Dangerous Cake

by

Nathan L. Atkinson

A PROJECT

submitted to

Oregon State University

University Honors College

in partial fulfillment of

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degree of

Honors Baccalaureate of Arts in Economics and Philosophy (Honors Scholar)

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Abstract

We explore the limitations of the CRRA utility specification as applied to cake-eating problems. We construct a set of conditions under which cake-eating problems have no optimal solution. Furthermore, we explore how agents choose their consumption path when an optimization problem has no solution, and suggest a method for ranking divergent utility streams.

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Contents

1 Introduction

Cake-eating problems comprise a central element of macroeconomics. The agent is faced with the decision of consuming now or consuming in the future. Cake-eating problems can be generalized to model a wide range of economic phenomena. In many macroeconomic models, agents are faced with a string of decisions that can be formulated as a recursive problem. Using the techniques of dynamic programming, we can solve these recursive problems.

Ramsey (1928) formulated the canonical dynamic programming problem, the infinite horizon deterministic growth problem. In this problem, the social planner maximizes the utility of households subject to the aggregate constraints of the economy. In the Ramsey problem, there is one infinitely-lived household maximizing their inter-temporal utility by choosing between consumption and savings, the household receives wages for hours worked and interest payments for money saved. Furthermore, in this model, households are restricted from accumulating infinite debt to fuel consumption. There are also firms in the economy that maximize profit by choosing quantities of capital and labor. Higher consumption results in greater immediate utility, whereas lower consumption results in higher savings and higher capital accumulation and ultimately increases in utility. The social planner's problem comes down to optimizing utility by balancing consumption and capital accumulation. Ramsey (1928) was able to use calculus of variations to solve for the optimal path of capital accumulation which yields optimal societal utility.

Formally, the Ramsey growth problem can be expressed

$$U = \max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$k_{t+1} = f(k_t) - c_t.$$

Where consumption is constrained by the law of motion of capital accumulation.

The cake-eating problem is a special case of the Ramsey problem. We can express a version of the cake-eating problem by,

$$U = \max_{0 \le c_t \le w_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

$$w_{t+1} = A(w_t - c_t)$$

$$w_0 > 0 \text{ given.}$$

There are two notable differences between these two problems. First, borrowing is prohibited in the cake-eating problem, whereas in the Ramsey problem it is not. Second, the production function in the cake-eating problem as exhibited by $w_{t+1} = A(w_t - c_t)$, is linear. In the most basic form of the problem, an agent is endowed with a finite amount of cake and must choose a consumption path that maximizes the utility derived from the cake over some time period.

The cake-eating problem is a special case of the Ramsey problem that has been studied in considerable detail. The cake-eating problem is a very useful tool in macroeconomics. Macroeconomic models rely on utility-maximizing consumers and profit maximizing firms. Cake-eating problems offer a mechanism for which agent's optimization can be understood. How can we best model an agent that we can place into a macroeconomic model? She is maximizing her utility based on her resource constraints- she has to choose how much to work, save, and spend. Furthermore she lives in an uncertain world. She may lose her job or receive a large inheritance, she may live until she is one hundred or she may die tomorrow. In order to answer macroeconomic questions via models, an economist must first model consumers with the desired level of complexity. Cake-eating problems are a way to model consumer behavior as it applies to macroeconomics. The cake-eating problem can be formulated in varying degrees of complexity to satisfactorily model consumers. After formulating and studying the general problem, a number of complexities can be added. This paper will focus on a general, deterministic case, but some extensions are outlined below.

The literature of cake-eating problems have explored a wide variety of topics, and have suggested means of optimizing consumption under a wide range of constraints. Stokey and Lucas (1989), Ljungqvist and Sargent (2000), and Adda and Cooper (2003) offer a comprehensive overview of applications and solutions of dynamic programming and cake-eating.

Adding stochastic elements to cake-eating problems can help explore a wide variety of problems. Adda and Cooper (2003) provide a useful survey of the ways that random events can be incorporated into cake-eating problems.

Suppose for example that we introduce taste shocks into the problem. In other words, there is uncertainty about the agent's appetite in the future. We can imagine that the consumer begins each period either hungry or already satiated. If she is hungry, her utility will be scaled up, whereas if she is already satiated, her utility will be scaled down. We can then express the agent's utility by

$$\epsilon u(c),$$

where ϵ is a positive random variable representing the magnitude of the taste shock. We can assume that ϵ satisfies the Markov property, which implies that the value of ϵ in any time period depends solely on the value of ϵ in the previous time period. The Markov property allows us to find the probability of ϵ changing between states.

In the nonstochastic cake-eating problem, the agent is concerned solely with the size of the cake, whereas now the agent needs to take taste shocks into account as well. Depending on the probabilities of the taste shocks in the next period, the agent may alter her consumption to maximize utility. Even with random taste shocks, we can solve for the agent's optimal policy function. The fact that a cake-eating problem can handle stochastic shocks allows it to more accurately model consumer behavior.

Research has also been done into optimal consumption given an unknown lifespan. Kumar (2005) explores the optimal horizon for cake eating problems. Given an unknown lifespan, how ought an agent consume her cake? Knowing the probability distribution, and using Bayesian updating, the agent can formulate optimal consumption in each period, given current information.

The usefulness of cake eating problems is not limited to modeling consumer behavior. Researchers have also used the cake-eating problem to explore the optimal consumption of an unknown quantity of a resource. Kemp (1976) first explored this question, and the discussion has been continued in Kemp (1977), Gilbert (1979), and Kemp and Long (2007), among others.

This paper explores the following question: under what conditions do cake-eating problems have no optimal solution? Furthermore, suppose that no optimal consumption path exists. How then should consumers choose their consumption path? In order to explore these questions, the remainder of this paper is structured as follows. In section 2 we develop the basic cake-eating problem. In section 3 we move on to solve cake-eating problems under specific constraints, including a set of constraints that yield no optimal solution. In section 4 we explore how consumers might behave when presented with utility streams that diverge to $-\infty$.

