AN ABSTRACT OF THE DISSERTATION OF

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Rebekah L. Elliott

Policy and research have increased attention in K12 mathematics education on core mathematical practices such as argumentation, justification, and proof, that are central to students learning and doing mathematics. One reason for this increased focus is the empirical evidence that engaging students in mathematical practices provide more equitable opportunities for student learning. Although mathematics disciplinary practices (MDPs) are found in postsecondary educational literature, they are often described in context to particular mathematical content areas in undergraduate mathematics. Research in undergraduate mathematics rarely takes mathematical practices as a focal point of research, even though these practices are essential to doing mathematics across multiple content areas. Because students in advanced mathematics courses are more explicitly engaged in the practices of the discipline, understanding how mathematicians report on using and teaching MDPs is an essential next step toward understanding the role that MDPs play across mathematical content domains.

My qualitative dissertation study presented in two manuscripts collectively examines mathematics disciplinary practices (MDPs) and their relation to teaching and learning in advanced undergraduate mathematics courses. Across two phases, I initially conducted an interview study with eight mathematicians that teach upper-division undergraduate mathematics courses to understand the ways that mathematicians use MDPs in their professional research work and how they approach teaching MDPs in their advanced mathematics courses. In the second phase, I identified two case studies from the first phase of the research of mathematicians’ advanced
undergraduate instruction that I investigated with the goal of better understanding the ways that MDPs might emerge in classroom settings. Across both manuscripts of the study, I provide a descriptive analysis of the MDPs that mathematicians use in research and teaching.

In the first manuscript, I highlight three themes that I identified through the process of my analysis of interviews with eight mathematicians. To situate the study within the educational literature, I trace the ways that mathematical practices were developed across policy documents. I offer a conceptualization of MDPs grounded in the research literature that provides the foundation for discussing MDPs in the professional work of mathematicians and how such practices are taught in advanced mathematics. The first theme outlines the MDPs that were reported by mathematicians and describes how these practices coalesced. The second theme reports on the ways that MDPs are used in mathematicians' research. The final theme examines mathematicians' reflections on teaching students MDPs in their advanced undergraduate mathematics courses. Within this theme, I explicate the social and analytical scaffolding mathematicians' reported to teach students MDPs in their advanced courses. When I pressed mathematicians for teaching practices that support students’ learning MDPs, they often drew on general teaching practices, including demonstrations with a variety of examples, employing group work, or assigning homework problems. Collectively, my analysis of the interview data highlights the difficulty of teaching towards MDPs and I discuss the complexity of MDPs in mathematicians' professional work and the ways MDPs might be taught in advanced mathematics settings. From my analysis, I suggest that teaching mathematical practices is a difficult endeavor.

In my second manuscript, I focus on an important MDP that I identified from my analysis of mathematicians' interviews. I describe how mathematicians attended to mathematical conditions, assumptions, and properties as they engage in mathematical work across content domains (which I call the CAPs practice). Although the CAPs practice appears in literature around mathematical modeling and to some extent in mathematical proof, the CAPs practice is treated by many researchers as a smaller component of these larger practices (e.g., creating mathematical models, proving mathematical statements). In this manuscript, I argue that attending to conditions, assumptions, and properties is an important cross-cutting practice of the discipline that mathematicians described as an important part of their research and in their teaching. This manuscript offers a framework for understanding four ways that the CAPs practice is used in the
discipline of mathematics, as well as how the CAPs practice extends to other STEM disciplines. The findings offer mathematician perspectives on the ways they learned to engage in the CAPs practice, how they currently engage in the practice professionally, and instances in which they discussed CAPs within their teaching. Implications from this work call attention to a disciplinary practice that is largely unexplored as an entity and offers an empirically grounded definition.

An implication of this overall study is for reframing the mathematics students should learn and ways to improve students' undergraduate experiences in advanced mathematics courses to support their learning. This implication reflects similar calls in K12 mathematics education for greater attention to mathematical practices. In particular, this study advances the importance of teaching towards mathematical practices across all content domains in postsecondary mathematics courses to support students' explicit engagement in MDPs in undergraduate mathematics classrooms. There is a real need to explore the relationships between the variety of professional mathematical practices and how mathematicians can employ specific instructional strategies to support students' engagement in these MDPs in their teaching. The second implication of this dissertation study is for the community of mathematics education researchers whose interests are in undergraduate mathematics. The results of this study indicate a need for more research on instructional approaches that support students in advanced mathematics to learn the constellation of MDPs that mathematicians use in their research. These two manuscripts describe the importance of MDPs in learning and doing mathematics. MDPs should not be treated implicitly as they are often subsumed in educational research focused on specific mathematical content areas. I look forward to advancing this work to explore MDPs in developmental courses and lower-division mathematics courses. A goal of future research will study specific classroom teaching practices that promote students' engagement in MDPs, as well as focus on student experiences as they learn to engage in MDPs in classroom settings.
An Exploratory Study of Mathematics Disciplinary Practices in Advanced Mathematics

by
Erin D. Glover

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APPROVED:

Major Professor, representing Mathematics Education

Dean of the College of Education

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes the release of my dissertation to any reader upon request.

Erin D. Glover, Author
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This is for my parents. I know they would be so proud of me and I wish they were still here to celebrate one of my greatest accomplishments.
CHAPTER 1 – Introduction

This dissertation is presented across two manuscripts and a conclusion chapter. I first outline my motivations for conducting this study and outline my research questions and methodological considerations and close this chapter with the significance of this dissertation.

Motivations for Research Study

The motivation to conduct this study began as a result of my early experiences mentoring in and later teaching the course, Inquiry into Mathematics and Mathematics Education in the College of Education at Oregon State University. To effectively mentor in and later teach this class, I had a crash-course in learning what “math practices” were. Not readily understanding the concept of a “math practice” felt quite strange that someone with a B.S. and an advanced degree in mathematics because shouldn’t I already understand and have internalized this mathematical content that allowed me to do mathematics? I was well out of K12 by the time the Common Core State Standards for Math (National Governors Association Center for Best Practices, 2010) came about, and yet, I never recall thinking about or communicating with my instructors, classmates, or study-friends about “making and testing conjectures” or “generalizing findings” or “validating this proof”. Certainly, I engaged in those practices throughout my mathematics courses; but, reflecting on my educational experiences, I wonder whether and how my undergraduate and advanced mathematics coursework could have been different had there been a name for these practices. Could I have developed more confidence in my mathematical understandings if there was a push to focus on practices that span all content domains in mathematics? Could mathematical practices have been the key for me to understand how mathematics as a connected subject instead of the way it was presented in higher education (10-week courses siloed by content domains)? Could I have been better at writing and validating proofs? It is no wonder I failed to see how connected mathematics really was until much later in my mathematical studies! These wonderings led to my fascination with mathematical practices and became the foundation of my research interests, as evidenced by my margin notes from 2017 in Figure 1.1.

While my dissertation study was not designed to answer my wonderings directly but engaging in this work has certainly refined my perspectives on mathematical practices. I find myself now explicitly teaching towards mathematical practices in my own mathematics courses at
the community college because I believe elevating mathematical practices to “thingness” could go a long way to support students engaging in novel mathematics.

Figure 1.1 Margins notes from reading in 2017.

**Rationale for Dissertation**

There is increased attention in K12 mathematics education to focus on core mathematical practices because they are central to learning and doing mathematics. The research community has argued for and found empirical evidence that instruction that engages students in mathematical practices provides more equitable opportunities for student learning. Although mathematics disciplinary practices (MDPs) are found in postsecondary educational literature, they are often described in relation to particular mathematical content areas in undergraduate mathematics. Research in undergraduate mathematics rarely takes MDPs as the focal point of research that are essential to *doing* mathematics across *all* content areas. Since students in postsecondary setting, and particularly those in advanced mathematics courses, are more explicitly engaging in the practices of the discipline, understanding how MDPs are taught in advanced mathematics courses is an essential next step in understanding the role that MDPs play across mathematical content domains and how these MDPs are taught in advanced undergraduate mathematics courses.

**Theoretical Framing**

Because researchers have not advanced a definitive conceptualization of the construct of *mathematical practice* across research literature and policy documents, it was necessary for me to theoretically ground what I mean by this phrase (Moschkovich, 2015). To understand the ways
that mathematicians think about and use mathematical practices in their research and how such practices show up in teaching, I take a situative perspective that conceptualizes learning as both a cognitive and social process (Greeno, 1998). As students and mathematicians come to understand and use mathematical practices, they are engaged in social settings in which mathematical practices are ways of acting, thinking, and talking with mathematics (Frank, Kazemi & Batty, 2007; Moschkovich, 2007; Sfard, 2012). In line with Moschkovich’s (2013) view:

Mathematical practices are social and cultural, because they arise from communities and mark membership in communities. They are also cognitive, because they involve thinking, and they are also semiotic, because they involve semiotic systems (signs, tools, and their meanings). Mathematical practices involve values, points of view, and implicit knowledge. (p. 264)

Moschkovich (2013) suggests practices are rooted within the mathematical discipline itself (ways that one engages in doing the work of the subject area) and are how one learns the discipline (ways that students individually and collectively participate in solving problems). Mathematical practices are the ways of engaging in disciplinary work and learning the subject. They are ways of reasoning that draw upon the system of terminology, symbols, and tools imbued with mathematical meaning (e.g., Boaler, 2002; Rogoff, 1990). To be a “practice” of the discipline it has to have utility across problems and sub-domains of the subject area while also being widely understood as normative in how disciplinary work is accomplished. Mathematical practices serve to organize ways of thinking, acting, and talking when doing mathematics across content and social spaces: in classrooms, in research presentations, and in mathematics seminars. Throughout this document, I refer to these practices as Mathematics Disciplinary Practices (MDPs) and define them to be the ways in which mathematicians do the work of mathematics in their research. Others have referred to these practices as disciplinary practices (Rasmussen, Wawro, & Zandieh, 2015) or academic mathematics to refer to the academic practices of mathematicians as opposed to everyday or workplace practices (e.g., Moschkovich, 2002).

**Research Methodology**

Advanced undergraduate mathematics was selected as the context for this study because students in these courses are expected to learn and engage in the professional practices of the discipline. With this in mind, I conjectured that the mathematicians working with students in these courses would employ a range of mathematical disciplinary practices. To understand the MDPs mathematicians reported using in their research and the function those practices play across
mathematical content domains, and how they teach those practices, I employed a two-pronged approach. I conducted semi-structured interviews with eight mathematicians to understand how MDPs are used in advanced mathematicians and how such practices are used in teaching MDPs. In particular, these interviews centered on what mathematicians reported about MDPs in the context of their research and how those same MDPs translate to their own teaching. I then conducted case studies in two of these mathematicians’ advanced undergraduate courses to see how MDPs might emerge in these classroom contexts. Collectively, the interviews and case studies were intended to shed light on what MDPs mathematicians report using in their research, and specifically, how they perceive of teaching practices to students.

In my first manuscript, I highlight three themes that I identified through the process of my thematic analysis of interviews with eight mathematicians and two classroom observational case studies. To situate the study within the educational literature, I trace the ways that mathematical practices were developed. I offer a conceptualization of practices that provide the foundation for discussing MDPs in mathematicians’ research and how such practices are taught in advanced mathematics. I begin by outlining the MDPs that were uncovered from my analysis of interviews with eight mathematicians and describe how the practices coalesced. The second theme I report on are the ways that MDPs are used in mathematics research. In the final theme, I outline the ways in which mathematician reflected on how they approach teaching students MDPs in their advanced undergraduate mathematic courses. Within this theme I explicate the ways that mathematicians drew on social and analytical scaffolding to support teaching students MDPs in their advanced courses. When pressed for teaching practices that support students’ learning MDPs, mathematicians often drew on general teaching practices including demonstrations with a variety of examples, employing group work, or assigning homework problems. Collectively, my analysis of the interview data highlights the challenges of teaching towards MDPs and makes a case for the need to better understand the complexity of MDPs in mathematicians’ research and the ways that MDPs might be taught in advanced mathematics settings. Despite being thoughtful, reflective practitioners, the data suggests that teaching MDPs is a difficult, but important endeavor.

My second manuscript investigates one MDP that describes the ways that mathematicians attend to conditions, assumptions, and properties during a mathematical endeavor that was
identified from the analysis of interviews with mathematicians in my study. The Conditions, Assumptions and Properties (CAPs) practice began as an emergent code to capture mathematicians’ utterances of conditions, assumptions, or properties in the early rounds of coding my data within my TA. My examination of this code gave led me to claim that attending to conditions, assumptions, and properties was actually a practice that all mathematicians reported using in their research or teaching. I wanted to explore this claim in more detail so the goal of my second manuscript was to answer the question: What are the characteristics of the CAPs practice? In what ways do mathematicians attend to conditions, assumptions, and properties in their research and in teaching? I did not select a preexisting analytic framework prior to analysis but drew upon my theoretical perspective on mathematical practices and a set of analytic questions to support my analysis for this manuscript. I use a situative perspective to coordinate an individual’s conceptions and behaviors with normative social and broader historical ways of participating in settings. Taking this approach means that I view learning mathematics as participating in the practices of the discipline of mathematics. I drew upon Sfard’s notion of “thinking as communicating” helpful because my dataset for the second manuscript was strictly from the interviews, where I viewed mathematician talk as their way of communicating their thinking about mathematical practices.

**Significance of Dissertation**

Focusing on mathematical practices as the focal object of study rather than the particular mathematical content domains in which practices are employed, my research offers additional empirical evidence on a small, but growing body of literature on the teaching of mathematical practices in postsecondary context. The first manuscript offers mathematicians' accounts of the MDPs they employ in their research and how these same practices show up in reports of their teaching. The second manuscript and offers an initial definition of an MDP that was identified through mathematicians’ reports of attending to mathematical conditions, assumptions, and properties. The two manuscripts in this dissertation are intended to complement one another and are meant to be stand-alone manuscripts. Collectively, these two manuscripts offer insights on MDPs mathematicians use in their research and how they talk about using such practices in their teaching. The second manuscript provides an in-depth description of one of these practices which have utility across mathematical research and advanced mathematics courses.
Chapter 2 – Mathematics Disciplinary Practices in Advanced Mathematics

Erin D. Glover

Mathematical practices such as justifying mathematical claims, using various mathematical representations, and making mathematical generalizations are vital aspects of learning and doing mathematics. A significant body of research in K-12 mathematics education examining mathematics teaching and learning has demonstrated the importance of supporting mathematical practices because they are vital to students developing conceptual understanding of mathematics (e.g., Boaler & Staples, 2008; Cobb et al., 1991; Ellis, 2007; Silver & Stein, 1996; Lannin, 2005; Selling, 2016; Staples, Bartlo, & Thanheiser, 2012; Stylianides, 2007). Policy guiding K12 mathematics learning has documented a growing emphasis on mathematical practices. This a shift was initiated in part by the National Council of Teachers of Mathematics (NCTM; 1989, 2000) process standards that highlight the ways of acquiring and applying mathematical content. A central feature of the K12 Common Core State Standards, the Standards for Mathematical Practice describe ways that students should engage in mathematics across the elementary, middle, and high school grades (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Given this ground swelling of research in K-12 mathematics education and policy, what can be said for teaching and learning of mathematical practices in more advanced mathematical settings?

The advanced mathematical thinking literature (e.g., Selden & Selden, 2005; Tall, 1991; Rasmussen, Zandieh, King, & Teppo, 2005) suggested that the kind of mathematics students do in their undergraduate courses is qualitatively different from the mathematics students encounter in their K12 education (Robert & Schwartzzenberger, 1991; Tall, 1991). Tall described how children are taught to ‘synthesize’ knowledge, whereas teaching at the university emphasizes the ‘analysis’ of knowledge. Synthesizing knowledge starts with “simple concepts, building up from experience and examples to more general concepts and the analysis of knowledge begins with ‘general abstractions and forming chains of deduction from them” (p.13). Earlier research in advanced mathematical thinking focused on the transition to formal mathematical proof, but more recent research includes attention to other important practices such as argumentation, defining, and
algorimatizing in addition to proof as vehicles for advanced mathematical thinking (Mamona-Downs & Downs, 2008; Rasmussen, Zandieh, King, & Teppo, 2005; Selden & Selden 2005).

It stands to reason that the teaching and learning demands for attending to analysis of knowledge and mathematical practices in advanced mathematics could be qualitatively different than what is called for in K12 mathematics teaching and learning. Furthermore, it is an open question whether and how the mathematical practices outlined in primary and secondary mathematics (e.g., CCSSM) overlap with the disciplinary practices of mathematicians (Rasmussen, Wawro, & Zandieh, 2015). Biza, Jaworski, and Hemmi (2014) even argued that “mathematical practices at university level are distinguished from those at secondary or primary level for reasons related to the mathematical content, the teachers and the students involved” (p. 161). Given there is not (yet) a clear alignment between mathematical practices outlined for primary and secondary mathematics and practices in university mathematics, the aim of this research is to better understand the landscape of mathematical practices in the context of advanced mathematics and how such practices are taught in advanced undergraduate courses.

**Literature Review and Theoretical Framing**

In the 1950s and 60s, the “new math” movement defined successful mathematics learning in terms of understanding the structure of mathematics together with its unifying ideas (e.g., Hekimoglu & Sloan, 2005; Schoenfeld, 2004; Wilson, 2008). This movement was a shift from focusing on computational skill to promoting students to think more like real scientists and mathematicians. This emphasis was followed by a “back to basics” movement in the 1970s that proposed returning to the view that success in mathematics meant being able to compute accurately and quickly. The reform movement of the 1980s and 1990s pushed the emphasis toward what was called the development of “mathematical power,” which involved reasoning, solving problems, connecting mathematical ideas, and communicating mathematics to others. (NRC, 2001). This was in part due to the introduction of the Curriculum and Evaluations Standards for School Mathematics (1989) and the Principles and Standards for School Mathematics (2000) by National Council of Teachers of Mathematics (NCTM). The “NCTM Standards,” to which they are commonly referred were put forth in an effort to revitalize teaching and learning of mathematical processes and reasoning important for doing mathematics (Heibert, 2003). What follows is a discussion of the ways in which mathematical practices have evolved across these documents to
illustrate the advances in K12 mathematics education research and policy. What follows is an examination of national policy documents which traces the evolution of mathematical practices to suggest how the field has changed its way of thinking about how mathematics gets done.

**National Policy Documents in K12**

NCTM’s *Curriculum and Evaluations Standards* (1989) segmented their set of standards into a set of five content domains and four process standards. The content standards described five mathematical content domains that students in primary and secondary mathematics should learn: number and operations, algebra, geometry, measurement, and data analysis and probability. The process standards highlight the ways of acquiring and applying mathematics content knowledge through four *processes*: problem solving, reasoning, communication, and connections. The *problem solving* and *reasoning* processes highlights the ways that students should reflect on their thinking during the problem-solving process so that they can apply and adapt the strategies they develop to other problems and in other contexts. While problem solving and reasoning are essential aspects of doing mathematics, as written, the communication and connections processes are general processes that could be applied to other disciplines. Communication includes the organizing of knowledge and the work of sharing knowledge and ideas with others. The process of *connections* describes how students should be able to recognize and use connections across ideas, the interplay among topics, in contexts that relate mathematics and other subjects. While important aspects of mathematics as a discipline, the processes of communication and connections do little to explicate how they are particular to mathematics and how the mathematical work gets done. For example, communicating mathematical ideas is an important aspect for learning mathematics in classrooms or communicating new mathematics to the field, the process itself it does not capture the relevant *mathematical work* that gets done. One could argue that other disciplines (e.g., biology, physics, chemistry) would also argue that communication and connections are important processes in their disciplines.

Expanding on the work from the 1989 *Curriculum and Evaluations Standards*, NCTM’s *Principles and Standards* (2000) continued their emphasis on the *processes* for doing mathematics but refined their previous processes and included a new process. The *reasoning* category was modified to include *proof* and introduced the process of *representation*. The process of *reasoning and proof* outlined how students should think analytically, tend to patterns, structure, or
regularities in both real-world and mathematical situations, and develop and evaluate mathematical arguments and proofs. The *Representations* process describes how students should be able to identify, select, develop and translate between various mathematical representations (e.g., pictures, concrete materials, tables, graphs, number and letter symbols, spreadsheet displays) and use such representations to model and interpret mathematical or real-world phenomena. The inclusion of new processes indicates how the field continued to refine what the essential components of doing mathematics that students should learn in across mathematical content domains.

Collectively the five processes outlined the ways that K12 student should be engaging with mathematics content. I note that reasoning and proof and problem solving are in line with the mathematical work that is done when *doing* in mathematics, but in terms of connections and communication, it is less clear what *mathematics* get done. Thus, I categorize the processes related to *connections* and *communication* as general practices of the discipline.

**Strands of mathematical proficiency.** In an effort to provide research-based recommendations and guidance to educators, researchers, and policymakers the National Research Council (2001) produced a report arguing that mathematical proficiency is more than understanding how to use and apply procedures in mathematics (as was at the “heart of the elementary school mathematics curriculum”) but rather, it is composed of five “interwoven and interdependent” strands (p. 144). What sets these strands apart from other conceptions of mathematics is that these strands take up different kinds of knowledge, skills, abilities, and beliefs in terms of doing mathematics than were outlined in NCTM’s process standards. The strategic competence and adaptive reasoning strands describe practices that are essential to doing mathematics. *Strategic competence* describes aspects of formulating, representing and solving mathematical problems, which is in line with NCTM’s problem solving process. *Adaptive reasoning* captures multiple practices around argumentation, justification, generalization, and proof which is reminiscent of NCTM’s process of reasoning and proof. However *procedural fluency* and *conceptual understanding* strands are not practices themselves, but rather, draw on practices in order to develop these proficiencies. For example, an awareness of the structure of a class of problems in which an algorithm or procedure works, one could employ practices to develop procedural fluency or gain conceptual understanding of that class of problems. The
productive disposition strand raises the importance of persistence in solving a problem in mathematics.

Of course, dispositions towards mathematics are an important aspect of doing mathematics, but does not describe the mathematical work that is done while solving problems. It could be argued that holding productive dispositions is an important aspect of any discipline. Taken together, NCTM’s processes and the NRC’s strands of mathematical proficiency outline that knowing mathematics is more than holding knowledge of concepts and procedures. Rather, mathematically proficient depends on the ways in which people “approach, think about, and work with mathematical tools and ideas [emphasis added]” (RAND, 2003, p. 32). In the same timeframe and centered on the work from the strands of mathematical proficiency, the RAND Mathematics Study Panel (2003) argued that focusing on mathematical practices is an essential aspect of research and development programs to revitalize teaching because they are a central aspect of mathematical proficiency which likely influenced the research and design of the set of mathematical practices which were outlined in the Common Core State Standards for Mathematics.

**Common Core standards for mathematical practice.** Given the momentum to revitalize teaching and learning in K12 mathematics and recommendations to include mathematical practices, the National Governor’s Association (2010) offered the CCSSM that were reminiscent of NCTM’s content and process standards (1989, 2000); however, the authors of the CCSSM used different terminology to highlight these ideas. The content standards were developed in line with the content domains from previous national documents but were written to provide even greater detail within each content domain and the content that appears across grade levels. In line with the NCTM’s process standards, a set of eight Standards of Mathematical Practice (SMPs) were included which outlined the ways that students ought to be engaging in mathematics, across content domains and grade levels which appear in Table 2.1.

<table>
<thead>
<tr>
<th>SMP 1. Make sense of problems &amp; persevere in solving them</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMP 2. Reason abstractly and quantitatively</td>
</tr>
<tr>
<td>SMP 3. Construct viable arguments &amp; critique the reasoning of others</td>
</tr>
<tr>
<td>SMP 4. Model with mathematics</td>
</tr>
<tr>
<td>SMP 5. Use appropriate tools strategically</td>
</tr>
<tr>
<td>SMP 6. Attend to precision</td>
</tr>
<tr>
<td>SMP 7. Look for and make use of structure</td>
</tr>
</tbody>
</table>
SMP 8. Look for and express regularity in repeated reasoning

The SMPs expanded from the set of five processes to eight practice standards and included new language and terminology, to more clearly articulate the mathematical work that is done as one engages in such practices. The shift towards further refining and explicating practices indicates there is not a clear consensus among the field about the salient practices of the discipline and what of those practices students are expected to learn to prepare for their future jobs or schooling. The collection of mathematical practices across these documents offer a useful set of constructs to think about the ways of thinking and doing mathematics, but it is important to note that these practices are organized differently because the field has honed its vision of mathematical practices.

Over time, these processes and practices have offered more details about what students should learn in primary and secondary mathematics. My review of the literature suggests that the field has changed its way of thinking about the salient features of doing mathematical work. Initially, they were processes for doing mathematics but later the language translated to calling these key processes mathematical practices. While the ways in which mathematical practices are organized, and the language used within them continues to evolve, what is consistent is the attention to how mathematical work is accomplished. For example, the NCTM began with four processes which evolved and expanded to a set of five processes. With the introduction of the SMPs, the set of mathematical practices expanded from five to eight, where all eight practices are specific to the work mathematics. This shift can be understood by the fact that mathematical practices are historically and culturally situated. They are created by the community of stakeholders and refined as the vision for the mathematical work students should learn in K12 mathematics evolves. The inclusion of the eight SMPs into the standards document indicates the important shift towards expanding the notion of what it means to do mathematics. These standards were built from an existing set of standards and were written with the intent to provide rigor, focus, and coherence and to provide students the necessary knowledge and skills to prepare for mathematics in college. What is less well understood is whether and how the mathematical practices outlined for primary and secondary mathematics overlap with the practices of academic mathematicians (Rasmussen, Wawro, & Zandieh, 2015; Rasmussen & Wawro, 2017).

Although much of the research on mathematical practices is situated in K12, investigations of postsecondary mathematical practices have emerged in research on content domains such as:
linear algebra (e.g., Wawro, Rasmussen, Zandieh, & Larson, 2013), differential equations (e.g., Rasmussen, Keene, Dunmyre, & Fortune, 2018), analysis (e.g., Roh, 2010), abstract algebra (Cook, 2014; Larsen, 2013), combinatorics and discrete mathematics (e.g., Lockwood, 2013; Hawthorne & Rasmussen, 2015). When MDPs are studied, often they are studied in isolation. As such, Rasmussen & Wawro (2017) argued for research to “exemplify and illuminate underlying processes key to student engagement in mathematical practices, both those that are tied to specific mathematical ideas and those that reflect the practices of the discipline of mathematics more broadly (p. 571). They recommend that research should transition away from focusing specifically on content domains or MDPs in isolation (Rasmussen & Wawro, 2017). My study examines the teaching and learning of MDPs across advanced mathematics, something long overdue in undergraduate mathematics according to Rasmussen and colleagues.

