The nature of flow in stratified reservoirs has been studied and a method developed, based on a laboratory model study, to predict the quality of the water discharged from the reservoir. The experimental data has been evaluated in dimensionless form, so that the results may be applied to actual reservoirs. The model was designed to simulate a relatively high head dam impounding a stratified reservoir in which the density gradient is approximately linear. The extent and magnitude of internal density currents can be determined, and the properties of the discharged water can be predicted from measurements of flow rate, density gradient, and depth. The temperature of the discharged water, effect on downstream environment, and change in the thermal structure of the reservoir are among the quantities which may be forecast. Illustrations have been provided to show the degree of control of water quality available by regulation of the reservoir water discharge.
INTERNAL DENSITY CURRENTS GENERATED IN A DENSITY STRATIFIED RESERVOIR DURING WITHDRAWAL

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INTERNAL DENSITY CURRENTS GENERATED IN A DENSITY STRATIFIED RESERVOIR DURING WITHDRAWAL

INTRODUCTION

Water is not an unlimited resource. Increased demand for water and hydroelectric power had led to intensive development of water resources with the construction of large dams and their attendant reservoirs. Most dams and reservoirs are now designed to serve multiple purposes. Water quality must be controlled for domestic use, for recreation, for the preservation of fish and wildlife, to maintain transport in rivers and canals, industrial processes, for irrigation, and many other uses. Storage or power structures built to serve only one purpose have often proved detrimental to other users. Many of the uses of the water depend upon water quality, as well as quantity of supply. To control the quality of the water discharged through a modern multi-purpose dam, requires an understanding of the structure of the reservoir and of the forces determining the flow and source of the discharged water within the reservoir.

Some of the problems associated with the use of reservoirs, which a better understanding of the role of stratification and density currents would help solve, are of very practical importance. It might be of interest to mention only a few of the many instances
where a knowledge of density currents could be valuable.

The period of retention in a reservoir is related to the extent of self purification. Water purity is important when a reservoir serves as a source of municipal water supply. If an organic, bacteriological, or chemical pollutant flows into a reservoir, it is essential to know the average retention time in the reservoir to estimate the amount of self purification. It has been found that a pollutant discharged from an industrial plant upstream flowed through the Cherokee Reservoir of the TVA system as a discrete flow. Because of low-temperature dissolved solids, this waste water had a significantly higher density than that of the main reservoir and moved along the bottom as an internal density current. There was only slight diffusion and dispersal, and the water was discharged through turbine outlets which drew from the bottom water after a minimum residence time. This resulted in poorer water quality for domestic use than had been estimated allowing for greater time for diffusion and recovery.

Water is released from Shasta Lake into the Sacramento River in California through low-level, hydroelectric power turbine outlets. This discharge water is stored snow melt, and is, particularly during late summer, much cooler than the temperatures which prevailed before the dam was built. The cooler summer water has had
a detrimental effect upon rice crops in the Sacramento Delta area
where this water is used for irrigation; however, the cooler down-
stream temperatures of the Sacramento River have resulted in a
substantial increase in the fishery resources of the river (10).

An opposite situation occurred in the Snake River in Idaho when
a series of reservoirs was constructed where the warm surface
layer of each reservoir is successively withdrawn by high level gates.
The cumulative effect has been an increase in the downstream tem-
peratures, which is probably associated with a decrease in the fish-
eries resource in the river. The temperature environment is related
to the productivity of salmon and other fish. Many activities of fish,
such as migration, spawning, incubation of eggs, and maturation are
directly affected by temperature. The incidence of parasites and
diseases is related to the stream temperature, and stream life in
general is affected by changing oxygen concentrations (2).

Water quality required for industrial needs varies according to
the intended use. Water of suitable temperature for cooling pur-
poses is essential in the design of steam plants. The study of con-
denser water temperatures for the large Kingston Steam Plant in
Tennessee showed the importance of utilizing density currents to
obtain water of desired temperature. Water of the proper tempera-
ture and in sufficient volume to meet plant needs has been assured
during those months when surface temperatures are unsuitable. (3).

Various suggestions have been made to utilize internal density currents to control or regulate sedimentation and filling of reservoirs. The proper use of density currents may sometimes substantially increase the useful life of reservoirs. The stream flow entering many reservoirs is loaded with sediment. Depending on the amount of sediment present and the temperature of the water, this water may form a density or turbidity current along the bottom. By encouraging discharge from this turbidity current, much of the sediment may be discharged before it has settled. In reservoirs which are also used for flood control the water in the fall must be drawn down to increase the reservoir storage capacity for spring floods. It is at this time, when waste water is being spilled, that the sediment laden water along the bottom could be discharged. It has been estimated that the useful life of Lake Mead could be lengthened by 20 percent in this manner (1).

These are only a few illustrations of the benefits and problems which may occur in stratified reservoirs because of internal density currents. While it is desirable to consider all potential reservoir uses, it has been impossible to provide special consideration for all cases. Most reservoirs have been constructed as if all the impounded water were of one quality and as if only the amount of
discharge should be controlled. Few dams have more than one withdrawal depth for discharged water. The Folsom Dam is one exception. After the dam was constructed, a set of louvered gates was installed to allow discharge water to be drawn from selected heights.

The present study has developed methods for predicting flow and withdrawal from density stratified reservoirs. A model study of two-dimensional, frictionless flow with linear density stratification was performed, and the results have been compared with the predictions from theoretical studies. Velocity profiles were measured in the model under various conditions of stratification. The thermal density stratification of an actual reservoir was stimulated by a saline density gradient in the model. Outlet height and rate of discharge were variable in the model. A large range of conditions, similar to those which might be expected in an actual reservoir, were tested. The test results were presented in non-dimensional form and compared with available prototype data. The results indicate that the model data can be used for direct extrapolation to various prototypes, as well as to evaluate constants and extend the use and applicability of the theoretical studies in determining the most efficient design and use of reservoirs.
INTERNAL CURRENTS IN RESERVOIRS

The temperature, density, and viscosity in an isothermal reservoir are constant from surface to bottom. Under these conditions, water is drawn toward a discharge outlet from all levels simultaneously. Conversely, when stratification exists, it may become possible to withdraw a single layer of water from within the reservoir. The flow of a single layer of water beneath the surface is termed an internal density current. An internal density current may be defined as the movement, without loss of identity by turbulent mixing at the bounding surfaces, of a stream of fluid through a miscible body of fluid due to density differences.

As yet, there is no complete solution for flow in stratified media, but solutions have been obtained for flow under various simplifying assumptions. The problem of stratified flow is one of continuing attention.

Craya (4) gave a mathematical discussion of flow in a particular two-layered system. He considered a two-layer flow with no mixing between the layers, and with discharge from the upper layer only. He provided criteria to determine the discharge rate necessary to raise the interface level to the outlet level so that subsequent discharge would be from both layers. Gariel (7) considered
the same problem and provided a series of experimental results to evaluate Craya's theoretical work.

