AN ABSTRACT OF THE THESIS OF

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(Major professor)

After defining $0^{o}_{(b)} = 1$ and $0^{n}_{(b)} = 0$ (where n is a natural number and b is a complex number called "the base"), the generalized power $x^{t}_{(b)}$ is defined recursively as $\sum_{r=0}^{t} N^{t}_{r}(b)(x-1)^{r}_{(b)}$ where the "extended binomial coefficients" $N^{t}_{r}(b)$ are functions of generalized factorials. After restricting x and t to non-negative integers, a natural extension of the theory permits them to assume complex values.

Some ordinary-looking results (e.g., a binomial theorem), as well as some results which look peculiar (e.g., a formula for negative powers of zero) arise. Other results include: (1) laws of exponents when the base is a root of unity; (2) the formula $(1+x)^{1-p}_{(b)} = 1$ when $|x| < 1$ and b has period p; (3) the existence of an $x \neq 0$ such that $x^{r}_{(b)} = 0$ for each integer $r \geq 2$, when $b^2 \neq 1$; (4) polynomials with some interesting properties (e.g., all zeros lie on the unit circle) which are generated by generalized powers; and (5) a formula for $\sum_{i=1}^{n} i^r$ when r is a natural number.

An immediate consequence of the development is that when b is a prime, $2^{r}_{(b)}$ represents the number of subgroups in an elememtary Abelian group of order b^r.

GENERALIZED POWERS

by

GLORIA OLIVE

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for Dean of Graduate School

Date thesis is presented [illegible]

Typed by Evelyn Bowen

A TRIPLE DEDICATION

I want to dedicate this thesis to the memory of my Mother (1889-1960), who gave so much - and expected nothing in return; to the memory of my Father (1876-1956), for his example of independence; and to the memory of Professor C. C. MacDuffee (1895-1961), for his willingness to serve as my sponsor.

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PREFACE

The following sequence of events has prompted the writing of this thesis: (1) the decision to work for a Ph.D. degree; (2) Professor R. H. Bruck's advice that I should try to find a topic of real interest and explore it - even if it is well known - for it may lead to new results; (3) Professor A. T. Lonseth's seminar on "Hilbert Space" which introduced an area of real interest to me. It was the original basis for my notes "Seminar in Vector Spaces" in both approach and content; and (4) Professor C. C. MacDuffee's inquiry on whether I had obtained any new results on quaternions in the notes I had written. Since this topic happened to be the one I particularly enjoyed, I was encouraged to explore it further when Professor Lonseth challenged me to write a thesis. It was while investigating the number of subgroups in a special group, which I used to extend the algebra of quaternions, that I was led to a concept of generalized powers.

It is my pleasure to express appreciation to (1) Professor MacDuffee; (2) Professor Lonseth; (3) Professor Bruck; and (4) Mrs. Evelyn Ward Bowen, President of the Anderson College Chapter of Kappa Mu Epsilon and Mathematics Department Assistant, 1960-1962, for typing this paper, drawing the symbols, and her fine spirit.

Gloria Olive

GENERALIZED POWERS

CHAPTER 1 PRELIMINARY DEVELOPMENT

1.1. Introduction. It is well known [2, p.111] that if b is a prime, then the number of subgroups of order b^g in an elementary Abelian group of order b^k is $\prod_{i=o}^{g-1} \frac{b^{k-i}-1}{b^{i+1}-1}$ if $g > 0$ and is 1 if $g = 0$. In this paper we shall generalize this result and then use the generalization to obtain "generalized powers." Although some of the basic properties of ordinary powers hold for "generalized powers", many curious-looking properties arise.

1.2. Notations and conventions. Unless otherwise indicated we will let a, b, t, x, y represent complex numbers; g, h represent integers; c, d, i, j, k, q, r represent non-negative integers; and n represent a natural number.

The following conventions will be adopted:

(1.2.1) $$\underline{\binom{t}{0}} = 1;$$

(1.2.2) $$\underline{\binom{t}{-n}} = 0;$$

(1.2.3) $$\text{if } x \neq 0,\ \underline{x^t} = e^{t \log x}$$

where the logarithm has its principal value [3, p.420];

(1.2.4) $$\underline{0^t} = 0 \quad \text{if } R(t) > 0$$

where R(t) is the real part of t;

(1.2.5) $$\underline{0^o} = 1;$$

and otherwise 0^t does not exist.

