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The solution set and its cardinality of the general linear equation in a real variable x and the greatest integer not exceeding x, denoted by [x], are found by a geometric argument and then by an algebraic argument. Then a general method is developed for finding the solution set and its cardinality of the equation $\prod_{i=1}^{m}(x-g_i([x])) = 0$, where $m \geq 1$ and g_i, for $1 \leq i \leq m$, is a real and single-valued function. This method is applied first to the general linear equation where the coefficient of x is not zero and then to the general quadratic equation. Some analogous results are obtained in the complex field and, in particular, the general method is extended and then used to find the solution set and its cardinality of the general linear equation.

THE SOLUTIONS OF CERTAIN EQUATIONS IN A VARIABLE AND ITS GREATEST INTEGER PART

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THE SOLUTIONS OF CERTAIN EQUATIONS IN A VARIABLE AND ITS GREATEST INTEGER PART

INTRODUCTION

On many occasions one considers the greatest integer not exceeding a real number. If x is a real number, then the bracket function of x, denoted by $[x]$, is defined to be the greatest integer not exceeding x. In this thesis we are concerned with equations in x and $[x]$ and more specifically, with the solution set and its cardinality of such equations.

The following problem from the Elementary Problem Section of the American Mathematical Monthly was proposed by R. G. Buschman (1, p. 667): " Let $N(a)$ be the number of solutions of the equation $[x] = ax$, where a and x are real and $[x]$ denotes the greatest integer not exceeding x. Find a simple formula for $N(a)$." A solution appears in a later issue (2, p. 439-40).

After certain preliminary results concerning the bracket function are developed in Chapter 1, the solution set and its cardinality of the general linear equation in x and $[x]$ are found in Chapter 2. This is done first by a geometric argument and then by an algebraic argument.

The geometric argument becomes too cumbersome and the algebraic argument, of Chapter 2, is not applicable, in general, for more complicated equations in x and $[x]$. In Chapter 3, a general algebraic method is developed for finding the solution set and its cardinality of the equation $\prod_{i=1}^{m}(x - g_i([x])) = 0$, where $m \geqslant 1$ and g_i, for $i = 1, 2, 3, \ldots, m$, is a real and single-valued function. This method is illustrated by applying it to the general linear equation where the coefficient of x is not zero, and is used, in Chapter 4, to find the solution set and its cardinality of the general quadratic equation in x and $[x]$.

In Chapter 5, certain results are extended to the complex field. In particular, the bracket function of the complex number $z = x + iy$, denoted by $[z]$, is defined to be $[x] + i[y]$, a partial ordering is established for the complex field, certain elementary properties of $[z]$ are developed, and the general method of Chapter 3 is extended and then used to find the solution set and its cardinality of the general linear equation.

The theorem giving the cardinality of the solution set usually preceeds the theorem giving the solution set. This is because the cardinality and the solution set are obtained independently and the former may be of more interest to the reader.

CHAPTER 1. PRELIMINARY RESULTS

Throughout this thesis the letters n and m will denote integral variables. Four elementary properties of $[x]$, which will be needed in this thesis, are given in the following theorem.

Theorem 1. The following properties hold for $[x]$:

(a) $x - 1 < [x] \leqslant x < [x] + 1$;

(b) if $x - 1 < n \leqslant x$, then $n = [x]$;

(c) if $n \leqslant x < n + 1$, then $n = [x]$;

(d) $[x + n] = [x] + n$.

Proof of (a). From the definition of $[x]$, we have $[x] \leqslant x$. If $[x] \leqslant x - 1$, then $[x] + 1 \leqslant x$ and so $[x]$ is not the greatest integer not exceeding x, a contradiction. Therefore, $x - 1 < [x]$, which implies $x < [x] + 1$.

Proof of (b) and (c). Suppose $x - 1 < n \leqslant x$ and either $n < [x]$ or $n > [x]$. If $n < [x]$, then $n \leqslant [x] - 1$; but $x - 1 < n$ which implies $x - 1 < [x] - 1$ and $x < [x]$, a contradiction. If $[x] < n$, then $[x]$ is not the greatest integer not exceeding x, a contradiction. Thus we have $n = [x]$, which proves (b). If $n \leqslant x < n + 1$, then $-x \leqslant -n < -x + 1$ and $x - 1 < n \leqslant x$; hence $n = [x]$ by (b), which proves (c).

Proof of (d). Since $x - 1 < [x] \leqslant x$ is equivalent to

$x - 1 + n < [x] + n \leqslant x + n$, we have $[x + n] = [x] + n$, by (b), which proves (d). This completes the proof of Theorem 1.

We will be concerned with the cardinality of sets; the notation to be used is given in Definition 1.

Definition 1. The cardinality of the set X will be denoted by (X). If X is denumerable, then we will write (X) = D; if X has the cardinality of the continuum, then we will write (X) = C.

We will be concerned with intervals of real numbers which will be denoted as follows: $[a, b] = \{x: a \leqslant x \leqslant b\}$; $[a, b) = \{x: a \leqslant x < b\}$; $(a, b] = \{x: a < x \leqslant b\}$; $(a, b) = \{x: a < x < b\}$; $(-\infty, a] = \{x: x \leqslant a\}$. The symbols $(-\infty, a)$, (a, ∞), $[a, \infty)$, and $(-\infty, \infty)$ are defined similarly. It will be necessary to know the cardinality of the set of integers, n, such that $n \in R$, where R is an interval. Definition 2 and Lemma 1 prepare us for Theorem 2 which provides this information.

Definition 2. If R is an interval of real numbers, then M(R) will denote the cardinality of the set of integers, n, such that $n \in R$.

Lemma 1. (a) The least integer not less than x is $-[-x]$; (b) the least integer exceeding x is $[x] + 1$; (c) the greatest integer not exceeding x is $[x]$; (d) the greatest integer less than x is $-[-x] - 1$.

Proof. By (a) and (b) of Theorem 1, we have $-x-1 < [-x] \leqslant -x$, $x < [x+1] \leqslant x+1$, and $-x < [-x+1] \leqslant -x+1$, which are equivalent to $x \leqslant -[-x] < x+1$, $x < [x]+1 \leqslant x+1$, and $x - 1 \leqslant -[-x] - 1 < x$, respectively, which in turn imply (a), (b), and (d), respectively. The definition of $[x]$ is (c).

<u>Theorem 2.</u> For $a \leqslant b$:

(a) $M([a,b]) = [b] + [-a] + 1$;

(b) $M([a,b)) = [-a] - [-b]$;

(c) $M((a,b]) = [b] - [a]$;

(d) $M((a,b)) = -[a] - [-b] - 1$;

(e) if R is an infinite interval, then $M(R) = D$.

Proof. If n is the greatest integer and m the least integer in an interval R, then $M(R) = n - m + 1$. By applying Lemma 1, we have:

(a) $M([a,b]) = [b] - (-[-a]) + 1 = [b] + [-a] + 1$;

(b) $M([a,b)) = (-[-b] - 1) - (-[-a]) + 1 = [-a] - [-b]$;

(c) $M((a,b]) = [b] - ([a] + 1) + 1 = [b] - [a]$;

(d) $M((a,b)) = (-[-b] - 1) - ([a] + 1) + 1 = -[a] - [-b] - 1$.

The fact that the set of all integers is denumerable implies (e).

CHAPTER 2. THE GENERAL LINEAR EQUATION

The general linear equation in x and $[x]$ is of the form

(1) $$dx + e[x] + f = 0,$$

where $d^2 + e^2 \neq 0$. If $e = 0$, then (1) becomes $dx + f = 0$, where $d \neq 0$, and there exists one and only one solution, namely $-f/d$. Hence it is assumed in the remainder of this chapter that $e \neq 0$ in which case (1) can be written in the form

(2) $$ax + b = [x].$$

The cardinality of the solution set of (2) is given in the following theorem.

