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THEORY TO THE EXISTENCE OF EIGENVALUES OF
INTEGRAL OPERATORS

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The existence of eigenvalues is shown for certain types of
integral equations with continuous kernels, the proofs utilizing some
basic results of collectively compact operator approximation theory.

Applications of Collectively Compact Operator Theory
to the Existence of Eigenvalues of Integral
Equations

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TABLE OF CONTENTS

<u>Chapter</u>	<u>Page</u>
I. INTRODUCTION	1
II. COLLECTIVELY COMPACT OPERATOR THEORY	3
Collectively Compact Sets of Operators	4
Reduction of Integral Operators to Finite Algebraic Systems	5
Iterated Kernels and Matrix Theory	8
III. THE EXISTENCE OF EIGENVALUES OF INTEGRAL OPERATORS WITH CONTINUOUS KERNELS	12
BIBLIOGRAPHY	24

APPLICATIONS OF COLLECTIVELY COMPACT OPERATOR THEORY TO THE EXISTENCE OF EIGENVALUES OF INTEGRAL OPERATORS

I. INTRODUCTION

In a recent paper by Russell and Shampine [7], the existence of eigenvalues for certain types of integral operators with continuous kernels is demonstrated. The proofs in [7] employ several basic concepts of numerical integration, finite matrix theory and calculus to derive the existence results for eigenvalues. One of the types of kernels treated in [7] is continuous and positive on the unit square. Russell and Shampine use several uniform continuity and compactness arguments to extend the eigenvalue properties of positive matrices to integral equations with positive, continuous kernels. Anselone and Lee [2] treat the same problem, but instead of continuity arguments, they employ collectively compact operator theory. Two additional classes of kernels discussed in [7] are:

- (a) $k(s, t)$ is complex-valued and continuous on the unit square; the trace of k^p , the p -th iterated kernel, is nonzero for some $p \geq 3$;

and

- (b) $k(s, t)$ is continuous and nonnegative on the unit square; the trace of k^p is nonzero for some $p \geq 1$.

The objective of this paper is to illustrate how collectively compact operator theory can be used to obtain simple derivations of existence and related results for eigenvalues of integral operators. Specifically, as in [7], it is shown here that integral operators with kernels (a) and (b) have nonzero eigenvalues. However, the collectively compact operator theory allows us to sharpen the results obtained in [7]. Namely, we prove that the spectral radius of an integral operator with kernel (b) is a nonzero eigenvalue with an associated nonnegative eigenfunction.

In Chapter II, some fundamental definitions and results of collectively compact operator theory and matrix theory, of use in the subsequent analysis, are reviewed. Then, in Chapter III, the theory is applied to the classes of integral equations with kernels described by (a) and (b). The kernels in this paper are defined on the closed unit interval for convenience, any interval $[a, b]$ would suffice.

II. COLLECTIVELY COMPACT OPERATOR THEORY

The following results of collectively compact operator theory, along with the accompanying notation, will be employed throughout this thesis. For proof of these results, the reader is referred to [1, Ch. 1, 2 and 4].

Let X be a complex Banach space, $\mathcal{B} = \{x \in X \mid \|x\| \leq 1\}$ the closed unit ball and $[X]$ the Banach space of bounded linear operators on X . Of particular importance in this paper are C and C' , the complex Banach spaces of continuous functions defined on the unit interval and the unit square, respectively. Both C and C' have the uniform norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|, \quad x \in C$$

$$\|k\| = \max_{0 \leq s, t \leq 1} |k(s, t)|, \quad k \in C'.$$

If $T_n, T \in [X]$ for $n = 1, 2, \dots$, then $T_n \rightarrow T$ denotes pointwise convergence on X , that is, $\|T_n x - Tx\| \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in X$. For $T \in [X]$, the spectrum of T is denoted by $\sigma(T)$,

$$\sigma(T) = \{\lambda \mid \text{there does not exist } (\lambda - T)^{-1} \in [X]\}.$$

The spectrum is compact and contains the eigenvalues of T . The

number $r(T) = \max\{|\lambda| \mid \lambda \in \sigma(T)\}$ is the spectral radius of T . In the case where T is a compact operator, $\sigma(T)$ has the following description.

