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Let L/K be a finite extension of global fields and suppose $A_1, \ldots, A_n$ are central simple L-algebras. In this dissertation we investigate minimal simultaneous embeddings of $A_1, \ldots, A_n$ into central simple K-algebras.

Let $d_K(A_1; \ldots; A_n)$ be the minimum degree of all central simple K-algebras into which each A_i simultaneously embeds. Here we compute $d_K(A_1; \ldots; A_n)$. The case $n = 1$ has been investigated previously in the literature and explicit algorithms are available for computing $d_K(A)$. Extending those results, we give an algorithm for determining $d_K(A_1; \ldots; A_n)$ when $n > 1$. This is done by first considering the case in which $\deg(A_i)$ is a power of a prime p for each $i = 1, \ldots, n$. Under these conditions we find a central simple L-algebra A called the p-embedder of $A_1, \ldots, A_n$ which has the property that $d_K(A_1; \ldots; A_n) = d_K(A)$. In the general case, we write $A_k = \bigotimes_{i=1}^{m} \Omega_{ki}$ for each $k = 1, \ldots, n$ where $\deg(\Omega_{ki})$ is a power of a prime p_i for each $k = 1, \ldots, n$ and $i = 1, \ldots, m$. We then find a p_i-embedder for $\Omega_{1i}, \Omega_{2i}, \ldots, \Omega_{ni}$ for each $i = 1, \ldots, m$ and show that if A is the central simple L-algebra isomorphic to the tensor product of these p_i-embedders, then $d_K(A_1; \ldots; A_n) = d_K(A)$.

Minimal Simultaneous Embeddings of Central Simple Algebras

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1. Introduction

Let L/K be a finite extension of fields and suppose A is a central simple L-algebra. In [4] the question of determining the "minimal" finite dimensional central simple K-algebra into which A embeds was studied. Two notions of minimality were studied; that of degree minimality of A and matrix size minimality of A. In this paper, the notion of degree minimality is extended to degree minimality of $A_1, A_2, \ldots, A_n$, where each A_i is a central simple L-algebra, and we determine the minimum degree of a central simple K-algebra into which each A_i embeds simultaneously, provided L/K is a finite extension of stable fields.

We begin with some terminology and a summary of results concerning central simple algebras. A more detailed discussion can be found in [3]. A ring A is called a K-algebra if there exists a monomorphism $\sigma: K \to A$ such that $\sigma(K)$ is contained in $\mathrm{Z}(A)$, the center of A. We say A is a <u>central simple K-algebra</u> (or A/K is <u>central simple</u>) if $\sigma(K) = \mathrm{Z}(A)$, A is simple, and $[A : K]$ is finite, where $[A : K]$ is the vector space dimension of A over K. If such an A is a division ring, we call it a K-division ring.

If A is a central simple K-algebra then $[A : K]$ is a square, and if $[A : K] = m^2$ we call m the degree of A and write $\deg(A) = m$. From Wedderburn's Theorem we know that $A \cong \mathrm{M}_n(D)$, the ring of $n \times n$ matrices with entries in a K-division ring D (unique up to isomorphism) and some suitable (unique) n. D is called the <u>skew field component of A</u> and n is called the <u>matrix size</u> of A.

The class of central simple K-algebras is closed under tensor product. Two central simple K-algebras A and B are called <u>similar</u>, denoted $A \sim B$, if there are integers s and t such that $A \otimes_K M_s(K) \cong B \otimes_K M_t(K)$. The relation $\sim$ is an equivalence relation, and if the skew field components of A and B are D and E, respectively, we have $A \sim B$ if and only if $D \cong E$. The equivalence class of A is denoted $[A]$ and the set of equivalence classes forms an abelian group, $\mathrm{B}(K)$, called the Brauer group of K, under the operation $[A] + [B] = [A \otimes_K B]$. In $\mathrm{B}(K)$, $0 = [K] = [\mathrm{M}_n(K)]$ and $-[A] = [A^{op}]$ where A^{op} is the opposite algebra of A, i.e. $A^{op} = (A, +, \circ)$ has multiplication $a \circ b = ba$. The p-primary component of $\mathrm{B}(K)$ is denoted $\mathrm{B}(K)_p$. If $[A] \in \mathrm{B}(K)$, then $A \cong \mathrm{M}_t(D)$ for some K-division ring D, and we define the index, $\mathrm{ind}(A)$, of A to be $\sqrt{[D : K]}$. Thus $\deg(A) = t \cdot \mathrm{ind}(A)$. By [1, (12.16)] $\deg(A \otimes_K B) = \deg(A) \cdot \deg(B)$. The order of $[A]$ in $\mathrm{B}(K)$ is called the exponent of A and is denoted $\exp(A)$. We say K is stable if $\mathrm{ind}(A) = \exp(A)$ for every central simple A/K. We call K a global field if it is either an algebraic number field or an algebraic function field in one variable over a finite constant field. By [5, 32.19], global fields are stable.

If L is an extension field of K and A is a central simple K-algebra then $A \otimes_K L$ is a central simple L-algebra and the restriction map $\mathrm{Res}_{L/K}{:}\mathrm{B}(K) \to \mathrm{B}(L)$ given by $[A] \mapsto [A \otimes_K L]$ is a homomorphism of groups. The relative Brauer group, denoted $\mathrm{B}(L/K)$, is the kernel of this homomorphism. Thus $[A] \in \mathrm{B}(L/K)$ if and only if $A \otimes_K L \cong \mathrm{M}_r(L)$ for some r. In this case, L is called a splitting field for A.

We summarize some results concerning central simple algebras which can be found in [3, §9].

Proposition 1. Let L/K be a finite extension of fields, and suppose A and B are central simple K-algebras. Then:

(a) $\mathrm{ind}(A \otimes_K L)$ divides $\mathrm{ind}(A)$.

(b) If $(\mathrm{ind}(A), [L : K]) = 1$ then $\mathrm{ind}(A \otimes_K L) = \mathrm{ind}(A)$.

(c) $\mathrm{ind}(A \otimes_K B)$ divides $\mathrm{ind}(A) \cdot \mathrm{ind}(B)$.

(d) If $(\mathrm{ind}(A), \mathrm{ind}(B)) = 1$ then $\mathrm{ind}(A \otimes_K B) = \mathrm{ind}(A) \cdot \mathrm{ind}(B)$.

(e) If D is a K-division ring and $\mathrm{ind}(D) = p_1^{e_1} \cdots p_n^{e_n}$ is the prime factorization of its index, then $D \cong D_{p_1} \otimes_K \cdots \otimes_K D_{p_n}$ with K-division rings D_{p_i} such that $\mathrm{ind}(D_{p_i}) = p_i^{e_i}$ for $i = 1, 2, \ldots, n$.

Proposition 1,(b) tells us that if D is a K-division ring and $(\text{ind}(D), [L:K]) = 1$, then $D \otimes_K L$ remains a division ring, and Proposition 1,(d) tells us that if D and E are K-division rings of coprime index, then $D \otimes_K E$ remains a division ring.

If n is a positive integer and p is a prime, we let n_p denote the p-part of n; $n = n_p n'$ where $(n_p, n') = 1$. If G is a group and $\alpha \in G$, we say α has order n and write $\text{ord}(\alpha) = n$, if $\alpha^n = 1$ and n is the smallest positive integer with this property. If α has order $n = n_p n'$, then α is uniquely expressible as $\alpha = \alpha_p \alpha'$ where α_p (the <u>p - component of α</u>) has order a power of p and α' (the <u>p - regular component of α</u>) has order relatively prime to p. Moreover, α_p and α' commute, for if $1 = un_p + vn'$ we have $\alpha_p = \alpha^{vn'}$ and $\alpha' = \alpha^{un_p}$. If A/L is central simple then $A \cong \otimes_p A_p$ where the tensor product is taken over L and over all primes p, and where $\deg(A_p)$ is a power of p [5, p. 256]. We have $[A_p] = [A]_p$. Also note that if ϕ is a homomorphism between groups H and G, and $\alpha \in \phi(H)$ then both α_p and $\alpha' \in \phi(H)$ since each is a power of α. In particular, if $[A] \in \text{Res}_{L/K}(\text{B}(K))$, then $[A_p] \in \text{Res}_{L/K}(\text{B}(K))$ and $[A'] \in \text{Res}_{L/K}(\text{B}(K))$.

Let $(G, +)$ be an abelian group in which each element has order a power of a prime p. Suppose $a, b \in G$ with $\text{ord}(a) = p^m$ and $\text{ord}(b) = p^n$ where $m \leq n$. We wish to determine what can be said about $\text{ord}(a+b)$. If $m < n$ then $p^n(a+b) = 0$ so $\text{ord}(a+b)$ divides p^n. But

$$p^{n-1}(a+b) = p^{n-1}a + p^{n-1}b = p^{n-1}b \neq 0,$$

so $\text{ord}(a+b) = p^n = \text{ord}(b)$. If $m = n$, we still have $\text{ord}(a+b)$ divides p^n so $\text{ord}(a+b) \leq p^n = \text{ord}(b)$, but nothing more may be said in general. We will make use of these facts repeatedly in the work which follows. In particular, if L is a stable field and A_1/L and A_2/L are central simple with $\text{ind}(A_1) = p^m$ and $\text{ind}(A_2) = p^n$ where $m < n$, then $\text{ind}(A_1 \otimes_L A_2) = p^n$, as $\text{ind}(A_i) = \exp(A_i)$ for $i = 1, 2$.

If A/L is central simple and L/K is finite dimensional, we say A/L is <u>embeddable</u> in a central simple B/K provided there exists a K-algebra monomorphism $\phi: A \to B$

with $\phi(1_A) = 1_B$. The condition that $\phi(1_A) = 1_B$ is required since we are concerned with finding the minimum of the degree of B, and by [4, Prop. 1] if $\phi(1_A) \neq 1_B$, this minimum will not be attained. If ϕ exists we usually identify A with its image in B.

Note that if r divides s then there exists an embedding from $\mathrm{M}_r(D)$ into $\mathrm{M}_s(D)$ given by sending the $r \times r$ matrix $[d_{ij}]$ to the $s \times s$ matrix which consists of blocks of the matrix $[d_{ij}]$ along its diagonal and 0 elsewhere.

Following the notation of [4] we now make precise the notion of "minimal" with the following definition:

Definition: Let L/K be a finite extension of fields and let A/L be central simple. Define

$$d_K(A) = \min\{\deg(B) \,|B/K \text{ is central simple and } A/L \text{ embeds into } B\}.$$

If A/L embeds in a central simple B/K and $\deg(B) = d_K(A)$, we say that B/K is degree minimal for A/L.

Next we extend this idea with the following definition.

Definition: Let L/K be a finite extension of fields and let $A_1, A_2, \dots A_n$ be central simple L-algebras. Define

$d_K(A_1; \dots ; A_n) =$

$\min\{\deg(B) \,|\, B/K$ is central simple and A_i/L embeds into B/K for $i = 1, \dots, n\}$.

Note that if A_i/L is central simple and L/K is finite, then A_i/L embeds into $B_i = \mathrm{M}_{u_i}(K)$ where $u_i = [A_i : K]$ via the left regular representation. Thus each A_i embeds into $B = \otimes_{i=1}^{n} B_i$, so that both $d_K(A)$ and $d_K(A_1; \dots ; A_n)$ exist.

In order to determine $d_K(A_1;\ldots;A_n)$ we must find embeddings of each A_i into a central simple K-algebra B. We see that if $A_1 \otimes_L \cdots \otimes_L A_n$ embeds into B, then each A_i also has an embedding into B so that

$$d_K(A_1;\ldots;A_n) \leq d_K(A_1 \otimes_L \cdots \otimes_L A_n).$$

We will show that equality does not hold in general. However it does hold if $(\deg(A_1),\ldots,\deg(A_n)) = 1$.