By use of the above conventions we now let

(1.2.6) $$\prod_{i=g}^{g-n} f(i) = 0^{n-1}$$

when $f(i)$ is a function of i;

(1.2.7) the superscript factorial of $f^j = [f^j]\& = \prod_{i=1}^{j} f^i$ when f^i is a function of the superscript i;

(1.2.8) $$L_g^t(a) = \prod_{i=0}^{g-1} \frac{a^{t-i}-1}{a^{i+1}-1}$$ if the product exists*;

and

(1.2.9) $$N_g^t(b) = \lim_{a\to b} L_g^t(a).$$

We note that the conventions regarding 0^t provide that $L_g^t(a)$ is continuous at a = 0 if $N_g^t(0)$ exists, and that otherwise $L_g^t(0)$ is non-existent.

1.3. Some properties of $N_g^t(b)$. The following five theorems follow directly from the above definitions.

THEOREM 1.3.1. If $a^n \neq 1$ when $n \leq \max(g, k-g)$,

$$L_g^k(a) = \frac{[a^k-1]\&}{[a^g-1]\&\ [a^{k-g}-1]\&} = \frac{[L_1^k(a)]\&}{[L_1^g(a)]\&\ [L_1^{k-g}(a)]\&} \text{ if } 0 \leq g \leq k.$$

$$L_g^k(a) = 0 \text{ if } g < 0 \text{ or } g > k.$$

THEOREM 1.3.2. $N_0^t(b) = 1$.

THEOREM 1.3.3. $N_{-n}^t(b) = 0$.

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* The product fails to exist if and only if $g > 0$ and either (1) $a^n = 1$ when $g \geq n$ or (2) $a = 0$ when both $g \geq 1 + R(t)$ and $g \neq 1 + t$.

THEOREM 1.3.4. $N_k^k(b) = 1$

THEOREM 1.3.5. $N_{k+n}^k(b) = 0$

The following theorem is a direct consequence of the theory of elementary Abelian groups [2, p. 111].

THEOREM 1.3.6. If b is a prime, $N_g^k(b)$ is the number of elementary Abelian groups of order b^g in an elementary Abelian group of order b^k.

By (1.2.8), $L_j^t(a)$ is defined for all j,t except when a = 0, and when a is a root of unity. We next investigate the existence of $N_j^t(b)$ when b = 0 and when b is a root of unity.

THEOREM 1.3.7. $N_j^t(0)$ exists for all j only if t is a non-negative integer.

PROOF. If t is not a non-negative integer and $j > |1 + R(t)|$, then $N_j^t(0)$ does not exist. The theorem follows.

The next theorem follows directly from (1.2.9).

THEOREM 1.3.8. $N_j^k(0) = 1$ if $0 \leq j \leq k$.

THEOREM 1.3.9. If b has period $p \neq 1$, and $j \geq p$, then $N_j^t(b)$ exists only if t is an integer.

PROOF. If $j \geq p$, $N_j^t(b)$ can exist only if $\prod_{i=0}^{j-1} (b^{t-i} - 1) = 0$. If b has period $p \neq 1$, $b^{t-i} - 1 = 0$ implies $t-i = hp$. Since i,h,p are all integers, t must be an integer.

THEOREM 1.3.10. If b has period p, $N_g^h(b)$ exists. Explicitly,

(1.3.1) $$N_g^h(b) = \binom{h_1}{g_1} N_{g_2}^{h_2}(b)$$

when $g = pg_1 + g_2$, $h = ph_1 + h_2$, and $0 \leq g_2, h_2 < p$.

PROOF. Since p is a positive integer g_1, g_2, h_1, h_2 can be found as specified by means of the Division Algorithm. If $g \leq 0$, (1.3.1) follows from (1.2.9), (1.2.1), (1.2.2). If $g > 0$, (1.3.1) follows by (1.2.9), the theory of limits and l'Hospital's rule. Since $N_{g_2}^{h_2}(b)$ exists when $g_2 < p$, by (1.2.9), the theorem follows.

Some interesting-looking Pascal-type triangles are a consequence of the above theorem. One of these triangles is exhibited in the Appendix.

The following theorem is a direct consequence of Theorems 1.3.3, 1.3.5, 1.3.8, 1.3.10.

THEOREM 1.3.11. $N_g^k(b)$ exists for all b.

The next theorem follows directly from l'Hospital's rule.

THEOREM 1.3.12. $N_g^t(1) = \binom{t}{g}$.

The above two theorems lead us into an investigation of combinatorial-like properties of $N_g^k(b)$ and some of their direct consequences.

THEOREM 1.3.13. $N_g^k(b) = N_{k-g}^k(b)$.

PROOF. Since $0 \leq g \leq k$ if and only if $0 \leq k-g \leq k$, the theorem follows from Theorems 1.3.1, 1.3.11.

We next establish some recursive relations.