Recall that an operator $K \in [X]$ is compact if $K\mathcal{B}$ is relatively compact, i. e., $\overline{K\mathcal{B}}$ is compact. The Arzelà-Ascoli theorem provides a means for identifying relatively compact sets in C and C' .

The Fredholm Alternative [1, p. 119] states that for $\lambda \neq 0$ and $K \in [X]$ a compact operator, there exists $(\lambda - K)^{-1}$ if and only if $(\lambda - K)X = X$, in which case $(\lambda - K)^{-1}$ is bounded. This implies that the nonzero elements of $\sigma(K)$ are all eigenvalues of K . If X is infinite dimensional, then $0 \in \sigma(K)$, and the eigenvalues of K are either finite in number or form an infinite sequence converging to zero [1, p. 120].

Collectively Compact Sets of Operators

A set of operators $\mathcal{K} \subset [X]$ is collectively compact if $\mathcal{K}\mathcal{B} = \{Kx \mid K \in \mathcal{K}, x \in \mathcal{B}\}$ is relatively compact. Every element of a collectively compact set is compact, and

- (1) if Λ is any bounded set of scalars, then $\Lambda\mathcal{K}$ is collectively compact.

Of interest to this thesis is the situation in which $K, K_n \in [X]$,

$n = 1, 2, \dots$, such that

(2) $\{K_n\}$ is collectively compact and

(3) $K_n \rightarrow K$ as $n \rightarrow \infty$.

Together, (2) and (3) imply that K is compact; and the following results concerning the spectral properties of the operators K_n and K hold [1, Ch. 4]:

(4) any neighborhood of $\sigma(K)$ contains $\sigma(K_n)$ for all n sufficiently large;

(5) if $\lambda_n \in \sigma(K_n)$ for $n = 1, 2, \dots$, and $\lambda_n \rightarrow \lambda$, then $\lambda \in \sigma(K)$;

(6) if $\lambda \in \sigma(K)$, then there exist $\lambda_n \in \sigma(K_n)$, $n = 1, 2, \dots$, such that $\lambda_n \rightarrow \lambda$.

(7) $r(K_n) \rightarrow r(K)$ as $n \rightarrow \infty$.

Reduction of Integral Operators to Finite Algebraic Systems

Let $k(s, t) \in C'$. The eigenvalue problem (EVP)

$$(8) \quad \int_0^1 k(s, t)\phi(t)dt = \lambda\phi(s), \quad 0 \leq s \leq 1$$

where $\phi \in C$, can be approximated by means of numerical integration using the rectangular quadrature formula:

$$(9) \quad \sum_{j=1}^n n^{-1} k(s, j/n) \phi_n(j/n) = \lambda_n \phi_n(s), \quad 0 \leq s \leq 1.$$

For $\lambda_n \neq 0$, (9) suggests an interpolation formula for ϕ_n in terms of the values $\phi_n(j/n)$, $j = 1, \dots, n$. In fact (see (17) below), when $\lambda_n \neq 0$, EVP (9) is equivalent to the matrix EVP

$$(10) \quad \sum_{j=1}^n n^{-1} k(i/n, j/n) \phi_n(j/n) = \lambda_n \phi_n(i/n), \quad i = 1, \dots, n.$$

This discretization procedure allows us to consider the EVP (8) by means of a passage to the limit as $n \rightarrow \infty$ in EVPs (9) and (10).

