}' \to \mathcal{K}$ and $s_\sigma : \mathcal{K}' \to \mathcal{K}'$ satisfying the

following properties.

(1) There are homeomorphisms

$$|\mathcal{K}'| \cong W, \quad |\mathcal{K}| \cong CP^n, \quad |\mathcal{L}'| \cong B_W \quad \text{and} \quad |\mathcal{L}| \cong B$$

(Here the symbol $|\mathcal{K}|$ denotes the space of $\mathcal{K}$.)

(2) The following diagrams commute

(3) For every simplex $\Delta \in \mathcal{K}$ there is a simplex $\Delta' \in \mathcal{K}'$ such that $s_\pi(\Delta') = \Delta$.

Let 1_W denote the identity map on W. The kernal of the homomorphism

$$1_{W*} - \sigma_* : H_*(W, Z) \to H_*(W, Z)$$

will be denoted by $H_*(W, Z)^+$, that is, the σ-invariant subgroup of $H_*(W, Z)$.

We use familiar notation found in simplicial homology theory. For example, the simplicial maps s_π and s_σ induce homomorphisms of simplicial chain complexes $\pi_\#$ and $\sigma_\#$, the homomorphisms of homology groups π_* and σ_*. The notation $H_*(\mathcal{K}', Z)$ and $H_*(\mathcal{K}, Z)$, being the homology groups of the simplicial complexes $\mathcal{K}'$ and $\mathcal{K}$, are used synonymously with $H_*(W, Z)$ and $H_*(CP^n, Z)$.

Theorem (2.3). There exists an injective homomorphism

$\pi_! : H_*(CP^n, Z) \to H_*(W, Z)^+$ such that the following diagram com-

mutes

(2.4)

$$
\begin{array}{ccc}
H_*(W, Z)^+ & \xrightarrow{\quad 2 \quad} & H_*(W, Z)^+ \\
\pi_* \downarrow \quad {}^{\pi_!}\nearrow & & \downarrow \pi_* \\
H_*(CP^n, Z) & \xrightarrow{\quad 2 \quad} & H_*(CP^n, Z),
\end{array}
$$

where the horizontal maps are multiplication by 2.

Proof. First of all we shall define $\pi_!$ on the chain level.

In this one case we will use the same symbol $\pi_!$ for a homomor-

phism of simplicial chain complexes as well as a homomorphism of

homology. Define

$$\pi_! : C_q(\mathcal{K}, Z) \to C_q(\mathcal{K}', Z)$$

by

$$\pi_! \left(\sum_i \Delta_i \right) = \sum_i (\Delta_i' + \sigma_\# \Delta_i')$$

where $\pi_\# \Delta_i' = \Delta_i$. The existence of Δ_i' is guaranteed by Lemma

(2.2), and it is clear $\pi_!$ is independent of any choice of Δ_i'. It is

also apparent $\pi_!$ is a homomorphism and commutes with the

boundary map ∂. Therefore, we have a homomorphism of

homology groups $\pi_! : H_*(\mathcal{K}, Z) \to H_*(\mathcal{K}', Z)$.

To establish the commuting of diagram (2.4) let $\psi = \sum_i \Delta_i$ be a q-cycle representing $[\psi] \in H_q(\mathcal{K}, Z)$. Then

$$\pi_!(\psi) = \sum_i (\Delta_i' + \sigma_\# \Delta_i')$$

and

$$\pi_\# \circ \pi_!(\psi) = \sum_i (\pi_\# \Delta_i' + \pi_\# \circ \sigma_\# \Delta_i')$$

$$= \sum_i 2\Delta_i$$

$$= 2\psi.$$

Conversely, let $\theta = \sum_i \Delta_i'$ represent $[\theta] \in H_q(\mathcal{K}', Z)^+$. Thus we can write $\theta - \sigma_\# \theta = \partial\varphi$ for some q+1-chain φ. Let $\Delta_i = \pi_\# \Delta_i'$, then

$$\pi_\#(\theta) = \sum_i \Delta_i$$

and

$$\pi_! \circ \pi_\#(\theta) = \sum_i (\Delta_i' + \sigma_\# \Delta_i')$$

$$= \theta + \sigma_\# \theta$$

$$= 2\theta + \partial(\sigma_\# \varphi).$$

Therefore $\pi_! \circ \pi_*$ is multiplication by 2.

Finally, we show that $\pi_!$ is injective. Let $\psi = \sum_i \Delta_i$ be a q-cycle representing $[\psi] \in H_q(\mathcal{K}, Z)$, and suppose

$$\pi_!(\psi) = \sum_i (\Delta_i' + \sigma_\# \Delta_i') = \partial\varphi$$

where φ is a q+1-chain. Suppose $\varphi = \sum_\alpha \gamma_\alpha$. Each simplex is the sum $\partial\varphi = \sum_i \Delta_i' + \sigma_\# \Delta_i'$, say Δ_i', is a q-face of some γ_α. It follows $\sigma_\# \Delta_i'$ is a q-face of either the same simplex γ_α or $\sigma_\# \gamma_\alpha$, depending upon whether γ_α belongs to $\mathcal{L}'$ or not (recall $|\mathcal{L}'|$ corresponds to the ramification locus of the covering). Therefore we can write

$$\varphi = 2\eta + (\nu + \sigma_\# \nu) + \rho$$

where $\partial\rho = 0$. Let $\varphi' = \varphi - \rho$ and note we still have $\partial\varphi' = \pi_!\psi$. Now

$$2\psi = \pi_\# \circ \pi_!\psi = \pi_\# \partial\varphi' = \partial\pi_\# \varphi'$$

$$= \partial\pi_\# (2\eta + \nu + \sigma_\# \nu)$$

$$= 2 \partial\pi_\# (\eta + \nu)$$

Therefore ψ bounds. This completes the theorem.

<u>Theorem (2.5)</u>. $\pi_* : H_*(W, Z) \to H_*(CP^n, Z)$ is a surjection.

<u>Proof</u>.   Let $[\xi] \in H_q(W, Z)$ be represented by $\displaystyle\sum_i \alpha_i \Delta_i$.   The

cycle $\displaystyle\pi_!(\xi) = \sum_i \alpha_i(\Delta_i' + \sigma_\# \Delta_i')$ can be written

$$\pi_!(\xi) = \sum_{\Delta_i' \in \mathscr{L}'} 2\alpha_i \Delta_i' + \sum_{\Delta_i' \notin \mathscr{L}'} \alpha_i(\Delta_i' + \sigma_\# \Delta_i')$$

Now the chain

$$\gamma = \sum_{\Delta_i' \in \mathscr{L}'} \alpha_i \Delta_i' + \sum_{\Delta_i' \notin \mathscr{L}'} \alpha_i \Delta_i'$$

is a cycle and $\pi_\# \gamma = \xi$.   Therefore π_* is onto.

By the Poincaré duality theorem there is an isomorphism

$$D_W : H^q(W, Z) \to H_{2n-q}(W, Z).$$

It is defined by the formula

$$D_W(x) = x \cap \mu_W$$

that is, by cap product with the fundamental class $\mu_W \in H_{2n}(W, Z)$.
Explicitly, for any $2n$-simplex Δ let $F_q(\Delta)$ denote the front
q-face of Δ and let $F^q(\Delta)$ denote the back q-face of Δ.

Now if ξ is a q-cocycle representing $[\xi] \in H^q(W, Z)$ and $\sum_i \Delta_i$ represents the fundamental class $\mu_W \in H_{2n}(W, Z)$ then

$$D_W[\xi] = (-1)^q [\sum_i <\xi, F_q(\Delta_i)> \cdot F^{2n-q}(\Delta_i)].$$

Using such duality isomorphisms, the composition

$$H_q(CP^n, Z) \xrightarrow{D^{-1}_{CP^n}} H^{2n-q}(CP^n, Z) \xrightarrow{\pi*} H^{2n-q}(W, Z) \xrightarrow{D_W} H_q(W, Z)$$

gives, a priori, a second homomorphism $H_*(CP^n, Z) \longrightarrow H_*(W, Z)$.

Theorem (2.6). $\pi_! = D_W \circ \pi* \circ D^{-1}_{CP^n}$.

Proof. If $\sum_i \Delta_i$ is the unique 2n-cycle representing $\mu_{CP^n} \in H_{2n}(\mathcal{K}, Z)$ then $\mu_W \in H_{2n}(\mathcal{K}', Z)$ is uniquely represented by $\sum_i \sum_{k=0}^{1} \sigma^k_{\#} \Delta'_i$, where again $\pi_{\#} \Delta'_i = \Delta_i$.

Letting ξ be a q-cocycle representing $[\xi] \in H^q(\mathcal{K}, Z)$ we have

$$D_W \pi^*[\xi] = (-1)^q \left[\sum_i \sum_{k=0}^1 <\pi^\# \xi, F_q(\sigma_\#^k \Delta_i')> \cdot F^{2n-q}(\sigma_\#^k \Delta_i')\right]$$

$$= (-1)^q \left[\sum_i \sum_{k=0}^1 <\xi, F_q(\pi_\# \sigma_\#^k \Delta_i')> \cdot F^{2n-q}(\sigma_\#^k \Delta_i')\right]$$

$$= (-1)^q \left[\sum_i \sum_{k=0}^1 <\xi, F_q(\Delta_i)> \cdot \sigma_\#^k (F^{2n-q}(\Delta_i)^i\right]$$

$$= \pi_! (-1)^q \left[\sum_i <\xi, F_q(\Delta_i)> \cdot F^{2n-q}(\Delta_i)\right]$$

$$= \pi_! D_{CP^n}[\xi]$$

Therefore $\pi_! = D_W \circ \pi^* \circ D_{CP^n}^{-1}$ as desired.

