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In this thesis we define two $n \times n$ matrices T and S to be isoconjunctive if there exists an $n \times n$ nonsingular hermitian matrix H such that $T = HSH$. We then proceed to establish necessary and sufficient conditions that two $n \times n$ matrices T and S be isoconjunctive in the case where both T and S are hermitian. The results are obtained by decomposing the problem into two cases. In Chapter II we consider the case where TS is nonsingular. In Chapter III we first obtain a canonical form, under contragradient conjunctivity, for a pair of $n \times n$ hermitian matrices whose product is nilpotent. This result is used to obtain the conditions in the nilpotent case. In Chapter IV we draw together the results of Chapter II and Chapter III. Also in Chapter IV we give necessary and sufficient conditions on a pair of $n \times n$ hermitian matrices T and S that there exist a positive definite matrix H such that $T = HSH$.

Isoconjunctivity of Hermitian Matrices

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ISOCONJUNCTIVITY OF HERMITIAN MATRICES

I. INTRODUCTION

Two $n \times n$ complex matrices T and S are said to be <u>conjunctive</u> if there exists a nonsingular $n \times n$ complex matrix C such that $T = C^*SC$, where C^* denotes the conjugate transpose of C. It is well known [5, p. 184] that if T and S are hermitian, then T and S are conjunctive if and only if T and S have the same rank and signature.

In this thesis we consider a property of pairs of matrices which is a special kind of conjunctivity. Accordingly we make the following definition.

<u>Definition 1.</u> Two $n \times n$ complex matrices T and S are <u>isoconjunctive</u> if there exists a nonsingular $n \times n$ hermitian matrix H such that $T = HSH$.

It is easy to see from Definition 1 that if two matrices are isoconjunctive, then they will also be conjunctive. Example 4 below will show that isoconjunctivity is not an equivalence relation (on the set of $n \times n$ complex matrices) since it is not transitive, whereas conjunctivity is an equivalence relation. Obviously isoconjunctivity is reflexive and symmetric. We will be interested in the isoconjunctivity of a pair of matrices in the case where one of the matrices (and hence the other) is hermitian. Before proceeding though, let us state and prove some of the more obvious consequences of Definition 1, some of which will be useful in later discussion.

<u>Theorem 1.</u> Let T and S be $n\times n$ complex matrices. Then the following are equivalent:

a) T and S are isoconjunctive

b) $T(ST)^p$ and $S(TS)^p$ are isoconjunctive for every nonnegative integer p

c) C^*TC and $C^{-1}SC^{*-1}$ are isoconjunctive for every nonsingular complex matrix C

d) there exist nonsingular complex matrices C and D such that $(CD)^* = CD$, $T = C^*SC$, and $T = D^*TD$.

Proof: We prove a)$\Longrightarrow$b)$\Longrightarrow$c)$\Longrightarrow$a) and a)$\Longleftrightarrow$d).

a)$\Longrightarrow$b): Assume $T = HSH$ with $H = H^*$ nonsingular. Then b) is true for $p = 0$. Now let $p \geq 1$. Then

$$T(ST)^p = HSH(ST)^{p-1}SHSH,$$

and

$$S(TS)^p = S(HSHS)^p$$
$$= SH(SHSH)^{p-1}SHS$$
$$= SH(ST)^{p-1}SHS.$$

Thus $T(ST)^p = HS(TS)^pH$ for every nonnegative integer p.

b)$\Longrightarrow$c): Assume b). Then in particular $T = HSH$ for some nonsingular $H = H^*$. Let C be an arbitrary $n\times n$ complex matrix. Then

$$C^*TC = C^*HSHC$$
$$= C^*H(CC^{-1})S(C^{*-1}C^*)HC$$
$$= (C^*HC)(C^{-1}SC^{*-1})C^*HC.$$

Since $(C^*HC)^* = C^*HC$ (and is nonsingular) and C is arbitrary this proves c).

c)$\Longrightarrow$a): Assume c). Take $C = I$ (the $n\times n$ identity matrix). Then c) says T and S are isoconjunctive, which is what we wanted.

a)$\Longleftrightarrow$d): Assume a). Then $T = HSH$ for some nonsingular $H = H^*$. In d) take $C = H$ and $D = I$ (the $n\times n$ identity matrix). Then $(CD)^* = CD$, $T = C^*SC$, and $T = D^*TD$. Now assume d). Then

$$(CD)^*S(CD) = D^*C^*SCD$$
$$= D^*TD$$
$$= T.$$

Hence T and S are isoconjunctive. This completes the proof of the theorem.

It should be noted at this point that the equivalence of a) and d) of Theorem 1 can be restated in an alternate manner using the concept of conjunctive automorphs (see[16]). Utilizing this idea the statement says that two conjunctive matrices T and S (i.e. $T = C^*SC$ for some nonsingular matrix C) are isoconjunctive if and only if there exists a conjunctive automorph D of T (i.e. $T = D^*TD$ with D nonsingular) such that CD is hermitian.

The next result addresses itself precisely to the case which is of primary interest here; namely, when T and S are both hermitian.

Theorem 2. Let T and S be hermitian matrices which are isoconjunctive. Then

a) TS is a square

b) $TS = A^2$ where A is similar to a real matrix

c) TS is similar to the square of a real matrix

d) there exists a nonsingular hermitian matrix K such that $TS = K^{-1}STK$.

Proof: Let $T = HSH$ for some nonsingular $H = H^*$. Then $TS = HSHS = (HS)^2$ where, by Theorem 2 of [3], HS is similar to a real matrix. Thus a) and b) are proved. Now c) follows from b), and d) may be observed by using Theorem 2 of [3].

Utilizing Theorem 2 of [3] once more let us note that if $\lambda \neq \bar{\lambda}$ is a characteristic root of TS (T and S are hermitian), then $\bar{\lambda}$ is also a characteristic root of TS with the same multiplicity. Furthermore the Jordan form of TS has the same number of Jordan blocks of each order at $\bar{\lambda}$ as at λ.

At this point it seems reasonable to show that there are, in fact, pairs of matrices which are nontrivially isoconjunctive, and also that isoconjunctivity does not coincide with conjunctivity. To this end we consider the following examples.

Example 1. Let T and S be $n\times n$ positive definite (hermitian) matrices. Then T and S are isoconjunctive. To see this we note that $T = HSH$ for some nonsingular $H = H^*$ if and only if

$$\begin{aligned} S^{1/2}TS^{1/2} &= (S^{1/2}HS^{1/2})(S^{1/2}HS^{1/2}) \\ &= KK \end{aligned}$$

for some nonsingular $K = K^*$. (Here $S^{1/2}$ denotes the positive definite square root of S.) Since $S^{1/2}TS^{1/2}$ is positive definite we may take $K = (S^{1/2}TS^{1/2})^{1/2}$, the positive definite (hermitian) square root of

$S^{1/2}TS^{1/2}$. Thus we may take $H = S^{-1/2}KS^{-1/2}$.

<u>Example 2.</u> Let

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that T and S both have rank two and signature zero (and hence they are conjunctive). Moreover

$$TS = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2,$$

so TS is the square of a real matrix (see b) of Theorem 2). Nevertheless T and S are not isoconjunctive. To see this we note that

$$TST = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } STS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus TST and STS cannot be isoconjunctive, and hence by Theorem 1 T and S are not isoconjunctive.

<u>Example 3.</u> Let

$$T = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Again it is easy to see that T and S are conjunctive. We ask if it is possible to find a nonsingular $H = H^*$

such that $T = HSH$. If we let

$$H = \begin{bmatrix} h_{11} & h_{12} \\ \bar{h}_{12} & h_{22} \end{bmatrix}$$

($\bar{h}_{12}$ denotes the conjugate of h_{12}), then

$$HSH = \begin{bmatrix} h_{12}h_{11}+h_{11}\bar{h}_{12} & h_{12}h_{12}+h_{11}h_{22} \\ h_{22}h_{11}+\bar{h}_{12}\bar{h}_{12} & h_{22}h_{12}+\bar{h}_{12}h_{22} \end{bmatrix}.$$

In order that $T = HSH$, we require (setting $h_{12} = a + bi$, where a and b are real)

$$h_{11}(2a) = 1 , \quad h_{22}(2a) = 0,$$

and

$$a^2 + b^2 + 2abi + h_{11}h_{22} = -1.$$

The first two equations imply $h_{22} = 0$ and $2a \neq 0$. Thus the third equation implies $b = 0$, so that $a^2 = -1$, which is impossible. Thus T and S are not isoconjunctive. Notice that in this example TS is not similar to the square of a real matrix (see c) of Theorem 1).

Example 4. Let

$$T = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1/2 & 1 \\ 1 & 0 \end{bmatrix} S \begin{bmatrix} 1/2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus T and S are both isoconjunctive with R. But Example 3 shows that T and S are not isoconjunctive. Thus isoconjunctivity cannot be an equivalence relation on the set of $n \times n$ complex matrices.

It is our intention to utilize the equivalence of a) and c) in Theorem 1 to split our search for necessary and sufficient conditions for isoconjunctivity into "simpler" cases. Before we can prove a theorem which will accomplish this, we will need some further basic results.

<u>Lemma 1.</u> Let T and S be $n \times n$ hermitian matrices and suppose that, for some nonsingular matrix C,

$$C^*TSC^{*-1} = \bigoplus_{i=1}^{m} M_i$$

where M_i has no characteristic roots in common with M_j^* when $i \neq j$. Then conformably, we must have

$$C^*TC = \bigoplus_{i=1}^{m} T_i, \quad \text{and} \quad C^{-1}SC^{*-1} = \bigoplus_{i=1}^{m} S_i.$$

Proof: We prove the case where $m = 2$, the general result following by induction using the case $m = 2$. Thus suppose there exists a nonsingular matrix C such that

$$C^*TSC^{*-1} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \tag{1.1}$$

where M_1 and M_2^* have no characteristic roots in common. Notice that $C^*TSC^{*-1} = (C^*TC)(C^{-1}SC^{*-1})$. Thus let

$$C^*TC = \begin{bmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{bmatrix}, \quad \text{and} \quad C^{-1}SC^{*-1} = \begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix},$$

where the blocks are conformable with those of (1.1). Thus (1.1) gives us

$$M_1 = T_1S_1 + T_2S_2^*, \tag{1.2}$$

$$M_2 = T_2^*S_2 + T_3S_3, \tag{1.3}$$

$$T_1S_2 = -T_2S_3, \tag{1.4}$$

and

$$T_2^*S_1 = -T_3S_2^*. \tag{1.5}$$

Calculating, we find

$$\begin{aligned} M_1T_2 &= (T_1S_1 + T_2S_2^*)T_2 \\ &= T_1S_1T_2 + T_2S_2^*T_2 \\ &= -T_1S_2T_3 + T_2S_2^*T_2 && \text{(by 1.5)} \\ &= T_2S_3T_3 + T_2S_2^*T_2 && \text{(by 1.4)} \\ &= T_2(S_3T_3 + S_2^*T_2) \\ &= T_2M_2^*. && \text{(by 1.3)} \end{aligned}$$

Thus by Theorem 46.2 of [10, p.90] we conclude that $T_2 = 0$ since M_1 and M_2^* have no characteristic roots in common. Similarly

$$\begin{aligned} M_2^*S_2^* &= (S_2^*T_2 + S_3T_3)S_2^* \\ &= S_2^*T_2S_2^* + S_3T_3S_2^* \\ &= S_2^*T_2S_2^* - S_3T_2^*S_1 \\ &= S_2^*T_2S_2^* + S_2^*T_1S_1 \\ &= S_2^*(T_2S_2^* + T_1S_1) \\ &= S_2^*M_1 . \end{aligned}$$

By the same reasoning as above we conclude that also $S_2^* = 0$, which is what we wanted to show.

Lemma 2. Let T and S be $n\times n$ hermitian matrices. Suppose that

$$T = \bigoplus_{i=1}^{m} T_i \quad \text{and} \quad S = \bigoplus_{i=1}^{m} S_i$$

so that T_iS_i has no characteristic root in common with T_jS_j for $i \neq j$. If there exists an $n\times n$ complex matrix C such that $T = C^*SC = CSC^*$, then, conformably, we must have

$$C = \bigoplus_{i=1}^{m} C_i .$$

Proof: As in Lemma 1 it suffices to prove the result for the case $m = 2$. Thus let

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

be partitioned conformably with

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix},$$

where $T = C^*SC = CSC^*$ and T_1S_1 has no characteristic roots in common with T_2S_2. With the matrices so partitioned, the equality $T = C^*SC = CSC^*$ implies

$$C^*SC = \begin{bmatrix} C^*_{11}S_1C_{11}+C^*_{21}S_2C_{21} & C^*_{11}S_1C_{12}+C^*_{21}S_2C_{22} \\ C^*_{12}S_1C_{11}+C^*_{22}S_2C_{21} & C^*_{12}S_1C_{12}+C^*_{22}S_2C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

and

$$CSC^* = \begin{bmatrix} C_{11}S_1C^*_{11}+C_{12}S_2C^*_{12} & C_{11}S_1C^*_{21}+C_{12}S_2C^*_{22} \\ C_{21}S_1C^*_{11}+C_{22}S_2C^*_{12} & C_{21}S_1C^*_{21}+C_{22}S_2C^*_{22} \end{bmatrix}$$

$$= \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

From these equalities we garner the following identities:

$$T_1 = C^*_{11}S_1C_{11}+C^*_{21}S_2C_{21} = C_{11}S_1C^*_{11}+C_{12}S_2C^*_{12}$$

$$T_2 = C^*_{12}S_1C_{12}+C^*_{22}S_2C_{22} = C_{21}S_1C^*_{21}+C_{22}S_2C^*_{22}$$

$$C^*_{11}S_1C_{12} = -C^*_{21}S_2C_{22}$$

$$C_{11}S_1C^*_{21} = -C_{12}S_2C^*_{22}.$$

Using these identities we find

$$
\begin{aligned}
T_1S_1C_{21}^* &= (C_{11}^*S_1C_{11}+C_{21}^*S_2C_{21})S_1C_{21}^* \\
&= C_{11}^*S_1C_{11}S_1C_{21}^* + C_{21}^*S_2C_{21}S_1C_{21}^* \\
&= -C_{11}^*S_1C_{12}S_2C_{22}^* + C_{21}^*S_2C_{21}S_1C_{21}^* \\
&= C_{21}^*S_2C_{22}S_2C_{22}^* + C_{21}^*S_2C_{21}S_1C_{21}^* \\
&= C_{21}^*S_2(C_{22}S_2C_{22}^* + C_{21}S_1C_{21}^*) \\
&= C_{21}^*S_2T_2.
\end{aligned}
$$

Since T_1S_1 and S_2T_2 have no characteristic roots in common the preceding equality implies, by Theorem 46.2 of [10, p.90], $C_{21}^* = 0$. Thus $C_{21} = 0$. Similarly using the above identities we find

$$
\begin{aligned}
T_2S_2C_{12}^* &= (C_{12}^*S_1C_{12} + C_{22}^*S_2C_{22})S_2C_{12}^* \\
&= C_{12}^*S_1C_{12}S_2C_{12}^* + C_{22}S_2C_{22}S_2C_{12}^* \\
&= C_{12}^*S_1C_{12}S_2C_{12}^* - C_{22}^*S_2C_{21}S_1C_{11}^* \\
&= C_{12}^*S_1C_{12}S_2C_{12}^* + C_{12}^*S_1C_{11}S_1C_{11}^* \\
&= C_{12}^*S_1(C_{12}S_2C_{12}^* + C_{11}S_1C_{11}^*) \\
&= C_{12}^*S_1T_1.
\end{aligned}
$$

Again since T_2S_2 and S_1T_1 have no characteristic roots in common, this equation implies $C_{12}^* = 0$ so $C_{12} = 0$. Thus the proof is complete.