**Accounts of Mathematicians’ Practice.** One way the field has come to understand the practices of the discipline comes from historical, autobiographical accounts, and research on mathematicians’ practices. Philosophers of mathematics provided insights into the practices of the mathematics discipline from a historical and philosophical standpoint (i.e., Lakatos, 1976; Polya, 1957). Much of what we know of mathematical practices of mathematicians originates from autobiographical accounts of their own practices (e.g., Davis & Hersch, 1982; Hardy, 1941; Schoenfeld, 1985.) These autobiographical accounts provided insights into mathematicians’ problems solving and proving processes, desire for efficient and elegant proofs. Others have researched mathematicians’ practices to understating mathematicians’ ways of thinking, working with others, and how the work of mathematics gets done. Burton (2004), who is well known for her research in studying mathematicians’ accounts of practices through her extensive interview studies with 70 mathematicians, identified that knowing mathematics contained personal, cultural, and social relatedness, aesthetics, intuition and insight, different approaches (especially in thinking) and connectivity (to other mathematics or with the real world). Byers’ (2010) work highlighted that ambiguity and conflict are important aspects of doing mathematics and is intimately tied to mathematical precision, rigor, and logic. Collectively, these philosophical, autobiographical accounts, and research on mathematicians’ practices highlight that mathematicians, when faced with challenges, ambiguity, or conflict in their work they bring as much “mathematical power” to the problem at hand, which include the collection of mathematical
practices they bring to bear across mathematical content. These practices involve aesthetic values like “elegance, simplicity, generalizability certainty, efficiency” (Moschkovich, 2002, p. 5). Such accounts of mathematicians highlight disciplinary ways of knowing mathematics but understanding and identifying the specific practices they bring to bear across content domains is less clear. What mathematical work is required to construct an elegant proof or make something generalizable? What mathematical practices do mathematicians employ in their research as they deploy intuition and seek elegant and simplistic solutions to their problems? The K12 standards, as written, do not describe how this mathematical work get done. Furthermore, how might such mathematical practices be taught to students? Pursuing these questions around doing mathematics in research and in the classroom is at the heart of this study.

**Conceptualizing Mathematical Practices**

Although the research and policy literature widely use the terms mathematical practice, mathematical processes, and proficiencies, there is not one shared conceptualization of such practices (Moschkovich, 2015). To understand the ways that mathematicians think about and use mathematical practices in their research and how such practices show up in teaching, I take a situative perspective that conceptualizes learning as both a cognitive and social process (Greeno, 1998). As students and mathematicians come to understand and use mathematical practices, they are engaged in social settings in which mathematical practices are ways of acting, thinking, and talking with mathematics (Frank, Kazemi & Batty, 2007; Moschkovich, 2007; Sfard, 2012). In line with Moschkovich’s (2013) view:

Mathematical practices are social and cultural, because they arise from communities and mark membership in communities. They are also cognitive, because they involve thinking, and they are also semiotic, because they involve semiotic systems (signs, tools, and their meanings). Mathematical practices involve values, points of view, and implicit knowledge (p. 264).

Moschkovich (2013) suggests practices are rooted within the mathematical discipline itself (ways that one engages in doing the work of the subject area) and are how one learns the discipline (ways that students individually and collectively participate in solving problems). Mathematical practices are the ways of engaging in disciplinary work and learning the subject. They are ways of reasoning that draw upon the system of terminology, symbols, and tools imbued with mathematical meaning (e.g., Boaler, 2002; Rogoff, 1990). To be a “practice” of the discipline, it has to have utility across problems and sub-domains of the subject area while also being widely understood as normative in
how disciplinary work is accomplished. Mathematical practices serve to organize ways of thinking, acting and talking when doing mathematics across content and social spaces: in classrooms, in research presentations, and in mathematics seminars. Throughout this document, I refer to these practices as Mathematics Disciplinary Practices (MDPs) and define them to be the ways in which mathematicians do the work of mathematics in their research. I use this terminology throughout this manuscript to highlight that the practices of this study are defined via the descriptions of mathematicians’ research and are not described in the context of K12 mathematics. Others have referred to these practices as disciplinary practices (Rasmussen, Wawro, & Zandieh, 2015) or academic mathematics to refer to the academic practices of mathematicians as opposed to everyday or workplace practices (e.g., Moschkovich, 2002).

I view MDPs as the most vital aspect of mathematics because MDPs are used across content domains and are the ways in which mathematics gets done. Rasmussen, Wawro, and Zandieh (2015) defined disciplinary practices as “the ways in which mathematicians go about their profession” and include such things as conjecturing, argumentation, and defining (p. 264). Their work and the work of other researchers suggest that disciplinary practices are the ways we come to know the discipline across mathematics content domains (e.g., Moschkovich, 2015). In line with these researchers, I define MDPs to be the ways in which the mathematics discipline is carried out by research mathematicians. Thus, the criteria for being an MDP in this study is a practice that is used in mathematics research work and share this practice in terms of their own professional practice or the practices of other research mathematicians. It would also be considered an MDP if the practice was applicable across different mathematical content domains, evidenced by multiple mathematicians discuss the same practice and focus on different content domains in their research. I conjectured that by investigating how mathematicians attend to MDPs in their research we may gain insights on what it means to know and teach mathematics. The focus of this study is to examine the ways that mathematicians describe their use of MDPs in their research and teaching.

**Teaching MDPs.** With the adoption of mathematical practices in primary and secondary mathematics education, teachers are expected to engage students in mathematical practices across grade levels. Selling (2016) points out that this is a difficult goal to achieve because K12 teachers (likely) had few experiences learning and engaging in these mathematical practices themselves. Considering that mathematicians have highly developed expert knowledge, they do not face the
same difficulties with engaging in the mathematics. Mathematicians have a different set of issues because they likely have few opportunities for professional development in teaching mathematics. Thus, it is an open question how mathematicians’ expert knowledge of mathematics content and the practices to engage with such content translate to teaching students in advanced mathematics.

**Teaching in advanced mathematics.** There exists a significant body of literature around the teaching and learning of proof because it is the purpose of advanced mathematics. Early studies focused on students’ abilities to engage in proof across advanced mathematical content domains (Harel & Sowder, 1998; Moore, 1994); whereas, later efforts sought to understanding pedagogical aspects of proof (Alcock, 2010; Lai & Weber, 2014; Lai, Weber, & Mejia-Ramos, 2012; Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, 2016; Mejia-Ramos; Selden & Selden, Weber, 2001; Yopp, 2011). Others have studied mathematical practices such as defining (Dawkins, 2012, 2014), symbolizing and algorithmatizing (Rasmussen et al., 2005), reasoning with examples (Aricha-Metzer & Zaslavsky 2017; Lockwood, Ellis, & Lynch, 2016, Fukawa-Connelly & Newton, 2014), and computing (Lockwood, DeJarnette, & Thomas, 2019). However, few of these researchers attend to pedagogical aspects of these practices, with the exception of proof. Collectively, the research more often reports on students' mathematical activity while engaging in practices in classroom teaching experiments or uses task-based interviews with students or mathematicians to understand ways that experts and novices engage in these practices. Far less is known about how such MDPs are taught or what instructional practices are effective for student learning of MDPs.

While Speer, Smith, & Horvath (2010) argued that few studies on teaching practices exist in the context of post-secondary mathematics, there is a growing body of research on designing, scaling up, and implementing inquiry oriented curriculum that focuses on MDPs such as argumentation, justification, and defining (e.g., Larson, 2013; Wawro et al., 2013). Despite the literature on inquiry-oriented teaching calling attention to important MDPs students should be learning, the curriculum is not widely adopted. Within this growing body of work in teaching in advanced mathematics, evidence suggests that mathematicians believe that content and MDPs are important considerations in teaching undergraduate mathematics, but they do not feel equipped to teach practices effectively (Lockwood & Weber, 2015). Given that the literature on teaching practices (outside of proof) is limited, and evidence suggests teaching MDPs is a significant
challenge, it is still an open question on how to effectively structure learning in a classroom to teach MDPs in advanced mathematics.

**Scaffolding in instruction.** Mathematics instruction policy and research report on the importance of tasks and discourse for effectively structuring classroom learning. (NCTM, 2000, 2014; Franke, Kazemi, & Battey, 2007; Larsen & Bartlo, 2009; Stein, Engle, Smith & Hughes, 2008). Research on discourse intensive teaching highlight the challenges of engaging students in discussing mathematical ideas that they are in the process of learning (c.f., Edwards & Mercer, 1987; Williams & Baxter, 1996). One construct widely examined to support effectively structuring classroom learning is *scaffolding* as a means for supporting students.

Scaffolding is rooted in Vygotsky’s notion of the zone of proximal development as the ways to support learning (Wertsch, 1985). Wood, Bruner and Ross (1976) defined scaffolding as “the process that enables a child or novice to solve a problem, carry out a task, or achieve a goal which would be beyond his unassisted efforts” (p. 90). Baxter and Williams (2010) note that the process of scaffolding “implies a dynamic relationship between the teacher and student in that the teacher’s support (i.e., the scaffolding) that enables the student to perform a particular task or solve a particular problem is gradually decreased until the student is functioning independently.

Collectively, there are many different scaffolds that have been identified in K12 mathematics teaching and include things as cognitive, metacognitive, and affective scaffolding (Leiss & Weigand, 2005) social and analytical scaffolds (e.g., González & DeJarnette, 2015; Sherin, 2002; Speer & Wagner, 2009; Williams & Baxter, 1996), scaffolding through carefully designed math tasks (e.g., Chen, Rovegno, Cone & Cone, 2012; Kajamies, Vaurus & Kinnunen, 2010); social scaffolding through group work (e.g., Calder, 2015; Schukajlow et al., 2012), scaffolding through questioning (e.g., Hunter, 2012; Moschkovich, 2015), and norm development for productive classroom engagement (e.g., Makar, Bakker, & Ben-Zvi, 2015). Specific to supporting mathematical practices in instruction, Moschkovich (2015) describes scaffolds to support learning mathematical practices in the context of primary and secondary mathematics. For example, she identified that *proleptic questions* (e.g., Now does that look like the right number? Do you have any sense of what the slope of that line should be?) scaffolds learning because the

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1 Bakker, Smith, and Wegerif’s (2015) offer a nice historical review overview scaffolding in addition to an extensive review of the literature regarding the concept of scaffolding in mathematics education. .
questions presuppose a task’s goal or expert thinking of how such tasks might be done. She also described that *revoicing* student contributions was a scaffold because it can support student engagement in discussion to “build on student’s own use of mathematical practices” or a student contribution can be revoiced to reflect new mathematical practices (p. 1074).

Because researchers explicate a variety of scaffolds teachers can use to structure learning in a classroom (e.g., Anghileri, 2006; Bakker, Smith, & Wegerif, 2015; Moschkovich, 2015) and since these definitions have been formulated and evolved overtime, in this study I stay close to the scaffolds outlined by Williams and Baxter (1996) and Baxter and Williams (2010) to describe the ways in which mathematicians may support their students in learning MDPs in their teaching. *Social scaffolding* refers to the kind of support that a teacher provides their students to learn together to “lay the foundation for the construction of mathematical understanding” (Baxter & Williams, 2010, p. 11). *Analytic scaffolding* refers to support offered by a teacher that is used to “help to focus attention and point out critical aspects of the mathematics being used” and include such things as providing physical manipulatives, models, metaphors, representations, explanations, and (p. 11).

The framework of social and analytical scaffolding is a useful lens for understanding the instructional moves a teacher makes while attending to social aspects of learning in the classroom as well as shaping learning of the mathematics. As such, it is taken up in undergraduate mathematics when Speer and Wagner (2009) conduct a case study of one mathematician using inquiry oriented curricular materials in a differential equations course. They reported that despite having extensive teaching experience and possessing strong content knowledge the instructor had difficulties in providing analytic scaffolding to move whole-class discussions toward a lesson’s mathematical goals. While their study was not specifically about MDPs, the curricular goals of the inquiry-oriented materials are “designed to challenge and encourage ways of thinking about the mathematics and to lead students to discover important ideas” (Speer and Wager, 2009, p. 538). Since mathematical practices are intimately tied to ways of thinking about mathematics, the social and analytical scaffolds constructs are helpful for investigating the ways that mathematicians can structure learning around MDPs in their advanced mathematics courses.

Since previous studies have not *explicitly* investigated MDPs, these study designs cannot fully address the extent of mathematicians’ awareness or intention to support MDPs in their
teaching. As such, my research aim was to better understand the landscape of MDPs in advanced undergraduate mathematics. The research question and sub-questions that drove this study were: *What MDPs do mathematicians discuss, either implicitly or explicitly, when explicating their research and teaching?*

a. In what ways do they report on using MDPs in their research?
b. How do they perceive teaching MDPs to students in advanced mathematics?

**Methods**

Advanced undergraduate mathematics was selected as the context for this study because students in these courses are expected to learn and engage in the professional practices of the discipline. With this in mind, I conjectured that the mathematicians working with students in these courses would employ a range of mathematical disciplinary practices. To understand the MDPs mathematicians reported using in their research and the function those practices play across mathematical content domains, and how such practices are taught, I employed a two-pronged approach. I conducted semi-structured interviews with eight mathematicians to understand how MDPs are used in advanced mathematicians and how those MDPs are used in teaching MDPs. In particular, these interviews centered on what mathematicians reported about MDPs in the context of their research and how those same MDPs translate to their own teaching. I conducted case studies in two of these mathematicians’ advanced undergraduate courses to see how MDPs might emerge in these classroom contexts. Collectively, the interviews and case studies were intended to shed light on what MDPs mathematicians’ report using in their research, and specifically, how they perceive of teaching practices to students.

**Data Sources**

Approximately one-hour long, semi-structured interviews with eight mathematicians served as the primary source of data for this study. The shortest interview was 58 minutes, and the longest interview was 83 minutes. The mathematicians were a convenience sample purposively selected for this study because they were employed as professors in mathematics departments, had earned a Ph.D. in mathematics, and taught advanced undergraduate mathematics courses within the last two years. As documented in Table 2.2, the mathematicians collectively had expertise in both applied and pure mathematics working in a range of mathematical domains; however, not all content domains are accounted for in this study. I interviewed each mathematician once using the
same interview protocol, but the nature of the open-ended interview questions allowed me to probe on what was reported and to link ideas across the protocol.

Due to the exploratory nature of this study, I designed the study to survey across mathematical domains in which mathematicians would conduct research and teach advanced mathematics. I deployed my efforts to capture the range of MDPs used in research and teaching in these domains as reported by a sample of eight mathematicians. My aim was to address the limited understanding in the literature of mathematical practices in *advanced mathematical thinking*. As such, I interviewed each mathematician once, resulting in 171 transcribed pages of interview data. I acknowledge that interviewing a mathematician once may be seen as a limitation of the study design in that I was not able to pursue multiple opportunities to engage with each mathematician and dig deeply into their work with MDPs. Future research would be warranted to take up this aim and could use this study as foundational to knowing what MDPs a sample of mathematicians working across domains report.

Table 2.2 Interview participants’ area of specialty and professor ranking

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Area of Specialty</th>
<th>Courses Taught</th>
<th>Professor Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam</td>
<td>Topology</td>
<td>Abstract Algebra, Topology</td>
<td>Late Career</td>
</tr>
<tr>
<td>Gloria</td>
<td>Abstract Algebra</td>
<td>Abstract Algebra</td>
<td>Late Career</td>
</tr>
<tr>
<td>Robert</td>
<td>Combinatorics</td>
<td>Group Theory, Discrete, Number Theory, Graph Theory</td>
<td>Tenured</td>
</tr>
<tr>
<td>Anthony</td>
<td>Control Theory</td>
<td>Group Theory, Analysis, Adv. Linear Algebra, ODE &amp; PDEs</td>
<td>Tenured</td>
</tr>
<tr>
<td>Sophie</td>
<td>Applied Numerical Methods</td>
<td>Numerical Analysis, ODE &amp; PDEs</td>
<td>Pre-Tenure</td>
</tr>
<tr>
<td>Ellen</td>
<td>Functional Analysis</td>
<td>Analysis</td>
<td>Tenured</td>
</tr>
<tr>
<td>Scott</td>
<td>Applied Numerical Methods</td>
<td>Numerical Analysis, ODE &amp; PDEs</td>
<td>Tenured</td>
</tr>
<tr>
<td>James</td>
<td>Number Theory</td>
<td>Number Theory, Abstract Algebra, Discrete</td>
<td>Pre-Tenure</td>
</tr>
</tbody>
</table>

**Interview Data.** The interviews served as the primary source of data to get a sense of the ways that MDPs are used in research and how those same practices are taught in advanced mathematics. After asking preliminary questions, I asked the following: *Can you briefly describe for me the kind of work you do as a research mathematician? What kinds of MDPs do you use to support you in that mathematical work you do? What do you do to try to help your students learn such practices?* Appendix A contains the full set of interview protocol prompts. This enabled me
to get a sense of their knowledge of MDPs, and time was allotted throughout the interview to regularly ask clarifying and follow-up questions so that I could understand the descriptions of the mathematics that came forward in the interview. It was particularly important during the interview process to ensure that the interviewee and I developed a shared understanding of MDPs that was rooted in each mathematician’s research experiences, therefore each interview included negotiation of the meaning of that term.

**Classroom observations.** After completing the mathematician interviews, I conducted case studies in two advanced mathematics courses taught by two of the interviewed mathematicians which served as another source of data for this study. Robert and Sophie, a convenience sample selected from the eight interviewed mathematicians, were chosen because they were teaching an advanced undergraduate mathematics course at the time of data collection and were amiable to having a researcher in their classroom observing their teaching. I attended, observed, and video recorded seven class meetings across Robert’s combinatorics class and Sophie’s numerical analysis course.

Each class observation was approximately one hour in length and took place during the middle of the course so I could observe classroom interactions after classroom routines were well established. I used a camera on a tablet that was set up to follow the mathematician as they moved about the classroom and used five audio recorders throughout the classroom to capture teacher and student dialogue. During my observations I was a passive participant observer and constructed fieldnote jottings which I later developed into fieldnotes (Emerson, Fretz, & Shaw, 2011). My fieldnotes were constructed chronologically and completed for each classroom observation. The fieldnotes included pictures of the classroom layout, images of mathematics problems recorded on the whiteboard to document the mathematical ideas that were displayed in the class, and any handouts provided to students. The fieldnotes also included details on the mathematical problems being discussed in class, summaries whole-class discussions, and descriptions of interactions between the students and teachers, if any. Finally, my fieldnotes included initial analytic notes and included reflective comments on teacher and students’ interactions that I conjectured related to MDPs unfolding in the classroom setting.
Data Analysis

The interview data were analyzed using the inductive thematic analysis (TA) procedure as described by Braun and Clarke (2006). Broadly, the process of conducting a TA includes six phases: familiarizing yourself with the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and lastly, producing the report. Using TA requires that three distinctions that need to be made. The first distinction to make is whether to take an inductive or deductive approach to analysis. A deductive analysis involves defining themes within a particular a priori category. In contrast, an inductive analysis involves defining themes without a specific coding scheme in mind. This study was designed to be exploratory and to gain a broad view of my dataset, I did not select a pre-existing coding scheme. Thus I orient myself with the inductive analytic approach.

The second distinction to make in thematic analysis is to decide whether the themes defined via the analysis are at the semantic (explicit) level or the latent (interpretive) level. The semantic level takes surface meanings of mathematicians’ responses versus the interpretive level that describes the “underlying ideas, assumptions, and conceptualizations” tied to a theme (p. 84). I take the interpretive approach because I construct meanings from the words and descriptions mathematicians use as they describe their research, MDPs, and their teaching. For example, when coding for MDPs in my study, mathematicians’ may not have used the same words as the code labeling, I made, and thus my analysis was based on my researcher interpretations.

A third and final distinction to make is to choose an epistemological stance on whether I align myself with a a realist (essentialist) or constructionist perspective. Someone orienting themselves in the realist paradigm believes there is an objective truth behind what mathematicians said and the themes would explicitly describe those assumed realities. I orient myself with the interpretivist epistemology because I use the mathematicians’ talk to develop themes about one possible reality of how mathematicians use MDPs and how they approach teaching such practices.

Researcher reflexivity. I chose TA because I did not want to be constrained by existing theory on MDPs, but rather, to be open to the themes I defined via the process of TA. I provide further details these methodological choices shortly, but first offer a statement on researcher reflexivity (e.g. Grey, 2017). How I approach my research is shaped by my identity, lived experiences, beliefs, and the culture in which I belong, which influences all aspects this work. As
a mathematics education researcher who holds an advanced degree in mathematics interested in studying MDPs, I am not able to completely remove my research interpretations from my notions about MDPs based on these experiences and readings around MDPs. For example, many of the MDPs that emerged in the interviews were familiar to be because of my knowledge of the existing literature, my use of MDPs when doing mathematics for myself, or when teaching mathematics to students. These prior experiences shape how I approach my research more broadly including the research design, questions, fieldnote construction, analysis, and reporting on findings.

**Thematic analysis.** Braun and Clarke’s (2006) TA was a sensemaking methodological choice because it is “a method for identifying, analyzing, and reporting patterns [emphasis added]” (p. 79). Since TA is a method to analyze data and not beholden to any particular a priori framework or coding scheme, interpretations of mathematicians’ responses in the interviews are the foundation of the themes presented in this manuscript I chose the inductive analysis path because I wanted to avoid pre-existing coding frames that could constrain what I learn from my dataset due to pre-existing theoretical commitments. However, my prior experiences in mathematics and knowledge of the literature on mathematical practices shaped my understanding of MDPs and how I interpret the ways in which others talk about them.

Consistent with the first phase of TA, I began by familiarizing myself with the data set. This process included transcribing each interview, followed by reading (and rereading) each transcript before early rounds of coding. The data excerpts presented in this manuscript exclude false starts, pauses, and use ellipses to indicate the removal of a portion of participants’ utterances. These features of speech were included in the transcriptions. TA is an iterative process that “involves a constant moving back and forward between the entire data set, the coded extracts of data that you are analyzing, and the analysis of the data you are producing” (p. 86). I wrote descriptive summaries for each interview that served as a form of early analysis and an aid to recall ideas that could be important to investigate into further. Within these summaries I began to notice similarities and differences across mathematician interviews in terms of the MDPs used in their research how such practices are taught to their students. These early descriptive summaries helped me think about how I might develop a coding scheme for the interview data. Once these initial readings and summaries were completed, the transcripts were uploaded into the qualitative data
analysis software program HyperResearch (ResearchWare, 2012) to begin the process of segmenting and coding the data set.

**Data segmenting and coding.** To avoid losing the context in which the data source emerged from I followed Braun and Clarke’s (2006) heuristic to keep some surrounding data to avoid this common criticism of TA. Finding boundaries for a codable unit was first given by the larger structural codes, and then within holistic codes nested inside structural codes. The maximum size of a codable unit is a structural code that captured the protocol interview prompt and the mathematician response. If there were multiple exchanges around the same question and me the entire exchange would be included in the codable unit. A new codable unit would occur when the mathematician changed the topic, I asked a follow up question or probing question, or I asked a new protocol prompt. After the data was segmented the second phase of TA can begin. The second phase of TA is to generate a list of initial codes. I drew on Miles and Huberman’s (1994) *inductive coding technique* to code interview data into categories that were meaningful for understanding the ways that mathematicians engage in MDPs and their relationship to teaching and learning advanced mathematics courses.

I began coding by identifying structural codes. Structural codes were developed by reading through the data set that previously chunked into codable units. Structural codes supported reducing my data so that I could “examine comparable segments’ commonalities, differences, and relationships” (Saldaña, 2012, p. 84). Examples of structural codes I identified captured instances of mathematicians’ *research*, mathematicians’ descriptions of *teaching and learning*, and mathematicians’ use of *MDPs in work and in teaching*. These codes specifically reflect the structure of my interview protocol examined mathematicians’ use of MDPs in their research, how they teach students such MDPs, and a math task example they offered to illustrate their reporting on teaching MDPs. Table 2.3 outlines the structural codes and the interview protocol question(s) that align to each structural code. Appendix A outlines the complete interview protocol.

I used structural codes to organize analyses in a way that allowed me to examine data across mathematicians’ responses within any of the structural codes. Structural codes do not overlap with other structural codes. This led to the second level of data reduction, identifying *holistic codes*. For example, I noticed that there were different “actors” at play within the larger structural codes. I created *holistic codes*, that “capture the sense of the overall content and possible
categories” (Saldaña, 2012, p. 142). For example, the structural code “Teaching and Learning” flagged all responses associated with how mathematicians learned MDPs, how a mathematician might approach teaching MDPs, and how students might learn MDPs. I selected the holistic codes to be able to differentiate the ‘actors’ that occurred within data identified within a structural code. Categorizing subsections of these data was helpful for understanding how MDPs were reported on in mathematical research and teaching.

Table 2.3 Structural codes with associated interview protocol prompt

<table>
<thead>
<tr>
<th>Structural Code</th>
<th>Associated Interview Prompts</th>
</tr>
</thead>
</table>
| Courses Taught           | Can you remind me what advanced undergraduate courses you teach?  
                          | What are your goals for students taking those classes?  
                          | What challenges do you foresee in teaching those courses?                                                                                                 |
| Research                 | Can you briefly describe for me the kind of work you do as a research mathematician?                                                                      |
| MDP used in Work         | What kinds of MDPs do you use to support you in the mathematical work you do?                                                                              |
| Math Example in Teaching | Please take a moment to think about a mathematical problem or topic your advanced mathematics course that you feel communicates mathematical content but also allows you to address mathematics disciplinary practices. Can you describe to me what the example, problem, or topic you were thinking about? |
| Teaching and Learning    | How did you learn these MDPs?  
                          | How do you think students might learn these same mathematical practices?  
                          | What are some things you do in your own teaching to help students learn this MDP?                                                                          |

The utility of having holistic and structural codes meant I could analyze intersections between them, or I could join codes. These efforts were aimed to reduce the data and create larger themes. For example, because I was interested in how MDPs were taught in advanced mathematics, I attended to the ways that mathematicians spoke about teaching MDPs. It was evident that their reports on teaching fell into two broad categories: teaching in general and
specifically teaching MDPs. It was pragmatic to distinguish between those two qualitatively
different types of responses because my study is designed to explore teaching MDPs, but any
aspects of teaching can provide important insights into teaching in advanced mathematics more
broadly. Examples of reports of teaching that were specifically related to MDPs (e.g., *You can tell
students to slow them down. ‘Hey, just run back to the definitions.’*) were coded with the holistic
code “Teaching MDP”. Reports of teaching that were general in nature and not specific to MDPs
(e.g., *I think I give back feedback on homework that they resent sometimes.*) were coded with the
holistic code “Teaching General”. A complete list of holistic codes is described with operational
definitions and examples in Table 2.4. The holistic codes were helpful to reduce the data into
potential themes to prepare for the third layer of coding in which finer-grained codes would be
applied to holistic codes to identify the MDPs that mathematicians used in their research and
teaching.