Long (11) extended the work of Craya and Gariel for the case of a continuous density gradient. He considered a two-dimensional flow of a stratified fluid with a linear density gradient, assuming both that the fluid was frictionless and steady state conditions. Yih (15) applied the second-order differential equation for motion in a stratified fluid obtained by Long to solve the problem of two-dimensional stratified flow with a line sink located at a bottom corner of the channel. To obtain an analytical solution to the equation, it was necessary to assume a constant vertical density gradient. Yih's solution, discussed in Appendix I (p. 66), provides a description of the flow before separation into layers, and also provides a criterion for determining under what discharge and density conditions withdrawal begins to occur from only discrete layers. However, the solution does not describe the flow to be expected, that is, the velocity distribution within the density current, the thickness of the layer, or changes which may occur in the nature of the density current for various density gradients.

When he investigated Yih's results in a model study, Debler (5) confirmed the existence of flow of limited vertical extent, and found a good correlation between model and theory for the criterion
for the development of internal density currents. He also obtained data on the thickness of the internal density current for various Froude numbers.

Harleman (9) applied Debler's analysis to the Fontana Reservoir in the TVA system, using temperature profiles which were previously compiled. He assumed a uniform velocity throughout the thickness of the interflowing layer and a rather arbitrary choice of layer thickness to estimate layer velocities. The results are consistent with the observed temperatures, but other choices for velocity distribution and layer thickness would also be consistent with the observed discharge temperatures. Unfortunately, the available temperature data cannot discriminate between various possibilities. The anticipated velocities, estimated to average 0.05 feet per second, are quite small and also difficult to measure.

The present study was designed to measure the actual velocity distribution within the layer, and to provide a relationship between layer thickness and density gradient, which could be extrapolated to reservoir prototypes.

To study adequately and understand the internal currents and flow established during drawdown or discharge conditions in a reservoir, it is necessary to consider the equations of fluid motion. However, before proceeding to a detailed study of hydrodynamics
and fluid motion, it is advantageous to have a general idea of the
distribution of properties in reservoirs, and of the general physical
and chemical conditions which may be expected to control the degree
of stratification and the type of internal flow.

Temperate lakes and reservoirs are normally thermally strat-
ified, and the water properties vary seasonally and with position in
the reservoir. The quality of the discharged water depends, not
only upon the time of year, but upon residence time within the res-
ervoir. The nature of the discharged water, in turn, affects the
properties of the residual water. Temperature is the most impor-
tant factor controlling density and, indirectly, the stability and circ-
ulation. Thus, comments here generally refer to reservoirs which
are thermally stratified, although the results would be valid where
density stratification was due to some other cause, such as salinity
gradients. This study has been designed to be applicable to a reser-
voir located in a temperate zone and of sufficient depth to allow ade-
quate development of a thermocline.

At the end of the winter, such a reservoir will normally have
been cooled to the temperature of maximum density. As the ice
melts, the reservoir becomes isothermal throughout its depth. When
the reservoir warms in the spring and summer, heat is absorbed at
the surface and mixed downwards, largely by wind action. At the
same time that the surface layers are becoming warmer and less
dense, river runoff will be entering the reservoir. This runoff is
often cooler than the reservoir water and will tend to sink toward the
bottom. As long as the density differences established daily are
small, the wind will be sufficient to mix the reservoir. However,
eventually the density difference will be great enough with a period
of weak wind, that the wind energy will be insufficient to mix the
reservoir throughout its depth, and the process of stratification will
be established. Depending upon the energy available for mixing and
the densities of the water at surface and at depth, a series of layers
may develop with a stepwise temperature and density structure. The
sharpest of these steps is normally the step nearest the surface, and
the sharp temperature gradient beneath the surface layer is called
the thermocline. Turbulence and internal currents may obscure the
secondary thermoclines, but the primary thermocline intensifies
during the summer and tends to insulate the rest of the reservoir
from the processes occurring at the surface.

Maximum stratification will normally occur in late summer.
In the fall, as the surface water is cooled, it becomes denser and
will sink and mix with the water beneath the surface. If the cooling
process continues long enough, the reservoir will again become iso-
thermal, and it will continue to cool isothermally until it reaches
the temperature of maximum density, $4^\circ\text{C}$. Further cooling will
again result in stratification with cooler water near the surface and
formation of an ice cover. This is a reasonable description of the
cycle of stratification in the reservoirs of interest, although factors
other than temperature can affect the stratification.

While dissolved oxygen concentrations do not affect the density
structure, they are affected by the stability and stratification which
control the density structure. The dissolved oxygen concentration is
one of the most important factors of water quality control. Oxygen
in the water is obtained from air at the surface and from the photo-
synthesis of the plants in the upper layers of the water. In a reser-
voir which has become stratified, there is little vertical circulation,
and as long as the reservoir remains stratified, no means of replen-
ishing the oxygen supply of the water below the surface layer. There
is however an oxygen demand upon this water from organisms living
in this layer and decomposition of organic material. In extreme
cases, the oxygen may be entirely depleted, and the lake or reser-
voir may become anaerobic at depth with consequent destruction of
the plant and animal life. One source of replenishment of oxygen at
depth is the cooler, highly oxygenated river water which flows into
the reservoir during the spring and summer, normally sinking to
depths below the thermocline. The distribution of oxygen will thus
be strongly dependent upon density currents which affect many of the problems presently associated with reservoir development.
THEORETICAL ANALYSIS OF THE FLUID FLOW

Hydrodynamics

Quasi-Potential Flow (Yih's Theory)

In the usual analysis of flow patterns in water, one of the first assumptions made is usually to consider that the fluid is of homogeneous density. In the present case, however, we are interested in just those effects which occur because of the small variations in density. While the fluid is nonhomogeneous, it will still be considered incompressible. That is, the density will be considered to have a spatial distribution \( \rho = \rho(x, y) \) throughout the fluid, but individual fluid elements will be supposed to conserve their volume as they move.