2 The Problem

Cake-eating problems are applications of dynamic programming to the consumption-savings problem. In the most basic form of the problem, an agent is endowed with a finite amount of cake and must choose a consumption path that maximizes the utility derived from the cake over some time period (In this paper we will focus on infinite-horizon problems). The amount of cake available in time t is denoted by w_t and utility in time t as a function of consumption is given by $u(c_t)$. Furthermore the agent is impatient, as is represented by the discount factor $\beta \in (0, 1)$. The consumer is thus faced with the following problem,

$$U = \max_{0 \leq c_t \leq w_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

$$w_{t+1} = A(w_t - c_t).$$

$$w_0 > 0 \text{ given.}$$

Note that in this version of the problem, borrowing of cake is prohibited, as evidenced by $0 \leq c_t \leq w_t$. Furthermore, the evolution of cake over time is governed by, $w_{t+1} = A(w_t - c_t)$, where A is the growth rate of the cake (which in the context of a consumption-savings problem often corresponds to the interest rate).

Now that we have a cake and a utility function, how do we go about solving the problem? We will look at two approaches, solving using the Euler Equations and the Transversality Condition, which Adda and Cooper (2003) refer to as the direct attack method, and secondly by using optimal control theory and the methods developed by Richard Bellman.

2.1 Direct Attack Approach

The direct attack approach relies on the relevant *Euler Equation*. The Euler Equation is a first order condition for optimality in a dynamic programming problem. It restricts consumption paths where we can increase utility by shifting consumption across time periods.

2.1.1 Derivation of the Euler Equation

We can derive the Euler Equation by considering a simple example and using an arbitrage argument in the vein of Dixit (1990). Consider a two period problem expressed by the following:

$$\max_{0 \leq c_t \leq w_t} \sum_{t=0}^{1} \beta^t u(c_t) \tag{4}$$

$$w_0 > 0 \text{ given}$$

$$w_1 = A(w_0 - c_0).$$

We can internalize the constraint and rewrite this problem as,

$$\max_{0 \leq c_0 \leq w_0} u(c_0) + \beta u(A(w_0 - c_0)) \tag{5}$$

$$w_0 > 0 \text{ given.}$$

We can now see that our two period utility is determined solely by the choice of c_0, since the agent will consume the entirety of the remaining cake in the second time period. Therefore the utility the agent derives at $t = 1$ is entirely dependent on her choice of c_0.

Suppose that our agent arbitrarily chooses his first period consumption, c_0, and contemplate a small change in consumption. If this new consumption path results in a higher utility level than the previous path, the agent shifts her consumption to the new path. She continues this process until she finds a consumption path for which there exists no other path that yields superior utility. The inability to find a superior path proves that the agent has found an optimal path.

Using the above method, suppose that we have found an optimal path. What characterizes the states of this path? We once again consider a small change. Let dc_0 represent an infinitesimally small change in our consumption of c_0. Consider consuming dc_0 less at $t = 0$. This frees up a bit more cake for consumption at $t = 1$. So at $t = 1$ the agent can consume Adc_0 more than on the previous path. How will this affect the agent's utility? Her previous utility was given by $U = u(c_0)+\beta u(A(w_0-c_0))$, whereas her new utility is be given by,

$$\hat{U} = u(c_0 - dc_0) + \beta u(A[w_0 - (c_0 + dc_0)])$$

.

Looking at the change between the two, the agent's utility decreased by the marginal utility multiplied by the decrease in consumption at $t = 0$, and increased by the discounted marginal utility multiplied by the product of the growth rate and the change in consumption at $t = 1$. In other words the change in utility was equal to

$$\Delta U = U - \hat{U} = -u'(c_0)dc_0 + A\beta u'(c_1)dc_0. \tag{6}$$

In order to be at an optimal solution, the agent can not increase her utility by changing c_0. Therefore it must the case that $-u'dc_0 + A\beta u'dc_0 = 0$, which implies the Euler Equation,

$$u'(c_0) = A\beta u'(c_1). \tag{7}$$

In other words, at optimality, the marginal utility of consumption at time t must be equal to the product of the growth rate and the discounted marginal utility at time $t+1$. If this condition does not hold, the agent can increase her utility by substituting consumption across time periods.

The above argument showed that the Euler Equation must hold in a two-period problem, however we can easily extend this condition to an n-period problem. Since the Euler equation must hold between adjacent time periods, all of the periods are linked together through a system of Euler equations. These equations can be combined to express the Euler equation in generality,

$$u'(c_0) = A^t \beta^t u'(c_t). \tag{8}$$

However it is important to note that as a first order condition, the Euler equation is necessary for an optimal solution, but it is not sufficient. In fact it is possible for a given objective function to have many consumption paths that satisfy the Euler equation. In order to identify which path is the optimal path, we rely on the *transversality condition*. The transversality condition limits our accumulation of wealth in the infinite horizon. In other words, no cake should be saved in the last period unless it is costless to do so. Formally the transversality condition for the cake-eating problem is,

$$\lim_{t \to \infty} \beta^t u'(c_t) w_t \leq 0. \tag{9}$$

The proof of the transversality condition lays beyond the scope of this paper, for a treatment see any advanced macroeconomics text or Kamihigashi (2001).The transversality condition and the Euler equations give us enough information to solve for the optimal consumption path, when one exists. The initial condition is given, the transversality condition is the terminal condition, and the Euler equations describe the path that links the initial and terminal conditions. We can use these conditions to solve recursively for the optimal solution when one exists.