Define operators $L, L_n \in [C]$, $n = 1, 2, \dots$, by

$$(11) \quad (L\phi)(s) = \int_0^1 k(s, t) \phi(t) dt$$

$$(12) \quad (L_n \phi)(s) = \sum_{j=1}^n n^{-1} k(s, j/n) \phi(j/n),$$

where $k(s, t) \in C'$. The convergence of the rectangular quadrature formula on C and the Arzelà-Ascoli theorem imply that the operators L and L_n , $n = 1, 2, \dots$, satisfy (2) and (3), so that

results (4)-(7) apply to the EVPs

$$(13) \quad L\phi = \lambda\phi$$

and

$$(14) \quad L_n \phi_n = \lambda_n \phi_n.$$

Define \tilde{L}_n as the $n \times n$ matrix

$$(15) \quad \tilde{L}_n = [n^{-1}k(i/n, j/n)], \quad 1 \leq i, j \leq n.$$

Let $\vec{\phi}_n = (\phi_n^1, \phi_n^2, \dots, \phi_n^n)$, where $\phi_n^i = \phi_n(i/n)$ for $i = 1, \dots, n$.

Then (10) can be rewritten as

$$(16) \quad \tilde{L}_n \vec{\phi}_n = \lambda_n \vec{\phi}_n.$$

Consider $\vec{\phi}_n$ to be an element of ℓ_n^∞ , the Banach space of n -tuples $\vec{v} = (v_1, \dots, v_n)$ with complex components, equipped with the uniform norm

$$\|\vec{v}\| = \max_{1 \leq j \leq n} |v_j|$$

Then $\tilde{L}_n \in [\ell_n^\infty]$, and the following easily verified relationship between EVPs (14) and (16) holds [1, p. 19]:

$$(17) \quad \text{for } \lambda_n \neq 0 \text{ and } \vec{v} = (v_1, \dots, v_n) \in \ell_n^\infty,$$

$$\tilde{L}_n \vec{v} = \lambda_n \vec{v} \quad \text{and} \quad \phi_n(s) = \lambda_n^{-1} \sum_{j=1}^n n^{-1} k(s, j/n) v_j$$

if and only if

$$L_n \phi_n = \lambda_n \phi_n \quad \text{and} \quad \vec{v} = (v_1, \dots, v_n)$$

where

$$v_j = \phi_n(j/n), \quad j = 1, \dots, n.$$

It follows that the nonzero eigenvalues of L_n and \tilde{L}_n are the same, hence

$$(18) \quad r(L_n) = r(\tilde{L}_n).$$

Along with results of certain compactness arguments, (18) will be used to extend several properties of specific types of matrices to the operators L_n , and ultimately in passing to the limit as $n \rightarrow \infty$, to the operator L .

Iterated Kernels and Matrix Theory

The following definitions and results of integral equations and matrix theory will be used in the subsequent analysis.

Let $k(s, t) \in C^1$. For a positive integer p , the p -th iterated kernel is defined recursively [5, p. 22] by

$$k^1(s, t) = k(s, t)$$

$$k^p(s, t) = \int_0^1 k^{p-1}(s, w)k(w, t)dw, \quad p > 1.$$

The trace of k^p is

$$\tau(k^p) = \int_0^1 k^p(s, s)ds, \quad p \geq 1.$$

The p -th iterated kernel is an element of C' , and τ is a bounded linear functional on C' .

The trace of a complex-valued $n \times n$ matrix $A = [a_{ij}]$ is defined to be

$$(19) \quad \tau(A) = \sum_{i=1}^n a_{ii}.$$

The Euclidean and uniform norms of A are defined respectively by

$$\|A\|_E = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}$$

and

$$\|A\|_\infty = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}| \right)$$

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A , then Schur's inequality

[8] states that

$$(20) \quad \sum_{i=1}^n |\lambda_i|^2 \leq \|A\|_E^2.$$

Let $\Delta_{p,n} = \{(t_1, \dots, t_p) \mid t_i = 1, \dots, n \text{ for } i = 1, 2, \dots, p\}$. Then, the p -th power of A is

$$A^p = [c_{ij}], \quad 1 \leq i, j \leq n,$$

where

$$(21) \quad c_{ij} = \sum_{\Delta_{p-1,n}} a_{it_1} a_{t_1 t_2} \dots a_{t_{p-1} j}.$$