THEOREM 1.3.14. If $a^n \neq 1$ when $n \leq g$, and if $a \neq 0$ when both $g \geq 1 + R(t)$ and $g \neq 1 + t$; then

$$L_g^{t+1}(a) = L_g^t(a) + a^{t-g+1} L_{g-1}^t(a). \tag{1.3.2}$$

PROOF. If $g \leq 0$, (1.3.2) follows from Theorems 1.3.2, 1.3.3. If $g > 0$, (1.3.2) is an identity by (1.2.8) and the hypothesis.

The following theorem is a direct consequence of the theory of limits and Theorems 1.3.11, 1.3.14.

THEOREM 1.3.15. $N_g^{k+1}(b) = N_g^k(b) + b^{k-g+1}N_{g-1}^k(b)$ if $g \leq k+1$.

THEOREM 1.3.16. If $a^n \neq 1$ when $n \leq g$, and $a \neq 0$ when either $g < 0$ or both $g \geq 1 + R(t)$ and $g \neq 1 + t$; then

(1.3.3) $$L_g^{t+1}(a) = a^g L_g^t(a) + L_{g-1}^t(a).$$

PROOF. If $g \leq 0$, (1.3.3) follows from Theorems 1.3.2, 1.3.3. If $g > 0$, (1.3.2) is an identity by (1.2.8) and the hypothesis.

The following theorem is a direct consequence of the theory of limits and Theorems 1.3.11, 1.3.16.

THEOREM 1.3.17. $N_j^{k+1}(b) = b^j N_j^k(b) + N_{j-1}^k(b)$.

We next use the above theorems to establish some special properties of $N_j^k(b)$ and $N_j^{-n}(b)$.

THEOREM 1.3.18. If $0 \leq j \leq k$, $N_j^k(b)$ is a polynomial in b of degree $j(k-j)$ with positive integral coefficients and leading coefficient equal to one.

PROOF. The theorem holds when $j = 0$ or k (by Theorems 1.3.2, 1.3.4). If we assume inductively that the theorem holds for $N_{j-1}^k(b)$ and $N_j^k(b)$, the theorem follows by Theorem 1.3.17 and finite induction.

The polynomial $\sum_{i=o}^{r} f(i)x^i$ will be called <u>coefficient symmetric</u> if $f(r-i) = f(i)$ for all i, and <u>coefficient anti-symmetric</u> if $f(r-i) = -f(i)$ for all i.

THEOREM 1.3.19. If $0 \leq j \leq k$, $N_j^k(b)$ is a coefficient symmetric polynomial.

PROOF. Since $N_j^k(b)$ is a polynomial in b of degree $j(k-j)$ (by Theorem 1.3.18), and

$$(1.3.4) \qquad N_j^k(b) = b^{j(k-j)} N_j^k\left(\frac{1}{b}\right)$$

is an identity by Theorems 1.3.1, 1.3.11; the theorem follows by elementary algebra.

As a direct consequence of Theorem 1.3.18 we have

THEOREM 1.3.20. If b is a non-negative integer, $N_j^k(b)$ is a positive integer when $0 \leq j \leq k$.

THEOREM 1.3.21. $N_j^k(-1)$ is a non-negative integer.

PROOF. By (1.3.1), $N_j^k(-1) = \binom{k_1}{j_1} N_{j_2}^{k_2}(-1)$ where $j = 2j_1 + j_2$, $k = 2k_1 + k_2$, and $0 \leq j_2, k_2 < 2$. Since $\binom{k_1}{j_1}$ is a non-negative integer, $N_0^0(-1) = N_0^1(-1) = N_1^1(-1) = 1$, and $N_1^0(-1) = 0$ (by Theorems 1.3.2, 1.3.4, 1.3.5), the theorem follows.

THEOREM 1.3.22. If $b > -1$, $N_j^k(b) > 0$ when $0 \leq j \leq k$.

PROOF. If b = 1, the theorem is an immediate consequence of Theorem 1.3.12. In all other cases the theorem follows from (1.2.9).

THEOREM 1.3.23. $N_j^{-n}(b) = (-1)^j \, b^{-nj-\binom{j}{2}} N_j^{n+j-1}(b)$ if $b \neq 0$.

PROOF. Since $L_j^{-n}(a) = (-1)^j \, a^{-nj-\binom{j}{2}} L_j^{n+j-1}(a)$ when $a \neq 0$ and <u>a</u> is not a root of unity, the theorem follows by the theory of limits and Theorem 1.3.10.