By combining Lemma 1 with c) of Theorem 1, it is clear that in searching for necessary and sufficient conditions that two hermitian matrices T and S be isoconjunctive, we may assume without loss of generality that

$$T = \bigoplus_{i=1}^{m} T_i \quad \text{and} \quad S = \bigoplus_{i=1}^{m} S_i$$

where T_iS_i has no characteristic roots in common with T_jS_j for $i \neq j$. It is with this idea in mind that we give our next result.

<u>Theorem 3.</u> Let $T = \bigoplus_{i=1}^{m} T_i$ and $S = \bigoplus_{i=1}^{m} S_i$ be $n \times n$ hermitian matrices with the property that T_iS_i has no characteristic roots in common with T_jS_j if $i \neq j$. Then T and S are isoconjunctive if and only if T_i and S_i are isoconjunctive for every $i = 1,\ldots,m$.

Proof: If $m = 1$ the result is obvious. As in Lemma 1 and Lemma 2 we will prove the result for $m = 2$, the result following for $m > 2$ by an easy induction using the result for $m = 2$.

Suppose then that there exist nonsingular matrices $H_1 = H_1^*$ and $H_2 = H_2^*$ such that

$$T_1 = H_1S_1H_1 \quad \text{and} \quad T_2 = H_2S_2H_2.$$

Let

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}.$$

Then $H = H^*$ and is nonsingular. Moreover

$$\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} = \begin{bmatrix} H_1S_1H_1 & 0 \\ 0 & H_2S_2H_2 \end{bmatrix}$$
$$= \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

Now suppose there exists a nonsingular $K = K^*$ such that $T = KSK$. We partition K conformably with S and T to write

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix}.$$

Since $K = K^*$ and T_1S_1 has no characteristic roots in common with T_2S_2 we may apply Lemma 2 to conclude $K_{12} = 0$ and thus $K_{12}^* = 0$. Since K is nonsingular and hermitian, K_{11} and K_{22} are also nonsingular and hermitian. Further the equality $T = KSK$ now implies

$$T_1 = K_{11}S_1K_{11} \quad \text{and} \quad T_2 = K_{22}S_2K_{22}.$$

Thus the proof is complete.

In accordance with this result, Chapters II and III will deal respectively with the investigation of necessary and sufficient conditions for isoconjunctivity of two hermitian matrices T and S when TS is nonsingular and when TS is nilpotent. In Chapter IV we draw these results together.

II. THE NONSINGULAR CASE

Before actually getting into the discussion of the nonsingular case we give a result which will be useful later in this chapter and also in Chapter IV.

<u>Proposition 1.</u> Let T and S be $n \times n$ nonsingular hermitian matrices. Then the following are equivalent:

a) there exists a nonsingular complex matrix C such that $T = C^*SC = CSC^*$

b) there exists a nonsingular complex matrix C such that $S^{-1} + iT = C^*(T^{-1} + iS)C$.

Proof: Under the hypothesis there exists a nonsingular complex matrix C such that $S^{-1} + iT = C^*(T^{-1} + iS)C$ if and only if $T = C^*SC$ and $S^{-1} = C^*T^{-1}C$. But this is equivalent to $T = C^*SC$ and $T = CSC^*$.

To deal with the question of the isoconjunctivity of two $n \times n$ hermitian matrices T and S whose product TS is nonsingular (thus both T and S are nonsingular) we will rely heavily on the following result which is a reformulation in matrix terms of results in [11, pp. 242-248].

<u>Theorem 4.</u> Let K and H be $n \times n$ hermitian matrices with K nonsingular. Then there exists a nonsingular

complex matrix C such that

$$C^*KC = \bigoplus_{i=1}^{p}\left[\begin{array}{ccc|ccc} & & & 0 & & 1\\ & 0 & & & \iddots & \\ & & & 1 & & 0\\ \hline & & 1 & & & \\ 0 & \iddots & & & 0 & \\ 1 & & 0 & & & \end{array}\right] \oplus \bigoplus_{j=1}^{q}\epsilon_j \begin{bmatrix} 0 & & 1\\ & \iddots & \\ 1 & & 0\end{bmatrix}, \qquad (2.1)$$

and

$$C^*HC = \bigoplus_{i=1}^{p}\left[\begin{array}{cccc|cccc} & & & & 0 & & 1 & \lambda_i\\ & 0 & & & & \iddots & \iddots & \\ & & & & 1 & \iddots & & 0\\ & & & & \lambda_i & & & \\ \hline & 0 & 1 & \bar{\lambda}_i & & & & \\ & \iddots & \iddots & & & & 0 & \\ 1 & \iddots & & & & & & \\ \bar{\lambda}_i & & 0 & & & & & \end{array}\right] \oplus \bigoplus_{j=1}^{q}\epsilon_j \begin{bmatrix} 0 & & 1 & \alpha_j\\ & \iddots & \iddots & \\ 1 & \iddots & & 0\\ \alpha_j & & & \end{bmatrix} \qquad (2.2)$$

where $\epsilon_j = \pm 1$, $j=1,\ldots,q$. The λ_i (and $\bar{\lambda}_i$), $i=1,\ldots,p$, and α_j, $j=1,\ldots,q$, are respectively the non-real and real characteristic roots of HK^{-1}. The corresponding direct summands in (2.1) and (2.2) are of equal size, being determined by the size of the Jordan blocks in the Jordan canonical form of HK^{-1}. For each real characteristic root α and each integer $r > 0$, the number of positive ϵ_j premultiplying the $r \times r$ summands associated with α is uniquely determined by H and K (and hence so is the number of such ϵ_j which are negative).

The problem of such simultaneous reduction of a pair of hermitian matrices by a conjunctivity has been the topic of much research, e.g. [8],[9],[14], and [19]. For some time it was thought by some writers that a necessary and sufficient condition that a pair of hermitian matrices (H_1, K_1) be conjunctive with the pair (H_2, K_2), where K_1 and K_2 are nonsingular, was that the elementary divisors of $\lambda I - H_1K_1^{-1}$ coincide with those of $\lambda I - H_2K_2^{-1}$, [9]. This <u>is</u> necessary but <u>not</u> sufficient. Turnbull [19] solved this problem by adding the "signature test" as did Muth [14] in the analagous real symmetric problem.

It will be useful later to know just how the ϵ_j (of (2.1) and (2.2)) premultiplying the $r \times r$ summands associated with a single real characteristic root α are determined. To this end let us assume HK^{-1} has only a single real characteristic root α. Suppose $(\alpha I - HK^{-1})^r$ and $(\alpha I - HK^{-1})^{r+1}$ have the same null space. By the Jordan decomposition this null space is HK^{-1}-invariant and has a HK^{-1}-invariant complement. Thus the hermitian form

$$x^*K^{-1}(\alpha I - HK^{-1})^{r-1}x \tag{2.3}$$

restricted to column vectors x in the null space of

$(\alpha I - HK^{-1})^r$ has a uniquely determined signature which depends only on H and K (and also has a uniquely determined signature on the complement). Thus the dimension of a maximal subspace (of the null space of $(\alpha I - HK^{-1})^r$) on which (2.3) is positive definite and the dimension of a maximal subspace on which (2.3) is negative definite are each uniquely determined by H and K so that the number of positive ϵ_j and the number of negative ϵ_j ((2.3) can be normalized) associated with a particular size block is unique. Notice in particular that if HK^{-1} is similar to a single Jordan block ($r \times r$ with real characteristic root α), then (2.3) has rank one and hence is semidefinite, so any column vector x which makes (2.3) non-zero will determine if the associated ϵ_j is positive or negative.

It should be noted that since the results of Theorem 4 are based on the Jordan canonical form of HK^{-1} which is unique only up to the ordering of the Jordan blocks, we cannot expect that the ordering of the direct summands in (2.1) and (2.2) is unique. Thus we will allow a permutation of the summands in either (2.1) or (2.2) if the corresponding summands of the two direct sums are permuted in exactly the same way. Let us see then how Theorem 4 applies to our situation.

<u>Corollary 1.</u> Let T and S be $n\times n$ hermitian matrices with S nonsingular. Then there exists a nonsingular complex matrix C such that

$$C^{-1}SC^{*-1} = \bigoplus_{i=1}^{p} \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{matrix} \\ \hline \begin{matrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{matrix} & 0 \end{array}\right] \oplus \bigoplus_{j=1}^{q} \epsilon_j \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \tag{2.4}$$

and

$$C^{*}TC = \bigoplus_{i=1}^{p} \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 & \lambda_i \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & \cdot^{\cdot^{\cdot}} & & 0 \\ \lambda_i & & & \end{matrix} \\ \hline \begin{matrix} 0 & & 1 & \bar{\lambda}_i \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & \cdot^{\cdot^{\cdot}} & & 0 \\ \bar{\lambda}_i & & & \end{matrix} & 0 \end{array}\right] \oplus \bigoplus_{j=1}^{q} \epsilon_j \begin{bmatrix} 0 & & 1 & \alpha_j \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & \cdot^{\cdot^{\cdot}} & & 0 \\ \alpha_j & & & \end{bmatrix} \tag{2.5}$$

where $\epsilon_j = \pm 1$, $j = 1,\ldots, q$. The λ_i (and $\bar{\lambda}_i$), $i = 1,\ldots, p$, and α_j, $j = 1,\ldots, q$, are respectively the non-real and real characteristic roots of TS. The corresponding direct summands in (2.4) and (2.5) are of equal size, being determined by the size of the Jordan blocks in the Jordan canonical form of TS. For each real characteristic root α and each integer $r > 0$

the number of positive ϵ_j premultiplying the $r \times r$ summands associated with α is uniquely determined by T and S (and hence so is the number of such ϵ_j which are negative).

Proof: If in Theorem 4 we let $T = H$ and $S^{-1} = K$, we have what we want except that in (2.4) we get $C^{-1}SC^{*-1}$ replaced by $C^*S^{-1}C$. For this choice of C, $(C^*S^{-1}C)^{-1} = C^*S^{-1}C$. Since also $(C^*S^{-1}C)^{-1} = C^{-1}SC^{*-1}$, the result follows.

Notice now that when T and S are $n \times n$ nonsingular hermitian matrices, Corollary 1 not only accomplishes the type of "splitting" indicated in Theorem 3 (which does not require nonsingularity) but also allows us to assume that T and S have very simple forms. Even so, in this situation we shall consider three cases:

<u>Case I:</u> TS has only non-real characteristic roots

<u>Case II:</u> TS has only positive-real characteristic roots

<u>Case III:</u> TS has only negative-real characteristic roots.

<u>Case I. T and S are $n \times n$ nonsingular hermitian matrices and TS has only non-real characteristic roots (which occur in conjugate pairs).</u> By Theorem 1 and

Corollary 1 we may assume

$$S = \bigoplus_{i=1}^{p} \left[\begin{array}{ccc|ccc} & & & 0 & & 1 \\ & 0 & & & \cdot & \\ & & & 1 & & 0 \\ \hline 0 & & 1 & & & \\ & \cdot & & & 0 & \\ 1 & & 0 & & & \end{array}\right] \quad \text{and} \quad T = \bigoplus_{i=1}^{p} \left[\begin{array}{cccc|cccc} & & & & 0 & \cdot & 1 & \lambda_i \\ & & 0 & & & \cdot & \cdot & \\ & & & & 1 & \cdot & & 0 \\ & & & & \lambda_i & & & \\ \hline 0 & \cdot & 1 & \overline{\lambda}_i & & & & \\ & \cdot & \cdot & & & & 0 & \\ 1 & \cdot & & & & & & \\ \overline{\lambda}_i & & 0 & & & & & \end{array}\right], \quad (2.6)$$

where λ_i and $\overline{\lambda}_i$, $i = 1, \ldots, p$, have the obvious meaning from Corollary 1. Note that the blocks in each of the direct summands are square so that each direct summand is of even order. By Theorem 1 of [7] we see that the signature of each direct summand in both T and S is zero. Thus since T and S are direct sums we may conclude that the signatures of T and S are zero. This allows us to deduce, since T and S are nonsingular, that T and S are conjunctive.

The following result will have an immediate application to our current considerations.