<u>Theorem 3.</u> The cardinality of the solution set S of the equation $ax + b = [x]$, where a and b are real, is given by:

$$(S) = \begin{cases} \left[-\frac{b}{a}\left|\frac{a}{1-a}\right|\right] - \left[-\frac{a+b}{a}\left|\frac{a}{1-a}\right|\right] & \text{if } a \neq 0 \text{ and } a \neq 1; \\ C & \text{if } a = 0 \text{ and } b \text{ is an integer}; \\ 0 & \text{if } a = 0 \text{ and } b \text{ is not an integer}; \\ D & \text{if } a = 1 \text{ and } -1 < b \leqslant 0; \\ 0 & \text{if } a = 1 \text{ and } b \leqslant -1 \text{ or } b > 0. \end{cases}$$

Theorem 3 is made quite intuitively apparent by a geometric argument which we now present. Later we give an algebraic proof.

If the graph of the equation $y = [x]$ and the graph of the equation $y = ax + b$ are constructed in the same $x - y$ coordinate system, then the solutions of (2) are the abcissas of the points of intersection of the two graphs. Note that in Figure 1, r_1 and r_2 are the solutions.

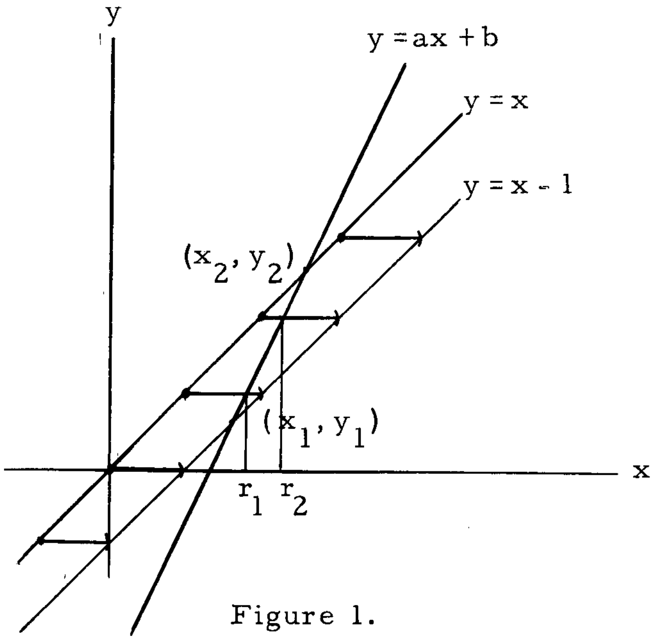

Figure 1.

With the aid of a figure such as that of Figure 1, we are able to make the following observations: If $a = 0$ and b is an integer, then each point in $[b, b + 1)$ is a point of intersection and $(S) = C$; if $a = 0$ and b is not an integer, then there are no points of intersection and $(S) = 0$; if $a = 1$ and $-1 < b \leqslant 0$, then there is a point of intersection on each step of the graph of $y = [x]$ and $(S) = D$; if $a = 1$ and $b > 0$ or $b \leqslant -1$, then there are no points of intersection and $(S) = 0$. It only remains to consider the case where $a \neq 0$ and $a \neq 1$. If $a \neq 0$ and $a \neq 1$, then the graph of $y = ax + b$ intersects the graph of $y = x - 1$ at (x_1, y_1) and the graph of

$y = x$ at (x_2, y_2). Since $y_1 = x_1 - 1 = ax_1 + b$, we have $y_1 = \frac{a+b}{1-a}$ and since $y_2 = x_2 = ax_2 + b$, we have $y_2 = \frac{b}{1-a}$. The graph of $y = ax + b$ will intersect the graph of $y = [x]$ on each step with ordinate n such that n is between y_1 and y_2 or $n = y_2$. If $a > 1$ or $a < 0$, then $y_2 > y_1$ and $(S) = M((y_1, y_2]) = [y_2] - [y_1] = [\frac{b}{1-a}] - [\frac{a+b}{1-a}]$, by Theorem 2. If $0 < a < 1$, then $y_1 > y_2$ and $(S) = M([y_2, y_1))$ $= [-y_2] - [-y_1] = [\frac{-b}{1-a}] - [-\frac{a+b}{1-a}]$, by Theorem 2. Therefore, if $a \neq 0$ and $a \neq 1$, then, in either case, $(S) = \left[-\frac{b}{a} \,\middle|\, \frac{a}{1-a}\right] - \left[-\frac{a+b}{a} \,\middle|\, \frac{a}{1-a}\right]$. This completes the geometric argument and now we proceed with the algebraic proof of Theorem 3.

Proof. If $a \neq 0$ and $a \neq 1$, then necessary and sufficient conditions for x to be a solution of (2) are (i) $x - 1 < ax + b \leqslant x$, and (ii) $ax + b$ is an integer, by parts (a) and (b) of Theorem 1. If $a > 1$, then the following conditions are equivalent to (i): $-(b+1) < (a-1)x \leqslant -b$, $\frac{b+1}{1-a} < x \leqslant \frac{b}{1-a}$, and $\frac{a+b}{1-a} < ax + b \leqslant \frac{b}{1-a}$. Therefore, if $a > 1$, then $(S) = M((\frac{a+b}{1-a}, \frac{b}{1-a}])$. Similarly, if $0 < a < 1$, then the following conditions are equivalent to (i): $-(b+1) < (a-1)x \leqslant -b$, $\frac{b}{1-a} \leqslant x < \frac{b+1}{1-a}$, and $\frac{b}{1-a} \leqslant ax + b < \frac{a+b}{1-a}$. Therefore, if $0 < a < 1$, then $(S) = M([\frac{b}{1-a}, \frac{a+b}{1-a}))$. If $a < 0$, then the following conditions are equivalent to (i): $-(b+1) < (a-1)x \leqslant -b$, $\frac{b}{1-a} \leqslant x < \frac{b+1}{1-a}$, and $\frac{a+b}{1-a} < ax + b \leqslant \frac{b}{1-a}$. Therefore, if $a < 0$ then $(S) = M((\frac{a+b}{1-a}, \frac{b}{1-a}])$. We now have $(S) = [\frac{b}{1-a}] - [\frac{a+b}{1-a}]$

for $a > 1$ or $a < 0$, and $(S) = [-\frac{b}{1-a}] - [-\frac{a+b}{1-a}]$ for $0 < a < 1$. These two expressions for (S) are special cases of the one given in Theorem 3.

If $a = 0$, then (2) becomes $b = [x]$. If b is not an integer, then $(S) = 0$. If b is an integer, then $b = [x]$ is equivalent to $b \leqslant x < b + 1$; therefore $(S) = C$.

If $a = 1$, then (2) can be written as $x = [x] - b$. Therefore, every solution is of the form $n - b$. By substituting $n - b$ into (2), we obtain $(n - b) + b = [n - b]$, which by Theorem 1 d is equivalent to $[-b] = 0$, which in turn is equivalent to $-1 < b \leqslant 0$. Therefore, if $-1 < b \leqslant 0$, then $(S) = D$ and if $b \leqslant -1$ or $b > 0$, then $(S) = 0$. This completes the proof of Theorem 3.

Two immediate results concerning the cardinality of the solution set of (2) are given in Corollaries 1 and 2.

<u>Corollary 1</u>. If $a < 0$, then $(S) = 0$ or $(S) = 1$.

Proof. Suppose $ax_1 + b = [x_1]$, $ax_2 + b = [x_2]$, and $x_1 < x_2$. Then $ax_2 + b < ax_1 + b$ and so $[x_2] < [x_1]$, which implys that $x_2 < x_1$, a contradiction.

<u>Corollary 2.</u> If $a \neq 1$, then $(S) = S(a, b)$ has a period of $|1 - a|$ in b.

