The construction of spinor fields on manifolds with smooth degenerate metrics

Jörg Schray,[a] Tevian Dray,[b] Corinne A. Manogue,[c] Robin W. Tucker,[d] and Charles Wang,[e]
School of Physics and Chemistry, Lancaster University, Lancaster LA1 4YB, United Kingdom

(Received 28 June 1995; accepted for publication 25 March 1996)

We examine some of the subtleties inherent in formulating a theory of spinors on a manifold with a smooth degenerate metric. We concentrate on the case where the metric is singular on a hypersurface that partitions the manifold into Lorentzian and Euclidean domains. We introduce the notion of a complex spinor fibration to make precise the meaning of continuity of a spinor field and give an expression for the components of a local spinor connection that is valid in the absence of a frame of local orthonormal vectors. These considerations enable one to construct a Dirac equation for the discussion of the behavior of spinors in the vicinity of the metric degeneracy. We conclude that the theory contains more freedom than the spacetime Dirac theory and we discuss some of the implications of this for the continuity of conserved currents. © *1996 American Institute of Physics.* [S0022-2488(96)01707-0]

I. INTRODUCTION

The interest in the influence of topology on physics is an old one. In recent times there has also been considerable debate on the influence of the geometrical structure of spacetime that may accompany a change in its overall topology. This has been partly motivated by the implications of the semi-classical theory of quantum gravity and partly by the interest in field theories on background spacetimes with interesting topologies. Further motivation arises from string theories in which string interactions arise from the topology of world sheets. In all these approaches fundamental assumptions about the signature of the spacetime metric are required. Such assumptions dictate the detailed behavior of both the causal structure of the theory and the selection rules for topology change. In the context of classical theory there are powerful constraints on the nature of such changes on manifolds with a global Lorentzian signature and a spinor structure.[1] To escape such constraints a number of authors have contemplated geometries in which the metric is allowed to become degenerate, particularly on hypersurfaces that partition the manifold into Lorentzian and Euclidean regions. Despite the obvious implications for causality there have been serious attempts to follow the consequences for physics associated with signature changing metrics. Despite the absence of a rigorous theory of second quantized fields on such a background, in Ref. 2 it was suggested that a quantized scalar field could exhibit spontaneous particle production even in the absence of gravitational curvature. This result relied on certain natural linear boundary conditions that were imposed on the scalar field at the hypersurface of signature change. Since there

[a] Permanent address: Department of Mathematics, Oregon State University, Corvallis, Oregon 97331; Electronic mail address: tevian@math.orst.edu

[b] Permanent address: Department of Physics, Oregon State University, Corvallis, Oregon 97331; Electronic mail address: corinne@physics.orst.edu

[c] Electronic mail address: J. Schray@lancaster.ac.uk

[d] Electronic mail address: r.w.tucker@lancaster.ac.uk

[e] Electronic mail address: cwang@lavu.lancs.ac.uk

0022-2488/96/37(8)/3882/15/$10.00

is no continuous orthonormal coframe in the presence of metric degeneracy and the field equations are themselves dependent on the metric one must rely on a prescribed differential structure in order to define the necessary limits of the gradients of the scalar field in the vicinity of the metric degeneracy. In practice this means one can always rely on a local coordinate coframe to effect one's calculations. Furthermore the differentiability class of all tensor fields is defined with respect to the differentiability of their components in an arbitrary coordinate (co-)frame independent of any metric structure.

Since matter in flat Lorentzian spacetime is also described in terms of various representations of the Lorentzian SPIN group it is natural to try and extend these considerations to the behavior of spinor fields on manifolds with a degenerate metric. In particular one may wish to formulate a dynamical theory of spinor fields and deduce from their field equations a class of natural boundary conditions at the hypersurface of signature change. However a number of interesting problems then arise that have no counterpart in the theory of tensor fields. The most obvious is that the dimensionality of the real irreducible SPIN representations is signature dependent so that it becomes meaningless to try and match spinor fields belonging to representations with different dimensions. If one persists with the search for matching conditions one must in general consider complex representations.

In a smooth local basis of spinor fields one can define the differentiability class of the components of a spinor field. Such a basis is a basis for a module carrying representations of the SPIN group, which is a double cover of the $SO(p,q)$ group associated with the signature of the underlying metric on the manifold. Clearly this procedure will fail at the hypersurface where the signature changes, since the SPIN groups differ across the hypersurface. In order to define continuous spinor fields on a neighborhood crossing the hypersurface, alternatives to the traditional reliance on lifting orthonormal frames to spinor frames must be pursued. Of necessity one must expect some arbitrariness in defining the notion of a continuous spinor field in the presence of signature change.

It is natural to subject local spinor fields to the appropriate Dirac equation in regions where the metric is non-degenerate. In such regions the conventional Dirac operator can be defined in terms of a spinor covariant derivative that is designed to satisfy the natural Leibniz rules on products of tensors and spinors. In this manner it can be made compatible with the natural linear connection on tensors. A unique Levi-Civita tensor covariant derivative is determined completely by the metric tensor. When this metric is non-degenerate one can exploit the existence of local orthonormal frames to uniquely fix the spinor connection that determines the spinor covariant derivative. It is important to stress that it is only the *existence* of a class of orthonormal frames that is necessary to effect this determination, since it provides a reference frame for normalization. The SPIN connection so defined is then compatible with a SPIN invariant inner product on spinors. If one attempts to define a spinor connection in the absence of a class of orthonormal frames then one must recognize the inherent arbitrariness that cannot be removed by normalization. Since we are interested in subjecting our spinor fields to the appropriate Dirac equation in regions where the metric is regular we must accommodate this freedom in the spinor connection if we wish to discuss the matching of spinor solutions at the hypersurface of degeneracy.

Little attention has been devoted to the formulation of spinor fields on spaces with degenerate metrics. Romano[3] recognized that the choice of spinor equation was not straightforward. His analysis was restricted to the case of a discontinuous change of signature, whereas in this article we restrict ourselves instead to the case of continuous degenerate metrics. It is our purpose to examine the essential arbitrariness inherent in a formulation of spinor theory on manifolds with such metrics.

In section II we offer a definition of complex spinors in terms of a *spinor fibration* over a manifold. Although our construction relies on the representation theory of Clifford algebras, we have translated our arguments into the traditional language of γ matrices. The essential novelty is that these are matrix representations of a set of coordinate vector fields that constitute a frame in the vicinity of the metric degeneracy. The representation structure is explicitly presented in terms of degenerate metrics in two and four dimensions.