2.2 Bellman Approach

Once again we start with a cake of size w_0. Bellman's insight into dynamic programming problems was that we can break an infinite-horizon problem into a two period problem. This can be done through the use of a *value function*. The value function $V(w_n) = \sum_{t=n}^{\infty} \beta^t u(c_t^*)$, gives the total discounted utility of an optimal stream of consumption c_t^*, given an amount of cake w_t. So instead of optimizing (3) directly, the agent now faces the following functional equation,

$$V(w) = \max_{0 \le c_t \le w_t} u(c_t) + \beta V(w_{t+1}), \tag{10}$$

$$w_{t+1} = A(w_t - c_t)$$

$$w_t > 0 given$$

where we are solving for the function V($\cdot$). Rather than finding the entire consumption path $\{c_t\}$, Bellman's method allows us to optimize our utility by choosing consumption in one period subject to the constraint of maximizing the value function in the next period. Bellman observed that the consumer doesn't need to know the entire consumption path $\{c_t\}$ in order to determine present consumption. Instead, she only needs to know that she will choose $\{c_t\}$ optimally given the state w_{t+1}. This is known as Bellman's principle of optimality, and from this we can derive an important first order condition.

Internalizing the constraint $w_{t+1} = A(w_t - c_t)$ into (10) yields,

$$V(w) = \max_{0 \leq c_t \leq w_t} u(c_t) + \beta V(A(w_t - c_t)), \tag{11}$$

we can maximize this value function by taking the derivative with respect to c_t and setting equal to zero (Assuming V is differentiable), which yields,

$$u'(c_t) = \beta V'(w_{t+1}). \tag{12}$$

However, we do not know what the derivative of value function, V', is. In order to take the derivative of V, we look back to (10), which holds for all values of w. Looking back to the derivation of the Euler Equation, we have that $u'(c_t) = A\beta u'(c_{t+1})$. Combining this with (12) yields

$$A\beta u'(c_{t+1}) = u'(c_t) = \beta V'(w_{t+1}),$$

which allows us to express the Euler Equation in terms of the value function,

$$Au'(c_{t+1}) = V'(w_{t+1}). \tag{13}$$

As in the direct attack approach, the sequence of Euler equations make up necessary conditions for w in each time period. The next step in finding a solution via the Bellman approach is to find the relevant *policy function*. The policy function tells a consumer how much cake they ought to consume as a function of their stock of cake. The policy function can be written in the form $c_t = f(w_t)$, and the stock of cake can be written in the form $w_{t+1} = A(w_t - f(w_t))$. Plugging these two values into the Euler equation allows us to express the problem in terms of policy functions alone,

$$u'(f(w_t)) = A\beta u'(f(Aw_t - Af(w_t))). \tag{14}$$

If we can solve for closed form solutions to the value and policy functions, our work is done. However, Adda and Cooper (2003) explain that generally, it is not possible to find these closed form solutions.

Stokey and Lucas (1989) offer a number of results on the existence of solutions to the value function. If we formulate the problem as they do, then we have,

$$V(s) = \max_{w_{t+1} \in \Gamma(w)} u(c_t) + \beta V(w_{t+1}), \text{for all w} \in W. \tag{15}$$

Where $\Gamma(w)$ is the constraint set. We can then determine whether there is a value function that solves (15).

Theorem 1 *Assume that $u(c_t)$ is real-valued, bounded, and continuous. Furthermore assume $\beta \in (0, 1)$ and that the constraint set is nonempty, continuous, and compact valued. Then there exists a unique value function that solves (15).*

Proof. See Stokey and Lucas (1989), theorem 4.6. ■

However we can see that our problem does not satisfy the condition that $u(c_t)$ is bounded. The CRRA utility function is not bounded above or below. It turns out that we do not have a solution due to the fact that $u(c) \to -\infty$ as $c \to 0$.

For a more complete analysis of when closed form solutions exist, see Stokey and Lucas (1989). In the cases when no closed form solution exits, we can attempt to solve numerically, or at the very least attempt to understand characteristics of the solution.

3 Applications with Functional Form

We will now explore actual applications of these methods by using utility functions with a given functional form and constraints on the discount rate β and the growth rate A. We will first solve for optimal consumption under logarithmic utility where $A\beta = 1$, and then we will explore the case where $A\beta < 1$ given a constant relative risk aversion utility function.

3.1 Logarithmic Utility

Suppose we are presented with the following problem,

$$U = \max_{0 \leq c_t \leq w_t} \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{16}$$

subject to the following constraints,

$w_{t+1} = A(w_t - c_t)$

$w_0 > 0$ given

$A\beta = 1$

$u(c) = log(c)$.

This problem has a clear solution, and we can easily solve for the optimal consumption path with either the direct attack or Bellman methods.

3.1.1 Solving Logarithmic Utility with Direct Attack

We begin by finding the relevant Euler Equations. We recall from (8) that the marginal utility must be equal across time periods. We can combine this with the fact that $\beta A = 1$, which implies that the discounting of cake is perfectly balanced out by the growth rate of cake. In other words, consumption must satisfy the following:

$$u'(c_0) = A^t \beta^t u'(c_t). \tag{17}$$

Substituting in for the logarithmic utility function and solving for c_t in terms of c_0, we have,

$$c_t = (\beta A)^t c_0.$$

And since $\beta A = 1$, it must be the case that $c_t = c_0$ for all time periods. So the correct consumption path is to consume a constant quantity c across all time periods. We can easily solve for this amount by considering three cases of consumption.

Suppose that given our initial endowment of w_0 in time t_0, we consume an amount of cake such that $c = c_0 > w_0(1-\beta)$. Then our stock of cake in time t_1 will be given by $w_1 = A(w_0 - c_0) < w_0$. It is clear that we can't consume a fixed quantity in every time period if that results in our stock shrinking down. If $c = c_0 > w_0(1-\beta)$, then our stock of cake will eventually go to 0, and the Euler Equations will not hold.

Suppose instead that we consume an amount of cake such that $c = c_0 < w_0(1-\beta)$. Clearly the stock of cake in period t_1 will be given by $w_1 = A(w_0 - c_0) > w_0$. In other words, our stock of cake increases in every time period. This would result in the stock of cake w_t increasing to infinity, but we would still only consume a fixed amount of cake. Clearly this can't be optimal, because we could do better by consuming a little bit more and not letting our stock increase towards infinity. So consuming $c = c_0 < w_0(1-\beta)$ can not be optimal. Formally, we can see that the Euler equations hold, but the transversality condition does not, as the stock of cake increases fast enough to offset the discount factor.