Proof. Since $b + 1$ is an integer if and only if b is an integer, we have $S(0, b) = S(0, b + 1) = C$ if b is an integer and

$S(0,b) = S(0, b+1) = 0$ if b is not an integer. Therefore, $S(0,b)$ has period 1.

If $a \neq 0$ and $a \neq 1$, then by using Theorem 1 d, we have

$$S(a, |1-a| + b) = \left[-\frac{|1-a|+b}{a} \;\Big|\; \frac{a}{1-a}\Big\|\right] - \left[-\frac{a+|1-a|+b}{a} \;\Big|\; \frac{a}{1-a}\Big\|\right]$$
$$= -\frac{|a|}{a} + \left[-\frac{b}{a} \;\Big|\; \frac{a}{1-a}\Big\|\right] + \frac{|a|}{a} - \left[-\frac{a+b}{a} \;\Big|\; \frac{a}{1-a}\Big\|\right] = \left[-\frac{b}{a} \;\Big|\; \frac{a}{1-a}\Big\|\right] -$$
$$\left[-\frac{a+b}{a} \;\Big|\; \frac{a}{1-a}\Big\|\right] = S(a,b).$$

The solution set of (2), is given in the following theorem.

<u>Theorem 4.</u> The solution set S of the equation $ax+b = [x]$, where a and b are real is given by the following statement:

(a) If $0 < a < 1$, then $S = \{\frac{n-b}{a} : \frac{b}{1-a} \leqslant n < \frac{a+b}{1-a}\}$;

(b) if $a > 1$ or $a < 0$, then $S = \{\frac{n-b}{a} : \frac{a+b}{1-a} < n \leqslant \frac{b}{1-a}\}$;

(c) if $a = 0$ and b is an integer, then $S = [b, b+1)$;

(d) if $a = 1$ and $-1 < b \leqslant 0$, then $S = \{n-b : n$ is an integer $\}$.

Proof. In the algebraic proof of Theorem 3, we found the following two necessary and sufficient conditions for x to be a solution of (2), where $a \neq 0$ and $a \neq 1$: (i) $\frac{b}{1-a} \leqslant ax+b < \frac{a+b}{1-a}$ for $0 < a < 1$, and $\frac{a+b}{1-a} < ax+b \leqslant \frac{b}{1-a}$ for $a > 1$ or $a < 0$, and (ii) $ax+b$ is an integer. This pair of conditions is equivalent to the following pair: (i) x is of the form $\frac{n-b}{a}$ and (ii) $\frac{b}{1-a} \leqslant n < \frac{a+b}{1-a}$ for $0 < a < 1$, and $\frac{a+b}{1-a} < n \leqslant \frac{b}{1-a}$ for $a > 1$ or $a < 0$, which proves (a) and (b).

If $a = 0$, then (2) becomes $b = [x]$. If b is an integer then $b = [x]$ is equivalent to $b \leqslant x < b + 1$, which proves (c).

In the algebraic proof of Theorem 3, it was shown that if $a = 1$ and $-1 < b \leqslant 0$, then x is a solution of (2) if and only if x is of the form $n - b$, which proves (d).

This completes the proof of Theorem 4.

CHAPTER 3. A GENERAL METHOD

The geometric method becomes too cumbersome and the algebraic method of Chapter 2 is not applicable, in general, for more complicated equations in x and [x]. We now develop a general algebraic method which can be applied to any equation in x and [x] which can be written in the form

(1) $$\prod_{i=1}^{m}(x - g_i([x])) = 0,$$

where $m \geq 1$ and g_i, for $i = 1, 2, 3, \ldots, m$, is a real and single-valued function.

The general method will be developed for $m = 1$ first, then applied to the general linear equation, and finally developed for $m \geq 1$.

If $m = 1$, then (1) can be written as

(2) $$x = g([x]).$$

If x_1 and x_2 are distinct solutions of (2), then $[x_1] \neq [x_2]$, for if $[x_1] = [x_2]$, then $g([x_1]) = g([x_2])$ and so $x_1 = x_2$, a contradiction. Therefore, there exists a one-to-one correspondence between the solution set of (2), which will be denoted by T, and the range of the bracket function with its domain restricted to T, which will be denoted by I. The set I is shown to be the set of all n such that

(3) $$n \leqslant g(n) < n + 1,$$

by the following argument. If $[x] \epsilon I$, then $[x] \leqslant g([x]) < [x] + 1$, by Theorem 1a. If n satisfies (3), then $n = [g(n)]$ and so $g(n) = g([g(n)])$. Therefore, if $x = g(n)$, then $x \epsilon T$ and $n = [x] \epsilon I$. Thus, we have $(T) = (I)$ and $T = \{g(n): n \epsilon I\}$, where I can be determined by solving (3) for n.

This algebraic method is now illustrated by applying it to the general linear equation, where the coefficient of x is not zero, which can be written as

(4) $$x = a[x] + b$$

Letting S denote the solution set of (4), we have $(S) = (I)$ and $S = \{an + b : n \epsilon I\}$, where I is the set of all n such that

(5) $$n \leqslant an + b < n + 1.$$

It is now necessary to solve (5) for n. In so doing, (5) becomes $-b \leqslant (a - 1)n < 1 - b$, which is equivalent to $\frac{-b}{a-1} \leqslant n < \frac{1-b}{a-1}$ if $a > 1$, to $0 \leqslant b < 1$ if $a = 1$, and to $\frac{1-b}{a-1} < n \leqslant \frac{-b}{a-1}$ if $a < 1$. Therefore, $(S) = M([\frac{-b}{a-1}, \frac{1-b}{a-1})) = [\frac{b}{a-1}] - [\frac{b-1}{a-1}]$ if $a > 1$; $(S) = D$ if $a = 1$ and $0 \leqslant b < 1$, $(S) = 0$ if $a = 1$ and $b < 0$ or $b \geqslant 1$, and $(S) = M((\frac{1-b}{a-1}, \frac{-b}{a-1}]) = [\frac{-b}{a-1}] - [\frac{1-b}{a-1}]$ if $a < 1$. Also, we have $S = \{an + b: \frac{-b}{a-1} \leqslant n < \frac{1-b}{a-1}\}$, if $a > 1$ $S = \{n + b: n \text{ is an integer}\}$ if $a = 1$ and $0 \leqslant b < 1$, $S = \{an + b: \frac{1-b}{a-1} < n \leqslant \frac{-b}{a-1}\}$ if $a < 1$.

Note that the general linear equation is reduced to the form $x = a[x] + b$ here in order to apply the algebraic method of this chapter, and to the form $ax + b = [x]$ in Chapter 2 in order to simplify the geometric and algebraic arguments appearing there. With this in mind, the agreement of the results can be verified.

The general method will now be developed for $m \geqslant 1$. Let T be the solution set of (1) and T_i the solution set of the equation $x - g_i([x]) = 0$, for $i = 1, 2, 3, \ldots, m$, so that $T = \bigcup_{i=1}^{m} T_i$. By a general combinatorial theorem (3, p. 105-107), we now have

$$(6) \qquad (T) = \sum_{i} (T_i) - \sum_{i,j} (T_i \cap T_j) + \sum_{i,j,k} (T_i \cap T_j \cap T_k) - \cdots ,$$

where the summation indices are distinct. Equation (6) can be written as

$$(T) = \sum_{A} (-1)^{(A)+1} \left(\bigcap_{i \in A} T_i \right),$$

where A varies over all nonempty subsets of the set of integers, 1 through m. Denoting $\bigcap_{i \in A} T_i$ by $T(A)$, we see that if $x_1 \in T(A)$, $x_2 \in T(A)$, and $x_1 \neq x_2$, then $[x_1] \neq [x_2]$, for if $[x_1] = [x_2]$, then $g_i([x_1]) = g_i([x_2])$ for each $i \in A$, and so $x_1 = x_2$, a contradiction. Therefore, there exists a one-to-one correspondence between $T(A)$ and the range of the bracket function with its domain restricted to $T(A)$, which will be denoted by $I(A)$. The set $I(A)$ is now shown to be the set of all n such that

(7) $$n \leqslant g_i(n) = g_j(n) < n + 1 \quad \text{for all } i \epsilon A \text{ and } j \epsilon A$$

If $[x] \epsilon I(A)$, then $x = g_i([x])$ for all $i \epsilon A$; consequently, $[x] \leqslant g_i([x]) = g_j([x]) < [x] + 1$ for all $i \epsilon A$ and $j \epsilon A$. If n satisfies (7), then $n = [g_i(n)]$ for all $i \epsilon A$, and so $g_i(n) = g_i([g_i(n)]$ for all $i \epsilon A$. Therefore, if $x = g_i(n)$ for all $i \epsilon A$, then $x \epsilon T(A)$ and $n = [x] \epsilon I(A)$.