<u>Lemma 3.</u> Let

$$T = \begin{bmatrix} 0 & L \\ L^* & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix},$$

where

$$L = \begin{bmatrix} & 0 & \cdot & 1 & \lambda \\ & & \cdot & \cdot & \\ 1 & \cdot & & & 0 \\ & \cdot & & & \\ \lambda & & & & \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{bmatrix}$$

are both $n\times n$, and $\lambda \neq \bar{\lambda}$. Then there exists a nonsingular hermitian matrix H such that $T = HSH$. In fact, we may take

$$H = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

where

$$B = \sqrt{\lambda}\begin{bmatrix} a_{n-1} & \cdots & a_1 & a_0 \\ \vdots & & \cdot & \\ \vdots & \cdot & & \\ a_1 & & & 0 \\ a_0 & & & \end{bmatrix}$$

is $n\times n$, $\overline{\sqrt{\lambda}} = \sqrt{\bar{\lambda}}$, $a_0 = 1$, $a_1 = \frac{1}{2\lambda}$

and

$$a_i = -1/2 \sum_{k=1}^{i-1} a_k a_{i-k} \qquad (i = 2,\ldots, n-1).$$

Proof: Let $n \geq 1$ be arbitrary. Let S and H be defined as above. Since

$$HSH = \begin{bmatrix} 0 & B \\ B^* & \end{bmatrix}\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & BKB \\ B^*KB^* & 0 \end{bmatrix}$$

and $K^* = K$, it is sufficient to prove that $L = BKB$. Define $P = BKB$. We denote the (i,j) entry of P by p_{ij}, the (i,j) entry of K by k_{ij}, and the (i,j) entry of B by b_{ij}. By definition

$$p_{ij} = \sum_{r=1}^{n} b_{ir} \sum_{s=1}^{n} k_{rs} b_{sj}$$

$$= \sum_{r=1}^{n} b_{ir} k_{r,n-r+1} b_{n-r+1,j}$$

$$= \sum_{r=1}^{n} b_{ir} b_{n-r+1,j}. \qquad (2.7)$$

We note that $b_{ij} = 0$ if $n + 1 < i + j \leq 2n$. Using this fact in (2.7) we find $p_{ij} = 0$ if $n + 1 < i + j \leq 2n$. Thus we know $p_{ij} = \ell_{ij}$ if $n + 1 < i + j \leq 2n$ (ℓ_{ij} is the (i,j) entry of L). So now let $1 < i + j \leq n + 1$. In this case (2.7) becomes

$$p_{ij} = \lambda \sum_{r=j}^{n+1-i} a_{n+1-i-r} a_{r-j}$$

$$= \lambda \sum_{r=0}^{n+1-i-j} a_{n+1-i-j-r} a_r. \qquad (2.8)$$

Notice that by definition

$$\sum_{k=0}^{i} a_k a_{i-k} = \begin{cases} 0, & i \geq 2 \\ \lambda^{-1}, & i = 1 \\ 1, & i = 0 \end{cases} .$$

Using this in (2.8) we see

$$p_{ij} = \begin{cases} 0 & , \ n+1-i-j \geq 2 \\ \lambda(\lambda)^{-1}, & n+1-i-j = 1 \\ \lambda & , \ n+1-i-j = 0 \end{cases} .$$

Thus $p_{ij} = \ell_{ij}$ for $1 < i + j \leq n + 1$, so that we conclude $p_{ij} = \ell_{ij}$ for $i, j = 1,\ldots, n$, which is what we wanted to show.

Theorem 5. Let T and S be $n\times n$ (nonsingular) hermitian matrices. If TS has only non-real characteristic roots, then T and S are isoconjunctive.

Proof: By Theorem 1 and Corollary 1 we may assume S and T are as in (2.6). Lemma 3 shows that the corresponding direct summands in (2.6) are isoconjunctive. Thus by Theorem 3 we conclude T and S are isoconjunctive.

Case II: T and S are $n\times n$ nonsingular hermitian matrices and TS has only positive-real characteristic roots. Again, as in Case I, by Theorem 1 and Corollary 1 we will assume

$$S = \bigoplus_{i=1}^{q} \epsilon_i \begin{bmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & 0 \end{bmatrix} \text{ and } T = \bigoplus_{i=1}^{q} \epsilon_i \begin{bmatrix} 0 & & 1 & \alpha_i \\ & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \\ \alpha_i & \cdot & & 0 \end{bmatrix} , \quad (2.9)$$

where $\epsilon_i = \pm 1$, $i = 1,\ldots, q$, and α_i, $i = 1,\ldots, q$ are the characteristic roots of TS as in Corollary 1. Since here all the α_i are positive we may again appeal to Theorem 1 of [7] to see that corresponding direct summands of S and T in (2.9) have equal signatures (0, if the summand has even order; 1, if the summand has odd order) so that T and S have equal signatures. Since T and S are nonsingular we may conclude that T and S are conjunctive. The next lemma is like Lemma 3.

<u>Lemma 4.</u> Let

$$S = \epsilon \begin{bmatrix} & & & 1 \\ & 0 & \cdot & \\ & \cdot & & \\ & \cdot & & 0 \\ 1 & & & \end{bmatrix}, \quad \text{and} \quad T = \epsilon \begin{bmatrix} & & & 1 & \alpha \\ & 0 & \cdot & \cdot & \\ & \cdot & \cdot & & \\ 1 & \cdot & & & 0 \\ \alpha & & & & \end{bmatrix}$$

be $n\times n$ matrices with $\bar{\alpha} = \alpha > 0$ and $\epsilon = \pm 1$. Then there exists a nonsingular hermitian matrix H such that $T = HSH$. In fact we may take

$$H = \sqrt{\alpha} \begin{bmatrix} a_{n-1} & \cdots & a_1 & a_0 \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ a_1 & \cdot & & \\ a_0 & & 0 & \end{bmatrix}$$

where $a_0 = 1$, $a_1 = \frac{1}{2\alpha}$, and

$$a_i = -1/2 \sum_{k=1}^{i-1} a_k a_{i-k} \qquad (i = 2, \ldots, n-1).$$

Proof: Let $n \geq 1$ be arbitrary and let S and H be defined as above. Define $P = HSH$. We denote the (i,j) entry of P by p_{ij}, the (i,j) entry of H by h_{ij}, and the (i,j) entry of S by s_{ij}. By definition

$$p_{ij} = \sum_{k=1}^{n} h_{ik} \sum_{m=1}^{n} s_{km}h_{mj}$$

$$= \sum_{k=1}^{n} h_{ik}s_{k,n-k+1}h_{n-k+1,j}$$

$$= \epsilon \sum_{k=1}^{n} h_{ik}h_{n-k+1,j}. \tag{2.10}$$

Since $h_{ij} = 0$ if $n + 1 < i + j \leq 2n$, (2.10) shows $p_{ij} = 0$ if $n + 1 < i + j \leq 2n$. Thus $p_{ij} = t_{ij}$ (the (i,j) entry of T) for $n + 1 < i + j \leq 2n$. Now let $1 < i + j \leq n + 1$. Then (2.10) becomes

$$p_{ij} = \alpha \cdot \epsilon \sum_{r=j}^{n+1-i} a_{n+1-i-r}a_{r-j}$$

$$= \alpha \cdot \epsilon \sum_{r=0}^{n+1-i-j} a_{n+1-i-j-r}a_{r}. \tag{2.11}$$

Notice that by definition

$$\sum_{k=0}^{i} a_k a_{i-k} = \begin{cases} 0, & i \geq 2 \\ \alpha^{-1}, & i = 1 \\ 1, & i = 0 \end{cases}.$$

Using this in (2.11) we see

$$p_{ij} = \begin{cases} 0 & , \quad n+1-i-j \geq 2 \\ \epsilon\alpha\alpha^{-1}, & \quad n+1-i-j = 1 \\ \epsilon\alpha & , \quad n+1-i-j = 0 \end{cases} .$$

Thus $p_{ij} = t_{ij}$ for $1 < i + j \leq n + 1$ and hence $p_{ij} = t_{ij}$ for $i, j = 1,\ldots, n$, which is what we wanted to show.

<u>Theorem 6.</u> Let T and S be $n\times n$ (nonsingular) hermitian matrices. If TS has only positive-real characteristic roots, then T and S are isoconjunctive.

Proof: By Theorem 1 and Corollary 1 we may assume S and T are as in (2.9). By Lemma 4 we know the corresponding direct summands of (2.9) are isoconjunctive. Thus by Theorem 3 we conclude that T and S are isoconjunctive.

<u>Case III:</u> <u>T and S are $n\times n$ nonsingular hermitian matrices and TS has only negative-real characteristic roots.</u> From Example 3 we see that we cannot expect conjunctivity and isoconjunctivity to be equivalent in this case. Thus our approach here will be different from that of the previous two cases. We first list a collection of necessary lemmas.

<u>Lemma 5.</u> Let T and S be $n\times n$ hermitian matrices. Suppose TS is similar to a single Jordan block

corresponding to a real characteristic root $\alpha \neq 0$. Then $S^{-1} + iT$ is conjunctive with

$$\epsilon \begin{bmatrix} & & & i & 1+i\alpha \\ & & \cdot & \cdot & \\ & 0 \; \cdot & \cdot & & \\ \cdot & & & 0 & \\ i \;\; \cdot & & & & \\ 1+i\alpha & & & & \end{bmatrix} \tag{2.12}$$

where $\epsilon = \pm 1$, and $T^{-1} + iS$ is conjunctive with

$$\epsilon \cdot \mathrm{sgn}\alpha \begin{bmatrix} & & & i & 1+i\alpha \\ & & \cdot & \cdot & \\ & 0 \; \cdot & \cdot & & \\ \cdot & & & 0 & \\ i \;\; \cdot & & & & \\ 1+i\alpha & & & & \end{bmatrix} \tag{2.13}$$

where $\mathrm{sgn}\ \alpha = \frac{|\alpha|}{\alpha}$.

Proof: Theorem 4 implies $S^{-1} + iT$ is conjunctive with (2.12). Without loss of generality we will assume $S^{-1} + iT$ equals (2.12). Since TS and ST are similar, ST is similar to a single Jordan block corresponding to the characteristic root α. Thus by Theorem 4 there exists a nonsingular complex matrix C such that $C^*(T^{-1} + iS)C$ is of the form of (2.13) with $\epsilon \cdot \mathrm{sgn}\ \alpha$ replaced by $\delta = \pm 1$. By the definition of T and S

$$v^*S(TS - \alpha I)^{n-1}v = \epsilon \tag{2.14}$$

if $v^* = [0,\ldots, 0, 1]$. But also

$$u^*C^*S(TS - \alpha I)^{n-1}Cu = u^*(C^*SC)[(C^{-1}TC^{*-1})(C^*SC) - \alpha I]^{n-1}u$$
$$= \delta \operatorname{sgn} \alpha$$

if $u^* = [|\alpha|^{-1/2}, 0,\ldots, 0]$ and C is as designated above. Since $S(TS - \alpha I)^{n-1}$ has rank one and the signature of $S(TS - \alpha I)^{n-1}$ equals the signature of $D^*S(TS - \alpha I)^{n-1}D$ for every nonsingular complex matrix D, the discussion immediately following Theorem 4 allows us to conclude that $\epsilon = \delta \cdot \operatorname{sgn} \alpha$. Thus $\delta = \epsilon \cdot \operatorname{sgn} \alpha$ and the result is proved.

<u>Lemma 6.</u> Let T and S be $n\times n$ hermitian matrices such that TS has only a single real characteristic root $\alpha \neq 0$. If the Jordan form of TS has Jordan blocks of sizes $m_1 \geq m_2 \geq \cdots \geq m_k > 0$, then $S^{-1} + iT$ is conjunctive with

$$\bigoplus_{j=1}^{k} \epsilon_j \begin{bmatrix} & & & i & 1+i\alpha \\ & 0 & & \cdot & \\ & & \cdot & & \\ i & \cdot & & 0 & \\ 1+i\alpha & & & & \end{bmatrix} \tag{2.15}$$

and $T^{-1} + iS$ is conjunctive with

$$\operatorname{sgn}\alpha \bigoplus_{j=1}^{k} \epsilon_j \begin{bmatrix} & & & & i & 1+i\alpha \\ & & & \cdot & \cdot & \\ & 0 & \cdot & \cdot & & \\ & \cdot & & & 0 & \\ i & \cdot & & & & \\ 1+i\alpha & & & & & \end{bmatrix} \tag{2.16}$$

where $\epsilon_j = \pm 1$, $j = 1, \ldots, k$ (equal in (2.15) and (2.16)), and the j^{th} direct summand in (2.15) and (2.16) is $m_j \times m_j$, $j = 1, \ldots, k$.

Proof: Apply Lemma 5 to the Jordan blocks of TS.

Lemma 7. Let T and S be $n \times n$ hermitian matrices such that TS has only a single negative-real characteristic root α. If there exists a nonsingular complex matrix C such that $S^{-1} + iT = C^*(T^{-1} + iS)C$, then $S^{-1} + iT$ is conjunctive with

$$\bigoplus_{j=1}^{p} \epsilon_j \begin{bmatrix} R_j & 0 \\ 0 & -R_j \end{bmatrix} \tag{2.17}$$

where, for $j = 1, \ldots, p$, $\epsilon_j = \pm 1$, and R_j is the $m_j \times m_j$ matrix

$$\begin{bmatrix} & & & & i & 1+i\alpha \\ & & & \cdot & \cdot & \\ & 0 & \cdot & \cdot & & \\ & \cdot & & & 0 & \\ i & \cdot & & & & \\ 1+i\alpha & & & & & \end{bmatrix}.$$

Proof: Note that (2.17) says that the blocks of each order "pair off" (the number of blocks of a given order which have $\epsilon = 1$ is equal to the number of blocks of the same order which have $\epsilon = -1$). Suppose then that this does not occur; that is, suppose $S^{-1} + iT$ is conjunctive with

$$\bigoplus_{i=1}^{q} \epsilon_j' R_j, \tag{2.18}$$

where for some size summand the number of positive ϵ_j' associated with that size summand does not equal the number of negative ϵ_j' associated with that same size direct summand. By Lemma 6 $T^{-1} + iS$ is conjunctive with

$$\bigoplus_{j=1}^{q} (-\epsilon_j') R_j. \tag{2.19}$$

Since (2.18) and (2.19) are in canonical form, conjunctivity of (2.18) with (2.19) (and thus of $S^{-1} + iT$ with $T^{-1} + iS$) requires that (up to permutation of the summands) (2.18) and (2.19) be equal. But under our assumption this is impossible. This contradiction proves the lemma.