Table 2.4 Holistic codes with operational definitions and example

<table>
<thead>
<tr>
<th>Holistic Code</th>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math Content</td>
<td>This code is used for instances in which particular mathematical content is discussed in relation to research, teaching and learning, or MDPs.</td>
<td>“…you can imagine if I have a set of size nine and I want to turn it into a commutative group...”</td>
</tr>
<tr>
<td>Social Practices</td>
<td>This code is used for instances in which mathematicians discussed in relation to doing mathematics that was not mathematical in nature (e.g., read literature) Responses related to doing mathematical work (e.g., collaboration with peers) that not doing mathematics.</td>
<td>“I like to have a coauthor that is more meticulous than I am.”</td>
</tr>
<tr>
<td>Mathematician Learning</td>
<td>This code is used for instances in which mathematicians described their learning of MDPs.</td>
<td>“…one of the things that helped me most is to be able to think about things like symbolically...”</td>
</tr>
<tr>
<td>Student Learning</td>
<td>This code is used for instances where responses related to the ways that students learn mathematical content and/or mathematical practices, goals for student learning, talking about student traits or abilities around content/practices.</td>
<td>“…it makes you realize that these students are not really secure in what they're writing down.”</td>
</tr>
</tbody>
</table>
Teaching General  This code is used to capture in which descriptions of teaching in general or teaching practices that may not be explicitly related to teaching particular mathematical content or practices.  “I think I give back feedback on homework that they resent sometimes.”

Teaching MDP  This code is used to capture instances of utterances describing teaching particular mathematical practices. Can be double coded with other fine-grained tags.  “But the feel of it – ‘how did you think about it? How did you think about it?.. How many think it’s this group?”

Coding for MDPs. Because my study was designed to explore MDPs, my next level of coding attends to the MDPs that appeared in mathematicians’ interviews. In coding for MDPs, I used in code labels that captured the words or phrases mathematicians used to describe the mathematical work in their research. For example, if a mathematician shared how they “developed a conjecture” in their work and I coded this as the practice of “conjecturing” because it includes the same root word that the mathematicians used, but MDPs are gerunds, I coded it as “conjecturing” to indicate that it is an aspect of ‘doing’ the work of mathematics. In the case of a mathematician describing an aspect of their research and not using the language from the existing literature, I took an interpretivist approach in coding. I note that many of the MDPs codes identified in this study follow a similar naming and meanings to existing disciplinary practices in the literature base in undergraduate mathematics. For example, the codable unit with the phrase “I try to play with the symbols until I get the quantity to the reverse side” was coded with the MDP “Syntactic Manipulation” to identify where a mathematician reported manipulating symbols, definitions, or facts; especially in proof. For example, I did not select “Syntactic Manipulation” as an a priori MDP code, but upon analyzing the dataset, I recognized the practice from the existing literature on syntactic proof productions (Weber & Alcock, 2004). Again, it was not the purpose of this study to provide new definitions of previously established MDPs. An MDP code would only be applied a single time within any holistic code, even if the MDP was mentioned several times within a holistic code, maintain the unit of analysis. Multiple MDP codes could be layered with other MDPs codes to indicate how MDPs appeared together.

The coding scheme was organized into a nested structure for the purpose of being able to analyze single codes across the dataset or intersections between codes within any codable unit.
Specifically, MDP codes were nested within a single holistic code and holistic codes were nested inside larger structural codes. The three layers of codes allowed for an organization for systematically running reports in the qualitative data analysis software HyperResearch. The nested structure of codes allows for analyzing intersections between structural codes and holistic codes in and join together with other codes to create larger themes. A simplified example of this systematic process is with the holistic code of “Teaching MDP”. When running a report on the holistic code “Teaching MDP” returned every instance of a mathematician reporting on their teaching specifically related to MDPs across the dataset. To identify the range of MDPs used in teaching, I examined within the holistic code to find the MDPs that mathematicians reported using in their teaching. Because an MDP code was marked only once within a single holistic code, I was also able to run counts on the number and kind of MDPs used in across the set of codable units.

**Problem Solving Practice.** One may also notice that “Problem Solving” was not coded for in this study. Problem solving in mathematics is a practice in the sense that “mathematician's main reason for existence is to solve problems” (Halmos, 1980, p. 519). Thus, when mathematicians shared instances of MDPs in their research and teaching, they were in the context of solving problems. Furthermore, one can also notice that various aspects of Polya’s (1957) second principle of problem solving, *devise a plan*, are reflected in the list of practices in Table 2.4 and include such strategies such as: make an orderly list, consider special cases, look for a pattern, draw a picture, use a model, solve a simpler problem. While problem solving is certainly a practice that mathematicians use in their research, it was intentional choice to not code for the problem solving because it is also a purpose of mathematics research. Essentially, any description of mathematicians’ research would be over-coded with *problem solving* and provide little insight into because of the sheer quantity of data coded with problem solving. Thus, it was an intentional methodological choice to omit problem solving an MDP in this study, but describe MDPs which were supportive mathematicians research, which inherently entailed the practice of problem solving.

**Searching for and defining themes.** The next phases of TA are to search for possible themes and define themes that appear across the data set. The interview data were coded sufficiently when the emergent codes began to get smaller in grain size that they began to overlap with other codes and were often absorbed into a larger-grained code. Those larger codes were then
combined to form themes. For example, the Conditions, Assumptions, and Properties practice (CAPs practice) MDP code originated from smaller, fine-grained codes and I created a larger code CAPs MDP which also became a theme (because codes can be themes themselves). There is no procedure which tells a researcher when coding is complete, so I took a pragmatic approach and followed Creswell’s (2015) heuristic of coding all the data into about 30 to 50 codes and before identifying codes that overlap or are redundant to reduce the overall number of codes. Subsequent organization of those codes then turn into larger themes (and headings in the results of the manuscript). Braun and Clarke (2006) highlight that the unit of analysis in TA is the themes identified in the analysis, which represent a patterned response in the data. They also note that themes are codes, so to understand these patterned responses, I focused on the relationship between codes across the entire dataset, rather than the codes themselves. Specifically, I tried to understand how different codes could combine to create a larger theme and how codes might become sub-themes of these larger codes. I searched for both confirming and disconfirming evidence of possible themes.

In the process of defining themes, I grappled with whether a theme overlapped with another theme. If there was some overlap, I had to examine whether it could be a new theme or a supporting subtheme. For example, the MDPs defined through the processes of TA became the unit of analysis for the first theme in this manuscript. Taking MDPs as the unit of analysis resulted in exploring the ways that MDPs are aspects of a mathematician’s research. Knowing that MDPs are interrelated and not mutually exclusive, exploring these relationships was a natural outcome of taking MDPs as the unit of analysis. In exploring these relationships, I defined sub-themes which organized my MDPs codes around four categories to describe ways that MDPs appeared together in the mathematicians’ reports of their research: Formalizing and Proving, Mathematizing and Computing, Structuring Mathematics, and Other Supporting Practices. Thus, TA is an iterative process in which I often revisited and reviewed codes themes, and sub-themes, and reread transcripts to ensure I was able to interpret and convey the mathematicians’ perspectives on MDPs as it relates to their research and teaching.

**Case study data.** After analyzing the interview dataset, I went back to my case study fieldnotes, analytic notes, and videos data. Reading and re-reading fieldnotes and reviewing video and audio recorded data from my observations allowed me to identify places in classroom episodes
that could be linked to the structural and holistic codes in mathematicians’ interview data. Initially, I extracted all instances of Robert and Sophie’s teaching from the interview data by attending to the holistic codes “Teaching General” and “Teaching MDPs”. I was able link their descriptions of teaching in their interview data with my fieldnotes and analytic notes. In my readings and re-readings of interview data and case study data, I began to notice a similarities between how Robert and Sophie described their research and MDPs (intersections between the structural code “Research” and the holistic code “MDP Used in Research”) with MDPs emerging in classroom episodes. Specifically, I was able to link their descriptions of in their interviews with the fieldnotes and analytic notes. In this manuscript I bring forward two classroom episodes that link structural and holistic codes with observations and analytic notes from the case studies. This is not necessarily representative of all mathematicians in this study because I only conducted case studies in two of their courses. It is not the goal of this study to generalize to every mathematicians teaching, but rather, show one possibility of how MDPs are taught in advanced mathematics.

**Results**

In an effort to describe the landscape of MDPs in advanced mathematics and answer my research question, I outline the results by the themes that I defined via the analyses of interview and case study data. My research questions were designed to reveal the MDPs mathematicians discuss, either implicitly or explicitly, when explicating their research and teaching. The results are framed by the following sub-questions: *In what ways do they report on using MDPs in their research?* And *How do they perceive teaching MDPs to students in advanced mathematics?* Collectively, the results provide a descriptive analysis of MDPs that mathematicians use in their research and in teaching in their advanced undergraduate mathematics courses. The first theme I present was the result of asking mathematicians to describe the MDPs used in their research. I also organize the MDPs defined through my analysis into four larger categories based on how they appeared to coalesce. The second theme defined via my analysis describes the MDPs mathematicians reported using in their research. The final theme I present is mathematicians’ descriptions of teaching MDPs in their advanced undergraduate mathematics courses. I describe these relationships through the lens of scaffolding (Williams & Baxter, 1996) to outline how mathematicians reported structuring the learning of MDPs in their courses. Within this theme, I outline instructional practices mathematicians reported deploying in instruction across two types
of scaffolding: analytic and social scaffolding. Collectively, the three themes presented in the results show that the landscape of teaching and learning MDPs is complex, and that mathematicians in this study hold very different perspectives on teaching MDPs.

**MDPs in Mathematical Research**

I begin by outlining the MDPs that I defined via my analysis of holistic codes across the data set of mathematicians’ interviews. I do this because it is important to link the range of MDPs to mathematicians’ reports on using in research and teaching. I organized the 25 MDPs identified across the full data set into four broad categories. I acknowledge that this categorization is one of many possible ways in which MDPs could be organized conceptually and that MDPs in one category might be connected to an MDP in another category. As Lockwood, DeJarnette & Thomas (2019) noted, MDPs are not necessarily mutually exclusive. Thus, I selected the four categories of MDPs solely based on the interviews with eight mathematicians and not some pre-existing categorization. I grouped MDPs codes defined through my analysis that appeared together within the same holistic codes. In developing the categories, I was influenced by Rasmussen and Stephan’s (2008) procedure to classify and organize classroom mathematical practices which they did by grouping them “according to the general mathematical activity.” The general mathematical activities that I used to group the MDP codes were Formalizing and Proving, Mathematizing and Computing, Structuring Mathematics, and Other Supporting Practices. I organized these four larger-sized ‘buckets’ in ways that I interpreted mathematicians’ descriptions of MDPs used in their research and teaching. What follows is a description of these four categories seen in Table 2.5 and offers illustrative examples of these categories through excerpts of mathematicians’ interview responses. For a full account of the MDPs, Appendix B offers operational definitions, examples, and number of mathematicians that reported each MDP.

---

2 Few categorizations exit. I have found only one example which organizes the Common Core SMPs in Kelemanik, Lucenta, and Creighton’s (2016).
Table 2.5. Four categories of mathematical practices.

<table>
<thead>
<tr>
<th>Formalizing &amp; Proving</th>
<th>Matematizing &amp; Computing</th>
<th>Structuring Mathematics</th>
<th>Other Supporting Practices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof &amp; Proving</td>
<td>Modeling</td>
<td>Leveraging</td>
<td>Precision</td>
</tr>
<tr>
<td>Argumentation and</td>
<td>Utilizing Technology</td>
<td>Mathematical Structure</td>
<td>Communication</td>
</tr>
<tr>
<td>Justification</td>
<td>Algorithm and Programming</td>
<td>Classifying</td>
<td>Conditions,</td>
</tr>
<tr>
<td>Conjecturing</td>
<td></td>
<td>Representing</td>
<td>Assumptions, &amp;</td>
</tr>
<tr>
<td>Syntactic Manipulation</td>
<td></td>
<td>Discovering Patterns</td>
<td>Properties</td>
</tr>
<tr>
<td>Logical structure</td>
<td></td>
<td>Generalizing</td>
<td>Intuition &amp; Creativity</td>
</tr>
<tr>
<td>Leveraging Examples</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Defining</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Formalizing and proving.** The MDPs that fall under the *Formalizing and Proving* category captures the MDP codes that were reported in mathematicians interviews on of proof and in their research. The MDPs within this category often appeared together within a single holistic code *across* the dataset. Anthony’s excerpt provides an illustrative example of how multiple MDPs can appear within a single holistic code. In this excerpt, Anthony discusses the mathematical work that he would like students in his advanced undergraduate courses to use as they learn to prove mathematical statements.

Anthony: Understanding the language of this sort of logic is helpful. You have to write everything only using symbols, but rather, this is the framework you need to think and set up to recognize that you are trying to prove something… As long as you didn't make any logical error that should be a proof. So, there's a structure to thinking, and they won't be able to do it as purely and cleanly every time, but they have something else besides just seeing mathematics, playing with symbols, if something comes up - like a structure, a scaffolding.

In this case, Anthony’s excerpt was coded with *Proof and Proving, Syntactic Manipulation, and Logical Structure*. In his excerpt, Anthony appeals to syntactic manipulation as a way to provide a structure to producing a logically-sound proof in his teaching. Anthony’s description provides one example of how multiple practices are important MDPs that can be used when construction proofs in advanced mathematics.

**Mathematizing and computing practices.** The second category *Mathematizing and Computing*, captures MDPs related applications in mathematics. MDPs in this category include *Modeling, Algorithm and Programming, and Utilizing Technology*. In using the term *matematize*, I mean, “making more mathematical” (Gravemeijer & Terwell, 2000, p. 781). This category of
MDP practices appeared most often in descriptions of applied mathematics research, though they are not *exclusive* to applied research. In my analysis across the holistic code “MDPs Used in Research” which also included the MDP code “Algorithm and Programming”, I noticed that the holistic code was frequently layered with other MDP codes like modeling, utilizing technology to perform calculations, or developing techniques to solve applied problems. Thus, I grouped these practices that were coded together across the mathematician interviews into the category of *Mathematizing and Computing*. For example, both pure and applied mathematicians in this study reported that writing computer programs was an important aspect of their research. An example of an excerpt that highlights MDPs within the *Mathematizing and Computing* category comes from Scott’s report on his research around designing algorithms in applied numerical methods.

Scott: So, these are some issues in algorithm design that are now more relevant when we scale things up. Because some algorithms just don’t scale. Other things are - this one is actually mathematicians should think about - which is why I was forced to write programs. Whatever I’m proposing, somebody ought to be able to read that paper and implement it.

As an academic researcher, Scott would like other researchers to use the mathematical algorithms he designs in his research. Thus, in designing algorithms, he writes computer programs that are comprehensible so others can implement or reproduce the algorithm.

**Structuring mathematics.** The *Structuring Mathematics* category captures MDPs used to examine the structure of the mathematical objects of interest to gain insights into the larger problem at hand. MDPs within this category include *Classifying, Leveraging mathematical structure, Representing*, and *Generalizing*. The MDPs in *Structuring Mathematics* category often appeared together within a single holistic code across the mathematician interviews. For example, when a holistic code such as “MDPs used in Research” was coded with the *Classifying* MDP code, and oftentimes the holistic code was also coded with the *Leveraging Mathematical Structure* code. I also note that these two MDP codes did not exclusively occur together across every holistic code with *Classifying*, indicating that they are *not* the same MDP, but are supporting practices within the *Structuring Mathematics* category. I bring forward Robert’s description of making mathematical progress in his combinatorial research because his excerpt describes multiple practices within this category. In particular, he uses examples to provide insights into the larger problem at hand.
Robert: I've read. I know what's out there and I do look at small examples and I do this by hand. I'll take small enough examples where I can just take blank paper and draw a bunch of examples and try to organize them in a way that makes sense to me. And then, I usually get some ideas from that - patterns that I'm seeing… It helps me test out baby ideas that I have to see whether they give some traction.

In this excerpt, Robert described how he creates and examines examples of mathematical objects (leveraging examples) that satisfy some mathematical properties of interest and then organizes the examples (Classifying) in to detect a pattern (Discovering patterns) which him to develop and test conjectures as a stepping stone to the larger problem he is trying to prove (Generalizing).

Other supporting practices. Lastly, the Other Supporting Practices category captures MDPs which have application across the other categories in supporting roles. For example, holistic codes that were coded with precision were also often coded with MDPs within each of the three categories. For example, the precision MDP code was coded with proof and proving, procedures and computations, and even when discovering patterns. Thus, I consider the MDPs in this category as supporting MDPs. By making this distinction, I am not implying that other MDPs are not supportive to MDPs that appear in the other three categories. Again, MDPs are not mutually exclusive, and these results describe the way that practices hung together in the set of eight interviews in this study.

I selected Sophie’s excerpt of her description of the mathematical work that students are expected to do in her numerical calculus courses to describe this category of practices. Namely, she described that students need to identify the conditions given by the problem (linear with boundary conditions) if they can to meet the specific properties (show it is well-posed) you can solve the problem using various methods (e.g., numerically, write a program).

Sophie:...Then you would at least have a statement of a problem that you know, that if its linear with these boundary conditions, and you can show that these are well-posed, then the theory says there exists a unique solutions. There's no point in solving it numerically unless you can do that. So then you still don't know the solution, but you know there exists a unique one... If you can't or its too difficult to write down the exact solution, we transform it and solve it on a computer and show that as you make your grid spacing smaller your solution is converging to something.

Sophie described one way that she engages in the Conditions, Assumptions, and Properties (CAPs) practice by coordinating the conditions given by the problem (linear with boundary conditions) with the desired mathematical properties, and then aligning those with particular methods for
finding a solution (transform and solve in computer). Since the CAPs practice is useful beyond the context of applied numerical analysis, I placed it in the Other Supporting Practices. I included “Intuition and Creativity” in this list of because it was an important aspect of three mathematicians’ research and influenced how they approached teaching in their advanced courses. I did not want to omit this the influence that Intuition and Creativity had on research and in teaching, and did code these responses as a general supporting practice of research and teaching.

Table 2.6 offers an inventory of how often MDPs in each of the four categories (from Table 2.4) appeared across the eight interviews. Each number in the table represents a count of the number of times any MDPs was coded within to a holistic code across each mathematician interview. For example, Robert reported MDPs under the Formalizing and Proving category (e.g., Proof & Proving, Argumentation and Justification, Conjecturing) a total of 30 times across his entire interview suggesting that many of the descriptions of MDPs were often centered on practices related to aspects of proof.

Table 2.6. Frequency of codes in four categories of MDPs in research and teaching

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Formalizing and Proving</th>
<th>Mathematizing and Computing</th>
<th>Structuring Mathematics</th>
<th>Other Supporting Practices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam (p)</td>
<td>7</td>
<td>0</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Anthony (a)</td>
<td>34</td>
<td>16</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>Ellen (p)</td>
<td>16</td>
<td>6</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>Gloria (p)</td>
<td>17</td>
<td>4</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>James (p)</td>
<td>30</td>
<td>12</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>Robert (p)</td>
<td>30</td>
<td>9</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>Scott (a)</td>
<td>12</td>
<td>17</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Sophie (a)</td>
<td>14</td>
<td>22</td>
<td>5</td>
<td>11</td>
</tr>
</tbody>
</table>

(a) Applied mathematician, (p) pure mathematician

Because advanced undergraduate mathematics is centered on rigorous mathematical proof and the context of this study is in advanced mathematics, unsurprisingly, the MDP category that was reported most (both the total instances and highest frequencies per mathematician) often was Formalizing and Proving. The Mathematizing and Computing practices (e.g., Modeling, Algorithm and Programming) appeared more often for the mathematicians whose research was in applied mathematics. For example, Anthony (16), Sophie (22), and Scott’s (17) research was in applied mathematics, and the other mathematicians’ research was in pure mathematics. James (12) and Robert (9) were interesting cases in this category because, while their research is in pure
mathematics, their research drew on several MDPs in the *Mathematizing and Computing* category because they designed algorithms and wrote computer programs to do mathematical work beyond what they could do by hand. The *Structuring Mathematics* and *Other Supporting Practices* were common amongst both pure and applied mathematicians in this study.

Reviewing frequencies across each row (representing all the codes for each mathematician), Adam had significantly fewer MDPs coded across his interviews. This can be explained in a few ways. Firstly, his jovial interview contained a wealth of stories about Adam’s research, but much of his reporting was not in terms of MDPs. Second, it also is evidence of the difficulty in coming to a shared understanding of MDPs. One could also notice that mathematicians who engaged in pure mathematics research more often engaged in practices in the *Formalizing and Proving* and *Structuring Mathematics* (e.g., Robert, Gloria). Collectively, mathematicians engaged in a variety of MDPs in their research, but in the context of mathematical research, MDPs related to proving were foundational aspects of doing mathematical research.

**Mathematicians’ Research and MDPs**

In order to address the research question: In what ways do mathematicians report on using MDPs in their research? I asked mathematicians the questions: *Can you briefly describe for me the kind of work you do as a research mathematician?* and *What MDPs do you use to support your research?* Their responses were coded with the structural Codes “Research” and “MDP used in Work” respectively. In their response to the first question about research, we had not (yet) discussed MDPs specifically because it came later in the interview protocol. The next set of results shows the variety of responses in which mathematicians described their research and MDPs used in that work, either from a direct question about MDPs or as a natural consequence of their descriptions of their research. The mathematicians’ responses are organized around how mathematicians use MDPs to make mathematical progress their efforts to create new mathematics. Collectively, their responses indicate that mathematicians draw in a variety of MDPs to make mathematical progress, often for the purpose of making new mathematics, or proving results using mathematical models.

**MDPs and making mathematical progress.** Mathematicians’ research work is centered on developing and proving *new* mathematical results. Two mathematicians (Gloria and Ellen) both described MDPs that support their work in creating new results in mathematics. When I asked
Gloria to describe the MDPs used in her abstract algebra research, she described how she attends to important elements of her problems and examines those elements across examples to align component in service of creating new mathematics. She shared, “Just a lot of reading the literature, talking with other mathematicians, working really, really hard to put the ideas together to figure out new mathematics.” When I probed Gloria by asking her for the activities which helped her make new mathematics, she responded with, “Looking for examples. Just trying out lots of different approaches… Seeing all the different components that might have something relevant and thinking about how they might fit together.” While Gloria engaged in exploring a variety of approaches to make new mathematics, Ellen provided more details about the mathematics behind the work of creating new mathematics in the domain of functional analysis. Despite different content domains, the conclusion was similar to that of Gloria because Ellen also detailed trying to fit components together to obtain mathematical results.

Ellen: So, you want to obtain this, there are certain steps and conditions that you need to have in place. So, what happens is my operator will do this type of things and behaves in this way… I cannot use the theory that they use here because my set of operators do not really go that way. Well, maybe can I develop something parallel to that that will work with my set of operators? Because in my theory, the operators were good for something, so I wanted to keep them… Can I modify this little bit and see if I can include more things that we work with this? Basically, you know what you want to obtain - Can I get there? What do I need to do in between? So I know my goal - I would like to have this solution. Okay, so let’s see if we can work in pieces… How do I move to the more general setting? You always try to see things in pieces that you can obtain results [emphasis added].

In this case, Ellen described how she also works in different pieces but focuses on making small modifications to make progress and obtain results (making new mathematics with her operators). Gloria and Ellen both described how they seek ways to relate components of their problem to make mathematical progress in the domain of abstract algebra and group theory, respectively. What is interesting is that MDPs which supported this work were not (necessarily) the same between Ellen and Gloria. Gloria reported that she creates and leverages different examples to gain perspective on how to put mathematical ideas together. In contrast, Ellen described how she modifies the conditions her set of operators to find how pieces can be put together to obtain mathematical results.

When describing their research, three mathematicians described a consequence of developing new mathematical results. Adam, Robert, and James reported that what they set out to
prove is not always known and they often end up proving a different, but related mathematical result. When I enquired about Adam’s research, algebraic topology, he reported that in his research he often has a goal for what he is trying to prove, but is also open to other possibilities, including that the theorem you want to prove may not actually be true.

Adam: Obviously you need a goal you're looking for. You need a theorem you want to prove, but that might not be the theorem you end up proving. Because you might discover the theorem that you wanted to prove is not true. So, you end up with another theorem or the technique you're using only works for a certain subclass of groups, for example, that you're trying to classify. So, you only do it for a subclass. Or there's a larger class that you hadn't realized that your proof was good for. You have to be able to be open to all those possibilities.

In describing the practice of proving mathematical results in algebraic topology, Adam’s description highlights that a consequence of proving something that, although different from the original research aim, can result in new mathematics (e.g., another theorem or technique) that may only work for certain cases (e.g., subclass of groups). Similarly, when I asked Robert to describe his research in graph theory (and not yet about the MDPs used in research) Robert reported that as he struggles to make progress, he modifies the problem at hand by adjusting the conditions to prove a narrower mathematical claim; thereby gaining insight on what he should be proving in the first place. The following excerpt is an example of how Robert engaged in the CAPs practice. Robert’s use of the CAPs practice is similar to how Ellen used her operators in functional analysis, but Robert reported that he used the CAPs practice to prove a narrower mathematical claim where Ellen engages in the CAPs practice to gain insights into how aspects of her problem might fit together. While Ellen and Robert deploy the CAPs in different ways in their research, they both use it to make mathematical progress. Robert’s following excerpt was the result of asking him to describe MDPs that support him in his research. Specifically, Robert attends to how he deploys the CAPs practices by adding additional assumptions to his problem in order to solve a conjecture he has a better chance of proving than the full-blown conjecture that will (hopefully) become new mathematical results.

Robert: That's really what its geared towards, is giving me an idea of what I'm even trying to prove. In other words, I don't know how to prove the conjecture. I don't know how to get from the assumption of the conjecture to the conclusion of it. So, instead, I'm throwing in a little extra assumptions in there and getting to a conclusion... I've re-tooled the original conjecture into a narrower mathematical claim, which still is relevant, and which I think I have a better chance of attacking...
I'm going to prove something, which is true about the technique I'm using - it doesn't successfully resolve the major question that was driving all of this, but it’s a true statement. It’s a true statement and handles some case of it.

Robert is able to prove a narrower mathematical claim through this modification of the assumptions in the problem he is working within, acknowledging that it does not solve the larger question driving his research. However, as Robert described, the new conjecture, while a narrower claim, allowed him to make progress towards the original conjecture, which would be a new result in mathematics. Solving the narrower claim does not solve the larger problem, but Robert reported that it does handle some cases. I interpreted that an important aspect of Robert’s research is to prove narrower mathematical claims because he is able to make progress. In describing MDPs used in his research, he shared that modifying the conditions of the problem is the key for him making progress.