As for the other coverings in (2.1) we get similar injections $(\pi_{2d})_! = D_{W'} \circ (\pi_{2d})^* \circ D_{CP^n}^{-1}$ and $(\pi_d)_! = D_{W'} \circ (\pi_d)^* \circ D_W^{-1}$ and commutative diagrams

$$
\begin{array}{ccc}
H_*(W', Z)_{2d}^+ & \xrightarrow{\quad 2d \quad} & H_*(W', Z)_{2d}^+ \\
(\pi_{2d})_* \downarrow \quad (\pi_{2d})_! \quad \nearrow & & \downarrow (\pi_{2d})_* \\
H_*(CP^n, Z) & \xrightarrow{\quad 2d \quad} & H_*(CP^n, Z)
\end{array}
$$

$$H_*(W', Z)^+_d \xrightarrow{\quad d \quad} H_*(W', Z)^+_d$$

$$(\pi_d)_* \downarrow \qquad (\pi_d')_! \qquad \qquad \downarrow (\pi_d)_*$$

$$H_*(W, Z) \xrightarrow{\quad d \quad} H_*(W, Z).$$

In this context $H_*(W', Z)^+_{2d}$ is the subgroup of $H_*(W', Z)$ invariant under the action of the cyclic group of order $2d$ induced from the covering $\pi_{2d} W' \to CP^n$; and $H_*(W', Z)^+_d$ is the subgroup of $H_*(W', Z)$ invariant under the action of the cyclic group of order d induced from the covering $\pi_d : W' \to W$.

Corollary (2.7). The homomorphisms

$$\pi^* : H^*(CP^n, Z) \to H^*(W, Z)$$

$$(\pi_{2d})^* : H^*(CP^n, Z) \to H^*(W', Z)$$

$$(\pi_d)^* : H^*(W, Z) \to H^*(W', Z)$$

are injections.

Theorem (2.8). The homology groups $H_*(W, Z)$ and $H_*(W', Z)$ have no torsion.

Proof. First of all $H_*(W', Z)$ has no torsion because W' is a nonsingular hypersurface in CP^{n+1}. Secondly, as we have noted,

$$(\pi_d)_! : H_*(W, Z) \to H_*(W', Z)$$

is injective. Therefore $H_*(W, Z)$ has no torsion.

__Theorem (2.9).__ $H_q(W, Z) \cong H_q(W, Z)^+$ for all $q \neq n$.

__Proof.__ Recall $W' \subseteq CP^{n+1}$. The automorphism $\tau : W' \to W'$, $\tau^{2d} = 1_{W'}$ extends to an automorphism $\tilde{\tau}$ of CP^{n+1} by the formula

$$\tilde{\tau}(z_0 : \ldots : z_n : z_{n+1}) = (z_0 : \ldots : z_n : e^{2\pi i/2d} \cdot z_{n+1})$$

Hence we have a commutative diagram

$$
\begin{array}{ccc}
W' & \overset{i}{\hookrightarrow} & CP^{n+1} \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tilde{\tau}} \\
W' & \overset{i}{\hookrightarrow} & CP^{n+1}
\end{array}
$$

The corresponding commutative diagram of homology groups is

$$
\begin{array}{ccc}
H_q(W', Z) & \overset{i_*}{\longrightarrow} & H_q(CP^{n+1}, Z) \\
{\scriptstyle \tau_*}\downarrow & & \downarrow{\scriptstyle \tilde{\tau}_*} \\
H_q(W', Z) & \overset{i_*}{\longrightarrow} & H_q(CP^{n+1}, Z)
\end{array}
$$

By the Lefschetz Hyperplane Theorem i_* is an isomorphism for $q < n$. Observe $\tilde{\tau}$ is holomorphic and takes hyperplanes to hyperplanes. It follows $\tilde{\tau}$ induces the identity on $H_*(CP^{n+1}, Z)$.

Therefore $\tau_* = i_*^{-1} \circ \tilde{\gamma}_* \circ i_* = 1_{W*}$ for $q < n$ which proves

$H_q(W', Z) = H_q(W', Z)_{2d}^+$ for $q < n$. By Poincaré Duality and

Theorem (2.8) there is equality for $q \neq n$. This together with the

fact $(\pi_d)_!$ is injective and the involution on W is induced from τ

implies the theorem is true for W.

Combining these facts together, we have the following descrip-

tion of the homology groups of double coverings of CP^n branched

along a nonsingular hypersurface

(2.10)
$$\pi_! : H_q(CP^n, Z) \cong H_q(W, Z), \quad \text{if } q \neq n$$
$$\pi_! : H_n(CP^n, Z) \cong H_n(W, Z)^+,$$
$$\pi_* : H_*(W, Z) \to H_*(CP^n, Z) \quad \text{is surjective,}$$
$$\pi_*[H_*(W, Z)^+] = 2H_*(CP^n, Z).$$

We can find $\dim H_n(W, Z)$ from the Euler-Poincaré characteristic

$\chi(W)$. From the pair (W, B_W) there is long exact sequence of

homology groups from which we conclude

$$\chi(W) = \chi(W, B_W) + \chi(B_W).$$

By Lefschetz Duality $\chi(W, B_W) = \chi(W - B_W)$. Substituting we have

$$\chi(W) = \chi(W - B_W) + \chi(B_W).$$

Since $W - B_W$ is a covering space of $CP^n - B$

$$\chi(W - B_W) = 2\chi(CP^n - B).$$

Therefore

$$\chi(W) = 2\chi(CP^n - B) + \chi(B_W)$$

$$= 2\chi(CP^n - B) + \chi(B)$$

$$= 2\chi(CP^n) - \chi(B)$$

$$= 2(n+1) - \chi(B).$$

$\chi(W)$ can also be computed from the highest Chern class of W which we compute in the next section. We conclude this section with

Theorem (2.11). If $\dim W \geq 2$ then W is simply connected.

Proof. Recall the cyclic, branched covering $\pi_d : W' \to W$ in (2.1). Let $x \in B_{W'}$. Every closed path in W with endpoints at $\pi_d(x)$ is the image of a closed path in W' with endpoints at x. Hence the map of pointed spaces

$$\pi_d : (W', x) \to (W, \pi_d(x))$$

induces a surjection of fundamental groups

$$(\pi_d)_* : \pi_1(W') \to \pi_1(W).$$

Again, since W' is a nonsingular hypersurface in CP^{n+1}, W' is simply connected if $n \geq 2$. Therefore $\pi_1(W) = 0$.

3. ADDITIONAL TOPOLOGICAL AND NUMERICAL INVARIANTS

Let X be a compact Kahler Manifold. Let T_X^* and $\overline{T}_X^*$ denote the holomorphic cotangent bundle and conjugate holomorphic cotangent bundle of X respectively. Introduce the following sheaves on X

$A^{p,q}$ = sheaf of germs of local differentiable sections of the vector bundle $\lambda^p T_X^* \otimes \lambda^q \overline{T}_X^*$.

Ω^p = sheaf of germs of local holomorphic sections of the vector bundle $\lambda^p T_X^*$.

= sheaf of germs of local holomorphic p-forms on X.

Note Ω^0 is by definition the sheaf $\mathcal{O}$, that is, the sheaf of germs of local holomorphic functions on X.

Some well known results from Hodge Theory are:

(1) The vector spaces $H^q(X, \Omega^p)$ are finite dimensional and vanish if $p+q > n$.

(2) $H^q(X, \Omega^p) \cong H^p(X, \Omega^q)$.

(3) (Serre Duality) $H^q(X, \Omega^p)$ is dual to $H^{n-q}(X, \Omega^{n-p})$.

(4) (Hodge Decomposition) $H^r(X, C) \cong \bigoplus_{p+q=r} H^q(X, \Omega^p)$.

(5) $H^p(X, \Omega^q)$ is computed by the complex $(\Gamma A^{p,*}, \overline{\partial})$ of Ω^p.

It is convenient to define the numbers $h^{p,q} = \dim H^q(X, \Omega^p)$. For example, (2) and (4) imply the useful fact

$$q = h^{1,0} = h^{0,1} = \dim H^1(X, \mathscr{O}) = 1/2 \dim H^1(X, C).$$

In particular, for the branched double covering $\pi : W \to CP^n$, this and (2.10) implies $H^1(W, \mathscr{O}) = 0$ if $n \geq 2$. This situation was mentioned in Section 1. We develope it further.

Consider the short exact sequence of sheaves on W

$$0 \to Z \to \mathscr{O} \to \mathscr{O}^* \to 0$$

where Z is the constant sheaf of integers and $\mathscr{O}^*$ is the sheaf of germs of local nowhere zero holomorphic functions on W. There is a similar short exact sequence of sheaves on CP^n. From each we get long exact sequences of cohomology groups, and from a portion of these long exact sequences we get a commutative diagram

$$\to H^1(W, \mathscr{O}) \quad \to H^1(W, \mathscr{O}^*) \quad \to H^2(W, Z) \quad \to H^2(W, \mathscr{O}) \to$$
$$\uparrow \pi_* \qquad\qquad \uparrow \pi_* \qquad\qquad \uparrow \pi_* \qquad\qquad \uparrow \pi_*$$
$$\to H^1(CP^n, \mathscr{O}) \to H^1(CP^n, \mathscr{O}^*) \to H^2(CP^n, Z) \to H^2(CP^n, \mathscr{O}) \to .$$

By the above result, this reduces to

(3.1)
$$0 \to H^1(W, \mathscr{O}^*) \quad \to H^2(W, Z) \to H^2(W, \mathscr{O}) \to$$
$$\uparrow \pi^* \qquad\qquad \uparrow \pi^* \qquad\qquad \uparrow \pi^*$$
$$0 \to H^1(CP^n, \mathscr{O}^*) \to H^2(CP, Z) \to H^2(CP^n, \mathscr{O}) \to .$$

<u>Lemma (3.2).</u> $\pi* : H^1(CP^n, \mathcal{O}*) \to H^1(W, \mathcal{O}*)$ is an injection.