2. Cyclic Algebras and Hasse Invariants

In this section we define cyclic algebras and Hasse invariants and gather some results needed in the following sections. We begin with the following definition.

Definition: Let L/K be a finite cyclic extension of fields with $\mathrm{Gal}(L/K) = \langle\sigma\rangle$ and suppose $[L : K] = n$. Let $a \in K^*$, let u be a symbol and form the associative K-algebra

$$A = (L/K, \sigma, a) = \sum_{i=0}^{n-1} Lu^i$$

where multiplication in A is defined by

$$u^n = a \qquad \text{and} \qquad u \cdot x = \sigma(x)u \qquad \text{for all } x \in L.$$

A is a finite dimensional central simple K-algebra and is called a <u>cyclic algebra</u>.

By [5, 32.20], every central simple K algebra is a cyclic algebra if K is a global field.

Now let R be a complete discrete valuation ring with maximal ideal $P = \pi R \neq 0$ and suppose K is the quotient field of R. Each $k \in K^*$ can be written in the form $k = \pi^r \epsilon$ where ϵ is a unit and r is an integer. Let ν_K be the exponential valuation on K, i.e. if $k = \pi^r\epsilon$ then $\nu_K(k) = r$ and $\nu_K(0) = \infty$. Let $\bar{K} = R/P$ be the residue class field and suppose $|\bar{K}| = q$.

Suppose D is a K-division ring with $\mathrm{ind}(D) = m$. Let W be the unique unramified extension of K of degree m. So $W = K(\omega)$ where ω is a primative $(q^m - 1)^{\text{th}}$ root of unity over K. W is a cyclic extension of K and $\mathrm{Gal}(W/K)$ is generated by the Frobenius automorphism of W/K, denoted $\sigma_{W/K}$, where $\sigma_{W/K}$ is defined by $\sigma_{W/K}(\omega) = \omega^q$ [6, 3-2-12].

In [5, §30] it is shown that $D \cong (W/K, \sigma_{W/K}, \pi^s)$ where $s \in \mathbb{Z}$.

Definition: Let D be a K-division ring of index m, so $D \cong (W/K, \sigma_{W/K}, \pi^s)$ where $W = K(\omega)$. The Hasse invariant of D is defined to be $\frac{s}{m} \in \mathbb{Q}/\mathbb{Z}$. If A/K is central simple, say $A \cong \mathrm{M}_t(D)$ with $D \cong (W/K, \sigma_{W/K}, \pi^s)$, we define the map

$$\mathrm{inv}\colon \mathrm{B}(K) \to \mathbb{Q}/\mathbb{Z} \text{ by } \mathrm{inv}[A] = \frac{s}{m} \pmod 1.$$

From [5, 31.8] we have the following.

Proposition 2. $\mathrm{inv}\colon \mathrm{B}(K) \to \mathbb{Q}/\mathbb{Z}$ is an additive isomorphism.

Now suppose K is an algebraic number field. A prime of K may be viewed as either a prime ideal in the ring of integers of K or as one of the equivalence classes of valuations on K. A prime is called finite or non-archimedean if it extends the p-adic valuation of $\mathbb{Q}$, and is called infinite or archimedean if it extends the usual absolute value on the rational field $\mathbb{Q}$. If L is an extension field of K and γ is a valuation defined on L such that $\gamma(k) = \pi(k)$ for every $k \in K$, where π is a valuation on K, we say that γ <u>divides</u> π or γ <u>extends</u> π.

If π is a prime of K, we let K_π denote the π-adic completion of K. If π is archimedean, K_π is either the real field $\mathbb{R}$, and π is called a real prime, or is the complex field $\mathbb{C}$, and π is called a complex prime.

If A/K is central simple we set $A_\pi = A \otimes_K K_\pi$. Then A_π is a central simple K_π-algebra so we have a homomorphism of Brauer groups given by

$$\mathrm{B}(K) \to \mathrm{B}(K_\pi)$$

$$[A] \mapsto [A_\pi].$$

Definition: Let K be an algebraic number field, A/K central simple, and π a prime of K (finite or infinite). The Hasse invariant of A at π is defined by

$$\mathrm{inv}_\pi[A] = \begin{cases} \mathrm{inv}[A_\pi]; & \text{if } \pi \text{ is finite;} \\ 0; & \text{if } \pi \text{ is complex;} \\ \frac{1}{2}; & \text{if } \pi \text{ is real and } A_\pi \sim \mathbb{H}; \\ 0; & \text{if } \pi \text{ is real and } A_\pi \sim \mathbb{R}; \end{cases}$$

where $\mathbb{H}$ denotes the division ring of real quaternions. The denominator of $\mathrm{inv}_\pi[A]$ as a fraction reduced to lowest terms is called the local index of A at π and is denoted $\mathrm{l.i.}_\pi(A)$.

We see if $A_\pi \cong \mathrm{M}_n(D_\pi)$ where D_π is a K_π-division ring, then $\mathrm{l.i.}_\pi(A) = \mathrm{ind}(D_\pi)$. In particular, if $\mathrm{inv}_\pi[A] = 0$ then $D_\pi \cong K_\pi$ so $\mathrm{ind}(D_\pi) = 1$ and $\mathrm{l.i.}_\pi(A) = 1$.

We summarize some properties of Hasse invariants which can be found in [5, §32].

Proposition 3. Let S be the set of primes of K, and let A/K and B/K be central simple. Then:

(a) $\mathrm{inv}_\pi[A] = 0$ for all but finitely many $\pi \in S$.

(b) $\sum_{\pi \in S} \mathrm{inv}_\pi[A] \equiv 0 \pmod 1$.

(c) $A \sim K$ if and only if $\mathrm{inv}_\pi[A] = 0$ for all $\pi \in S$.

(d) $A \sim B$ if and only if $\mathrm{inv}_\pi[A] = \mathrm{inv}_\pi[B]$ for all $\pi \in S$.

(e) $\exp(A) = \mathrm{l.c.m.}\{\mathrm{l.i.}_\pi(A) \mid \pi \in S\}$.

We also state the following two propositions whose proofs are found in [2, Satz 4, p. 113] and [2, Satz 9, p. 119], respectively.

Proposition 4. Suppose L/K is a finite extention of fields, π a prime of K and γ a prime of L dividing π. Then, if D is a K-division ring,

$$[L_\gamma : K_\pi] \cdot \mathrm{inv}_\pi[D] \equiv \mathrm{inv}_\gamma[D \otimes_K L] \pmod 1.$$

PROPOSITION 5. Let $\pi_1, \pi_2, \ldots, \pi_n$ be a set of primes of K, $u_1, u_2, \ldots, u_n$ rational numbers in lowest terms such that

$$0 \leq u_i < 1, \qquad \sum_{i=1}^{n} u_i \equiv 0 \pmod{1},$$

$$u_j = 0 \text{ or } \frac{1}{2} \text{ if } \pi_j \text{ is real, and } u_j = 0 \text{ if } \pi_j \text{ is complex.}$$

Then there exists a K-division ring D with $\mathrm{inv}_{\pi_j}[D] = u_j$ for all j and $\mathrm{inv}_{\pi}[D] = 0$ for all other primes π of K.

3. Computing d_K(A)

We begin this section by gathering some standard results about the centralizer of a simple subalgebra of B/K. We denote the centralizer in B of a subalgebra E by

$$C_B(E) = \{x \in B \mid xy = yx \text{ for all } y \in E\}.$$

Proposition 6. Let L/K be a finite extension of fields and suppose A/L is central simple. Suppose A/L embeds into a central simple B/K. Let Y be the centalizer of A in $C_B(L)$. Then:

(a) Y/L is central simple such that $C_B(L) \cong A \otimes_L Y$.
(b) $B \otimes_K L \cong M_r(C_B(L))$ where $r = [L : K]$.
(c) $\deg(B) = \deg(A) \cdot \deg(Y) \cdot [L : K]$ and $[A] + [Y] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$.

proof: This follows from [5, pages 94–96]. ∎

Now we summarize some results from [4] dealing with the computation of $d_K(A)$. If $A \cong \mathrm{M}_t(\Delta)$ where Δ is an L-division ring, then by [4, Cor. 3]

$$d_K(A) = t \cdot d_K(\Delta).$$

Moreover, if A/L embeds into a central simple B/K then $d_K(A)$ divides $\deg(B)$ [4, Thm. 12]. In particular, this shows that $d_K(A_i)$ divides $d_K(A_1; \ldots; A_n)$ for each $i = 1, \ldots, n$, if $A_1, \ldots, A_n$ are central simple L-algebras.

We will also state the following proposition which is proven in [4, Prop. 5].

Proposition 7. Let L/K be a finite extension of fields and suppose A/L is central simple. If $[A] = [D \otimes_K L]$, D a K-division ring, then there exists an integer w such that A/L embeds into $B = \mathrm{M}_w(D)$ so that $A \cong C_B(L)$ and

$$\deg(B) = \deg(A) \cdot [L : K] = d_K(A).$$

A numerical invariant μ of a central simple L-algebra A is said to localize if $\mu(A)_p = \mu(A_p)$. Since $d_K(A)$ does not localize, a new invariant of A/L is introduced. Proposition 6 shows that if A/L embeds into B/K and Y is the centralizer of A in $C_B(L)$ then $\deg(B) = \deg(A) \cdot \deg(Y) \cdot [L:K]$ so $\deg(B)$ will be minimal provided $\deg(Y)$ is minimal. Thus, following the notation of [4], we define:

Definition: Let L/K be finite and A/L central simple. Define

$$r_K(A) = \min\{\mathrm{ind}(Y) \mid Y \text{ is an } L-\text{division ring with } [A]+[Y] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))\}.$$

By [4, Prop. 8], $r_K(A)$ localizes and by [4, Thm. 6], we have the following.

Proposition 8. Let L/K be a finite extension of fields and let A/L be central simple. Then $d_K(A) = \deg(A) \cdot r_K(A) \cdot [L:K]$.

From [4, Cor. 11] we also have

Proposition 9. Let L/K be a finite extension of fields and suppose A/L is central simple. Then $r_K(A)$ divides $\mathrm{ind}(A)$.

4. Computing $d_K(A_1; \ldots ;A_n)$

Next we turn our attention to computing $d_K(A_1;\ldots;A_n)$ where each A_i is a central simple L-algebra. Throughout this section let L/K be a finite extension of stable fields. We will use freely the fact that if A/L is central simple then $\mathrm{ind}(A) = \exp(A)$. Given central simple L-algebras $A_1,\ldots,A_n$ we will give an algorithm for determining a central simple L-algebra A such that $d_K(A_1;\ldots;A_n) = d_K(A) = \deg(A)\cdot r_K(A)\cdot [L:K]$.

One might assume that a candidate for such an L-algebra A would be one of minimum degree into which each A_i embeds. We begin with an example to show that this is, in general, not the case.

EXAMPLE 1. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{2})$. Then a prime p ramifies if and only if $p = 2$, p remains prime if and only if $p \equiv \pm 3 \pmod 8$, and p splits if and only if $p \equiv \pm 1 \pmod 8$ [6, 6-2].

Let $p_1 \equiv 1 \pmod 8$ and let α_1 and α_2 be primes of L extending p_1. Let $p_2 \equiv 3 \pmod 8$ and let β be the prime of L extending p_2. By Proposition 5, there exists L-division rings Δ_1 and Δ_2 such that

$$\mathrm{inv}_{\alpha_1}[\Delta_1] = \mathrm{inv}_{\alpha_2}[\Delta_1] = \mathrm{inv}_{\beta}[\Delta_1] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Delta_1] = 0$$

for all other primes ρ of L, and

$$\mathrm{inv}_{\alpha_1}[\Delta_2] = \frac{2}{9}, \quad \mathrm{inv}_{\beta}[\Delta_2] = \frac{7}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Delta_2] = 0$$

for all other primes ρ of L.