Having defined the notion of spinor continuity in terms of a spinor fibration, we turn to the notion of the spinor covariant derivative in section III. We show how this can be determined to be both compatible with a SPIN invariant inner product and to commute with the complex structure ("charge conjugation"). In section IV we write down and solve the two dimensional Dirac equation written in terms of this spin connection, making explicit the dependence of the singularity structure of these solutions on both the spin metric and the metric on the underlying manifold. We conclude with a brief discussion of the $U(1)$ currents associated with these solutions and offer some speculations on alternative approaches.

II. SPINORS

In $n=2m$ dimensions we consider the manifold $M=\mathrm{R}^{2m}$ with metric

$$g=h(t)dt\otimes dt+\hat{g}_{ij}(\vec{x})dx^i\otimes dx^j \tag{1}$$

in a chart $(t,x^i)=(x^\mu)$, $i=1,\ \ldots,n-1$, where $\hat{g}$ is assumed to be positive definite. h is a smooth function which may have zeroes (at most countably many that are nowhere dense). However, we require that h changes sign at zeroes of h. None of the crucial steps of the development below rely on the topological triviality of this particular manifold. Although the discussion applies to complex spinors on any even dimensional manifold with signature change, we will pay particular attention to the cases $n=4$ and $n=2$.

Kossowski and Kriele have shown[4] under fairly general conditions that, at any zero of h where $\dot{h}\neq 0$, one can switch to coordinates (t',x^i) in a neighborhood of the zero such that $h(t)dt^2=t'dt'^2$. However, the precise nature of the signature change is not of importance within the scope of this article.

To define Dirac spinors on a manifold M of constant signature one usually[5] considers local irreducible representations $\boldsymbol{\gamma}$ of the complex Clifford algebra bundle

$$\mathscr{C}\ell(M)=\cup_{p\in M}\mathscr{C}\ell(T_pM,g_p), \tag{2}$$

i.e, $\boldsymbol{\gamma}$ is a fiber preserving homomorphism,

$$\boldsymbol{\gamma}:\pi^{-1}(U)\subset\mathscr{C}\ell(M)\to M_k(\mathrm{C})\times U, \tag{3}$$

where $\pi:\mathscr{C}\ell(M)\to M$ is the bundle projection, U is an open subset of M, and $M_k(\mathrm{C})$ is the set of complex $k\times k$ matrices ($k=2^m$). We will assume for now that $\boldsymbol{\gamma}$ is at least continuous. If the representation $\boldsymbol{\gamma}$ is also faithful, which is the case for even dimension of M, then $\boldsymbol{\gamma}$ is just a local trivialization of $\mathscr{C}\ell(M)$. In particular, for vector fields X and Y, $\boldsymbol{\gamma}$ satisfies

$$\{\boldsymbol{\gamma}(X),\boldsymbol{\gamma}(Y)\}=2\,g(X,Y)\mathbf{1}. \tag{4}$$

With respect to a local coordinate chart $\boldsymbol{\gamma}$ is given by its components

$$\gamma_\mu:=\boldsymbol{\gamma}(\partial_\mu). \tag{5}$$

(Note that we use bold-faced $\boldsymbol{\gamma}$ for the representation map, and light-faced symbols for particular images under a representation. Both kinds of symbols may appear in a single expression, in which case a map defined by pointwise multiplication is described as in $[\gamma_\mu\boldsymbol{\gamma}](a)=\gamma_\mu[\boldsymbol{\gamma}(a)]$, $a\in\mathscr{C}\ell(M)$.) With this definition we obtain the familiar relationship

$$\{\gamma_\mu,\gamma_\nu\}=2\,g_{\mu\nu}\mathbf{1}. \tag{6}$$

The Dirac spinor bundle $S(M)$ is a vector bundle carrying such a representation γ, i.e., there is a chart for $S(M)$ such that the Clifford action of $\mathscr{C}\ell(M)$ on $S(M)$ is given by multiplication of the γ-matrices with column spinors. If $\mathscr{C}\ell(M)$ transforms under a product of tensor representations of the orthogonal group and the Clifford action is covariant under this transformation, then $S(M)$ transforms under a spin representation of the orthogonal group.

Except for regions that contain zeroes of h it is straightforward to generalize these ideas to our signature changing spacetime M. The crucial question is how to link the spinor bundles across hypersurfaces of signature change. In the following exposition we will use the fact that the Clifford bundles are linked and lift this link to the spinor bundles. Specifically we will consider an algebra fibration which coincides with the Clifford bundles where the metric is non-degenerate and representations of this fibration which are continuous across a hypersurface of signature change. A detailed study of such representations suggests certain additional conditions which are sufficient to ensure the invariance of the resulting structure under appropriate changes of representations and/or coordinates. Since the group of transition functions is different for different signature we will adopt the term "fibration" for $\mathscr{C}\ell(M)$ and $S(M)$ instead of "bundle," but we will still refer to this object as the "Clifford" and "spinor" fibration, although we use these expressions in a non-traditional context.

The following example will illustrate some of the key issues we have to face.

A. An example in two dimensions

In $n=2$ dimensions we consider $M=\mathbb{R}^2$ with coordinates (t,x) and metric

$$g=h(t)dt\otimes dt+dx\otimes dx, \tag{7}$$

i.e., $\hat{g}=1$. Then the following γ-matrices:

$$\gamma_x=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\gamma_t=\begin{pmatrix}0 & 1\\ h(t) & 0\end{pmatrix}, \tag{8}$$

define a continuous representation γ on all of M which is faithful and irreducible for $h(t)\neq 0$. Note that γ_t is necessarily degenerate at zeroes of h, where the matrix algebra generated by these matrices actually reduces to upper-triangular matrices. Therefore, this representation is neither faithful nor irreducible at metric degeneracies. This behavior is generic because of an incompatibility of representations of degenerate and non-degenerate Clifford algebras: Since the Clifford algebra is no longer semi-simple for $h=0$, the dimension of an irreducible representation is smaller by a factor of two. The irreducible representation of the degenerate algebra is in fact just an irreducible representation of its non-degenerate "spatial" part, i.e., the part corresponding to the "spatial" γ_x. For the representation to remain faithful it would have to double its dimension in order to accommodate the whole nilpotent ideal generated by the degenerate direction. (Note that half of the algebra, namely the ideal generated by γ_t, is nilpotent of order 2 at the degeneracy.)

B. The general case

For a precise description of the behavior of a representation around a metric degeneracy we examine the behavior of $\gamma(\partial_\mu)|_p$ as p approaches a hypersurface $H=\{t=t_0\}$, where $h(t_0)=0$.