We will also consider the flow to be steady, frictionless, and two-dimensional. The coordinate system is shown in Figure 1. (x, y) are Cartesian coordinates, with y positive upward from the bottom of the reservoir and x the horizontal coordinate positive in the direction of flow through the outlet. \( u \) and \( v \) are the x and y components of the velocity, respectively, and \( \rho(x, y) \) is the density. With this notation, the condition for incompressible flow is:
Figure 1. Schematic diagram of flow system.
\[
\frac{dp}{dt} = 0
\]  \hspace{1cm} (1)

Expanding, and setting \( \frac{\partial p}{\partial t} = 0 \), since we have specified a steady state:

\[
u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} = 0
\]  \hspace{1cm} (2)

This may be substituted in the general equation of continuity:

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0
\]  \hspace{1cm} (3)

to give:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  \hspace{1cm} (4)

The equation of motion, which represents the balance of forces acting in the fluid, may be written as force per unit volume:

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x}
\]  \hspace{1cm} (5)

\[
\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} - \rho g
\]  \hspace{1cm} (6)

The terms may be identified in the following verbal statement:

that the

\textbf{Inertial term} = - \textbf{Pressure gradient term} + \textbf{Gravitational term}

Since the flow is both steady and incompressible, a stream function will exist such that:
The pressure terms can be eliminated by cross differentiating equations (5) and (6) and subtracting. \( \psi \) is substituted for \( u \) and \( v \) using equations (7) and (8). By using equation (2) and an integration, Long (11) obtained the following equation for the stream function:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{2 \rho} \frac{\partial \rho}{\partial \psi} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + 2gy \right] = H(\psi) \\
\text{or} \quad \nabla^2 \psi + \frac{1}{2 \rho} \frac{\partial \rho}{\partial \psi} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + 2gy \right] = H(\psi)
\]

where \( H(\psi) \) is a constant of integration to be determined by the upstream boundary condition.

We observe that if the fluid is homogeneous, \( \frac{\partial \rho}{\partial \psi} = 0 \), equation (9) reduces to the familiar Laplace's equation, and the flow will be potential flow:

\[
\nabla^2 \psi = H(\psi)
\]

From the form of equation (9), when the inertial effects are strong and the stratification weak, we can expect the effects of stratification to appear as a perturbation upon potential flow.
Potential flow in a reservoir is characterized by essentially uniform withdrawal from all levels of the reservoir and uniform velocities throughout the reservoir. Figure 2 shows a predicted pattern of potential flow taken from the study of the Lake Mead Reservoir (8). For potential flow, the discharged water would have the same composition, and withdrawal should occur from the same depths of the reservoir, irrespective of the depth, size or position of the outlet. However, observation of the discharge temperatures from Lake Mead indicate that stratification is sufficient that undisturbed potential flow does not occur.

Before equation (9) can be solved analytically, the constant of integration $H(\psi)$ must be evaluated, and some means must be found to remove or circumvent the non-linear terms, such as $\left(\frac{\partial \psi}{\partial x}\right)^2$, in equation (9).

Long (11) found a transformation which changes equation (9) into a linear equation. An associated stream function, $\psi'$, is defined such that:

$$d\psi' = \rho \frac{\partial}{\partial x} \psi$$

With this relationship, equation (9) becomes:

$$\nabla^2 \psi' + gy \frac{d\rho}{d\psi'} = H(\psi')$$
Figure 2. Two-dimensional potential flow in a reservoir.
The constant of integration, $H(\psi')$, may be evaluated from upstream conditions. If, at some distance, $-x_0$, sufficiently upstream from the outlet, the flow is sufficiently horizontal that the streamlines lie in equipotential surfaces, then the motion will be irrotational, and:

$$\nabla^2 \psi' = 0$$  \hspace{1cm} (13)

Equation (12) may be written, for these conditions:

$$H(\psi') = gy \frac{d\rho}{d\psi'}$$  \hspace{1cm} (14)

This expression may be evaluated provided that $\frac{d\rho}{d\psi'}$ may be determined. Yih (14) used two assumptions. First, he assumed that the density gradient with depth was linear, and second that the velocity sufficiently upstream was essentially uniform from top to bottom in the fluid. The second condition is a critical one, since it precludes any use of Yih's solution for flow which is limited to a particular layer within the fluid. Under these conditions, $\frac{d\rho}{d\psi'}$, becomes a constant and Yih has been able to solve equation (12). The solution is given in Appendix I. Yih's solution cannot, of course, provide a solution for the case of separation of flow, but by examining the range in which the solution is valid, gives an indication of the effects and extent of stratification, during which the flow can be considered as quasi-potential and essentially drawn from all levels of the
reservoir. The critical values for Yih's solution occur in terms of
the densimetric Froude number. The densimetric Froude number is
defined as:

\[ F = \frac{u}{\sqrt{g \Delta \rho \cdot d}} \]  \hspace{1cm} (15)

The solution of equation (12) is valid, provided that:

\[ F > F_{cr} = \frac{1}{\pi} = 0.318 \]  \hspace{1cm} (16)

Where \( F_{cr} \) is the critical value of the densimetric Froude number.
For values of \( F \) greater than the critical value, there will be quasi-
potential flow. For values of \( F \) less than the critical value, the solu-
tion fails and separation of the flow presumably occurs. Solutions
given by Yih for several values of \( F \) are shown in Figure 3.

Yih considered the outlet located as a line sink at the bottom of
the reservoir. Harleman (9) has adapted Yih's solution to locate the
outlet at any position on the face of a vertical wall. Aside from some
singular solutions, apparently without physical significance, the solu-
tion and the critical value of \( F \) found by Harleman are the same as
those determined by Yih.

While Yih's theory does provide criteria in terms of a densi-
metric Froude number beyond which separated flow cannot occur, it
Figure 3. Yih's Solution for Densimetric Froude Numbers 0.5 and 0.32.
does not provide solutions for the observed separated flow. The problem here becomes more difficult than the problem of quasi-potential flow and generally requires the solution of non-linear second order partial differential equations. It is helpful to examine some of the conditions under which the linearized equation (12) fails to provide solutions. Yih (16) has derived an expression for the general upstream stream function if it is to be a solution of the linearized equation (12). His result is:

\[
\psi'_0 = \frac{A}{Bb} y + C \sinh \sqrt{B} \frac{y}{b} + D \cosh \sqrt{B} \frac{y}{b}
\]

(17)

A, B, b, C and D are constants. It can be seen by examination of the stream function that it can never represent separated flow (since \( \psi \) can never approach a constant value). Even though the equations may be non-linear, it may be possible to find solutions. Several possibilities have been investigated.

Yih (16) considered a variable density and was able to obtain a solution in terms of modified Bessel functions for an exponential density variation. The solution, however, has the same difficulty as (17) and does not seem applicable to layered flow. Trustrum (13) has attempted to solve the equations by a perturbation method. She was able to reproduce Yih's solution (17) but encountered the same difficulty of being unable to match the boundary conditions upstream with
the equation for the case of layered flow.

Another possibility is to approximate the solution from the observed velocity profiles and stream lines. If the approximate solution is substituted in the approximate equations, the nature of the restraints required to match the assumed solution to the equation may provide information on the final type of solution to be sought. This procedure will be investigated here.