We will show that the minimum degree of any L-algebra A into which both Δ_1 and Δ_2 embed is 27, but if Δ_1 and Δ_2 embed into A where $\deg(A) = 27$, then $d_K(A) > d_K(\Delta_1;\Delta_2)$.

By Proposition 3,(c), $\mathrm{ind}(\Delta_1) = 3$ and $\mathrm{ind}(\Delta_2) = 9$. Suppose Δ_1 and Δ_2 embed into a central simple L-algebra A. Let Y_1 and Y_2 be the centralizers of Δ_1 and Δ_2 in A, respectively. Then by Proposition 6,(a), $A \cong \Delta_1 \otimes_L Y_1 \cong \Delta_2 \otimes_L Y_2$ so $\deg(A) = 3 \cdot \deg(Y_1) = 9 \cdot \deg(Y_2)$. So 9 divides $\deg(A)$. Suppose $\deg(A) = 9$. Then $\deg(Y_1) = 3$ and $\deg(Y_2) = 1$. But $\deg(Y_2) = 1$ implies $Y_2 \cong L$, so $A \cong \Delta_2$. Then $Y_1 \sim \Delta_2 \otimes_L \Delta_1^{op}$ so $\mathrm{ind}(Y_1) = \mathrm{ind}(\Delta_2 \otimes_L \Delta_1^{op})$. But since $\mathrm{ind}(\Delta_2) = 9$ and $\mathrm{ind}(\Delta_1) = \mathrm{ind}(\Delta_1^{op}) = 3$ we have $\mathrm{ind}(\Delta_2 \otimes_L \Delta_1^{op}) = \mathrm{ind}(\Delta_2) = 9$, contrary to $\deg(Y_1) = 3$. So Δ_1 and Δ_2 do not embed into any L-algebra of degree 9.

Note, however, that if Ω is the L-division ring defined by

$$\mathrm{inv}_{\alpha_1}[\Omega] = \frac{5}{9}, \quad \mathrm{inv}_{\alpha_2}[\Omega] = \frac{1}{3}, \quad \mathrm{inv}_{\beta}[\Omega] = \frac{1}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Omega] = 0$$

for all other primes ρ of L, we have $\Delta_1 \otimes_L \Delta_2 \sim \Omega$, by Proposition 3,(d). Then since $\deg(\Delta_1 \otimes_L \Delta_2) = \deg(\Delta_1) \cdot \deg(\Delta_2) = 27$ we have $\Delta_1 \otimes_L \Delta_2 \cong \mathrm{M}_3(\Omega)$. Thus Δ_1 and Δ_2 embed into $\mathrm{M}_3(\Omega)$ and $\deg(\mathrm{M}_3(\Omega)) = 27$, so the minimum degree L-algebra into which both Δ_1 and Δ_2 embed has degree 27.

Next, let us compute $d_K(\Delta_1; \Delta_2)$. We begin by finding $d_K(\Delta_1)$ and $d_K(\Delta_2)$. By [6, 2-4-4 (iv)], $[L_{\alpha_1} : K_{p_1}] = [L_{\alpha_2} : K_{p_1}] = 1$ and $[L_\beta : K_{p_2}] = 2$ so by Proposition 4 $D \otimes_K L \sim \Delta_1$ where D is the K-division ring with

$$\mathrm{inv}_{p_1}[D] = \frac{1}{3}, \quad \mathrm{inv}_{p_2}[D] = \frac{2}{3}, \quad \text{and} \quad \mathrm{inv}_q[D] = 0$$

for all other primes q of K, whose existence is guaranteed by Proposition 5. Thus $\Delta_1 \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$ and $r_K(\Delta_1) = 1$. So

$$d_K(\Delta_1) = \deg(\Delta_1) \cdot r_K(\Delta_1) \cdot [L : K] = 6.$$

Now suppose X is an L-division ring such that $[\Delta_2 \otimes_L X] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$, $r_K(\Delta_2) = \mathrm{ind}(X)$, and suppose E is a K-division ring such that $E \otimes_K L \sim \Delta_2 \otimes_L X$. Then by Proposition 4

$$\mathrm{inv}_{p_1}[E] \equiv \mathrm{inv}_{\alpha_1}[\Delta_2 \otimes_L X] \equiv \mathrm{inv}_{\alpha_2}[\Delta_2 \otimes_L X]$$

where the congruences are mod 1. So

$$\frac{2}{9} + \mathrm{inv}_{\alpha_1}[X] \equiv \mathrm{inv}_{\alpha_2}[X].$$

But this forces $\mathrm{l.i.}_{\alpha_1}[X]$ or $\mathrm{l.i.}_{\alpha_2}[X]$ to be 9, so by Proposition 3,(e) $\mathrm{ind}(X) \geq 9$. But $r_K(\Delta_2) \leq \mathrm{ind}(\Delta_2)$ since $[\Delta_2 \otimes_L \Delta_2^{op}] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$ and $\mathrm{ind}(\Delta_2^{op}) = \mathrm{ind}(\Delta_2) = 9$. So $r_K(\Delta_2) = 9$ and $d_K(\Delta_2) = 2 \cdot 3^4$.

By Proposition 5 there exists L-division rings Γ_1 and Γ_2 such that

$$\mathrm{inv}_{\alpha_1}[\Gamma_1] = \mathrm{inv}_{\alpha_2}[\Gamma_1] = \frac{8}{9}, \quad \mathrm{inv}_{\beta}[\Gamma_1] = \frac{2}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Gamma_1] = 0$$

for all other primes ρ of L, and

$$\mathrm{inv}_{\alpha_2}[\Gamma_2] = \frac{2}{9}, \quad \mathrm{inv}_{\beta}[\Gamma_2] = \frac{7}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Gamma_2] = 0$$

for all other primes ρ of L. Then by Proposition 3,(d) $\Delta_1 \otimes_L \Gamma_1 \sim \Delta_2 \otimes_L \Gamma_2 \sim \Psi$ where Ψ is the L-division ring with

$$\mathrm{inv}_{\alpha_1}[\Psi] = \mathrm{inv}_{\alpha_2}[\Psi] = \frac{2}{9}, \quad \mathrm{inv}_{\beta}[\Psi] = \frac{5}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Psi] = 0$$

for all other primes ρ of L. But $\deg(\Delta_1 \otimes_L \Gamma_1) = 27$ and $\deg(\Delta_2 \otimes_L \Gamma_2) = 81$ so $\Delta_1 \otimes_L \Gamma_1 \cong \mathrm{M}_3(\Psi)$ and $\Delta_2 \otimes_L \Gamma_2 \cong \mathrm{M}_9(\Psi)$. Then, since $\mathrm{M}_3(\Psi)$ embeds into $\mathrm{M}_9(\Psi)$, Δ_1 and Δ_2 embed into $\mathrm{M}_9(\Psi)$. Thus $d_K(\Delta_1;\Delta_2) \leq d_K(\mathrm{M}_9(\Psi))$. Now by Proposition 4, $F \otimes_K L \sim \Psi$ where F is the K-division ring such that

$$\mathrm{inv}_{p_1}[F] = \frac{2}{9}, \quad \mathrm{inv}_{p_2}[F] = \frac{7}{9}, \quad \text{and} \quad \mathrm{inv}_{q}[F] = 0$$

for all other primes q of K. So $r_K(\Psi) = 1$ and $d_K(\mathrm{M}_9(\Psi)) = 2 \cdot 3^4$. But

$$d_K(\Delta_1;\Delta_2) \not< 2 \cdot 3^4$$

since $d_K(\Delta_2) = 2 \cdot 3^4$ and $d_K(\Delta_2)$ divides $d_K(\Delta_1;\Delta_2)$. Thus $d_K(\Delta_1;\Delta_2) = 2 \cdot 3^4$.

Finally we show that there is no central simple A/L into which both Δ_1 and Δ_2 embed with $\deg(A) = 27$ and $d_K(A) = 2 \cdot 3^4$. Suppose A/L has these properties.

Then we have $r_K(A) = 3$ by Proposition 8. Let Y_1 and Y_2 be the centralizers of Δ_1 and Δ_2 in A, respectively. So by Proposition 6,(a) $A \cong \Delta_1 \otimes_L Y_1 \cong \Delta_2 \otimes_L Y_2$ and hence $\deg(Y_1) = 9$ and $\deg(Y_2) = 3$. Since $\deg(Y_2) = 3$ we have either $\operatorname{ind}(Y_2) = 1$ or $\operatorname{ind}(Y_2) = 3$. If $\operatorname{ind}(Y_2) = 1$ then $A \sim \Delta_2$ so $r_K(A) = r_K(\Delta_2) = 9$, contrary to $r_K(A) = 3$. If $\operatorname{ind}(Y_2) = 3$ then $\text{l.i.}_{\alpha_1}(Y_2) = 1$ or $\text{l.i.}_{\alpha_1}(Y_2) = 3$, so

$$\operatorname{inv}_{\alpha_1}[Y_2] = 0, \frac{1}{3}, \text{ or } \frac{2}{3}.$$

In any case, $\text{l.i.}_{\alpha_1}[\Delta_2 \otimes_L Y_2] = 9$ so $\text{l.i.}_{\alpha_1}(A) = 9$. Also $\text{l.i.}_{\alpha_2}(Y_2) = 1$ or $\text{l.i.}_{\alpha_2}(Y_2) = 3$, and therefore $\text{l.i.}_{\alpha_2}(A) = 1$ or $\text{l.i.}_{\alpha_2}(A) = 3$. Thus we have two possibilities. First is the case in which $\text{l.i.}_{\alpha_1}(A) = 9$ and $\text{l.i.}_{\alpha_2}(A) = 1$. Then $\operatorname{inv}_{\alpha_1}[A] = \frac{a}{9}$ where $(a, 9) = 1$ and $\operatorname{inv}_{\alpha_2}[A] = 0$. Then if Z is a central simple L-algebra with $r_K(A) = \operatorname{ind}(Z)$, and H is a K-division ring such that

$$H \otimes_K L \sim A \otimes_L Z,$$

by Proposition 5,

$$\operatorname{inv}_{p_1}[H] \equiv \operatorname{inv}_{\alpha_1}[A \otimes_L Z] \equiv \operatorname{inv}_{\alpha_2}[A \otimes_L Z]$$

so

$$\frac{a}{9} + \operatorname{inv}_{\alpha_1}[Z] \equiv \operatorname{inv}_{\alpha_2}[Z].$$

But then either $\text{l.i.}_{\alpha_1}[Z] = 9$ or $\text{l.i.}_{\alpha_2}[Z] = 9$ and in either case $\operatorname{ind}(Z) = r_K(A) = 9$, a contradiction. Secondly, we may have $\text{l.i.}_{\alpha_1}(A) = 9$ and $\text{l.i.}_{\alpha_2}(A) = 3$. But again, by an argument similar to that given above, this forces $r_K(A) = 9$. So there is no A/L of degree 27 into which both Δ_1 and Δ_2 embed with $d_K(A) = 2 \cdot 3^4 = d_K(\Delta_1; \Delta_2)$. This concludes the example.

Again let $A_1, \ldots, A_n$ be central simple L-algebras. We wish to determine a central simple L-algebra A with $d_K(A_1; \ldots; A_n) = d_K(A)$. Initially we will consider the case in which $\deg(A_i)$ is a power of a prime p for each $i = 1, \ldots, n$. We will give an algorithm for finding a central simple L-algebra called the p-embedder of $A_1, \ldots, A_n$, and then show that the minimum degree central simple K-algebra into which it embeds is $d_K(A_1; \dots; A_n)$.