<u>Proof</u>.   This follows from the above diagram and the fact $\pi* : H^2(CP^n, Z) \to H^2(W, Z)$ is injective.

Now we compute the total Chern class of   W,   where again $\pi : W \to CP^n$ is double covering of CP^n,   branched along a non-singular hypersurface of degree   2d.

Suppose $p \in B_W$.   We can choose local coordinates $\tilde{z}_1, \ldots, \tilde{z}_n$ centered at   p   with B_W defined locally by $\tilde{z}_n = 0$; and local coordinates $z_1, \ldots, z_n$ centered at $\pi(p) \in B$ with   B defined locally by $z_n = 0$;   and such that π has the local description

$$\pi(\tilde{z}_1, \ldots, \tilde{z}_{n-1}, \tilde{z}_n) = (\tilde{z}_1, \ldots, \tilde{z}_{n-1}, \tilde{z}_n^2).$$

Let T_W and T_{CP^n} denote the holomorphic tangent bundles of   W and CP^n respectively.   There is a short exact sequence of sheaves

$$0 \to \mathcal{O}(T_W) \xrightarrow{(d\pi)_*} \mathcal{O}(\pi*T_{CP^n}) \to \mathcal{J} \to 0$$

where $\mathcal{J}$ is supported on B_W.   In terms of local coordinates at p   the map

$$(d\pi)_* : \mathcal{O}(T_W)_p \to \mathcal{O}(\pi^* T_{\mathbb{CP}^n})_p$$

is given by

$$\left(\sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial \tilde{z}_j} \right)_p \mapsto \left(\sum_{j=1}^{n-1} \alpha_j \frac{\partial}{\partial z_j} + 2\alpha_n \tilde{z}_n \frac{\partial}{\partial z_n} \right)_p$$

There is another exact sequence of sheaves

$$0 \to \mathcal{O}([B_W]) \to \mathcal{O}([2B_W]) \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is concentrated on B_W. In terms of local coordinates, the first map is defined at each stalk over $p \in B_W$ by

$$(\alpha)_p \mapsto (\alpha \cdot \tilde{z}_n)_p .$$

This implies the quotient sheaf $\mathcal{F}$ is isomorphic to $\mathcal{S}$. Since these sheaves are coherent, the Chern character is defined. From the two exact sequences we get

$$\text{ch}\, \mathcal{O}[2B_W] = \text{ch}\, \mathcal{O}[B_W] + \text{ch}\, \mathcal{S}$$

$$\text{ch}(\mathcal{O}\pi^* T_{\mathbb{CP}^n}) = \text{ch}(\mathcal{O}T_W) + \text{ch}\, \mathcal{S}$$

Eliminating $\text{ch}\, \mathcal{S}$ we obtain the formula

$$\text{ch}(\mathcal{O}T_W) = \text{ch}(\mathcal{O}\pi^* T_{\mathbb{CP}^n}) + \text{ch}(\mathcal{O}[B_W]) - \text{ch}(\mathcal{O}[2B_W]).$$

The corresponding total Chern class is

$$c(T_W) = c(\pi^* T_{CP^n}) \cdot c([B_W]) \cdot c([2B_W])^{-1} .$$

If we let h_n denote the dual cohomology class of a hyperplane H in CP^n we have

$$c(T_W) = (1+\pi^* h_n)^{n+1}(1+d\pi^* h_n)(1+2d\pi^* h_n)^{-1}$$

since $\pi^*[dH] = [B_W]$, by Theorem (2.3) and Lemma (3.2).

Next we relate the Canonical Bundle of W to the canonical bundle of CP^n. By definition, the canonical bundle of W is the line bundle $\lambda^n T_W^*$. If ω is a meromorphic section of Ω^n then ω defines a divisor K_W on W and $[K_W] = \lambda^n T_W^*$. Let φ be a meromorphic n-form on CP^n. We can pull φ back to a meromorphic n-form $\pi^* \varphi$ on W. In terms of local coordinates $\tilde{z}_1, \ldots, \tilde{z}_n$ centered at $p \in B_W$ and $z_1, \ldots, z_n$ centered at $\pi(p) \in B$

$$\varphi = f(z_1, \ldots, z_n) dz_1 \wedge \ldots \wedge dz_n$$

$$\pi^* \varphi = 2\tilde{z}_n f(\tilde{z}_1, \ldots, \tilde{z}_n^2) d\tilde{z}_1 \wedge \ldots \wedge d\tilde{z}_n .$$

Therefore we have the following formula of divisors

$$K_W = \pi^* K_{CP^n} + B_W .$$

Therefore

(3.3)
$$[K_W] = \pi*[-(n+1)H] \otimes \pi*[dH]$$
$$= \pi*[(d-n-1)H]$$

and

(3.4)
$$h^{n,0} = \dim H^0(W, \Omega^n) = \dim H^0(W, \mathcal{O} \pi*[(d-n-1)H]).$$

4. THE SHEAF $\pi_* \mathcal{O}_W$

We continue our investigation of double coverings $\pi : W \to CP^n$ with nonsingular branch locus B in CP^n. We can apply the push down functor π_* to the structure sheaf $\mathcal{O}_W$ of W and obtain the sheaf $\pi_* \mathcal{O}_W$ on CP^n.

Lemma (4.1). $\pi_* \mathcal{O}_W$ is locally free.

Proof. Let $x \in CP^n$ and suppose x does not belong to the branch locus B. Then there is a small neighborhood U of x such that U is evenly covered by $\pi^{-1}(U)$. Therefore

$$\Gamma(U, \pi_* \mathcal{O}_W) = \Gamma(\pi^{-1}U, \mathcal{O}_W) \cong \Gamma(U, \mathcal{O}_{CP^n}) \oplus \Gamma(U, \mathcal{O}_{CP^n}).$$

Now suppose $x \in B$. We can choose a neighborhood U of x with local coordinates $z_1, \dots z_n$ centered at x such that B is defined locally by $z_n = 0$, and local coordinates $\tilde{z}_1, \dots, \tilde{z}_n$ centered at $\pi^{-1}(x)$ such that B_W is defined locally by $\tilde{z}_n = 0$, and π is given locally by

$$(\tilde{z}_1, \dots, \tilde{z}_n) \mapsto (\tilde{z}_1, \dots, \tilde{z}_{n-1}, \tilde{z}_n^2).$$

Any $f \in \Gamma(U, \pi_* \mathcal{O}_W) = \Gamma(\pi^{-1}U, \mathcal{O}_W)$ can be expressed as a power series

$$f(\tilde{z}) = \sum_{\ell=0}^{\infty} a_\ell(\tilde{z}')\tilde{z}_n^\ell$$

where $\tilde{z}' = (\tilde{z}_1, \ldots, \tilde{z}_{n-1})$. Let

$$f_0(\tilde{z}) = \sum_{\ell=0}^{\infty} a_{2\ell}(\tilde{z}')\tilde{z}_n^\ell$$

$$f_1(\tilde{z}) = \sum_{\ell=0}^{\infty} a_{2\ell+1}(\tilde{z}')\tilde{z}_n^\ell .$$

It follows

$$f(\tilde{z}) = f_0(\tilde{z}', \tilde{z}_n^2) + \tilde{z}_n f_1(\tilde{z}', \tilde{z}_n^2)$$

$$= f_0(z) + \tilde{z}_n \cdot f_1(z).$$

Therefore

$$\Gamma(\pi^{-1}U, \, \mathcal{O}_W) \cong \Gamma(U, \, \mathcal{O}_{CP^n}) \oplus \tilde{z}_n \cdot \Gamma(U, \, \mathcal{O}_{CP^n}),$$

which proves $\pi_* \mathcal{O}_W$ is locally free.

Let I denote the trivial line bundle on CP^n.

<u>Theorem (4.2)</u>. $\pi_* \mathcal{O}_W \cong \mathcal{O}(I \oplus [-dH])$, where the direct sum means Whitney sum of line bundles.

Proof. Since $\pi_* \mathcal{O}_W$ is locally free, it is the sheaf of germs of local holomorphic sections of a vector bundle E of rank two. The involution $\sigma : W \to W$ defines a vector bundle involution $\hat{\sigma}$ of E, and each fibre of E is the direct sum of the eigensubspaces of $\hat{\sigma}$. To see that E is a Whitney sum of line bundles let $\mathcal{U} = (U_i)_{i=1}^N$ be a covering of CP^n by coordinate neighborhoods such that E is defined by the $\mathcal{U}$-cocycle $\{g_{ij}\}$. Locally $\hat{\sigma}$ is given by a collection $\{\hat{\sigma}_i\}$ of holomorphic functions

$$\hat{\sigma}_i : U_i \times C^2 \to U_i \times C^2$$

satisfying

$$\hat{\sigma}_i^2 = 1_{U_i \times C^2}$$

$$\hat{\sigma}_i g_{ij} = g_{ij} \hat{\sigma}_j$$

on $U_i \cap U_j$.

There is a holomorphic change of coordinates, after possibly choosing a smaller open covering of CP^n, such that on each fibre $\hat{\sigma}_i$ is a diagonal matrix consisting of the eigenvalues of $\hat{\sigma}$. Since σ is sheet interchanging the eigenvalues are 1 and -1. More precisely, there is a collection $\{T_i\}_{i=1}^N$ of local sections $T_i \in \Gamma(U_i, GL(2, C))$ such that

$$T_i \hat{\sigma}_i T_i^{-1} = \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for all $x \in U_i$, $1 \leq i \leq N$. Let $g'_{ij} = T_i g_{ij} T_j^{-1}$, then at each point of $U_i \cap U_j$ we have

$$
\begin{aligned}
\Lambda g'_{ij} \Lambda^{-1} &= \Lambda T_i g_{ij} T_j^{-1} \Lambda \\
&= T_i \hat{\sigma}_i g_{ij} \hat{\sigma}_j^{-1} T_j^{-1} \\
&= T_i g_{ij} T_j^{-1} \\
&= g'_{ij} .
\end{aligned}
$$

This implies $g'_{ij}(x)$ is a diagonal matrix for each $x \in U_i \cap U_j$. Therefore, the structure group of E can be reduced to $\mathcal{O}^* \oplus \mathcal{O}^*$. This is equivalent to E being the Whitney sum of two line bundles.