Robert: If I'm sitting there thinking about the full-blown conjecture and I don't really understand whether it’s true or false - If I don't tinker it down to where I can attack it, I don’t think I'm going to have any chance of making progress on the problem.”

When I asked Robert to provide more details about the mathematical work that he does to gain insights on the larger problem, he shares how the MDP of leveraging examples is important because “tinkering” with the assumptions and analyzing these changes via examples can illuminate patterns that are helpful for developing new conjectures but are still related to the original conjecture he sought to prove. He explains that the newly discovered patterns provide productive insights into possible smaller conjectures (“test out baby ideas”).

Robert: I do look at small examples and I do this by hand… I can just take blank paper and draw a bunch of examples and try to organize them in a way that makes sense to me. I usually get some ideas from that - patterns that I'm seeing. It helps me clarify … It helps me kind of test out baby ideas that I have to see whether they are stupid or whether they give some traction.

While not the full-blown conjecture that Robert was originally trying to prove, the practice of leveraging examples and discovering patterns was an important part of making progress by proving smaller, but related conjecture. Similarly, when I asked James to describe MDPs that support him in his research in number theory and graph theory, he shared, like Robert, that he too gains insight into what he should be attempting to prove through the practice of looking for patterns in constructed examples.
James: One practice that mathematicians use is construction of small examples, right? I think it’s what everybody does… And keep an eye out for patterns. Based on the examples and patterns you've been noticing… Like when we were computing the statistics of these wreath products - Oh, I wonder if the statistics just match the statistics of the symmetric groups since the fixed points are the fixed point of the symmetric groups? And then you can ask your computer to look at the statistics for like some large wreath product and see if you get the number that is the same number as the statistics the symmetric group… Now I know what I should try and prove.

James designed a program to compute statistics for particular examples of wreath products because “It’s hard to prove something you don’t know it’s true or suspect that it’s true… I don't even know what I was trying to prove at that point. I had this big messy expression and I did it for a few examples”. The practice of constructing examples and leveraging such examples to discern patterns was a salient aspect of mathematicians’ research that helped them identify new insights and led many toward constructing new mathematics.

Taken together, the mathematicians described a variety of mathematical practices that were helpful for mathematicians to make progress in their research. In the case of Gloria and Ellen, leveraging examples were useful for being able to put mathematical ideas together to “obtain mathematical results” or “make new mathematics.” In the case of Adam, Robert and James, they made mathematical progress through their examples, even when they proved something different than what they originally set out to prove. James’ excerpt highlights that exploring with examples gives him insights into what he can try and prove, whereas Robert

The previous set of responses came from reports on mathematicians whose research primarily fell into pure mathematics. What follow are responses specifically address MDPs from mathematicians whose research is in applied mathematics research.

Mathematizing and computing MDPs in applied mathematics research. Both pure and applied mathematicians’ reports of their research were centered on proving and was a salient MDPs code across the interview dataset. The set previous responses around MDPs in support of making mathematical progress towards the goal of obtaining results, making new mathematics, or even revealing what could be proved was centered on mathematicians whose research is primarily in pure mathematics. However, mathematicians that employed MDPs in the Mathematizing and Computing category provided insights into different ways that mathematicians contribute to creating new mathematics in the field. To target the MDPs Anthony used in his work, I asked him
to describe in more detail about the “activities” he does while engaged in his research. He responded by juxtaposing the kind of applied mathematics research he does with pure mathematics research. Anthony described pure mathematics as building theory through rigorous proof but that in applied mathematics research, theorems are often not yet discovered.

Anthony: It’s all about proving theorems. Well, try to find the theorems you can prove which means building the theory. Building the theory means propose certain, even define new combinations of objects that can be defined mathematically and find what properties they have, what interactions they have. In order to actually build theory you have to show that these interactions exist… So, technically you should be proving theorems all the time.

Anthony described his understanding of how applied mathematics is set apart from pure mathematics. Specifically, he identified that fewer theorems are available to prove because the theorems have not yet been discovered. Thus part of his work in applied mathematical research is to explore what can be proven in applied settings. In the following excerpt, Anthony highlights that the development of computer programs to devise possible solution(s) is an essential aspect of his work.

Anthony: In applied mathematics, it’s a little different. There are fewer theorems being proven because they haven't discovered them. There is a lot to be explored and understood, so people who work in that area do a lot of computer simulations. They set up some structure, say it’s a very complicated differential equation and they don't know if they have any solutions, or they have multiple solutions, or what the solutions will look like. They can write that in a computer program, do a numerical approximation, and see what the numerical approximation is telling you. If it’s carefully done analytically well, then you will track whether for this method for approximate in the differential equation preserves the inherent properties.

When I asked Anthony to describe in more detail the mathematical work that goes into developing computer programs and analyzing the solutions. He explained, “So I sit down and try to write the equations for it, try to see can I solve this.” He went on to say that that he first attempts all the techniques he knows, and if those don’t work, he attempts a small transformation to see if he can find a way to solve a more simple, related problem.

Anthony: Well, you try everything you know. Apply it and see if it works. So, if I can write a differential equation there are a number of techniques for trying to solve differential equations. I try it. I see if some of those work. If they don't, I see if this differential equation has a particular unique structure that people write books about them. But I could make a little transformation, all of a sudden it looks so simple. I can transform it to a different problem.
In his research, Anthony described how he engaged in the practice of applying techniques and when these techniques do not work, he described how he engages in the CAPs practice by applying a small transformation to the conditions of the problem to make mathematical progress; much like how Robert did in his graph theory research. When Robert struggled to prove a mathematical conjecture, he described how he makes small adjustments to the assumptions in the conjecture in order to prove a narrower mathematical, thereby making progress on his research. Despite proving a different conjecture than Robert’s original aim, he gains insights into the larger aim because he has proved something about a subset of cases of the original conjecture. Both Robert and Anthony suggest that making adjustments to the conditions or assumptions of the problems is an important aspect part of their research. Furthermore, every mathematician in this study described how they engaged the CAPs practice in their research. This suggests that the CAPs practice is useful across pure and applied domains mathematical research. Furthermore, the CAPs practice is useful for making progress towards creating new mathematics.

Asking Sophie to detail her research in applied numerical methods resulted in her describing the application of her research in earth science. To get a more targeted response, I probed further and asked Sophie to specifically describe MDPs that she uses in her own research by asking, “What sort of things do you do to make progress on your problems?” and Sophie subsequently rooted her response around her modeling practice.

Sophie: For the modeling, what I would do first is outline what it is that I want to model. If I'm doing research, there are sort of known standards on how to approach certain things. So I would take what's known about it first, as like at least a platform. And then I would think about what assumptions am I making in this work… And making sure that the problem is stated in a way such that there could be one. Maybe it doesn't have to be unique, like you want to be able to look at like your problem statement and say that these are meaningful expressions… If you write down a partial differential equation and specify boundary conditions you want to know that those boundary conditions make sense for the problem.

Sophie’s response provided insights into how Sophie’s modeling practice. Like Anthony, Sophie also first identifies known standards and techniques available which work in the context of her work. As Sophie develops her mathematical models, she then considers what assumptions that will be built into her model, all the while considering whether the model she is developing is even feasible. She does this by selecting appropriate boundary conditions for the problem and ensuring they make sense for her model. Sophie’s description of her process provides evidence of the
complexity of the modeling process. Interestingly, while Sophie and Scott’s research interests are both in applied numerical methods, Scott’s responses around the MDPs used in his research were focused on algorithms, algorithmic thinking, and algorithm design. When I asked him to describe in more detail that work by asking “I’m curious to know like what kind of mathematical work goes into algorithm design?” his reply indicated that finding an efficient and adoptable design was important.

Scott: I want to minimize not only for the sake of cost, minimize the number of operations that I’m doing also because every time I do an operation, there could be some rounding error. And that could snowball on me. So you want to know, okay, what would be the effect of getting some computations wrong. Other things are, this one is actually mathematicians should think about it - which is why I was forced to write programs as well - Is whatever I’m proposing, somebody ought to be able to read that paper and implement it. So there are certain algorithms that could work very well, but are so technical that they are not going to be widely adopted.

While Sophie and Scott’s research are in a similar mathematical content domain, applied numerical methods, yet, their interviews suggested that the MDPs central to their research were different. In the case of Sophie’s research, the practice of modeling was central for her interdisciplinary work. On the other hand, Scott’s research was focused on algorithm design. That is not to say that Sophie and Scott exclusively use different MDPs in their research. Instead, it indicates that mathematicians employ a range of MDPs in their research, even within similar research domains.

**Teaching MDPs in Advanced Mathematics Courses**

In this section, I set out to answer the research sub-question: *How do mathematicians perceive teaching MDPs to students in advanced mathematics?* I begin this section by offering Robert’s perspective on the goals for students in advanced courses because it provides some evidence for how a mathematician perceives MDPs as an important aspect of advanced courses. While Robert’s response was unique among the participants of this study, I selected his excerpt because it provides explicit evidence of how mathematicians view MDPs as a vital aspect of in advanced mathematics. When I probed Robert to explain in more detail why MDPs are important in his teaching, he identified that MDPs including *defining, conjecturing, and proving* are salient MDPs across content domains in advanced coursework.
Robert: Ideally [in] these classes, the topic changes as you go from course to course... I think they can see how mathematical things get defined, how these definitions determine unintended properties... Then that carries with it all these implications... You're doing a double victory for students because those are the common language of most problems that they're going to involve in math... the practices that they're learning. Hopefully they are engaging with baby examples of conjecturing and verifying or disproving these little claims based on definitions or theorems.

Robert’s excerpt highlights that MDPs like verifying and disproving conjectures and using examples, are important aspects of what they should be learning across all their advanced undergraduate coursework. The following set of responses in this section were an outcome of my analysis to outline the instructional practices that mathematician described in their accounts of teaching MDPs in their advanced undergraduate mathematics courses. Their responses were a result of asking, “What are some things you do in your own teaching to help students learn this mathematical practice?” It is important to note that the discussion around teaching MDPs was rooted in their previous descriptions of MDPs to ensure a shared understanding of MDPs and their approaches in teaching such practices.

I organize the next set of responses drawing on the construct of scaffolding, which has its roots in the Vygotskian literature to describe ways to support learning (Bruner & Sherwood, 1975; Wood, Bruner, & Ross, 1976). While researchers have explicated a variety of scaffolds teachers can use to structure learning in a classroom (Anghileri, 2006; Moschkovich, 2015), I use social and analytic scaffolds outlined by Baxter and Williams (2010) to describe the ways in which these mathematicians reported supporting their students in learning MDPs in their teaching. As a reminder, social scaffolding refers to the scaffolding of norms for social behavior and expectations around mathematics learning. Analytic scaffolding refers to support offered by a teacher that is used to draw attention to critical aspects of the mathematics and to scaffold the ideas for students.

**Social scaffolding.** Mathematicians described a variety of ways to support students’ learning MDPs through social means. One of the ways they reported was by structuring their class to communicate important aspects of the discipline. Robert’s excerpt highlights one way to structure learning of MDPs is to develop an atmosphere where it was normative for students to ask questions and be aware of the mathematics that might not yet be fully understood. Specifically, after I asked Robert, “what challenges do you find as you try and teach students about these
practices that we've talked about?” followed by “how do you sort of deal with that?” he described that he structures his course in a way that he can model what doing mathematics entails.

Robert: By example. Which is I try to ask questions, and I try to tell people that the core of this - its not a dumb question to ask - what does onto mean? And so I try to just model that partly because that worked for me... Being careful. There's no shame in doing that. And in fact, you're not doing math if you are not being that careful. So, I just try to make that be the atmosphere of class. Which is like, what do you really know? It's not wrong to - you're better off knowing what you don't know and going slow and carefully. And being safe and confident about what you know then asking questions you haven’t unpacked yet.

Specifically, Robert’s attention to cultivating a classroom atmosphere around the MDP of being mathematically precise (coded as mathematical precision) is an example of social scaffolding that specifically addressed MDPs teaching. However, this was not the only example of scaffolding learning. The next set of responses are specific to teaching the practice of proof in advanced undergraduate courses. It was not surprising that mathematicians focused their responses on proof because it is a central practice of the discipline that that students are expected to learn in advanced courses. However, Anthony, Adam, and Gloria all shared how they use a different kind of social scaffolding to demonstrate and present multiple ways of constructing and validating mathematical proof in their teaching.

Anthony reported carefully selecting which proof attempts to bring forward to the class, which I interpreted as a way that he developed norms for using student work as a launching point learning about what constitutes a proof. Anthony’s response was a result of me posing the question, “I'm curious what kind of mathematical work students are doing to learn proof?” to which he responded with a variety of instructional approaches. Specifically Anthony noted that he will often use student-constructed proofs as the focal point of a classroom discussion.

Anthony: Different approaches, different proof, different writings. In my [advanced undergraduate]-level class... I photocopied and I put four answers to the same problem. Discussion of this in class, and presenting them, and having people comment on them is another way for them to see. Seeing other people's approaches is helpful.”

In his description of teaching, Anthony described how he uses student work as a launching for discussions on what constitutes a proof. Anthony’s description of how he supported students in proof during classroom instruction is one way to use a social scaffold to structure learning of MDPs. Both Gloria and Adam described a similar approach to Anthony’s. They reported asking
students write out their proofs on whiteboards to unpack, validate, and walk through the proofs for students. It was not clear from their descriptions whether students were expected to engage in discourse around what was on the board, or whether it was a moment where the mathematician would impart their knowledge to their students. Gloria, however, reported using students’ proofs to highlight efficient and elegant proofs, or proofs that had extraneous components so that students could be exposed to different ways of proving (or not proving) a mathematical argument. Her response was the result of asking, “How do you teach students to construct proofs? Her response was:

Gloria: So, they write proofs on the board from their homework, I have them write proofs in the in-class activities. Again, they put them up on the board, and I point out what, you know, this is an example of a really well-written proof cause it doesn't have any irrelevant information, and it, you know, uses the definitions or the theorems and points out when you're using theorems.

Similarly, Adam begins his group theory class with having students write proofs of selected problems on the board, and he validates and discusses the components of the students’ proof productions. After he shared with me some of the difficulties students’ face when learning group theory proofs, I probed, “In your teaching, how do you address that difficulty that students have?” Adam’s response was:

Adam: I give them homework sets that the class starts with them writing it all up on the board - not presenting, but just all at once writing things on the board. And then I'll go over what's good and bad - you know, kind of critique each thing that's written on the board. And if they write something really egregious on the board, I won't let them erase it. ‘You made the classic mistake! We've got to warn everyone about this.’ I'm not identifying who wrote it up there and I'm not asking them to justify what they wrote up there.

Both Adam and Gloria, and to some extent Anthony, have attempted to normalize using student work as a launching point for teaching students construct and validate proof across mathematical content domains. Where Anthony carefully selects what student proofs to use in advance, providing him with a bit more control of what his students are exposed to during class, Gloria and Robert will have students construct proof in class or hold class discussions where they have less control of the student contributions. Regardless of the social scaffold used in teaching proof, mathematicians in this study were able to deploy social scaffolds to structure learning of MDPs in their advanced undergraduate course.
Normalizing group work. Another aspect of social scaffolding happens when mathematicians create an atmosphere where it is normative for students to work productively together. Working in concert with others was an important social practice of academic mathematicians in my study and others (i.e., Bass, 2011), and there appeared to be some indications of supporting similar activities in their own courses. For example, Scott highlighted that oftentimes in his research he looks beyond his area of expertise because “It’s very rare that you find someone that is truly an expert in more than one of those links. You have to work in a team.” Gloria approached teaching in her advanced mathematics courses by writing open-ended questions on the board and having students explore problems in smaller groups during class.

Gloria: I do as much cooperative learning as I possibly can. And as much open-ended questions to get students to explore. I bite my tongue and let them struggle... And if they raise their hand and say, "I'm just really stuck!" I'll go over and ask some leading questions.

Ellen and Anthony reported doing group-oriented work in their advanced courses to support students in MDPs. However, they described tensions due to the time investment of students engaging in problems together during class.

Anthony: So I see [advanced undergraduate math classes] as a place where you should really pull up your sleeves and get this kind of understanding of how to put scaffolding on things... So, how do I get them to learn this? In the classroom I try to have them, depending on the schedule, time to work on things right there.

Much like Robert, Anthony acknowledged that advanced undergraduate students should become more independent with their learning. In contrast, Ellen highlighted how students struggle with this. For instance, when Ellen reflected on instances where she had asked students to engage in a problem in class, she noted how students often looked to the course textbook for answers rather than trying to construct the proof for themselves.

Ellen: Takes forever. Why? Let's prove this. I did exactly one similar thing now and changing a little thing so they need to adapt to whatever we had before. And the students are looking, passing the pages in the book. The students did not pay attention. And the student that was paying attention did it and it worked. So, let's try to understand this - and sometimes the class is too short.

When I asked Adam to describe how he Adam described that he uses mathematical discourse in his advanced undergraduate courses (e.g., group theory, topology) because he believes that discourse can promote students’ engagement in mathematical practices.
Adam: At least in terms of undergraduate teaching, how to engage students in mathematical discourse? If you focus on student discourse, then the mathematical practices sort of naturally arise out of that.

Earlier in Adam’s interview, when I asked Adam to describe his research in algebraic topology, he reflected on his use of discourse in his research. Adam said, “Engaging in mathematical discourse is fun. And productive. And it is how I think about mathematics.” He noted that he had “tried a lot of techniques” and “engaged in professional development over the last three years” in order to implement instructional practices to support students’ mathematical discourse.

Adam did not discuss the techniques he used to engage students in mathematical discourse, but he was the only mathematician in this study to discuss professional development in any capacity. Adam has knowledge of engaging students in MDPs but acknowledged being challenged when faced with trying to support mathematical discourse in his class.

Adam: ...you sort of have to organize the room in such a way that when someone needs assistance with the problem, then maybe someone else can give them that, ‘Oh, what if you try this?’ Rather than, ‘Well the answer is -’ You don't want to go there. You have to structure the room so that works.

When a student offers a verbal or written proof in class, one of his challenges is when students are missing a step is to not “telegraph to them that it is correct or incorrect” when asking questions about the missing link in the proof he asks the class, "Do you agree with that? Is it accurate? Can you reproduce it?" Adam provides evidence of his knowledge of content and teaching (KCT) when trying to support student’s engagement in discourse during class to support learning mathematical practices.

I did not conduct case studies in Anthony’s, Adam’s, or Gloria’s classroom and have no way to substantiate their self-reports. Collectively, they described using various social scaffolds in their teaching in an effort to support students learning MDPs. This suggests that mathematicians who teach advanced mathematics do have knowledge of instructional practices beyond traditional lecture. I would also point out that by drawing on these social scaffolds, students are exposed to the ways in which the discipline is organized. For example, Gloria and Anthony elicit student-generated proofs to highlights how one proof might be more efficient (or elegant) than another.

**Analytic scaffolding.** Analytic scaffolding refers to the ways in which mathematicians scaffold mathematical ideas for students, and in the context of this study, the ways that mathematicians scaffold learning of MDPs. One of my interview questions asked mathematicians
to share how they support students learning MDPs and resulted in a variety of responses around how to scaffold mathematics in a way to make practices visible to students.

**Scaffolding by tasks.** Half of the mathematicians reported supporting students learning MDPs by providing students with problems. James shared, “I give my students problems because that's how I learned. I try to give them problems where it’s not ‘prove that this thing is true’, but ‘prove or disprove’. I selected James’ instructional practices which align with how he learned to engage in the practices of the discipline himself:

James: I just did everything. I did way more problems than anybody else. I did all of them because I was tired of failing my exams… Doing the problems was the right thing to do. Not just like reading the problems and feeling like I understood them. I did all of the problems and that's when I realized that doing stuff is useful for understanding.

James provided evidence that he coordinates how he learned practices with instructional practices to support his students learning practices, which indicates that he might be drawing on his KC to support students in MDPs. However, Sophie highlights that the quantity of problems was not an essential aspect of giving students problems to help them learn MDPs. When Sophie reflected on how she could teach practices more effectively, she described that even just a single exciting task could be effective in teaching mathematics practices in her numerical analysis course.

Sophie: I did an application problem yesterday in my class and it was about bike traffic flow around [a city] and they loved it. Because we'd been doing nothing but reduced row echelon form for two weeks. My god, they hated me! And so we did this application and we talked about which streets would have the most bike traffic given these data points. And they got so into it! And a student in the back who has been really quiet raised his hand - this is a really good problem because this could inform me where to set up my food cart to get the most flow. And they were really into it.”

Our interview did not unpack the difficulty of this task or what MDPs students engaged as they modeled traffic flow, it was clear that Sophie recognized the advantage of interesting problems. She noted, “This sounds so cheesy, but if you're passionate about something or if the problem is exciting, you’re going to engage in it more. Maybe we create better more engaging examples?” This could be an example of Sophie being constrained by her KC. She validated that exciting examples can help her in modeling applications, but it could be the case that curricular materials not readily available.
When asking Robert how he teaches MDPs in the context of his advanced courses, Like Sophie, shared that he also offered assigning problems to students but also noted that he selects difficult problems because challenging problems are how students begin to think about MDPs.

Robert: It’s kind of important, at least in a course like this, to throw them a little bit into the deep end. Go ahead and assign some of the harder problems… Beyond just the choosing problems that aren't over-scaffolded, I do think there are classes in which if you aren’t careful, you aren't even assigning problems that lend themselves to deep thinking or mathematical practices. There has to be a certain difficulty to the work you're asking them to do. Then they have to write proofs and then that's where it happens… That's what these courses are about.

Collectively, mathematicians described the importance of giving students problems and shared different perspectives about the quantity and quality of problems students should experience in class. I admit that giving students’ problems is low-hanging fruit in the context of analytic scaffolding because homework problems are an obvious way to get students to engage in solving problems. However, there is no guarantee that students recognize or think about MDPs as they engage in such problems, However, mathematicians think about the quality and quantity of problems they give students and even held different perspectives on assigned problems. While James argued that practicing lots of problems supports learning MDPS because that is how he learned, Robert suggested that assigning more difficult problems is how he supports students to learn practices (which aligns with his perspective that students should be developing mathematical maturity in advanced mathematics courses). Sophie shared a unique perspective among the mathematicians in my study which indicated that even a single exciting task could engage students in unexpected ways.

Scaffolding feedback. Mathematicians viewed feedback as an important aspect of supporting students in learning MDPs. Three mathematicians described how they provide feedback to students on their written assignments to help them learn to engage in practices, but that feedback is not always received positively. James said, “I think I give back feedback on homework that they resent sometimes.” While we did not go into great detail about what kind of feedback he gives and why James thought students felt this way about his feedback, Ellen and Anthony describe in more detail about the nature of written feedback in the context of their advanced courses.

Anthony: A lot of these classes have a lot of grading of homework, and the grading for me doesn't mean just 'no, right or wrong' it's a lot of feedback on do you really
mean that? You probably meant this. When you say this wrong, but you probably meant this. What you said here, like this, then this, then this, is where all three are true, but this doesn't follow from the one above.

Anthony described how he brings forward the logical structure of students’ proof and note where the mathematical argument broke in the chain of logical reasoning for incorrect proof productions. Ellen also provides written feedback to her students on homework, but she reported on the use of examples (and counterexamples) as being important in her feedback.

Ellen: I can give the feedback - the student needs to read it. That's the problem. It’s important to give feedback because you need to learn how to write mathematics too. The feedback is well, let me give you a counterexample. You need to think of an example to say 'see, consider this. It's not true'. And the point of that is well, when you claim something make sure that claim is true! It doesn't take much but just check. And it happens with all the inequalities with epsilon and delta. Students claim okay this is less than that. Except that it is not. Why is that you didn't take a minute of your time to check that because it is just the cross product and you do it? And then when I grade and then I do the cross product and a has to be greater than this number. I did the calculation… Hours. it takes. And I wish that would show the students that they need to really put more time in and do it better.

Ellen has a strong sense of the common student errors in her introductory analysis but one of the challenges she faces teaching advanced undergraduates is they might not yet know the importance of checking and testing claims and conjectures. Analyzing, validating, and constructing counterexamples in students’ proof requires strong content knowledge, as evidenced by Ellen and Anthony’s reports of providing feedback to students. Both Anthony and Ellen describe that written feedback is an important aspect of their teaching, but in addition to providing written feedback to students, Adam noted that he supported students’ learning of MDPs through feedback during classroom instruction as they engaged in mathematical discourse.

Adam: Practice and feedback. That's really what it requires. I am getting better at giving students written feedback. In-class feedback, that's easy. (Pause) Well, maybe it's not easy. That's part of what that discourse is doing - is that when you have students tell you a proof.

Adam suggested an alternative option for providing student feedback, a perspective that was unique among mathematicians in my study. He reported that supporting mathematical discourse in the classroom was important for students to learn to engage in the practice of proof in his introductory group theory course as well as providing feedback to students. Here, Adam describes how this plays out in his classroom:
Adam: One of the things I ask them - What is a proof? A good proof, what's it doing?... In the mathematical community, a good proof is one that the community accepts. So, I try to push them to try to write proofs that their classmates will accept. Once a student provides a proof in class, either written or oral, I will ask the class, ‘Okay, do you agree with that? Is it accurate? Can you reproduce it?’ And sometimes they might not have elucidated a piece, so I ask about that specific piece, and so, even if it is correct - hopefully I will not telegraph to them that it is correct or incorrect. That's the hardest part.

Feedback was an important way that mathematicians communicated MDPs to their students. Understanding how students engage with and perceive teacher feedback is something that is relatively under-researched in education broadly (Weaver, 2006). However, Hattie and Timperley (2007) argue that “feedback is one of the most powerful influences on learning and achievement”(p. 81). It is an open question whether and how teachers’ written feedback supports students learning in advanced mathematics, and in particular, students’ learning of MDPs. However, Adam’s attempt to engage students in discourse around the mathematics is an existence proof that feedback is not limited to just written feedback on assessments.

Analytic scaffolding by providing a window into expert thinking. In sharing his own experiences as an advanced undergraduate mathematics student, Scott recalled coveting the intuitions that his professors held. He claimed that he did not struggle to follow the logic of mathematical arguments, rather “the whole idea on how on earth did you come up with that argument? Or whoever came up with that argument, what inspired them to come up with that augment? That's what I coveted.” When I asked, “How do you think students learn how to do algorithm design and leveraging examples?” he described that during classroom instruction he wants students to engage in problems “so they can actually get realistic experiences” because the mathematical intuition is developed. Specifically, in his teaching Scott shares his insights into the problems:

Scott: …Comes through experience and seeing the tricks of others, and hopefully professors try to open up the brain and ‘this is what I was thinking’. And don’t just give you the finished product. I give them as much insight into, what was I thinking.