Order the L-algebras $A_1, \dots, A_n$ so that

$$\deg(A_1) \le \deg(A_2) \le \cdots \le \deg(A_n)$$

and suppose $\deg(A_i) = p^{r_i}$ for $i = 1, \dots, n$. Let Γ_n be an L-division ring such that

$$A_n \otimes_L \Gamma_n \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad \mathrm{ind}(\Gamma_n) = r_K(A_n).$$

Let Ψ be the skew field component of $A_n \otimes_L \Gamma_n$ so

$$A_n \otimes_L \Gamma_n \cong \mathrm{M}_{t_n}(\Psi) \quad \text{with} \quad t_n = \frac{\deg(A_n) \cdot \mathrm{ind}(\Gamma_n)}{\mathrm{ind}(\Psi)}.$$

Note that $\mathrm{ind}(\Psi) = \mathrm{ind}(A_n \otimes_L \Gamma_n)$, so $\mathrm{ind}(\Psi)$ is a power of p and hence t_n is a power of p.

For $i = 1, \dots, n-1$, let Γ_i be the L-division ring such that $A_i \otimes_L \Gamma_i \sim \Psi$. We see $\mathrm{ind}(\Gamma_i)$ is also a power of p. Then for $i = 1, \dots, n$ we have

$$A_i \otimes_L \Gamma_i \cong \mathrm{M}_{t_i}(\Psi) \quad \text{with} \quad t_i = \frac{\deg(A_i) \cdot \mathrm{ind}(\Gamma_i)}{\mathrm{ind}(\Psi)}.$$

Each t_i is also a power of p so if $t = \max\{t_1, \dots, t_n\}$, t_i divides t for each $i = 1, \dots, n$ and each $A_1, \dots, A_n$ simultaneously embeds into $\mathrm{M}_t(\Psi)$. We call $\mathrm{M}_t(\Psi)$ a <u>p-embedder</u> of $A_1, \dots, A_n$. Note that a p-embedder of $A_1, \dots, A_n$ is not unique but depends on the choice of the L-division ring Γ_n. We will show that

$$d_K(A_1; \dots; A_n) = d_K(\mathrm{M}_t(\Psi))$$

where $\mathrm{M}_t(\Psi)$ is any p-embedder of $A_1, \dots, A_n$, and use this result to prove our main theorem which we now state.

Theorem A. Let $A_1, \ldots, A_n$ be central simple L-algebras and suppose $p_1, \ldots, p_m$ are primes dividing $\deg(A_k)$ for any $k = 1, \ldots, n$. Suppose $A_k \cong \bigotimes_{i=1}^m \Omega_{ki}$ where Ω_{ki} is a central simple L-algebra with $\deg(\Omega_{ki})$ a power of p_i for each $k = 1, \ldots, n$ and $i = 1, \ldots, m$. Let $\mathrm{M}_{t_i}(\Psi_i)$ be a p_i-embedder for $\Omega_{1i}, \Omega_{2i}, \ldots, \Omega_{ni}$ for each $i = 1, \ldots, m$ and let $A = \bigotimes_{i=1}^m \mathrm{M}_{t_i}(\Psi_i)$. Then $d_K(A_1; \ldots; A_n) = d_K(A)$. Moreover, $r_K(A) = 1$ so $d_K(A_1; \ldots; A_n) = [L : K] \cdot \prod_{i=1}^m t_i \cdot \mathrm{ind}(\Psi_i)$.

We note that even though $r_K(A) = 1$, in order to determine a p_i-embedder of $\Omega_{1i}, \Omega_{2i}, \ldots, \Omega_{ni}$, it is necessary to determine $r_K(\Omega_{ji})$ for some $j = 1, \ldots, n$. Techniques for computing these values are illustrated in Example 1.

To prove these results we begin with the following three lemmas. Recall that L/K is a finite extension of stable fields throughout this section.

Lemma 10. Let $A_1, \ldots, A_n$ be central simple L-algebras with $\deg(A_i) = p^{r_i}$, p a prime. Suppose that each A_i embeds into a central simple K-algebra B, where $B \cong \mathrm{M}_t(D)$, D a K-division ring. For each $i = 1, \ldots, n$, let Y_i be the centralizer of A_i in $C_B(L)$. If q is a prime, $q \neq p$, and q divides $\deg(Y_i)$ for each i, then each A_i embeds into a central simple K-algebra B' where $\deg(B') < \deg(B)$.

proof: Since q divides $\deg(Y_i)$ we may write $Y_i \cong Y_{iq} \otimes_L Y_i'$ where $\deg(Y_{iq})$ is a power of q and $(\deg(Y_i'), q) = 1$. By Proposition 6,(c)

$$\deg(B) = \deg(A_i) \cdot \deg(Y_{iq}) \cdot \deg(Y_i') \cdot [L : K] \quad \text{for each } i = 1, \ldots, n.$$

So

$$\deg(A_j) \cdot \deg(Y_{jq}) \cdot \deg(Y_j') = \deg(A_i) \cdot \deg(Y_{iq}) \cdot \deg(Y_i') \quad \text{for } i, j = 1, \ldots, n.$$

Equating the q-regular components yields

$$\deg(A_j) \cdot \deg(Y_j') = \deg(A_i) \cdot \deg(Y_i') \quad \text{for } i, j = 1, \ldots, n. \tag{3}$$

Also we see $q|\deg(B)$ so either $q|t$ or $q|\mathrm{ind}(D)$.

If $q|\mathrm{ind}(D)$ write $D \cong D_q \otimes_K D'$ where D_q and D' are K-division rings with $\mathrm{ind}(D_q)$ a power of q and $(\mathrm{ind}(D'), q) = 1$. By Proposition 6

$$D \otimes_K L \sim A_i \otimes_L Y_i \quad \text{for} \quad i = 1, \ldots, n.$$

So

$$(D_q \otimes_K L) \otimes_L (D' \otimes_K L) \sim A_i \otimes_L Y_{iq} \otimes_L Y_i'$$

and equating the q-regular components of $[D \otimes_K L]$ in $\mathrm{B}(L)$, we have

$$D' \otimes_K L \sim A_i \otimes_L Y_i'.$$

Then by Proposition 7, A_i embeds into $\mathrm{M}_{v_i}(D')$ with

$$v_i = \frac{\deg(A_i) \cdot \deg(Y_i') \cdot [L : K]}{\mathrm{ind}(D')} \quad \text{for} \quad i = 1, \ldots, n.$$

But then by (3), $v_1 = v_2 = \cdots = v_n$. Let v be this common value so that A_i embeds into $\mathrm{M}_v(D')$ for each i and

$$\deg(\mathrm{M}_v(D')) = \deg(A_i) \cdot \deg(Y_i') \cdot [L : K]$$

which is strictly less than $\deg(B)$ unless $Y_{iq} \cong L$, which is contrary to q dividing $\deg(Y_{iq})$.

Next suppose $q|t$ and $q \not| \mathrm{ind}(D)$. We have

$$D \otimes_K L \sim A_i \otimes_L Y_{iq} \otimes_L Y_i' \quad \text{for} \quad i = 1, \ldots, n$$

so

$$\mathrm{ind}(D \otimes_K L) = \mathrm{ind}(A_i \otimes_L Y_{iq} \otimes_L Y_i') \quad \text{for} \quad i = 1, \ldots, n.$$

By Proposition 1,(a), $\mathrm{ind}(D \otimes_K L)|\mathrm{ind}(D)$ so $q \not| \mathrm{ind}(D \otimes_K L)$. Thus

$$q \not| \mathrm{ind}(A_i \otimes_L Y_{iq} \otimes_L Y').$$

But by Proposition 1,(d), $\mathrm{ind}(A_i \otimes_L Y_{iq} \otimes_L Y') = \mathrm{ind}(A_i \otimes_L Y') \cdot \mathrm{ind}(Y_{iq})$ so

$$q \nmid \mathrm{ind}(Y_{iq}).$$

But $q | \deg(Y_{iq})$ so $Y_{iq} \cong \mathrm{M}_{u_i}(L)$ where $q|u_i$. But then

$$D \otimes_K L \sim A_i \otimes_L Y_i'.$$

So A_i embeds into $\mathrm{M}_{w_i}(D)$ with

$$w_i = \frac{\deg(A_i) \cdot \deg(Y_i') \cdot [L:K]}{\mathrm{ind}(D)} \quad \text{for} \quad i = 1, \ldots, n.$$

Again by (3), $w_1 = w_2 = \cdots = w_n$. Let w be this common value so that A_i embeds into $\mathrm{M}_w(D)$ for $i = 1, \ldots, n$, and $\deg(\mathrm{M}_w(D)) = \deg(A_i) \cdot \deg(Y_i') \cdot [L:K]$ which is strictly less than $\deg(B)$ unless $Y_{iq} \cong L$, which is contrary to q dividing $\deg(Y_i)$. ■

Lemma 11. Suppose $A_1, \ldots, A_n$ are central simple L-algebras with $\deg(A_i) = p^{r_i}$, p a prime. If q is prime, $q \neq p$, such that $q \nmid [L:K]$, then $q \nmid d_K(A_1; \ldots; A_n)$.

proof: Let B/K be central simple such that $d_K(A_1; \ldots; A_n) = \deg(B)$ and suppose q divides $\deg(B)$. Let Y_i be the centralizer of A_i in $C_B(L)$ for each $i = 1, \ldots, n$. By Proposition 6,(c)

$$\deg(B) = \deg(A_i) \cdot \deg(Y_i) \cdot [L:K] \quad \text{for} \quad i = 1, \ldots, n.$$

Since $(q, \deg(A_i)) = (q, [L:K]) = 1$, q divides $\deg(Y_i)$ for each i, and thus by Lemma 10, A_i embeds into B'/K for each i, where $\deg(B') < \deg(B)$, contrary to $\deg(B) = d_K(A_1; \ldots; A_n)$. ■

Lemma 12. Suppose $A_1, \ldots, A_n$ are central simple L-algebras with $\deg(A_i) = p^{r_i}$. Then $d_K(A_1; \ldots; A_n) = p^t \cdot [L:K]$ for some t.

proof: Without loss of generality we may assume that the L-algebras are ordered so

$$\deg(A_1) \leq \deg(A_2) \leq \cdots \leq \deg(A_n).$$

Let B/K be central simple such that $d_K(A_1; \ldots ; A_n) = \deg(B)$ and suppose

$$[L : K] = p^a \cdot q_1^{a_1} \cdot q_2^{a_2} \cdots q_l^{a_l}$$

is the prime factorization. By Proposition 9, $r_K(A_i)$ divides $\deg(A_i)$ for each $i = 1, \ldots , n$ so we may write $r_K(A_i) = p^{s_i}$ for some s_i, and since $r_K(A) \leq \deg(A)$, $s_i \leq r_i$. By Proposition 8,

$$d_K(A_i) = p^{r_i + s_i} \cdot [L : K] \quad \text{for} \quad i = 1, \ldots , n.$$

We know $d_K(A_i)$ divides $d_K(A_1; \ldots ; A_n)$ for each i, so

$$d_K(A_1; \ldots ; A_n) = m \cdot p^v \cdot [L : K]$$

where $v = \max\{r_i + s_i \mid i = 1, \ldots , n\}$. By Lemma 11, the only primes which may divide m are $p, q_1, \ldots , q_l$. To prove the lemma we must show that $q_j \not| \, m$ for $j = 1, \ldots , l$.

Let Y_i be the centralizer of A_i in $C_B(L)$ for $i = 1, \ldots , n$. Then

$$\deg(B) = \deg(A_i) \cdot \deg(Y_i) \cdot [L : K]$$

so

$$m \cdot p^v \cdot [L : K] = p^{r_i} \cdot \deg(Y_i) \cdot [L : K]$$

or

$$m \cdot p^{v - r_i} = \deg(Y_i).$$

Suppose $q_j | m$ for some $j = 1, \ldots , l$. Then q_j divides $\deg(Y_i)$ for each $i = 1, \ldots n$. So by Lemma 10, each A_i embeds into a central simple B'/K where $\deg(B') < \deg(B)$, contrary to $d_K(A_1; \ldots ; A_n) = \deg(B)$. ∎

We are now in a position to compute $d_K(A_1; \ldots ; A_n)$ for the case in which each central simple L-algebra A_i has degree a power of a prime p. We will show

$d_K(A_1; \ldots; A_n) = d_K(\mathrm{M}_t(\Psi))$ where $\mathrm{M}_t(\Psi)$ is any p-embedder of $A_1, \ldots, A_n$. Let us recall how we determine a p-embedder of $A_1, \ldots, A_n$.