We now have

(8) $$(T) = \sum_A (-1)^{(A)+1} (I(A)).$$

If (7) can be solved for n for all A then Theorem 2 can be applied to determine $(I(A))$. If (7) can not be solved algebraically for n for some A, then the aid of a high speed electronic computer may still enable one to obtain $I(A)$ for all A for an explicit equation of the form (1), which may be simpler than using the computer directly on (1).

The solutions of (1) are now easily found. Let $I(A) = I_i$ if $A = \{i\}$ for $i = 1, 2, 3, \ldots, m$. Since $n \epsilon I_i$ if and only if $g_i(n) \epsilon T_i$, we have $T_i = \{g_i(n) \colon n \epsilon I_i\}$, where $I_i = \{n \colon n \leqslant g_i(n) < n+1\}$, for $i = 1, 2, 3, \ldots, m$. Consequently,

(9) $$T = \bigcup_{i=1}^{m} \{g_i(n) \colon n \epsilon I_i\}$$

The procedure for finding T and (T) by this general method is given in the following theorem:

<u>Theorem 5.</u> The following algorithm gives a procedure for determining the solution set T and its cardinality of an equation that can be written in the form $\prod_{i=1}^{m}(x - g_i([x])) = 0$ where $m \geq 1$ and g_i, for $i = 1, 2, 3, \ldots, m$, is a real and single valued function: Determine Ω, the set of all nonempty subsets of the integers, 1 through m. Let A be a variable with Ω as its range; for each value of A determine $(I(A))$ where $I(A) = \{n: n \leq g_i(n) = g_j(n) < n+1$ for all $i \in A$ and $j \in A\}$. Also, determine I_i where $I_i = \{n: n \leq g_i(n) < n+1\}$ and for each $n \in I_i$ determine $g_i(n)$, for $i = 1, 2, 3, \ldots, m$. Then (T) and T can be found from (8) and (9) respectively.

CHAPTER 4. THE GENERAL QUADRATIC EQUATION

The general quadratic equation in x and $[x]$ is of the form

$$ax^2 + b[x]^2 + cx[x] + dx + e[x] + f = 0, \tag{1}$$

where $a^2 + b^2 + c^2 \neq 0$. We will also assume that $a \geqslant 0$, since, if the coefficient of x^2 is negative then multiplication by -1 gives an equivalent equation where the coefficient of x^2 is positive. The cardinality of the solution set of (1) is given in the following theorem.

<u>Theorem 6.</u> The cardinality of the solution set U of the equation $ax^2 + b[x]^2 + cx[x] + dx + e[x] + f = 0$, where $a^2 + b^2 + c^2 \neq 0$, $a \geqslant 0$, and a, b, c, d, e, and f are real, is given by the following statement:

(a) If $a > 0$, then $(U) = (J_1 \cap J_2 \cap J_7 \cap J_8) + (J_1 \cap \overline{J}_2 \cap J_4 \cap J_8) + (J_1 \cap J_3 \cap J_4 \cap J_6) + (J_1 \cap \overline{J}_4 \cap J_6 \cap J_9) - (J_0 \cap J_3 \cap J_4)$, where the sets J_i are given in Table 1;

(b) if $a = 0$ and there exists an integer n such that both $cn + d = 0$ and $bn^2 + en + f = 0$, then $(U) = C$;

(c) if $a = 0$ and there does not exist an integer n such that both $cn + d = 0$ and $bn^2 + en + f = 0$, then $(U) = (J_4 \cap J_7 \cap J_8) + (J_5 \cap J_6 \cap J_9)$, where again the sets J_i are given in Table 1.

$$J_i = \{n: n \in R_i\}$$

TABLE 1

Condition on coefficients	R_o	R_1
$c^2 - 4ab > 0$		
$E \geqslant 0$	$[r_1, r_1] \cup [r_2, r_2]$	$(-\infty, r_2] \cup [r_1, \infty)$
$E < 0$	ϕ	$(-\infty, \infty)$
$c^2 - 4ab = 0$		
$cd - 2ae > 0$	$[r, r]$	$[r, \infty)$
$cd - 2ae = 0$		
$d^2 - 4af > 0$	ϕ	$(-\infty, \infty)$
$d^2 - 4af = 0$	$(-\infty, \infty)$	$(-\infty, \infty)$
$d^2 - 4af < 0$	ϕ	ϕ
$cd - 2ae < 0$	$[r, r]$	$(-\infty, r]$
$c^2 - 4ab < 0$		
$E \geqslant 0$	$[r_1, r_1] \cup [r_2, r_2]$	$[r_1, r_2]$
$E < 0$	ϕ	ϕ

Condition on coefficients	R_2	R_3
$2a + c > 0$	$(-\infty, -\frac{d}{2a+c}]$	$[-\frac{d}{2a+c}, \infty)$
$2a + c = 0$		
$d \geqslant 0$	$(-\infty, \infty)$	ϕ
$d < 0$	ϕ	$(-\infty, \infty)$
$2a + c < 0$	$[-\frac{d}{2a+c}, \infty)$	$(-\infty, \frac{-d}{2a+c}]$

Table 1 (Continued)

Condition on coefficients	R_4	R_5
$2a + c > 0$	$(-\infty, -\frac{2a+d}{2a+c})$	$(-\frac{2a+d}{2a+c}, \infty)$
$2a + c = 0$		
$2a + d \geqslant 0$	$(-\infty, \infty)$	ϕ
$2a + d < 0$	ϕ	$(-\infty, \infty)$
$2a + c < 0$	$(-\frac{2a+d}{2a+c}, \infty)$	$(-\infty, -\frac{2a+d}{2a+c})$

Condition on coefficients	R_6	R_7
$a + b + c > 0$		
$F \geqslant 0$	$(-\infty, s_2] \cup [s_1, \infty)$	$[s_2, s_1]$
$F < 0$	$(-\infty, \infty)$	ϕ
$a + b + c = 0$		
$d + e > 0$	$[-\frac{f}{d+e}, \infty)$	$(-\infty, -\frac{f}{d+e}]$
$d + e = 0$		
$f > 0$	$(-\infty, \infty)$	ϕ
$f \leqslant 0$	ϕ	$(-\infty, \infty)$
$d + e < 0$	$(-\infty, -\frac{f}{d+e}]$	$[-\frac{f}{d+e}, \infty)$
$a + b + c < 0$		
$F \geqslant 0$	$[s_1, s_2]$	$(-\infty, s_1] \cup [s_2, \infty)$
$F < 0$	ϕ	$(-\infty, \infty)$

Table 1 (Continued)