Observation 1: *If a continuous local representation γ satisfies Eq. (4) on an open set $U\subset M$ intersecting H and is faithful and irreducible on $U\setminus H$, then γ is a faithful representation of the "spatial subalgebra" $\mathscr{C}\ell^{sp}(M)$ generated by $\{\partial_i\}_{i=1,\ldots,n-1}$ on all of U. Furthermore, $\mathscr{C}\ell^{sp}(M)$ contains central orthogonal idempotents $P_\pm$ which effect a Pierce decomposition of $\mathscr{C}\ell(M)$ and a corresponding decomposition of γ. Given a particular form of $\gamma(P_\pm)$, this decomposition is reflected in a block structure of the matrix representation.*

From the previous example we infer that $\gamma(\partial_t)|_p$ becomes degenerate as $p \to H$. For the other coordinate vector fields this is not the case, since $\gamma(\partial_i)^2 = \mathbf{1}$ everywhere in U. This corresponds to the fact that the algebra generated by $\{\partial_i\}_{i=1,\ldots,n-1}$, which we call the "spatial subalgebra" $\mathscr{C}\ell^{sp}(M)$ remains non-degenerate on H, whence γ restricted to $\mathscr{C}\ell^{sp}(M)$ remains a faithful representation. Therefore, this spatial subalgebra does not "notice" the metric degeneracy and will provide the link that constrains the behavior of γ_t as we pass through H. $\mathscr{C}\ell^{sp}(M)$ contains central orthogonal idempotents,

$$P_{\pm} := \frac{1}{2}(1 \pm z), \tag{9}$$

where z is the normalized dual of the volume element of H, whence $z^2 = 1$, $P_{\pm}^2 = P_{\pm}$, and $P_{\pm}P_{\mp} = 0$. For example, $z = \partial_x$ for the metric given by Eq. (7), whereas $z = i\det\hat{g}^{-1/2}\partial_1 \wedge \partial_2 \wedge \partial_3$ in four dimensions with metric given by Eq. (1). The idempotents or projectors $P_{\pm}$ split $\mathscr{C}\ell^{sp}(M)$ into a direct sum of simple components,

$$\mathscr{C}\ell^{sp}(M) = \mathscr{C}\ell_{+}(M) \oplus \mathscr{C}\ell_{-}(M), \tag{10}$$

where

$$\mathscr{C}\ell_{\pm}(M) := P_{\pm}\mathscr{C}\ell(M)P_{\pm}. \tag{11}$$

Therefore γ induces inequivalent representations

$$\boldsymbol{\gamma}_{\pm} := \gamma_{\pm}\boldsymbol{\gamma}\gamma_{\pm} \tag{12}$$

of $\mathscr{C}\ell^{sp}(M)$, where

$$\gamma_{\pm} := \boldsymbol{\gamma}(P_{\pm}). \tag{13}$$

So we get the following Pierce decomposition with respect to the idempotents $P_{\pm}$:

$$\begin{aligned}
\mathscr{C}\ell(M) &= (P_{+} + P_{-})\mathscr{C}\ell(M)(P_{+} + P_{-}) \\
&= P_{+}\mathscr{C}\ell(M)P_{+} \oplus P_{+}\mathscr{C}\ell(M)P_{-} \oplus P_{-}\mathscr{C}\ell(M)P_{+} \oplus P_{-}\mathscr{C}\ell(M)P_{-} \\
&= \mathscr{C}\ell_{+}(M) \oplus P_{+}\mathscr{C}\ell(M)P_{-} \oplus P_{-}\mathscr{C}\ell(M)P_{+} \oplus \mathscr{C}\ell_{-}(M),
\end{aligned} \tag{14}$$

which translates into representations

$$\boldsymbol{\gamma} = \gamma_{+}\boldsymbol{\gamma}\gamma_{+} + \gamma_{+}\boldsymbol{\gamma}\gamma_{-} + \gamma_{-}\boldsymbol{\gamma}\gamma_{+} + \gamma_{-}\boldsymbol{\gamma}\gamma_{-} = \boldsymbol{\gamma}_{+} + \gamma_{+}\boldsymbol{\gamma}\gamma_{-} + \gamma_{-}\boldsymbol{\gamma}\gamma_{+} + \boldsymbol{\gamma}_{-}. \tag{15}$$

Since $\mathscr{C}\ell^{sp}(M)$ commutes with $P_{\pm}$ and

$$\mathscr{C}\ell(M) = \mathscr{C}\ell^{sp}(M) \oplus \mathscr{C}\ell^{sp}(M)\partial_t = \mathscr{C}\ell^{sp}(M) \oplus \partial_t\mathscr{C}\ell^{sp}(M), \tag{16}$$

the cross terms in Eq. (14) come from ∂_t:

$$P_{\pm}\mathscr{C}\ell(M)P_{\mp} = P_{\pm}(\mathscr{C}\ell^{sp}(M)\partial_t)P_{\mp} = \mathscr{C}\ell_{\pm}(M)\partial_t = P_{\pm}(\partial_t\mathscr{C}\ell^{sp}(M))P_{\mp} = \partial_t\mathscr{C}\ell_{\mp}(M). \tag{17}$$

[Note that $P_{\pm}\partial_t = \partial_t P_{\mp}$ and $P_{\pm}\mathscr{C}\ell^{sp}(M)P_{\mp} = 0$.] This can also be seen from the decomposition of γ_t:

$$\gamma_t = \gamma_{+}\gamma_t\gamma_{-} + \gamma_{-}\gamma_t\gamma_{+}. \tag{18}$$

If $\gamma_{\pm}$ takes the form

$$\gamma_+=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \quad \gamma_-=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \tag{19}$$

in terms of $2^{m-1}\times 2^{m-1}$ unit and zero matrices, which can always be achieved by an equivalence transformation pointwise on U (even on H), then the Pierce decomposition is reflected in a block structure of the matrix representation $\gamma(\mathscr{C}\ell(M))$. In particular, the induced representations $\gamma_\pm$ only have one non-zero block, namely in the upper left (lower right) corner. Denoting the non-zero blocks of the corresponding matrices by overlined symbols, for example,

$$\gamma_+(\mathscr{C}\ell_+(M))=\begin{pmatrix}\overline{\gamma_+(\mathscr{C}\ell_+(M))} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \tag{20}$$

we have the following block structure of $\gamma(\mathscr{C}\ell(M))$:

$$\begin{aligned}\gamma(\mathscr{C}\ell(M))&=\begin{pmatrix}\overline{\gamma_+(\mathscr{C}\ell_+(M))} & \overline{\gamma_+\gamma_t\gamma_-(\mathscr{C}\ell_-(M))}\\ \overline{\gamma_-\gamma_t\gamma_+(\mathscr{C}\ell_+(M))} & \overline{\gamma_-(\mathscr{C}\ell_-(M))}\end{pmatrix}\\ &=\begin{pmatrix}\overline{\gamma_+(\mathscr{C}\ell_+(M))} & \overline{\gamma_+(\mathscr{C}\ell_+(M))\gamma_t\gamma_-}\\ \overline{\gamma_-(\mathscr{C}\ell_-(M))\gamma_t\gamma_+} & \overline{\gamma_-(\mathscr{C}\ell_-(M))}\end{pmatrix}.\end{aligned} \tag{21}$$