Suppose $A_1, \ldots, A_n$ are central simple L-algebras with $\deg(A_i) = p^{r_i}$ ordered so

$$\deg(A_1) \leq \deg(A_2) \leq \cdots \leq \deg(A_n),$$

and suppose $r_K(A_i) = p^{s_i}$ for $i = 1, \ldots, n$. Then by Proposition 8

$$d_K(A_i) = p^{r_i + s_i} \cdot [L : K] \quad \text{for} \quad i = 1, \ldots, n.$$

Let Γ_n be an L-division ring such that

$$A_n \otimes_L \Gamma_n \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad \mathrm{ind}(\Gamma_n) = r_K(A_n) = p^{s_n}.$$

Let Ψ be the skew field component of $A_n \otimes_L \Gamma_n$ so

$$A_n \otimes_L \Gamma_n \cong \mathrm{M}_{t_n}(\Psi) \quad \text{with} \quad t_n = \frac{\deg(A_n) \cdot \mathrm{ind}(\Gamma_n)}{\mathrm{ind}(\Psi)}.$$

Note that $\Psi \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$ so $r_K(\Psi) = 1$ and $d_K(\Psi) = \mathrm{ind}(\Psi) \cdot [L : K]$.

For $i = 1, \ldots, n-1$, let Γ_i be the L-division ring such that $A_i \otimes_L \Gamma_i \sim \Psi$. Say $\mathrm{ind}(\Gamma_i) = p^{u_i}$. Then for $i = 1, \ldots, n$ we have

$$A_i \otimes_L \Gamma_i \cong \mathrm{M}_{t_i}(\Psi) \quad \text{with} \quad t_i = \frac{\deg(A_i) \cdot \mathrm{ind}(\Gamma_i)}{\mathrm{ind}(\Psi)}.$$

A p-embedder of $A_1, \ldots, A_n$ is $\mathrm{M}_t(\Psi)$ where $t = \max\{t_1, \ldots, t_n\}$, and, as noted previously, $A_1, \ldots, A_n$ simultaneously embed into $\mathrm{M}_t(\Psi)$, so

$$d_K(A_1; \ldots; A_n) \leq d_K(\mathrm{M}_t(\Psi)). \tag{4}$$

THEOREM 13. Let the context be as above. Then $d_K(A_1; \ldots; A_n) = d_K(\mathrm{M}_t(\Psi))$.

proof: We consider two cases.

Case 1. $t = t_n$.

In this case $d_K(A_1; \ldots ; A_n) \leq t_n \cdot d_K(\Psi) = \deg(A_n) \cdot \mathrm{ind}(\Gamma_n) \cdot [L : K]$, and $d_K(A_1; \ldots ; A_n) \not< \deg(A_n) \cdot \mathrm{ind}(\Gamma_n) \cdot [L : K]$ since $d_K(A_n)$ divides $d_K(A_1; \ldots ; A_n)$, and $d_K(A_n) = \deg(A_n) \cdot \mathrm{ind}(\Gamma_n) \cdot [L : K]$.

Case 2. $t = t_j$ for some $j < n$.

In this case we may assume that $t_n < t_j$ so

$$\deg(A_n) \cdot \mathrm{ind}(\Gamma_n) < \deg(A_j) \cdot \mathrm{ind}(\Gamma_j)$$

which yields $\mathrm{ind}(\Gamma_n) < \mathrm{ind}(\Gamma_j)$ as $\deg(A_j) \leq \deg(A_n)$. In particular we have

$$\mathrm{ind}(\Gamma_j \otimes_L \Gamma_n^{op}) = \mathrm{ind}(\Gamma_j) = p^{u_j}.$$

By (4)

$$\begin{aligned} d_K(A_1; \ldots ; A_n) &\leq d_K(\mathrm{M}_{t_j}(\Psi)) \\ &= \deg(A_j) \cdot \mathrm{ind}(\Gamma_j) \cdot [L : K] \\ &= p^{r_j + u_j} \cdot [L : K]. \end{aligned}$$

We wish to show that equality holds.

Suppose $d_K(A_1; \ldots ; A_n) < p^{r_j + u_j} \cdot [L : K]$. Using Lemma 12 and the fact that $d_K(A_j)$ divides $d_K(A_1; \ldots ; A_n)$ we know

$$d_K(A_1; \ldots ; A_n) = p^{r_j + s_j + k} \cdot [L : K]$$

for some k between 0 and $u_j - s_j - 1$, inclusive.

Let B/K be central simple such that $A_1, \ldots , A_n$ each embed into B and

$$\deg(B) = d_K(A_1; \ldots ; A_n) = p^{r_j + s_j + k} \cdot [L : K].$$

Let Y_j be the centralizer of A_j in $C_B(L)$ and let Y_n be the centralizer of A_n in $C_B(L)$. Then

$$\begin{aligned} \deg(B) &= \deg(A_j) \cdot \deg(Y_j) \cdot [L : K] \\ &= \deg(A_n) \cdot \deg(Y_n) \cdot [L : K] \end{aligned}$$

or

$$p^{s_j+k} = \deg(Y_j) \quad \text{and} \quad p^{r_j+s_j+k-r_n} = \deg(Y_n).$$

By Proposition 6

$$B \otimes_K L \sim A_j \otimes_L Y_j \sim A_n \otimes_L Y_n$$

so

$$A_n \otimes_L A_j^{op} \sim Y_j \otimes_L Y_n^{op}.$$

Thus $\mathrm{ind}(A_n \otimes_L A_j^{op}) = \mathrm{ind}(Y_j \otimes_L Y_n^{op})$. But $r_j \leq r_n$ so

$$p^{r_j+s_j+k-r_n} \leq p^{s_j+k}$$

and thus

$$\mathrm{ind}(A_n \otimes_L A_j^{op}) = \mathrm{ind}(Y_j \otimes_L Y_n^{op}) \leq p^{s_j+k} < p^{u_j}.$$

But $A_n \otimes_L A_j^{op} \sim \Gamma_j \otimes_L \Gamma_n^{op}$ so

$$\mathrm{ind}(A_n \otimes_L A_j^{op}) = \mathrm{ind}(\Gamma_j \otimes_L \Gamma_n^{op}) = \mathrm{ind}(\Gamma_j) = p^{u_j},$$

a contradiction. ■

As noted previously, if A/L embeds into B/K then $d_K(A)$ divides $\deg(B)$. We have an analogous theorem in our situation.

THEOREM 14. Let $A_1, \ldots, A_n$ be central simple L-algebras with $\deg(A_i) = p^{r_i}$ for each $i = 1, \ldots, n$. If $A_1, \ldots, A_n$ embed into a central simple B/K, then

$$d_K(A_1; \ldots; A_n) \quad \text{divides} \quad \deg(B).$$

proof: Again we may assume the L algebras are ordered so

$$\deg(A_1) \leq \deg(A_2) \leq \cdots \leq \deg(A_n).$$

Let Γ_n be an L-division ring such that

$$A_n \otimes_L \Gamma_n \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad r_K(A_n) = \mathrm{ind}(\Gamma_n).$$

Say $\operatorname{ind}(\Gamma_n) = p^{s_n}$. Let Ψ be the skew field component of $A_n \otimes_L \Gamma_n$ and suppose for $i = 1, \ldots, n-1$, Γ_i is an L-division ring such that $A_i \otimes_L \Gamma_i \sim \Psi$. Say $\operatorname{ind}(\Gamma_i) = p^{u_i}$. Let

$$t_i = \frac{\deg(A_i) \cdot \operatorname{ind}(\Gamma_i)}{\operatorname{ind}(\Psi)},$$

so for $i = 1, \ldots, n$ we have $A_i \otimes_L \Gamma_i \cong \mathrm{M}_{t_i}(\Psi)$ and by Theorem 13 we know $d_K(A_1; \ldots; A_n) = d_K(\mathrm{M}_t(\Psi))$ where $t = \max\{t_1, \ldots, t_n\}$.

In particular, if $t = t_n$ then

$$d_K(A_1; \ldots; A_n) = \deg(A_n) \cdot \operatorname{ind}(\Gamma_n) \cdot [L : K] = p^{r_n + s_n} \cdot [L : K]$$

and if $t = t_j$ for some $j < n$ then

$$d_K(A_1; \ldots; A_n) = \deg(A_j) \cdot \operatorname{ind}(\Gamma_j) \cdot [L : K] = p^{r_j + u_j} \cdot [L : K].$$

Suppose $A_1, \ldots, A_n$ each embed into a central simple K-algebra $B \cong \mathrm{M}_a(D)$ where D is a K-division ring. If

$$d_K(A_1; \ldots; A_n) = p^{r_n + s_n} \cdot [L : K]$$

then the result follows easily as

$$d_K(A_n) = p^{r_n + s_n} \cdot [L : K]$$

and we know $d_K(A_n)$ divides $\deg(B)$.

Suppose $d_K(A_1; \ldots; A_n) = p^{r_j + u_j} \cdot [L : K]$. We must show that $p^{r_j + u_j} \cdot [L : K]$ divides $\deg(B)$ or, equivalently, that $p^{r_j + u_j}$ divides $\dfrac{\deg(B)}{[L : K]}$.

Suppose $p^{r_j + u_j}$ does not divide $\dfrac{\deg(B)}{[L : K]}$. We know $d_K(A_j) = p^{r_j + u_j} \cdot [L : K]$ divides $\deg(B)$, so p^{r_j} divides $\dfrac{\deg(B)}{[L : K]}$. Let v be the greatest integer such that $p^{r_j + v}$

divides $\dfrac{\deg(B)}{[L:K]}$, say $m \cdot p^{r_j+v} = \dfrac{\deg(B)}{[L:K]}$ where $(m,p)=1$ and $v < u_j$.

Let Y_i be the centralizer of A_i in $C_B(L)$ for each $i = 1, \ldots, n$. Then

$$\deg(B) = \deg(A_i) \cdot \deg(Y_i) \cdot [L:K]$$

or

$$m \cdot p^{r_j+v-r_i} = \deg(Y_i) \quad \text{for} \quad i = 1, \ldots n.$$

Thus m divides $\deg(Y_i)$ for each i. Let

$$m = q_1^{e_1} \cdot q_2^{e_2} \cdots q_l^{e_l}$$

be the prime factorization. So for each i we may write

$$Y_i \cong Y_{iq_1} \otimes_L \cdots \otimes_L Y_{iq_l} \otimes_L Y_{ip}$$

where Y_{iq_k} and Y_{ip} are central simple L-algebras with

$$\deg(Y_{iq_k}) = q_k^{e_k} \quad \text{and} \quad \deg(Y_{ip}) = p^{r_j+v-r_i} \quad \text{for} \quad k = 1, \ldots, l \quad \text{and} \quad i = 1, \ldots, n.$$

In particular note that $\deg(Y_{jp}) = p^v$, and

$$\deg(A_i) \cdot \deg(Y_{ip}) = p^{r_j+v} \quad \text{for each} \quad i = 1, \ldots, n.$$

Thus the product $\deg(A_i) \cdot \deg(Y_{ip})$ does not depend on i.