Condition on coefficients	R_8	R_9
$a + b + c > 0$		
$G \geq 0$	$(-\infty, t_2) \cup (t_1, \infty)$	(t_2, t_1)
$G < 0$	$(-\infty, \infty)$	ϕ
$a + b + c = 0$		
$2a + c + d + e > 0$	(t, ∞)	$(-\infty, t)$
$2a + c + d + e = 0$		
$a + d + f > 0$	$(-\infty, \infty)$	ϕ
$a + d + f \leq 0$	ϕ	$(-\infty, \infty)$
$2a + c + d + e < 0$	$(-\infty, t)$	(t, ∞)
$a + b + c < 0$		
$G \geq 0$	(t_1, t_2)	$(-\infty, t_1) \cup (t_2, \infty)$
$G < 0$	ϕ	$(-\infty, \infty)$

Notation:

$$E = 16a\,(ae^2 - cde + cf^2 + bd^2 - 4abf)$$

$$F = (d + e)^2 - 4f(a + b + c)$$

$$G = (2a + c + d + e)^2 - 4(a + d + f)(a + b + c)$$

$$r = \frac{4af - d^2}{2(cd-2ae)}$$

$$r_1 = \frac{-2cd + 4ae + \sqrt{E}}{2(c^2 - 4ab)}$$

Table 1 (Continued)

$$r_2 = \frac{-2cd + 4ae - \sqrt{E}}{2(c^2 - 4ab)}$$

$$s_1 = \frac{-d - e + \sqrt{F}}{2(a + b + c)}$$

$$s_2 = \frac{-d - e - \sqrt{F}}{2(a + b + c)}$$

$$t = -\frac{a + d + f}{2a + c + d + e}$$

$$t_1 = -\frac{2a + c + d + e - \sqrt{G}}{2(a + b + c)}$$

$$t_2 = -\frac{2a + c + d + e + \sqrt{G}}{2(a + b + c)}$$

ϕ denotes the empty set

In the proof of Theorem 6, it will be necessary to solve inequalities of the form (i) $Q(n) \geq 0$, (ii) $Q(n) \leq 0$, (iii) $Q(n) > 0$, and (iv) $Q(n) < 0$, where $Q(n) = \alpha n^2 + \beta n + \gamma$, $\alpha \neq 0$. Lemma 2 provides the solutions of (i) and (ii), from which the solutions of (iii) and (iv) are easily obtained.

<u>Lemma 2.</u> If $Q(n) = \alpha n^2 + \beta n + \gamma$, $\alpha \neq 0$, $x_1 = \frac{1}{2\alpha}\left(-\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)$, and $x_2 = \frac{1}{2\alpha}\left(-\beta - \sqrt{\beta^2 - 4\alpha\gamma}\right)$, then:

(a) If $\alpha > 0$ and $\beta^2 - 4\alpha\gamma \geq 0$, then $Q(n) \geq 0$ if and only if $n \geq x_1$ or $n \leq x_2$;

(b) if $\alpha > 0$ and $\beta^2 - 4\alpha\gamma \geqslant 0$, then $Q(n) \leqslant 0$ if and only if $x_2 \leqslant n \leqslant x_1$;

(c) if $\alpha > 0$ and $\beta^2 - 4\alpha\gamma < 0$, then $Q(n) > 0$ for all n;

(d) if $\alpha < 0$ and $\beta^2 - 4\alpha\gamma \geqslant 0$ then $Q(n) \geqslant 0$ if and only if $x_1 \leqslant n \leqslant x_2$;

(e) if $\alpha < 0$ and $\beta^2 - 4\alpha\gamma \geqslant 0$, then $Q(n) \leqslant 0$ if and only if $n \leqslant x_1$ or $n \geqslant x_2$;

(f) if $\alpha < 0$ and $\beta^2 - 4\alpha\gamma < 0$, then $Q(n) < 0$ for all n.

Proof. $Q(n) = \alpha n^2 + \beta n + \gamma = \alpha\left(n^2 + \frac{\beta}{\alpha} n + \frac{\gamma}{\alpha}\right)$ $= \alpha\left(n^2 + \frac{\beta}{\alpha} n + \frac{\beta^2}{4\alpha^2} - \frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha}\right) = \alpha\left(\left(n + \frac{\beta}{2\alpha}\right)^2 - \frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}\right)$. Therefore, if $\beta^2 - 4\alpha\gamma \geqslant 0$, then $Q(n) = \alpha\left(\left(n + \frac{\beta}{2\alpha}\right)^2 - \left(\frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\right)^2\right)$ $= \alpha(n - x_2)(n - x_1)$. If $\alpha > 0$, then $x_1 \geqslant x_2$ which implies that $Q(n) \geqslant 0$ if and only if $n \geqslant x_1$ or $n \leqslant x_2$ and $Q(n) \leqslant 0$ if and only if $x_2 \leqslant n \leqslant x_1$. If $\alpha > 0$ and $\beta^2 - 4\alpha\gamma < 0$, then $Q(n) = \alpha\left(\left(n + \frac{\beta}{2\alpha}\right)^2 + \frac{4\alpha\gamma - \beta^2}{4\alpha^2}\right) > 0$ for all n.

Parts (a), (b), and (c) have now been proved, and parts (d), (e), (f) are proved in a similar manner.

Proof of Theorem 6. In order to apply Theorem 5, we must write (1) in the form $\prod_{i=1}^{m}(x - g_i([x])) = 0$ and, therefore, must consider the cases (i) $a > 0$ and (ii) $a = 0$ separately.

If $a > 0$, then by means of the quadratic formula, we see that

$$(2) \qquad (x - \frac{1}{2a}(c[x] + d - \sqrt{D([x])}))(x - \frac{1}{2a}(c[x] + d + \sqrt{D([x])})) = 0,$$

where $D([x]) = (c[x] + d)^2 - 4a(b[x]^2 + e[x] + f)$, is equivalent to (1). Therefore, we are able to apply Theorem 5, and in so doing, we use the notation of Chapter 3. Since $m = 2$, the values of A are $\{1, 2\}$, $\{1\}$, and $\{2\}$. The corresponding values of $I(A)$ will be denoted by I_{12}, I_1, and I_2, respectively.

We now have

$$(3) \qquad \begin{cases} g_1(n) = \frac{1}{2a}(-cn - d + \sqrt{D(n)}), \\ g_2(n) = \frac{1}{2a}(-cn - d - \sqrt{D(n)}), \end{cases}$$

and

$$(4) \qquad \begin{cases} I_1 = \{n : n \leqslant g_1(n) < n + 1\}, \\ I_2 = \{n : n \leqslant g_2(n) < n + 1\}, \\ I_{12} = \{n : n \leqslant g_1(n) = g_2(n) < n + 1\}. \end{cases}$$

Substition of (3) into (4) leads to

$$(5) \qquad \begin{cases} I_1 = \{n : (2a + c)n + d \leqslant \sqrt{D(n)} < (2a + c)n + 2a + d\} \\ I_2 = \{n : (2a + c)n + d \leqslant -\sqrt{D(n)} < (2a + c)n + 2a + d\}, \\ I_{12} = \{n : (2a + c)n + 2a + d > 0,\ (2a + c)n + d \leqslant 0,\ D(n) = 0\}. \end{cases}$$