<u>Lemma 8.</u> Let

$$T = \epsilon \begin{bmatrix} L & 0 \\ 0 & -L \end{bmatrix} \qquad \text{and} \qquad S = \epsilon \begin{bmatrix} K & 0 \\ 0 & -K \end{bmatrix},$$

where

$$L = \begin{bmatrix} & & & 1 & \alpha \\ & 0 & \cdot & \cdot & \\ & \cdot & \cdot & & \\ 1 & \cdot & & 0 & \\ \alpha & & & & \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} & & & 1 \\ & 0 & \cdot & \\ & & \cdot & \\ & \cdot & & 0 \\ 1 & & & \end{bmatrix}$$

are $n \times n$ with $\alpha < 0$ and $\epsilon = \pm 1$. Then there exists a nonsingular hermitian matrix H such that $T = HSH$. In fact, we may take

$$H = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$$

where

$$B = \sqrt{-\alpha} \begin{bmatrix} a_{n-1} & \cdots & a_1 & a_0 \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ a_1 & \cdot & & 0 \\ a_0 & & & \end{bmatrix},$$

$a_0 = 1$, $a_1 = \frac{1}{2\alpha}$, and

$$a_i = -1/2 \sum_{k=1}^{i-1} a_k a_{i-k}, \qquad (i=2,\ldots, n-1).$$

Proof: Let $n \geq 1$ be arbitrary and let H and S be defined as above. Since

$$HSH = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & -K \end{bmatrix} \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} = \begin{bmatrix} -BKB & 0 \\ 0 & BKB \end{bmatrix}$$

it is sufficient to prove $L = -BKB$. Define $P = -BKB$.

We denote the (i,j) entry of P by p_{ij}, the (i,j) entry of B by b_{ij}, and the (i,j) entry of K by k_{ij}. By definition

$$p_{ij} = -\sum_{r=1}^{n} b_{ir} \sum_{s=1}^{n} k_{rs} b_{sj}$$

$$= -\sum_{r=1}^{n} b_{ir} k_{r,n-r+1} b_{n-r+1,j}$$

$$= -\sum_{r=1}^{n} b_{ir} b_{n-r+1,j} \ .$$

Now a repeat of the last part of the proof of Lemma 3, carefully noting the differences in definition, gives the result. Thus we are equipped to prove the final result of this chapter.

<u>Theorem 7.</u> Let T and S be $n\times n$ hermitian matrices. Suppose TS has only negative-real characteristic roots. Then T and S are isoconjunctive if and only if there exists a nonsingular complex matrix C such that $T = C^*SC = CSC^*$.

Proof: If T and S are isoconjunctive then by definition there exists a nonsingular matrix $H = H^*$ such that $T = HSH$. Thus $T = H^*SH = HSH^*$.

Now suppose there exists a nonsingular complex matrix C such that $T = C^*SC = CSC^*$. By Lemma 2 it suffices to consider the case where TS has only a single negative-real characteristic root α. Now Proposition 1 and Lemma 7 imply $S^{-1} + iT$ is conjunctive with a matrix of the form of (2.17). Thus there exists a nonsingular complex matrix D such that

$$D^*S^{-1}D = \bigoplus_{i=1}^{q} \epsilon_i \left[\begin{array}{c|c} \begin{matrix} & 0 & & 1 \\ & & \cdot & \\ & \cdot & & 0 \\ 1 & & & \end{matrix} & 0 \\ \hline 0 & \begin{matrix} & & & -1 \\ & 0 & \cdot & \\ & \cdot & & 0 \\ -1 & & & \end{matrix} \end{array}\right] \tag{2.20}$$

and

$$D^*TD = \bigoplus_{i=1}^{q} \epsilon_i \left[\begin{array}{c|c} \begin{matrix} & & & 1 & \alpha \\ & 0 & \cdot & \cdot & \\ & \cdot & \cdot & & \\ 1 & \cdot & & 0 & \\ \alpha & & & & \end{matrix} & 0 \\ \hline 0 & \begin{matrix} & & & -1 & -\alpha \\ & 0 & \cdot & \cdot & \\ & \cdot & \cdot & & \\ -1 & \cdot & & 0 & \\ -\alpha & & & & \end{matrix} \end{array}\right] \tag{2.21}$$

where the size of the blocks are as determined in Lemma 7, and $\epsilon_i = \pm 1$, $i = 1, \ldots, q$. By Lemma 8 the (respective) direct summands in (2.20) and (2.21) are isoconjunctive.

Thus by Theorem 3 $D^*S^{-1}D$ and D^*TD are isoconjunctive. But $(D^*S^{-1}D)^{-1} = D^*S^{-1}D$. Thus we have $(D^*S^{-1}D)^{-1} = D^{-1}SD^{*-1}$ and D^*TD are isoconjunctive, which by Theorem 1 implies T and S are isoconjunctive.

Looking at Theorem 5, Theorem 6, and Theorem 7, we see that the only time there is any problem in deciding whether or not two $n \times n$ nonsingular hermitian matrices T and S are isoconjunctive is when TS has negative-real characteristic roots. If we knew these characteristic roots we could obtain the Jordan blocks of TS associated with them and use the method outlined immediately following Theorem 4 to determine whether or not we get the "pairing" indicated in Lemma 6 at each negative-real characteristic root. Obviously if there were an odd number of Jordan blocks of a given size associated with some negative-real characteristic root a negative answer could be given immediately. On the other hand it should not be concluded that T and S are necessarily isoconjunctive even if the number of Jordan blocks of each size associated with each negative-real characteristic root is even.

III. THE NILPOTENT CASE

In this chapter we will have to employ a different procedure than that of Chapter II. Here we consider the problem of isoconjunctivity when TS is nilpotent. Since we want T and S to be isoconjunctive we would be foolish to assume one of the matrices was nonsingular while the other was (necessarily) singular. Thus the decomposition afforded in Theorem 4 will not be applicable here. We do, however, take a hint from Theorem 4 to come up with a canonical form which looks somewhat like the one given there and seems to be new.

To obtain the above-mentioned canonical form we will view the hermitian matrix S as a map from a (n-dimensional) vector space V (the space of $n \times 1$ column vectors) into V^* (the *-dual) and the matrix T as a map from V^* into V. Thus TS (in left hand notation) is a map from V into V. Thus no confusion should arise if we write "$S : V \to V^*$ is hermitian." Since this viewpoint is seldom used, let us see in more detail how we are going to operate in this setting.

If V^* is to be called "dual" it should fulfill the requirement, namely if we operate on a vector in V with a vector in V^* we should obtain a complex number. Thus if we take V^* also to be the space of $n \times 1$ column vectors, we define a complex valued function $\langle \cdot,\cdot \rangle$ which indicates the action of a vector $y \in V^*$ on a vector $x \in V$ by

$$\langle x,y \rangle = x^*y,$$

where x^* is the conjugate transpose of x. If $S : V \to V^*$ is the linear map corresponding to the $n\times n$ matrix S, then $S^* : V^{**} \to V^*$, where S^* is the conjugate transpose of S. Thus if we identify V^{**} with V we have for $x,y \in V$

$$\langle y,Sx \rangle = y^*Sx = (S^*y)^*x = \overline{\langle x,S^*y \rangle}.$$

Thus our function $\langle \cdot,\cdot \rangle$ together with a hermitian map $S : V \to V^*$ gives rise naturally to the hermitian form

$$x^*Sx \quad (= \langle x,Sx \rangle)$$

and the sesquilinear form

$$y^*Sx \quad (= \langle y,Sx \rangle)$$

on V. Thus if we denote by V_1^o the annihilator of V_1 [5, p. 63] (V_1^o is a subspace of V^* if $V_1 \subseteq V$, and V_1^o is a subspace of $V^{**} = V$ if $V_1 \subseteq V^*$), questions about subspaces and their annihilators will be discussed using the above forms.

In connection with the above ideas let us observe the matrix interpretation of some statements which will be used often in the sequel. Let $S : V \to V^*$ be linear with $\dim V = n$. We denote by $\mathcal{N}S$ the null space of S. If we take V_1 to be the subspace of V spanned by the first k unit column vectors (the r^{th} unit vector has a 1 in the r^{th} position and zeros elsewhere), the statement that $\eta S \cap V_1 = 0$ means that the first k columns of S are linearly independent. Similarly the statement $SV_1 \subseteq V_1^o$ says that the leading $k \times k$ principal submatrix of S is zero. Further if we let $V = V_1 \oplus V_2$ where V_2 is the subspace spanned by the remaining $n-k$ unit column vectors, the statement $(SV_2)^o \cap V_1 = 0$ indicates that the rows of the $k \times (n-k)$ matrix S_{12}, where

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

are linearly independent. Finally the statement $SV_2 \subseteq V_1^o$ (where V_1 and V_2 are as before) means $S_{12} = 0$.

Before proceeding we list some facts which will be used later. For these considerations we let V be an n-dimensional vector space and let V^* be its *-dual.

<u>Fact A.</u> If $K : V \to V^*$ and $H : V^* \to V$ are linear and U is a subspace of V such that $\eta K(HK)^p \cap U = 0$ and $(HK)^{p+1}U = 0$, then $\dim U = \dim HKU = \ldots = \dim (HK)^p U = \dim KU = \ldots = \dim K(HK)^p U$. Further the subspaces $(HK)^i U$ $(i = 0, \ldots, p)$ are independent and the subspaces $K(HK)^i U$ $(i = 0, \ldots, p)$ are independent.

<u>Fact B.</u> If $H : V \to V^*$ is hermitian and U_1, U_2, and

U_3 are subspaces of V, then

1) $U_2 \subseteq (HU_1)^o \Longleftrightarrow HU_2 \subseteq U_1^o$

2) $(HU_1)^o \cap U_2 = 0 \Longrightarrow U_1^o \cap HU_2 = 0$

3) $(HU_1)^o \cap U_2 = 0 \Longleftrightarrow U_1^o \cap HU_2 = 0$ and $U_2 \cap \eta H = 0$

4) $(HU_1)^o \cap U_2 = 0$ and $\dim U_2 \geq \dim U_1 \Longrightarrow U_1 \cap \eta H = 0$

5) $HU_1 \subseteq U_1^o \cap U_2^o$, $HU_2 \subseteq U_2^o$, and $U_3 \subseteq U_1 + U_2 \Longrightarrow$ $HU_3 \subseteq U_3^o$.

Fact C. Let V and W be finite-dimensional vector spaces and let $K : V \to W$ be linear. Let U and $\hat{U}$ be respectively k-dimensional subspaces of V and W^*. Then $(KU)^o \cap \hat{U} = 0 \Longleftrightarrow (K^*\hat{U})^o \cap U = 0$.

Now in preparation for the proof of our decomposition theorem in the nilpotent case we present some preliminary results, some of which are interesting in their own right.

Lemma 9. Let $S : V \to V^*$ be hermitian. Let V_1 and V_2 be (independent) k-dimensional subspaces of V such that $SV_1 \subseteq V_1^o$ and $(SV_2)^o \cap V_1 = 0$. Then there exists a k-dimensional subspace V_3 such that $V_1 \oplus V_3 = V_1 \oplus V_2$, $SV_3 \subseteq V_3^o$, and $(SV_3)^o \cap V_1 = 0$.

Proof: Let $\{x_1, \ldots, x_k\}$ be a basis for V_2. Since $(SV_2)^o \cap V_1 = 0$, there exist $y_1, \ldots, y_k \in V_1$ such that

$$y_i^* S x_j = \delta_{ij} \quad (i,j = 1, \ldots, k), \tag{3.1}$$

where δ_{ij} denotes the Kronecker delta. It is easy to see that $\{y_1, \ldots, y_k\}$ is linearly independent and thus is a basis for V_1. Let

$$z_i = x_i + \sum_{m=1}^{k} a_{im} y_m \quad (i = 1, \ldots, k), \tag{3.2}$$

where for the moment the a_{ij}, $i, j = 1, \ldots, k$, are undetermined complex numbers. By (3.1), (3.2), and the fact that $SV_1 \subseteq V_1^\circ$ we find

$$\begin{aligned} z_i^* S z_j &= \left(x_i + \sum_{m=1}^{k} a_{im} y_m \right)^* S \left(x_j + \sum_{n=1}^{k} a_{jn} y_n \right) \\ &= x_i^* S x_j + a_{ji} + \bar{a}_{ij}. \end{aligned} \tag{3.3}$$

By proper choice (for example: $a_{ji} = -x_i^* S x_j$, $i < j$; $a_{ji} = 0$, $i > j$; and $a_{ii} + a_{ii} = -x_i^* S x_i$) of the a_{ij}, $i, j = 1, \ldots, k$, (3.3) can be made zero for $i, j = 1, \ldots, k$. Let $V_3 = \operatorname{span} \{z_1, \ldots, z_k\}$ with such a proper choice. Then (3.3) is zero for $i, j = 1, \ldots, k$, so $SV_3 \subseteq V_3^\circ$. Since $V_1 \oplus V_2 \subseteq V_1 + V_3$ and $\dim V_3 \leq \dim V_2$, the latter sum must be direct. By (3.2) and the fact that $SV_1 \subseteq V_1^\circ$ we find $y_i^* S z_j = \delta_{ij}$, $i, j = 1, \ldots, k$. Thus we conclude $(SV_3)^\circ \cap V_1 = 0$ and the proof is complete.

The matrix interpretation of Lemma 9 (in the case $V = V_1 \oplus V_2$) is that if there exists a basis for V such that $S : V \to V^*$ is represented by a matrix of the form

$$\begin{bmatrix} 0 & S_1 \\ S_1^* & S_2 \end{bmatrix}$$

where each block is $k \times k$ and S_1 is nonsingular, then there exists a basis for V such that S is represented by a matrix of the form

$$\begin{bmatrix} 0 & S_0 \\ S_0^* & 0 \end{bmatrix}$$

where S_0 is $k \times k$ and nonsingular.