CHAPTER 2 GENERALIZED POWERS OF THE FORM $c^k_{(b)}$

2.1. An extension of the binomial theorem. We note that the $N^t_g(b)$ possess various binomial coefficient type properties (e.g. Theorems 1.3.13, 1.3.17). If we also note that the number of subsets of k elements $= \sum_{j=0}^{k} N^k_j(1) = (1+1)^k$ (by Theorem 1.3.12) and when b is a prime the number of subgroups in an elementary Abelian group of order $b^k = \sum_{j=0}^{k} N^k_j(b)$ (by Theorem 1.3.6), then it does not seem unreasonable to introduce a symbol such as $(1+1)^k_{(b)}$ to represent the latter sum. It is an extension of this concept which leads to our development of "generalized powers."

Our objective is to create and investigate generalized powers of the form $x^t_{(b)}$ with medial x, exponent t, base b which have the property indicated in the above paragraph. In order to embark upon this expedition we will establish generalized powers with both medials and exponents restricted to non-negative integers by means of

(2.1.1) $$\underline{0^0_{(b)}} = 1,$$

(2.1.2) $$\underline{0^n_{(b)}} = 0,$$

and

(2.1.3) $$\underline{(c+1)^k_{(b)}} = \sum_{j=0}^{k} N^k_j(b)\, c^j_{(b)}.$$

If we let c = 0 in (2.1.3) and apply (2.1.1), (2.1.2), and Theorem 1.3.2; we have

THEOREM 2.1.1. $1^k_{(b)} = 1$.

If we now let c=1 in (2.1.3) and apply the above theorem we have

THEOREM 2.1.2. $2^k_{(b)} = \sum_{j=o}^{k} N^k_j(b)$.

The above theorem together with Theorem 1.3.6 yields

THEOREM 2.1.3. If b is a prime, $2^k_{(b)}$ represents the number of subgroups in an elementary Abelian group of order b^k.

By use of (2.1.3) we could now recursively find $3^k_{(b)}$, $4^k_{(b)}$, etc. However, since we have found this process to be tedious and the results to be uninteresting, we shall not proceed in this direction. Instead, we shall investigate some general properties of the generalized powers which have been defined.

THEOREM 2.1.4. If b is a non-negative integer, $n^k_{(b)}$ is a positive integer.

PROOF. $1^k_{(b)} = 1$ by Theorem 2.1.1. If we assume inductively that $n^j_{(b)}$ is a positive integer, and let $c = n$ in (2.1.3), the theorem follows by Theorem 1.3.20 and finite induction.

THEOREM 2.1.5. If $-1 \leq b \leq 1$, then $n^k_{(b)} > 0$.

PROOF. $1^k_{(b)} = 1$ by Theorem 2.1.1. If we assume inductively that $n^j_{(b)} > 0$, and let $c = n$ in (2.1.3), the theorem follows by Theorems 1.3.21, 1.3.22, and finite induction.

THEOREM 2.1.6.* $c^o_{(b)} = 1$.

PROOF. Since $(c+1)^o_{(b)} = c^o_{(b)}$ [by (2.1.3) and Theorem 1.3.2] and $0^o_{(b)} = 1$ [by (2.1.1)], the theorem follows by finite induction.

THEOREM 2.1.7.* $c^1_{(b)} = c$.

PROOF. Since $0^1_{(b)} = 0$ [by (2.1.2)], and $(c+1)^1_{(b)} = c^1_{(b)} + 1$ [by (2.1.3) and Theorems 1.3.2, 2.1.6, 1.3.4], the theorem follows by finite

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* Theorem 3.1.3 validates this theorem when c is a complex variable.

induction.

As may be suspected from Theorems 1.3.12, 2.1.2, we have

THEOREM 2.1.8.* $c^k_{(1)} = c^k$.

PROOF. $0^k_{(1)} = 0^k$ by (1.2.5), (2.1.1), (2.1.2). If we assume inductively that $c^j_{(1)} = c^j$, then (2.1.3), Theorem 1.3.12, and the binomial theorem yield $(c+1)^k_{(1)} = \sum_{j=o}^{k} \binom{k}{j} c^j = (c+1)^k$. The theorem now follows by finite induction.

THEOREM 2.1.9.* $(c+d)^k_{(b)} = \sum_{j=o}^{k} N^k_j(b)\ c^{k-j}_{(b)}\ d^j_{(b)}$.

PROOF. In order to establish this theorem by finite induction we will show that it holds for $d = 0$, and for $d + 1$ whenever it holds for d.

If $d = 0$, the theorem is a direct consequence of (2.1.1), (2.1.2), and Theorem 1.3.2.