We may write

$$D \cong D_{q_1} \otimes_K \cdots \otimes_K D_{q_l} \otimes_K D'$$

where D_{q_k} and D' are K-division rings with $\mathrm{ind}(D_{q_k})$ a power of q_k and

$$(\mathrm{ind}(D'), q_k) = 1 \quad \text{for} \quad k = 1, \ldots, l.$$

By Proposition 6

$$D \otimes_K L \sim A_i \otimes_L Y_{iq_1} \otimes_L \cdots \otimes_L Y_{iq_l} \otimes_L Y_{ip} \quad \text{for each} \quad i = 1, \ldots, n.$$

So

$$(D_{q_1} \otimes_K L) \otimes_L \cdots \otimes_L (D_{q_l} \otimes_K L) \otimes_L (D' \otimes_K L) \sim A_i \otimes_L Y_{iq_1} \otimes_L \cdots \otimes_L Y_{iq_l} \otimes_L Y_{ip}$$

for $i = 1, \ldots, n$. Equating the q_k-regular components of $[D \otimes_K L]$ in $\mathrm{B}(L)$ for each $k = 1, \ldots, l$, we see that

$$D' \otimes_K L \sim A_i \otimes_L Y_{ip} \quad \text{for each} \quad i = 1, \ldots, n.$$

Thus by Proposition 7, A_i embeds into $\mathrm{M}_{w_i}(D')$ where

$$w_i = \frac{\deg(A_i) \cdot \deg(Y_{ip}) \cdot [L : K]}{\mathrm{ind}(D')} \quad \text{for} \quad i = 1, \ldots, n.$$

But then $w_1 = w_2 = \cdots = w_n$ since $\deg(A_i) \cdot \deg(Y_{ip})$ is constant for all i. Let w be this common value so A_i embeds into $\mathrm{M}_w(D')$ and

$$\deg(\mathrm{M}_w(D')) = \deg(A_i) \cdot \deg(Y_{ip}) \cdot [L : K] \quad \text{for each} \quad i = 1, \ldots, n.$$

In particular

$$\begin{aligned}\deg(\mathrm{M}_w(D')) &= \deg(A_j) \cdot \deg(Y_{jp}) \cdot [L : K] \\ &= p^{r_j + v} \cdot [L : K] \\ &< p^{r_j + u_j} \cdot [L : K] \\ &= d_K(A_1; \ldots; A_n),\end{aligned}$$

a contradiction. Thus $d_K(A_1; \ldots; A_n)$ divides $\deg(B)$. ■

We are now in a position to compute $d_K(A_1; \ldots; A_n)$ in the general case.

Theorem 15. Let $A_1, \ldots, A_n$ be central simple L-algebras and suppose that

$$p_1, \ldots, p_m$$

are primes dividing $\deg(A_k)$ for any $k = 1, \ldots, n$. Suppose

$$A_k = \bigotimes_{i=1}^{m} \Omega_{ki}$$

where Ω_{ki} is a central simple L-algebra with $\deg(\Omega_{ki})$ a power of p_i for each

$$k = 1, \ldots, n \quad \text{and} \quad i = 1, \ldots, m.$$

Then

$$d_K(A_1; \ldots; A_n) = [L : K] \cdot \prod_{i=1}^{m} \frac{d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni})}{[L : K]}$$

$$= \text{l.c.m.}\{d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni}) | i = 1, \ldots, m\}.$$

proof: The second equality follows from Lemma 12 which assures us that

$$d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni}) = p_i^{t_i} \cdot [L : K]$$

for some t_i. So

$$\text{l.c.m.}\{d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni}) | i = 1, \ldots, m\} = [L : K] \cdot \prod_{i=1}^{m} p_i^{t_i}$$

$$= [L : K] \cdot \prod_{i=1}^{m} \frac{d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni})}{[L : K]}.$$

For $i = 1, \ldots, m$ let B_i be a central simple K-algebra such that Ω_{ki} embeds into B_i for each $k = 1, \ldots, n$ and

$$\deg(B_i) = d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni}),$$

and suppose $B_i \cong \mathrm{M}_{u_i}(D_i)$ where D_i is a K-division ring. Let Y_{ki} be the centralizer of Ω_{ki} in $C_{B_i}(L)$ for $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Then

$$\deg(B_i) = \deg(\Omega_{ki}) \cdot \deg(Y_{ki}) \cdot [L : K]$$

and

$$D_i \otimes_K L \sim \Omega_{ki} \otimes_L Y_{ki} \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad k = 1, \ldots, n.$$

Let D be the skew field component of $\bigotimes_{i=1}^{m} D_i$. Then

$$D \otimes_K L \sim \bigotimes_{i=1}^{m} (\Omega_{ki} \otimes_L Y_{ki}) \sim A_k \otimes_L (\bigotimes_{i=1}^{m} Y_{ki}) \quad \text{for each} \quad k = 1, \ldots, n.$$

Then by Proposition 7, A_k embeds into $\mathrm{M}_{t_k}(D)$ for each $k = 1, \ldots, n$ with

$$t_k = \frac{\deg(A_k) \cdot \left(\prod_{i=1}^{m} \deg(Y_{kp_i})\right) \cdot [L:K]}{\operatorname{ind}(D)}$$

$$= \frac{\left(\prod_{i=1}^{m} \deg(\Omega_{ki}) \cdot \deg(Y_{ki})\right) \cdot [L:K]}{\operatorname{ind}(D)}.$$

Now

$$\deg(B_i) = \deg(\Omega_{ki}) \cdot \deg(Y_{ki}) \cdot [L:K] \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad k = 1, \ldots, n$$

so

$$\prod_{i=1}^{m} \deg(B_i) = \prod_{i=1}^{m} (\deg(\Omega_{ki}) \cdot \deg(Y_{ki}) \cdot [L:K])$$

and thus

$$[L:K] \cdot \prod_{1=1}^{m} (\deg(\Omega_{ki}) \cdot \deg(Y_{ki})) = \frac{\prod_{i=1}^{m} \deg(B_i)}{[L:K]^{m-1}}.$$

So the product

$$[L:K] \cdot \prod_{1=1}^{m} (\deg(\Omega_{ki}) \cdot \deg(Y_{ki}))$$

is the same for each $k = 1, \ldots, n$. But then $t_1 = t_2 = \cdots = t_n$. Let t be this common value, so for each $k = 1, \ldots, n$, A_k embeds into $\mathrm{M}_t(D)$ and

$$\deg(\mathrm{M}_t(D)) = \left(\prod_{i=1}^{m} \deg(\Omega_{ki}) \cdot \deg(Y_{ki})\right) \cdot [L:K]$$

$$= \frac{\prod_{i=1}^{m} \deg(B_i)}{[L:K]^{m-1}}$$

$$= \frac{\prod_{i=1}^{m} d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni})}{[L:K]^{m-1}}$$

$$= [L:K] \cdot \prod_{i=1}^{m} \frac{d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni})}{[L:K]}.$$

Thus

$$d_K(A_1;\ldots;A_n) \le \deg(\mathrm{M}_t(D))$$

$$= [L:K] \cdot \prod_{i=1}^{m} \frac{d_K(\Omega_{1i};\Omega_{2i};\ldots;\Omega_{ni})}{[L:K]}$$

$$= \text{l.c.m.}\{d_K(\Omega_{1i};\Omega_{2i};\ldots;\Omega_{ni}) | \ i = 1,\ldots,m\}.$$

To complete the proof, we need to show that equality holds. Suppose B/K is central simple such that $A_1,\ldots,A_n$ embed into B and $\deg(B) = d_K(A_1;\ldots;A_n)$. Then Ω_{ki} embeds into B for $k = 1,\ldots,n$ and $i = 1,\ldots,m$, so by Theorem 14 $d_K(\Omega_{1i};\Omega_{2i};\ldots;\Omega_{ni})$ divides $\deg(B)$ for $i = 1,\ldots,m$.. But then

$$\text{l.c.m.}\{d_K(\Omega_{1i};\Omega_{2i};\ldots;\Omega_{ni}) | \ i = 1,\ldots,m\} \quad \text{divides} \quad \deg(B).$$

That is $\deg(\mathrm{M}_t(D))$ divides $\deg(B)$ and thus $d_K(A_1;\ldots;A_n) = \deg(\mathrm{M}_t(D))$. ■

We are now in a position to prove our main theorem.

THEOREM A. Let $A_1,\ldots,A_n$ be central simple L-algebras and suppose $p_1,\ldots,p_m$ are primes dividing $\deg(A_k)$ for any $k = 1,\ldots,n$. Suppose $A_k \cong \bigotimes_{i=1}^m \Omega_{ki}$ where Ω_{ki} is a central simple L-algebra with $\deg(\Omega_{ki})$ a power of p_i for each $k = 1,\ldots,n$ and $i = 1,\ldots,m$. Let $\mathrm{M}_{t_i}(\Psi_i)$ be a p_i-embedder for $\Omega_{1i},\Omega_{2i},\ldots,\Omega_{ni}$ for each $i = 1,\ldots,m$ and let $A = \bigotimes_{i=1}^m \mathrm{M}_{t_i}(\Psi_i)$. Then $d_K(A_1;\ldots;A_n) = d_K(A)$. Moreover $r_K(A) = 1$, so $d_K(A_1;\ldots;A_n) = [L:K] \cdot \prod_{i=1}^m t_i \cdot \mathrm{ind}(\Psi_i)$.

proof: By Theorem 13 $d_K(\Omega_{1i};\Omega_{2i};\ldots;\Omega_{ni}) = d_K(\mathrm{M}_{t_i}(\Psi_i))$ where $\mathrm{M}_{t_i}(\Psi_i)$is a p_i-embedder of $\Omega_{1i},\Omega_{2i},\ldots,\Omega_{ni}$ for $i = 1,\ldots,m$. Also

$$d_K(\mathrm{M}_{t_i}(\Psi_i)) = t_i \cdot \mathrm{ind}(\Psi_i) \cdot [L:K]$$

by Theorem 8. Let

$$A = \bigotimes_{i=1}^m \mathrm{M}_{t_i}(\Psi_i).$$

Since Ω_{ki} embeds into $\mathrm{M}_{t_i}(\Psi_i)$ for $k = 1, \ldots, n$, $\Omega_{k1} \otimes_L \Omega_{k2} \otimes_L \cdots \otimes_L \Omega_{km}$ embeds into

$$\mathrm{M}_{t_1}(\Psi_1) \otimes_L \mathrm{M}_{t_2}(\Psi_2) \otimes_L \cdots \otimes_L \mathrm{M}_{t_m}(\Psi_m).$$

That is, A_k embeds into A for each $k = 1, \ldots, n$.

Now $d_K(A) = \deg(A) \cdot r_K(A) \cdot [L : K]$. But $r_K(A) = 1$ since

$$A \sim \Psi_1 \otimes_L \cdots \otimes_L \Psi_m$$

and each $\Psi_i \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$. So

$$\begin{aligned}
d_K(A) &= [L : K] \cdot \deg(A) \\
&= [L : K] \cdot \prod_{i=1}^{m} \deg(\mathrm{M}_{t_i}(\Psi_i)) \\
&= [L : K] \cdot \prod_{i=1}^{m} t_i \cdot \mathrm{ind}(\Psi_i) \\
&= [L : K] \cdot \prod_{i=1}^{m} \frac{d_K(\mathrm{M}_{t_i}(\Psi_i))}{[L : K]} \\
&= [L : K] \cdot \prod_{i=1}^{m} \frac{d_K(\Omega_{1i}; \Omega_{2i}; \ldots ; \Omega_{ni})}{[L : K]} \\
&= d_K(A_1; \ldots ; A_n).
\end{aligned}$$

■

We conclude this section with the following corollary which generalizes Theorem 14.