Scott views that giving students opportunities to engage with expert thinking can support their learning to develop mathematical intuition. This provides further evidence that mathematicians coordinate their own experiences learning to engage in MDPs with how they approach teaching their students.
James shared that if students in his advanced courses ask about homework problems he “presents” the answer to the class and “explain the false-starts, and the examples that they can do. I try and explain like a realistic thought process rather than just writing down the answer on the board.” He explains that he wants students to “see that I didn't just memorize the answer and the answer takes work. And you don't know what to think at the beginning until you mess around - I want that to all be normal.” Other times James constructs the proof starting using the students’ ideas instead of his own thought processes. He says:

James: I write this bubble on the board and I just write whatever the students say in my scratch work bubble. It’s supposed to be what you'd write down before you know what the answer is. And outside of I'll write the formal proof based on the scratch work.

James reported that he provides the analytic scaffold to structure learning about constructing a rigorous proof using a ‘scratch work bubble’ for his students when presenting proofs from homework sets. Both James and Scott reported using analytic scaffolds by explaining their own ideas about the mathematicians to their students with the goal of supporting them in developing mathematical intuition for mathematics that does not have a straightforward solution.

**Analytic scaffolding MDPs via demonstrations.** Another way that mathematicians reported teaching students MDPs was through demonstrating the mathematical work to solve a problem during class, which is another way to provide insights into expert thinking. In these examples, I highlight the importance of the mathematical representations in these demonstrations because they are an essential component of learning and doing mathematics, and “people have access to mathematical ideas only through the representations of those ideas” (National Research Council, 2001, p. 94). The following excerpts show the ways in which mathematicians used analytics scaffolding in their demonstrations as they connected mathematical representations, which was an MDP that seven mathematicians in this study described in their research. In this excerpt, Anthony described how he uses technology to highlight the relationships between mathematical representations in his applied courses. His response was a result of asking how he supports students in learning the practices he engaged in his research (e.g., proof, utilizing technology, and modeling).

Anthony: In applied mathematics, we learn it in the classroom, a lot of it. I bring up the computer every other day. And I tell them, this is what the solution should look like, what does it say about that interaction we are talking about? That kind of
interplay - About different ways to approach understanding concepts, analytically, numerically, graphically... Multiple representations of the same issue.

Robert also described why he brings forward the idea of multiple representations in his advanced courses. When I asked him to describe a mathematical task that communicates MDPs in his discrete course Robert described that problems related to equivalence relations and partitions are helpful for communicating MDPs because “not every topic in math has multiple representations, but when it’s there it’s a great way to coax people into thinking a little more deeply about a topic”.

I bring forward Anthony and Robert’s excerpts to highlight how mathematicians bring forward the practice of leveraging mathematical representations through their demonstrations in class. Robert and Anthony described the value of using multiple representations as a way for students to think more deeply about mathematics concepts. Anthony demonstrated relationships between mathematical representations because it viewed it as supporting the learning of mathematical concepts, but it is not known whether and how students make sense of the practice of connecting mathematical representations and the analytic scaffolding to move between them given the fact that I did not observe Anthony’s teaching directly.

Collectively, mathematicians use various instructional practices to structure learning of MDPs in their advanced undergraduate mathematics course. In their teaching, mathematicians reported they structured their classrooms to support learning of MDPs through normalizing group work and promoting mathematical discourse. They also reported using various analytic scaffolds to support students learning of MDPs in their advanced undergraduate courses. Specifically, mathematicians provided oral and written feedback, carefully selected mathematical tasks, and provided a view in their expert thinking as ways to structure learning of MDPs for their students.

**Discussion**

The broad research question that framed this study was: *what MDPs do mathematicians discuss, either implicitly or explicitly, when explicating their research and teaching?* I answered this question by presenting three themes which provided deep insights on a range of MDPs for building on and creating new mathematics and insights into the MDPs, which are salient in mathematics research. Across the interviews, there was no indication that some MDPs are specific to particular research domains and quite the opposite. Mathematicians in this study engaged in a variety of pure and applied mathematics research and often drew on similar MDPs despite having very different research interests (e.g., CAPs practice). This result provides additional empirical
evidence that MDPs are interrelated and not mutually exclusive (Lockwood, DeJarnette, & Thomas, 2019).

In trying to understand the ways that MDPs are taught in advanced courses, the results indicated that mathematicians draw on general instructional practices in support of teaching MDPs, typically falling into structuring the classroom through social means (e.g., group work) or analytic means (e.g., feedback in discussions). It should be noted that the construct of scaffolding has generally been used when directly observing classroom teaching, and I used this to organize mathematicians’ self-report of teaching. Despite the fact scaffolding is used as a way to observe teaching directly, it was helpful for making sense of the instructional practices mathematicians reported using in their courses in service of teaching MDPs.

**Relating MDPs for Research and Teaching.**

Given that mathematicians reported that they use a wide variety of MDPs in their research, a natural question is how they might approach teaching this constellation of MDPs their own advanced courses. It is an open question whether and how mathematicians might coordinate between the MDPs for their research and instructional approaches, but there is some evidence from this study that indicates the possibility that mathematics might do this. Although it was not consistent across each mathematician interview and thus, was not considered a ‘theme’ I noticed similarities between mathematicians’ approach in teaching MDPs with how they engage in such practices themselves. What was apparent in this research of mathematicians’ reports on teaching is that, like Lockwood & Weber (2015), there is still more to understanding how to teach MDPs in advanced mathematics. I offer two classroom episodes to illustrate this point, but note that my study was not explicitly designed to explore whether and how mathematicians coordinate their use of MDPs in research with their teaching. However, these classroom episodes provide insights into how mathematicians’ research and teaching can be related.

**Classroom Episode 1: Robert.** Earlier, when Robert was asked to describe the MDPs which support his research in graph theory, he explained that he creates and organizes examples for the purpose of detecting patterns to gain traction on his problem at hand.

Robert: I'll take small enough examples where I can just draw a bunch of examples and try to organize them in a way that makes sense to me. And then, I usually get some ideas from patterns that I'm seeing. It helps me kind of test out baby ideas that I have to see whether they give some traction.
While observing Robert’s classroom during the case study portion of this study, I noticed that he offered a similar practice to his students. During this particular classroom episode, Robert was navigating a lively classroom discussion to demonstrate a solution to a combinatorics homework problem in his discrete class. After eliciting two unsuccessful student solution strategy attempts, Robert paused and offered the following anecdote:

Robert: Did anybody just try what happens when \( n = 1 \)? What happens when \( n = 2 \)? What happens when \( n = 3 \)? I’d like to just make a short pitch for that as a reasonable approach to problems when you don’t know what else to do. Plug in just a few small examples. Sometimes you will see a pattern… We haven’t solved the problem at that point. It's just like we would have a guess as to what the expression would be. It's not a waste of time to do that, because you might be able to then check that your pattern continues by using induction. Don’t view experimenting with small numbers as sort of beneath dignity or not worth doing. It is an important technique [emphasis added].

Following this, Robert asked students to try this solution strategy as a part of their in-class activities before resuming his demonstration of the solution. What I found to be interesting about this exchange was that it was reflective of how he described making progress on his research during the interview portion of the study. In this case, the analytic scaffold to demonstrate how students should approach solving the problem by testing small cases, which is similar to trying out smaller examples in his research. In both cases, the ability to detect a pattern is the result, and Robert notes that this kind of work is being mathematics and is a valued technique. It could be interpreted mathematicians coordinate their knowledge of MDPs used in their research with their instructional approaches in their advanced mathematics courses.

**Classroom Episode 2: Sophie.** To offer a second example of how mathematicians might coordinate their knowledge of MDPs for research with their instructional approaches, I first offer Sophie’s reflection on how she uses multiple mathematical representations to approach challenging proofs. Sophie’s response was the result of asking her to share the difficulties students faced when learning to use mathematical models in her numerical analysis course and provided some insight into her teaching.

Sophie: “There's no logical path to the conclusion that I can see - and I wrote down what the assumptions imply. And those problems are harder. Like that involves creativity… And that's really hard to teach.”

Juxtaposing the challenges in teaching MDPs related to modeling, Sophie went on to describe in more detail about how she expresses assumptions through multiple mathematical representations.
In particular, she finds that drawing visual representation is helpful for her gaining insight into a proof and how a proof can be constructed via mathematical representations.

Sophie: I think one of the things that helped me most is to be able to think about things symbolically, and then I always have to draw pictures. I think expressing the assumptions geometrically with a figure has always helped me. Maybe gain some insight into how this proof could possibly be true. Or how it could possibly be proved.

I offer this classroom episode to demonstrate how Sophie provided analytic scaffolding in her class, coordinating in an effort to describe how Sophie might coordinate her knowledge of MDPs for research, specifically leveraging multiple representations, with her instructional approaches in her numerical analysis course. In this episode, Sophie demonstrates for her students by making a graphical representation of the phase portrait for a Lotka-Voltera predator-prey model. She says, “If you actually wanted to draw a phase portrait, you’ll have to do it qualitatively or numerically. Okay, so things to think about. So we’ve got some parameters here...” Beginning with the parameters of the equations, Sophie describes the contextual representation given by the symbolic equations.

![Fig 2.1. Sophie’s whiteboard with symbolic equations.](image)

Sophie: Now, what I would do is look at the first equation. This is the rate of change of the prey population and it’s got these two terms involved. [pointing to the αx term]. This term is a little bit harder to interpret [pointing to the βxy-term]. But I want you to tell me what you think α represents. [short pause] I want you to think about it this way. If y is zero, so if there are no predators at all, this term is not here. So you just have the \( \dot{x} \) αx. What kind of situation would you be in?”
I omit the mathematical work within Sophie’s demonstration to move from the symbolic Lotka-Volterta equations to the phase portrait in Figure 2.1. The analytic scaffolding in the demonstration to move students from the symbolic equations to the phase portrait was provided to students by Sophie. During this demonstration, Sophie asked students to interpret changes in the parameters in terms of the predator/prey context. Students were expected to perform computations (e.g., test the stability of equilibrium points by evaluating the Jacobian matrix at those points) after being provided the analytic scaffolding to do such a task. For example, Sophie prompted, “tell me what you think $\alpha$ represents,” and after a short pause went on to say, “I want you to think about it this way.” I see this as evidence of scaffolding learning for students by eliciting facts and telling how students how orient themselves to the problem at hand.

While Sophie carefully selected the example to explore in class, her demonstration did not allow for student contributions or allow students to explore the content on their own (outside of computations they were directed to perform). Sophie’s demonstration included the idea of testing out changes in the parameters (much like Robert did in his work by testing smaller cases) for the purpose of identifying what those changes mean in the context of a predator-prey model. Eventually, the demonstration led to the sketch of the visual representation of the model in Figure 2.2 through the testing of different values of parameters analyzing those changes.

![Figure 2.2. Whiteboard with visual representation of solutions of the model.](image)

Given Lew, Fukawa-Connelly, Mejía-Ramos, and Weber’s (2016) findings that students might not understand the main points of what their professors are trying to convey, it is not known whether students in Sophie’s class understood the mathematics behind analytic scaffolding (and the methodologies of this study were not designed to study this). However, it does provide a second
example of how a mathematician might coordinate their knowledge of MDPs in their research with their instructional approaches in their advanced courses. In particular, Sophie’s demonstration to sketch a visual representation of the model aligns closely with how she approaches solving challenging problems in her research.

This study was not designed explicitly to study whether and how mathematicians coordinate their use of MDPs in research with their instructional approaches. However, the two classroom episodes of Sophie and Robert’s instruction shed some light into how it could possibly be the case that that mathematicians coordinate the MDPs used in research with their instructional practices advanced courses. It is also outside to scope of this study (and also not my desire) to evaluate effectiveness of mathematicians’ teaching in this study. My aim was to investigate what mathematicians report about teaching to provide a descriptive foundation of teaching MDPS in advanced mathematics. My analysis uncovered that the mathematicians in my sample appeal to general teaching strategies as they report on teaching. The general strategies included what policy in collegiate teaching call “high engagement strategies” such as group work and opportunities for students to ask questions. Not surprising, most mathematicians were focused on strategies that support students’ understanding of proving and proofs. This study contributes an initial taxonomy of strategies that this sample of mathematicians employ, answering Speer, Smith, & Horvath’s (2010) the call for more research on instructional practices in collegiate mathematics that Speer, Smith, & Horvath (2010)

I also make no claims whether employing general instructional practices is helpful or unhelpful in the context of teaching advanced mathematics. Furthermore, it is very likely that mathematicians’ general responses about their instruction are a result of not (yet) having the language for communicating about instruction in ways that a mathematics educator interested in teaching in undergraduate mathematics might. Furthermore, it is very likely that mathematicians’ general responses about their instruction are a result of not (yet) having the language for communicating about instruction in ways that a mathematics educator interested in teaching in undergraduate mathematics might. Why would a mathematician possess the language to communicate about teaching with so few opportunities for professional development focused on teaching? Selling (2016) made the point that K12 teachers will be challenged teaching math practices because they haven’t’ experienced them as learners. This work suggests that
mathematicians do not have normative ways of teaching MDPs, even though they have extensive experience working with MDPs and developing new mathematical knowledge with them.

National documents call for improving instruction, which poses a significant problem for academic mathematicians. They rarely have opportunities to engage in professional development around their instructional practice, and even fewer opportunities exist to improve instruction around teaching MDPs. Despite this conundrum, I see a glimmer of hope. If it is the case that mathematicians coordinate their use of MDPs with their instructional approaches in their advanced courses, it stands to reason that reframing learning in advanced mathematics is one way that could make teaching towards MDPs more accessible in teaching (later). Thus, changing early experiences is one avenue to explore in the effort to revitalize teaching and learning in advanced mathematics in the future.

The results also indicated that while mathematicians may have the same goal (e.g. put the pieces together to obtain mathematical results), they reported different MDPs to arrive to the same goal (e.g., leveraging examples versus modifying conditions CAPs practice). This result suggests that mathematicians deploy MDPs in different ways and for different purposes, and how MDPs are brought forward is context-dependent. As such, teaching MDPs to students is a significant challenge because different MDPs can be used to reach a single mathematical aim, and often multiple MDPs are brought to bear simultaneously. Much like mathematics educators have argued for bringing forward ways of thinking, and moving away from teaching rote mathematical procedures, a proactive approach to revitalizing teaching in advanced settings could benefit from avoiding teaching specific MDPs. Instead, instructional practices that support students to flexibly engage in multiple MDPs, even for a single mathematical aim, should be encouraged. It stands to reason that it would be helpful for students learning to more explicitly engage in the practices of academic mathematicians, but it could even change the ways that future mathematicians approach teaching MDPs.

Conclusions

The research aim of this paper was to understand the landscape of MDPs in the context of advanced mathematics from the perspective of mathematicians. In particular, I sought to understand the ways that MDPs are used in research and the ways that mathematicians approach teaching such practices in their advanced undergraduate courses. This was an exploratory study
the provided descriptive analyses of MDP use in research and teaching. The first theme of this manuscript outlined 25 MDPs, which were defined in my analysis of mathematicians’ responses around their research and descriptions of their teaching in advanced undergraduate courses. It is notable that many of the MDPs from the set of eight interviews are reflective of the current set of MDPs in the existing literature on mathematical practices. The purpose of this study was not designed to create an objective taxonomy of MDPs or identify new MDPs. Rather, this study was designed to understand better those MDPs which are used in the work of academic mathematicians’ research and teaching. In coding for MDPs, it was inevitable to create a taxonomy of MDPs codes (for those which were defined via the set of eight interviews in this study) which resulted in a significant number of MDPs which is difficult to comprehend, collectively. One of the difficulties of organizing a list of MDPs in a way to make it more comprehensible as a collection of practices is that because they are brought to bear in different ways across mathematical context, there is no one single categorization that represents some objective truth about how MDPs are related to each other. My study offers empirical evidence of how MDPs are interrelated (and not mutually exclusive). What is clear is that when one categorization strategy is selected, an argument could be made that a different categorization is possible. While this seems like a hopeless situation, it suggests that MDPs really are the ties that bind mathematics as a discipline together. While it was unexceptional (and some might interpret as unfortunate) that the list of MDPs defined in my analysis in this study was reflective of the existing literature base, it does provide corroborating evidence that the results around MDPs in my study are reasonable.

The second theme in this manuscript outlined the ways that mathematicians used MDPs in their research work to answer the question In what ways do they report on using MDPs in their research? Mathematicians highlighted a variety of ways that MDPs appear across a range of domains of mathematics research. The results also indicated that while mathematicians may have the same goal (e.g., put the pieces together to obtain mathematical results), they reported different MDPs to arrive to the same goal (e.g., leveraging examples versus modifying conditions CAPs practice). This result suggests that mathematicians use of MDPs is varied and likely context-dependent. As such, teaching MDPs to students must be difficult because different practices could be used to reach the same mathematical goal. Much like mathematics educators moved away from teaching rote mathematical procedures, it would be proactive to take a similar approach to teaching
MDPs. Specifically, it would make more sense to avoid teaching specific practices for particular domains of mathematics or classes of problems, but rather, but train students to flexibly engage in multiple practices for a single mathematical aim.

The third theme in this manuscript answered the question: *How do mathematicians perceive teaching MDPs to students in advanced mathematics?* In service of helping students learn MDPs, mathematicians employed various social and analytical scaffolds that were coordinated to their knowledge of content (MDPs). Furthermore, I suggested that mathematicians’ orientations towards teaching MDPs might be reflective of the way they experienced learning the practices themselves. Overall, the results of this study suggest that mathematicians’ strong knowledge of mathematics content and ways of working with such content via MDPs does not necessarily translate into being able to easily articulate the MDPs which support their research, or ways they teach such practices. Pragmatically, teachers make instructional decisions based on the instructional practices available to them and described general teaching practices likely because they have not developed the language of teaching in ways that mathematics educators and mathematics teacher educators might. If we are to revitalize teaching and learning in collegiate mathematics, mathematicians need opportunities for professional development, and I argue that these opportunities should include instructional practices that specifically in support of teaching MDPs.

**Limitations and Avenues of Future Research**

This manuscript has contributed to the literature in several ways. Firstly, this manuscript identified a list of MDPs and offered additional corroborating evidence of the nature of the mathematical work of research mathematicians. Second, this manuscript provided additional empirical evidence on mathematicians’ perspectives on the teaching and learning of in collegiate mathematics, and more specifically, the ways that mathematicians report teaching MDPs in advanced undergraduate mathematics. Although this study offered additional empirical evidence of MDPs in collegiate mathematics, because it was an exploratory study, the research findings are bounded in particular ways.

Firstly, the data in this study is the result of interviews with eight mathematicians and two case studies in two of those mathematicians’ advanced-undergraduate classrooms. The participants selected for this study do not fully represent every content domain in advanced mathematics. This
means that the results and subsequent findings may not generalize to all mathematicians who teach advanced mathematics. Contrary to this criticism, there is no reason not to believe that the results from case studies will not generalize beyond the cases studied (Maxwell, 2013). Pragmatically, the intention of this study was never to provide some objective truth about MDPs. Rather, this study was designed to offer one possible reality of how MDPs are used in advanced mathematics and ways that teaching might support students in leaning those practices.

A second limitation is that the analysis did not include a second coder, so it is possible that some MDPs were not captured within the interviews and case study data or other interpretations of larger themes within mathematician reports could have been found. For example, the categorization of MDPs offered in Table 2.4 organized the 25 practices into four categories. While these four larger categories described the ways that MDPs often appeared together across eight interviews, other researchers may find other ways to hang practices together based on other data sets. Currently, few conceptual models which describe how MDPs are connected exist. Future empirical studies could include research designs that explicitly study the ways that MDPs are interrelated. One avenue for this work that might be fruitful is drawing on social network methodologies to examine what MDPs coalesce. Furthermore, it would be interesting to see whether and how these MDPs hang together differently across various mathematical content domains. Understanding the ways that MDPs are connected could have implications for how to best approach teaching practices across grade levels.

A third limitation is that because this was an exploratory study, it was not able to identify instructional practices that best support student engagement and learning of MDPs. Understanding how MDPs are taken up by students as they engage in more advanced mathematics. As such, future research could examine and identify instructional practices that specifically target learning of MDPs and promote those practices in national documents for various audiences (e.g., mathematicians, mathematics educators, policymakers). In coordination with policy documents Professional Development opportunities for mathematicians to support taking up teaching practices specifically in support of MDPs.
References


CHAPTER 3 – Attending to Conditions, Assumptions, and Properties as a Mathematics Disciplinary Practice

Erin Glover

What does it mean to do mathematics and who gets to define this? Philosophers of mathematics have provided insights into the practices of the mathematics discipline from a historical and philosophical standpoint (i.e., Lakatos, 1976; Polya, 1957). Mathematicians have written autobiographical accounts on the nature of their own academic mathematical practice (e.g., Davis & Hersh 1982; Hardy, 1941; Schoenfeld, 1992). Mathematics educators have studied how mathematicians think about and engage in mathematics (e.g., Burton, 1999a, 1999b; Mason, Burton, & Stacey, 2011; Schoenfeld, 1985). Others have outlined mathematical ways of thinking and reasoning including habits of mind such as pattern-sniffing, experimenting, tinkering, and inventing (Cuoco, Goldenberg, & Mark, 1996; Schoenfeld, 1992). Much of this work provides insights into key ways of reasoning in the mathematics discipline.

Mathematical Practices

These ideas about reasoning and habits of mind central to mathematical practices have manifest in K12 and post-secondary educational research and policy documents for educators because practices are rooted within how the discipline is organized and how knowledge is developed ((National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA), 2010; RAND Mathematics Study Panel, 2003). Attention to mathematical practices in advanced mathematics has focused mostly on proof and problem-solving in the last thirty years. Though studies related to teaching and learning of mathematical practices such as algorithmatizing, defining, and symbolizing (e.g., Rasmussen, Wawro, & Zandieh, 2015) do appear in the literature in undergraduate mathematics educations. However, my review of the existing educational literature on mathematical practices in the context of advanced mathematics suggests that much of this work focused on explicating important issues related to teaching and learning of particular mathematical content. It is not clear whether these studies were focused on mathematical practices or the content itself.

Some of the work that does explicitly identify mathematical practices in research in undergraduate mathematics includes argumentation, symbolizing, and algorithmatizing
(Rasmussen, Zandieh, King, & Teppo, 2005), defining (Dawkins, 2014), use of examples in proving (Aricha-Metzer & Zaslavsky, 2017; Lockwood, Ellis, & Lynch, 2016; Lynch & Lockwood, 2017), computing (Lockwood, DeJarnette, & Thomas, 2019); justifying, modeling, and problem solving (Lockwood & Weber; 2015). There is a growing interest in mathematical practices in the context of undergraduate mathematics. And while educational researchers view mathematical practices as integral to learning and doing mathematics, I posit that few research studies explicitly treat mathematical practices as the focal object of study in the undergraduate context. I argue that mathematical practices are complex phenomena deserving of being a focal topic of study and which was the driving force of my study.

To attend to the paucity of research in undergraduate mathematics designed to study mathematical practices, I explicitly set out to study these practices by interviewing and observing the kinds of mathematical practices that were employed by mathematicians within their research and teaching. The research aim of this study is to characterize the entailments of central or common mathematical practices that mathematicians draw upon in their research and teaching. In this paper, I investigate the characteristics of the practice of mathematicians in which they attend to mathematical conditions, assumptions, and properties in their research and in teaching. I call this practice the Conditions, Assumptions, and Properties (CAPs) practice. The research questions which motivated this paper were: What are the characteristics of the CAPs practice? In what ways do mathematicians attend to conditions, assumptions, and properties in their research and in teaching? This manuscript identifies and outlines the ways in which mathematicians described attending to conditions, assumptions, and properties through four activities: identifying, coordinating, aligning, and modifying.

**Literature Review**

To understand the mathematics practices students should learn, I briefly review what K-12 mathematics literature says about MDPs as the foundation to my argument because few national documents exist which outline the role that mathematical practices play in advanced mathematics. For the past forty years, mathematics education policymakers and researchers have focused their attention on portraying mathematics as a compilation of content and processes. The National Council of Teachers of Mathematics (2000) provided standards documents highlighting processes that learners use to engage with mathematical content: problem solving, reasoning and proof,
representation, communication, and connection. The committee writing these standards suggested that these processes are essential to engaging with mathematics across K12. The five processes differ in how they are rooted in the discipline. For example, proof and problem-solving have clear connections to how the discipline is structured, while communication and connections are more generic and are processes found across many disciplines. The framing of mathematics process standards highlights the interconnected nature between the multiple activities learners might engage in as they do mathematics.

To further explicate the discipline, the National Research Council described mathematics as five intertwined strands of mathematical proficiency (procedural and conceptual knowledge, strategic competence, adaptive reasoning, and productive dispositions) to frame the nature of the content, ways of thinking and reasoning, and dispositional aspects of mathematical engagement, which make doing mathematics possible (NRC, 2001). While they do not specify “practices” of the discipline, I argue that these five strands illuminate the ways that mathematics is done via their attention to thinking and reasoning. For example, they identify adaptive reasoning as the capacity to engage in “logical thought, reflection, explanation, and justification” and strategic competence as the “ability to formulate, represent, and solve mathematical problems” (p. 116). In service of those proficiencies they outline procedural fluency and conceptual understanding as ways of understanding how mathematics is organized. Lastly, they highlight dispositional aspects of doing mathematics, which may not be strictly mathematical in nature but they argue that the “habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” is an important aspect of mathematical work (p. 116). It wasn’t until 2003 that “mathematical practices” were foregrounded K12 when the RAND mathematics study panel argued for research efforts around the teaching and learning of mathematical practices to improve the quality of mathematics education. They argued that research efforts should investigate mathematical practices to

1. Develop a fuller understanding of specific mathematical practices, including how they interact and how they matter in different mathematical domains.

2. Examine the use of these mathematical practices in different settings (e.g., practices that are used in various aspects of schooling, students’ out-of-school.
3. Investigate ways in which these specific mathematical practices can be developed in classrooms and the role these practices play as a component of a teacher’s mathematical resources (RAND, 2003, p. xix).

Significant research efforts to transform teaching and learning in K12 mathematics resulted in the Common Core State Standards for Mathematics outlined the mathematics concepts and procedures students need to know in addition to eight mathematical practices that K12 students should engage in while doing mathematics (NGA, 2010). The Standards for Mathematical Practice (SMPs) were designed for K12 and it is still an open question about whether and how the SMPs overlap with advanced mathematical practices (Rasmussen et al., 2015). One could argue that the SMPs are reflective of the discipline because the SMPs were designed by a group of mathematicians and mathematics educators. It is more likely that the SMPs are reflective of the discipline but are incomplete, as there are practices that mathematicians would consider to be practices that are not specifically named as SMPs in the Common Core documents (e.g., algorithmatizing, defining). As the RAND mathematics study panel suggested, how these mathematical practices get used outside of K12 school setting is still not well known.