Since we have shown that it isn't optimal to consume $c = c_0 < w_0(1-\beta)$ or $c = c_0 > w_0(1-\beta)$, the only remaining candidate is consuming such that $c = c_0 = w_0(1-\beta)$. Since $\beta A = 1$, our stock will be replenished in each period, so that $w_0 = w_t$. This consumption path alone satisfies the Euler equations and the transversality condition. So the optimal policy is to consume a constant amount of cake, determined by

$$c = w_0(1-\beta).$$

Given that the above maximized (16), the maximum value of U is therefore given by,

$$V(w_0) = \max_{0 \leq c_t \leq w_t} \sum_{t=0}^{\infty} \beta^t log(c_t) = \frac{log(c)}{1-\beta} = \frac{log(w_0(1-\beta))}{1-\beta}.$$

We can then check whether our solution satisfies the transversality condition, (9). Plugging in our calculated value of c, and taking the limit, we have,

$$\lim_{t\to\infty} \beta^t u'(c_t) w_t = \lim_{t\to\infty} \beta^t \left(\frac{1}{w_0(1-\beta)}\right) w_t = 0.$$

Clearly this limit goes to zero because $\beta \in (0,1)$, so β^t goes to zero, while $\left(\frac{1}{w_0(1-\beta)}\right) w_t$ is constant. So our solution satisfies the transversality condition.

We can also arrive at the same solution using the value function approach of Richard Bellman.

3.1.2 Solving Logarithmic Utility with the Bellman Approach

Adda and Cooper (2003) explain that it is often not possible to find policy functions or closed form solutions for most utility specifications. In the cases where no closed form solution exists, we can still attempt to solve numerically or to characterize aspects of the solution. The logarithmic utility function is one of the specifications for which we can solve for the closed form solution.

To solve for the policy function, we will use the method of undetermined coefficients. Adda and Cooper (2003) suggest that we guess and verify that the solution to the functional equation is of the form,

$V(w) = M + Nlog(w)$, for all w.

Now if we can solve for the values of M and N, we will have our solution. However it is not immediately clear that we can find values of M and N that will satisfy the functional equation. Often times we cannot, but in this case we will assume that we can for now. Plugging our guess of the form into (10) yields,

$$M + Nlog(w_t) = \max_{0 \leq c_t \leq w_t} log\left(c_t\right) + \beta\left(M + Nlog\left(w_{t+1}\right)\right). \qquad (18)$$

Taking the first order condition we have that,

$$w_{t+1} = \frac{\beta N}{1 + \beta N} w_t. \qquad (19)$$

We can plug this into (18), which gives,

$$M + Nlog(w_t) = log\left(\frac{w_t}{1 + \beta N}\right) + \beta\left(M + Nlog\left(\frac{\beta N}{1 + \beta N} w_t\right)\right). \qquad (20)$$

We can group terms and use the fact that the functional equation (18) holds for w_t for all t, and work through some messy algebra to solve for the value N,

$N = \frac{1}{1-\beta}$.

Then we can plug this value of N back into (20), and solve for M.

$M = \frac{\beta log(\beta) + (1-\beta) log(1-\beta)}{(1-\beta)^2}$

Since we can solve for M and N, we can be confident that (18) holds for all t. We can then plug the value of N that we calculated into the first order condition, (19), to solve for the decision rule

$w_{t+1} = \beta w_t.$

Once again we find that the agent's optimal solution is to consume a constant fraction of the cake,

$c = (1 - \beta) w_t.$

3.2 CRRA Utility

The Constant Relative Risk Aversion utility function (also known as the isoelastic or power utility function) exhibits some special properties that make it a logical choice for many economic models. Since it is so widely used, we explore some of its limitations here.

We begin by looking at the functional form of CRRA Utility,

$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}.$

an important feature of CRRA utility is that continuity is still preserved as $\sigma \to 1$. Using l'Hopital's rule, we can see that the limit as $\sigma \to 1$=log(c). It is also useful to note that at $\sigma = 0$ corresponds to risk neutrality, and as $\sigma \to \infty$, the agent approaches infinite risk aversion.

Suppose we are presented with the following problem,

$$U = \max_{0 \le c_t \le w_t} \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{21}$$

subject to the following constraints,

$w_{t+1} = A(w_t - c_t)$

$w_0 > 0$ given

$A\beta < 1$

$\sigma > 0$

$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$.

Assuming this maximization problem has a solution, the Euler equation must hold, and tells us that $u'(c_t) = \beta A u'(c_{t+1})$, which assumes an interior solution. It follows that if we let $\psi = (\beta A)^{1/\sigma}$ then,

$$c_t = \psi^t c_0. \tag{22}$$

.

Therefore if c_t satisfies (22), then we can solve recursively for w_t,

$$w_t = A^t \left(w_0 - c_0 \left(\frac{1 - \hat{\psi}^t}{1 - \hat{\psi}} \right) \right), where\ \hat{\psi} = \frac{\psi}{A}. \tag{23}$$

The Euler equation has allowed us to solve for consumption and the stock of cake in any time period. We will now use the transversality condition to narrow the paths that satisfy the Euler equation down to the single optimal path. Recall that the transversality condition states $\lim_{t\to\infty} \beta^t u'(c_t) w_t = 0$. By plugging in equations (22) and (23) into the transveraliy condition, we get

$$\lim_{t\to\infty} \beta^t (\psi^{-\sigma})^t A^t \left(w_0 - c_0 \left(\frac{1 - \hat{\psi}^t}{1 - \hat{\psi}} \right) \right) = 0. \tag{24}$$

Looking at the first part of this equation, we can see that $\beta^t (\psi^{-\sigma})^t A^t = 1$. So it must be the case that,

$$\lim_{t\to\infty}\left(w_0 - c_0\left(\frac{1-\hat{\psi}^t}{1-\hat{\psi}}\right)\right) = 0. \tag{25}$$

(25) is a necessary condition for optimality. We can learn about the conditions under which there exists an optimal consumption path by looking at the value of $\hat{\psi}$. First note that when $\hat{\psi} = 1$, the solution is indeterminate. Consider instead $\hat{\psi} < 1$. This implies that $w_0 - \frac{c_0}{1-\hat{\psi}} \to 0$ as t $\to \infty$. We can then see that $c_0 = (1-\hat{\psi})w_0$. Letting $\alpha = 1 - \hat{\psi}$, and extending this relationship across all time periods yields, $c_t = \alpha w_t$. In other words, when $\hat{\psi} < 1$, the optimal policy is to consume a constant fraction α of the cake.