It is now necessary to solve the inequalities in (5) for n. It may be verified in a straight forward but tedious manner that

$$(6)\quad \begin{cases} I_1 = \{J_1 \cap J_2 \cap J_7 \cap J_8\} \cup \{J_1 \cap \bar{J}_2 \cap J_4 \cap J_8\}, \\ I_2 = \{J_1 \cap J_3 \cap J_4 \cap J_6\} \cup \{J_1 \cap \bar{J}_4 \cap J_6 \cap J_9\}, \\ I_{12} = \{J_0 \cap J_3 \cap J_4\}, \end{cases}$$

where,

$$(7)\quad \begin{cases} J_0 = \{n : D(n) = 0\}, \\ J_1 = \{n : D(n) \geqslant 0\}, \\ J_2 = \{n : (2a + c)n + d \geqslant 0\}, \\ J_3 = \{n : (2a + c)n + d \leqslant 0\}, \\ J_4 = \{n : (2a + c)n + 2a + d > 0\}, \\ J_5 = \{n : (2a + c)n + 2a + d < 0\}, \\ J_6 = \{n : h_1(n) \geqslant 0\}, \\ J_7 = \{n : h_1(n) \leqslant 0\}, \\ J_8 = \{n : h_2(n) > 0\}, \\ J_9 = \{n : h_2(n) < 0\}, \end{cases}$$

where,

$$\begin{aligned} h_1(n) &= (((2a + c)n + d)^2 - D(n))/4a \\ &= ((4a^2 + 4ac + c^2)n^2 + (4ad + 2cd)n + d^2 - e^2n^2 - 2cdn \\ &\quad -d^2 + 4abn^2 + 4aen + 4af)/4a \\ &= ((4a^2 + 4ab + 4ac)n^2 + 4a(d + e + f)n + 4af)/4a \\ &= (a + b + c)n^2 + (d + e)n + f, \end{aligned}$$

and

$$h_2(n) = (((2a + c)n + 2a + d)^2 - D(n))/4a$$
$$= ((4a^2 + 4ac + c^2)n^2 + (8a^2 + 4ac + 4ad + 2cd)n$$
$$+ 4a^2 + 4ad + d^2 - e^2n^2 - 2cdn - d^2 + 4abn^2 + 4aen$$
$$+ 4af)/4a$$
$$= (a + b + c)n^2 + (2a + c + d + e)n + a + d + f.$$

From (6), we see that I_1 and I_2 are each the union of two disjoint sets and it follows that $(U) = (I_1) + (I_2) - (I_{12}) = (J_1 \cap J_2 \cap J_7 \cap J_8) + (J_1 \cap \overline{J}_2 \cap J_4 \cap J_8) + (J_1 \cap J_3 \cap J_4 \cap J_6) + (J_1 \cap \overline{J}_4 \cap J_6 \cap J_9) - (J_0 \cap J_3 \cap J_4)$.

The inequalities in (7) can be solved for n thus making it possible to describe the sets J_i, for $i = 1, 2, 3, \ldots, 9$, in terms of intervals of real numbers. For example, J_1 is the set of all n such that $D(n) = (cn + d)^2 - 4a(bn^2 + en + f) = (c^2 - 4ab)n^2 + (2cd - 4ae)n + d^2 - 4af \geqslant 0$. If $c^2 - 4ab \neq 0$ and $E = 4(cd - 2ae)^2 - 4(c^2 - 4ab)(d^2 - 4af) = 4c^2d^2 - 16acde + 16a^2e^2 - 4c^2d^2 + 16afc^2 + 16abd^2 - 64a^2bf = 16a(ae^2 - cde + f^2c + bd^2 - 4abf) \geqslant 0$ then let

$$r_1 = \frac{-2cd + 4ae + \sqrt{E}}{c^2 - 4ab} \quad \text{and} \quad r_2 = \frac{-2cd + 4ae - \sqrt{E}}{c^2 - 4ab}.$$

By Lemma 2, we see that if $c^2 - 4ab > 0$, then $D(n) \geqslant 0$ if and only if $n \leqslant r_2$ or $n \geqslant r_1$ if $E(n) \geqslant 0$, and $D(n) \geqslant 0$ for all n if $E(n) < 0$. Similarly, if $c^2 - 4ab < 0$, then $D(n) \geqslant 0$ if and only if $r_1 \leqslant n \leqslant r_2$

if $E(n) \geqslant 0$ and $D(n) < 0$ for all n if $E(n) < 0$. If $c^2 - 4ab = 0$, then $D(n) = 2(cd - 2ae)n + d^2 - 4af$. Therefore, $D(n) \geqslant 0$ if and only if $n \geqslant \frac{4af - d^2}{2(cd - 2ae)}$, if $cd - 2ae > 0$, or $n \leqslant \frac{4af - d^2}{2(cd - 2ae)}$ if $cd - 2ae < 0$. If $c^2 - 4ab = 0$ and $cd - 2ae = 0$, then $D(n) \geqslant 0$ for all n if $d^2 - 4af \geqslant 0$ and for no n if $d^2 - 4af < 0$. Such results from solving the inequalities in (7) for n are given in Table 1. This completes the proof of (a).

If $a = 0$, then (1) becomes

(8) $$b[x]^2 + cx[x] + dx + e[x] + f = 0,$$

where $b^2 + c^2 \neq 0$. If there exists an integer n such that $cn + d = 0$ and $bn^2 + en + f = 0$, we see that x is a solution of (8) if $n \leqslant x < n+1$, and so $(U) = C$, which proves (b). If there does not exist such an integer n then $c[x] + d \neq 0$ for all x, and (8) is equivalent to

(9) $$x = - \frac{b[x]^2 + e[x] + f}{c[x] + d}$$

Applying Theorem 5 to (9), we have $(U) = (I)$, where $I = \{n : n \leqslant g(n) < n + 1\}$ and $g(n) = - \frac{bn^2 + en + f}{cn + d}$. It may be shown in a straight forward but tedious manner that $I = \{n : cn + d > 0, h_2(n) > 0, h_1(n) \leqslant 0\} \cup \{n : cn + d < 0, h_2(n) < 0, h_1(n) \geqslant 0\} = \{J_4 \cap J_7 \cap J_8\} \cup \{J_5 \cap J_6 \cap J_9\}$, where h_1, h_2, J_4, J_5, J_6, J_7, J_8, and J_9 are defined in (7). Since J_4 and J_5 are disjoint, I is the union of two disjoint sets and $(I) = (J_4 \cap J_7 \cap J_8) + (J_5 \cap J_6 \cap J_9)$, which proves (a).

The solution set of (1) is given in the following theorem.

Theorem 7. The solution set U of the equation $ax^2 + b[x]^2 + cx[x] + dx + e[x] + f = 0$, where $a^2 + b^2 + c^2 \neq 0$, $a \geqslant 0$, and a, b, c, d, e, and f are real, is given by the following statement, where the sets J_i, $i = 1, 2, 3, \ldots, 9$, and $D(n)$ are given in Table 1:

(a) If $a > 0$, then $U = \{\frac{1}{2a}(-cn - d + \sqrt{D(n)})$: $n \in \{J_1 \cap J_2 \cap J_7 \cap J_8\} \cup \{J_1 \cap \overline{J}_2 \cap J_4 \cap J_8\}\} \cup \{\frac{1}{2a}(-cn - d - \sqrt{D(n)})$: $n \in \{J_1 \cap J_3 \cap J_4 \cap J_6\} \cup \{J_1 \cap \overline{J}_4 \cap J_6 \cap J_9\}\}$;

(b) if $a = 0$, then $U = \{[n, n + 1)$: $cn + d = 0$, and $bn^2 + en + f = 0\} \cup \{-\frac{bn^2 + en + f}{cn + d}$: $n \in \{J_4 \cap J_7 \cap J_8\} \cup \{J_5 \cap J_6 \cap J_9\}\}$.

Proof. If $a > 0$, by Theorem 5, we have $U = \{g_1(n): n \in I_1\} \cup \{g_2(n) : n \in I_2\}$, which is equivalent to (a). If $a = 0$, then $U = \{[n, n + 1)$:$cn + d = 0$, and $bn^2 + en + f = 0\} \cup S$, where S is the solution set of (9). By Theorem 5, $S = \{g(n): n \in I\}$ and (b) follows immediately.

CHAPTER 5. EXTENSIONS TO THE COMPLEX FIELD

The domain of the bracket function is extended to the complex field by Definition 3.

Definition 3. If z is the complex number $x + iy$, where x and y are real, then $[z]$ is defined to be the complex number $[x] + i[y]$.