Let L_1 and L_2 denote these two line bundles. We shall consider their pull back to W. The new local coordinates are similar to those we used above:

$$
\tilde{z}_1^i, \ldots, \tilde{z}_n^i \qquad \text{in } \pi^{-1}(U_i)
$$

$$
z_1^i, \ldots, z_n^i \qquad \text{in } U_i
$$

such that $\tilde{z}_n^i = 0$ defines B_W in $\pi^{-1}(U_i)$, $z_n^i = 0$ defines B in U_i and

$$
\pi(\tilde{z}_1^i, \ldots, \tilde{z}_n^i) = (\tilde{z}_1^i, \ldots, \tilde{z}_{n-1}^i, (\tilde{z}_n^i)^2).
$$

Recall from the proof of Lemma (4.1) that in these coordinates the local sections in $\Gamma(\pi^{-1}U, \mathcal{O}_W)$ have the form

$$f_0(z^i) + z_n^i \cdot f_1(z^i).$$

Hence the $\pi^{-1}\mathcal{U}$-cocycle $\{\sigma^* g'_{ij}\} = \{g'_{ij} \circ \pi\}$ has the form

$$\begin{bmatrix} 1 & 0 \\ 0 & z_n^j / z_n^i \end{bmatrix}.$$

Therefore, one of these line bundles, say L_1, is trivial when pulled back to W; and the other is the dual to the line bundle of the divisor B_W, that is, $\pi^* L_2 = [-B_W] = \pi^*[-dH]$. Since π^* is injective this proves the theorem.

Corollary (4.3). $\dim H^0(W, \mathcal{O}\pi^*[mH]) = \binom{m+n}{n} + \binom{m-d+n}{n}$.

Proof. The functor π_* does not effect cohomology so

$$H^0(W, \mathcal{O}\pi^*[mH]) \cong H^0(\mathbb{CP}^n, \pi_* \mathcal{O}\pi^*[mH]).$$

By the projection formula

$$\pi_* \mathcal{O}\pi^*[mH] = \mathcal{O}[mH] \otimes \pi_* \mathcal{O}_W.$$

Therefore

$$\pi_* \mathcal{O}\pi*[mH] \cong \mathcal{O}[mH] \otimes \mathcal{O}(I \oplus [-dH]) .$$

$$\cong \mathcal{O}[mH] \oplus \mathcal{O}[(m-d)H].$$

It follows

$$\dim H^0(CP^n, \pi_* \mathcal{O}\pi*[mH]) = \binom{m+n}{n} + \binom{m-d+n}{n}.$$

Let f be the homogeneous polynomial of degree 2d that defines the branch locus B. Let $\mathcal{U} = (U_i)_{i=1}^n$ be the covering of CP^n by affine open sets $U_i = \{(z_0 : \ldots : z_n) | z_i \neq 0\}$. The $\mathcal{U}$-cocycle $\{(z_j/z_i)^m\}$ defines the line bundle [mH] on CP^n. If we regard z_j/z_i as a function on $\pi^{-1}(U_i \cap U_j)$ then $\{(z_j/z_i)^m\}$ is a $\pi^{-1}\mathcal{U}$-cocycle representing the line bundle $\pi*[mH]$.

There are holomorphic functions h_i on $\pi^{-1}(U_i)$ such that

$$h_i^2(z_0/z_i, \ldots, z_n/z_i) = f(z_0/z_i, \ldots z_n/z_i)$$

and $h_i/h_j = (z_j/z_i)^d$ since $f(z_0/z_i, \ldots, z_n/z_i)$ has a holomorphic square root on $\pi^{-1}(U_i)$. Let $\sqrt{f}$ denote the collection $\{h_i\}$, and we shall call $\sqrt{f}$ a form of degree d on W.

If m is a positive integer then by the above discussion homo-geneous forms of degree m define sections of $\pi*[mH]$.

Corollary (4.4). Every section of $\pi*[mH]$ is defined by a homogeneous form of the type

$$R\sqrt{f} + Q \quad \text{if} \quad m \geq d$$

$$Q \quad\quad\quad \text{if} \quad m < d,$$

where R and Q are homogeneous polynomials in CP^n of degree $m-d$ and m respectively.

Proof. As R and Q vary we get a subspace of sections of dimension

$$\binom{m+n}{n} + \binom{m-d+n}{n}.$$

So by Corollary (4.3) this subspace must be all of $H^0(W, \mathcal{O}\pi*[mH])$.

Corollary (4.3) also provides a way of computing the topological index of W when $n = 2$. The quadratic form, induced from the intersection pairing on $H^2(W, R)$, has p^+ positive eigenvalues and p^- negative eigenvalues. The index $\tau(W)$ is defined to be the signature of this quadratic form

$$\tau(W) = p^+ - p^-.$$

Using the highest Chern class of W one computes

$$\dim H^2(W, R) = p^+ + p^- = 2 + (2d-1)(2d-2).$$

By the Hodge index theorem [Hirzebruch; 5]

$$p^+ = 2h^{2,0} + 1.$$

By (3.4), $h^{2,0} = \dim H^0(W, \mathcal{O}\pi*[(d-3)H])$, so by Corollary (4.3) $h^{2,0} = \frac{1}{2}(d-1)(d-2)$. Combining, we arrive at the formula

$$\tau(W) = 2 - 2d^2.$$

An alternate method of computing the index of W is by Hirzebruch's index theorem [Hirzebruch; 5] using the Chern classes of W which we have computed:

$$\tau(W) = \frac{1}{3}(c_1^2 - 2c_2),$$

or by the G-signature theorem given in [Hirzebruch; 6].

Additional properties of branched coverings of CP^n which includes a useful characterization of the Hodge filtration of W can be found in [C. H. Clemens; 2].

5.   THE PICARD GROUP

In this section we investigate the Picard group of   W   where $\pi : W \to CP^n$ is a double covering of CP^n,   branched over a smooth hypersurface   B   of degree   2d.   From diagram (3.1) we have an injection

(5.1)
$$H^1(W, \mathcal{O}^*) \to H^2(W, Z)$$

and by (2.10)

$$H^2(W, Z) \cong Z \quad \text{if} \quad n \geq 3.$$

This implies   Pic $W \cong Z$ and is generated by $\pi^*[H]$ if $n \geq 3$. Therefore, for topological reasons, the Picard group of these branched double covers have a simple description if   dim $W \geq 3$. On the other hand, we have found that the rank of $H^2(W, Z)$ can be large when   W   is a surface.   There is of course no guarantee that Pic W is large on this basis alone.

The approach taken here is to describe the Picard group of   W in terms of properties of the branch curve   B.   This is a natural idea because specifying a branch curve   B   completely determines the covering $\pi : W \to CP$,   therein, one expects special properties of the surface   W   to reflect themselves in special of the branch curve B.   We have seen this is so for the topology of   W,   and because of

(5. 1) some of the topological properties of the covering $\pi : W \to CP^2$ will be useful here.

Fix such a double covering $\pi : W \to CP^2$, with a branch locus B and involution $\sigma : W \to W$. Let us agree to write the group operation of line bundles additively, thus if L and L' are two line bundles, we write $L + L'$ instead of $L \otimes L'$. Suppose L is a line bundle on W such that $\sigma^* L = L$. By (5. 1) we can regard L as a cohomology class. By Theorem (2. 3) the homomorphism $\pi_! \circ \pi_* : H_2(W, Z)^+ \to H_2(W, Z)^+$ is multiplication by 2. Dual to this is $\pi^* \circ \pi^! : H^2(W, Z)^+ \to H^2(W, Z)^+$, which is also multiplication by 2. Since $L \in H^2(W, Z)^+$ we have

$$2L = \pi^* \circ \pi^! L = \pi^*[mH]$$

for some integer m. Using the intersection pairing on $H^2(W, Z)$

$$4L^2 = (\pi^*[mH])^2 = 2[mH]^2 = 2m^2$$

Therefore m is divisible by 2. This implies $L = \pi^*[(m/2)H]$ because there is no torsion in $H^2(W, Z)$. Thus we have shown

Lemma (5. 2). Every σ-invariant line bundle on W is the pull back of a line bundle on CP^2.

If L is an arbitrary line bundle on W we get a σ-invariant line bundle by forming the sum $L + \sigma^* L$. Therefore, characterizing

the Picard group of W divides into two problems: (1) Describing

the irreducible curves in CP^2 which have a reducible inverse

image in W, and (2) finding the linear equivalence relations

among the components of their pre-images.

Generators. By Lemma (3.2) and the above discussion we can

always take $\pi^*[H]$ to be a generator of Pic W. We will character-

ize the existence of additional generators.

Let C be an irreducible curve in CP^2.

Lemma (5.3). A necessary condition that $\pi^{-1}(C)$ is

reducible is that the local intersection

$$(C \cdot B)_x = 0 \mod 2 \quad \text{for all} \quad x \in B.$$

Proof. Suppose $\pi^*(C) = A + A^*$ where $\sigma^*(A) = A^*$. Let

$x \in B$ and set $y = \pi^{-1}(x)$. Then

$$(C \cdot B)_x = (\pi^*C \cdot B_W)_y$$

$$= (A \cdot B_W)_y + (A^* \cdot B_W)_y$$

$$= 2(A \cdot B_W)_y$$

since

$$(A \cdot B_W)_y = (\sigma^*A \cdot \sigma^*B_W)_y = (A^* \cdot B_W)_y .$$

Let $j : B \to CP^2$ denote the inclusion of B in CP^2. If C is a curve in CP^2 (not necessarily irreducible) that satisfies $(C \cdot B)_x = 0 \bmod 2$ for all $x \in B$ we can define the following effective divisor

$$D = \sum_{x \in B} \frac{1}{2} (C \cdot B)_x \cdot x.$$

Thus $2D = j{*}C$.