Fully Separated Flow

Let us assume an incompressible, inviscid fluid with a linear density gradient. Let us further assume that at a distance sufficiently upstream the stream function is given by:

\[ \psi_o = \frac{A}{2b} \left[ y \left( b^2 - y^2 \right)^{1/2} + b^2 \sin^{-1} \frac{y}{b} \right] \]  (18)

The velocity upstream will be given by:

\[ u_o = \frac{A}{b} \left( b^2 - y^2 \right)^{1/2} = -\psi/y \]  (19)

The density upstream is given by:

\[ \rho = \rho_c - \beta y \]  (20)
And
\[ \frac{dp}{d\psi} = -\beta / \frac{d\psi'}{dy} = \beta / u \]  \hspace{1cm} (21)

\( H(\psi) \) may now be evaluated in equation (12)
\[ \frac{2}{\sqrt{\psi'}} = \frac{Ay}{b} (b^2 - y^2)^{-\frac{1}{2}} \]  \hspace{1cm} (22)

\[ gy \frac{dp}{d\psi} = g \frac{\beta b}{A} y (b^2 - y^2)^{-\frac{1}{2}} \]  \hspace{1cm} (23)

(18) and (19) define a flow limited between \( y = \pm b \).

Table 1 gives numerical values of the constants for problems of interest.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Dimensions</th>
<th>Value in MKS units</th>
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<tr>
<td>A</td>
<td>( M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} )</td>
<td>0.1 to 1.0</td>
</tr>
<tr>
<td>b</td>
<td>L</td>
<td>10</td>
</tr>
<tr>
<td>g</td>
<td>L \ T^{-2}</td>
<td>10</td>
</tr>
<tr>
<td>( \beta )</td>
<td>M \ L^{-4}</td>
<td>0.1 to 0.1</td>
</tr>
<tr>
<td>( \frac{g\beta b}{A} )</td>
<td>( M^{\frac{1}{2}} L^{-\frac{3}{2}} T^{-1} )</td>
<td>1 to 100</td>
</tr>
<tr>
<td>( \frac{A}{b} )</td>
<td>( M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} )</td>
<td>0.1 to 0.1</td>
</tr>
</tbody>
</table>

Thus, we may neglect (22) in comparison with (23), and the expression for \( H(\psi) \) becomes:
\[ H(\psi) = \frac{g\beta b}{A} y \left( b^2 - y^2 \right)^{\frac{1}{2}} \]  

(24)

It is difficult to substitute for \( y(\psi) \), but a substitution may be made in terms of \( \frac{\partial \psi}{\partial y} \), using (19) and (21).

\[ H(\psi) = \frac{g\beta b}{A} \left( A^2 - \left( \frac{\partial \psi}{\partial y} \right)^2 \right)^{\frac{1}{2}} \]  

(25)

Finally, equation (12) may be written

\[ \left( \nabla \cdot \left( \frac{\partial \psi}{\partial y} \right) \right) + K \frac{\partial \psi}{\partial y} \left[ \left( \frac{1}{2} \right) b \left( b^2 - y^2 \right)^{-\frac{1}{2}} \right] = K \left( A^2 - \left( \frac{\partial \psi}{\partial y} \right)^2 \right)^{\frac{1}{2}} \]  

(26)

where \( K = \frac{g\beta b}{A} \).

No exact solution of equation (26) has been found, but an approximate solution may be investigated.

An approximation to the observed velocity profile may be given by:

\[ u = \frac{A}{b} \left( b^2 - y^2 \right)^{\frac{1}{2}} \frac{y}{\beta} \frac{x}{b} - \frac{A}{b} \left( b^2 - y^2 \right)^{\frac{1}{2}} \frac{c \alpha}{xb^2} \left( b^2 - y^2 \right) - \frac{c \alpha}{2x} \]  

(27)

While \( \psi (x, y) \) is not generally derivable in closed form from (27), it may be evaluated for any \( u(y) \) for any chosen value of \( (x) \).

If \( C \) is chosen as \( C = \frac{4}{3} \frac{Ab}{\pi} \), then (27) satisfies the transport condition
\[ \int_0^b u \, dx = \frac{\pi Ab}{4} \quad (28) \]

in the vicinity of the point \( x = -\alpha \), and at all points upstream. At this point, \( \psi \) may be approximated

\[ \psi(y) = \frac{A}{2} \left( \frac{4+3\pi}{3\pi b} \right) \left[ y \left( b^2 - y^2 \right) + b \sin \frac{-1}{b} y \right] - \frac{A}{b} \left\{ - \frac{1}{3} (b^2 - y^2) \frac{3}{2} \right\} \quad (29) \]

\( x = -1 \)

When equation (29) is substituted into equation (26), the coefficients of powers of \( y \) may be equated, since the equation now is an identity.

The result for the zero order term is:

\[ \left( \frac{gbb^2}{A^2} \right) \left[ 1 - \frac{4+3\pi}{6\pi} - \left( \frac{4+3\pi}{24\pi} \right)^2 \right] = \frac{4+3\pi}{3\pi} \]

\[ \frac{gbb^2}{A^2} \sim 5.5 = \frac{1}{F^2} \quad (30) \]

But \( \frac{gbb^2}{A^2} \) is the reciprocal square of a Froude number

\[ F \sim 0.4 \]

The nature of the approximations used requires that any conclusions deduced from this value must be speculative, but it at least suggests that the solution to (12) and (26) of physical significance for
separated flow may be a singular solution with a constant value of $F$. Since $F > \frac{1}{\pi}$ for quasi-potential flow, the form and result of (30) suggest that $F = \frac{1}{\pi} \sim 0.32$ is a reasonable restraint condition to investigate.

It should be noted that the densimetric Froude number, $F_b$, as used in equation (30), differs slightly from that defined in equation (15). The term for depth, $b$, in equation (30) no longer represents the depth from surface to outlet depth, but is now one-half the thickness of the interflowing layer. As long as $F > 0.32$, the densimetric Froude number will be the same, whether defined by (15) or (30).

Since $d$, the depth to outlet, is generally known, we will normally calculate and refer to the densimetric Froude number as defined by equation (15). It is convenient to redefine $F$, however, in terms of discharge rather than velocity:

$$F = \frac{u}{\sqrt{g \frac{4\rho}{\rho} d}}$$

And $\beta = \frac{4\rho}{d}$  $u = \frac{Q}{wb}$ where $w$ is the width

$$F = \frac{Q}{d^2 w \sqrt{g\beta/\rho}}$$

(31)
In this form the densimetric Froude number is easily calculated from the available parameters. If \( F < \frac{1}{\pi} \), we know that the flow is separated, but we do not immediately determine the thickness of the interflowing layer. To obtain this thickness, we use the fact that for the densimetric Froude number using the depth of the interflowing layer in place of total depth, the value remains constant for separated flow. If we call this constant densimetric Froude number \( F_b \), then it too may be written in terms of discharge:

\[
F_b = \frac{A}{b \sqrt{g \beta}} = \frac{1}{\pi}
\]

\[
\frac{Q}{b^2 w \sqrt{g \beta / \rho}} = \frac{1}{\pi}
\]

\[
\frac{Q}{w \sqrt{g \beta / \rho}} = \frac{b^2}{\pi}
\]

If this is substituted in our previous expression for \( F \), we obtain a relationship between the densimetric Froude number, depth to outlet, and thickness of interflowing layer:

\[
F = \frac{1}{\pi} \left( \frac{b^2}{d^2} \right)
\]  

(32)

Values of the densimetric Froude number have been calculated using equation (31) for various flows in the model. Corresponding
values of \( \frac{b_i}{d} \) have been used to experimentally check the validity of equation (32).