Next consider the case $\hat{\psi} > 1$. It is immediately clear that (25) does not hold when $\hat{\psi} > 1$. This is because their is no optimal solution to the objective function (21) when $\hat{\psi} > 1$. We will prove this rigorously and discuss the implications of this in the next section.

3.3 A Functional Form without an Optimal Solution

In this section, we explore the conditions under which a cake-eating problem given a CRRA utility function has no optimal solution. To refresh the reader, the problem is given by,

$$U = \max_{0<c_t<w_t} \sum_{t\geq 0} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma}\right) \tag{26}$$

where the evolution of cake over time is governed by,

$w_{t+1} = A(w_t - c_t)$, w_0 given,

and subject to the following constraints:

$A < 1$

$\beta < 1$

$\sigma > 0.$

We know that the Euler equation is a necessary condition for an optimal solution. As such, we have incorporated the Euler equation into our calculation of c_t and w_t as demonstrated by equations (22) and (23). However we can see that when $\hat{\psi} > 1$, we eventually have a negative stock of cake.

Lemma 2 *If c_t satisfies (22), w_t satisfies (23), and $\hat{\psi} > 1$, then there exists a T such that $w_T < 0$ and $c_0 > 0$*

Proof. $w_t < 0 \Leftrightarrow w_0 < c_0 \left(\frac{1-\hat{\psi}^t}{1-\hat{\psi}}\right) \equiv c_0 \left(\frac{\hat{\psi}^t - 1}{\hat{\psi} - 1}\right)$ ■

Corollary 1 *If $0 < c_t < w_t$ and $w_{t+1} = A(w_t - c_t)$ then $c_t \neq \psi^t c_0$*

This then leads us to the fact that if $\hat{\psi}$ is positive, then our maximization problem has no interior solution.

Theorem 3 *If $\hat{\psi} > 1$, then (26) has no solution.*

Proof. Consider an optimal consumption path $\hat{c}_t$. By Corollary (1), (22) does not hold. Therefore it must be that either $\hat{c}_t > \psi \hat{c}_{t-1}$ or $\hat{c}_t < \psi \hat{c}_{t-1}$. To prove these cases, we will consider a slight deviation from the optimal path $\hat{c}_t$, and show that this deviation results in a greater utility, so therefore by contradiction there does not exist an optimal path.

Case 1. $\hat{c}_t > \psi \hat{c}_{t-1}$

$\hat{c}_t > \psi \hat{c}_{t-1} \Leftrightarrow \hat{c}_t^{-\sigma} < \psi^{-\sigma} \hat{c}_{t-1}^{-\sigma} \Leftrightarrow \hat{c}_{t-1}^{-\sigma} > \beta A \hat{c}_t^{-\sigma} \Rightarrow u'(\hat{c}_{t-1}) > \beta A u'(\hat{c}_t).$

If we let $c_{t-1} = \hat{c}_{t-1} + dc_{t-1} \Rightarrow c_t = \hat{c}_t - Adc_{t-1}$. Thus $\{c_t\}$ is feasible. So the total change in utility by contemplating a small change in the consumption path $\hat{c}_t$ is given by,

$$dU = u'(\hat{c}_{t-1})dc_{t-1} + \beta u'(\hat{c}_t)(-Adc_{t-1}) = (u'(\hat{c}_{t-1}) - \beta Au'(\hat{c}_t))\, dc_{t-1} > 0.$$

However the change in utility cannot be positive since $\hat{c}_t$ was defined to be an optimal solution. Therefore there is a contradiction, and it cannot be the case that $\hat{c}_t > \psi \hat{c}_{t-1}$.

Case 2. $\hat{c}_t < \psi \hat{c}_{t-1}$

$$\hat{c}_t < \psi \hat{c}_{t-1} \Leftrightarrow \hat{c}_t^{-\sigma} > \psi^{-\sigma} \hat{c}_{t-1}^{-\sigma} \Leftrightarrow \hat{c}_{t-1}^{-\sigma} < \beta A \hat{c}_t^{-\sigma} \Rightarrow u'(\hat{c}_{t-1}) < \beta A u'(\hat{c}_t).$$

We once again consider a small change in the time periods c_{t-1} and c_t. If we let $c_{t-1} = \hat{c}_{t-1} - dc_{t-1} \Rightarrow c_t = \hat{c}_t + Adc_{t-1}$. Once again the path $\{c_t\}$ is feasible. So the total change in utility by contemplating a small change in the consumption path $\hat{c}_t$ is given by,

$$dU = u'(\hat{c}_{t-1})(-dc_{t-1}) + \beta u'(\hat{c}_t)(Adc_{t-1}) = (-u'(\hat{c}_{t-1}) + \beta Au'(\hat{c}_t))\, dc_{t-1} > 0.$$

However, once again the change in utility from an optimal path $\hat{c}_t$ cannot be positive. Therefore there is a contradiction and it cannot be the case that $\hat{c}_t < \psi \hat{c}_{t-1}$.