In the real field, we were concerned with integers. Correspondingly, in the complex field we will be concerned with Gaussian integers which are complex numbers of the form $n_1 + i\,n_2$, where n_1 and n_2 are real integers. In this chapter, n will denote a Gaussian integer.

In the real field, we saw that the order relation played an important role; therefore, an order relation is established for the complex field. This is done in Definition 4.

Definition 4. If a and b are the complex members $a_1 + ia_2$ and $b_1 + ib_2$, where a_1, a_2, b_1, and b_2 are real, then $a \leqslant b$ if and only if $a_1 \leqslant b_1$ and $a_2 \leqslant b_2$; $a < b$ if and only if $a \leqslant b$ and $a \neq b$; $a \geqslant b$ if and only if $b \leqslant a$; $a > b$ if and only if $b < a$.

It is a simple matter to show that $a \leqslant b$ is equivalent to $a + c \leqslant b + c$ or $a < b$ is equivalent to $a + c < b + c$. However, if a and b are ordered by one of the order relations of Definition 4, then it does not necessarily follow that ac and bc are ordered unless c is real. This dissimilarity will prove to be the only source of difficulty in the extensions to the complex field.

Two elementary properties of $[z]$ are given in Theorem 8.

Theorem 8. (a) If z is a complex number, then $z - (1 + i) < [z] \leqslant z < [z] + 1 + i$; (b) if $n \leqslant z < n + 1 + i$, then $[z] = n$.

Proof. Let $z = x + iy$. We have by Theorem 1a, $x - 1 < [x] \leqslant x < [x] + 1$ and $y - 1 < [y] \leqslant y < [y] + 1$. From Definition 3, we have $iy - i < i[y] \leqslant iy < i[y] + i$. Therefore, $x + iy - (1 + i) < [x] + i[y] \leqslant x + iy < [x] + i[y] + 1 + i$ which is equivalent to (a). Let $n = n_1 + in_2$. Then we have $n_1 \leqslant x < n_1 + 1$ $n_2 \leqslant y < n_2 + 1$. By Theorem 1c, $[x] = n_1$ and $[y] = n_2$. Therefore, $n = [x] + i[y] = [z]$.

In the real field, $[x]$ was defined to be the greatest integer not exceeding x. Let us see what analogous result we have in the complex field. Suppose there exists an n such that $[z] = [x] + i[y] < n = n_1 + in_2 \leqslant z = x + iy$. Either $n_1 > [x]$ or $n_2 > [y]$, which implies that $n_1 > x$ or $n_2 > y$, a contradiction. Therefore, $[z]$ is the greatest Gaussian integer less than or equal to z. However, the greatest Gaussian integer not exceeding z does not exist since $[z] + m$, where m is a positive real integer, is unbounded and not ordered with respect to z.

With these results an analog of Theorem 5 for the complex field is easily proved. This analog is identical to Theorem 5 except that 1 is replaced by $1 + i$ in the condition $n \leqslant g_i(n) = g_j(n) < n + 1$.

We now proceed to find the solution set and its cardinality, of the general linear equation in z and $[z]$, which is of the form

$$dz + e[z] + f = 0, \tag{1}$$

where $d^2 + e^2 \neq 0$. If $d = 0$, then (1) becomes $e[z] + f = 0$, where $e \neq 0$. In this case, if f/e is a Gaussian integer, then the solution set is $\{z : \frac{-f}{e} \leqslant z < -\frac{f}{e} + 1 + i\}$ and its cardinality is C; if $\frac{f}{e}$ is not a Gaussian integer, then there are no solutions. Hence, it is assumed from here that $d \neq 0$, in which case, (1) can be written in the form

$$z = a[z] + b. \tag{2}$$

For the equation $x = a[x] + b$, in the real field, we saw in Chapter 3 that there exists an interval R such that x is a solution if and only if x is of the form $an + b$, where $n \in R$. The analogous result for (2), in the complex field, is given in the following theorem.

<u>Theorem 9.</u> The equation $z = a[z] + b$, where $a = a_1 + ia_2$ and $b = b_1 + ib_2$ are complex, holds if and only if z is of the form $an + b$, where $n = n_1 + in_2$ lies in the rectangular region C

in the complex plane which is defined by

$$(3)\qquad \begin{aligned} -b_1 &\leqslant (a_1 - 1)\,n_1 - a_2 n_2 < 1 - b_1, \\ -b_2 &\leqslant a_2 n_1 + (a_1 - 1)\,n_2 < 1 - b_2. \end{aligned}$$

Proof. Letting V denote the solution set of (2), we have, by the complex analog of Theorem 5, that $(V) = (I)$ and $V = \{ an + b : n \in I \}$, where I is the set of all n such that

$$(4)\qquad n \leqslant an + b < n + 1 + i \; .$$

It is now necessary to solve (4) for n. In so doing, (4) becomes $-b \leqslant (a - 1)\,n < -b + 1 + i$, which can be written as

$$-b_1 - ib_2 \leqslant (a_1 - 1)n_1 - a_2 n_2 + (a_2 n_1 + (a_1 - 1)n_2)\,i < 1 - b_1 + (1 - b_2)\,i,$$

which, in turn, is equivalent to the simultaneous pair of inequalities (3).

It will now be shown that (3) defines a rectangular region in the complex plane. Let ℓ_1 and ℓ_2 be the parallel linear graphs of the equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ respectively, where $c_2 > c_1$. Then $n_1 + in_2$ lies in the same open half plane which is formed by ℓ_1 as ℓ_2 if and only if $an_1 + bn_2 + c_1$ has the same sign as $ax_1 + by_1 + c_1$ where (x_1, y_1) is a point on ℓ_2. If $b \neq 0$, then $(0, -\frac{c_2}{b})$ is a point on ℓ_2 and $a \cdot 0 + b \cdot (\frac{-c_2}{b}) + c_1 = c_1 - c_2 < 0$; if $b = 0$, then $(-\frac{c_2}{a}, 0)$ is a point on ℓ_2 and $a \cdot (\frac{-c_2}{b}) + b \cdot 0 + c_1 = c_1 - c_2 < 0$. Thus, $n_1 + in_2$ lies in the same open half plane as ℓ_2 if and only if $an_1 + bn_2 + c_1 < 0$.

Similarly, it is shown that $n_1 + i n_2$ lies in the same closed half plane which is formed by ℓ_2 as ℓ_1 if and only if $an_1 + bn_2 + c_2 \geqslant 0$. Thus, $n_1 + i n_2$ lies between ℓ_1 and ℓ_2 or on ℓ_2 if and only if $-c_2 \leqslant a n_1 + b n_2 < -c_1$. Let ℓ_3 and ℓ_4 be the parallel linear graphs of the equations $ay - bx + c_3 = 0$ and $ay - bx + c_4 = 0$, respectively, where $c_4 > c_3$. Then ℓ_3 and ℓ_4 are perpendicular to ℓ_1 and ℓ_2. Also, $n_1 + i n_2$ lies between ℓ_3 and ℓ_4 or on ℓ_4 if and only if $-c_4 \leqslant - b n_1 + a n_2 < - c_3$. Therefore, $n_1 + i n_2$ lies within or on two of the boundaries of the rectangle formed by ℓ_1, ℓ_2, ℓ_3 and ℓ_4 if and only if

$$(5) \qquad \begin{cases} -c_2 \leqslant a n_1 + b n_2 < -c_1, \\ -c_4 \leqslant -bn_1 + a n_2 < -c_3. \end{cases}$$

Finally, we see that (3) is of the form (5) since $-b_1 + 1 > -b_1$ and $-b_2 + 1 > -b_2$. This completes the proof of Theorem 9.

We have shown that (V) is the number of Gaussian integers n such that $n \epsilon C$. A very complicated formula for this number can be obtained by a straight forward counting method. It is only when either $a_2 = 0$ or $a_1 = 1$, that (3) can be easily solved for n_1 and n_2 thereby giving a simple expression for (V). These two cases correspond to the case where C is oriented such that its boundaries are parallel to the coordinate axes.