<u>Lemma 10.</u> Let $S : V \to V^*$ be hermitian. Let V_1, V_2, and V_3 be k-dimensional subspaces of V such that $(SV_1)^\circ \cap V_2 = 0$ and $V_2 \cap V_3 = 0$. Then there exists a k-dimensional subspace V_4 such that $V_2 \oplus V_4 = V_2 \oplus V_3$ and $SV_1 \subseteq V_4^\circ$.

Proof: Let V_1 and V_3 have respective bases $\{x_1, \ldots, x_k\}$ and $\{z_1, \ldots, z_k\}$. Since $(SV_1)^\circ \cap V_2 = 0$, there exist $y_1, \ldots, y_k \in V_2$ such that

$$y_i^* S x_j = \delta_{ij} \qquad (i, j = 1, \ldots, k). \tag{3.4}$$

As in Lemma 9, $\{y_1, \ldots, y_k\}$ is linearly independent and thus is a basis for V_2. Let

$$w_i = z_i + \sum_{m=1}^{k} a_{im}y_m \qquad (i = 1, \ldots, k), \tag{3.5}$$

where for the moment the a_{ij}, $i, j = 1, \ldots, k$, are arbitrary. By (3.4)

$$\begin{aligned} w_i^* S x_j &= \left(z_i + \sum_{m=1}^{k} a_{im}y_m\right)^* S x_j \\ &= z_i^* S x_j + \bar{a}_{ij}. \end{aligned} \tag{3.6}$$

If we choose $\bar{a}_{ij} = -z_i^* S x_j$, $i, j = 1, \ldots, k$, (3.6) can be made zero for $i, j = 1, \ldots, k$. Thus if we let $V_4 = \text{span}\,\{w_1, \ldots, w_k\}$, the fact that $w_i^* S x_j = 0$ for $i, j = 1, \ldots, k$ tells us $SV_1 \subseteq V_4^o$. Since $V_2 \oplus V_3 \subseteq V_2 + V_4$ and $\dim V_4 \leq \dim V_3$, the latter sum must be direct so that $V_2 \oplus V_3 = V_2 \oplus V_4$. Thus the result is proved.

The matrix interpretation of Lemma 10 (in the case $V = V_1 \oplus V_2 \oplus V_3$) is that if there exists a basis for V such that $S : V \to V^*$ is represented by a matrix of the form

$$\begin{bmatrix} S_1 & S_4 & S_6 \\ S_4^* & S_2 & S_5 \\ S_6^* & S_5^* & S_3 \end{bmatrix}$$

where each block is $k \times k$ and S_4 is nonsingular, then there exists a basis for V such that S is represented by a matrix of the form

$$\begin{bmatrix} \hat{S}_1 & \hat{S}_4 & 0 \\ \hat{S}_4^* & \hat{S}_2 & \hat{S}_5 \\ 0 & \hat{S}_5^* & \hat{S}_3 \end{bmatrix}$$

where again each block is $k \times k$ and $\hat{S}_4$ is nonsingular.

<u>Lemma 11.</u> Let $S : V \to V^*$ and $T : V^* \to V$ be hermitian. Let W_1 and $\hat{W}_1$ be k-dimensional subspaces of V and V^* respectively such that

a) $\left((TS)^{p-1}W_1\right)^{\circ} \cap \hat{W}_1 = 0$

b) $T(ST)^{p-1}\hat{W}_1 = 0.$

Then there exists a subspace W of V having the properties of W_1 above and also $S(TS)^i W \subseteq W^{\circ}$ $(i = 0, \ldots, p-2)$.

Proof: Let W_1 and $\hat{W}_1$ be as in the hypothesis. We assume $p \geq 2$. Notice first that by condition a) (and a standard dimensionality argument) $\dim \left|(TS)^{p-1}W_1\right|^{\circ} \leq n-k$. Thus $\dim (TS)^{p-1}W_1 \geq k$. Since $\dim W_1 = k$ this shows $W_1 \cap \eta(TS)^{p-1} = 0$. Also since $\eta(ST)^{p-1} = \left|(TS)^{p-1}V\right|^{\circ} \subseteq \left((TS)^{p-1}W_1\right)^{\circ}$ we conclude by condition a) that $\hat{W}_1 \cap \eta(ST)^{p-1} = 0$. We apply Lemma 9 with $V_1 = T\hat{W}_1$, $V_2 = W_1$, and the hermitian map $S(TS)^{p-2}$. To see that the hypotheses of Lemma 9 are satisfied we note first that $\dim T\hat{W}_1 = \dim \hat{W}_1 = \dim W_1$ since $\hat{W}_1 \cap \eta(ST)^{p-1} = 0$. If $y \in T\hat{W}_1 \cap W_1$, then $y = Tx$ where $x \in \hat{W}_1$. Thus $(TS)^{p-1}y = T(ST)^{p-1}x$. Since $x \in \hat{W}_1$, condition b) implies $0 = T(ST)^{p-1}x = (TS)^{p-1}y$. Since $y \in W_1$ and $W_1 \cap \eta(TS)^{p-1} = 0$, this implies $y = 0$. Thus $T\hat{W}_1 \cap W_1 = 0$. Therefore $T\hat{W}_1$ and W_1 are independent k-dimensional subspaces. By condition b) we know $T(ST)^{p-1}\hat{W}_1 \subseteq \hat{W}_1^{o}$. Thus using Fact B.1 we find $S(TS)^{p-2}T\hat{W}_1 \subseteq (T\hat{W}_1)^{o}$. Finally since by hypothesis $\left(TS(TS)^{p-2}W_1\right)^{o} \cap \hat{W}_1 = 0$, we use Fact B.2 to conclude $\left(S(TS)^{p-2}W_1\right)^{o} \cap T\hat{W}_1 = 0$. Thus the hypotheses of Lemma 9 are satisfied so there exists a subspace W_2 of V such that

$$T\hat{W}_1 \oplus W_1 = T\hat{W}_1 \oplus W_2,$$

$$S(TS)^{p-2}W_2 \subseteq W_2^{o}, \tag{3.7}$$

and

$$\left(S(TS)^{p-2}W_2\right)^{o} \cap T\hat{W}_1 = 0$$

Since $T\hat{W}_1 \oplus W_2 = T\hat{W}_1 \oplus W_1$ and $T(ST)^{p-1}\hat{W}_1 = 0$, we conclude $(TS)^{p-1}W_2 = (TS)^{p-1}W_1$. Thus by condition a) we conclude $\left((TS)^{p-1}W_2\right)^{o} \cap \hat{W}_1 = 0$. Thus W_2 has the desired properties of W_1 and also (3.7) holds. If $p = 2$, W_2 is the desired subspace. If $p > 2$ we apply Lemma 9 again but this time with $V_1 = TST\hat{W}_1$, $V_2 = W_2$, and the hermitian map $S(TS)^{p-3}$. The fact that the hypotheses are satisified is confirmed as above. Thus we obtain a subspace W_3 with the property that $S(TS)^{p-3}W_3 \subseteq W_3^o$ and also the properties of W_1 (in the hypothesis). But also since $W_3 \subseteq TST\hat{W}_1 \oplus W_2$, Fact B.5 implies $S(TS)^{p-2}W_3 \subseteq W_3^o$. Thus this crucial property of W_2 is also possessed by W_3. Thus after p-1 repetitions of this argument we obtain a subspace W_p such that

$$S(TS)^i W_p \subseteq W_p^o \quad (i = 0, \ldots, p-2),$$

and also having the properties of W_1 stated in the hypothesis. Obviously W_p is the desired subspace.

<u>Lemma 12.</u> Let $S : V \to V^*$ and $T : V^* \to V$ be hermitian. Let W_1 and $\hat{W}_1$ be k-dimensional subspaces of V and V^* respectively such that

a) $\left((TS)^{p-1}W_1\right)^o \cap \hat{W}_1 = 0$

b) $T(ST)^{p-1}\hat{W}_1 = 0$

c) $S(TS)^{p-1}W_1 = 0$.

Then there exist subspaces W and $\hat{W}$ of V and V^* respectively having the properties of W_1 and $\hat{W}_1$ above, and also

a) $S(TS)^h W \subseteq W^o$ $\quad (h = 0, \ldots, p-2)$

b) $T(ST)^i \hat{W} \subseteq \hat{W}^o$ $\quad (i = 0, \ldots, p-2)$

c) $(TS)^j W \subseteq \hat{W}^o$ $\quad (j = 1, \ldots, p-2)$.

Proof: Let T, S, W_1, and $\hat{W}_1$ be as in the hypothesis. Let $n = \dim V = \dim V^*$. By Fact C we see that condition a) implies $\left((ST)^{p-1}\hat{W}_1\right)^{\circ} \cap W_1 = 0$. Hence in our hypotheses the hypothesis of Lemma 11 is satisfied (for V, V^*, W_1, $\hat{W}_1$, S, and T, and also for V^*, V, $\hat{W}_1$, W_1, T, and S). Thus we apply Lemma 11 twice (once to replace W_1 and once to replace $\hat{W}_1$) and thereby may assume W_1 and $\hat{W}_1$ have also the respective properties of W and $\hat{W}$ of conclusion a) and conclusion b). Thus all we need to establish is that conclusion c) holds after suitable replacement is made for $\hat{W}_1$.

To do this we appeal to Lemma 10. We begin by applying Lemma 10 with $V_1 = SW_1$, $V_2 = ST\hat{W}_1$, $V_3 = \hat{W}_1$, and the hermitian map $T(ST)^{p-3}$ (this part of Lemma 12 has no content if $p \leq 2$). To see that Lemma 10 is applicable we note first that by Fact A, $\hat{W}_1$ and $ST\hat{W}_1$ are independent subspaces (of the same dimension). Since, as in the proof of Lemma 11, condition a) implies $\mathcal{N}(TS)^{p-1} \cap W_1 = 0$, we conclude $\dim SW_1 = \dim W_1 = \dim \hat{W}_1$. Using Fact B.2 we see $\left((TS)^{p-1}W_1\right)^{\circ} \cap \hat{W}_1 = 0$ implies $\left(T(ST)^{p-3}(SW_1)\right)^{\circ} \cap ST\hat{W}_1 = 0$. Thus the hypotheses of Lemma 10 are satisfied so there exists a subspace $\hat{W}_2$ such that

$$ST\hat{W}_1 \oplus \hat{W}_1 = ST\hat{W}_1 \oplus \hat{W}_2 \qquad (3.8)$$

and

$$T(ST)^{p-3}(SW_1) = (TS)^{p-2}W_1 \subseteq \hat{W}_2^{o}. \tag{3.9}$$

From (3.8) it is obvious that $T(ST)^{p-1}\hat{W}_2 = 0$. By hypothesis $T(ST)^{i}\hat{W}_1 \subseteq \hat{W}_1^{o}$ $(i = 0, \ldots, p-2)$. Since also $T(ST)^{p-1}\hat{W}_1 = 0$, we conclude $T(ST)^{i}(ST\hat{W}_1) \subseteq \hat{W}_1^{o}$ and $T(ST)^{i}(ST\hat{W}_1) \subseteq (ST\hat{W}_1)^{o}$ for $i = 0, \ldots, p-2$. Thus since $\hat{W}_2 \subseteq ST\hat{W}_1 + \hat{W}_1$, by Fact B.5 we conclude $T(ST)^{i}\hat{W}_2 \subseteq \hat{W}_2^{o}$ $(i=0, \ldots, p-2)$. Since $(TS)^{p}W_1 = 0$, $ST\hat{W}_1 \subseteq \left((TS)^{p-1}W_1\right)^{o}$. Thus

$$\begin{aligned} ST\hat{W}_1 &= \left((TS)^{p-1}W_1\right)^{o} \cap [(ST)\hat{W}_1 \oplus \hat{W}_1] \\ &= \left((TS)^{p-1}W_1\right)^{o} \cap [ST\hat{W}_1 \oplus \hat{W}_2] \\ &= ST\hat{W}_1 \oplus \left[\left((TS)^{p-1}W_1\right)^{o} \cap \hat{W}_2\right] \end{aligned}$$

by hypothesis a), two applications of the modular law, and (3.8). Therefore we conclude $\left((TS)^{p-1}W_1\right)^{o} \cap \hat{W}_2 = 0$ so $\hat{W}_2$ has the same necessary properties as $\hat{W}_1$ and also (3.9) is true.

If $p = 3$, W_1 and $\hat{W}_2$ are the desired subspaces. Otherwise we apply Lemma 10 again, but this time with $V_1 = SW_1$, $V_2 = (ST)^2\hat{W}_2$, $V_3 = \hat{W}_2$, and the hermitian map $T(ST)^{p-4}$. Thus we obtain a subspace $\hat{W}_3$ such that

$$(ST)^2\hat{W}_2 \oplus \hat{W}_2 = (ST)^2\hat{W}_2 \oplus \hat{W}_3 \tag{3.10}$$

and

$$T(ST)^{p-4}SW_1 = (TS)^{p-3}W_1 \subseteq \hat{W}_3^{o}. \qquad (3.11)$$

Since $\hat{W}_3 \subseteq (ST)^2\hat{W}_2 \oplus \hat{W}_2$, $(TS)^{p-2}W_1 \subseteq \hat{W}_2^{o}$, and $(TS)^{p-2}W_1 \subseteq \left((ST)^2\hat{W}_2\right)^{o}$, we have $(TS)^{p-2}W_1 \subseteq \hat{W}_2^{o} \cap \left((ST)^2\hat{W}_2\right)^{o} = \left(\hat{W}_2 \oplus (ST)^2\hat{W}_2\right)^{o} \subseteq \hat{W}_3^{o}$. The verifications that $T(ST)^{p-1}\hat{W}_3 = 0$, $T(ST)^{i}\hat{W}_3 \subseteq \hat{W}_3^{o}$ $(i = 0, \ldots, p-2)$, and $\left((TS)^{p-1}W_1\right)^{o} \cap \hat{W}_3 = 0$ are similar to those of the previous step. Thus $\hat{W}_3$ has the desired properties of $\hat{W}_2$, but also (3.11) is true. Therefore applying Lemma 10 p-2 times we finally obtain $\hat{W}_{p-1}$ with the desired properties. Obviously W_1 and $\hat{W}_{p-1}$ are the desired subspaces.