Few documents or studies exist which outline the kinds of mathematical practices undergraduate or advanced mathematics students in mathematics should be learning. One such document was produced by the Mathematical Association of America’s (MAA) Committee on the Undergraduate Program in Mathematics (CUMP; Schumacher & Siegel, 2015) who offered a set of recommendations outlining the mathematical content and cognitive recommendations that are reflective of the mathematical habits of mind that students should learn. The last CUPM guide (MAA, 2015) outlined that undergraduate students should learn to:

- state problems carefully, articulate assumptions, understand the importance of precise definition, and reason logically to conclusions;
- identify and model essential features of a complex situation, modify models as necessary for tractability, and draw useful conclusions, assess the correctness of solutions, create and explore examples,
- carry out mathematical experiments, and devise and test conjectures, recognize and make mathematically rigorous arguments.
These cognitive outcomes are reminiscent of the proficiencies outlined by the K12 national documents and are certainly reflective of important practices of the discipline. However, the CUPM guide lacks specification of what these practices entail, the relationships between them, and how to approach teaching towards these cognitive recommendations. If mathematicians are expected to engender these important practices in their teaching to support students’ learning, there is a clear need to articulate what these practices entail in the post-secondary context.

**Conceptualizing Mathematical Practices**

There is no single agreed-upon definition of mathematical practices (Moschkovich, 2013) and it is even more difficult to define the boundaries of these practices (Lockwood et al., 2019). While Cobb, Stephan, McClain, and Gravemeijer (2001) coordinated cognitive and sociocultural perspectives to frame collective mathematical progress of a classroom community, Rasmussen, Wawro, and Zandieh (2015) expanded upon their framework to include *disciplinary practices* to more fully account for the activity in advanced mathematics students, as they are more explicitly engaging in the professional practices of the discipline. They defined disciplinary practices as “the ways in which mathematicians go about their profession” (Rasmussen et al., 2015, p. 264). I align myself with Rasmussen and colleagues, but I highlight that mathematical practices refer to the activities that are mathematical in nature and do not include professional activities such as committee work or student advising. I refer to these practices as *mathematics disciplinary practices* (MDPs). While not mathematical in nature, habits of mind such as creativity and aesthetics have been discussed at length, and while not “mathematical,” they are regarded as important aspects of *doing* mathematics (e.g., Alcock, 2010; Bass, 2011; Savic, Karakok, Tang, El Turkey, & Naccarato, 2017).

**Theoretical Perspective**

To understand the ways that mathematicians think about and use mathematical practices in their research and how they show up in teaching, I take a situative perspective that conceptualizes learning as both a cognitive and social process (Greeno, 1998). As students and mathematicians come to understand and use mathematical practices, they are engaged in social settings in which mathematical practices are ways of acting, thinking, and talking with mathematics (Frank, Kazemi & Batty, 2007; Moschkovich, 2007; Sfard, 2012). They are practices rooted within the mathematical discipline itself (ways that one engages in doing the work of the subject area) and
for how one learns the discipline (ways that students individually and collectively participate in solving problems). For mathematical ways of engaging in the work and learning the subject to be a “practice,” it has to have utility across problems and across sub-domains of the subject area. There needs to be a sense that the practice is normative because of the way mathematics is structured. Thus, mathematical practices serve to organize the ways of thinking, acting, and talking when doing mathematics across social spaces: in classrooms, in research presentations, and in mathematics seminars.

A situative perspective coordinates individuals’ conceptions and behaviors with normative social and broader historical ways of participating in settings. Coordinating the individual within the setting when considering learning is key. To make sense of the role of practices from this perspective, I consider practices as the normative ways individuals and communities go about doing the work of the discipline whether that the work of learning the discipline within a classroom or the work of research mathematician in the mathematician’s lab, office, or research symposium. This resonates with the work of Cobb, Stephan, McClain, and Gravemeijer (2003), which outlined the emergent perspective framework to analyze classroom teaching and learning that coordinates the individual and social aspects of engaging in classroom mathematical practices. However, this work was never intended to describe the emergence of normative ways of reasoning about disciplinary practices (Stephan & Cobb, 2003). From this work Rasmussen, Wawro, and Zandieh (2015) added the disciplinary practices to the interpretive framework to more fully account for the kinds of collective mathematical activity in undergraduate classrooms as they relate to the practices of the mathematics discipline. While I do not use the emergent perspective interpretive framework to make sense of my dataset because the primary source of data for this manuscript comes from interviews, it describes one possible methodology for studying MDPs at the classroom level.

Since I was interested in understanding the nature of MDPs in context to social settings, I was also influenced by Sfard’s (2008) notion of thinking as communicating. She defined thinking as “an individualized version of interpersonal communication” and communication as the “collectively performed patterned activity” which belong to some well-defined repertoire of actions that are communicational (p. 86). The repertoire of actions are the ways of acting, thinking, and talking that are cognitive routines employed when engaged in classroom mathematics or mathematical research. The cognitive routines of interest in this study are mathematical practices.
MDPs are also social routines with historical roots because they are ways that groups of learners and practicing mathematicians engage in the work of mathematics that for some MDPs have endured across centuries (e.g., proving). Because I view mathematics disciplinary practices as communication acts (i.e., cognitive routines) I designed my study to explore mathematicians' conceptions of MDPs through the ways they communicated their thinking.

**Methods**

Advanced undergraduate mathematics was selected as the context for this study because students in these courses are expected to learn and engage in the professional practices of the discipline. With this in mind, I conjectured that the mathematicians working with students in these courses would employ a range of mathematical disciplinary practices. To understand the MDPs mathematicians reported using in their research and the function those practices play across mathematical content domains, and how such practices are taught, I employed a two-pronged approach. I conducted semi-structured interviews with eight mathematicians to understand how MDPs are used in advanced mathematicians and how those MDPs are used in teaching MDPs. In particular, these interviews centered on what mathematicians reported about MDPs in the context of their research and how those same MDPs translate to their own teaching. I conducted case studies in two of these mathematicians’ advanced undergraduate courses to see how MDPs might emerge in these classroom contexts. Collectively, the interviews and case studies were intended to shed light on what MDPs mathematicians’ report using in their research, and specifically, how they perceive of teaching practices to students.

**Data Sources**

Approximately one-hour long, semi-structured interviews with eight mathematicians served as the primary source of data for this study. The shortest interview was 58 minutes and the longest interview was 83 minutes. The mathematicians were a convenience sample purposively selected for this study because they were employed as professors in mathematics departments, had earned a Ph.D. in mathematics, and taught advanced undergraduate mathematics courses within the last two years. As documented in Table 3.1, the mathematicians collectively had expertise in both applied and pure mathematics working in a range of mathematical domains; however, not all content domains are accounted for in this study. I interviewed each mathematician once using the
same interview protocol, but the nature of the open-ended interview questions allowed me to probe on what was reported and to link ideas across the protocol.

Due to the nature of this study being exploratory, I designed the study to survey across mathematical domains in which mathematicians would conduct research and teach advanced mathematics. I deployed my efforts to capture the range of MDPs used in research and teaching in these domains as reported by a sample of mathematicians. My aim was to address the limited understanding in the literature of mathematical practices in advanced mathematical thinking. As such, I interviewed each mathematician once resulting in 171 transcribed pages of interview data. I acknowledge that interviewing a mathematician once may be seen as a limitation of the study design in that I was not able to pursue multiple opportunities to engage with each mathematician and dig deeply into their work with MDPs. Future research would be warranted to take up this aim and could use this study as foundational to knowing what MDPs a sample of mathematicians working across domains report.

Table 3.1. Interview participants’ area of specialty and professor ranking

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Area of Specialty</th>
<th>Courses Taught</th>
<th>Professor Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam</td>
<td>Topology</td>
<td>Abstract Algebra, Topology</td>
<td>Late Career</td>
</tr>
<tr>
<td>Gloria</td>
<td>Abstract Algebra</td>
<td>Abstract Algebra</td>
<td>Late Career</td>
</tr>
<tr>
<td>Robert</td>
<td>Combinatorics</td>
<td>Group Theory, Discrete, Number Theory, Graph Theory</td>
<td>Tenured</td>
</tr>
<tr>
<td>Anthony</td>
<td>Control Theory</td>
<td>Group Theory, Analysis, Adv. Linear Algebra, ODE &amp; PDEs</td>
<td>Tenured</td>
</tr>
<tr>
<td>Sophie</td>
<td>Applied Numerical Methods</td>
<td>Numerical Analysis, ODE &amp; PDEs</td>
<td>Pre-Tenure</td>
</tr>
<tr>
<td>Ellen</td>
<td>Functional Analysis</td>
<td>Analysis</td>
<td>Tenured</td>
</tr>
<tr>
<td>Scott</td>
<td>Applied Numerical Methods</td>
<td>Numerical Analysis, ODE &amp; PDEs</td>
<td>Tenured</td>
</tr>
<tr>
<td>James</td>
<td>Number Theory</td>
<td>Number Theory, Abstract Algebra, Discrete</td>
<td>Pre-Tenure</td>
</tr>
</tbody>
</table>

**Interview Data.** The interviews served as the primary source of data to get a sense of the ways that MDPs are used in research and how those same practices are taught in advanced mathematics. After asking preliminary questions, I asked the following: *Can you briefly describe for me the kind of work you do as a research mathematician? What kinds of MDPs do you use to support you in that mathematical work you do? What do you do to try to help your students learn such practices?* Appendix A contains the full set of interview protocol prompts. This enabled me to get a sense for their knowledge of MDPs and time was allotted throughout the interview to
regularly ask clarifying and follow-up questions so that I could understand the descriptions of the mathematics that came forward in the interview. It was particularly important during the interview process to ensure that the interviewee and I developed a shared understanding of MDPs that was rooted in each mathematician’s research experiences, therefore each interview included a negotiation of the meaning of that term.

**Classroom observations.** After completing the mathematician interviews, I conducted case studies in two advanced mathematics courses taught by two of the interviewed mathematicians, which served as another source of data for this study. Robert and Sophie, a convenience sample selected from the eight interviewed mathematicians, were chosen because they were teaching an advanced undergraduate mathematics course at the time of data collection and were amiable to having a researcher in their classroom observing their teaching. I attended, observed, and video recorded seven class meetings across Robert’s combinatorics class and Sophie’s numerical analysis course.

Each class observation was approximately one hour in length and took place during the middle of the course so I could observe classroom interactions after classroom routines were well established. I used a camera on a tablet that was set up to follow the mathematician as they moved about the classroom and used five audio recorders throughout the classroom to capture teacher and student dialogue. During my observations, I was a passive participant observer and constructed fieldnote jottings that were later developed into fieldnotes (Emerson, Fretz, & Shaw, 2011). My fieldnotes were constructed chronologically and completed for each classroom observation. The fieldnotes included pictures of the classroom layout, images of mathematics problems recorded on the whiteboard to document the mathematical ideas that were displayed in the class, and any handouts provided to students. The fieldnotes also included details on the mathematical problems being discussed in class, summaries whole-class discussions, and descriptions of interactions between the students and teachers, if any. Finally, my fieldnotes included initial analytic notes and included reflective comments on teacher and students’ interactions that I conjectured related to MDPs unfolding in the classroom setting.

**Data Analysis**

The interview data were analyzed using the inductive thematic analysis (TA) procedure, as described by Braun and Clarke (2006). Broadly, the process of conducting a TA includes six
phases outlined by Braun and Clark (2006) include: familiarizing yourself with the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and lastly, producing the report. Using TA requires that three distinctions that need to be made. Firstly, whether I was going to and inductive versus deductive approach. To gain a birds-eye view of my dataset I did not use a pre-existing coding scheme and thus, I chose the inductive approach. Second, I had to decide whether my themes were going to be at the semantic (explicit) level or the latent (interpretive) level. The semantic level would take the surface meanings of mathematicians’ responses versus the interpretive level which describes the “underlying ideas, assumptions, and conceptualizations” to the theme (p. 84). I took the interpretive approach because I constructed meanings about mathematicians' talk about MDPs in their research and in their teaching. For example, when I coded for *Conditions, Assumptions, and Properties* across the data set, I did not code for conditions, assumptions, or properties separately. Instead, I took an interpretive approach and used their talk to describe a mathematical practice of attending to conditions, assumptions, and properties (the CAPs practice) as it related to their research. It is interpretive in the sense that there is no current body of work around such a practice, so my coding for the CAPs practices was a researcher interpretation. Lastly, I had to take an epistemological stance on whether I was taking a realist (essentialist) or constructionist perspective. Someone in the realist paradigm would argue that there is an objective truth behind what mathematicians said, and the themes would explicitly describe those assumed realities. I orient myself with the interpretivist epistemology because I used the mathematicians’ talk to develop themes about one possible reality of how mathematicians use MDPs and how they approach teaching such practices.

**Researcher Reflexivity.** I chose TA because I did not want to be constrained by existing theory on MDPs, but rather, to be open to the themes I defined via the process of TA. I go into further detail about how I made these methodological choices shortly, but first offer a statement of researcher reflexivity (e.g., Grey, 2017). How I approach my research is shaped by my identity, lived experiences, beliefs, and the culture in which I belong and influences all aspects of this work. As a mathematics education researcher that holds an advanced degree in mathematics interested in studying MDPs, I am not able to completely remove my research interpretations from my notions about MDPs based on these experiences and readings around MDPs. For example, many of the MDPs that emerged in the interviews were familiar to be because of my knowledge of the
existing literature and my use of such MDPs in my own mathematics and teaching mathematics. These prior experiences shaped how I approached my research more broadly, from the research questions, research design, analysis, and reporting of my research findings.

**Thematic analysis.** Braun and Clarke’s (2006) TA was a sensemaking methodological choice because it is “a method for identifying, analyzing, and reporting patterns [emphasis added]” (p. 79). Since TA is a method to analyze data and not beholden to any particular a priori framework or coding scheme, interpretations of mathematicians’ responses in the interviews are the foundation of the themes presented in this manuscript I chose the inductive analysis path because I wanted to avoid pre-existing coding frames that could constrain what I learn from my dataset due to pre-existing theoretical commitments. However, my prior experiences in mathematics and knowledge of the literature on mathematical practices shaped my understanding of MDPs and how I interpret the ways in which others talk about them.

Consistent with the first phase of TA, I began by familiarizing myself with the data set. This process included transcribing each interview, followed by reading (and rereading) each transcript before early rounds of coding. The data excerpts presented in this manuscript exclude false starts, pauses, and use ellipses to indicate the removal of a portion of participants’ utterances. These features of speech were included in the transcriptions. TA is an iterative process that “involves a constant moving back and forward between the entire data set, the coded extracts of data that you are analyzing, and the analysis of the data you are producing” (p. 86). I wrote descriptive summaries for each interview that served as a form of early analysis to aid in recalling ideas that could be important to investigate further. Within these summaries, I began to notice similarities and differences across mathematician interviews in terms of the MDPs used in their research how such practices are taught to their students. These early descriptive summaries helped me think about how I might develop a coding scheme for the interview data. Once these initial readings and summaries were completed, the transcripts were uploaded into the qualitative data analysis software program HyperResearch (ResearchWare, 2012) to begin the process of segmenting and coding the data set.

**Data segmenting and coding.** To avoid losing the context in which the data source emerged from I followed Braun and Clarke’s (2006) heuristic to keep some surrounding data to avoid this common criticism of TA. Finding boundaries for a codable unit was first given by the
larger structural codes, and then within holistic codes nested inside structural codes. The maximum size of a codable unit is a structural code that captured the protocol interview prompt and the mathematician response. If there were multiple exchanges around the same question the entire exchange would be included in the codable unit. A new codable unit would occur when the mathematician changed the topic, I asked a follow up question or probing question, or I asked a new protocol prompt. After the data was segmented the second phase of TA can begin. The second phase of TA is to generate a list of initial codes. I drew on Miles and Huberman’s (1994) *inductive coding technique* to code interview data into categories that were meaningful for understanding the ways that mathematicians engage in MDPs and their relationship to teaching and learning advanced mathematics courses.

I began coding by identifying structural codes. Structural codes were developed by reading through the data set that previously chunked into codable units. Structural codes supported reducing my data so that I could “examine comparable segments’ commonalities, differences, and relationships” (Saldaña, 2012, p. 84). Examples of structural codes I identified captured instances of mathematicians’ research, mathematicians’ descriptions of *teaching and learning*, and mathematicians’ use of *MDPs in work and in teaching*. These codes specifically reflect the structure of my interview protocol examined mathematicians’ use of MDPs in their research, how they teach students such MDPs, and a math task example they offered to illustrate their reporting on teaching MDPs. Table 3.2 outlines the structural codes and the interview protocol question(s) that align to each structural code. Appendix A outlines the complete interview protocol.

I used structural codes to organize analyses in a way that allowed me to examine data *across* mathematicians’ responses within any of the structural codes. Structural codes do not overlap with other structural codes. This led to the second level of data reduction, identifying *holistic codes*. For example, I noticed that there were different “actors” at play within the larger structural codes. I created holistic codes, that “capture the sense of the overall content and possible categories” (Saldaña, 2012, p. 142). For example, the structural code “Teaching and Learning” flagged all responses associated with how mathematicians learned MDPs, how a mathematician might approach teaching MDPs, and how students might learn MDPs. I selected the holistic codes to be able to differentiate the ‘actors’ that occurred within data identified within a structural code.
Categorizing subsections of these data were helpful for understanding how MDPs were reported on in mathematical research and teaching.

Table 3.2 Structural codes with associated interview protocol prompt

<table>
<thead>
<tr>
<th>Structural Code</th>
<th>Associated Interview Prompts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courses Taught</td>
<td>Can you remind me what [advanced undergraduate] classes you teach?</td>
</tr>
<tr>
<td></td>
<td>What are your goals for students taking those classes?</td>
</tr>
<tr>
<td></td>
<td>What challenges do you foresee in teaching those courses?</td>
</tr>
<tr>
<td>Research</td>
<td>Can you briefly describe for me the kind of work you do as a research mathematician?</td>
</tr>
<tr>
<td>MDP used in Work</td>
<td>What kinds of mathematics disciplinary practices do you use to support you in the mathematical work you do?</td>
</tr>
<tr>
<td>Math Example in Teaching</td>
<td>Please take a moment to think about a mathematical problem or topic in a discrete or other advanced math course that you feel communicates mathematical content but also allows you to address MDPs. Can you describe to me what the example, problem, or topic you were thinking about?</td>
</tr>
<tr>
<td>Teaching and Learning</td>
<td>How did you learn these MDPs?</td>
</tr>
<tr>
<td></td>
<td>How do you think students might learn these same MDPs</td>
</tr>
<tr>
<td></td>
<td>What are some things you do in your own teaching to help students learn this mathematical practice?</td>
</tr>
</tbody>
</table>

The utility of having holistic and structural codes meant I could analyze intersections between them, or I could join codes. These efforts were aimed to reduce the data and create larger themes. For example, because I was interested in how MDPs were taught in advanced mathematics, I attended to the ways that mathematicians spoke about teaching MDPs. It was evident that their reports on teaching fell into two broad categories: teaching in general and specifically teaching MDPs. It was pragmatic to distinguish between those two qualitatively different types of responses because my study is designed to explore teaching MDPs but any aspects of teaching can provide important insights into teaching in advanced mathematics more broadly. Examples of reports of teaching that were specifically related to MDPs (e.g., You can tell
student and try to slow them down. ‘Hey, just run back to the definitions.’) were coded with the holistic code “Teaching MDP”. Reports of teaching that were general in nature and not specific to MDPs (e.g., I think I give back feedback on homework that they resent sometimes.) were coded with the holistic code “Teaching General”. A complete list of holistic codes is described with operational definitions and examples in Table 3.3. The holistic codes were helpful to reduce the data into potential themes and preparing for the third layer of coding in which finer-grained codes would be applied to holistic codes to identify the MDPs that mathematicians used in their research and teaching.

Table 3.3 Holistic codes with operational definitions and example

<table>
<thead>
<tr>
<th>Holistic Code</th>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math Content</td>
<td>This code is used for instances in which particular mathematical content is discussed in relation to teaching and learning, or MDPs. This code CAN be double coded with fine-grained codes.</td>
<td>“…you can imagine if I have a set of size nine and I want to turn it into a commutative group...”</td>
</tr>
<tr>
<td>Social Practices</td>
<td>This code is used for instances in which mathematicians discussed in relation to doing mathematics that was not mathematical in nature (e.g., read literature) Responses related to doing mathematical work (e.g., collaboration with peers) that not doing mathematics.</td>
<td>“I like to have a coauthor that is more meticulous than I am.”</td>
</tr>
<tr>
<td>Mathematician Learning</td>
<td>This code is used for instances in which mathematicians described their learning of MDPs.</td>
<td>“…one of the things that helped me most is to be able to think about things like symbolically...”</td>
</tr>
<tr>
<td>Student Learning</td>
<td>This code is used for instances where responses related to the ways that students learn mathematical content and/or mathematical practices, goals for student learning, talking about student traits or abilities around content/practices.</td>
<td>“…it makes you realize that these students are not really secure in what they're writing down.”</td>
</tr>
<tr>
<td>Teaching General</td>
<td>This code is used to capture in which descriptions of teaching in general or teaching practices that may not be explicitly related to teaching particular mathematical content or practices.</td>
<td>“I think I give back feedback on homework that they resent sometimes.”</td>
</tr>
</tbody>
</table>
Teaching MDP

This code is used to capture instances of utterances describing teaching particular mathematical practices. Can be double coded with other fine-grained tags.

“But the feel of it – ‘how did you think about it? How did you think about it?.. How many think it’s this group?”

**Coding for MDPs.** Because my study was designed to explore MDPs, my next level of coding attends to the MDPs that appeared in mathematicians’ interviews. In coding for MDPs, I used in code labels that captured the words or phrases mathematicians used to describe the mathematical work in their research. For example, if a mathematician shared how they “developed a conjecture” in their work and I coded this as the practice of “conjecturing” because it includes the same root word that the mathematicians used, but MDPs are gerunds, I coded it as “conjecturing” to indicate that it is an aspect of ‘doing’ the work of mathematics. In the case of a mathematician describing an aspect of their research and not using the language from the existing literature, I took an interpretivist approach in coding. I note that that many of the MDPs codes identified in this study follow a similar naming and meanings to existing disciplinary practices in the literature base in undergraduate mathematics. For example, the codable unit with the phrase “I try to play with the symbols until I get the quantity to the revere side” was coded with the MDP “Syntactic Manipulation” to identify where a mathematician reported manipulating symbols, definitions, or facts; especially in proof. For example, I did not select “Syntactic Manipulation” as an a priori MDP code, but upon analyzing the dataset, recognized the practice from the exiting literature on syntactic proof production (Weber & Alcock, 2004). Again, it was not the purpose of this study to provide new definitions to previously established MDPs. An MDP code would only be applied a single time within any holistic code, even if the MDP was mentioned several times within a holistic code, maintain the unit of analysis. Multiple MDP codes could be layered with other MDPs codes to indicate how MDPs appeared together.

The coding scheme was organized into a nested structure for the purpose of being able to analyze single codes across the dataset or intersections between codes within any codable unit. Specifically, MDP codes were nested within a single holistic code and holistic codes were nested inside larger structural codes. The three layers of codes allowed for an organization for systematically running reports in the qualitative data analysis software HyperReserch. The nested structure of codes allows for analyzing intersections between structural codes and holistic codes.
in and join together with other codes to create larger themes. A simplified example of this systematic process is with the holistic code of “Teaching MDP”. When running a report on the holistic code “Teaching MDP” returned every instance of a mathematician reporting on their teaching specifically related to MDPs across the dataset. To identify the range of MDPs used in teaching, I examined within the holistic code to find the MDPs that mathematicians reported using in their teaching. Because an MDP code was marked only once within a single holistic code, I was also able to run counts on the number and kind of MDPs used in across the set of codable units.

Coding for the CAPs practice. As an example of how MDPs were coded as practices follows. In early rounds of coding for MDPs, I had codes such as conditions, assumptions, and properties, which indicated mathematicians’ use of mathematical conditions, assumptions, and properties in their research or teaching. What I began to notice is that these initial in vivo codes often occurred within the same holistic codes. Through multiple passes of the data, I realized that because conditions, assumptions, and properties happened within the same holistic code, I was faced with an MDP which has not yet been explicated in the literature. The operational definition given for the Conditions, Assumptions, and Properties (CAPs) practice was “utterances about conditions, assumptions or properties in relation to mathematics content; imposing, modifying, or introducing conditions, assumptions, or properties.” With this operational definition in mind, I was able to determine what was an example of the practice and what would not be an example. For example, Sophie said, “Its linear hyperbolic which means I have all these tools for solving hyperbolic PDEs. If you pose it that way and recognize that that's what it is, then you can analyze it using sort of known practices.” This was coded with the “Conditions Assumptions and Properties” MDP code because Sophie shared how she selects particular mathematical tools based on the structural properties (being a linear-hyperbolic partial differential equation). Later, Sophie was again talking about mathematical tools for solving applied problems and said, “it could be when you look at it, it’s not something you've seen before, so you might have to not use a tool in your back pocket, but go find other tools.” While Sophie described two ways in which she uses mathematical tools, the latter was not coded with the “Conditions Assumptions and Properties” MDP code because she did not describe how she used conditions, assumptions, and properties.

Searching for and defining themes. The next phases of TA are to search for possible themes and define themes that appear across the data set. The interview data were coded
sufficiently when the emergent codes began to get smaller in grain size that they began to overlap with other codes and were often absorbed into a larger-grained code. Those larger codes were then combined to form themes. For example, the Conditions, Assumptions, and Properties practice (CAPs practice) MDP code originated from smaller, fine-grained codes, and I created a larger code CAPs MDP which also became a theme (because codes can be themes themselves). There is no procedure that tells a researcher when coding is complete, so I took a pragmatic approach and followed Creswell’s (2015) heuristic of coding all the data into about 30 to 50 codes and before identifying codes that overlap or are redundant to reduce the overall number of codes. Subsequent organization of those codes then turn into larger themes (and headings in the results of the manuscript). Braun and Clarke (2006) highlight that the unit of analysis in TA is the themes identified in the analysis, which represent a patterned response in the data. They also note that themes are codes, so to understand these patterned responses, I focused on the relationship between codes across the entire dataset, rather than the codes themselves. Specifically, I tried to understand how different codes could combine to create a larger theme and how codes might become sub-themes of these larger codes. I searched for both confirming and disconfirming evidence of possible themes.