Both equations (31) and (32) have been compared against field data from actual reservoir operation.
EXPERIMENTAL WORK

Model Design

The design criteria for the experimental apparatus are maintenance of basic assumptions, versatility, and practicality. The experiment is conducted in an open channel flume 192 inches long, 24 inches high, and 13 inches wide, constructed of 3/8-inch acrylic plastic, commonly known as Plexiglass. For fluid model studies, Plexiglass has the desirable qualities of ease of construction, visibility of stream flow, and low drag characteristics. The sides and bottom of the flume are joined tightly together and di-Chloromethane is injected along the joint with a hypodermic needle. Capillary action draws the solution evenly throughout the joint. The di-Chloromethane dissolves the edges of the Plexiglass to be joined. The edges, upon rehardening, form a bond that makes a unified assemblage that is strong and watertight.

The dimensions of the flume are determined by the type of flow desired and practicality. The width of the flume is chosen to minimize side effects in maintaining the assumption of two-dimensional flow, and to allow as large a volume as practical to minimize the decrease in head pressure at discharge and maintain the assumption of steady flow conditions. The discharge orifice has a rounded entrance
to eliminate the formation of eddies about the orifice. The orifice height and width are adjustable to allow variability in height and quantity of discharge. Thus, the basic assumptions of two-dimensional, steady flow are maintained within the limits of practicality.

**Test Procedure**

To achieve the density gradient, the reservoir is filled to the desired overall height with distinct layers of salt water of decreasing densities. The layers are dyed alternately with Rodimeine B-X and blue with Erroglaucine A Supia.

The first layer of water is poured into the reservoir and salt and dye are added and thoroughly mixed in. Spill-boards, designed to float so that their top surfaces will be level with the water surface, are used to reduce turbulence and vertical mixing when additional layers are added (Figure 4). The second layer, containing a predetermined amount of water, salt, and dye to obtain a lighter density, is mixed in a mixing tank and introduced into the reservoir at a constant flow rate onto the spill-boards floating on the surface of the first layer. The lighter density second layer flows horizontally off the spill-boards, resulting in a distinct interface between the layers. To inhibit convective mixing, each new layer is acclimated in a
Schematic Plan of Apparatus

Method of Layering Different Density Fluids

Figure 4. Schematic diagram of experimental apparatus.
reserve tank to the environmental temperature prevailing about the reservoir, thereby maintaining a system as isothermal as possible. This layering procedure is repeated until 15 four-centimeter layers are added to the reservoir. The process requires about 30 hours.

A staircase density profile exists immediately upon completion of the filling process. An 18-hour period is allowed to lapse between the time the reservoir is filled and the experiment is performed. This time is assumed to be sufficient to permit the diffusion process to smooth out abrupt density differences between adjacent layers and thereby promote establishment of the desired linear density distribution. The bottom and top layers are made three-fourths the height of the intermediate layers to compensate for the absence of diffusion at the upper and lower surfaces of the fluid.

The density profile is determined by temperature and salinity readings. The temperature of the fluid is measured in situ with a thermometer, and 200-ml. samples of the salt solution are taken vertically every three centimeters. The salinity of the samples is measured by determining conductivity of the solutions and correlating these readings to conductivities for known salinities. Density as a function of salinity and temperature can then be determined.

After this data is taken, the experiment is ready to be performed. The spill-boards are carefully removed. The orifice,
which was previously adjusted to the desired height, is opened to the
desired discharge. The discharge rate is determined by observing
the time required for the free surface in the reservoir to drop one
centimeter in height. Cameras placed opposite the orifice and eight
feet upstream are used to record the flow patterns. To obtain the
velocity profile, vertical dye streaks of concentrated Erroglaucine
A Supia dye are injected with hypodermic needles from surface to
bottom at the centerline of the reservoir. The dye comes in pow-
dered form and is prepared by mixing with water. The density of
the dye solution is approximately that of the fluid in the reservoir
and, therefore, does not move vertically after being introduced.
Erroglaucine A Supia is particularly well suited for dye streak use
because of its controllable density, low diffusity rate, and very dis-
tinct dark blue color. The horizontal movement of the vertical dye
streaks at discharge are recorded by the cameras taking pictures at
15-second intervals. A grid on the side of the reservoir gives a
measure of the relative displacements. By projecting frame by
frame, the successive lateral positions of the dye streak profiles are
analyzed to give the velocity profile. Care is taken to analyze only
the dye streaks in the center of the photographs so that any effects
of parallax are eliminated.
The vertical flow patterns and separation heights are determined in a similar manner. The interfaces of the layers allow observation of vertical movement of the fluid. The photographs of the flow pattern in the vicinity of the orifice are used to determine the height of the internal density current.

The model allows variability of all the independent variables, and these variables can be measured directly. The flow rate can be changed by adjusting the gate opening. The orifice height is adjustable. The density gradient is varied by changing the salt concentrations of the various layers. The density gradient and orifice height are measured before running the experiment, and the flow rate is measured during the run. The dependent variables, velocities and heights of internal density current, are determined by analyzing the pictures after the experiment. Thus, the model allows a complete variation of conditions, and this complete range of conditions can be determined experimentally.

Results

The experimental data is summarized in Table 2, showing that the depth of flow varies from 54.6 cm in Test 1 to 31.0 cm in Test 15. The densities in the experiment vary from 0.99760 g/cc to
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1.00728 g/cc, with the density gradients varying from $1.94 \times 10^{-5}$ g/cc/cm. The flow rate per unit width varies from $19.6 \text{ cm}^2/\text{sec}$ to $75.4 \text{ cm}^2/\text{sec}$.

Figure 5 shows a photograph of the flow during the experiment. The upper and lower boundaries of the internal density currents are determined by studying the photographs of the flow patterns. The different colored layers of varying density indicate the streamlines of the flow. All the streamlines of the internal density current converge to the outlet, clearly indicating the extent of the internal density current. The streamlines above and below the internal density current form long vertical eddies instead of converging to the outlet. Thus, the boundaries are determined from the photographs by measuring the height of the streamline dividing the two types of flow.