Therefore if $\hat{\psi} > 1$, our maximization problem has no solution. ■

We can now ask the question why is there no optimal solution when $\hat{\psi} > 1$? What characterizes our utility stream when $\hat{\psi} > 1$? We can in fact see that when $\hat{\psi} > 1$, our infinite horizon utility, U, necessarily diverges to $-\infty$. To show that this is the case, we will construct a consumption path that bounds any feasible consumption path from above, and show that this path diverges to $U = -\infty$.

Theorem 4 *If $\hat{\psi} > 1$, then $U = -\infty$.*

Proof. Suppose to the contrary that there exists a consumption path such that we could consume the entire stock of cake in each time period, and still be left with cake in the next period. In effect, we would be consuming one unit of cake multiple (and perhaps infinite) times. Let this consumption path be denoted $\{\bar{c}_t\}$, and be governed by the following:

$$\bar{c}_t = w_t = Aw_{t-1} = A\bar{c}_{t-1}.$$

In each period we consume the entire stock of cake, and in the next period our stock of cake is equal to the previous period's stock cake multiplied by the growth factor, A. We can then solve recursively to get $\bar{c}_t$ in terms of $\bar{c}_0$,

$$\bar{c}_t = A^t \bar{c}_0.$$

Under the path $\{\bar{c}_t\}$, the agent is allowed to consume one piece of cake multiple times. For this reason, the consumption path $\{\bar{c}_t\}$ would be clearly superior to any feasible consumption path in which the agent is allowed to consume each piece of cake only once. As such $\{\bar{c}_t\}$ acts as an upper bound on all of the feasible consumption paths. Substituting this value of $\bar{c}_t$ into the objective function yields the following life time utility, $\bar{U}$ under the path $\bar{c}_t$:

$$\bar{U} = \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t) = \frac{\bar{c}_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} \beta^t A^{t(1-\sigma)} = \frac{\bar{c}_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta A^{(1-\sigma)})^t. \tag{27}$$

Since $\sigma > 1$, it is clear that $\frac{\bar{c}_0^{1-\sigma}}{1-\sigma} < 0$. Furthermore, we can see that the summation term diverges to ∞, because $\hat{\psi} = (\beta A^{1-\sigma})^{1/\sigma} > 1$ implies that $\beta A^{1-\sigma} > 1$, which implies that $\sum_{t=0}^{\infty} (\beta A^{(1-\sigma)})^t = \infty$. Putting these two results together, we have that,

$$\bar{U} = \frac{\bar{c}_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta A^{(1-\sigma)})^t = -\infty. \tag{28}$$

So the path $\{\bar{c}_t\}$ yields total utility $\bar{U} = -\infty$. Due to the specification of $\{\bar{c}_t\}$, the path $\{\bar{c}_t\}$ necessarily results in a higher lifetime utility than any feasible consumption path. In fact for any feasible consumption path $\{c_t\}$, it must be the case that for any time period, $c_t \leq \bar{c}_t$. Since the agent must consume less on $\{c_t\}$ than on $\{\bar{c}_t\}$, the utility derived from $\{c_t\}$ is bounded above by that of $\{\bar{c}_t\}$. Since $\{c_t\}$ is bounded above by $-\infty$, $U(c_t) = -\infty$. Therefore there is no feasible path for which a consumer can achieve an optimal solution. ■

Since the agent's infinite horizon utility necessarily diverges to $-\infty$, does she have any decisions to make? Does it make a difference to her whether she consumes all of her cake in the first period or never consumes it at all? In the next section we suggest a method for comparing divergent utility streams in order to answer these questions.

4 Comparing Divergent Utility Streams

In this section we explore the question of how we might create a preference ordering among divergent utility streams. We continue to look at the problem given in the last section, CRRA utility and $\hat{\psi} > 1$.

We begin with the simplest case. Suppose that an agent has the choice between two consumption paths that yield the utility streams $u(z_t)$ and $u(y_t)$, both of which diverge to $-\infty$ as $t \to \infty$. If $u(z_t) > u(y_t)$ for all t, then surely the agent should prefer the utility stream $u(z_t)$ to that of $u(y_t)$. In fact under these conditions, the agent should strictly prefer the consumption stream $\{z_t\}$ to that of $\{y_t\}$.

An example of a path being strictly preferable to another can be easily constructed. Adopting the CRRA utility function used in the previous section, we can allow the consumption path z_t to be governed by $z_t = \alpha w_t$ where $\alpha \in (0, 1)$, and allow the consumption path y_t to be governed by $y_t = \frac{z_t}{2}$. We can then clearly see that the path z_t is strictly preferrable to y_t. This follows from the fact that along z_t, the agent is consuming a positive amount of cake in each time period, and therefore receives a finite amount of utility in each period. Along y_t the agent consumes less in each period than she would on the path z_t. So the utility in each period generated by y_t is strictly less than that generated by z_t. Therefore the consumption path z_t is strictly preferred to y_t. Even though the utility streams from both of the possible consumption paths converge to $-\infty$, we can say that z_t is preferred to y_t.

Next we consider the case of an agent weakly preferring one path to another. Once again consider two paths, $\{z_t\}$ and $\{y_t\}$. If $u(z_t) \geq u(y_t)$ for all t, then the agent should weakly prefer the consumption stream $\{z_t\}$ to that of $\{y_t\}$. It could be the case that $u(z_t) = u(y_t)$ for all t, in which case the agent would be indifferent between the two. However if $u(z_t) \geq u(y_t)$ for all t, then we can say that $\{z_t\}$ is at least as good as $\{y_t\}$.

The above two instances are the simple cases. However, how do we rank consumption paths in which the utility from one path is at some points greater than the other path, and at other times less than that path? If it is the case that there exist points in time such that $u(z_t) > u(y_t)$ and other points where $u(z_t) < u(y_t)$, we cannot simply declare one to be preferable to the other. In order to examine these cases, we can instead look at the partial sums of the utility streams. Let $P(y, n)$ be the partial sum of the utility stream of y_t at time n, so that we have,

$$P(y, n) = \sum_{t=0}^{n} \beta^t u(c_t). \tag{29}$$

We can use this partial sum to compare utility streams. If at a given time t, $P(y, t) > P(z, t)$, then we can say that up to time t, y_t is preferable to z_t. By using partial sums, we can easily compare any two finite period consumption paths.