It is an elementary result of matrix theory [3, p. 95] that

$$(22) \quad \tau(A^p) = \sum_{i=1}^n (\lambda_i)^p; \quad p \geq 1.$$

The $n \times n$ matrix $P = [p_{ij}]$ is said to be positive if $p_{ij} > 0$ for all (i, j) . Therefore,

$$0 < \alpha = \min_{i,j} p_{ij} \leq \max_{i,j} p_{ij} = \beta.$$

The properties of positive matrices that will be useful in the subsequent analysis are the following:

(23) The positive $n \times n$ matrix P has a unique eigenvalue λ of maximum modulus; $\lambda > 0$ (hence $\lambda = r(P)$) and simple; and λ has a corresponding eigenvector with positive components, (positive eigenvector) [6];

(24) If $A = [a_{ij}]$ is an $n \times n$ matrix with $|a_{ij}| \leq p_{ij}$ for all (i, j) , then $r(A) \leq r(P)$. (In fact, the inequality holds for any nonnegative matrix P) [9, p. 47].

(25) If $\vec{v} = (v_1, \dots, v_n)$ is any positive vector, then [4]

$$\min_i \frac{(P\vec{v})_i}{v_i} \leq r(P) \leq \max_i \frac{(P\vec{v})_i}{v_i}.$$

For $\vec{v} = (1, \dots, 1)$, the above inequalities become

$$0 < na \leq \min_i \sum_{j=1}^n p_{ij} \leq r(P) \leq \max_i \sum_{j=1}^n p_{ij} \leq n\beta.$$

III. THE EXISTENCE OF EIGENVALUES OF INTEGRAL OPERATORS WITH CONTINUOUS KERNELS

The results developed in the previous chapter will now be applied to prove two theorems concerning the existence of eigenvalues of specific types of integral operators. The operators L , L_n and \tilde{L}_n referred to below are defined by (11), (12) and (15). The letter 'p' always denotes a positive integer.

Theorem 1. Let $k(s, t) \in C^1$. If $\tau(k^p)$ is not zero for some $p \geq 3$, then there exists a nonzero eigenvalue of the operator L .

Proof. Let $M = \max_{(s,t)} |k(s, t)|$, $0 \leq s, t \leq 1$.

Consider the matrix EVP (16). Let λ_{n1} be the eigenvalue of maximum modulus for \tilde{L}_n , with corresponding eigenvector $\vec{\phi}_n$, $n = 1, 2, \dots$. The basic idea of the proof is to establish that the sequence $\{\lambda_{n1}\}$ is bounded away from zero and ∞ for all n sufficiently large. Then, by means of a simple collectively compact operator approximation theory argument, it will follow that L has a nonzero eigenvalue.

From the definitions of \tilde{L}_n , λ_{n1} and $\vec{\phi}_n$, we have

$$\lambda_{n1} \vec{\phi}_n = \tilde{L}_n \vec{\phi}_n.$$

Thus,

$$|\lambda_{n1}| \leq \|\tilde{L}_n\|_{\infty} \leq M, \quad n \geq 1.$$

To show that the sequence $\{\lambda_{n1}\}$ is bounded away from zero for all but a finite number of n , we make use of the following lemma.

Lemma 1. If $k(s, t) \in C'$ then $\tau[(\tilde{L}_n)^p] \rightarrow \tau(k^p)$ as $n \rightarrow \infty$, for $p \geq 1$.