The measured vertical velocity profiles are given in Figures 6 through 11. The maximum velocity attained in the experiment is $2.7 \text{ cm/sec}$ in Test 6. A definite velocity gradient exists. It can be seen that the direction of flow is not all towards the outlet. The flow reverses direction above and below certain heights in all the velocity profiles except Test 15. This means that only the interior layer of water flowing towards the orifice, an internal density current, is discharged. This shows that an internal density current can develop when a density gradient is present.
Figure 5. Photograph of flow during experiment.
The divisions of flow are horizontal and extend the length of the tank. The eddies above and below these boundaries account for the reversal in flow. It was not determined if the eddies extended the length of the tank or consisted of several discrete eddies. All of the fluid within the internal density current flows toward the orifice and is the only fluid discharged. The heights of the boundaries are checked by applying the equation of continuity to the velocity profiles.

The vertical velocity profiles of the internal density currents of the tests are summarized in Figure 12. These are plotted using the dimensionless parameters $\frac{V}{V_{\text{max}}}$ vs. $\frac{d}{d_{1,3}}$. $d$ is the height above or below the orifice. $d_{1,3}$ is the height of the boundaries of the internal density current above or below the orifice. $\frac{V}{V_{\text{max}}}$ is the ratio of the velocity to the maximum velocity for each profile. It should be noted that the velocity at the boundaries is not zero, but has some positive value. The average velocity profile is indicated by the dashed line.

Figure 12 is plotted using dimensionless parameters for purposes of comparison, but distorts the actual velocity profiles. Hence, Figure 13 shows an undistorted average velocity profile of the internal density currents. This shows that when a linear density gradient is present, about 60 percent of the height of the internal density current is above the orifice and about 40 percent below the orifice.
Figure 6. Measured velocity profiles for tests 1, 2, and 3.
Figure 7. Measured velocity profiles for tests 4, 5, and 12.
Figure 8. Measured velocity profiles for tests 6, 7, and 8.
Figure 9. Measured velocity profiles for tests 9 and 13.
Figure 10. Measured velocity profiles for tests 10 and 11.
Figure 11. Measured velocity profiles for tests 14 and 15.
Figure 12. Summary of vertical velocity profiles of the internal density currents.
Figure 13. Mean vertical velocity profiles of the internal density currents.
The two-dimensional characteristics of the flow are shown in the observations of the horizontal velocity profiles. Figure 14 shows the velocities given in the ratio $V/V_{\text{max}}$ as a function of position. The data shows that the horizontal velocity profile is nearly uniform to within a relatively small distance from the walls. Thus, it appears that the horizontal velocity gradient is negligible over most of the width. Since stress is a function of the velocity gradient, it can be concluded that, except very near the walls, the flow is dynamically two-dimensional.

The extent of the internal density current as a function of the densimetric Froude number is shown in Figure 15. This shows that the extent of the internal density current decreases as the densimetric Froude number decreases. The internal density current is divided at the height of the orifice into two separate flows, and a densimetric Froude number is determined for each. The upper densimetric Froude number is given as

$$F_u = \left( \frac{q}{d_1+d_3} \right) \frac{1}{d_2^2} \sqrt{\frac{\rho_u - \rho}{\beta_u g}}$$

(33; c.f. 31)

and the lower densimetric Froude number as

$$F_L = \left( \frac{d_3}{d_1+d_3} \right) \frac{1}{d_2^2} \sqrt{\frac{\rho_L - \rho}{\beta_L g}}$$

(34; c.f. 31)
Figure 14. Horizontal velocity profiles.
Figure 15. Extent of internal density current as a function of the densimetric Froude number.
The density gradient has been divided into two separate density gradients for the upper and lower flows, $\beta_u$ and $\beta_l$. This is done because the density gradient varies slightly as the fluid discharges. The quantity of discharge for each flow is determined by multiplying the total discharge by the ratio of the heights of the internal density current measured from the orifice, $d_1$ and $d_3$, and the total height of the internal density current, $d_1 + d_3$. It is desired to present data of a form that can be utilized in an actual reservoir. Thus, the surface height measured from the orifice, $d_2$, is present in the ratio for both the upper and lower flows.

The experimental value of the critical densimetric Froude number is 0.274 determined in Test 15. $F_{cr}$ occurs when the height of the internal density current extends to the surface, $d_1/d_2 = 1.0$, with the velocity at the surface equal to zero. Below the value of $F_{cr}$ an internal density current develops.

The dashed lines in Figure 15 represent the parabolic curve $(d_1/d_2)^2 = F$. It is proposed that this curve describes the fluid. The upper and lower end points are justified by the theoretical analysis. The end points are the value of the critical densimetric Froude number, $F_{cr} = 0.318$. This is the theoretical maximum value of the densimetric Froude number for which an internal density current can develop. At $F_{cr}$, the internal density current extends the full depth.
of the fluid. Drawing the curve through the origin is justified experimentally by the fact that as the flow approaches zero, the height of the internal density current also approaches zero. The shape of the curve approximates the experimental data. Hence, the parabolic curve is assumed, and equation (32) is seen to be satisfied.
DISCUSSION

**Experimental Results**

The experimental results confirmed a critical value of the densimetric Froude number such that for $F > F_{cr} = 0.27$, the entire fluid is in motion. For $F < 0.27$, the thickness of the interflowing layer is limited and becomes thinner as $F$ becomes smaller. In a related experiment, Debler (5) also obtained a value of $F_{cr} = 0.27$. Yih (14, 15, 16, 17) has given a theoretical derivation of $F_{cr} = 0.32$.

The observed value of $F_{cr}$ is consistently less than the theoretically expected result. This is very likely due to viscous effects within the fluid. For $F$ just slightly less than 0.32, we should expect separation and layering. However, the flowing layer will exert a frictional stress upon the adjacent, non-flowing fluid because of the viscous forces which have been neglected in the theoretical analysis. The practical result will be to induce a flow in the adjacent layer because of the stress at the boundary and to extend the thickness of the interflowing layer. Thus, we should expect $F_{cr}$ to be somewhat less than the theoretical value of 0.32 before complete separation of the flow occurs. The observed value of $F_{cr} = 0.27$ is close to the predicted value, and indicates that the assumption of a non-viscous fluid is a good approximation.
The depth of maximum flow velocity, \( V_{\text{max}} \), was sometimes observed to be slightly below orifice level. This effect is probably the result of the model design and operation. As the flow is initiated, it assumes a steady velocity distribution along an axis at orifice depth. As the water level drops, the pressure distribution will tend to maintain \( V_{\text{max}} \) at orifice level. However, since water is withdrawn from below orifice depth, there will also be a tendency for the axis to drop, and the resulting flow to become asymmetric. Extended discharge will also change the density distribution. To reduce these effects, only those data have been considered and used, which were measured shortly after the initiation of flow, during which the steady state was well approximated in the model.