<u>Definition 2.</u> If $R : V \to V$ and W is a subspace of V we define

$$\mathcal{J}_R W \equiv \sum_{k=0}^{\infty} R^k W,$$

i.e. $\mathcal{J}_R W$ is the smallest R-invariant subspace of V containing W.

<u>Lemma 13.</u> Let $T : V^* \to V$ and $S : V \to V^*$ be hermitian with $(TS)^p V = 0$. Let $\dim V = n$, and let W be a subspace of V satisfying

$$V = \mathcal{N} S(TS)^{p-1} \oplus W.$$

Then

a) $V = \mathcal{J}_{TS}W \oplus (\mathcal{J}_{ST}SW)^{\circ}$ b) $V^* = \mathcal{J}_{ST}SW \oplus (\mathcal{J}_{TS}W)^{\circ}$

c) $S(\mathcal{J}_{ST}SW)^{\circ} \subseteq (\mathcal{J}_{TS}W)^{\circ}$ d) $T(\mathcal{J}_{TS}W)^{\circ} \subseteq (\mathcal{J}_{ST}SW)^{\circ}$

e) $S\mathcal{J}_{TS}W \subseteq \mathcal{J}_{ST}SW$ f) $T\mathcal{J}_{ST}SW \subseteq \mathcal{J}_{TS}W$.

Proof: We will prove only a), c), and e) since b) follows immediately from a) and the proofs of d) and f) are analogous to those of c) and e). To see that the sum in a) is direct, suppose $y \in (\mathcal{J}_{TS}W) \cap (\mathcal{J}_{ST}SW)^{\circ}$. By the definition of $\mathcal{J}_{TS}W$ and the fact that $(TS)^pV = 0$, there exist $x_0, \ldots, x_{p-1} \in W$ such that $y = \sum_{j=0}^{p-1} (TS)^j x_j$. Since $y \in (\mathcal{J}_{ST}SW)^{\circ}$ we also have

$$0 = y^*(ST)^iSW = \sum_{j=0}^{p-1} x_j^*(ST)^{i+j}SW \qquad (i = 0, 1, \ldots).$$

For $i = p-1$ this gives

$$\begin{aligned} 0 = x_0^*(ST)^{p-1}SW &= x_0^*S(TS)^{p-1}[\eta S(TS)^{p-1} \oplus W] \\ &= x_0^*S(TS)^{p-1}V. \end{aligned}$$

Thus $x_0^*S(TS)^{p-1} = 0$ so $x_0 \in \eta S(TS)^{p-1}$. But also $x_0 \in W$. Therefore by the hypothesis on W, $x_0 = 0$. Similarly one proves $x_1 = \ldots = x_{p-1} = 0$. Thus $y = 0$ so $(\mathcal{J}_{TS}W) \cap (\mathcal{J}_{ST}SW)^{\circ} = 0$. To see that the sum in a) is all of V we notice that, by Fact A, $\dim \mathcal{J}_{TS}W =$

$\dim \mathcal{J}_{ST}SW = pk$, where $k \equiv \dim W$. Thus $\dim (\mathcal{J}_{ST}SW)^{o} = n-pk$. Thus the sum of the dimensions of the summands is n. Therefore the result follows.

To see that c) holds, let $y \in (\mathcal{J}_{ST}SW)^{o}$. Then by definition

$$0 = y^*S(TS)^{i}W = (Sy)^*(TS)^{i}W \qquad (i = 0, 1, \ldots).$$

Thus $Sy \in (\mathcal{J}_{TS}W)^{o}$ which is what we wanted to show. The truth of e) is observed by definition. (Notice also that from c), d), e), and f) we can conclude that the decompositions a) and b) are respectively TS-invariant and ST-invariant.)

The matrix interpretation of this lemma will be useful later. It says that there exists a nonsingular matrix C (with the first pk columns of C a basis for $\mathcal{J}_{TS}W$ and the remaining $n-pk$ columns a basis for $(\mathcal{J}_{ST}SW)^{o}$) such that $C^{-1}TC^{*-1} = T_1 \oplus T_2$ and $C^*SC = S_1 \oplus S_2$, where the matrices T_1 and S_1 (both $pk \times pk$) are matrices of the restriction maps of T and S to $\mathcal{J}_{ST}SW$ and $\mathcal{J}_{TS}W$ respectively. Similarly T_2 and S_2 are matrices of the restriction maps of T and S, respectively, to $(\mathcal{J}_{TS}W)^{o}$ and $(\mathcal{J}_{ST}SW)^{o}$.

<u>Theorem 8.</u> Let S and T be $n\times n$ hermitian matrices such that $(TS)^{p} = 0$. Then there exists a nonsingular complex matrix C such that

$$C^{-1}TC^{*-1} = \bigoplus_{q=1}^{p}\left[\bigoplus_{h=1}^{k_q} \epsilon_{hq}F_q \bigoplus_{i=1}^{\ell_q} \rho_{iq}E_q \bigoplus_{j=1}^{m_q} G_{2q-1}\right] \qquad (3.12)$$

and

$$C^{*}SC = \bigoplus_{q=1}^{p}\left[\bigoplus_{h=1}^{k_q} \epsilon_{hq}E_q \bigoplus_{i=1}^{\ell_q} \rho_{iq}F_q \bigoplus_{j=1}^{m_q} F_{2q-1}\right] \qquad (3.13)$$

where E_q, F_q, and G_q are the $q\times q$ matrices

$$E_q = \begin{bmatrix} & & & 1 \\ 0 & & \cdot & \\ & \cdot & & \\ \cdot & & 0 & \\ 1 & & & \end{bmatrix}, F_q = \begin{bmatrix} & 0 & \cdot^{1} & \cdot & 0 \\ & & \cdot & & \\ & \cdot & & \cdot & \\ 1 & & & & 0 \\ 0 & \cdot & & & \end{bmatrix}, G_q = \begin{bmatrix} & & & & 0 \\ 0 & & & \cdot & \\ & & \cdot & & 1 \\ & \cdot & & \cdot & 0 \\ 0 & 1 & & & \end{bmatrix},$$

and for each h, i, q we have $\epsilon_{hq} = \pm 1$ and $\rho_{iq} = \pm 1$ arranged so that

$$\epsilon_{1q} \geq \epsilon_{2q} \geq \cdots \geq \epsilon_{k_q q}$$

and

$$\rho_{1q} \geq \rho_{2q} \geq \cdots \geq \rho_{\ell_q q}.$$

The numbers k_q, ℓ_q, and m_q are uniquely determined by S and T. In fact if we let

$$\nu_q = \text{nullity } (TS)^q,$$

$$\mu_q = \text{nullity } S(TS)^q,$$

and

$$\mu'_q = \text{nullity } T(ST)^q,$$

then

$$m_q = \mu_{q-1} + \mu'_{q-1} - \nu_q - \nu_{q-1},$$

$$k_q = 2\nu_q - \mu_{q-1} - \mu'_q, \tag{3.14}$$

and

$$\ell_q = 2\nu_q - \mu'_{q-1} - \mu_q,$$

for $0 < q \leq p$. If we let

$$\epsilon_q = \sum_{h=1}^{k_q} \epsilon_{hq}$$

and

$$\rho_q = \sum_{i=1}^{\ell_q} \rho_{iq},$$

then the numbers ϵ_q and ρ_q (and hence the ϵ_{hq} and ρ_{iq}) are uniquely determined by T and S. In fact if we let

$$\sigma_r = \text{sig } S(TS)^r \quad \text{(the signature of } S(TS)^r\text{)},$$

and

$$\sigma'_r = \text{sig } T(ST)^r \quad \text{(the signature of } T(ST)^r\text{)},$$

then

$$\epsilon_{p-i} = \sigma_{p-i-1} - \sigma'_{p-i},$$

and (3.15)

$$\rho_{p-i} = \sigma'_{p-i-1} - \sigma_{p-i},$$

for $0 \leq i < p$.

Proof: We consider $T : V^* \to V$ and $S : V \to V^*$. Let W_1 be a subspace of V satisfying

$$V = \mathcal{N} S(TS)^{p-1} \oplus W_1 .$$

We let $k \equiv \dim W_1$. By Lemma 13

$$V = \mathcal{J}_{TS}W_1 \oplus (\mathcal{J}_{ST}SW_1)^{\circ} \tag{3.16}$$

and

$$V^* = \mathcal{J}_{ST}SW_1 \oplus (\mathcal{J}_{TS}W_1)^{\circ} . \tag{3.17}$$

Thus there exists a nonsingular matrix C_0 (with first kp columns a basis for $\mathcal{J}_{TS}W_1$ and remaining n-pk columns a basis for $(\mathcal{J}_{ST}SW_1)^{\circ}$) such that

$$C_0^*SC_0 = S_0 \oplus S_1 \quad \text{and} \quad C_0^{-1}TC_0^{*-1} = T_0 \oplus T_1$$

where S_0 and T_0 are $pk \times pk$. Since S_0 is the matrix of the restriction of S to $\mathcal{J}_{TS}W_1$ (in a suitable basis), and $\dim S(\mathcal{J}_{TS}W_1) = \dim \mathcal{J}_{ST}SW_1 = kp$, by Fact A and the initial assumption on W_1, we conclude S_0 is nonsingular. Since $S_0(T_0S_0)^{p-1}$ is the matrix of the restriction of $S(TS)^{p-1}$ to $\mathcal{J}_{TS}W_1$ and, by our assumption about W_1, $\dim S(TS)^{p-1}\mathcal{J}_{TS}W_1 = \dim S(TS)^{p-1}W_1 = k$, we conclude rank $S_0(T_0S_0)^{p-1} = k$. Thus since $(T_0S_0)^p = 0$, rank $(T_0S_0)^{p-1} = k$, and T_0S_0 is $pk \times pk$, the Jordan Decomposition Theorem implies T_0S_0 has Jordan form consisting of k $p \times p$ blocks (with characteristic root zero). Thus by Corollary 1 there exists a nonsingular complex matrix C_1 such that

$$C_1^*S_0C_1 = \bigoplus_{h=1}^{k} \epsilon_h E_p$$

and (3.16)

$$C_1^{-1}T_0C_1^{*-1} = \bigoplus_{h=1}^{k} \epsilon_h F_p$$

where $\epsilon_h = \pm 1$, $h = 1, \ldots, k$, $\epsilon_1 \geq \ldots \geq \epsilon_k$, and E_p and F_p are defined as in the hypothesis.

Now consider the $(n - pk) \times (n - pk)$ matrices S_1 and T_1. Since rank $S_0(T_0S_0)^{p-1} = k =$ rank $S(TS)^{p-1}$, we have rank $S_1(T_1S_1)^{p-1} = 0$ and hence $(S_1T_1)^p = 0$. Thus by the same reasoning as above there exists a non-singular complex matrix C_2 (combining the steps above) such that

$$C_2^*S_1C_2 = \bigoplus_{i=1}^{\ell} \rho_i F_p \oplus S_2$$

and (3.17)

$$C_2^{-1}T_1C_2^{*-1} = \bigoplus_{i=1}^{\ell} \rho_i E_p \oplus T_2$$

($\ell =$ rank $T_1(S_1T_1)^{p-1}$), where $T_2(S_2T_2)^{p-1} = 0$. Moreover $S_2(T_2S_2)^{p-1} = 0$ since $S_1(T_1S_1)^{p-1} = 0$. So now consider the $(n - pk - p\ell) \times (n - pk - p\ell)$ matrices T_2 and S_2. Let $n_1 \equiv n - pk - p\ell$. If we let V_1 be the space of $n_1 \times 1$ complex matrices and V_1^* be its *-dual, then $T_2 : V_1^* \to V_1$, and $S_2 : V_1 \to V_1^*$. Moreover $T_2(S_2T_2)^{p-1}V_1^* = 0$ and $S_2(T_2S_2)^{p-1}V_1 = 0$. Now let W_2 and $\hat{W}_2$ be subspaces of V_1 and V_1^* respectively such that

$$V_1 = \mathcal{N}(T_2S_2)^{p-1} \oplus W_2$$

and (3.18)

$$V_1^* = \mathcal{N}(S_2T_2)^{p-1} \oplus \hat{W}_2.$$

Since T_2 and S_2 are hermitian this implies $\dim W_2 = \dim \hat{W}_2$. Suppose $y \in \left((T_2S_2)^{p-1}W_2\right)^{\circ} \cap \hat{W}_2$. Then $0 = y^*(T_2S_2)^{p-1}W_2 = y^*(T_2S_2)^{p-1}\left(W_2 \oplus \mathcal{N}(T_2S_2)^{p-1}\right) = y^*(T_2S_2)^{p-1}V_1$.