[To arrive at this equation apply γ to Eq. (14) using Eqs. (12), (17), (19) and inserting projectors $P_\pm$ when appropriate.] This block structure helps us to understand what happens to a representation when we cross H. The blocks on the diagonal make up the spatial subalgebra and do not contain γ_t. Therefore, these blocks remain non-degenerate throughout U. The off-diagonal blocks show that γ_t intertwines γ_+ and γ_-.

Observation 2: *The inequivalent faithful representations $\gamma_\pm$ of $\mathscr{C}\ell^{sp}(M)$ have equivalent restrictions $\gamma_\pm^+$ to the even subalgebra $\mathscr{C}\ell^+(M)\subset\mathscr{C}\ell^{sp}(M)$. Furthermore, the restrictions $\gamma_\pm^+$ are intertwined by γ_t, which implies that for any $p\in H$, one of the off-diagonal blocks of $\gamma_t|_p$ in the previously discussed block structure vanishes and the other either vanishes or is regular. (The diagonal blocks are trivially zero.)*

Even though the representations $\gamma_\pm$ vanish on one of the simple components, $\gamma_\pm(\mathscr{C}\ell_\mp(M))=0$, they are equivalent when restricted to the even part $\mathscr{C}\ell^+(M)$ of $\mathscr{C}\ell^{sp}(M)$, which is a simple algebra isomorphic to $\mathscr{C}\ell_\pm(M)$. Applying $\gamma_i\gamma_t=-\gamma_t\gamma_i$ twice, we have $\gamma_i\gamma_j\gamma_t=+\gamma_t\gamma_i\gamma_j$, which implies that the restrictions $\gamma_\pm^+$ of $\gamma_\pm$ to $\mathscr{C}\ell^+(M)$ are intertwined by γ_t:

$$\gamma_\pm^+\gamma_t=\gamma_t\gamma_\mp^+. \tag{22}$$

In the block structure (21) the non-zero blocks $\overline{\gamma_\pm^+(\mathscr{C}\ell^+(M))}$ induce irreducible representations $\overline{\gamma_\pm^+}:\pi^{-1}(U)\cap\mathscr{C}\ell^+(M)\rightarrow M_{k/2}(\mathbb{C})\times U$. Since an intertwiner of two irreducible representations is determined up to a scale, with the intertwiner being non-singular unless the scale is zero, we see from the non-zero blocks associated with Eq. (22) that the two blocks of $\gamma_t=\gamma_+\gamma_t\gamma_-+\gamma_-\gamma_t\gamma_+$ are determined by Eq. (22) up to a scale. Since $(\gamma_\pm\gamma_t\gamma_\mp)\times(\gamma_\mp\gamma_t\gamma_\pm)=h(t)\gamma_\pm$, in fact, only a relative scale remains undetermined. Therefore at least one entire block of γ_t has to vanish for $h(t)\rightarrow 0$, so that we are left with a block triangular or block diagonal matrix algebra on H.

Even though we may not be able to achieve this block structure on all of U at the same time, this argument still shows that γ_t is determined up to a relative scale between $\gamma_+\gamma_t\gamma_-$ and $\gamma_-\gamma_t\gamma_+$ and that $\gamma(\mathscr{C}\ell(M))$ is isomorphic to a block triangular or block diagonal matrix algebra at each point of H.

Observation 3: *Two continuous local representations $\gamma_{(r)}$, $r=1, 2$, satisfying Eq. (4) on an open set $U\subset M$ intersecting H and faithful irreducible on $U\backslash H$, are equivalent if and only if the block structures of $\gamma_{(r)}(\partial_t)|_H$ agree. Furthermore, the intertwiner is guaranteed to be continuous across H if one block of $\gamma_{(r)}(\partial_t)$ stays regular.*

Given two overlapping local representations, we can use the same decomposition to show that it is a necessary condition that γ_t has the same behavior on H for both representations if they are related by a non-singular intertwiner. Conversely, if the behavior of γ_t is different for two local representations, the intertwiner necessarily becomes singular on H. Not only the agreement in block structure but its particular form on H is of importance. If both blocks of γ_t vanish, i.e., γ_t vanishes entirely for both overlapping local representations, their intertwiner may be discontinuous. If on the other hand only one block of γ_t vanishes then the intertwiner inherits the smoothness properties of the local representations, in particular it is at least continuous. In this case the non-zero block of γ_t serves as a link across H and no additional requirement of continuity of the intertwiner is needed to ensure that the gluing together of local representations is well-defined. Of course, the transition functions can be restricted to lie in the appropriate spin groups away from H, which requires the transition functions on H to continuously connect both spin groups.

C. Criteria for a spinor fibration

It can be shown that if γ is assumed to be not only C^1 away from H (this is required in order to define a spin connection as we will see in section IIIC) but also to have bounded partial derivatives on any bounded set, then exactly one of $\gamma_\pm \gamma_t \gamma_\mp$ vanishes on all of H and the other one does not. Therefore, a simple smoothness assumption gains the desired control over the block structure. Since the minor technical difference between requiring bounded partial derivatives on bounded sets and C^1, namely that the partial derivatives have limits on H, does not affect the continuity structure of the spinor fibration in question, we will use the more intuitive condition of continuous differentiability. Allowing the partial derivatives of γ to be locally unbounded relinquishes any control over the block structure, e.g., in the two dimensional example:

$$\gamma_x=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \gamma_t=\begin{cases}\begin{pmatrix}0 & |h(t)|^{1/2}+x^2\\ h(t)[|h(t)|^{1/2}+x^2]^{-1} & 0\end{pmatrix}, & \text{for } x>0.\\ \begin{pmatrix}0 & |h(t)|^{1/2}\\ h(t)|h(t)|^{-1/2} & 0\end{pmatrix}, & \text{for } x\leq 0.\end{cases} \tag{23}$$

Piecing γ's like this one together we can get any behavior of γ_t on H we (do not) like.