In the process of defining themes, I grappled with whether a theme overlapped with another theme. If there was some overlap, I had to examine whether it could be a new theme or a supporting subtheme. For example, the MDPs defined through the processes of TA became the unit of analysis for the first theme in this manuscript. Taking MDPs as the unit of analysis resulted in exploring the ways that MDPs are aspects of a mathematician’s research. Knowing that MDPs are interrelated and not mutually exclusive, exploring these relationships was a natural outcome of taking MDPs as the unit of analysis. In exploring these relationships, I defined sub-themes which organized my MDPs codes around four categories to describe ways that MDPs appeared together in the mathematicians’ reports of their research: Formalizing and Proving, Mathematizing and Computing, Structuring Mathematics, and Other Supporting Practices. Thus, TA is an iterative process in which I often revisited and reviewed codes themes, and sub-themes, and reread transcripts to ensure I was able to interpret and convey the mathematicians’ perspectives on MDPs as it relates to their research and teaching.
Results

I report on the ways that mathematicians communicated about attending to conditions, assumptions, and properties with respect to their research and teaching for the purpose of defining the entailments of the CAPs practice. I do this by providing illustrative examples to the CAPs practice thorough two significant ideas. First, I give a working definition and features of the CAPs practice and provide examples of how mathematics described the features of the MDP in their own words. In this case, the CAPs practice is viewed as a localized practice when engaging in specific mathematical work like in the example in the introduction of proof and modeling. Second, I elucidate how the CAPs practice can also be used as a cross-cutting practice of the discipline because it can be used as a cognitive routine with wide applicability across various mathematical contexts. I am not suggesting that the CAPs practice should be conceptualized as a dichotomy. Instead, the CAPs practice can be deployed across a spectrum between more localized mathematical contexts and cross-cutting in the sense that it can be used as a general-orienting perspective towards mathematics across a variety of mathematical contexts, and across content domains.

The CAPs Practice: Features and Engagement

What follows is a definition of the CAPs practice to explicate what engaging in the CAPs practice entails. The CAPs MDP comprises an interrelated set of activities that involve attending to mathematical conditions, assumptions, and properties when engaged in a mathematical endeavor. I characterize the CAPs practice by four activities that the set of eight mathematicians described as they attended to conditions, assumptions, and properties in their descriptions of research and teaching. The four ways mathematicians described attending to conditions, assumptions, and properties is through identifying, coordinating, aligning, and modifying. Given this conceptualization of the practice, an individual (or group) is engaged in the CAPs practice if they are engaged in one or more of these activities. However, as my results will reveal, these activities are interrelated and engaging in one of the activities often requires (re)engaging in another activity. This definition was an outcome of my analysis, but I present the definition upfront and present corroborating evidence from the interviews to provide context to the definition in the results with discussion. Thus, my results will be organized around how I have conceptualized the
CAPs practice and will focus on the four defining activities that a person (or group) could engage in as they attend to conditions, assumptions, and properties in a mathematical endeavor.

**Defining characteristics of the CAPs practice.** I outline activities that can occur when attending to conditions, assumptions, or properties during a mathematical endeavor. A necessary activity of the CAPs practice is to identify the conditions, assumptions, and properties that are relevant in a given mathematical situation. The activity of identifying conditions, assumptions, and properties could include acknowledging the assumptions that are needed in a particular mathematical situation or attending to the conditions that need to be met. Identifying could also include attending to the properties that the involved mathematical objects possess or need to possess. Any subsequent activity in which mathematicians attend to conditions, assumptions, and properties requires that they first engage in identifying, but any further activities can happen iteratively and in any order.

A second activity of the CAPs practice is coordinating conditions, assumptions, and properties with one another. Coordinating conditions, assumptions, and properties include attending to whether and how the underlying assumptions give rise to conditions that are to be satisfied or properties that need to be attributed to relevant mathematical objects. Coordinating can also include attending to how the conditions, assumptions, and properties are related logically (e.g., which conditions are independent, and which are consequences of others). Coordinating can also include attending to which properties are given and which properties are to be established in a mathematical situation.

A third activity included in the CAPs practice is aligning conditions, assumptions, or properties with the techniques and tools that can be deployed to address the task at hand. Aligning conditions, assumptions, and properties with appropriate techniques and tools will typically require coordinating them correctly. For example, when two conditions linked by “or” this admits a different set of possible proof techniques than if they are linked by “if…then.” Aligning conditions, assumptions, and properties would include attending to what proof techniques are appropriate given the conditions that need to be satisfied. Aligning includes attending to the algorithms which are appropriate given the assumptions that have been made. Aligning could also include attending to what kinds of mathematical models are appropriate or give the best fit in light of the conditions, assumptions, and properties.
A fourth activity included in the CAPs practice is *modifying* conditions, assumptions, or properties in a mathematical endeavor. Modifications can include eliminating (or disregarding), weakening, or strengthening any of the conditions, assumptions, or properties involved in a mathematical endeavor. Engaging in the *modifying* activity can include attending to whether a stronger property can be proving or whether an existing property can be proven using weaker conditions. Modifying is a relevant activity of the CAPs practices because it also includes attending to how the complexity of a mathematical model changes in light of adding or removing assumptions. Modifying also includes attending to how the difficulty of a problem changes when conditions are added or removed. *Modifying* conditions, assumptions, or properties will invariably necessitate re-aligning and re-coordinating, highlighting that the CAPs practice is not (necessarily) a linear or hierarchical process.

The CAPs practice minimally must include the activity of *identifying* but can also draw on one or more of the remaining three activities: *coordinating*, *aligning*, or *modifying*. What follows is an elaboration of the four activities of the CAPs practice shown in Figure 3.1.

![Figure 3.1. Activities of the CAPs Practice](image)

**Identifying.** The foundational activity of the CAPs practice is *identifying* the conditions, assumptions, and the mathematical properties at play. Mathematicians across their disciplinary specialties regarded *identifying* the conditions, assumptions, and properties in light of definitions and theorems and was an essential aspect of proving mathematical results. Ellen described the aspect of identifying in the CAPs practice as it relates to proving statements in her introductory analysis. Specifically, she described the fundamental need to identify the mathematical properties
given by definitions and theorems when I asked her to describe the difficulties that students have in learning MDPs.

Ellen: Today we needed to prove that if a function is continuous on a closed and bounded interval then its integrable. Then we needed to use extreme value theorem, intermediate value theorem, uniform continuity… What do we know about that? … If you don't have that information in your head, you're not going to prove the result.

Ellen highlighted that students must identify the properties at play that are given by the theorems to be able to construct a proof about integration on function that is continuous on a closed and bounded interval. Sophie that the work of identifying conditions, assumptions, and properties is particularly challenging for students in her applied numerical methods course. She described that her students have a hard time translating assumptions into mathematical statements to be proven.

Sophie: Assume this function is continuous, then prove this. Well, if it's continuous, what does that mean? I think that they just lack experience… They don't know how to translate assumptions into mathematical statements.

Sophie’s excerpt illustrated that students in her class struggle to identify the properties given by the assumption of a function being continuous. Both Sophie and Ellen described the ways in which students struggle with writing down mathematical properties given by a definition or theorem for the purposes of strategically translating those properties to construct a proof. I interpret that their statements suggest that learning to identify conditions, assumptions, and properties is an important practice that students in advanced-undergraduate courses should be learning. When I asked Sophie to describe how she helps students translate assumptions into mathematical statements, she shared what often happens in her office hours when students are struggling with constructing a proof in her numerical analysis course:

Sophie: “What I tend to do is look at the result of the problem statement and look at the assumption again. If your assumption is that this function is continuously differentiable - what does that imply? What's the definition of continuity? I ask them that and still takes them a while to write down – ‘If it’s continuously differentiable then for all epsilon…”

The ability to identify conditions, assumptions, and properties is a central feature of the CAPs practice because it allows one to engage in other activities of the CAP practice. In the context of the practice of proving both Ellen and Sophie provided evidence of challenges in translating assumptions (and interpreting the mathematical properties gained by those assumptions) into mathematical statements for the goal of constructing a proof was the ability to identify the
conditions, assumptions, and properties. The next section will expand on this idea and provide evidence of how mathematicians are adept at identifying conditions, assumptions, and properties and strategically leveraging conditions assumptions and properties by coordinating them as they engage in research and the importance of these activities in their teaching.

*Coordinating.* The second way that one can attend to conditions, assumptions, and properties are to coordinate them with one another. *Coordinating* conditions, assumptions, and properties allow a mathematician to reveal underlying assumptions that give rise to conditions that must be satisfied or properties that need to be attributed to relevant objects. I offer Adam’s description of his research as an example of how attending to conditions, assumptions, and properties by *coordinating* in the context of topology. He first *identifies* the conditions of the topological space (given by its definition) and then coordinates those conditions to be able to *identify* and associate those conditions to an algebraic structure.

Adam: You know what a group presentation is, right? Associate to a group presentation is a topological space. For each one of the generators, you create a loop, and each one of the relations you sew on a disk. That's a topological space and it turns out that you can get the group back algebraically, so the fundamental group of that space is going to be the group that the presentation represents. So, you have a presentation of a group. Can you identify the group? Suppose you had the same group and you have two different presentations - How are they related to one another? Are the spaces homeomorphic, or are the spaces homotopic equivalent to one another?

Adam’s excerpt describes the mathematical work of coordinating the algebraic properties of a group with the properties of a topological space. Specifically, Adam associates the group presentation to its topological space by constructing a one-to-one mapping between each of the generators of the algebraic group with a loop of the topological space.

*Aligning.* A third way that one can engage in the CAPs practice is *aligning* conditions, assumptions, or properties with the techniques and tools that can be deployed to address the task at hand. By aligning conditions, assumptions, and properties mathematicians can uncover what proof techniques could be appropriate given the statements they are trying to prove, what algorithms could be used given particular assumptions, or selecting a type of mathematical model in light of the conditions, assumptions, or properties. Mathematicians shared that identifying the conditions, assumptions, and properties in a mathematical situation gave them perspective on how to align those with the kind of mathematical techniques or tools could be brought to bear.
Half of the mathematicians described that they coordinate properties of the mathematical object(s) of interest and align those with possible mathematical tools and techniques. This is also a practice that mathematicians want students to engage in as well. Adam described how he encourages students in his group theory courses to attend to the mathematical properties so they may determine what techniques that could be brought to bear on the problem at hand. Adam asks questions like “What is it about a multiplication that allows you to use the commutative property?” In order to help his students to understand how to engage in the aspect of the CAPs practice that requires structural thinking he says, “I would actually push different ways of doing a problem and elucidate what techniques you could use, what structures were you taking advantage of.” Adam describes how, in his teaching, he advances the idea of examining the structures to identify which mathematical techniques might be useful.

Seven mathematicians identified aligning as an important aspect of their research. Scott, interested in the accumulation of errors of particular numerical methods, shared how he coordinates mathematical properties given by the structure of an existing mathematical model to help select particular methods of analysis. He said, “Typically, I work on existing model and just say if... the structure looks like this mathematically - use these kinds of methods and you can expect this kind of error.” Beyond simply identifying the properties and selecting methods, Scott aligns the properties of particular numerical methods to the characteristic errors that arise from those methods, which is central to making advances in his research. Sophie, also interested in numerical methods, describes how she uses the conditions of a “problem” to determine her approach to analyzing a situation using a mathematical model.

Sophie: In doing research, I realize that the type of problem is linear hyperbolic which means I have all these tools for solving hyperbolic PDEs. And if you pose it that way and recognize that that's what it is, then you can analyze it using sort of known practices... It could be when you look at it it's not something you've seen before. You might have to maybe not use a tool in your back pocket, but go find other tools.

Sophie uses methods that are already known to her, but also may have to go find other tools. Five of the mathematicians interviewed made some connection between the conditions and properties of a mathematical situation and the selection of a particular method or tool as one aspect of their research. The mathematicians in this study reported that they align the conditions and assumptions to help select particular methods for solving the problem.
**Modifying.** Half of the mathematicians interviewed shared how they attended to conditions assumptions and properties by modifying them in support of developing insights into their proving. Anthony, in describing his research around scheduling streetlights to maximize urban traffic, he first begins by reducing the assumptions down to a city with only a few streets to start with a smaller problem. He will then run a simulation modeling traffic lights the smaller city with fewer streets before scaling up to something more reflective of the urban setting.

Anthony: We don't want to wait more than anybody else in line at the streetlight… And I discuss with, of course, applied people because they need to tell me if this makes any sense. And then, I write a big MATLAB simulation for a simple urban collection of three or four or five streets to see how this would play out.

In terms of solving a more complex problem, Anthony described that reducing the conditions and assumptions allows him to write a program to simulate what properties arise from the conditions before he attempts to simulate traffic flow in the complete urban setting. Other mathematicians reported engaging in similar practices, suggesting that the CAPs practice is not unique to a particular content domain. Thus, the CAPs practice is used by the discipline more broadly.

In the following excerpts, Sophie and Scott also describe how they translate down to the most straightforward conditions in order to solve a simpler problem before scaling back up to the more complex problem at hand. Sophie emphasizes, “When you're trying to solve the hardest problem and it’s going wrong and you have no insight into why” she outlined that she modifies the condition and assumption to make progress on difficult problems. In this case, she translated her applied problem from a 2-dimensional problem to a 1-dimensional or 0-dimensional problem in order to solve it in the easier case. Only once Sophie solved her problem in the lower-dimensional cases, was she able to move back up to the more complicated situation in which her problem originated.
Sophie: You have to simply if down to a simpler analog of the same problem and analyze that. If that's still too hard to analyze, you simplify it again. I ended up going from a two-dimensional problem down to a zero-dimensional problem and analyzing it there. He taught me to translate everything down the simplest version it and build it back up from there. If you can solve it on that level, you complexify [sic] it a little bit, solve it there. If you can analyze it in zero or one-dimension and say, this is the mathematical statement of the problem in zero dimensions. Then you have confidence.

In terms of advancing students from computation to proof in his numerical analysis course, Scott summarized the process he uses to help students learn to prove statements by first selecting examples to understand what a theorem is actually conveying mathematically.

Scott: They're starting to move from compute this to compute that, to prove this, disprove that. There is this theorem and I'm not even going to allow you to try and prove it first. Let's come up with some examples. They usually say, if this is so, then this is so. Well, let's just start with some examples where this is so. And something small enough that we can talk about but not completely trivial that you could make any claim... You always make something as simple as possible, but not simpler. Then it puts you in a place to actually make that more abstract statement.

Scott suggests that he wants students to learn how to explore a mathematical statement of a theorem in more general terms by drawing on various examples that have ‘simpler’ conditions and assumptions. He also notes that there is a delicate balance to selecting appropriate examples because he wants students to select examples that are not so simple they become mathematically trivial, and thus, do not provide insight into proving the theorem. In this case, Scott highlights the activity of modifying as an aspect of what he would like students to learn as they attempt to make sense of a theorem.

In order to make more abstract statements, mathematicians described how they relaxed the conditions and assumptions to work a simpler problem, thereby modifying the CAPs of the original problem at hand, which then necessitates re-coordinating and re-aligning conditions, assumptions and properties. Ellen described how she modifies the properties of the mathematical objects that are a ‘parallel’ to the properties of the set of function operators in her research.
Ellen: So, you only have an idea - you want to obtain this - there are certain steps and conditions that you need to have in place. Maybe can I develop something parallel to that that will work with my set of operators? I wanted to keep them. I didn't want to modify them. But then, what else can I do with those types of objects? Can I modify this little bit and see if I can produce more things that will work with this? Basically, you know what you want to obtain. Can I get there? What do I need to do in between?

Developing things in ‘parallel’ would require that Ellen coordinates the properties across mathematical objects in such a way that they satisfy some of the conditions that will work with her sets of operators. Ellen describes that in working to align these objects, she will make small modifications to conditions or assumptions help in her make progress, which is also something that other mathematicians shared. When faced with a conjecture in which there is no obvious way to align the assumptions of the statement to the conclusion he is after, one of Robert's habits is to add additional assumptions to the conjecture. While this changes what he is now actually proving, it gives him some perspective on the larger problem at hand.

Robert: Proof. That's really what it's geared towards. Giving me an idea of what I'm even trying to prove. I don't know how to prove the conjecture. I don't know how to get from the assumption of the conjecture to the conclusion of it. So, instead, I'm throwing in little extra assumptions in there and getting to a conclusion. I've retooled the original conjecture into a narrower mathematical claim which still is relevant, and which I think I have a better chance of attacking. It doesn't successfully resolve the major question that was driving all of this, but it's a true statement and handles some cases of it.”

Robert provides an example of modifying because he changes the assumptions of the problem to prove a narrower claim, and by doing so, Robert gained insight into the larger problem at hand. By solving an easier problem, he makes progress on the larger problem and simultaneously reaps the benefit of handing a subset of the total cases. The act of modifying conditions and assumptions is one entailment of the CAPs practice, which is useful for doing research, as described by Ellen and Robert. However, mathematicians also described that modifying conditions, assumptions, and properties was also a useful pedagogical tool.

In the following excerpt, Anthony who is very attuned to the logical structure and precision needed for rigorous proof, described how he strategically selects examples to highlight slight variations on conditions and assumptions. He described that, “more than half the course is devoted
to just logical structure of arguments and how to prove logical statements” in his introduction to proof course.

Anthony: I first present some examples. Usually, you start with examples. Depending on the level, less theory quickly, and just some basic definitions and illustrative examples. Counterexamples, examples - situations that looks like we are in there - but not quite. There is something that is missing. This is not quite the same as before. Oh, and then you don't expect the same conclusion. You shouldn’t. The proofs are completely different.”

This is relevant to the CAPs practice because Anthony carefully selects examples to bring forward to the class, which illuminates the relationship between the logical structures of arguments and proof. And in particular, that these modifications on the conditions given by a mathematical statement have consequences for the structure of the proof.

Ellen shared that students in her introductory analysis course often struggle with the aspect of modifying conditions, assumptions, and properties. She explicates that students that are used to procedural computations (for example, in calculus) often struggle to understand how to coordinate the conditions of a hypothesis and with the properties required to get to the conclusion in the context of proving mathematical statements.

Ellen: The goal is to learn how to prove things in mathematics. What a mathematician does is that. To understand why this structure works. How you can get conclusions. How to extend the result. How to change the hypothesis. Can we reduce the hypothesis? In analysis especially you can impose many different levels of conditions… The first thing I tell the students is, okay, this is a theorem. Why do we need these conditions? Could we have an example in which this fails? And do you get a result? If you want to obtain a result from something else there are conditions that you need to have to validate the conclusion.

Ellen highlights the transition to formal, rigorous mathematics that is particular to advanced-undergraduate mathematics courses is difficult for her students. One aspect of this difficulty could be their lack of experience in engaging in the CAPs practices, or even understanding how it could be helpful.

Both Ellen and Anthony described the finer-grained activity of modifying that they leverage in their teaching by demonstrating how minor variations of conditions or assumptions of a hypothesis can have important mathematical consequences. Anthony described via examples how these variations impact the proof structure, whereas Ellen described how she wants students
to see what happens when conditions are imposed or whether conditions of a hypothesis can be reduced.

**Conditions, Assumptions, and Properties across Content Domains**

The previous section outlined ways that the CAPs practice can be used to do specific fine-grained mathematical work within in service of research and in teaching. I transition to the ways that the CAPs practice is a cross-cutting practice. As I indicated in my theoretical perspective, the mathematical ways of engaging in the work and learning the subject to be a “practice,” it has to have utility across problems and across sub-domains of the subject area. As such, I will organize this section around ways the CAPs practice translates across mathematical contexts. I will describe how the CAPs practice can be used between two related mathematical content domains. Then I will describe how the CAPs practice can be used to translate between a real-world context and a mathematical domain, describing the relationship between CAPs and mathematical modeling.

The ways that mathematicians described attending to conditions, assumptions, and properties communicated how they use the CAPs practice as a way to orient themselves in a larger mathematical problem, but they also shared how the CAPs practice could be a general-orienting perspective for working across different contexts that are supported by mathematical ways of thinking and reasoning. My goal is to provide further evidence that the CAPs practice is a cognitive routine that provides a general-orienting perspective to approaching mathematical problems across various contexts.

**CAPs practice: Translating between related content domains.** I noticed that Gloria and James discussed how they translate between two related mathematical content domains in their research. In light of their descriptions of their research, I hypothesized that because mathematics is a connected subject, there might be a natural correspondence between conditions, assumptions, and properties across related mathematical content domains. When describing how she approaches making new mathematics in abstract algebra, Gloria often translates her problem from the domain of abstract algebra back down to linear algebra.
Gloria: Well, you always want to try to look at your problem or reduce your problem to linear algebra... How can I look at this question in terms of linear algebra? It’s mostly when you can get it down to linear algebra, then you can use the linear algebra tools.

Given that abstract algebra is a generalization and abstraction of linear algebra, Gloria provided some evidence how the CAPs practice could be used in attending to conditions, assumptions, and properties between the two related mathematical contexts. In particular, she talks about how she “looks” at her problem in a different way, which would require attending to the conditions and properties from the abstract algebra domain and coordinate those properties with the linear algebra domain. Moreover, she does this in order to use the ‘tools’ from linear algebra. However, Gloria’s example of coordinating between two related content domains does not provide explicit evidence of how she attends to the mathematical conditions and properties behind this work. I turn to James to shed light on how a mathematician can attend to conditions, assumptions, and properties across related mathematical domains. In the discussion of his research, James portrays how he aligns conditions and properties between number theory concepts (dynamical systems) and group theory concepts (symmetric groups) in his research.

James: If we can prove things about the wreath products then we can prove things about this map... Knowing that this family of Galois groups are all wreath products is something specific to people who do dynamical systems. So, our training as number theorists helped us in this dynamical system question. Then we just had to study the wreath products really hard. And so now you start studying fixed points of $S_3$ instead of fixed points of the wreath product.

This excerpt illustrates how James translates between the number theory and dynamical systems subdomains and the group theory domain. Specifically, he aligns the mathematical properties between $S_3$ and wreath products. It was more efficient to work with the fixed points of the symmetric group than the wreath products themselves. Thus, an important aspect of James’ research how to coordinate and align properties between these two, structurally related subdomains. I take this as one aspect of the CAPs practice because he actively seeks to find more efficient ways to do his research by way of relating conditions, assumptions, and properties across domains.

On the other hand, Adam, a trained topologist, explained how he translates between the domain of topology and the domain of abstract algebra. Revisiting his excerpt from earlier, Adam highlighted how he coordinates a topological space with its respective algebraic group presentation
in his research. He first identifies the conditions of a group presentation of a topological space (given by its definition) and then coordinates those conditions with the algebraic properties of a group.

Adam: … Associate to a group presentation is a topological space. For each one of the generators, you create a loop, and each one of the relations you sew on a disk. That's a topological space and it turns out that you can get the group back algebraically, so the fundamental group of that space is going to be the group that the presentation represents… Are the spaces homeomorphic, or are the spaces homotopic equivalent to one another?… what spaces are associated to it?

Adam is described how he coordinates the algebraic properties of a group with the properties of a topological space. Specifically, to coordinate between the generators of the group with a loop of the topological space. This illustrates how he uses the CAPs practice to navigate between two related mathematical domains, as evidenced by identifying and coordinating between the structural properties of an algebraic group (its generators) and the respective topological spaces (discs).

While Gloria, James, and Adam do not explicitly use the words ‘conditions’, ‘assumptions’, or ‘properties’ to describe the ways they identify, coordinate, align, or modify between two related content domains, it appears that in order for them to make progress in their research, they identify, coordinate, align, or modify conditions, assumptions, and properties, as a ways of thinking about navigating between two different sub-disciplines. Because mathematics is a highly connected subject which has been developed historically, over time, and builds upon previous work, it makes intuitive sense that coordinating conditions, assumptions, and properties across these two different contexts. This is evidenced by an entire sub-field of mathematics devoted to using tools from algebra in topology (algebraic topology). Adam’s excerpt highlights how the CAPs practice is useful thinking about how to translate between mathematical sub-disciplines and is a general-orienting perspective to do mathematical work across related mathematical content sub-domains.

CAPs translate between the real world and mathematical domains: The case of modeling. One of the ways in which mathematicians attended to conditions, assumptions, and properties is during the process of mathematical modeling. While there are many perspectives on modeling in mathematics and no single way that it is defined (Abassian, Safi, Bush, & Bostic, 2019), informally mathematical modeling is an act in which mathematics is used to describe,
analyze, and make predictions about real-world phenomena. In order to describe how CAPs are used in the context of mathematical modeling, I offer Zbiek and Connor’s (2006) description of the modeling process. They elaborated on places where learning might occur during the processes and sub-processes of generating and validating mathematical models, as described in Figure 3.2.

Figure 3.2. Modeling process diagram from Zbiek and Connor (2006).

I focus on a few key constructs from their framework with the goal to describe the entailments of CAPs practice within the disciplinary practice of mathematical modeling. Zbiek and Connor described specifying as identifying the conditions and assumptions from the real-world context that will be used in the resulting mathematical model. They describe mathematizing when “a modeler creates or acknowledges mathematical properties and parameters (P & P) that correspond to the situational conditions and assumptions that have been specified” which represents the mathematical work to move across “the bridge” between the real-world situation and the mathematical world of the model (p. 99).
Because I take a situative perspective, the bridge is not treated a noun, but rather a practice that allows a modeler to coordinate and align conditions, assumptions, and properties between the real-world phenomena and the domain of mathematics. The mathematical work that describes movement across “the bridge” aligns with how I conceptualize the CAPs practice, which is a vital aspect of mathematical modeling.

Three mathematicians discussed how developing or using mathematical models was an important aspect of their research and teaching. The interview excerpts presented here are intended to show how different aspects of the CAPs practice which appears in the modeling process, and how it relates to mathematicians’ research and teaching. Modeling was central to Sophie’s research in numerical analysis, and when I asked her to describe her modeling process revealed many complexities inherent to mathematical modeling. In terms of the relationship between her modeling process and the CAPs practice, the following excerpt highlights that she went beyond identifying the conditions and assumptions at play. Sophie determined whether the conditions given by those assumptions at play align with a valid problem statement that was even solvable.
Sophie: For the modeling, what I would do first is outline what it is that I want to model... I would take what's known about it first. Then I would think about what assumptions am I making in this work - they would be built into the model. And making sure that the problem is stated in a way such that there could be one. You want to be able to look at your problem statement and say that these are meaningful expressions. For instance, if we have four unknowns, we should probably have four equations. And we might know that just from our standard linear algebra. If you write down a partial differential equation and specify boundary conditions, you want to know that those boundary conditions make sense for the problem.

Sophie uses conditions given by the physical context and identifies the assumptions to be included in her model. She coordinates those conditions and assumptions to then devise a problem that is actually solvable. Then she aligns the condition and assumptions to the mathematical properties to select proper tools to prove something about the problem statement. In this case, she noted that if there are four unknown variables at play, she knows she will set up for equations. Identifying the real-world conditions and the mathematical world properties at play is central to modeling, but the ability to coordinate and align between the real-word conditions and the mathematical properties is what the CAPs practice entails.