COROLLARY 16. Let $A_1, \ldots, A_n$ be central simple L-algebras which each embed into a central simple B/K. Then $d_K(A_1; \ldots ; A_n)$ divides $\deg(B)$.

proof: Suppose

$$A_k = \bigotimes_{i=1}^{m} \Omega_{ki}$$

where Ω_{ki} is a central simple L-algebra with $\deg(\Omega_{ki})$ a power of p_i for each

$$k = 1, \ldots, n \quad \text{and} \quad i = 1, \ldots, m.$$

Since each A_i embeds into B, we have that Ω_{ki} embeds into B for each $k = 1, \ldots, n$ and $i = 1, \ldots, m$, so by Theorem 14, $d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni})$ divides $\deg(B)$ for each $i = 1, \ldots, m$, and hence l.c.m.$\{d_K(\Omega_{1i}; \Omega_{2i}; \ldots; \Omega_{ni}) \mid i = 1, \ldots, m\}$ divides $\deg(B)$, that is $d_K(A_1; \ldots; A_n)$ divides $\deg(B)$. ■

5. An Example

In this section we define two central simple L-algebras A_1 and A_2 and compute $d_K(A_1; A_2)$. We do this by decomposing A_1 and A_2 into their p-components and then determining p-embedders for these components. Then, using Theorem A, we produce a central simple L-algebra $A \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$ such that $d_K(A_1; A_2) = d_K(A) = \deg(A) \cdot [L : K]$.

Example 2. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{2})$, and let $p_1 \equiv 1 \pmod 8$ and $p_2 \equiv 3 \pmod 8$ be primes and suppose α_1 and α_2 are primes of L extending p_1, and β is the prime of L extending p_2.

Let $A_1 \cong \mathrm{M}_5(\Delta_1)$ where Δ_1 is the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Delta_1] = \frac{7}{12}, \quad \mathrm{inv}_{\alpha_2}[\Delta_1] = \frac{1}{3}, \quad \mathrm{inv}_{\beta}[\Delta_1] = \frac{1}{12}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Delta_1] = 0$$

for all other primes ρ of L. Let $A_2 \cong \mathrm{M}_3(\Delta_2)$ where Δ_2 is the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Delta_2] = \frac{1}{6}, \quad \mathrm{inv}_{\alpha_2}[\Delta_2] = \frac{1}{2}, \quad \mathrm{inv}_{\beta}[\Delta_2] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Delta_2] = 0$$

for all other primes ρ of L. We wish to compute $d_K(A_1; A_2)$. Note that $\deg(A_1) = 60$ and $\deg(A_2) = 18$ so the primes dividing $\deg(A_1)$ or $\deg(A_2)$ are 2, 3, and 5.

First we decompose A_1 and A_2. We see

$$A_1 \cong \Upsilon_2 \otimes_L \Upsilon_3 \otimes_L \mathrm{M}_5(L)$$

and

$$A_2 \cong \Omega_2 \otimes_L \mathrm{M}_3(\Omega_3)$$

where $\Upsilon_2, \Upsilon_3, \Omega_2$ and Ω_3 are defined as follows:

$$\mathrm{inv}_{\alpha_1}[\Upsilon_2] = \frac{1}{4}, \quad \mathrm{inv}_{\beta}[\Upsilon_2] = \frac{3}{4}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Upsilon_2] = 0$$

for all other primes ρ of L,

$$\mathrm{inv}_{\alpha_1}[\Upsilon_3] = \mathrm{inv}_{\alpha_2}[\Upsilon_3] = \mathrm{inv}_{\beta}[\Upsilon_3] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Upsilon_3] = 0$$

for all other primes ρ of L,

$$\mathrm{inv}_{\alpha_1}[\Omega_2] = \mathrm{inv}_{\alpha_2}[\Omega_2] = \frac{1}{2}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Omega_2] = 0$$

for all other primes ρ of L, and

$$\mathrm{inv}_{\alpha_1}[\Omega_3] = \frac{2}{3}, \quad \mathrm{inv}_{\beta}[\Omega_3] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Omega_3] = 0$$

for all other primes ρ of L.

We will find

$$\mathrm{M}_{t_1}(\Psi_1), \quad \text{a 2-embedder of} \quad \Upsilon_2, \Omega_2,$$

$$\mathrm{M}_{t_2}(\Psi_2), \quad \text{a 3-embedder of} \quad \Upsilon_3, \mathrm{M}_3(\Omega_3),$$

and

$$\mathrm{M}_{t_3}(\Psi_3), \quad \text{a 5-embedder of} \quad \mathrm{M}_5(L), L.$$

Then by Theorem A, if

$$A = \mathrm{M}_{t_1}(\Psi_1) \otimes_L \mathrm{M}_{t_2}(\Psi_2) \otimes_L \mathrm{M}_{t_3}(\Psi_3)$$

we will have $d_K(A_1; A_2) = d_K(A)$.

Now let us find $\mathrm{M}_{t_1}(\Psi_1)$. By applying techniques similar to the previous examples we find $r_K(\Upsilon_2) = 4$. Since $\deg(\Omega_2) = 2 \leq \deg(\Upsilon_2) = 4$ we begin by finding an L-division ring Γ_2 such that

$$\Upsilon_2 \otimes_L \Gamma_2 \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad \mathrm{ind}\,(\Gamma_2) = r_K(\Upsilon_2) = 4.$$

For example, let Γ_2 be such that

$$\mathrm{inv}_{\alpha_2}[\Gamma_2] = \frac{1}{4}, \quad \mathrm{inv}_{\beta}[\Gamma_2] = \frac{3}{4}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Gamma_2] = 0$$

for all other primes ρ of L. Then $\Upsilon_2 \otimes_L \Gamma_2 \cong M_4(\Psi_1)$ where Ψ_1 is the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Psi_1] = \mathrm{inv}_{\alpha_2}[\Psi_1] = \frac{1}{4}, \quad \mathrm{inv}_{\beta}[\Psi_1] = \frac{1}{2}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Psi_1] = 0$$

for all other primes ρ of L.

Now let Γ_1 be the L-division ring such that $\Omega_2 \otimes_L \Gamma_1 \sim \Psi$, i.e. Γ_1 is such that

$$\mathrm{inv}_{\alpha_1}[\Gamma_1] = \mathrm{inv}_{\alpha_2}[\Gamma_1] = \frac{3}{4}, \quad \mathrm{inv}_{\beta}[\Gamma_1] = \frac{1}{2}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Gamma_1] = 0$$

for all other primes ρ of L. Then $\Omega_2 \otimes_L \Gamma_1 \cong M_2(\Psi_1)$. So $M_4(\Psi_1)$ is a 2-embedder of Υ_2, Ω_2.

Next let us find $M_{t_2}(\Psi_2)$. Again applying techniques similar to the previous examples we find $r_K(M_3(\Omega_3)) = 3$. Since $\deg(\Upsilon_3) = 3 \le \deg(M_3(\Omega_3)) = 9$ we begin by finding an L-division ring Θ_2 such that

$$M_3(\Omega_3) \otimes_L \Theta_2 \in \mathrm{Res}_{L/K}(B(K)) \quad \text{and} \quad \mathrm{ind}\,(\Theta_2) = r_K(M_3(\Omega_3)) = 3.$$

For example, let Θ_2 be such that

$$\mathrm{inv}_{\alpha_2}[\Theta_2] = \frac{2}{3}, \quad \mathrm{inv}_{\beta}[\Theta_2] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Theta_2] = 0$$

for all other primes ρ of L. Then $M_3(\Omega_3) \otimes_L \Theta_2 \cong M_9(\Psi_2)$ where Ψ_2 is the L-division ring with

$$\mathrm{inv}_{\alpha_1}[\Psi_2] = \mathrm{inv}_{\alpha_2}[\Psi_2] = \mathrm{inv}_{\beta}[\Psi_2] = \frac{2}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Psi_2] = 0$$

for all other primes ρ of L. Now let Θ_1 be the L-division ring such that

$$\Upsilon_3 \otimes_L \Theta_1 \sim \Psi_2,$$

i.e. Θ_1 is such that

$$\mathrm{inv}_{\alpha_1}[\Theta_1] = \mathrm{inv}_{\alpha_2}[\Theta_1] = \mathrm{inv}_{\beta}[\Theta_1] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Theta_1] = 0$$

for all other primes ρ of L. Then $\Upsilon_3 \otimes_L \Theta_1 \cong \mathrm{M}_3(\Psi_2)$, so $\mathrm{M}_9(\Psi_2)$ is a 3-embedder of Υ_3, Ω_3.

Finally we see that $\mathrm{M}_5(L)$ is a 5-embedder of $\mathrm{M}_5(L), L$ so if

$$A = \mathrm{M}_4(\Psi_1) \otimes_L \mathrm{M}_9(\Psi_2) \otimes_L \mathrm{M}_5(L)$$

then

$$\begin{aligned} d_K(A_1; A_2) &= d_K(A) \\ &= \deg(\mathrm{M}_4(\Psi_1) \otimes_L \mathrm{M}_9(\Psi_2) \otimes_L \mathrm{M}_5(L)) \cdot [L : K] \\ &= 2^5 \cdot 3^3 \cdot 5. \end{aligned}$$

6. Remarks on $d_K(A_1; \ldots; A_n)$

In the proof of Theorem 13 it was necessary to order the central simple L-algebras $A_1, \ldots, A_n$ so that

$$\deg(A_1) \leq \deg(A_2) \leq \cdots \leq \deg(A_n)$$

in order to determine the L-division ring Γ_n with

$$A_n \otimes_L \Gamma_n \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad r_K(A_n) = \mathrm{ind}(\Gamma_n).$$

This in turn allowed us to find the skew field component Ψ of the L-algebra $\mathrm{M}_t(\Psi)$ into which each A_i was embeddable with the property that

$$d_K(A_1; \ldots; A_n) = d_K(\mathrm{M}_t(\Psi)).$$

Suppose we began by picking an L-algebra A_j which does not have maximum degree and then find an L-division ring Ω_j such that

$$A_j \otimes_L \Omega_j \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad r_K(A_j) = \mathrm{ind}(\Omega_j).$$

If Λ is the skew field component of $A_j \otimes_L \Omega_j$, and Ω_i is the L-division ring with

$$A_i \otimes_L \Omega_i \sim \Lambda \quad \text{for} \quad i = 1, \ldots, n,$$

and

$$t_i = \frac{\deg(A_i) \cdot \mathrm{ind}(\Omega_i)}{\mathrm{ind}(\Lambda)}$$

then $A_i \otimes_L \Omega_i \cong \mathrm{M}_{t_i}(\Lambda)$ for $i = 1, \ldots, n$. So if $t = \max\{t_1, \ldots, t_n\}$ each A_i is embeddable into $\mathrm{M}_t(\Lambda)$ and

$$d_K(A_1; \ldots; A_n) \leq d_K(\mathrm{M}_t(\Lambda)).$$

However we do not necessarily have equality in this instance, as the following example illustrates.