We can also use partial sums to compare infinite horizon consumption paths. If there exists a point in time, T, such that for all $T < t$ it is the case that $P(y, n) > P(z, n)$, then we can say that as $t \to \infty$, $\{y_t\}$ is preferred to $\{z_t\}$. Since we are attempting to maximize infinite horizon utility, we are only interested in the comparison between utility streams as $t \to \infty$. Alternatively we can compare the two streams as follows,

Definition *If $\exists T$ such that $\forall n > T \Rightarrow P(y, n) - P(z, n) > 0$, then $\{y_t\}$ is preferred to $\{z_t\}$ as $t \to \infty$.*

This method works as long as one of the consumption streams has a finite number of partial sums of utility that are less than the other stream. In other words, it does not matter if one stream alternates between being the superior and inferior stream for a finite period, as long as ultimately there is a time period after which one of the streams is always above the other.

For example if we allow y and z to be governed by the following: $y_t = \alpha w_t$ and $z_t = \gamma w_t$ for $\alpha, \gamma \in (0, 1), \alpha > \gamma$. Since α is greater than γ, it follows that initially the consumption under z_t is greater than that under y_t. This results in the stock of cake under y_t to decay faster than the stock under w_t, which leads to some point n, after which the consumption y_t is always greater than that z_t. Therefore if the partial sums of utility derived from y_t ever surpass z_t, then the path y_t will be preferable as $t \to \infty$.

By looking at partial sums of utility, we can compare any two paths as long as there exists a point T, after which all of the partial sums of one of the streams is superior to the other. So what we are truly trying to find is the path that diverges most slowly to $-\infty$ utility.

Lemma 5 *Given $\hat{\psi} > 1$ if $y_t = \alpha w_t$ and $z_t = \gamma w_t$ and $\alpha, \gamma \in (0, 1)$ with $\alpha < \gamma$ Then* *there exists an N such that $n > N$ implies that* $\sum_{t=0}^{n} \beta^t u(y_t) > \sum_{t=0}^{n} \beta^t u(z_t)$

Proof. Plugging in the CRRA utility function into (29), and noting that consuming a constant fraction of cake α, implies that $w_t = A^t(1-\alpha)^t w_0$. So we can calculate the utility at time n as a function of α,

$$U_n(\alpha) = \frac{(\alpha w_0)^{1-\sigma}}{1-\sigma} \sum_{t=0}^{n} ((\beta A^{1-\sigma})(1-\alpha))^t. \tag{30}$$

We can then separate $U_n(\alpha)$ into two component parts. Let

$$U_n(\alpha) = f(\alpha) \sum_{t=0}^{n} R(\alpha)^t, \tag{31}$$

where $f(\alpha) = \frac{(\alpha w_0)^{1-\sigma}}{1-\sigma}$, and $R(\alpha) = ((\beta A^{1-\sigma})(1-\alpha))^t$.

Looking at $f(\alpha)$, we can se that since $\hat{\psi} > 1$, it must be that $\sigma > 1$, so for any $\alpha \in (0,1)$, $-\infty < f(\alpha) < 0$. Furthermore, $f(\alpha)$ does not depend on t, so its magnitude will be muted by the size of $\sum_{t=0}^{n} R(\alpha)^t$. Looking at $R(\alpha)$, we have that

$$R(\alpha) = ((\beta A^{1-\sigma})(1-\alpha))^t = (\hat{\psi}^{\sigma}(1-\alpha)^{1-\sigma})^t > 1.$$

Therefore $U_n(\alpha) \to -\infty$ as $t \to \infty$.

We can see further properties by computing the derivatives of $R(n)$ with respect to α,

$$R'(\alpha) = \beta(\sigma-1)A^{1-\sigma}(1-\alpha)^{-\sigma} > 0, \tag{32}$$

$$R''(\alpha) = \sigma A^{1-\sigma}\beta(\sigma-1)(1-\alpha)^{-\sigma-1} > 0. \tag{33}$$

Therefore $R(\alpha)$ is increasing and concave up with respect to α. So for any $\alpha < \gamma$, it must be the case that

$$\sum_{t=0}^{n}(R(\gamma)^t - R(\alpha)^t) \to \infty, \text{ as } t \to \infty. \tag{34}$$

So even though it is the case that for $\alpha < \gamma$ that $f(\alpha) < f(\gamma)$, since $f(*) < 0$, we can choose a time N large enough such that $n > N$ implies $U_n(\alpha) > U_n(\gamma)$.

■

So by Lemma 5, given $\hat{\psi} > 1$, and considering any two proportions of the cake consume α and γ, the lesser of the two will always be preferable past some point n. This leads us to the following theorem,

Theorem 6 *There is no most-preferable consumption path of the form* $c_t = \alpha w_t, \alpha \in (0, 1)$.

Proof. Follows directly from Lemma 5.

■

Since there is no most-preferred consumption path of the form $c_t = \alpha w_t, \alpha \in (0, 1)$, we next consider the case of whether there is one of the form $c_t = \alpha_t w_t$.

Lemma 7 *For any path* $c_t = \alpha_t w_t$*, there exists a path* $\hat{c}_t = \gamma_t w_t$*, that is superior.*

Proof.