These observations lead us to a set of criteria for local representations which ensure that they are related by C^1 equivalence transformations:

(i) γ is C^1 satisfying Eq. (4).

(ii) γ is faithful irreducible for $h(t) \neq 0$.

(iii) $\gamma_- \gamma_t \rightarrow 0$ for $h(t)\rightarrow 0$.

[Of course, the γ-matrices given by Eq. (8) satisfy these criteria.] Condition (iii) singles out one class of representations with a certain behavior for $h(t)\rightarrow 0$. Equally well, one could require

(iii′) $\gamma_+ \gamma_t \rightarrow 0$ for $h(t)\rightarrow 0$,

or even a mixture of both, fixing the behavior of the representation for each hypersurface of metric degeneracy separately. In this paper we focus on the issues arising from just one zero of h. In this case (iii′) is obtained from (iii) under a spatial inversion.

D. A possible generalization

We can relax the assumption of a metric of the form Eq. (1) if we assume the existence of a local frame of non-zero vector fields $\{X_\mu\}$ on any open set intersecting a hypersurface H of signature change, such that $X_i \in T(H)$ satisfies $g(X_i,X_j)|_H=\delta_{ij}$ and $g(X_0,X_\mu)|_H=0$ and construct $\gamma(X_\mu)$ instead of $\gamma(\partial_\mu)$. The spatial subalgebra $\mathscr{C}\ell^{sp}(M)$ of $\mathscr{C}\ell(M)$ generated by $\{X_i\}$ coincides with the appropriate extension of $\mathscr{C}\ell(H)=\cup_{p\in H}\mathscr{C}\ell(T_pH,H^*g_p)$, which is really the only intrinsic structure in the vicinity of H. It is essential that the pullback metric H^*g_p be non-degenerate. It is then straightforward to retrace the steps we followed above and come to the same conclusions. Of course, the existence of a global fibration $S(M)$ will depend on the topology of M and possibly on the topology of hypersurfaces of metric degeneracy.

III. THE SPINOR COVARIANT DERIVATIVE

Having defined a spinor fibration $S(M)$ we have a notion of continuity of a spinor field. Namely, a spinor field is continuous if its component sections are continuous with respect to a bundle chart. In other words, given a set of γ-matrices satisfying appropriate conditions, a continuous spinor field is given by a column of continuous functions on which these γ-matrices act.

In order to write down a Dirac equation on M, we need a notion of covariant differentiation of a spinor field. However, given a linear connection on M, the spinor connection is not uniquely determined unless it is also required to be compatible with both a choice of spinor metric and a notion of charge conjugation. Furthermore, the traditional construction of a spinor connection relies on the existence of a non-degenerate metric. In the following we discuss these separate aspects in regions where the metric is manifestly non-degenerate. In section IV the interrelation between these different aspects will be examined in the vicinity of a hypersurface of signature change.

Authors of other literature on this subject usually work in orthonormal frames (see for example Ref. 6) with the notable exception of an early review[7] which also contains references to most of the original work and notes the scaling freedom in the spinor metric discussed below.

A. The spinor metric

In order to discuss the Dirac equation below we introduce the notion of a spinor metric. In particular, we adopt a Hermitian symmetric spin invariant bilinear form on Dirac spinors,

$$S(M)\times S(M)\rightarrow\mathfrak{F}(M),$$

$$\Psi,\Xi\mapsto(\Psi,\Xi)=\Psi^\dagger C\Xi, \tag{24}$$

where $\mathfrak{F}(M)$ denotes the space of functions on M and C is chosen to satisfy

$$C=C^\dagger, \tag{25}$$

$$C\gamma_\mu=-\gamma_\mu^\dagger C, \tag{26}$$

on M. The familiar Dirac adjoint is then given by

$$\bar{\Psi}=\Psi^\dagger C. \tag{27}$$

For our example, Eq. (8),

$$C_1=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{28}$$

satisfies Eqs. (25) and (26). However, the spinor metric C is only determined up to a real scalar at each point of the manifold. Therefore $C_f=fC_1$ could equally well be chosen as a spinor metric, where $f=f^*\in\mathfrak{F}(M)$. Usually the spinor metric is required to be smooth and non-degenerate, which restricts f to be smooth and non-zero. This is one of the reasons why the choice of spinor metric does not usually appear in the standard discussion of the Dirac equation. The scaling function f is normalized to make the equation simple, i.e., C is chosen to be constant for constant

γ-matrices. (Note that the γ-matrices cannot be constant across a hypersurface of signature change.) The behavior of f where the spacetime metric is degenerate must be postulated separately, and it can not *a priori* be ruled out that f may be zero or singular there.

B. Charge conjugation

Charge conjugation can be defined as a map,

$$S(M)\to S(M),$$

$$\Psi\mapsto\Psi^c:=B^*\Psi^*, \tag{29}$$

where B satisfies

$$B\gamma_\mu=\gamma_\mu^* B, \tag{30}$$

$$B^*B=\pm\mathbf{1}=\beta\mathbf{1}. \tag{31}$$

These conditions determine B up to a phase which may vary over M. The sign in the second condition depends on the signature. $\beta=+1$ if there exists a real representation $\beta=-1$ otherwise. Defining the index ν of a metric to be the (signed) difference of the number of positive and negative eigenvalues of the metric, we note that

$$\beta=\begin{cases}+1, & \text{for } \nu\equiv 0,\ 2 \bmod 8,\\ -1, & \text{for } \nu\equiv 4,\ 6 \bmod 8.\end{cases} \tag{32}$$

Therefore, β changes sign and B is necessarily discontinuous if the signature changes from $(-+++)$, i.e., $\nu=2$, to $(++++)$, i.e., $\nu=4$, in four dimensions, while for the change of signature $(-+)\to(++)$ in two dimensions $\beta=1$ in both regions. Since β also determines the periodicity of the charge conjugation operation, namely

$$(\Psi^c)^c=\beta\Psi, \tag{33}$$

continuity of a spinor is only compatible with continuity of its charge conjugate if β is the same in Euclidean and Lorentzian regions. [This observation warrants an investigation of alternative spinor metrics and notions of charge conjugation for the opposite metrics, i.e., signature changing from $(+---)$ to $(----)$, in four dimensions. The reader is invited to pursue these technical aspects which lie outside the main thrust of this article. Note that the standard definitions for opposite Lorentzian metrics differ by signs in Eqs. (26) and (30). For completeness, one may also consider the inclusion of spinors with Grassmann-valued components or even non-standard versions of Eqs. (26) and (30).]