On the other hand, an effective divisor D which satisfies $2D \underset{\ell}{\sim} j{*}mH$ ($\underset{\ell}{\sim}$ denotes linear equivalence of divisors) implies the existence of a curve in CP^2 satisfying the necessary conditions of Lemma (5.3).

There is some classical language that is useful here, called residuation [Basset; 1].

Definition (5.4). A divisor D on B is said to have zero residual, written $\text{res}_m[D] = 0$ if D is the restriction of a divisor of degree m in CP^2. If D and D' are divisors on B we say D is residual to D' if $\text{res}_m[D+D'] = 0$.

We state some basic properties of divisors on B using the language of residuation:

(a) If $\text{res}_m[D] = 0$ and $\text{res}_n[D'] = 0$ then
$\text{res}_{m+n}[D+D'] = 0$.

(b) If $res_m[D] = 0$ and k is an integer then $res_{km}[kD] = 0$.

The subscript m in the definition, referring to the degree of the plane divisor, is not always important. So in some cases m is suppressed. Thus $res[D] = 0$ simply means D is cut out on B by a divisor of CP^2.

We now turn to sufficient conditions. Let C_m be an irreducible curve of degree m in CP^2 such that $(C_m \cdot B)_x = 0 \bmod 2$ for all $x \in B$. The sum

$$\sum_{x \in B} \frac{1}{2} (C_m \cdot B)_x \cdot x$$

can be regarded as either a divisor D_B on B or a divisor D_{C_m} on C_m.
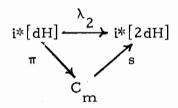
Recall the construction of the covering W branched along B.

(5.5)
$$[dH] \xrightarrow{\lambda_2} [2dH]$$
$$\pi \searrow \quad \nearrow s$$
$$CP^2$$

where s is a section of $[2dH]$ which defines B and $W = \lambda_2^{-1} \circ s(CP^2)$.

Letting $i: C_m \to CP^2$ denote the inclusion of C_m in CP^2 we can restrict diagram (5.5) to C_m

$$i*[dH] \xrightarrow{\lambda_2} i*[2dH]$$

with π and s mapping down to C_m.

__Theorem (5.6)__. $\pi*(C_m)$ is reducible if and only if $[D_{C_m}] = i*[dH]$.

__Proof.__ If $[D_{C_m}] = i*[dH]$ then there is a homogeneous polynomial of degree d in CP^2 which when restricted to C_m vanishes on D_{C_m} and nowhere else. Hence, this polynomial defines a section $s': C_m \to i*[dH]$ such that

$$\lambda_2 \circ s' = (\text{constant}) \cdot s$$

Therefore, $\pi*(C_m)$ is reducible.

Conversely, if $\pi*(C_m) = A + A*$, then there is a section $s': C_m \to i*[dH]$ such that $s'(C_m) = A$. Now A defines a divisor on the zero section of $i*[dH]$. By identifying C_m with the zero section of $i*[dH]$, the line bundle of this divisor is clearly equal to D_{C_m} and $i*[dH]$.

We can use this theorem to provide sufficient conditions for the existence of such curves. We state the conditions in terms of divisors on the branch curve B. Let H_B denote the divisor on B obtained by restricting a hyperplane H in CP^2 to B.

Theorem (5.7). Let D be a divisor on B of degree md such that

(i) $\text{res}_m[2D] = 0$ but $\text{res}[D] \neq 0$.

(ii) $\ell(D) > \ell((m-d)H_B)$.

(iii) $\ell(dH_B - D) > \ell((d-m)H_B)$.

Then there is a curve C of degree at most m such that each component of C has reducible inverse image. We abbreviate $\ell(D)$ for $\dim H^0(B, \mathcal{O}[D])$.

Proof. Conditions (i) and (ii) imply that D, or a divisor linearly equivalent to D, is effective and there is a curve C of degree m satisfying (1) $D = \sum_{x \in B} \frac{1}{2}(C \cdot B)_x \cdot x$, (2) not all components of C are multiple curves.

Condition (iii) says there is a polynomial Q of degree d in CP^2 that vanishes on D and that Q does not vanish identically on C. If however Q vanishes on some component of C, say C_1, then we define $D_1 = \sum_{x \in B} \frac{1}{2}(C - C_1 \cdot B)_x \cdot x$ and apply the theorem to D_1. This is why the conclusion asserts the curve C has degree at most m.

The existence of the polynomial Q implies $[D] = i*[dH]$ where $i : C \rightarrow CP^2$ is an inclusion. Therefore by Theorem (5.6)

each component of the curve   C   (of degree at most   m)   has

reducible inverse image in   W.

There are some interesting observations from Theorem (5.6)

and Theorem (5.7).   First of all, if the conditions of Theorem (5.6)

are satisfied for a curve   C,   then   deg $C \leq 2d$.   This does not mean

the only irreducible curves in CP^2 that have reducible inverse

image in   W   must have degree at most   2d.   There is a way of

characterizing all irreducible curves in CP^2 that have reducible

inverse image.

Let   g   be a homogeneous polynomial of degree   m   that

defines an irreducible curve   C   in CP^2 such that $\pi*C$ is

reducible.   Write $\pi*C = A + A*$.   Note that $A \cdot B_W = A* \cdot B_W = md$.

Now   A   meets   A*   in at least   md   points (counting multiplicities),

that is, over the points where   C   meets   B.   If $A \cdot A* = md$ then

A   defines a section $s : C \rightarrow i*[dH]$ as in Theorem (5.6)   (A*

defines   -s).   Hence $m = \deg C \leq 2d$.

In general $A \cdot A* > md$,   so we define the following divisor   E

on   C:

$$E = \frac{1}{2} \sum_{y \, \epsilon \, \pi^{-1}(C) - B_W} (A \cdot A*)_y \cdot \pi(y)$$

The divisor is nonzero if and only if   A   meets   A*   away from the

ramification locus B_W. If it is nonzero, neither A nor A*
defines a section of $i^*[dH]$.

Suppose the branch curve B is defined by a homogeneous
polynomial f of degree 2d, and let R be a homogeneous poly-
nomial of sufficiently large degree, say ℓ, so that R vanished
on the divisor E. We will use Theorem (5.6) to obtain a relation
between the polynomials g, f and R. The idea is to form a new
double covering such that the degree of C is less than the degree
of branch locus.

There is commutative diagram of holomorphic maps between
the total spaces of line bundles

(5.8)
$$
\begin{array}{ccc}
[dH] & \xrightarrow{\widehat{R}} & [(d+\ell)H] \\
\lambda_2 \downarrow & & \downarrow \lambda_2 \\
[2dH] & \xrightarrow{\widehat{R}^2} & [2(d+\ell)H]
\end{array}
$$

The map λ_2 has been defined in Section 1. The map
$\widehat{R} : [dH] \to [(d+\ell)H]$ is given locally by a collection

$$\widehat{R}_i : U_i \times C \to U_i \times C$$

by

$$(z, t) \mapsto (z, R(z_0/z_i, \ldots, z_n/z_i) \cdot t)$$

where $U_i = \{(z_0 : \ldots : z_n) | z_i \neq 0\}$. It is clear diagram (5.8) also com-
mutes with projection to CP^2.

We can form a double covering $\widetilde{\pi} : \widetilde{W} \to CP^2$ branched along the reducible curve $\widetilde{B}$ of degree $2(\ell + d)$ defined by the zeros of $R^2 f$. This covering is different from the other coverings we have considered in that $\widetilde{W}$ is a singular algebraic variety, however, it possesses many of the properties that the nonsingular double coverings have. In particular, Theorem (5.6) is applicable.

If C is as above, then $\widehat{R}(\pi^{-1}C) = \widehat{\pi}^{-1}C$. Therefore $\widetilde{\pi}*C$ is reducible in $\widetilde{W}$. Write $\widetilde{\pi}*C = A + A*$. Note that we can form the divisor

$$D = \sum_{x \in B} \frac{1}{2}(C \cdot \widetilde{B})_x \cdot x.$$

We can apply Theorem (5.6) and conclude there is a homogeneous polynomial Q of degree $d+\ell$ in CP^2 that vanishes on D.

Let F be the unique effective divisor on B such that $D + F$ is defined by $Q = 0$; that is, F is the other points where Q meets B. Since $res_{2(d+\ell)}[2(D+F)] = 0$ and $res_m[2D] = 0$ it follows $res_{2(d+\ell)-m}[2F] = 0$. Therefore, there is a homogeneous polynomial h of degree $2(d+\ell)-m$ such that $h = 0$ defines $2F$ on B.

Lemma (5.9). We have $gh = R^2 f - Q^2$, after possibly multiplying h and Q by some appropriate complex numbers.

Proof. Since $gh = 0$ defines $2(D+F)$ on $\tilde{B}$ and $Q^2 = 0$ also defines $2(D+F)$ on $\tilde{B}$ it follows some linear combination of the two vanished identically on $\tilde{B}$. Since $\tilde{B}$ is defined by $R^2 f = 0$ and

$$\deg gh = \deg Q^2 = \deg R^2 f$$

we get the desired equation.

Theorem (5.10). If C is an irreducible curve in CP^2 such that $\pi^{-1}(C)$ is reducible and C has only locally irreducible singularities away from B, then one of the following holds.

(i) $\deg C < 2d$, and neither of the two components of $\pi^{-1}(C)$ is linearly equivalent to a divisor pulled back from CP^2.

(ii) $\deg C = 2d$, and both components of $\pi^{-1}(C)$ is linearly equivalent to $\pi^* dH$.