For \( F < 0.27 \), the flow agrees very well with the relation predicted in equation (32). This relationship is also shown in Figure 15. Here, again, the experimental values obtained for the thickness of the layer, appearing as \( \frac{b_1^2}{d_1} \) or \( \frac{d_1^2}{d_2} \), are slightly greater than the predicted values. This, however, is to be expected, since the prediction was based on the assumption that the water was non-viscous. In general, the observed data seem to be in very good agreement with the predicted values, and confirm the assumption of equation (30), and equation (32). Equation (32) may now form a basis for predicting thickness of interflowing layers.
Application to Prototype

The application of the model study to a prototype is by the law of similarity. If similarity is to exist between flow in the prototype and flow in its model, all of the significant dimensionless parameters in the model must have the same numerical value as the corresponding parameter in the prototype. In the type of flow in this study, similarity laws require the model and prototype to be similar geometrically, kinematically, and dynamically.

In this model study, we have considered only two-dimensional flow, with an infinite upstream extent. The significant geometric dimensions are the vertical direction and the horizontal direction parallel to the direction of flow. Since the flow is infinite in the horizontal direction, only the vertical dimensions are pertinent in this case. Thus, for the model and prototype to be geometrically similar, it is sufficient that one can be obtained from the other by changing all vertical linear dimensions in the same ratio. This is equivalent to saying that ratios of vertical dimensions must have the same value in prototype and model. In particular:

\[
\begin{bmatrix}
  b' \\
  d' 
\end{bmatrix}_{\text{model}} = \begin{bmatrix}
  b \\
  d 
\end{bmatrix}_{\text{prototype}}
\] (35)
Kinematic similarity exists when the streamline patterns in the model and prototype are the same. Since laminar, non-turbulent flow is expected under normal conditions in both model and prototype, we can expect the streamlines upstream to be essentially parallel and horizontal in both the model and prototype.

Dynamic similarity exists when the ratios of forces at corresponding points in the flow have equal values in both model and prototype. Since there are normally three forces—inevita, gravity, and friction—to consider in normal flow, there will be normally two independent parameters to scale. These are usually taken to be the Froude number, the ratio of inertial force; and the Reynolds number, the ratio of inertial to viscous forces.

In a stratified fluid, it is also necessary to consider the effects of variable density. Thus, the density gradient, \( \beta \), is incorporated into the Froude number, and the non-dimensional parameter is called the densimetric Froude number. This may be written from equation (31) as:

\[
F_b = d \frac{u}{\sqrt{g\beta/\rho}}
\]  

(36)

In similar flow, the Froude numbers will be the same in model and prototype. In particular, then, since \( g \) and \( \beta \) will have the same value in model and prototype:
The Reynolds number is usually given in the form:

\[
\text{Re} = \frac{\rho q}{\mu}
\]  \hspace{1cm} (38)

For model studies using the same fluid, water, in both prototype and model, the viscosity and density will be the same in model and prototype. For similarity, it is then required that the discharge per unit width, \( q \), be the same in model and prototype. Since prototype discharges are very high, this becomes impossible for the model. An alternative is to use a highly viscous material in the model. However, the nature of the problem being investigated had indicated that inertial and gravity effects are more significant than viscous effects. Hence, the viscous effects and the Reynolds number have been neglected. The strong agreement between the theoretical results for an inviscid fluid and the experimental results confirm that viscous effects are not significant in the case of the model. Since the velocity gradients are much greater in the model than in the prototype, we may assume that viscous effects may also be neglected in the prototype.
Direct testing of equation (32) is difficult, since the velocities within the reservoir, which would have to be measured, are very small. It is possible, however, to check indirectly the applicability of equation (32) by temperature effects. That the temperature of the discharged water must agree with a temperature calculated using withdrawal according to equation (32) is a necessary, but not a sufficient condition, since other flows might also produce the "correct" temperature of discharged water. It is a useful procedure, however, since one of the proposed uses for multiple outlet reservoirs is to control downstream temperatures. Discharge temperatures have been calculated for several reservoirs. The results are presented in Table 3. In general, they agree very well with the observed temperatures and provide confidence that this method may be used to estimate temperatures of withdrawn water under various conditions.
<table>
<thead>
<tr>
<th>Date</th>
<th>Q</th>
<th>d</th>
<th>W</th>
<th>$\beta \times 10^5$</th>
<th>F</th>
<th>b/d</th>
<th>b</th>
<th>$T_{\text{calc}}$</th>
<th>$T_{\text{obs}}$</th>
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<td>Hoover</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9/29/48</td>
<td>31,600</td>
<td>136</td>
<td>757</td>
<td>3.10</td>
<td>0.138</td>
<td>0.658</td>
<td>89</td>
<td>64.2</td>
<td>64</td>
</tr>
<tr>
<td>7/24/48</td>
<td>33,000</td>
<td>146</td>
<td>757</td>
<td>3.18</td>
<td>0.093</td>
<td>0.583</td>
<td>85</td>
<td>61.9</td>
<td>61</td>
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</tr>
<tr>
<td>7/24/47</td>
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<td>198</td>
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<td>1.20</td>
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<td>1.48</td>
<td>0.0045</td>
<td>0.119</td>
<td>29</td>
<td>57.0</td>
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<td>0.150</td>
<td>32</td>
<td>58.1</td>
<td>56.8</td>
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<tr>
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<td>200</td>
<td>1250</td>
<td>1.69</td>
<td>0.0069</td>
<td>0.169</td>
<td>34</td>
<td>57.2</td>
<td>59.0</td>
</tr>
</tbody>
</table>

- Q - discharge (ft$^3$/sec)
- d - depth of outlet (ft)
- W - width of reservoir at mid-depth (ft)
- $\beta$ - density gradient (slug/ft$^2$/ft)
- F - densimetric Froude number
- b - height (depth) of internal density current from outlet elevation (ft)
- $T_{\text{calc}}$ - calculated discharge temperature (°F)
- $T_{\text{obs}}$ - observed discharge temperature (°F)
SUMMARY AND CONCLUSIONS

The internal density currents which discharge in a stratified reservoir during withdrawal have been investigated using a model constructed to allow variable heights, rates of discharge, and density gradients. The actual currents observed have been used to check Yih's theoretical analysis of the effect of density variation on steady fluid flow in a two-dimensional, frictionless fluid. Velocity profiles photographed in the model have shown that a velocity gradient exists with the maximum velocity at approximately the height of the orifice, and with a net positive velocity at the upper and lower boundaries of the internal density current. The velocity profiles to be expected in an actual reservoir should approximate the model velocity profiles.