For our 2 dimensional example, we may take

$$B=e^{i\theta}\mathbf{1}, \tag{34}$$

where $\theta=\theta^*\in\mathfrak{F}(M)$.

C. The spinor connection

Given a spinor metric the spinor covariant derivative S_μ with respect to a vectorfield ∂_μ is given by

$$S_\mu=\partial_\mu+\Sigma_\mu, \tag{35}$$

where the spinor connection Σ_μ has to be determined such that the axioms for a spinor covariant derivative are satisfied:

$$S_\mu(a^\nu \gamma_\nu \Psi) = (\nabla_\mu a^\nu)\gamma_\nu \Psi + a^\nu \gamma_\nu (S_\mu \Psi), \tag{36}$$

$$\partial_\mu(\Psi,\Xi) = (S_\mu \Psi,\Xi) + (\Psi, S_\mu \Xi), \tag{37}$$

$$S_\mu(\Psi^c) = (S_\mu \Psi)^c. \tag{38}$$

$\nabla_\mu a^\nu := a^\nu{}_{;\mu} := \partial_\mu a^\nu + \Gamma^\nu{}_{\mu\rho} a^\rho$ denotes the components of the covariant derivative of the vector field given by a^ν, where $\Gamma_{\rho\mu\nu}$ are the spacetime connection coefficients, i.e., for the Levi-Civita connection $\Gamma_{\rho\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$. These axioms ensure compatibility of covariant differentiation of tensors and spinors, Eq. (36), and compatibility of the spinor covariant derivative with the spinor metric and charge conjugation, Eqs. (37) and (38). Using the defining properties, Eqs. (35), (25), (26), (30), and (31), in Eqs. (36)–(38) we get the following conditions:

$$\partial_\mu \gamma_\nu - \Gamma^\rho{}_{\mu\nu}\gamma_\rho = [\gamma_\nu, \Sigma_\mu] = \gamma_\nu \Sigma_\mu - \Sigma_\mu \gamma_\nu, \tag{39}$$

$$C^{-1}\partial_\mu C = \Sigma_\mu + C^{-1}\Sigma_\mu^\dagger C, \tag{40}$$

$$B^{-1}\partial_\mu B = \Sigma_\mu - B^{-1}\Sigma_\mu^* B. \tag{41}$$

In order to give an explicit expression for Σ_μ we expand it in a basis of the Clifford algebra:

$$\Sigma_\mu = \sum_I \sigma_{\mu I}\gamma^I, \tag{42}$$

where the sum is taken over the set of ordered indices $\{(i_1,\ldots,i_p): 1 \le i_1 < \cdots < i_p \le n, 0 \le p \le n-1\}$, with $n = \dim M$, where also $\gamma^{(i_1 \cdots i_p)} = \gamma^{i_1} \ldots \gamma^{i_p}$ and $\gamma^{\emptyset} = \mathbf{1}$ are understood. (Note that the superscript is the empty set $\emptyset$ *not* 0 in the last equation.) In particular $\{\gamma^I\}$ is a basis for the Clifford algebra in the representation $\boldsymbol{\gamma}$.

We first solve for the components of Σ_μ using Eq. (39):

$$[\gamma_\nu, \Sigma_\mu] = \left[\gamma_\nu, \sum_I \sigma_{\mu I}\gamma^I\right] = 2 \sum_{\substack{\nu \notin I \\ |I| \text{odd}}} \sigma_{\mu I}\gamma_\nu \gamma^I + 2 \sum_{\substack{\nu \in I \\ |I| \text{even}}} \sigma_{\mu I}\gamma_\nu \gamma^I, \tag{43}$$

where $|I|$ denotes the length of the multi index. Thus all but the scalar part of Σ_μ is determined:

$$\sigma_{\mu I} = \frac{1}{2^{N+1}} \mathrm{tr}[\gamma_{I^r}\gamma^\nu(\partial_\mu \gamma_\nu - \Gamma^\rho{}_{\mu\nu}\gamma_\rho)] \quad (\text{no sum over } \nu), \tag{44}$$

where for given I one may choose any ν such that for $|I|$ even $\nu \in I$ while for $|I|$ odd $\nu \notin I$. (I^r denotes indices in reversed order, $N = 2^{n/2}$.) For example, to calculate $\sigma_{\mu(0,1,2,3)}$ in four dimensions we may take any $\nu \in \{0,1,2,3\}$, the result is guaranteed to be the same.

We solve for the scalar part of Σ_μ using Eqs. (40) and (41):

$$\sigma_{\mu\emptyset} = \frac{1}{2^{N+1}} \mathrm{tr}(C^{-1}\partial_\mu C + B^{-1}\partial_\mu B). \tag{45}$$

Thus Σ_μ is completely determined. Eq. (45) is derived from the general conditions arising from Eqs. (40) and (41):

$$\mathrm{Re}\ \sigma_{\mu I} = \frac{1}{2^{N+1}} \mathrm{tr}(\gamma_{I^r} C^{-1}\partial_\mu C) \quad (|I| \text{ even}), \tag{46}$$

$$\text{Im}\ \sigma_{\mu I} = \frac{1}{2^{N+1} i}\ \text{tr}(\gamma_{I'} B^{-1} \partial_\mu B). \tag{47}$$

Again these expressions are guaranteed to be real and compatible with Eq. (44). In some instances it may actually be more convenient to use these latter relationships to solve for various components of Σ_μ.

Applying Eqs. (44) and (45) to Eqs. (8), (25), (26), (30), and (31), we obtain for the spinor connection for our 2 dimensional example,

$$\Sigma_x = \frac{1}{2} f^{-1} \partial_x f + \frac{1}{2} i \partial_x \theta,$$

$$\Sigma_t = \frac{1}{2} f^{-1} \partial_t f + \frac{1}{2} i \partial_t \theta + \frac{1}{4} h^{-1} \partial_t h \gamma_x. \tag{48}$$

In the case of a local orthonormal frame $\{X_a\}$ with constant γ-matrices and constant matrices C and B, the familiar solution for Σ_a is purely a bivector

$$\Sigma_a = \frac{1}{4} \omega_{abc} \gamma^b \gamma^c, \tag{49}$$

where $\omega_{abc} = g(X_b, \nabla_{X_a} X_c)$ are the connection coefficients. (Note that the metric compatibility of the connection implies $\omega_{abc} = -\omega_{acb}$.)