Proof. Define $T, T_n \in [C']$, $n = 1, 2, \dots$, by

$$(Tf)(s, t) = \int_0^1 k(s, w)f(w, t)dw$$

$$(T_n f)(s, t) = \sum_{j=1}^n n^{-1} k(s, j/n)f(j/n, t), \quad f \in C'.$$

Letting $f = k$, we obtain for any $p \geq 1$

$$(26) \quad (T^{p-1}k)(s, t) = k^p(s, t)$$

and

$$(27) \quad (T_n^{p-1}k)(s, t) = \sum_{\Delta_{p-1,n}} n^{-(p-1)} k(s, t_1/n)k(t_1/n, t_2/n) \dots \dots k(t_{p-1}/n, t)$$

$(T^0 = T_n^0 = I$, the identity operator on C').

From (21) and (27), we see that the p -th power of \tilde{L}_n is

$$(28) \quad (\tilde{L}_n)^p = [n^{-1}(T_n^{p-1}k)(i/n, j/n)], \quad 1 \leq i, j \leq n.$$

The operators T, T_n are bounded linear operators, hence continuous, on C' . Also, the convergence of the rectangular quadrature formula on C implies that $T_n \rightarrow T$ as $n \rightarrow \infty$.

Hence, for any $p \geq 1$, $T_n^p, T^p \in [C']$, $n = 1, 2, \dots$, and

$$(29) \quad T_n^p \rightarrow T^p \text{ as } n \rightarrow \infty.$$

By (26) and (29) we obtain

$$(30) \quad \|T_n^{p-1} k - k^p\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, by the continuity of the trace function,

$$(31) \quad |\tau(T_n^{p-1} k) - \tau(k^p)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Convergence of the rectangular quadrature formula on C implies

$$(32) \quad \left| \sum_{j=1}^n n^{-1} (T_n^{p-1} k)(j/n, j/n) - \int_0^1 (T_n^{p-1} k)(s, s) ds \right| \rightarrow 0$$

as $n \rightarrow \infty$.

However, by (19)

$$(33) \quad \sum_{j=1}^n n^{-1} (T_n^{p-1} k)(j/n, j/n) = \tau[(\tilde{L}_n)^p]$$

and

$$(34) \quad \int_0^1 (T_n^{p-1} k)(s, s) ds = \tau(T_n^{p-1} k).$$

Making the substitutions of (33) and (34) into (32),

$$(35) \quad |\tau[(\tilde{L}_n)^p] - \tau(T_n^{p-1}k)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Together, (31) and (35) give the desired result

$$\tau[(\tilde{L}_n)^p] \rightarrow \tau(k^p) \quad \text{as } n \rightarrow \infty.$$

Now, suppose there exists a subsequence of the eigenvalues $\{\lambda_{n1}\}$, denoted by $\{\lambda_{n'1}\}$, such that $|\lambda_{n'1}| \rightarrow 0$ as $n' \rightarrow \infty$.

Denote the eigenvalues of \tilde{L}_n by $\lambda_{n1}, \dots, \lambda_{nn}$. Let $p \geq 3$.

By (22)

$$(36) \quad |\tau[(\tilde{L}_{n'})^p]| = \left| \sum_{i=1}^{n'} (\lambda_{n'i})^p \right| \leq |\lambda_{n'1}|^{p-2} \sum_{i=1}^{n'} |\lambda_{n'i}|^2.$$

Schur's inequality implies

$$\sum_{i=1}^{n'} |\lambda_{n'i}|^2 \leq \|\tilde{L}_{n'}\|_E^2 \leq M^2,$$

so that (36) becomes

$$|\tau[(\tilde{L}_{n'})^p]| \leq |\lambda_{n'1}|^{p-2} M^2.$$

Hence,

$$(37) \quad \tau[(\tilde{L}_{n'})^p] \rightarrow 0 \quad \text{as } n' \rightarrow \infty.$$

Consequently, Lemma 1 and (37) imply that $\tau(k^p) = 0$ for all $p \geq 3$, which contradicts the hypothesis of Theorem 1.