Proof. By Lemma (5.9) there is an equation relating f and g

(5.11) $$gh = R^2 f - Q^2$$

Examination of (5.11) implies if C has only locally irreducible singularities away from B then $\deg R = 0$. Hence

(5.12) $$gh = f - Q^2$$

so $\deg g \leq 2d$.

If   deg C = 2d,   then   deg h = 0   and (5.12) becomes

$$g = f - Q^2$$

It follows the two components of $\pi^{-1}(C)$ are defined by zeros of

$\sqrt{f} - Q$ and $\sqrt{f} + Q$.   Therefore by Corollary (4.4) if   deg C = 2d

then each component of $\pi^{-1}(C)$ is linearly equivalent to $\pi^* dH$.

This proves (ii).

Next assume   m = deg C < 2d.   In this case,   C   satisfies the

conditions of Theorem (5.6) because   Q   is a polynomial of degree

d   that vanished on $D = \sum_{x \in B} \frac{1}{2} (C \cdot B)_x \cdot x$.   As pointed out above,

when this condition is satisfied, the two components of $\pi^{-1}(C)$

meet only on the ramification locus B_W in   W.   Thus

(5.13) $A \cdot A^* = md$

$$2m^2 = [\pi^* C]^2 = (A + A^*)^2 = 2A^2 - 2A \cdot A^*$$

Assume, for a contradiction, that either   A   or   A*   is

linearly equivalent to the pull back of a divisor on CP^2.   Now the

rank of the subgroup of σ-invariant line bundles on   W   is one.

Hence $mA \underset{\ell}{\sim} nA^*$,   but $A^2 = A^{*2}$ and both   A   and   A*   are

effective, so we can conclude $A \underset{\ell}{\sim} A^*$,   which in turn implies

$A^2 = A \cdot A^*$.   But this together with (5.13) forces   deg C = 2d   which

we have assumed is less than   2d.

We can say more about the singular set of these curves.

Corollary (5.14). Suppose C satisfies the hypothesis of Theorem (5.10), then C is nonsingular at each point of $B \cap C$, that is C is totally tangent to the branch locus B.

Proof. If F is a homogeneous polynomial in CP^2 let ∇F denote $(\partial F / \partial z_0, \partial F / \partial z_1, \partial F / \partial z_2)$. Thus F is nonsingular if ∇F never vanishes.

Again let $f(z_0 \ z_1, z_2) = 0$ define B and $g(z_0, z_1, z_2) = 0$ define C in CP^2. By Lemma (5.9) we have an equation

$$gh = f - Q^2$$

Applying ∇ we have

$$g\nabla h + h\nabla g = \nabla f - 2Q\nabla Q.$$

At each point of $B \cap C$ this becomes

$$h\nabla g = \nabla f$$

because $Q = g = 0$ on $B \cap C$. Therefore, $\nabla g \neq 0$ on $B \cap C$ since B is nonsingular.

When we relax the assumption that a curve C has only locally irreducible singularities away from B there is no bound on the degree of C. We shall investigate this situation, but first we discuss
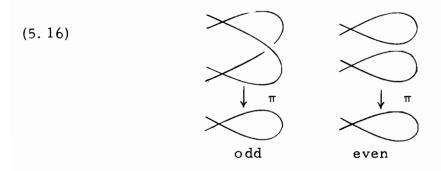
another interesting observation of Theorem (5.6) and Theorem (5.7).

Suppose D is a divisor on the branch curve B that satisfies conditions (i), (ii) and (iii) of Theorem (5.7). There is a certain symmetry to these conditions, because the divisor $\tilde{D} = dH_B - D$ also satisfies conditions (i), (ii), and (iii). Therefore, to the divisor D there corresponds a curve C of degree at most m such that $\pi^{-1}(C)$ is reducible and to $\tilde{D}$ there corresponds a curve $\tilde{C}$ of degree at most 2d-m such that $\pi^{-1}(\tilde{C})$ is reducible. We make the following definition.

Definition (5.15). Suppose C and C' are curves in CP^2 such that each irreducible component of C and C' has reducible inverse image in W. We say C is residual to C' if the divisors

$$D = \sum_{x \in B} \frac{1}{2} (C \cdot B)_x \cdot x \quad \text{and} \quad D' = \sum_{x \in B} \frac{1}{2} (C' \cdot B)_x \cdot x \quad \text{are residual}$$

divisors on B, that is $res[D+D'] = 0$.

According to this definition C and $\tilde{C}$ are residual curves. This observation and language will be useful when relations are discussed.

We return to the general case of describing the irreducible curves in CP^2 such that their inverse image is reducible. It may happen that a totally tangent curve of arbitrarily large degree may possess reducible singularities (e.g. double points) that forces

$\pi^{-1}(C)$ to be reducible.

(5. 16)

odd   even

Topologically, the singularity creates a loop in C which 'winds round' the branch curve an odd number of times (see diagram (5. 16)). More precisely, suppose C is irreducible, $\pi^*C = A + A^*$, and C does not satisfy the conditions of theorem (5. 6). As we have pointed out, when this occurs A meets A^* away from the ramification locus B_W. If $(A \cdot A^*)_y \neq 0$ for $y \notin B_W$ then $(A \cdot A^*)_{\sigma(y)} \neq 0$ so the additional contribution to $A \cdot A_*$ is divisible by 2, so we can write

(5. 17) $$A \cdot A^* = md + 2\lambda.$$

Theorem (5. 18). Let C be as above and suppose that A (and hence A^*) is not linearly equivalent to the pull back of a divisor on CP^2. Then

$$\lambda > \frac{1}{4}(m^2 - 2md).$$

Proof. By the Nakai-Moisizon criterion [Hartshorne; 4], the ramification locus B_W is ample. Clearly $(A-A*) \cdot B_W = 0$, so by the Hodge index theorem

$$0 \geq (A-A*)^2 = 2A^2 - 2A \cdot A*,$$

with equality if and only if $A \sim_\ell A*$ (since $H^1(W, \mathcal{O}) = 0$). By (5.13) $A^2 = m^2 - A \cdot A*$. Combining this with (5.17) we get

$$\lambda > \frac{1}{4} (m^2 - 2md).$$

We have given necessary and sufficient conditions for the existence of additional generators of Pic W. The previous theorem enables us to divide them into two types. Those arising from curves satisfying the conditions of Theorem (5.6) constitutes the first type. By Theorem (5.7) these relate nicely to the branch curve. The other type arises from factors of polynomials of the form $R^2 f - Q^2$, and we may assume they have degree greater than the degree of the branch locus. Relating the second type of curves to the branch locus may require more than simple conditions similar to those in Theorem (5.7).

Relations. Suppose C and $\widetilde{C}$ are irreducible plane curves satisfying the conditions of Theorem (5.6). Then C and $\widetilde{C}$ lift to reducible divisors in W. We can write $\pi*(C) = A + A*$ and $\pi*\widetilde{C} = \widetilde{A} + \widetilde{A}*$. There is of course no natural way to distinguish A

from A* except that A* = σ*A, likewise for $\widetilde{A}$ and $\widetilde{A}$*.

Nevertheless, we can specify linear equivalence relations between the

divisors A, A*, $\widetilde{A}$ and $\widetilde{A}$*. In some cases, they are determined

by linear equivalence relations between the divisors

$$D_C = \sum_{x \in B} \frac{1}{2}(C \cdot B)_x \cdot x \quad \text{and} \quad D_{\widetilde{C}} = \sum_{x \in B} \frac{1}{2}(\widetilde{C} \cdot B)_x \cdot x \quad \text{on} \quad B.$$

<u>Theorem (5.19)</u>. Let C and $\widetilde{C}$ be as above. If

$\text{res}_d[D_C + D_{\widetilde{C}}] = 0$ then, after possibly relabeling, we have

$$A + \widetilde{A} \sim_\ell \pi^* dH.$$

<u>Proof</u>. Suppose g = 0 defines C, h = 0 defines $\widetilde{C}$ and

f = 0 defines the branch curve B. If $\text{res}_d[D_C + D_{\widetilde{C}}] = 0$ there is a

polynomial Q of degree d that vanished precisely on $D_C + D_{\widetilde{C}}$.

The polynomial g·h of degree 2d vanished precisely on

$2D_C + 2D_{\widetilde{C}}$, hence after possibly multiplying h and Q by

appropriate constatns we have

$$gh = f - Q^2$$

Using this, we can label the components of π*C in such a way

that if π*(C) = A + A* then A is defined by g = √f + Q = 0.

Also, we can arrange it so π*($\widetilde{C}$) = $\widetilde{A}$ + $\widetilde{A}$* and $\widetilde{A}$ is defined by

h = √f + Q = 0. Thus A + $\widetilde{A}$ is defined in W by the zeros of

$\sqrt{f} + Q$. But $\sqrt{f} + Q$ defines a section of $\pi^*[dH]$. Therefore $A + A \underset{\ell}{\sim} \pi^* dH$.

According to Definition (5.15), C and $\widetilde{C}$ are residual curves. We have proven there is a relation between the inverse image of residual curves. As an example, suppose $\pi : W \twoheadrightarrow CP^2$ is a two sheeted covering of CP^2 branched along a smooth quartic curve B_4. A quartic always admits a bitangent L. L must satisfy the conditions of Theorem (5.6) because it is simply connected. This implies there is a plain conic passing through the points P_1 and P_2 where L is tangent to B_4. This conic also passes through six other points of B_4 say $p_3, \ldots, p_8$. There is a cubic curve C totally tangent to B_4 at the points $p_3, \ldots, p_8$ and C is a residual curve to L. Now Theorem (5.19) implies we can write $\pi^*(L) = A + A^*$ and $\pi^* C = B + B^*$ and there is a relation

$$A + B \underset{\ell}{\sim} \pi^* 2H$$

As pointed out above, a curve C of degree m that satisfies the conditions of Theorem (5.6) has a residual curve of degree at most $2d-m$. This implies we need only consider such curves of degree up to half the degree of the branch curve.