IV. THE MASSLESS DIRAC EQUATION IN TWO DIMENSIONS

With the definition (35) of the spinor covariant derivative the massless Dirac equation in arbitrary dimensions takes the form

$$\not{S}\Psi \equiv \gamma^\mu S_\mu \Psi = 0. \tag{50}$$

In two dimensions for the spinor connection (48) we obtain a family of equations depending on the two real functions f and θ:

$$\left[\gamma^\mu \left(\partial_\mu + \frac{1}{2} f^{-1} \partial_\mu f + \frac{1}{2} i \partial_\mu \theta \right) + \gamma^t \frac{1}{4} h^{-1} \partial_t h \gamma_x \right] \Psi = 0. \tag{51}$$

A. Solution for the massless Dirac equation in two dimensions

We solve this equation for regions where it is regular. It is easy to check that Eq. (51) is equivalent to

$$\gamma^\mu [(f^{-1/2} e^{-1/2 i \theta} D^{-1}) \partial_\mu (f^{1/2} e^{1/2 i \theta} D)] \Psi = 0, \tag{52}$$

where the matrix D must satisfy

$$\partial_t D = D \frac{1}{4} h^{-1} \partial_t h \gamma_x, \tag{53}$$

$$\partial_x D = 0. \tag{54}$$

Thus, up to an unimportant constant factor,

$$D=\frac{1}{2}[|h|^{1/4}(1+\gamma_x)+|h|^{-1/4}(1-\gamma_x)]=\begin{pmatrix}|h|^{1/4} & 0\\ 0 & |h|^{-1/4}\end{pmatrix}, \tag{55}$$

with

$$D^{-1}=\frac{1}{2}[|h|^{-1/4}(1+\gamma_x)+|h|^{1/4}(1-\gamma_x)]=\begin{pmatrix}|h|^{-1/4} & 0\\ 0 & |h|^{1/4}\end{pmatrix}. \tag{56}$$

The plane wave ansatz,

$$\Psi=(f^{-1/2}e^{-1/2i\theta}D^{-1})\psi_0 e^{-i(k_\tau\tau-k_x x)}, \tag{57}$$

where $\tau=\int\sqrt{|h(t)|}\,dt$, leads to

$$(-\gamma^t\sqrt{|h|}k_\tau+\gamma^x k_x)D^{-1}\psi_0=0. \tag{58}$$

For non-trivial solutions we need

$$\det(-\gamma^t\sqrt{|h|}k_\tau+\gamma^x k_x)=-k_x^2-h^{-1}|h|k_\tau^2=0, \tag{59}$$

which gives the dispersion relation

$$k_\tau=\begin{cases}\pm k_x, & \text{for } h<0,\\ \pm ik_x, & \text{for } h>0,\end{cases} \tag{60}$$

and corresponding solutions for ψ_0,

$$\psi_0=\begin{pmatrix}1\\ \mp 1\end{pmatrix},\quad \text{for } h<0,\ k_\tau=\pm k_x, \tag{61}$$

$$\psi_0=\begin{pmatrix}1\\ \mp i\end{pmatrix},\quad \text{for } h>0,\ k_\tau=\pm ik_x. \tag{62}$$

Thus the general solutions for regions where $h\neq 0$ and $f\neq 0$ are

$$\Psi_L=(f^{-1/2}e^{-(1/2)i\theta}D^{-1})\sum_{k>0}\left[\left(a_k^+e^{ikx}\begin{pmatrix}1\\ -1\end{pmatrix}+a_k^-e^{-ikx}\begin{pmatrix}1\\ 1\end{pmatrix}\right)e^{-ik\tau}\right|$$
$$+\left(b_k^+e^{ikx}\begin{pmatrix}1\\ 1\end{pmatrix}+b_k^-e^{-ikx}\begin{pmatrix}1\\ -1\end{pmatrix}\right)e^{ik\tau}\Bigg]\quad (h<0). \tag{63}$$

$$\Psi_E=|(f^{-1/2}e^{-(1/2)i\theta}D^{-1})\sum_{k>0}\left[\left(c_k^+e^{ikx}\begin{pmatrix}1\\ -i\end{pmatrix}+c_k^-e^{-ikx}\begin{pmatrix}1\\ i\end{pmatrix}\right)e^{k\tau}\right|$$
$$+\left(d_k^+e^{ikx}\begin{pmatrix}1\\ i\end{pmatrix}+d_k^-e^{-ikx}\begin{pmatrix}1\\ -i\end{pmatrix}\right)e^{-k\tau}\Bigg]\quad (h>0), \tag{64}$$

where $a_k^\pm$, $b_k^\pm$, $c_k^\pm$, and $d_k^\pm$ are arbitrary complex constants. (We omit the zero frequency solution.)

B. Asymptotic behavior and continuity of solutions

Assuming the Fourier sums above are convergent then the singularity structure of these solutions in the vicinity of the degeneracy is determined by $f^{-1/2}D^{-1}$:

$$\Psi_{L/E} \simeq \begin{pmatrix} O(f^{-1/2}|h|^{-1/4}) \\ O(f^{-1/2}|h|^{1/4}) \end{pmatrix}. \tag{65}$$

In particular, solutions are bounded if $f \simeq O(|h|^{-1/2})$. Thus one cannot have both bounded solutions and a bounded spinor metric at the degeneracy hypersurface. One possible choice is $f=|h|^{-1/2}$, in which case a continuous match of a Lorentzian and Euclidean solution would imply

$$a_k^+ + b_k^+ = c_k^+ + d_k^+ , \quad a_k^- + b_k^- = c_k^- + d_k^- , \tag{66}$$

where $\tau(t_0)=0$ is assumed. With this choice, requiring continuity does not induce a bijective map between Lorentzian and Euclidean solutions.

V. CURRENTS

There are two important currents that are locally conserved for solutions to the massless Dirac equation above. In regular domains the current

$$j_D^\mu[\Psi,\Xi] = \mathrm{Im}(\Psi,\gamma^\mu \Xi) \tag{67}$$

is conserved for solutions Ψ,Ξ:

$$\begin{aligned} \nabla_\mu(\Psi,\gamma^\mu\Xi) &= \partial_\mu(\Psi,\gamma^\mu\Xi) + \Gamma^\mu{}_{\mu\rho}(\Psi,\gamma^\rho\Xi) \\ &= (S_\mu\Psi,\gamma^\mu\Xi) + (\Psi,S_\mu(\gamma^\mu\Xi)) + \Gamma^\mu{}_{\mu\rho}(\Psi,\gamma^\rho\Xi) \\ &= -(\gamma^\mu S_\mu\Psi,\Xi) + (\Psi,\gamma^\mu S_\mu\Xi), \end{aligned} \tag{68}$$

using $[S_\mu,\gamma^\nu] = -\Gamma^\nu{}_{\mu\rho}\gamma^\rho$ and $(\Psi,\gamma^\mu\Xi) = -(\gamma^\mu\Psi,\Xi)$ which follow from the definitions and properties of the spinor covariant derivative and spinor metric (see section III). For a massless theory the axial vector current is also conserved,

$$j_A^\mu[\Psi,\Xi] = \mathrm{Re}(\Psi,\mathfrak{z}\gamma^\mu\Xi), \tag{69}$$

where $\mathfrak{z} = \sqrt{|h|}\gamma^t\gamma^x$, since

$$\nabla_\mu(\Psi,\mathfrak{z}\gamma^\mu\Xi) = (\gamma^\mu S_\mu\Psi,\mathfrak{z}\Xi) + (\Psi,\mathfrak{z}\gamma^\mu S_\mu\Xi). \tag{70}$$