Therefore, there exists $m > 0$ and a positive integer N such that

$$(38) \quad m \leq |\lambda_{n1}| \leq M \quad \text{for all } n \geq N.$$

(Without loss of generality, let $N = 1$.) Thus, there exists a scalar λ and a subsequence $\{\lambda_{n'1}\}$ such that $\lambda_{n'1} \rightarrow \lambda$ and $|\lambda| \geq m > 0$. For each n' , let $\phi_{n'}(t)$ be the eigenfunction of the operator $L_{n'}$ corresponding to $\lambda_{n'1}$. Assume $\|\phi_{n'}\| = 1$ for all n' .

The following argument employs the results of collectively compact operator theory to show that there exists a subsequence of $\{\phi_{n'}\}$ that converges in norm to an eigenfunction ϕ of the operator L , with associated eigenvalue λ .

By (38) the sequence $\{(\lambda_{n'1})^{-1}\}$ is bounded; hence (1) implies that the set

$$\mathcal{L} = \{(\lambda_{n'1})^{-1} L_{n'}\}$$

is collectively compact. Thus, \mathcal{LB} is relatively compact. Since $\phi_{n'} = (\lambda_{n'1})^{-1} L_{n'} \phi_{n'} \in \mathcal{LB}$ for all n' , there exists a subsequence of $\{\phi_{n'}\}$, denoted by $\{\phi_{n''}\}$, and a function $\phi \in C$ such that

$$(39) \quad \|\phi_{n''} - \phi\| \rightarrow 0 \quad \text{as } n'' \rightarrow \infty.$$

Also, successive application of the triangle inequality and the Banach-Steinhaus theorem reveals that

$$(40) \quad \|\phi_{n''} - \lambda^{-1} L\phi\| = \|(\lambda_{n''1})^{-1} L_{n''} \phi_{n''} - \lambda^{-1} L\phi\| \rightarrow 0$$

as $n'' \rightarrow \infty$.

Together, (39) and (40) imply

$$L\phi = \lambda\phi.$$

The function ϕ is nontrivial because for all n''

$$\|\phi_{n''}\| = 1$$

and

$$|\|\phi_{n''}\| - \|\phi\|| \leq \|\phi_{n''} - \phi\| \rightarrow 0 \quad \text{as } n'' \rightarrow \infty.$$

Hence

$$\|\phi\| = 1.$$

Therefore, λ is a nonzero eigenvalue of L , which is the desired result.

Notation. In the next theorem, the following notation will be used:

S is the unit square;

$(\tilde{L}_n)^p$ is defined by (28).

The p -th power of the integral operator L is denoted by L^p .

It is easily verified that L^P is an integral operator on C with kernel k^P , the p -th iterated kernel. Thus,

$$(L^P \phi)(s) = \int_0^1 k^P(s, t) \phi(t) dt, \quad \phi \in C.$$

Let $(\widetilde{L^P})_n$ be the $n \times n$ matrix

$$(\widetilde{L^P})_n = [n^{-1} k^P(i/n, j/n)], \quad i, j = 1, \dots, n.$$

Theorem 2. Let $k \in C'$ such that $k \geq 0$ on S , and $\tau(k^P) > 0$ for some $p \geq 1$. Then $r(L)$ is a positive eigenvalue of L , with corresponding nontrivial eigenfunction $\phi \geq 0$ on $[0, 1]$.

Proof. Because k is continuous and nonnegative on S , and

$$\tau(k^P) = \int_0^1 k^P(s, s) ds > 0 \quad \text{for some } p \geq 1,$$

it follows that there exists a point $(s_0, s_0) \in S$ and a neighborhood Ω of (s_0, s_0) in S such that $k^P(s, t) > 0$ on Ω . Let $0 < \alpha < k^P(s_0, s_0)$. Without loss of generality, Ω can be chosen to be a closed square, symmetric about the diagonal $s = t$, with sides parallel to the sides of S , such that $k^P(s, t) \geq \alpha$ for all $(s, t) \in \Omega$.