To view the problem of describing all linear equivalence relations in W, let $\mu : B_W \twoheadrightarrow W$ be the inclusion of the ramification

locus in W. The restriction of line bundles on W to B_W gives a homomorphism

$$\mu* : \text{Pic } W \to \text{Pic } B_W .$$

Since the involution $\sigma : W \to W$ is the identity on B_W the subgroup of line bundles of the form $L - \sigma*L$ is contained in the kernel of $\mu*$. Because of this, we shall mean by $\mu*$ the homomorphism

(5.20)
$$\mu* : \frac{\text{Pic } W}{\{L - \sigma*L\}} \to \text{Pic } B_W .$$

The precise form of the kernel of $\mu*$ is unknown. As a starting point, I make the following conjecture and then investigate to what extent it is true.

Conjecture. The homomorphism $\mu*$ in (5.20) is injective.

A convenient way of looking at this is to return to the use of divisors. We can choose a set of divisors $\{A_1, \ldots, A_N\}$ of W so that the line bundles

$$[A_1], \ldots, [A_N] \quad \text{and} \quad \pi*[H]$$

form a basis of Pic W. We can also choose $A_1, \ldots, A_N$ to be very ample. There is a positive integer m_i assigned to each A_i because

$$\pi^* m_i H \underset{\ell}{\sim} A_i + A_i^*$$

for $1 \leq i \leq N$.

Thus every divisor D in W is linearly equivalent to a divisor of the form $\sum_i \alpha_i A_i - m\pi^* H$; and since $2A_i \underset{\ell}{\sim} A_i - A_i^* + m_i \pi^* H$ we may replace the coefficients α_i by 1. So we write

$$D \underset{\ell}{\sim} \sum_i \epsilon_i A_i - m\pi^* H \mod\{E - \sigma^* E\}$$

where ϵ_i is either 1 or 0.

Suppose D is a divisor and $[D] \epsilon \ker \mu^*$. We claim $D \underset{\ell}{\sim} -\sigma^* D$. To see this, observe $D + \sigma^* D \underset{\ell}{\sim} r\pi^* H$ for some integer r, but r must be zero because $0 = (D + \sigma^* D) \cdot \pi^* H = 2r$. Therefore line bundles in the kernel of μ^* are skew invariant. Conversely, a skew invariant line bundle L satisfies

$$2\mu^* L = \mu^* (L - \sigma^* L) = 0$$

and so determines a point of order 2 in $\mathrm{Pic}^0 B_W$. To show μ^* is an injection, one must show that skew invariant line bundles, not of the form $L - \sigma^* L$, restricts to nontrivial half-periods in $\mathrm{Pic}^0 B_W$.

Define a skew invariant divisor to be a divisor D such that $[D] = -\sigma^* [D]$. If D is a skew invariant divisor then clearly

$D \cdot \pi^*H = 0$. By replacing D by its linear equivalent representative

$$\sum_i \epsilon_i A_i - m\pi^*H \text{ modulo} \{E - \sigma^*E\} \quad \text{it follows}$$

$$0 = D \cdot \pi^*H = \sum_i \epsilon_i A_i \cdot \pi^*H - m\pi^*H \cdot \pi^*H$$

$$= \sum_i \epsilon_i m_i - 2m.$$

This motivates the following definition. Define the group

$$\mathscr{S} = \{ \sum_i \alpha_i A_i \mid \sum_i \alpha_i m_i = 0 \bmod 2 \}$$

Provided at least one of the m_i are odd, we have an exact sequence

$$0 \to \frac{\mathscr{S}}{2\mathscr{S}} \to \frac{Z^N}{2Z^N} \to \frac{Z}{2Z} \to 0$$

where the first map is $\sum_i \alpha_i A_i \mapsto (\alpha_1, \ldots, \alpha_N)$ and the second

$(\alpha_1, \ldots, \alpha_N) \mapsto \sum_i \alpha_i m_i$. Note that $\mathscr{S}/2\mathscr{S}$ is isomorphic to

$\left(\dfrac{\text{Pic } W}{\{L - \sigma^*L\}} \right)^{\text{skew}}$, the skew invariant subgroup of the involution

$\sigma^* : \dfrac{\text{Pic } W}{\{L - \sigma^*L\}} \to \dfrac{\text{Pic } W}{\{L - \sigma^*L\}}$. We have a homomorphism

(5.21)
$$\frac{\mathscr{S}}{2\mathscr{S}} \to \text{Pic}^0 B_W$$

given by

$$\sum A_i \longmapsto \mu * \left[\sum A_i - \left[\frac{\sum m_i}{2} \right] \pi * H \right].$$

Its image is contained in the half-periods of $\mathrm{Pic}^0 B_W$.

At this point, we mention a topological analogue of (5.21).

Recall that $H_2(W, Z)^+$ denotes the σ-invariant subgroup of

$H_2(W, Z)$. It is useful to define the skew invariant classes as elements

of

$$\frac{H_2(W, Z)}{H_2(W, Z)^+} \otimes Z_2$$

Now there is a way of assigning to each skew invariant class an

element of $H_1(B, Z_2)$, which topologically can be identified with the

points of order 2 in $\mathrm{Pic}^0 B$. Interestingly, the point of order 2

will be zero if and only if the class is of the form $\xi - \sigma * \xi$.

This is done as follows. Let $\{D_s\}_{s \in CP^1}$ be the pencil in

CP^2 spanned by the branch locus $D_0 = B$ and the divisor $D_\infty = 2C$,

where C is a nonsingular curve of degree d that meets B

transversally. If C is sufficiently general the pencil has irre-

ducible singular fibres $D_{t_1}, \ldots, D_{t_r}$ consisting of only one ordinary

node. Associated to the singular fibres are so called vanishing cycles

$\{\gamma_i\}$ in B that generate $H_1(B, Z)$, where each of the vanishing

cycles bound 2-cells $\Sigma_1, \ldots, \Sigma_r$ in CP^2.

Corresponding to this pencil, there is a pencil $\{\tilde{D}_s\}_{s \in CP^1}$ in W, spanned by $\tilde{D}_0 = B_W$ and $\tilde{D}_\infty = \pi^{-1}(C)$. This pencil is actually parameterized by a two sheeted covering $CP^1 \to CP^1$, branched over 0 and ∞. The covering is defined by $(s_0 : s_1) \to (s_0^2, s_1^2)$.

Lemma (5.22). $\{\tilde{D}_s\}_{s \in CP^1}$ is a Lefschetz pencil.

Proof. The ramification locus B_W is defined by the zeros of the form $\sqrt{} f$ and $\pi^{-1}C$ is defined by the zeros of a homogeneous polynomial Q (regarded as a form on W). If $(s_0, s_1) \in CP^1$ then $\tilde{D}_s = s_1 \sqrt{} f + s_0 Q$ and $\tilde{D}_{-s} = s_1 \sqrt{} f - s_0 Q$ are members of this pencil lying over $D_t = t_1 f - t_0 Q^2$ where $(t_0 : t_1) = (s_0^2 : s_1^2)$. Now $\tilde{D}_s$ has one ordinary node if and only if D_t does. Also, since $\pi^{-1}(C)$ is nonsingular, we conclude $\{\tilde{D}_s\}_{s \in CP^1}$ is a Lefschetz pencil.

This pencil has singular fibres

$$\tilde{D}_{s_1}, \ldots, \tilde{D}_{s_r}, \tilde{D}_{-s_1}, \ldots, \tilde{D}_{-s_r}$$

lying over $D_{t_1}, \ldots, D_{t_r}$. The 2-cells $\Sigma_1, \ldots, \Sigma_r$ lift to 2-cells

$$\Gamma_1, \ldots, \Gamma_r, \Gamma'_1, \ldots, \Gamma'_r$$

with the same vanishing cycles $\{\gamma_i\}$ (using the isomorphism between B_W and B). The skew invariant classes can be represented by cycles of the form $\sum_i (\Gamma_i - \Gamma_i')$.

The homomorphism $\pi_* : H_2(W, Z) \to H_2(CP^2, Z)$ is onto by Theorem (2.5), and $H_2(W, Z)^+$ maps isomorphically onto $2H_2(CP^2, Z)$ by Theorem (2.3). This implies the homomorphism

$$\pi_* : \frac{H_2(W, Z)}{H_2(W, Z)^+} \otimes Z_2 \to H_2(CP^2, Z_2)$$

is a surjection.

Using the pencil $\{\widetilde{D}_s\}_{s \in CP^1}$, C. H. Clemens [3] has proven there is an exact sequence

$$(5.23) \qquad 0 \to H_1(B_W, Z_2) \xrightarrow{\alpha} \frac{H_2(W, Z)}{H_2(W, Z)^+} \otimes Z_2 \xrightarrow{\pi_*} H_2(CP^2, Z_2) \to 0$$

where the injection α is given by $\{\gamma_i\} \mapsto \{\Gamma_i - \Gamma_i'\}$.

Returning to the group $\mathcal{J}/2\mathcal{J}$ we clearly have an injection

$$\frac{\mathcal{J}}{2\mathcal{J}} \to \frac{H^2(W, Z)}{H^2(W, Z)^+} \otimes Z_2$$

This homomorphism is given by first forming the line bundle associated to a representative divisor in $\mathcal{J}$, then assigning to this

line bundle its first Chern class in $H^2(W, Z)$.

Lemma (5.24). Using the duality between

$$\frac{H_2(W, Z)}{H_2(W, Z)^+} \otimes Z_2 \quad \text{and} \quad \frac{H^2(W, Z)}{H^2(W, Z)^+} \otimes Z_2,$$

the image of $\mathscr{L}/2\mathscr{L}$ is contained in the kernel of π_*.