Anthony described part of his modeling process as running a simulation to model a situation and using that to attempt to prove something about that real-world situation. He provided an explanatory example of modeling a real-world situation of railroad crossing gate that is open or closed, depending on whether or not a train is crossing the road. “You check some conditions. Can you guarantee these things? How complicated of a system? Can you have a way to verify that it will never in trouble? And we try to get very abstract on that.” To understand his process of using mathematical models in his work, I asked Anthony to describe the relationship between the model and how he proves with his model. In this description, Anthony discusses the relationship between the parameter space and the conditions given by the real-world context, all in relation to proving with solutions given by the model.
Anthony: You run a simulation to see what the solutions look like. It looks like it's going up here but eventually going down. Can I prove that that happens to this? So I sit there and try to prove it - that's what seems to be happening. This is tricky. I realize that I cannot prove that - Why can I not prove it? And then I might say because here it's never guaranteed that these qualities are bigger than these other ones, so I try to play with the symbols until I get the quantity to the reverse side. Sometimes, a solution totally different comes up. I wasn't looking in a good enough parameter space to get a rich picture. So, I go back. Not prove a theorem but enforces what I - so that's the closest analysis that I can do. That happens often.

Anthony describes a situation where he takes the conditions (“these qualities are never bigger than these others”) and describes the mismatch with the mathematical properties he is after (“wasn’t looking at a good enough parameter space”). This is an example that engaging in the CAPs practice does not guarantee a solution or correct answer. Rather, the CAPs practice can provide insight into the modeling processes more broadly.

Given that mathematicians discuss modeling in their own work, I directed our conversation to discussing their perspectives about teaching those same practices to their students. In particular to modeling in numerical analysis, Sophie shared how one of the real challenges that students face is being able to translate between the conditions given by the assumptions and translate that to the specific problem they are working within.

Sophie: We definitely did prove that existence-uniqueness theorem. And the student would be given an ODE and asked if there exists a unique solution. And it could be a really complex ODE, and like the theory says the right-hand side of your ODE has to be continuously differentiable on some open set surrounding your initial condition. So, they have to take all that language and translate it to their specific problem. Like they have to think about, what is the initial condition? Is my function continuously differentiable in some open set around this?

Sophie noted that identifying the conditions at play and what assumptions they are making and then reasoning about the mathematical consequences of those conditions and assumptions is a challenge for her students in numerical analysis. In particular, they might struggle with identifying what the initial conditions mean, how to coordinate those conditions to the symbolic form of the ODE, and how those relate to a unique solution of the ODE or align with a particular solution strategy. Sophie and Anthony highlighted how translating conditions and assumptions in the context of modeling was an important part of their research, but, as Sophie indicated, it can be challenging for students. It is clear from Zbiek and Connor’s (2006) framework that the collective process of modeling in mathematics is composed of many different complex activities.
Discussion

The results section offered a definition of the CAPs MDP through four types of activities (identifying, coordinating, aligning, and modifying) that mathematicians described as aspects of their research. At this point, it will be useful to further situate my results in the literature by briefly discussing some previous work that describes aspects of the entailments of the CAPs practice.

**CAPs and proof.** Because the context of this study was in advanced-undergraduate mathematics where students are expected to learn rigorous mathematics, proof is at the center of the ways in which they mathematicians communicated their thinking. There is an inextricable link between the CAPs practice and the practice of rigorous proving because considering initial conditions and assumptions and the existing or desired mathematical properties is an essential aspect of rigorous proof (e.g., Tall, 1991). It is difficult to disentangle the CAPs practice from that of proof, which means and defining the borders of the CAPs practice is difficult. One could argue that CAPs is simply sub-practice of proof and is not worthy of being a practice itself. However, the evidence provided by mathematicians talk about their research and teaching illuminated that the CAPs practice is not exclusive to the practice of proving. The results also outline how the CAPs practice is an important aspect of the mathematical modeling process and connected to other important practices of the discipline as well. Despite the clear connection between CAPs practice and proving practice, I did not treat proof as the central practice. Rather, I sought to expose how the CAPs practice could extend beyond rigorous mathematical proof and is useful in multiple mathematical contexts.

**CAPs practice within MDPs.** Lockwood et al., (2019) argued that defining mathematical practices is difficult because they are not singular practices, but rather entail “webs of interrelated activity, and practices are not necessarily mutually exclusive” (p. 4). While the mathematicians’ talk about how they think about mathematics or their process of doing mathematical research was imbued with multiple MDS, but the results of this study did not explicitly describe the interrelation between the CAPs practice and other MDPs. However, I draw on existing literature to suggest how aspects of the CAPs practice interact with other mathematical practices.

Kobiela and Lehrer’s (2015) proposed an analytic framework with eight aspects of defining, which included describing the properties or relations of the mathematical object being defined. I view which is in alignment with the activity of identifying in the CAPs practice because
describing properties requires identifying the properties. Freudenthal (1973) distinguished between two different defining activities. Descriptive defining “outlines a known object by singling out a few characteristic properties,” that a person could use properties of the mathematical object of interest. I align descriptive defining with the activity of identifying. Constructive defining describes how a person “models new objects out of familiar ones” using known properties of one object and coordinate those with a new mathematical object of interest (p. 457). I view this type of defining in alignment with the coordinating activity of the CAPs practice because defining new mathematical objects should require the definer to connect properties across the objects of interest. While the CAPs practice is a supporting practice defining practices, but an important point of this paper is to identify that the CAPs practice is a cross-cutting practice because it can be used as a supporting practice of many different MDPs.

Mason, Stephens, and Watson (2009) described a way of thinking that attends to aspects of the CAPs practice. They defined structural thinking as being able to identify general properties of related mathematical objects, and central to structural thinking is an “awareness of the use of properties [emphasis added]” (p. 10). I view this as being related to multiple activities of the CAPs practice because it requires the identification of the properties but also an awareness of how to use them, which could indicate aspects of coordinating and aligning. When mathematicians described the use of properties or how conditions and assumptions related to mathematical properties, implicitly, they were communicating their thinking via the mathematical work of the CAPs practice. Given the link between activities of the CAPs practice and ways of thinking, it stands to reason that the CAPs practice takes the role of a cognitive routine.

**CAPs practice across disciplines.** In their recent handbook chapter, Rasmussen and Wawro (2017) argued for investigations into the ways in which mathematics disciplinary practices might be commensurate with discipline-specific practices other STEM fields (e.g., physics, biology, chemistry). Hearing the ways in which mathematicians communicated about their research inspired me to think about whether and how CAPs could be used outside of mathematics. While my dataset did not provide explicit examples of this because of the nature of my interview questions, I found one example of the CAPs practice in the chemistry education literature. The structure and symmetry of a molecule are directly related to its chemical properties, and chemists use group theory concepts to classify chemically important molecules like water, ammonia, or
ethane (Bergman & French, 2019). Bergman conducted a teaching experiment in which two mathematics education graduate students with little knowledge of chemistry developed a classification system for the shape of molecular structures as described by their symmetries. Given the graduate students’ previous experiences in group theory, the pair were able to develop and describe a classification system for finding the symmetry group of a molecule, and in doing so, appeared to engage in the CAPs practice.

As the pair were developing their classification algorithm, they identified the properties of the physical ball-and-stick representation of the molecule and *identified* and *coordinated* the physical properties to the symmetries they defined: rotations, planes, and slices. For example, while the graduate students were developing the algorithm for finding and classifying all the symmetries of a molecule, they made statements like, “We need to count rotations first”, “What if there are no slices?” “What if there are no slices or planes?” What I found interesting in their exchanges as they developed their classification algorithm was their ability to *modify* the properties to understand possible cases of symmetries. This resonates with ways that mathematicians described modifying and adjusting conditions and assumptions. For example, they started with identifying the property of “no slices” and then restricted the conditions further by having “no slices or planes”. Because the participants were mathematics graduate students, it is not surprising that they were able to develop a classification system because of their previous experiences with abstract algebra. It is not known whether and how chemistry students would have attempted such a task. Bergman and French’s study highlights one cross-disciplinary case of the CAPs practice could be was used outside mathematics, though I admit there are likely other cross-disciplinary examples given that other disciplines rely on mathematics.

**Conclusion**

I began by asking the question about what it means to do mathematics? I argued that MPDs are the essential to make mathematical progress. This paper offered an initial definition of the CAPs practice through four activities in which the mathematicians attend to conditions, assumptions, and properties. The results described the entailments of CAPs practice, which showed that engaging in CAPs practice (minimally) requires *identifying* conditions, assumptions, and properties. One could also engage in *coordinating, aligning, or modifying* conditions, assumptions, and properties, but this is not linear are hierarchical and can be done iteratively. In
articulating the CAPs practice, I made connections to the existing literature to situate my results within broader research literature to illustrate how the CAPs are useful across a variety of mathematical settings. The results were intended to show the ways that CAPs practice is a useful tool for doing mathematics at a local level, but also as a cognitive tool for thinking about mathematics across mathematical contexts.

**Issues related to teaching and learning.** The eight mathematicians in this study shared various ways they attended to conditions, assumptions, and properties in their research and, to some extent, how CAPs practice is helpful in their own teaching. It is important to note that this was a small sample of mathematicians, which may not reflect the perspectives of all mathematicians that do research and teach advanced mathematics. However, these results of this study provided an existence proof that attending to conditions, assumptions, and properties in a mathematical endeavor is a disciplinary practice of mathematics. The CAPs practice has the potential for being useful in teaching students mathematics, but teaching towards MDPs is one of the many challenges teachers of advanced mathematics face when supporting student learning in advanced mathematics. Lockwood and Weber’s (2015) interviews with academic mathematicians found that mathematicians regarded MDPs as essential to teaching mathematics but felt they did not know how to teach mathematical practices effectively.

Regardless of the difficulties that students face in learning MDPs and the role that teaching plays in mediating these issues, the CAPs practice can be used as a versatile conceptual tool for learners of mathematics. The results indicated that the CAPs practice was used as an important way of thinking through a variety of mathematical spaces. The CAPs practice can also be conceptualized as a global practice because it has the power to connect domains within and outside of the mathematics discipline. I contend that the CAPs practice is worthy of further research because, as the results indicated, mathematicians engage in the CAPs practice in their research, and to some extent, in their teaching to foreground aspects mathematics content to their students. Simply put, the CAPs practice could support students learning to engage in complex mathematical ideas across their advanced undergraduate courses. Certainly, more research is needed to understand the CAPs practice from the student perspective in order to be able to teach towards this practice. For example, how might mathematicians support students in learning to strategically modify CAPs to gain insight into novel problems as they are accustomed? I wonder if students
learn to identify and learn how to engage with CAPs in their early undergraduate experiences, might they be able to engage in more complicated aspects of the CAPs practice as mathematicians do?

**Limitations and Avenues of Future Research**

This paper has contributed to the literature in several ways. Firstly, the results provided an existence proof of the CAPs disciplinary practice of mathematics, contributing to the body of literature on the professional work of mathematicians. Secondly, I offered an initial framing of the CAPs mathematics disciplinary practice and provided insights into ways that mathematicians engage use the practice in their research and teaching. A limitation of this study was that it was not explicitly designed to study the CAPs disciplinary practice. Rather, the practice emerged from my discussions with eight mathematicians to understand MDPs broadly in relation to teaching and learning. Follow-up studies could further refine the CAPs disciplinary practice and how this practice could be taken up by students. I imagine that cognitive studies designed exclusively to study the CAPs practice with professional mathematicians or students would be important to explicate its entailments further. A third limitation of this study is that it was not designed to specifically address the CAPs practice from the student perspective. An open question is how students learn the CAPs practice or how educational experiences might support students learning the CAPs practice.

This context of this study was in advanced-undergraduate mathematics, and future studies could address the CAPs in other educational contexts, including K12 or community college courses. A limitation of this study is that only eight mathematicians were interviewed for this study, which limits the quantity of data that was used defined the CAPs disciplinary practice. The sample of mathematicians interviewed in this study may not generalize to every mathematician that teaches advanced-undergraduate mathematics courses, though there is no reason not to believe that these results will not generalize more broadly (Maxwell, 2013).
References


CHAPTER 4 – Conclusion

My qualitative dissertation study presented in two manuscripts collectively examined mathematics disciplinary practices (MDPs) and their relation to teaching and learning in advanced undergraduate mathematics courses. My goal was to understand the roles that MDPs play across content domain and teaching and how mathematicians talk about MDPs either implicitly or explicitly when explicating their research and teaching. Across two phases, I initially conducted an interview study with eight mathematicians that teach upper-division undergraduate mathematics courses to understand the ways that mathematicians use MDPs in their own professional research and how they approach teaching MDPs in their advanced mathematics courses. After concluding the interview phase of the study, two case studies of mathematicians’ advanced undergraduate instruction were identified from the first phase of the research with the goal of better understanding the ways that MDPs might emerge in classroom settings.

Main Findings and Implications

This dissertation study contributed to the literature in several ways. Firstly, this manuscript identified a list of MDPs and offered additional corroborating evidence of the nature of the mathematical work of research mathematicians. Second, it provided additional empirical evidence on mathematicians’ perspectives on the teaching and learning of in collegiate mathematics, and more specifically, the ways that mathematicians report teaching MDPs in advanced undergraduate mathematics. One implication of this study is to argue for reframing the kinds of mathematics students should learn, and ways to support that learning is essential to improve undergraduate experiences for students in advanced mathematics courses, much like the shift that has already taken hold in K12 mathematics education. In particular, this study advances the importance of teaching towards mathematical practices across all content domains in postsecondary mathematics courses for the purpose of supporting students’ explicit engagement in MDPs in undergraduate mathematics classrooms. There is a real need to explore the relationships between the variety of professional mathematical practices and how mathematicians can employ specific instructional strategies to support students’ engagement in these MDPs in their teaching. Mathematicians in this study reported deploying a variety of instructional practices in service of helping students learn MDPs and coordinate their knowledge of MDPs used their research with how they approach teaching MDPs in their advanced courses. Mathematicians’ experiences in professional
development around teaching could explain why their descriptions of pedagogical practices to support students in learning MDPs were general (e.g., assign problems, demonstrations, group work). This finding suggests an area ripe for exploration – what instructional practices could mediate between a mathematician content knowledge of MDPs and the knowledge of MDPs they want their students to learn.

A second implication of this dissertation study is for the community of mathematics education researchers whose interests are in undergraduate mathematics. Both manuscripts describe the importance of MDPs are in learning and doing mathematics and should not be treated implicitly as it is often concealed in educational research around specific mathematical content areas. I look forward to advancing this work to explore MDPs in developmental courses and lower-division mathematics courses. Reframing the kinds of mathematics students should learn, and ways to support that learning is essential to improve undergraduate experiences for students in advanced mathematics courses (much like the shift that has already taken hold in K12 mathematics education). These two manuscripts describe the importance of MDPs are in learning and doing mathematics and should not be treated implicitly as it is often concealed in educational research around specific mathematical content areas. This study advanced the importance of teaching towards mathematical practices across all content domains in postsecondary mathematics courses for the purpose of supporting students’ explicit engagement in MDPs in undergraduate mathematics classrooms. However, because this was an exploratory study, there is still much to learn about the relationships between the variety of MDPs used in mathematicians’ research and how mathematicians can employ specific instructional strategies to support students’ engagement in these MDPs in their teaching.

Limitations and Considerations for Future Research

Although this study offered additional empirical evidence of MDPs in collegiate mathematics, because it was an exploratory study, the research findings are bounded in particular ways. Firstly, the data in this study is the result of interviews with eight mathematicians and two case studies in two of those mathematicians’ advanced-undergraduate classrooms. Furthermore, the participants selected for this study did not fully represent every content domain in advanced mathematics. This means that the results and subsequent findings may not generalize to all
mathematicians who teach advanced mathematics. Contrary to this criticism, there is no reason not to believe that the results from case studies will not generalize beyond the cases studied (Maxwell, 2013). Pragmatically, the intention of this study was never to provide some objective truth about MDPs. Instead, this study was designed to offer one possible reality of how MDPs are used in advanced mathematics and ways that teaching might support students in leaning those practices.

A second limitation is that the analysis did not include a second coder, so it is possible that some MDPs were not captured within the interviews and case study data or other interpretations and larger themes within mathematician reports could have been found. For example, the categorization of MDPs offered in Table 2.4 organized the 25 practices into four categories. While these four larger categories described the ways that MDPs often appeared together across eight interviews, other researchers could find other ways to organize MDPs together based on other data sets. Future empirical studies could include research designs that explicitly study the ways that MDPs are interrelated. One avenue for this work that might be fruitful is drawing on social network methodologies to examine what MDPs coalesce. Furthermore, it would be interesting to see whether and how these MDPs hang together differently across various mathematical content domains. Understanding the ways that MDPs are connected could have implications for how to best approach teaching practices across grade levels.

A third limitation is that because this was an exploratory study, it was not able to identify instructional practices that best support student engagement and learning of MDPs. Understanding how mathematical practices are taken up by students as they engage in more advanced mathematics. Future research could examine and identify instructional practices that specifically target learning of MDPs and promote those practices in national documents for various audiences (e.g., mathematicians, mathematics educators, policymakers). Furthermore, the field could benefit from additional efforts to develop policy documents, and coordinated professional development opportunities for mathematicians could be helpful in supporting mathematicians taking up instructional practices that support student engagement and learning of MDPs.

Because this was an exploratory study, I wasn’t able to write about every possible theme that could be found in my data set. I regularly went back to transcripts to substantiate claims to find corroborating and disconfirming evidence within a single interview and across all interviews.
Reliability in this study would have been improved by employing a second coder and member checks with mathematicians.

I have three ideas that I would like to pursue further. The first is to draw on social network methodologies to examine the ways in which MDPs are interrelated to provide additional empirical evidence on how the ways that MDPs. There are some significant methodological issues to address (e.g., what is the size of a chunk and how are codes applied in a chunk) in order to conduct such an analysis. Second, I would suggest future work to include a research design that explicitly explores the CAPs practice in more detail because it was an emergent practice from the interview phase of my study. In particular, I think task-based interview drawing on constructivist methodologies could be useful in understanding how mathematicians and students attend to conditions, assumptions, and properties in a mathematical endeavor. Lastly, I would like to examine the classroom case studies using the expanded emergent interpretive framework (Rasmussen, Wawro, and Zandieh, 2015) to explore the classroom mathematical progress around MDPs. Since my data collection for this study was over the course of three and four days in my case studies, the current data set could be used as a pilot for a more substantive data collection in the future. It would be interesting to see additional studies on MDPs in the context of undergraduate mathematics, and in particular, how students take up these practices and what mediates their learning practices.
BIBLIOGRAPHY


Dawkins, P. C. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *Journal of Mathematical Behavior, 31,* 331–343


Wawro, M., Rasmussen, C., Zandieh, M., & Larson, C. (2013). Design research within undergraduate mathematics education: An example from introductory linear algebra. In T. Plomp, & N. Nieveen (Eds.), *Educational design research – Part B: Illustrative cases (pp. 905-925)*. Enschede, the Netherlands: SLO.


APPENDICES

Appendix A. Mathematician interview protocol

Introduction
Can you remind me what [advanced-undergraduate] classes you teach?
- What are your goals for students taking those classes?
- What challenges do you foresee in teaching those courses?

Research
Can you briefly describe for me the kind of work you do as a research mathematician?
What kinds of mathematics disciplinary practices do you use to support you in the mathematical work you do?
- Are these practices specific to your work as a research mathematician or do you see these practices show up in other contexts?
- How did you learn these mathematical practices?
- How do you think students might learn these same mathematical practices?

Teaching
What do you do try to help your students learn how to do what you do as a (research) mathematician?
- How do you teach students about these mathematics disciplinary practices?
- What challenges do you find when you teach students about disciplinary practices?
- Do you think these are important disciplinary practice of mathematics broadly, or specific to content in advanced mathematics courses?

Content & Practices
Please take a moment to think about a mathematical problem or topic in a discrete or other advanced math course that you feel communicates mathematical content but also allows you to address mathematics disciplinary practices.
- Can you describe to me what the example, problem, or topic you were thinking about?
- What kind of mathematical practices would you use to solve problems within that topic or the problem you identified?
- What are some things you might do in your own teaching to help students learn this mathematical practice?
- Can you talk a bit about how you think students learn this mathematical practice? [Probe: What is the mathematical work that students do while they are learning this practice?]
**Appendix B.** Mathematics Disciplinary Practice (MDP) codes generated by mathematician interviews.

<table>
<thead>
<tr>
<th>MDP Code</th>
<th>Operational Definition</th>
<th>Example</th>
<th>Number of mathematicians</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Formalizing and Proving</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proof &amp; Proving</td>
<td>Utterances about proof and proving, creating, validating, or unpacking a mathematical proof.</td>
<td>…you can imagine if I have a set of size nine and I want to turn it into a commutative group...</td>
<td>8</td>
</tr>
<tr>
<td>Argumentation &amp; Justification</td>
<td>Any utterance about developing, using, or interpreting and argument or justification.</td>
<td>they're writing some kind of mathematical argument - whether or not it is a formal proof or just some lone computation.</td>
<td>6</td>
</tr>
<tr>
<td>Conjecturing</td>
<td>Making conjectures as it relates to proof, proving, argumentation, or justification. This is not the same as Questioning which captures more general questions relate to the Problem Solving code.</td>
<td>I don’t know how to prove the conjecture. I don’t know how to get from the assumption of the conjecture to the conclusion of it.”</td>
<td>4</td>
</tr>
<tr>
<td>Syntactic Manipulation</td>
<td>References to manipulating symbols, definitions, or facts, esp. in proof. Can be double coded with Procedures and Computations, but is specific to manipulation or “symbol pushing”.</td>
<td>I try to play with the symbols until I get the quantity to the revere side.</td>
<td>2</td>
</tr>
<tr>
<td>Logical structure</td>
<td>Utterances about developing or drawing on logical structure (especially in proof).</td>
<td>Expressing yourself logically, I mean, I do it. When I write my proofs. I try and explain every step.</td>
<td>6</td>
</tr>
<tr>
<td>Leveraging Examples</td>
<td>This code captures utterances about using, leveraging, creating, unpacking, organizing (etc) examples when learning, teaching, or doing mathematics.</td>
<td>There is this theorem and I’m not even going to allow you to try and prove it first. Lets come up with some examples, you know, what is this actually saying</td>
<td>8</td>
</tr>
<tr>
<td>Defining</td>
<td>References to applying or unpacking definitions, creating definition(s), connecting definitions to mathematical properties. Does not require providing definition.</td>
<td>Which means building the theory, building the theory means propose certain, even define new combinations of objects that can be defined mathematically and find what properties they have.”</td>
<td>6</td>
</tr>
</tbody>
</table>
### Mathematizing and Computing

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
<th>Example</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Modeling</strong></td>
<td>Talk about describing, using, developing, analyzing, proving things about mathematical models.</td>
<td>So mathematically it means that models are usually differential equations, or they could be discrete equations.</td>
<td>3</td>
</tr>
<tr>
<td><strong>Utilizing Technology</strong></td>
<td>References to using technology as a tool for doing mathematics. Includes things like using CAS, Calculators, Desmos, Computers, writing codes using Maple or other systems</td>
<td>If you can't or it's too difficult to write down the exact solution, we transform it and solve it on a computer.</td>
<td>5</td>
</tr>
<tr>
<td><strong>Algorithm and Programming</strong></td>
<td>Instances of talk about coding, programming, or algorithm design, or algorithmic thinking.</td>
<td>I then try to think how I'm going to program, say maple, to run some loops and generate examples.</td>
<td>5</td>
</tr>
<tr>
<td><strong>Develop &amp; Use Techniques</strong></td>
<td>References to applying known mathematical techniques and tools, or developing new techniques as new mathematics</td>
<td>References to applying known mathematical techniques and tools, or developing new techniques as new mathematics.</td>
<td></td>
</tr>
<tr>
<td><strong>Theory Use &amp; Development</strong></td>
<td>References to using current theory, or developing new theory, or learning mathematical theory.</td>
<td>... mathematical structures that help you study these systems and their mathematical properties of these. And so you can study all the theory there.</td>
<td>2</td>
</tr>
<tr>
<td><strong>Procedures &amp; Computations</strong></td>
<td>Refers to use or selection of procedures, performing computations. (Not rote)</td>
<td>So this is where we drew our pictures and did computations.</td>
<td>7</td>
</tr>
</tbody>
</table>

### Structuring Mathematics

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
<th>Example</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Leveraging Mathematical Structure</strong></td>
<td>Utterances regarding structure, leveraging or using mathematical structure.</td>
<td>If the structure looks like this mathematically, use these kinds of methods and you can expect this kind of error.</td>
<td>7</td>
</tr>
<tr>
<td><strong>Classifying</strong></td>
<td>References to classifying, listing, or organizing mathematical objects based on some characteristics, properties, or structures.</td>
<td>You end up with another theorem or the technique you're using only works for a certain subclass of groups, for example, that you're trying to classify.</td>
<td>4</td>
</tr>
</tbody>
</table>
Appendix B (continued)

<table>
<thead>
<tr>
<th>Practice</th>
<th>Utterances</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representing</td>
<td>Utterances about using, describing, leveraging, comparing, coordinating mathematical representations. Can be double coded with other things, like Patterns, Mathematical structure, et.</td>
<td>7</td>
</tr>
<tr>
<td>Discovering Patterns</td>
<td>Utterances describing heuristics around patterns, looking for or leveraging patterns. Used as a finer-grained code (for instance with Problem Solving or Proof) or with other fine-grained codes.</td>
<td>2</td>
</tr>
<tr>
<td>Generalizing</td>
<td>Utterances about “generalizing’, or abstracting, and can be double coded with pattern.</td>
<td>4</td>
</tr>
<tr>
<td>Other Practices</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Precision</td>
<td>Utterances related to being mathematically precise (in computations, applications, theory, etc) or using/applying mathematical language correctly. Can be double coded with communication &amp; procedures and computations</td>
<td>5</td>
</tr>
<tr>
<td>Communication</td>
<td>Instances of talk around communication mathematics, understanding what is being communicated mathematically.</td>
<td>2</td>
</tr>
<tr>
<td>Conditions, Assumptions, &amp; Properties</td>
<td>Utterances about conditions, assumptions or properties in relation to math content; imposing, modifying, introducing CAPs.</td>
<td>8</td>
</tr>
</tbody>
</table>
that... I think I have a better chance of attacking.”

Appendix B (continued)

<table>
<thead>
<tr>
<th>Intuition &amp; Creativity</th>
<th>Utterances that relate to the mathematics as an artistic, creative pursuit, can also capture mathematical intuition, hunches, etc.</th>
<th>Any budding mathematician should want that intuition.</th>
<th>3</th>
</tr>
</thead>
</table>