EXAMPLE 3. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{2})$. Let p_1 and p_2 be primes with $p_1 \equiv 1 \pmod 8$ and $p_2 \equiv 3 \pmod 8$. Suppose α_1 and α_2 are primes of L extending p_1 and β is the prime of L extending p_2. Let A_1 and A_2 be L-division rings such that

$$\mathrm{inv}_{\alpha_1}[A_1] = \frac{1}{9}, \quad \mathrm{inv}_{\alpha_2}[A_1] = \frac{2}{9}, \quad \mathrm{inv}_{\beta}[A_1] = \frac{2}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[A_1] = 0$$

for all other primes ρ of L, and

$$\mathrm{inv}_{\alpha_1}[A_2] = \frac{1}{27}, \quad \mathrm{inv}_{\alpha_2}[A_2] = \frac{10}{27}, \quad \mathrm{inv}_{\beta}[A_2] = \frac{15}{27}, \quad \text{and} \quad \mathrm{inv}_{\rho}[A_2] = 0$$

for all other primes ρ of L. Using techniques similar to those in the previous examples we find $r_K(A_1) = 9$ and $r_K(A_2) = 3$. Ordering the L algebras by degree, we begin by finding an L-division ring Γ_2 such that

$$A_2 \otimes_L \Gamma_2 \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad r_K(A_2) = \mathrm{ind}(\Gamma_2).$$

For example, let Γ_2 be such that

$$\mathrm{inv}_{\alpha_1}[\Gamma_2] = \frac{1}{3}, \quad \mathrm{inv}_{\beta}[\Gamma_2] = \frac{2}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Gamma_2] = 0$$

for all other primes ρ of L. Then $A_2 \otimes_L \Gamma_2 \cong \mathrm{M}_3(\Psi)$ where Ψ is the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Psi] = \mathrm{inv}_{\alpha_2}[\Psi] = \frac{10}{27}, \quad \mathrm{inv}_{\beta}[\Psi] = \frac{7}{27}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Psi] = 0$$

for all other primes ρ of L. Then let Γ_1 be the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Gamma_1] = \frac{7}{27}, \quad \mathrm{inv}_{\alpha_2}[\Gamma_1] = \frac{4}{27}, \quad \mathrm{inv}_{\beta}[\Gamma_1] = \frac{15}{27}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Gamma_1] = 0$$

for all ther primes ρ of L. So $A_1 \otimes_L \Gamma_1 \cong \mathrm{M}_9(\Psi)$. Then by Theorem 13

$$d_K(A_1; A_2) = d_K(\mathrm{M}_9(\Psi)) = 2 \cdot 3^5.$$

Now suppose we first find an L-division ring Ω_1 such that

$$A_1 \otimes_L \Omega_1 \in \mathrm{Res}_{L/K}(\mathrm{B}(K)) \quad \text{and} \quad r_K(A_1) = \mathrm{ind}(\Omega_1) = 9.$$

Say Ω_1 is the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Omega_1] = \frac{1}{9}, \quad \mathrm{inv}_{\beta}[\Omega_1] = \frac{8}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Omega_1] = 0$$

for all ther primes ρ of L. Let Λ be the skew field component of $A_1 \otimes_L \Omega_1$. So

$$\mathrm{inv}_{\alpha_1}[\Lambda] = \mathrm{inv}_{\alpha_2}[\Lambda] = \frac{2}{9}, \quad \mathrm{inv}_{\beta}[\Lambda] = \frac{5}{9}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Lambda] = 0$$

for all other primes ρ of L. Then $A_1 \otimes_L \Omega_1 \cong \mathrm{M}_9(\Lambda)$. Now let Ω_2 be the L-division ring such that

$$\mathrm{inv}_{\alpha_1}[\Omega_2] = \frac{5}{27}, \quad \mathrm{inv}_{\alpha_2}[\Omega_2] = \frac{23}{27}, \quad \mathrm{inv}_{\beta}[\Omega_2] = \frac{26}{27}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Omega_2] = 0$$

for all other primes ρ of L. Then $A_2 \otimes_L \Omega_2 \cong \mathrm{M}_{81}(\Lambda)$. So A_1 and A_2 both embed into $\mathrm{M}_{81}(\Lambda)$, but $d_K(\mathrm{M}_{81}(\Lambda)) = 2 \cdot 3^6 \neq d_K(A_1; A_2)$.

As noted at the end of Chapter 1, we will have a much simpler means of computing $d_K(A_1; \ldots; A_n)$ provided the degrees of the central simple L-algebras are pairwise relatively prime, that is,

$$d_K(A_1; \ldots; A_n) = d_K(A_1 \otimes_L \cdots \otimes_L A_n).$$

In order to show that this is the case, we first prove the following lemma.

Lemma 17. Let L/K be a finite extension of fields and suppose $A_1, \ldots, A_n$ are central simple L-algebras with $(\mathrm{ind}(A_1), \mathrm{ind}(A_2), \ldots, \mathrm{ind}(A_n)) = 1$. Then

$$r_K(A_1 \otimes_L A_2 \otimes_L \cdots \otimes_L A_n) = r_K(A_1) \cdot r_K(A_2) \cdots r_K(A_n).$$

proof: We prove the lemma by induction on n.

For $n = 2$ we know $r_K(A_1)$ divides $\mathrm{ind}(A_1)$ and $r_K(A_2)$ divides $\mathrm{ind}(A_2)$ by Proposition 9, and therefore $(r_K(A_1), r_K(A_2)) = 1$.

Let X, Y, and Z be L-division rings such that

$$[A_1 \otimes_L X] \in \mathrm{Res}_{L/K}(\mathrm{B}(K)), \quad [A_2 \otimes_L Y] \in \mathrm{Res}_{L/K}(\mathrm{B}(K)),$$

$$[A_1 \otimes_L A_2 \otimes_L Z] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$$

and

$$r_K(A_1) = \mathrm{ind}(X), \quad r_K(A_2) = \mathrm{ind}(Y) \quad \text{and} \quad r_K(A_1 \otimes_L A_2) = \mathrm{ind}(Z).$$

We see $[A_1 \otimes_L A_2 \otimes_L X \otimes_L Y] = [A_1 \otimes_L X] + [A_2 \otimes_L Y] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$ so

$$r_K(A_1 \otimes_L A_2) \leq \mathrm{ind}(X \otimes_L Y) = \mathrm{ind}(X) \cdot \mathrm{ind}(Y) = r_K(A_1) \cdot r_K(A_2).$$

Also

$$[A_1 \otimes_L A_2 \otimes_L Z] \in \mathrm{Res}_{L/K}(\mathrm{B}(K))$$

so $r_K(A_1)$ divides $\mathrm{ind}(A_2 \otimes_L Z)$ [4, Cor. 10]. But

$$\mathrm{ind}(A_2 \otimes_L Z) \quad \text{divides} \quad \mathrm{ind}(A_2) \cdot \mathrm{ind}(Z)$$

by Proposition 1,(c), so $r_K(A_1)$ divides $\mathrm{ind}(A_2) \cdot \mathrm{ind}(Z)$. But $(r_K(A_1), \mathrm{ind}(A_2)) = 1$, so $r_K(A_1)$ divides $\mathrm{ind}(Z)$. Similarly, $r_K(A_2)$ divides $\mathrm{ind}(Z)$. Then, since

$$(r_K(A_1), r_K(A_2)) = 1,$$

we have $r_K(A_1) \cdot r_K(A_2)$ divides $\mathrm{ind}(Z)$, i.e. $r_K(A_1) \cdot r_K(A_2)$ divides $r_K(A_1 \otimes_L A_2)$. So $r_K(A_1) \cdot r_K(A_2) \leq r_K(A_1 \otimes_L A_2)$, and thus we have equality.

Assume now that the lemma is valid for $n-1$ central simple L-algebras. Let $A_1, \ldots, A_n$ be central simple L-algebras with $(\mathrm{ind}(A_1), \ldots, \mathrm{ind}(A_n)) = 1$. Let

$$A = A_1 \otimes_L \cdots \otimes_L A_{n-1}.$$

By Proposition 1,(d),

$$\mathrm{ind}(A) = \prod_{i=1}^{n-1} \mathrm{ind}(A_i) \quad \text{so} \quad (\mathrm{ind}(A), \mathrm{ind}(A_n)) = 1.$$

Then

$$\begin{aligned} r_K(A_1 \otimes_L \cdots \otimes_L A_n) &= r_K(A \otimes_L A_n) \\ &= r_K(A) \cdot r_K(A_n) \\ &= r_K(A_1) \cdots r_K(A_{n-1}) \cdot r_K(A_n) \end{aligned}$$

■

THEOREM 18. Let L/K be a finite extension of stable fields and suppose $A_1, \ldots, A_n$ are central simple L-algebras with $(\deg(A_1), \deg(A_2), \ldots, \deg(A_n)) = 1$. Then $d_K(A_1; \ldots; A_n) = d_K(A_1 \otimes_L \cdots \otimes_L A_n)$.

proof: Suppose $p_{k1}, p_{k2}, \ldots, p_{km_k}$ are the primes dividing $\deg(A_k)$ for $k = 1, \ldots, n$. Write

$$A_k \cong \bigotimes_{i=1}^{m_k} \Omega_{ki}$$

where Ω_{ki} is a central simple L-algebra with $\deg(\Omega_{ki})$ a power of p_{ki}. Then by Theorem 15

$$\begin{aligned} d_K(A_1; \ldots; A_n) = [L:K] \cdot \left(\prod_{i=1}^{m_1} \frac{d_K(\Omega_{1i}; L; \ldots; L)}{[L:K]} \right) \cdot \left(\prod_{i=1}^{m_2} \frac{d_K(\Omega_{2i}; L; \ldots; L)}{[L:K]} \right) \cdots \\ \cdot \left(\prod_{i=1}^{m_n} \frac{d_K(\Omega_{ni}; L; \ldots; L)}{[L:K]} \right) \end{aligned}$$

$$\begin{aligned} &= [L:K] \cdot \left(\prod_{i=1}^{m_1} \frac{d_K(\Omega_{1i})}{[L:K]} \right) \cdot \left(\prod_{i=1}^{m_2} \frac{d_K(\Omega_{2i})}{[L:K]} \right) \cdots \left(\prod_{i=1}^{m_n} \frac{d_K(\Omega_{ni})}{[L:K]} \right) \\ &= [L:K] \cdot \left(\prod_{i=1}^{m_1} \deg(\Omega_{1i}) \cdot r_K(\Omega_{1i}) \right) \cdot \left(\prod_{i=1}^{m_2} \deg(\Omega_{2i}) \cdot r_K(\Omega_{2i}) \right) \cdots \\ &\qquad \cdot \left(\prod_{i=1}^{m_n} \deg(\Omega_{ni}) \cdot r_K(\Omega_{ni}) \right) \end{aligned}$$

$$= [L:K]\cdot(\deg(A_1)\cdot r_K(A_1))\cdot(\deg(A_2)\cdot r_K(A_2))\cdots(\deg(A_n)\cdot r_K(A_n))$$
$$= [L:K]\cdot\deg(A_1\otimes_L\cdots\otimes_L A_n)\cdot r_K(A_1\otimes_L\cdots\otimes_L A_n)$$
$$= d_K(A_1\otimes_L\cdots\otimes_L A_n).$$

■

Note that the conclusion of Theorem 18 is not valid if we only require that $(\mathrm{ind}(A_1),\mathrm{ind}(A_2)) = 1$ rather than the stronger condition $(\deg(A_1),\deg(A_2)) = 1$. We illustrate this with the following example.

EXAMPLE 4. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{2})$, and suppose

$$p_1 \equiv 1(\mathrm{mod}\ 8), \quad p_2 \equiv 3(\mathrm{mod}\ 8)$$

and α_1 and α_2 are primes of L extending p_1, and β is the prime of L extending p_2. Let $A_1 = \mathrm{M}_2(\Delta_1)$ where Δ_1 is such that

$$\mathrm{inv}_{\alpha_1}[\Delta_1] = \mathrm{inv}_{\alpha_2}[\Delta_1] = \mathrm{inv}_{\beta}[\Delta_1] = \frac{1}{3}, \quad \text{and} \quad \mathrm{inv}_{\rho}[\Delta_1] = 0$$

for all other primes ρ of L, and let $A_2 = \mathrm{M}_2(L)$. Then $A_1 \otimes_L A_2 \cong \mathrm{M}_4(\Delta_1)$, so

$$d_K(A_1\otimes_L A_2) = 4\cdot d_K(\Delta_1) = 2^3\cdot 3.$$

But $A_1 \cong \Delta_1 \otimes_L \mathrm{M}_2(L)$ so by Theorem 15

$$d_K(A_1;A_2) = [L:K]\cdot\frac{d_K(\mathrm{M}_2(L);\mathrm{M}_2(L))}{[L:K]}\cdot\frac{d_K(\Delta_1;L)}{[L:K]} = 2^2\cdot 3.$$

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