Note that $\nabla\mathfrak{z}=0$, since the connection is metric compatible and $\mathfrak{z}$ is the metric dual of the metric volume element.

Given

$$\Psi = (f^{-1/2}e^{-1/2i\theta}D^{-1})\psi, \quad \Xi = (f^{-1/2}e^{-(1/2)i\theta}D^{-1})\xi, \tag{71}$$

which are defined piecewise on the non-degenerate parts of M, where they satisfy the massless Dirac equation, we obtain for the components of the Dirac current,

$$j_D^t[\Psi,\Xi] = -|h|^{-1/2}\mathrm{Re}\ \psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{sgn}\ h \end{pmatrix}\xi, \ j_D^x[\Psi,\Xi] = \mathrm{Re}\psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\xi, \tag{72}$$

and for the components of the axial current

$$j_A^t[\Psi,\Xi]=h^{-1}|h|^{1/2}\text{Im }\psi^\dagger\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\xi,\quad j_A^x[\Psi,\Xi]=\text{Im }\psi^\dagger\begin{pmatrix}1 & 0\\ 0 & -\text{sgn }h\end{pmatrix}\xi. \tag{73}$$

The continuity of these currents depends on the assumptions made for the continuity of the spinor components. From our discussion above it is clear that this requires some assumptions about the behavior of the spinor metric in the vicinity of the signature change.

However some purely signature dependent effects can be seen by considering the coordinate independent contractions,

$$g_{\mu\nu}j^\mu_{D/A}[\Psi,\Xi]j^\nu_{D/A}[\Psi,\Xi]\simeq O(1), \tag{74}$$

which stay bounded near the hypersurface of signature change but contain terms which depend on sgn h. [Note that $\psi\simeq O(1)\simeq\xi$.] Thus the currents do not exhibit any divergences which depend on the choice of spinor metric or on h, although they can be seen to be discontinuous in general for any linear prescription relating spinor data across the hypersurface of signature change.

VI. CONCLUSION

We have drawn attention to some of the subtleties involved in discussing spinor fields in the presence of a smooth metric degeneracy. By insisting on interpolating smoothly (C^1) between the representations on either side of the degeneracy, we have been able to derive a number of interesting results. In particular, we have introduced the notion of a spinor fibration and used this to give a natural interpolation between the notions of a spinor on the two sides of the degeneracy. This enables one to discuss the concept of continuity of a spinor field in this context. Despite the absence of a continuous field of local orthonormal frames we have shown how a local massless Dirac equation can be constructed, albeit in terms of a class of spinor metrics equivalent up to local scalings and a phase freedom associated with charge conjugation. We have shown that the singularity structure of the solutions at metric degeneracies depends on the choice of spinor metric. An important conclusion of our work is that it is impossible to have both a continuous spinor metric and continuous solutions to the Dirac equation. Researchers studying spinor fields on manifolds with smooth degenerate metrics will be forced to make a choice. Furthermore, our formalism allows one to determine explicitly how various assumptions regarding the continuity of the spinor components affect the continuity of the Dirac current.

A dynamic theory of spinors on a degenerate background geometry may require a dynamical prescription to remove the freedom inherent in the construction of the spinor connection. One way to implement this idea would be to promote the scaling degree of freedom in the spinor metric to an independent scalar field and include this in the dynamical theory. A less radical suggestion might be to relinquish completely the irreducible spinor representations for matter by embedding a multiplet of spinor fields into a single Kähler field. The natural dynamics of such a multicomponent tensor field depends only on the metric structure of the manifold which is no longer required to sustain a spinor structure.

Relinquishing the assumption of a smooth interpolation of representations on either side of the metric degeneracy may lead to an alternative construction of a spinor fibration. However, it is unlikely to circumvent the discontinuity of the currents which was found to be purely an algebraic effect of the signature change.

ACKNOWLEDGMENTS

T.D., C.A.M., and J.S. would like to thank the School of Physics and Chemistry at Lancaster University for kind hospitality. Maple was used extensively to manipulate Clifford algebra expressions. This work was partially supported by NSF Grant No. PHY-9208494 (C.A.M., T.D., and J.S.), a Fulbright Grant (C.A.M.), and the Human Capital and Mobility Programme of the European Union (R.W.T.). C.W. is grateful to the Committee of Vice-Chancellors and Principals for an Overseas Research Studentship, Lancaster University for a Peel Studentship, and the School of Physics and Chemistry at Lancaster University for a School Studentship.

[1] G. W. Gibbons and S. W. Hawking, ''Selection rules for topology change,'' Commun. Math. Phys. **148**, 345 (1992).
[2] T. Dray, C. A. Manogue, and R. W. Tucker, ''The scalar field equation in the presence of signature change,'' Phys. Rev. D **48**, 2587 (1993); ''Particle production from signature change,'' Gen. Rel. Grav. **23**, 967 (1991).
[3] J. D. Romano, ''Scalar and spinor fields in signature changing spacetimes,'' Phys. Rev. D **47**, 4328 (1993).
[4] M. Kossowski and M. Kriele, ''Transverse, type changing, pseudo Riemannian metrics and the extendability of geodesics,'' Proc. R. Soc. London Ser. A **444**, 297 (1994).
[5] I. M. Benn and R. W. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics* (Adam Hilger, Bristol, 1987).
[6] R. Penrose and W. Rindler, *Spinors and Space-time*, Vol. 1: ''Two-spinor calculus and relativistic fields,'' Vol. 2: ''Spinor and twistor methods in space-time geometry'' (Cambridge University, Cambridge, 1984, 1985).
[7] W. L. Bade and H. Jehle, ''An Introduction to Spinors,'' Rev. Mod. Phys. **25**, 714 (1953).