Proof. If $\sum_i \alpha_i A_i$ satisfies $\sum_i \alpha_i m_i \equiv 0 \bmod 2$ then

$$\sum_i \alpha_i A_i + \sum_i \alpha_i A_i^* \sim_\ell \pi^* 4mH.$$

It follows π_* sends the homology class of $\sum_i \alpha_i A_i$ to the homology class of a curve of degree $2m$ in CP^2, which is zero in $H_2(CP^2, Z_2)$.

Thus, there is a topological way of assigning to each skew invariant line bundle a point of order two in $Pic^0 B_W$ which is zero if and only if the line bundle is of the form $L - \sigma^* L$.

It is an open question whether the homomorphism (5.21) is dual to the homomorphism α.

We can summarize the entire problem via the homomorphism in (5.21). The problem of finding generators reduces to finding half-periods in the image of (5.21), and the problem of describing

relations reduces to describing the kernel of (5.21). The results

given here deliver a partial answer to each, their greatest asset

being their constructive nature.

6. APPLICATIONS

The direct application of the results in Section 5 depends of course on how much is known about the branch curve. To illustrate how the geometry of the branch locus relates to curves on the surface W we will apply these results to a double covering branched along a smooth quartic curve. In this case, much is known about the branch curve.

Also, some properties of the branch locus translate to intrinsic properties of the surface. For example, we will give a criterion for a double covering branched over a nonsingular sextic to be elliptic. Before giving these examples, we introduce some additional structure of Riemann surfaces.

Let C be a Riemann surface of genus $g \geq 2$. We define the Jacobian variety of C to be

$$J(C) = \frac{H^1(C, \mathcal{O})}{H^1(C, Z)} \cong \begin{array}{l} \text{group of line bundles on C} \\ \text{of degree zero.} \end{array}$$

Choose a basis $\omega_1, \ldots, \omega_g$ of Abelian differentials for C and consider the complex torus obtained by forming the quotient of C^g by the lattice Λ consisting of the vectors

$$\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \qquad \gamma \in H_1(C, Z).$$

After choosing a base point b_0 in C, Abels theorem and the Riemann-Roch theorem implies every line bundle in $J(C)$ is the line bundle of a divisor of the form $p_1 + \ldots + p_g - g \cdot b_0$. Using this representation of line bundles, we get an isomorphism $J(C) \to C^g/\Lambda$ given by

$$[p_1 + \ldots + p_g - g \cdot b_0] \mapsto \sum_{i=1}^{g} \left(\int_{b_0}^{p_i} \omega_1, \ldots, \int_{b_0}^{p_i} \omega_g \right).$$

The usefulness of $J(C)$ includes an imbedding

$$\psi : C \to J(C)$$

given by

$$\psi(p) = \left(\int_{b_0}^{p} \omega_1, \ldots, \int_{b_0}^{p} \omega_g \right)$$

which extends to the rth symmetric product $C^{(r)}$ by

$$\psi\left(\sum_i p_i \right) = \sum_i \psi(p_i),$$

where $C^{(r)}$ can be identified with effective divisors of degree r on C.

$J(C)$ also has the structure of a principally polarized Abelian variety whose polarization explicitly defines the Riemann theta

function θ of C. Its zero locus defines a periodic divisor on C^g (with respect to Λ) which defines a divisor Θ in $J(C)$, called the theta divisor.

The concluding examples will depend upon some facts pertaining to Riemann's solution to the Jacobi inversion problem, which relates properties of the theta divisor to special effective divisors on C. A detailed treatment can be found in [Lewittes; 7].

These facts center around the existence of a point $\mathcal{K}_{b_0}$ in $J(C)$, depending only upon the base point b_0, called a Riemann constant. For a given $e \in J(C)$ we have

(1) If $e \notin \Theta$, then there is a unique divisor $D \in C^{(g)}$ such that $e = \psi(D) + \mathcal{K}_{b_0}$.

(2) If $e \in \Theta$ there is an effective divisor $D \in C^{(g-1)}$ such that $e = \psi(D) + \mathcal{K}_{b_0}$ and $\ell(D)$ equals the multiplicity of Θ at e.

(3) $-2\mathcal{K}_{b_0} = \psi(K)$, where K is a canonical divisor on C.

By (2) and (3) if $e \in \Theta$ is a half-period in $J(C)$, then there is an effective divisor D of degree $g-1$ such that $2D$ is a canonical divisor. Such divisors are called <u>theta characteristics</u>. Relative to the basis of periods, the half-periods in $J(C)$ are given by $e = \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$ where ϵ and δ are $1 \times g$ vectors whose entries are either 0 of $1/2$. A theta characteristic is even or odd depending upon whether $4\epsilon \cdot \delta$ is even or odd. Some additional results found in [7] are

(4) The multiplicity of Θ at an even (resp. odd) half-period is even (resp. odd).

(5) All the odd half-periods lie on Θ and there are $2^{g-1}(2^g-1)$ of them.

Let B_4 be a nonsingular quartic curve in CP^2 and $\pi : W \to CP^2$ a double covering branched along B_4. We shall consider the class of curves that satisfy the conditions of Theorem (5.6) that lead to additional generators of Pic W.

First we make a reduction. According to the remarks which follow Theorem (5.7) we need not consider cubic curves because they are residual to lines. Every conic that satisfies the required conditions also has a residual curve. In fact, in this case, a one dimensional family of residual conics that contains a degenerate one, consisting of two lines. So again we have reduced the problem to studying totally tangent lines to B_4.

Bitangents of B_4 are in one-to-one correspondence with effective square roots of a canonical divisor, i.e. theta characteristics. A result of Lewittes [7] states a Riemann surface of genus 3 has an even theta characteristic if and only if it is hyperelliptic. A plane quartic is never hyperelliptic, so we conclude B_4 has only odd theta characteristics. By property (5) above, there are $2^2(2^3-1) = 28$ odd theta characteristics; and therefore 28 bitangents

of B_4. The 28 bitangents lift to 56 exceptional curves

$E_1, \ldots, E_{28}, E_1^*, \ldots, E_{28}^*$ in W.

Let us consider the linear equivalence relations among these

exceptional curves. We have the obvious

$$E_i + E_i^* \sim_\ell E_j + E_j^* \qquad 1 \le i \ne j \le 28$$

giving 27 relations. Next suppose that among the 28 half-periods

$\{e_i\}_{i=1}^{28}$ corresponding to the 28 odd theta characteristics, we have a

relation

$$e_{i_1} + e_{i_2} + e_{i_3} + e_{i_4} = 0$$

in $J(C)$. Then by property (2) we have

$$\sum_{k=1}^{4} \psi(D_{i_k}) + 4 \mathcal{H}_{b_0} = 0.$$

Thus $D_{i_1} + D_{i_2} + D_{i_3} + D_{i_4} \sim_\ell$ canonical divisor of B_4

$$\sim_\ell 2 H_{B_4}.$$

By Theorem (5.19) the bitangents $L_{i_1}, \ldots, L_{i_4}$ are residual curves,

and therefore give rise to a relation between $E_{i_1}, \ldots, E_{i_4}, E_{i_1}^*, \ldots, E_{i_4}^*$.

A computation shows there are precisely 19 such independent relations

between $\{e_i\}_{i=1}^{28}$. Therefore we have

$$\rho = \text{rank Pic } W \geq 56 - 27 - 19 = 8.$$

But since W is rational, it is well known $\rho = 8$. Therefore we have given an explicit description of the entire Picard group of W.

For a second application, let $\pi: W \to CP^2$ be a double covering branched along a nonsingular sextic B_6. W is a Kummer surface.

<u>Proposition (6.1)</u>. If B_6 has an even theta characteristic, then W is an elliptic surface.

<u>Proof</u>. An even theta characteristic corresponds to half-canonical divisor D such that $\ell(D)$ is even, so in particular $\ell(D) \geq 2$. By Theorem (5.7) there is a totally tangent cubic curve C such that $\pi*(C)$ is reducible. C has a residual curve $\widetilde{C}$. We can write $\pi*C = A + A*$ and $\pi*\widetilde{C} = A + \widetilde{A}*$; and by Theorem (5.19) we have a relation $A + \widetilde{A} \sim_{\ell} \pi*3H$. Using the intersection pairing on W and formulas (5.13) we have

$$(A)^2 = \frac{1}{2}(\pi*3H)^2 - A \cdot A*$$

$$= 9 - 9$$

$$= 0,$$

and similarly $(A*)^2 = 0$, $(\widetilde{A})^2 = 0$ and $(\widetilde{A}*)^2 = 0$. Next,

$$18 = (\pi*3H)^2 = (A+\widetilde{A})^2$$

$$= (A)^2 + 2A \cdot \widetilde{A} + (\widetilde{A})^2$$

$$= 2A \cdot \widetilde{A}$$

so $A \cdot \widetilde{A} = 9$. Next,

$$18 = (\pi*3H)^2 = (A+A*) \cdot \pi*3H$$

$$= 2A \cdot \pi*3H$$

and

$$9 = A \cdot \pi*3H$$

$$= A \cdot (\widetilde{A}+\widetilde{A}*)$$

$$= A \cdot \widetilde{A} + A \cdot \widetilde{A}*$$

$$= 9 + A \cdot \widetilde{A}*.$$

Therefore $A \cdot \widetilde{A}* = 0$.

These intersection numbers imply (1) A is linearly equivalent to $\widetilde{A}*$ because $(A-\widetilde{A}*)^2 = 0$ and $(A-\widetilde{A}*) \cdot \pi*H = 0$ so by the Hodge index theorem, A is numerically equivalent to $\widetilde{A}*$. This implies A is linearly equivalent to $\widetilde{A}*$ because $H^1(W,\mathcal{O}) = 0$. (2) Both A and $\widetilde{A}*$ have arithmetic genus equal to one. Therefore, the linear system spanned by A and $\widetilde{A}*$ defines an elliptic fibration $\mu : W \to CP^1$.

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