Thus $(S_2T_2)^{p-1}y = 0$ so $y \in \mathcal{N}(S_2T_2)^{p-1}$. Since also $y \in \hat{W}_2$, this implies $y = 0$. Thus $\left((T_2S_2)^{p-1}W_2\right)^{\circ} \cap W_2 = 0$. Hence the hypotheses of Lemma 12 are satisfied, so we may assume further (by Lemma 12) that

$$S_2(T_2S_2)^h W_2 \subseteq W_2^{\circ} \qquad (h = 0, \ldots, p-2),$$

$$T_2(S_2T_2)^i \hat{W}_2 \subseteq \hat{W}_2^{\circ} \qquad (i = 0, \ldots, p-2), \qquad (3.19)$$

and

$$(T_2S_2)^j W_2 \subseteq \hat{W}_2^{\circ} \qquad (j = 1, \ldots, p-2).$$

Thus if we let $m = \dim W_2 = \dim \hat{W}_2$, using (3.19) we can select a basis $\{x_1, \ldots, x_m\}$ for W_2 and a basis $\{y_1, \ldots, y_m\}$ for $\hat{W}_2$ such that

$$y_i^*(T_2S_2)^{p-1}x_j = \delta_{ij} \qquad (i, j = 1, \ldots, m),$$

$$x_i^*S_2(T_2S_2)^r x_j = 0 \qquad \begin{pmatrix} i, j = 1, \ldots, m \\ r = 0, \ldots, p-2 \end{pmatrix},$$

$$y_i^*T_2(S_2T_2)^r y_j = 0 \qquad \begin{pmatrix} i, j = 1, \ldots, m \\ r = 1, \ldots, p-2 \end{pmatrix},$$

and

$$y_i^*(T_2S_2)^r x_j = 0 \qquad \begin{pmatrix} i,\ j = 1,\ \ldots,\ m \\ r = 1,\ \ldots,\ p-2 \end{pmatrix}.$$

Notice now that the subspaces W_2, $T_2S_2W_2$, ..., $(T_2S_2)^{p-2}W_2$, $T_2\hat{W}_2$, ..., $T_2(S_2T_2)^{p-2}\hat{W}_2$ are independent and each has dimension m. That each has dimension m follows since $\eta(T_2S_2)^{p-1} \cap W_2 = 0$ and $\eta(S_2T_2)^{p-1} \cap \hat{W}_2 = 0$. To see that the subspaces are independent, let x_0, ..., $x_{p-1} \in W_2$ and y_0, ..., $y_{p-2} \in \hat{W}_2$. Then

$$\sum_{i=0}^{p-1} (T_2S_2)^i x_i \;+\; \sum_{j=0}^{p-2} (T_2S_2)^j T_2 y_j = 0$$

implies

$$0 = \sum_{i=0}^{p-1} x_i^*(S_2T_2)^i (S_2T_2)^h \hat{W}_2 + \sum_{j=0}^{p-2} y_j^* T_2 (S_2T_2)^j (S_2T_2)^h \hat{W}_2$$

$$= x_{p-1-h}^*(S_2T_2)^{p-1}\hat{W}_2 \qquad (h = 0,\ \ldots,\ p-1)$$

Thus $x_{p-1-h} \in W_2 \cap \left((S_2T_2)^{p-1}\hat{W}_2\right)^o = 0$ for $h = 0, \ldots, p-1$. Hence $x_i = 0$ for $i = 0, \ldots, p-1$. Similarly $y_j = 0$ for $j = 0, \ldots, p-2$. Thus the subspaces are independent. By the same type of reasoning the subspaces $\hat{W}_2$, $S_2T_2\hat{W}_2$, ..., $(S_2T_2)^{p-1}\hat{W}_2$, S_2W_2, ... $S_2(T_2S_2)^{p-2}W_2$ are independent and each has dimension m. Thus with our bases for W_2 and $\hat{W}_2$ chosen as above,

$$\left\{\bigcup_{i=0}^{p-1} \left\{(T_2S_2)^i x_j\right\}_{j=1}^{m}\right\} \bigcup \left\{\bigcup_{i=0}^{p-2} \left\{T_2(S_2T_2)^i y_j\right\}_{j=1}^{m}\right\} \qquad (3.20)$$

and

$$\left\{\bigcup_{i=0}^{p-1}\left\{(S_2T_2)^i y_j\right\}_{j=1}^{m}\right\}\bigcup\left\{\bigcup_{i=0}^{p-2}\left\{S_2(T_2S_2)^i x_j\right\}_{j=1}^{m}\right\}, \quad (3.21)$$

respectively, form bases for the $(2p-1)m$-dimensional subspaces $\mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2)$ and $\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)$.

From (3.19) it follows that

$$\left(\mathcal{J}_{T_2S_2}(W_2 \oplus T\hat{W}_2)\right)^{\circ} \cap \mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2) = 0$$

and

$$\left(\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)\right)^{\circ} \cap \mathcal{J}_{T_2S_2}(W_2 \oplus T\hat{W}_2) = 0.$$

Thus

$$V_1 = \mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2) \oplus \left(\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)\right)^{\circ}$$

and

$$V_1^* = \mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2) \oplus \left(\mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2)\right)^{\circ}.$$

By definition T_2 maps $\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)$ into $\mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2)$ and S_2 maps $\mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2)$ into $\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)$. By an argument like the one establishing c) of Lemma 13 we also find S_2 maps $\left(\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)\right)^{\circ}$ into $\left(\mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2)\right)^{\circ}$ and T_2 maps $\left(\mathcal{J}_{T_2S_2}(W_2 \oplus T_2\hat{W}_2)\right)^{\circ}$ into $\left(\mathcal{J}_{S_2T_2}(\hat{W}_2 \oplus S_2W_2)\right)^{\circ}$.

For each vector x_i in our basis for W_2 let

$X_i = \text{span}\,\{x_i\}$. Similarly we let $Y_i = \text{span}\,\{y_i\}$ for each vector y_i in our basis for $\hat{W}_2$. For each j $(j = 1, \ldots, m)$ we order the basis for $\mathcal{J}_{T_2S_2}X_j \oplus \mathcal{J}_{T_2S_2}T_2Y_j$ as follows:

$$x_j,\ T_2y_j,\ T_2S_2x_j,\ \ldots,\ (T_2S_2)^{p-2}T_2y_j,\ (T_2S_2)^{p-1}x_j.$$

For each corresponding j $(j = 1, \ldots, m)$ we order the basis for $\mathcal{J}_{S_2T_2}Y_j \oplus \mathcal{J}_{S_2T_2}S_2X_j$ as follows:

$$(S_2T_2)^{p-1}y_j,\ (S_2T_2)^{p-2}S_2x_j,\ \ldots,\ S_2T_2y_j,\ S_2x_j,\ y_j.$$

Thus there exists a nonsingular complex matrix C_3 such that

$$C_3^*S_2C_3 = \bigoplus_{j=1}^{m} G_{2p-1} \oplus S_3$$

and (3.22)

$$C_3^{-1}T_2C_3^{*-1} = \bigoplus_{j=1}^{m} F_{2p-1} \oplus T_3$$

Hence we have obtained

$$D^*SD = \bigoplus_{h=1}^{k} \epsilon_h E_p \bigoplus_{i=1}^{\ell} \rho_i F_p \bigoplus_{j=1}^{m} G_{2p-1} \oplus S_3$$

and

$$D^{-1}TD^{*-1} = \bigoplus_{h=1}^{k} \epsilon_h F_p \bigoplus_{i=1}^{\ell} \rho_i E_p \bigoplus_{j=1}^{m} F_{2p-1} \oplus T_3$$

for some nonsingular complex matrix D.

Now observe that $\text{rank}\,(F_{2p-1}G_{2p-1})^{p-1} = 1$ and therefore $\text{rank} \bigoplus_{j=1}^{m} (F_{2p-1}G_{2p-1})^{p-1} = m = \dim W_2 = \text{rank}\,(T_2S_2)^{p-1}$. Thus by (3.18) and (3.22) we conclude

$(T_3S_3)^{p-1} = 0$. Hence if we consider the $\left(n-pk-p\ell-(2p-1)m\right) \times \left(n-pk-p\ell-(2p-1)m\right)$ matrices T_3 and S_3 we have returned to the situation of the first part of the proof with p replaced by $p - 1$. Consequently after a finite number of repetitions of the above process the required forms for C^*SC and $C^{-1}TC^{*-1}$ are obtained.

Now it remains to show that (3.14) and (3.15) hold. Since the rank and signature of a matrix are preserved by conjunctivity, and the rank of a matrix is preserved by similarity, it will be sufficient to prove (3.14) and (3.15) are true assuming T and S are respectively equal to (3.12) and (3.13). To do this let us observe the following (where $\nu(E)$ will denote the nullity of the matrix E):

For $0 \leq r < q$,

$$\nu(F_qE_q)^r = r, \qquad \nu(E_qF_q)^r = r,$$

$$\nu(G_{2q-1}F_{2q-1})^r = 2r, \qquad \nu(E_q[F_qE_q]^r) = r,$$

$$\nu(F_q[E_qF_q]^r) = r + 1, \qquad \nu(F_{2q-1}[G_{2q-1}F_{2q-1}]^r) = 2r + 1,$$

and
$$\nu(G_{2q-1}[F_{2q-1}G_{2q-1}]^r) = 2r + 1.$$

(For $r \geq q$ the preceding nullities equal the respective orders.) Thus for $0 \leq j \leq p$

$$\nu_j = \sum_{q>j}\left(jk_q + j\ell_q + 2jm_q\right) + \sum_{q\leq j}\left(qk_q + q\ell_q + (2q-1)m_q\right)$$

$$\mu_{j-1} = \sum_{q\geq j-1}\left((j-1)k_q+j\ell_q+(2j-1)m_q\right)+\sum_{q<j-1}\left(qk_q+q\ell_q+(2q-1)m_q\right)$$

$$\mu'_{j-1} = \sum_{q\geq j-1}\left(jk_q+(j-1)\ell_q+(2j-1)m_q\right)+\sum_{q<j-1}\left(qk_q+q\ell_q+(2q-1)m_q\right),$$

where we agree $k_q = \ell_q = m_q = 0$ if $q > p$. Using these equations we find that

$$\nu_j - \nu_{j-1} = \sum_{q\geq j} m_{q+1} + k_q + \ell_q + m_q$$

$$= m_j + \left[k_j + \sum_{q>j}(m_q+k_q)\right] + \left[\ell_j + \sum_{q>j}(m_q+\ell_q)\right],$$

$$\nu_j - \mu_{j-1} = k_j + \sum_{q>j}(m_q + k_q),$$

and

$$\nu_j - \mu'_{j-1} = \ell_j + \sum_{q>j}(m_q + \ell_q).$$

From these equations the expressions for k_q, ℓ_q and m_q can easily be obtained.

To confirm (3.15) we use Theorem 1 of [7] to see that

$$\operatorname{sig} F_{2q-1}(G_{2q-1}F_{2q-1})^r = \operatorname{sig} G_{2q-1}(F_{2q-1}G_{2q-1})^r = 0$$

for $r, q = 0, \ldots, p$. Similarly we find

$$\operatorname{sig} E_q(F_qE_q)^r = \begin{cases} 1 & r = q - 2i - 1 \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

and

$$\operatorname{sig} F_q(E_qF_q)^r = \begin{cases} 1 & r = q - 2i - 2 \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

where $i = 0, 1, \ldots$. Using these facts and the fact that the signature of a direct sum of matrices is the sum of the signatures of the direct summands, we obtain

$$\sigma_{p-2r-1} = \epsilon_p + \epsilon_{p-2} + \cdots + \epsilon_{p-2r} + \rho_{p-1} + \cdots + \rho_{p-2r+1},$$

$$\sigma'_{p-2r-1} = \rho_p + \rho_{p-2} + \cdots + \rho_{p-2r} + \epsilon_{p-1} + \cdots + \epsilon_{p-2r+1},$$

$$\sigma_{p-2r} = \epsilon_{p-1}+\epsilon_{p-3}+\cdots+\epsilon_{p-2r+1}+\rho_p+\rho_{p-2}+\cdots+\rho_{p-2r+2},$$

and

$$\sigma'_{p-2r} = \rho_{p-1}+\rho_{p-3}+\cdots+\rho_{p-2r+1}+\epsilon_p+\epsilon_{p-2}+\cdots+\epsilon_{p-2r+2}.$$

From these expressions the explicit formulas of (3.15) are obtained. Thus the proof is complete.

<u>Theorem 9.</u> Let T and S be $n\times n$ hermitian matrices with $(TS)^p = 0$. Then T and S are isoconjunctive if and only if $S(TS)^i$ and $T(ST)^i$ are conjunctive for $i = 0, \ldots, p$.

Proof: If T and S are isoconjunctive, Theorem 1 shows $S(TS)^i$ and $T(ST)^i$ are isoconjunctive (and hence conjunctive) for $i = 0, \ldots, p$.

To prove the converse, by Theorem 1 we may assume T and S are of the form (3.12) and (3.13) respectively. Since $S(TS)^{p-1}$ and $T(ST)^{p-1}$ are conjunctive we conclude (in Theorem 8) that $\mu_{p-1} = \mu'_{p-1}$ and thus $k_p = \ell_p$. Similarly we conclude $k_q = \ell_q$, $q = 1, \ldots, p-1$. Thus also $\epsilon_q = \rho_q$ for all q. Therefore (because of the ordering) $\epsilon_{iq} = \rho_{iq}$ for every i and q. Further since $E_q F_q E_q = G_q$ and $E_q E_q E_q = E_q$ we may assume (after suitable reordering of the blocks using simultaneous permutation conjunctivities)

$$T = \bigoplus_{q=1}^{p} \left[\bigoplus_{h=1}^{k_q} \epsilon_{hq}(G_q \oplus E_q) \bigoplus_{j=1}^{m_q} G_{2q-1} \right]$$

and

$$S = \bigoplus_{q=1}^{p} \left[\bigoplus_{h=1}^{k_q} \epsilon_{hq}(E_q \oplus F_q) \bigoplus_{j=1}^{m_q} F_{2q-1} \right].$$

Since $(G_q \oplus E_q)$ and $(E_q \oplus F_q)$ are isoconjunctive $(G_q \oplus E_q = E_{2q}(E_q \oplus F_q)E_{2q})$, and since G_{2q-1} and F_{2q-1} are isoconjunctive $(G_{2q-1} = E_{2q-1}F_{2q-1}E_{2q-1})$ we conclude by Theorem 4 that T and S are isoconjunctive.

IV. SUMMARY

Now that we have found necessary and sufficient conditions that two $n \times n$ hermitian matrices be isoconjunctive in both the case where their product is nonsingular and in the case where their product is nilpotent, it seems reasonable to try to put the pieces together to obtain a single criterion in the general case. Unfortunately this condition can be no better than the irrational condition derived in the case where the product TS has only negative-real characteristic roots. However in certain special cases we can do better than this.

Theorem 10. Let T and S be $n \times n$ nonsingular hermitian matrices. Suppose none of the characteristic roots of TS are negative-real. Then T and S are isoconjunctive.