The most significant result of the experiment was the demonstration that internal density currents develop for densimetric Froude numbers below a certain critical value. This critical value of the densimetric Froude number was experimentally determined to be 0.274. This agrees well with the theoretical value of 0.318 proposed by Yih. Above the critical value, quasi potential flow is found throughout the fluid.

The result suggested by an approximate solution for bounded flow that $F = 1/\pi$ for all bounded flows has been shown to be verified
experimentally, and a relation developed to allow the determination of thickness and extent of the flowing layer under various conditions. The results have been presented graphical form using dimensionless parameters. Flow conditions in an actual reservoir can be predicted using these graphs provided the geometry, density gradient, and rate of discharge are known. Similitude imposes the restrictions on the prototype that the streamlines must be essentially horizontal upstream of the dam, that the flow is steady and can be approximated as two-dimensional, and that an approximate linear density gradient exists.

In nature, this prototype is commonly found in high head dams, where the relatively deep reservoir allows thermal stratification to develop. These constitute, however, a majority of the system of interest. A comparison with available prototype data indicates that the model results should have wide application and be useful under many situations.
BI B L I O G R A P H Y


APPENDIX
Appendix 1. Yih's Theoretical Analysis

Starting with equation (9), Yih (15) has shown that the terms involving the squares of the derivatives of \( \psi \) can be eliminated if the new stream function \( \psi' \) is used:

\[
\psi' = \int_0^{\psi} \sqrt{\rho} \, d\psi \tag{1-A}
\]

Using this transformation and the fact that for steady flow streamlines are isopycnic lines, density a function of \( \psi \) only, equation (9) can be reduced to the simpler form

\[
\nabla^2 \psi' + gy \frac{d\rho}{d\psi} = H_1(\psi') \tag{2-A}
\]

If the fluid is considered to originate at infinity where the velocity is zero, and starts to flow far upstream, the flow is irrotational far upstream from the outlet with streamlines, \( \psi \) constant, horizontal and uniformly spaced. Since the streamlines are imbedded in horizontal planes here, the gravity effects can be neglected. Thus, if \( \psi'_o \) is the stream function for the flow far upstream,

\[
\psi'_o = -Ay, \tag{3-A}
\]

where \( A \) is a positive constant. From equations (7) and (1-A), we find that the vertical velocity distribution of the actual flow far upstream is given by

\[
u_o = \frac{A}{\sqrt{\rho}}
\]
If the density far upstream is assumed to decrease linearly with depth, then

\[ \rho = \rho_0 - \beta y \quad \quad \beta = \frac{\rho_0 - \rho}{d} \quad (5-A) \]

where \( \rho_0 \) is the density at the bottom of the reservoir, where \( y = 0 \), and \( d \) is the distance between the surface and the bottom. Since the fluid density is constant along a streamline \( \psi'_0 \), substituting equation (3-A) into (5-A), the expression for the density at any point is given by

\[ \rho = \rho_0 - \beta y = \rho_0 + \frac{\beta \psi'}{A} \]

Thus,

\[ \frac{d\rho}{d\psi'} = \frac{\beta}{A} \]

and equation (2-A) becomes

\[ \nabla^2 \psi' + \frac{g\beta}{A} y = H_1(\psi') \quad (6-A) \]

Since far upstream \( \psi' \) is \(-Ay\), from equations (4-A) and (6-A) \( H(\psi') \) may be evaluated

\[ H(\psi') = \frac{g\beta}{A} y = -\frac{g\beta}{A^2} \psi' \quad (7-A) \]

Therefore, equation (6-A) may be written

\[ \nabla^2 \psi' + \left(\frac{g\beta}{A^2}\right) \psi' = \left(-\frac{g\beta}{A}\right) y \quad (8-A) \]
which is a linear equation in $\psi'$ with constant coefficients. Linearization of equation (9) has been achieved by transformation so that terms involving squares of the dependent variable, $\psi$, have been eliminated.

Equation (8-A) may be put in dimensionless form with the transformations

$$
\frac{y}{d} = \eta \quad \frac{x}{d} = \zeta \quad \frac{\psi'}{Ad} = \theta
$$

(9-A)

equation (8-A) becoming

$$
\frac{\partial^2 \theta}{\partial \zeta^2} + \frac{\partial^2 \theta}{\partial \eta^2} + F^{-2}(\theta + \eta) = 0
$$

(10-A)

In which

$$
F^{-2} = \frac{d^2 g \beta}{A^2}
$$

(11-A)

The coefficient $F$ may be given a physical interpretation by noting that

$$
A^2 = \rho u^2
$$

(12-A)

and

$$
\beta = \frac{\rho_o - \rho}{d} = \frac{\Delta \rho}{d}
$$

(13-A)

where $\rho_o - \rho = \Delta \rho$ is the total difference of density in the vertical direction. Thus,
which is the inverse square of the densimetric Froude number of the flow. The densimetric Froude number being

\[ F = \frac{u_o}{\sqrt{\frac{g}{\rho} \frac{d}{\delta}}} \]  

Equation (10-A) may be solved for \( \theta \) in terms of \( \eta \), \( \zeta \), and \( F \) by Fourier series. The boundary conditions are

\[ \theta = -1 \text{ at } \eta = 1, \text{ and } \zeta = 0 \quad \text{ - conditions at surface} \]
\[ \theta = 0 \text{ at } \eta = 0 \quad \text{ - condition at bottom} \]
\[ \theta = -\eta \text{ at } \quad \zeta = -\infty \quad \text{ - condition at } x = . \]

If we set

\[ \theta = -\eta + f \]

then equation (10-A) becomes

\[ \frac{\partial^2 f}{\partial \zeta^2} + \frac{\partial^2 f}{\partial \eta^2} + F^{-2} f = 0 \]  

and the boundary conditions are

\[ f = 0 \text{ at } \eta = 1, \]  
\[ f = -1 + \eta \text{ at } \zeta = 0, \]
\[ f = 0 \quad \text{at} \quad \eta = 0, \quad (20-A) \]
\[ f = 0 \quad \text{at} \quad \zeta = - \quad (21-A) \]

Equations (18-A), (19-A), (20-A), and (21-A) are satisfied by

\[ f = \sum_{n=1}^{\infty} B_n e^{i a_n} \sin n \pi \eta \quad (22-A) \]

in which

\[ a_n^2 = n \pi^2 - F^{-2} \quad (23-A) \]

In order to satisfy equation ( ), the \( B \)'s must be chosen to be the Fourier coefficients of \( \eta - 1 \):

\[ F_n = - \frac{2}{n\pi} \quad (24-A) \]

Yih (15) obtained the solution provided

\[ F > F_{cr} = \frac{1}{\pi} = 0.318 \quad (25-A) \]

in which \( F_{cr} \) is the critical Froude number.