Our consumption is given by $c_t = \alpha_t w_t$, and our stock of cake is given by $w_t = A(w_{t-1} - c_{t-1})$. If we let $\{c_t\}$ be a consumption path such that $c_t = f(w_t)$, and we let $\{\alpha_t\}$ be a sequence where $\{\alpha_t\} \in (0, 1)$, where,

$$c_t = \alpha_t w_t \qquad \text{and} \qquad w_{t+1} = A(w_t - c_t),$$

then we can use these equations to find c_{t+1} in terms of c_t.

$$c_{t+1} = \alpha_{t+1} w_{t+1} = \alpha_{t+1} A(w_t - c_t) = \alpha_{t+1} A \left(\tfrac{1-\alpha_t}{\alpha_t}\right) c_t$$

We can then extend this recursive process to find c_{t+1} in terms of c_0, and we get the following:

$$c_{t+1} = A^{t+1} c_0 \left(\frac{\alpha_{t+1}}{\alpha_0} \prod_{k=0}^{t} (1 - \alpha_k) \right) \tag{35}$$

Our utility up to time n is therefore given by,

$$U_n(\alpha) = \frac{c_0^{1-\sigma} - n - 1}{1-\sigma} + \frac{c_0^{1-\sigma}}{1-\sigma} \sum_{t=1}^{n} \left(\beta A^{1-\sigma}\right)^t \left(\frac{\alpha_t}{\alpha_0} \prod_{k=0}^{t-1} (1 - \alpha_k) \right)^{1-\sigma}. \tag{36}$$

We can then consider an alternative consumption path $\hat{c}_t$. If we let $\hat{c}_t = \gamma_t w_t$, where $\gamma_t = \frac{\alpha_t}{2}$, then the utility up to time n is given by,

$$U_n(\gamma) = U_n(\frac{\alpha}{2}) = \frac{(\frac{c_0}{2})^{1-\sigma} - n - 1}{1-\sigma} + \frac{(\frac{c_0}{2})^{1-\sigma}}{1-\sigma} \sum_{t=0}^{n} \left(\beta A^{1-\sigma}\right)^t \left(\frac{\alpha_t}{\alpha_0} \prod_{k=0}^{t-1} \left(1 - \frac{\alpha_k}{2}\right) \right)^{1-\sigma}. \tag{37}$$

Comparing $U_n(\alpha)$ and $U_n(\frac{\alpha}{2})$, we can observe three things. First looking at the terms $\frac{c_0^{1-\sigma}-n-1}{1-\sigma}$ and $\frac{(\frac{c_0}{2})^{1-\sigma}-n-1}{1-\sigma}$, we can see that both are finite-valued for any $0 < c_0 \leq w_0$.

Next, we can see that because $\sigma > 1$, it must be that

$$-\infty < \frac{(\frac{c_0}{2})^{1-\sigma}}{1-\sigma} < \frac{c_0^{1-\sigma}}{1-\sigma} < 0 \tag{38}$$

for $0 < c_0 \leq w_0$. This leaves us the terms in the summation to consider. Since

$1-\alpha_k < 1-\frac{\alpha_k}{2}$ and $\sigma > 1$, we can observe that

$$\sum_{t=0}^{n}\left(\beta A^{1-\sigma}\right)^t \frac{\alpha_t}{\alpha_0}\left(\left(\prod_{k=0}^{t-1}(1-\alpha_k)\right)^{1-\sigma}-\left(\prod_{k=0}^{t-1}\left(1-\frac{\alpha_k}{2}\right)\right)^{1-\sigma}\right) \to \infty, \text{ as } n \to \infty. \tag{39}$$

Since the difference between these values diverges to infinity, we can choose an N large enough such that $n > N$ implies that $U_n(\alpha_t) < U_n(\frac{\alpha_t}{2})$.

■

Theorem 8 *There is no most preferable consumption path of the form $c_t = \alpha_t w_t, \alpha_t \in (0,1)$.*

Proof. Follows directly from Lemma 7. ■

So, we have that there is no most preferable consumption path, but given any consumption path there exists a path that is preferable to that one. This allows us to rank paths in order of preference, but it doesn't allow us to decide upon a most preferable path. How then does the consumer choose her consumption path? If she ever decides on a path, she can easily find one that is superior to that one. The consumer can always do better by consuming a smaller fraction of cake in each time period.

5 Conclusion

We have shown that given,

$$U = \max_{0<c_t<w_t} \sum_{t\geq 0} \beta^t \left(\frac{c_t^{1-\sigma}-1}{1-\sigma}\right)$$

$$w_{t+1} = A(w_t - c_t),\ w_0 \text{ given},$$

and subject to the following constraints:

$A < 1$

$\beta < 1$

$\sigma > 0.$

$\frac{(\beta A)^{1/\sigma}}{A} > 1,$

there is no optimal consumption path. Furthermore we have shown that the consumer's infinite-horizon utility will necessarily diverge to $-\infty$. To deal with this problem, we have offered a means of partially ranking divergent utility streams. We have assumed that *If* $\exists T$ *such that* $\forall n > T \Rightarrow P(y, n) - P(z, n) > 0$*, then* $\{y_t\}$ *is preferred to* $\{z_t\}$ *as* $t \to \infty$. Under this convention, we have shown that we can rank consumption paths, and furthermore that given any consumption path, we can find an alternative consumption path that is superior.

We can use these results to examine aspects of cake-eating problems. Given a CRRA utility function, we can use the test of whether $\hat{\psi} > 1$ to determine if there is no solution to a given problem. Given that there is no solution to a particular problem raises the question of whether or not the chosen parameters were a good choice for the model. If we are attempting to model a consumer, and they necessarily end up at $-\infty$ utility, we must ask why that is the case. Does the consumer truly achieve infinitely negative utility, or would a better choice of parameters have resulted in a measurable amount of utility?

Having utility diverge to $-\infty$ poses a number of problems for an agent or for a modeling economist, but we have suggested a means for a partial ranking of divergent utility streams. Using this convention, the agent still has reason to think about her choice of consumption path. This ranking of utility streams is only a partial ranking. Our definition requires that the two utility streams that we are comparing do not oscillate infinitely many times. Instead we need a point after which one utility stream is always above the other. We believe that this is a natural way of ranking divergent streams, but it is still limited by the type of utility streams that may be ranked. In the future we would like to explore the question of how to more completely rank divergent utility streams.

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