Proof: Suppose T and S are nonsingular hermitian matrices such that TS has no negative-real roots. By the Jordan Decomposition Theorem, there exists a nonsingular matrix C such that

$$C^{-1}TSC = \bigoplus_{i=1}^{p} M_i$$

where M_i has only a single positive-real characteristic root or M_i has only a single pair of conjugate nonreal characteristic roots and M_i has no characteristic roots in common with M_j^* if $i \neq j$. By Theorem 1 and Lemma 1 we may assume

$$T = \bigoplus_{i=1}^{p} T_i \quad \text{and} \quad S = \bigoplus_{i=1}^{p} S_i$$

where the set of matrices $T_iS_i (i = 1,\dots, p)$ has the properties given above for the set of matrices $M_i (i = 1,\dots, p)$. By Theorem 3 T and S are isoconjunctive if and only if T_i and S_i are isoconjunctive for $i = 1,\dots, p$. But the isoconjunctivity of T_i with S_i, $i = 1,\dots, p$, has been established by Theorem 5 and Theorem 6. Thus the theorem is proved.

<u>Theorem 11.</u> Let T and S be $n\times n$ hermitian matrices. Suppose none of the characteristic roots of TS are negative-real (the roots may be zero). Then T and S are isoconjunctive if and only if for every nonnegative integer i there exists a nonsingular complex matrix C_i such that

$$T(ST)^i = C_i^* S(TS)^i C_i$$

(i.e. for every integer $i \geq 0$, $T(ST)^i$ and $S(TS)^i$ have the same rank and signature).

Proof: If T and S are isoconjunctive, then by definition there exists a nonsingular hermitian matrix H such that $T = HSH$. Now the proof of Theorem 1 shows we may take $C_i = H$, $i = 0, 1,\dots$.

Suppose now that for every nonnegative integer i there exists a nonsingular complex matrix C_i such that

$$T(ST)^i = C_i^* S(TS)^i C_i .$$

Again by the Jordan Decomposition Theorem, Theorem 1, and Lemma 1 we may assume

$$T = T_1 \oplus T_2 \quad \text{and} \quad S = S_1 \oplus S_2$$

where T_1S_1 is nonsingular with no negative-real characteristic roots and T_2S_2 is nilpotent (say of degree p). Since by hypothesis $T(ST)^p$ and $S(TS)^p$ are conjunctive, the hermitian matrices $T_1(S_1T_1)^p$ and $S_1(T_1S_1)^p$ are conjunctive $((S_2T_2)^p = (T_2S_2)^p = 0)$. Since T_1 and S_1 are nonsingular, this implies $T_1(S_1T_1)^i$ and $S_1(T_1S_1)^i$ are conjunctive for $i = 0, 1, \ldots$. But this, together with the hypothesis and the fact that T and S are hermitian, implies $T_2(S_2T_2)^i$ and $S_2(T_2S_2)^i$ are conjunctive for $i = 0, 1, \ldots$. Now applying Theorem 9 and Theorem 10 we deduce that T_2 and S_2 are isoconjunctive and also T_1 and S_1 are isoconjunctive. Thus by Theorem 3 T and S are isoconjunctive.

Notice that in both Theorem 10 and Theorem 11 we need to know something about the set of characteristic roots of TS before the theorems can be used. It should also be noted that in applying Theorem 11 we need only calculate the rank and signature of $T(ST)^i$ (and $S(TS)^i$) for $i = 0, \ldots, k$ (hopefully k is small) since the sequence

$$\{0\} \subseteq \eta TS \subseteq \eta(TS)^2 \subseteq \ldots \subseteq \eta(TS)^m \subseteq \ldots$$

is eventually stationary. In fact we may take k to be the smallest integer p such that $\eta(TS)^p = \eta(TS)^{p+1}$. Note also that our discussion in no way claims a "workable" method to construct the hermitian matrix H which implements the isoconjunctivity of two hermitian matrices. However given all the characteristic roots of TS we can calculate the canonical form for (T,S) under contragradient conjunctivity (see Theorem 4 and Theorem 8), the matrix implementing this contragradient conjunctivity, and thus the hermitian matrix H displaying the isoconjunctivity, as was done in the proofs.

To get our next result we need to observe two simple facts.

Lemma 14. Let T and S be $n\times n$ complex matrices. If there exists a complex matrix C such that $T = C^*SC = CSC^*$, then $T(ST)^i = C^*S(TS)^iC$, $i = 0, 1, \ldots$.

Proof: By hypothesis

$$\begin{aligned} T(ST)^i &= C^*SC(SC^*SC)^i \\ &= C^*S(CSC^*S)^iC \\ &= C^*S(TS)^iC. \end{aligned}$$

Lemma 15. Let T and S be $n\times n$ complex matrices such that for some nonsingular complex matrix C, $T = C^*SC = CSC^*$. Then for every $n\times n$ nonsingular complex matrix D there exists a nonsingular complex matrix B such that

$$D^*TD = B^*(D^{-1}SD^{*-1})B = B(D^{-1}SD^{*-1})B^*.$$

Proof: Let $T = C^*SC = CSC^*$ where C is nonsingular. Let D by any $n\times n$ nonsingular matrix. Then

$$\begin{aligned} D^*TD &= D^*C^*SCD = D^*CSC^*D \\ &= (D^*C^*D)(D^{-1}SD^{*-1})(D^*CD) = (D^*CD)(D^{-1}SD^{*-1})(D^*C^*D) \\ &= (D^*CD)^*(D^{-1}SD^{*-1})(D^*CD) = (D^*CD)(D^{-1}SD^{*-1})(D^*CD)^*. \end{aligned}$$

Thus take $B = D^*CD$ (which is nonsingular) and the proof is complete.

<u>Theorem 12.</u> Let T and S be $n\times n$ hermitian matrices. Then T and S are isoconjunctive if and only if there exists a nonsingular complex matrix C such that

$$T = C^*SC = CSC^*. \tag{4.1}$$

Proof: Suppose T and S are isoconjunctive. Then by definition there exists a nonsingular hermitian matrix H such that $T = HSH$. Thus in (4.1) take $C = H$.

Suppose now that there exists a nonsingular matrix C satisfying (4.1). By the Jordan Decomposition Theorem TS is similar to a matrix of the form $M \oplus N$ where M has no negative-real characteristic roots and N has only negative-real characteristic roots. Thus by Theorem 1, Lemma 2, and Lemma 16 we may assume further that

$$T = T_1 \oplus T_2 \quad \text{and} \quad S = S_1 \oplus S_2$$

where T_1S_1 has no negative-real characteristic roots and T_2S_2 has only negative-real characteristic roots. By Lemma 2 we must have, conformably,

$$C = C_1 \oplus C_2 \quad (C_1 \text{ and } C_2 \text{ nonsingular}).$$

Thus $T_1 = C_1^*S_1C_1 = C_1S_1C_1^*$ and $T_2 = C_2^*S_2C_2 = C_2S_2C_2^*$. By Lemma 14 and Theorem 11, T_1 and S_1 are isoconjunctive. By Theorem 7 T_2 and S_2 are isoconjunctive. Thus by Theorem 3, T and S are isoconjunctive.

It is well to point out the analogy between Theorem 12 and Theorem 2 of [3]. There Carlson shows that a complex matrix A is similar to A^* ($A^* = C^{-1}AC$ for some nonsingular matrix C) if and only if A is hermitian-similar to A^* ($A^* = K^{-1}AK$ for some nonsingular $K = K^*$). Since condition (4.1) can be written

$$\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}^* = \begin{bmatrix} C^{-1} & 0 \\ 0 & C^* \end{bmatrix} \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & C^* \end{bmatrix}^{-1}, \tag{4.2}$$

our Theorem 12 (like Carlson's) says that (4.2) holds if and only if there exists a matrix $H = H^*$ such that

$$\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}^* = \begin{bmatrix} H^{-1} & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} \begin{bmatrix} H^{-1} & 0 \\ 0 & H \end{bmatrix}^{-1}.$$

Notice however that our similarities are very special, namely block diagonal.

Finally, as a consequence of our results and Corollary 8 of [3] we can find necessary and sufficient conditions on a pair of $n \times n$ hermitian matrices T and S such that there exist a positive definite (hermitian) matrix H such that $T = HSH$. Noting that C^*HC is positive definite for every nonsingular matrix C if and only if H is positive definite, we can use Theorem 1, Lemma 1, and Theorem 3 to split the problem into the nonsingular and nilpotent parts. Accordingly we give the next two lemmas.

Lemma 16. Let T and S be $n \times n$ nonsingular hermitian matrices. Then there exists a positive definite matrix H such that $T = HSH$ if and only if TS is similar to a diagonal matrix with all positive-real characteristic roots.

Proof: Let $T = HSH$ with H positive definite. Then $T = HSHS$. By Corollary 8 of [3], HS is similar to a diagonal matrix with all non-zero real characteristic roots. Thus $TS = (HS)^2$ is similar to a diagonal matrix with all positive-real characteristic roots.

Now let TS be similar to a diagonal matrix with all positive-real characteristic roots. By Corollary 1 and our observations previous to this lemma we may assume

$$T = \mathrm{diag}(\epsilon_1\beta_1, \ldots, \epsilon_n\beta_n)$$

and

$$S = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$$

where $\epsilon_i = \pm 1$ $(i = 1, \ldots, n)$ and $\beta_1, \ldots, \beta_n$ are the characteristic roots of TS. Thus if we take $H = \mathrm{diag}(\sqrt{\beta_1}, \ldots, \sqrt{\beta_n})$ (which is positive definite) the result follows.

<u>Lemma 17.</u> Let T and S be $n\times n$ hermitian matrices such that TS is nilpotent. Then $T = HSH$ for some positive definite matrix H if and only if $T = S = 0$.

Proof: If $T = S = 0$ take $H = I$ (the $n\times n$ identity matrix). Now let $T = HSH$ with H positive definite. By Corollary 8 of [3], $TS = HSHS$ is similar to a diagonal matrix. Thus since all the characteristic roots of TS are zero, $TS = 0$. Since $T = H(SHS)$ and H is nonsingular this implies $SHS = 0$. Thus in particular $s^*Hs = 0$ for every column s of S. Since H is positive definite this implies every column of S is zero. Thus $S = 0$. But $T = HSH$ so $T = 0$. Hence the proof is complete.

<u>Theorem 13.</u> Let T and S be $n\times n$ hermitian matrices. Then the following are equivalent

a) $T = HSH$ for some positive definite H

b) $C^{-1}TC^{*-1} = C^*SC$ for some nonsingular complex matrix C

c) TS is similar to a diagonal matrix with all nonnegative-real characteristic roots and rank T = rank S = rank TS.

Proof: We prove a)$\Longleftrightarrow$b) and a)$\Longleftrightarrow$c).

<u>a)$\Longleftrightarrow$b):</u> Since H is positive definite if and only if $H = CC^*$ for some nonsingular matrix C, $T = HSH \Longleftrightarrow T = CC^*SCC^* \Longleftrightarrow C^{-1}TC^{*-1} = C^*SC$. Thus a)$\Longleftrightarrow$b).

<u>a)$\Longleftrightarrow$c):</u> Let $T = HSH$ with H positive definite. By Corollary 8 of [3] $TS = HSHS$ is similar to a diagonal matrix with nonnegative real characteristic roots. Since H is nonsingular, rank TS = rank $HSHS$ = rank SHS. Obviously rank $SHS \leq$ rank S. But also $SHSx = 0 \Longrightarrow x^*SHSx = 0 \Longrightarrow (Sx)^*H(Sx) = 0$. Since H is positive definite this implies $Sx = 0$. Thus nullity $SHS \geq$ nullity S, so rank $S \geq$ rank SHS. Therefore rank TS = rank SHS = rank S. Since $T = HSH$, rank T = rank S. Thus a)$\Longrightarrow$c). Now let TS be similar to a diagonal matrix with all nonnegative-real characteristic roots. Assume also rank T = rank S = rank TS. Thus there exists a nonsingular complex matrix C such that $C^{-1}TSC = \mathrm{diag}(\beta_1, \ldots, \beta_m, 0, \ldots, 0)$, where $\beta_i \neq 0$ $(i = 1, \ldots, m)$. By Lemma 1 we must have $C^{-1}TC^{*-1} = T_1 \oplus T_2$ and $C^*SC = S_1 \oplus S_2$ where T_1 and S_1 are $m \times m$. Since $T_1S_1 = \mathrm{diag}(\beta_1 \ldots \beta_m)$

we conclude rank T_1 = rank S_1 = rank $T_1S_1 = m$. Since rank $C^{-1}TSC$ = rank $T_1S_1 = m$ and by hypothesis rank T = rank S = rank TS, we conclude rank T_2 = rank S_2 = rank $T_2S_2 = 0$. Thus $T_2 = S_2 = 0$ so $T_2 = IS_2I$ (where I is the $(n-m)\times(n-m)$ identity matrix, which is positive definite). Since T_2S_2 is diagonal with all positive-real characteristic roots, Lemma 16 implies there exists a positive definite matrix K_1 such that $T_1 = K_1S_1K_1$. Let $K = K_1 \oplus I$. Then K is positive definite and $T_1 \oplus T_2 = K(S_1 \oplus S_2)K$. Thus

$$\begin{aligned} T &= C(T_1 \oplus T_2)C^* \\ &= CK(S_1 \oplus S_2)KC \\ &= CKC^*C^{*-1}(S_1 \oplus S_2)C^{-1}CKC^* \\ &= (CKC^*)S(CKC^*). \end{aligned}$$

Since CKC^* is positive definite this proves c)$\Longrightarrow$a) so the proof is complete.

As a final comment we note that this isoconjunctivity by positive definite matrices (like isoconjunctivity) is not an equivalence relation on the set of $n\times n$ hermitian matrices since it is not transitive (although it is obviously symmetric and reflexive) as the following example indicates.

Example 5. Let

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then T and S are not isoconjunctive by a positive definite matrix since rank T = rank $S = 1$ while rank $TS = 0$. However

$$R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} S \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}.$$

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