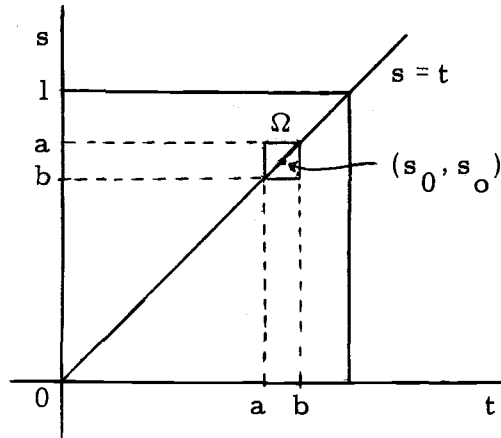


Figure 1

Let A_n be the set of points in S defined by

$$A_n = \{(i/n, j/n) \mid i = 1, \dots, n \text{ and } j = 1, \dots, n\}$$

Then, the image of A_n under $n^{-1}(T_n^{p-1}k)$, arranged in the appropriate square array, is the matrix $(\tilde{L}_n)^p$. It is easy to see that there exists a positive integer N_1 such that for all $n \geq N_1$, A_n intersects Ω . Therefore, for $n \geq N_1$, define \tilde{P}_n to be the $m_n \times m_n$ submatrix of $(\tilde{L}_n)^p$ obtained by deleting all entries $n^{-1}(T_n^{p-1}k)(i/n, j/n)$ of $(\tilde{L}_n)^p$ such that $(i/n, j/n)$ is not an element of Ω . By (28) and (30), the difference between the i - j th entries of $(\tilde{L}_n)^p$ and $(\tilde{L}^p)_n$ goes to zero, uniformly in i - j , as $n \rightarrow \infty$. Thus, there exists a positive integer $N_2 \geq N_1$ such that

$$p_{ij} \geq \frac{1}{n} \left(\frac{a}{2} \right) > 0 \text{ for all entries } p_{ij} \text{ of } \tilde{P}_n,$$

when $n \geq N_2$.

By simultaneously permuting the rows and columns of $(\tilde{L}_n)^P$, we see that $(\tilde{L}_n)^P$ is similar to

$$\tilde{Q}_n = \left(\begin{array}{c|c} \tilde{P}_n & A \\ \hline C & B \end{array} \right),$$

where A , B and C are nonnegative block matrices, and both \tilde{P}_n and B are square. Thus,

$$r((\tilde{L}_n)^P) = r(\tilde{Q}_n).$$

Let \tilde{Q}_n^0 be the $n \times n$ matrix obtained from \tilde{Q}_n by replacing all the entries of matrices A , B and C by zeros. Then,

$$\tilde{Q}_n^0 = \left(\begin{array}{c|c} \tilde{P}_n & 0 \\ \hline 0 & 0 \end{array} \right).$$

Note that triangularization of \tilde{Q}_n^0 is essentially triangularization of \tilde{P}_n , so that

$$r(\tilde{Q}_n^0) = r(\tilde{P}_n).$$

Application of (24) and (25) yields

$$r((\tilde{L}_n)^P) = r(\tilde{Q}_n) \geq r(\tilde{Q}_n^0) = r(\tilde{P}_n) \geq m_n\left(\frac{a}{2n}\right)$$

for $n \geq N_2$.

Also, it is not difficult to see from Figure 1 that

$$\lim_{n \rightarrow \infty} m_n \left(\frac{1}{n} \right) = b - a.$$

Hence, the above limit and the fact that

$$r((\tilde{L}_n)^p) = (r(\tilde{L}_n))^p$$

imply

$$\lim_{n \rightarrow \infty} (r(\tilde{L}_n))^p \geq \frac{a(b-a)}{2} > 0.$$

Therefore,

$$(41) \quad r(L) = \lim_{n \rightarrow \infty} r(L_n) = \lim_{n \rightarrow \infty} r(\tilde{L}_n) \geq \left(\frac{a(b-a)}{2} \right)^{1/p} > 0.$$

It remains to be shown that $r(L)$ is an eigenvalue of L . The proof will be simplified by the following lemma.