The following two theorems give (V) and V for $a_2(a_1 - 1) = 0$.

<u>Theorem 10.</u> The cardinality of the solution set of the equation $z = a[z] + b$, where $a = a_1 + ia_2$ and $b = b_1 + ib_2$ are complex and $a_2(a_1 - 1) = 0$, is given by:

$$(V) = \begin{cases} \left(\left[\frac{b_1}{|a_2|}\right] - \left[\frac{b_1 - 1}{|a_2|}\right]\right)\left(\left[\frac{b_2}{|a_2|}\right] - \left[\frac{b_2 - 1}{|a_2|}\right]\right) & \text{if } a_2 \neq 0 \text{ and } a_1 = 1; \\ \left(\left[\frac{b_1}{|a - 1|}\right] - \left[\frac{b_1 - 1}{|a - 1|}\right]\right)\left(\left[\frac{b_2}{|a - 1|}\right] - \left[\frac{b_2 - 1}{|a - 1|}\right]\right) & \text{if } a_2 = 0 \text{ and } a \neq 1; \\ D \quad \text{if } a = 1,\ 0 \leqslant b_1 < 1, \text{ and } 0 \leqslant b_2 < 1; \\ 0 \quad \text{if } a = 1 \text{ and } b_1 < 0 \text{ or } b_2 < 0 \text{ or } b_1 \geqslant 1 \text{ or } b_2 \geqslant 1 \end{cases}$$

Proof. If $a_2 \neq 0$ and $a_1 = 1$, then inequalities (5) become $-b_1 \leqslant -a_2 n_2 < 1 - b_1$ and $-b_2 \leqslant a_2 n_1 < 1 - b_2$, which are equivalent to $-\frac{1 - b_1}{a_2} < n_2 \leqslant \frac{b_1}{a_2}$ and $\frac{-b_2}{a_2} \leqslant n_1 < \frac{1 - b_2}{a_2}$ if $a_2 > 0$, and to $\frac{b_1}{a_2} \leqslant n_2 < -\frac{1 - b_1}{a_2}$ and $\frac{1 - b_2}{a_2} < n_1 \leqslant -\frac{b_2}{a_2}$ if $a_2 < 0$. Therefore, if $a_2 \neq 0$ and $a_1 = 1$, then $(V) = M((-\frac{1 - b_1}{a_2}, \frac{b_1}{a_2}]) \cdot M([-\frac{b_2}{a_2}, \frac{1 - b_2}{a_2})) = ([\frac{b_1}{a_2}] - [-\frac{1 - b_1}{a_2}])([\frac{b_2}{a_2}] - [-\frac{1 - b_2}{a_2}])$ if $a_2 > 0$,

and $(V) = M([\frac{b_1}{a_2}, -\frac{1-b_1}{a_2})) \cdot M((\frac{1-b_2}{a_2}, -\frac{b_2}{a_2}]) = ([-\frac{b_1}{a_2}]$
$- [\frac{1-b_1}{a_2}])([-\frac{b_2}{a_2}] - [\frac{1-b_2}{a_2}])$ if $a_2 < 0$. These two expressions for (V) are special cases of the one given above.

If $a_2 = 0$ and $a \neq 1$, then inequalities (5) become $-b_1 \leqslant (a-1)n_1 < 1 - b_1$ and $-b_2 \leqslant (a-1)n_2 < 1 - b_2$, which are equivalent to $\frac{-b_1}{a-1} \leqslant n_1 < \frac{1-b_1}{a-1}$ and $\frac{-b_2}{a-1} \leqslant n_2 < \frac{1-b_2}{a-1}$ if $a > 1$, and to $\frac{1-b_1}{a-1} < n_1 \leqslant \frac{-b_1}{a-1}$ and $\frac{1-b_2}{a-1} < n_2 \leqslant \frac{-b_2}{a-1}$ if $a < 1$. Therefore, if $a_2 = 0$ and $a \neq 1$, then $(V) = M([\frac{-b_1}{a-1}, \frac{1-b_1}{a-1})) \cdot$
$M([\frac{-b_2}{a-1}, \frac{1-b_2}{a-1})) = ([\frac{b_1}{a-1}] - [-\frac{1-b_1}{a-1}])([\frac{b_2}{a-1}] - [-\frac{1-b_2}{a-1}]$ if $a > 1$, and $(V) = M((\frac{1-b_1}{a-1}, \frac{-b_1}{a-1}]) \cdot M((\frac{1-b_2}{a-1}, \frac{-b_2}{a-1}]) = ([\frac{-b_1}{a-1}] - [\frac{1-b_1}{a-1}]) \cdot$
$([\frac{-b_2}{a-1}] - [\frac{1-b_2}{a-1}]$ if $a < 1$. These two expressions for (V) are special cases of the one given above.

If $a = 1$, then inequalities (5) become $-b_1 \leqslant 0 < 1 - b_1$ and $-b_2 \leqslant 0 < 1 - b_2$. Therefore, if $0 \leqslant b_1 < 1$ and $0 \leqslant b_2 < 1$, then all Gaussian integers satisfy (5) and $(V) = D$; if $b_1 < 0$ or $b_2 < 0$ or $b_1 \geqslant 1$ or $b_2 \geqslant 1$, then no Gaussian integer satisfies (5) and $(V) = 0$. This completes the proof of Theorem 10.

Theorem 11. The solution set of the equation $z = a[z] + b$, where $a = a_1 + i a_2$ and $b = b_1 + ib_2$ are complex and $a_2(a_1 - 1) = 0$ is given by the following statement:

(a) If $a_2 > 0$ and $a_1 = 1$, then $V = \{a(n_1 + in_2) + b:$

$$\frac{-b_2}{a_2} \leqslant n_1 < \frac{1 - b_2}{a_2} \text{ and } \frac{b_1 - 1}{a_2} < n_2 \leqslant \frac{b_1}{a_2} \};$$

(b) if $a_2 < 0$ and $a_1 = 1$, then $V = \{ a(n_1 + in_2) + b:$

$$\frac{1 - b_2}{a_2} < n_1 \leqslant \frac{-b_2}{a_2} \text{ and } \frac{b_1}{a_2} \leqslant n_2 < \frac{b_1 - 1}{a_2} \};$$

(c) if $a_2 = 0$ and $a > 1$, then $V = \{ a(n_1 + in_2) + b:$

$$\frac{-b_1}{a - 1} \leqslant n_1 < \frac{1 - b_1}{a - 1} \text{ and } \frac{-b_2}{a - 1} \leqslant n_2 < \frac{1 - b_2}{a - 1} \};$$

(d) if $a_2 = 0$ and $a < 1$, then $V = \{ a(n_1 + in_2) + b:$

$$\frac{1 - b_1}{a - 1} < n_1 \leqslant \frac{-b_1}{a - 1} \text{ and } \frac{1 - b_2}{a - 1} < n_2 \leqslant \frac{-b_2}{a - 1} \};$$

(e) if $a = 1$, $0 \leqslant b_1 < 1$, and $0 \leqslant b_2 < 1$, then $V = \{ n + b:$ n is a Gaussian integer$\}$.

Proof. By the complex analog of Theorem 5, $V = \{ an + b: n \epsilon I \}$, where I is the set of all Gaussian integers satisfying (5). Thus, Theorem 11 follows directly from results in the proof of Theorem 10.

Frequently, the application of the analog of Theorem 5 fails because an algebraic solution of the inequalities, $n \leqslant g_i(n) = g_j(n) < n + 1 + i$ for all $i \in A$ and $j \in A$, can not be found. For the general quadratic equation, in the real field, it was necessary to "square each side" in order to solve inequalities of the above type. However, in so doing in the complex field, an ordering is not preserved.

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