Lemma 2. Let B be an $n \times n$ matrix with nonnegative entries. Then $r(B)$ is an eigenvalue of B , having a corresponding nontrivial eigenvector \vec{x} , such that the coordinates of \vec{x} are nonnegative.

Proof. Let B_m denote the matrix obtained from B by adding $1/m$ to each entry of B , $m = 1, 2, \dots$. Then

$$(42) \quad \|B - B_m\|_{\infty} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

B_m is a positive matrix for all $m \geq 1$. Therefore, by (23) B_m has a positive eigenvalue $\lambda_m = r(B_m)$, and λ_m has an associated positive eigenvector $\vec{x}_m = (x_{m1}, \dots, x_{mn})$. Assume that

$$(43) \quad \|\vec{x}_m\| = \max_{1 \leq i \leq n} |x_{mi}| = 1, \quad m = 1, 2, \dots$$

By (24)

$$\lambda_1 = r(B_1) \geq \lambda_2 = r(B_2) \geq \dots \geq r(B) \geq 0.$$

Therefore, there exists a nonnegative number λ such that

$$(44) \quad \lambda_m \rightarrow \lambda \geq r(B) \geq 0 \quad \text{as } m \rightarrow \infty.$$

Also, (43) implies that there exists a subsequence $\{\vec{x}_{m'}\}$ and a vector \vec{x} such that

$$(45) \quad \|\vec{x}_{m'} - \vec{x}\| \rightarrow 0 \quad \text{as } m' \rightarrow \infty.$$

Hence, $\|\vec{x}\| = 1$ and the coordinates of \vec{x} are nonnegative. By

(42) and (45) and the fact that $B_{m'} \vec{x}_{m'} = \lambda_{m'} \vec{x}_{m'}$, we have

$$\|\lambda_{m'} \vec{x}_{m'} - B \vec{x}\| = \|B_{m'} \vec{x}_{m'} - B \vec{x}\| \rightarrow 0 \quad \text{as } m' \rightarrow \infty,$$

and

$$\|\lambda_{m'} \vec{x}_{m'} - \lambda \vec{x}\| \rightarrow 0 \quad \text{as } m' \rightarrow \infty.$$

Therefore $B \vec{x} = \lambda \vec{x}$ and recalling (44), we see that $\lambda = r(B)$ is an eigenvalue of B .

Thus, by Lemma 2, $r(\widetilde{L}_n) = \lambda_{n1}$ is an eigenvalue of \widetilde{L}_n with corresponding nonnegative eigenvector $\vec{\phi}_n$ such that $\|\vec{\phi}_n\| = 1$, for all $n \geq 1$. Let $r(L) > \beta > 0$. Then, from (17) and (41) it follows that there exists a positive integer N , such that for all $n \geq N$, the following results are obtained:

- i) $r(L_n) = r(\widetilde{L}_n) = \lambda_{n1} > \beta$;
- ii) $r(L_n)$ is an eigenvalue of L_n , with corresponding eigenfunction $\phi_n \geq 0$, defined by (17), such that $\|\phi_n\| = 1$;
- iii) $L_n \rightarrow L$ as $n \rightarrow \infty$;
- iv) $r(L_n) \rightarrow r(L)$, hence $r(L) = \lambda$ is an eigenvalue of L .

In addition, application of the compactness argument used in the proof of Theorem 1 yields a subsequence $\{\phi_{n'}\}$ and a function $\phi \in C$ such that

$$\|\phi_{n'} - \phi\| \rightarrow 0 \text{ as } n' \rightarrow \infty,$$

$$\phi \geq 0, \quad \|\phi\| = 1,$$

and

$$L\phi = \lambda\phi.$$

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