By (3.1.10), (2.1.3), the inductive assumption on d, and a rearrangement of terms [4, p. 245] we have

$$(2.1.4) \qquad [c + (d+1)]^k_{(b)} = [(c+d)+1]^k_{(b)} = \sum_{j=o}^{k} N^k_j(b)\ (c+d)^j_{(b)}$$

$$= \sum_{g=o}^{k} \sum_{j=g}^{k} N^k_j(b)\ N^j_g(b)\ c^{j-g}_{(b)}\ d^g_{(b)}.$$

If we now let $h = k - j + g$, (2.1.4) becomes

$$(2.1.5) \qquad [c + (d+1)]^k_{(b)} = \sum_{g=o}^{k} \sum_{h=g}^{k} N^k_h(b)\ N^h_g(b)\ c^{k-h}_{(b)}\ d^g_{(b)}.$$

After a rearrangement of terms [4, p. 245] and application of (2.1.3), (2.1.5) reduces to

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* Theorem 3.1.3 validates this theorem when c and d are complex variables.

$$(2.1.6) \qquad [c + (d+1)]^k_{(b)} = \sum_{h=0}^{k} \sum_{g=0}^{h} N^k_h(b)\, N^h_g(b)\, c^{k-h}_{(b)}\, d^g_{(b)}$$
$$= \sum_{h=0}^{k} N^k_h(b)\, c^{k-h}_{(b)}\, (d+1)^h_{(b)}$$

The theorem now follows by finite induction.

Since the above theorem reduces to the binomial theorem when $b = 1$ (by Theorems 1.3.12, 2.1.8) it is an extended binomial theorem.

2.2. Some laws of exponents. After observing the various ordinary-looking properties of generalized powers in the last section, one may suspect that the ordinary laws of exponents will also be inherited. In this section we shall establish some laws of exponents by restricting the base to a root of unity.

THEOREM 2.2.1.* If b has period p, then

$$(2.2.1) \qquad c^{qp+r}_{(b)} = c^q \cdot c^r_{(b)}.$$

PROOF. We will consider two cases: (1) $r < p$ and (2) r is any non-negative integer.

In order to establish Case (1) by finite induction we note that (2.2.1) holds for $c = 0$ [by (1.2.5), (2.1.1), (2.1.2)] and proceed to show that (2.2.1) holds for $(c+1)$ whenever it holds for c.

By (2.1.3), $\quad (c+1)^{qp+r}_{(b)} = \sum_{j=0}^{qp+r} N^{qp+r}_j(b)\, c^j_{(b)},$ and

by (1.3.1), $\quad N^{qp+r}_j(b) = \binom{q}{h} N^r_i(b)$ where $j = hp + i$ and $i, r < p$.

Thus when $r < p$ we have

$$(c+1)^{qp+r}_{(b)} = \sum_{h=0}^{q} \sum_{i=0}^{r} \binom{q}{h} N^r_i(b)\, c^{hp+i}_{(b)} \qquad \text{by Theorem 1.3.5.}$$

* Theorem 3.1.3 validates this theorem when c is a complex variable.

If we now use the inductive assumption on c, the latter equation becomes $(c+1)^{qp+r}_{(b)} = \sum_{h=0}^{q} \binom{q}{h} c^h \cdot \sum_{i=0}^{r} N_i^r(b)\, c^i_{(b)} = (c+1)^q \cdot (c+1)^r_{(b)}$ by (2.1.3) and the binomial theorem. Case (1) now follows by finite induction.

To establish Case (2) we let $r = jp + i$ with $i < p$. We thus obtain $c^{qp+r}_{(b)} = c^{(q+j)p+i}_{(b)} = c^{q+j} \cdot c^i_{(b)} = c^q (c^j \cdot c^i_{(b)}) = c^q \cdot c^r_{(b)}$ by Case (1) and elementary algebra. The theorem follows.

THEOREM 2.2.2.* If b has period p, then

(2.2.2) $$c^{qp}_{(b)} = c^q.$$

PROOF. If we let $r = 0$ in (2.2.1), the theorem follows by Theorem 2.1.6.

THEOREM 2.2.3.* If b has period p, then

(2.2.3) $$c^{qp}_{(b)} \cdot c^r_{(b)} = c^{qp+r}_{(b)},$$

(2.2.4) $$(c^q)^{rp}_{(b)} = (c^{qr})^p_{(b)} = c^{qrp}_{(b)},$$

(2.2.5) $$(cd)^{qp}_{(b)} = c^{qp}_{(b)} \cdot d^{qp}_{(b)}.$$

PROOF. $c^{qp+r}_{(b)} = c^q \cdot c^r_{(b)} = c^{qp}_{(b)} \cdot c^r_{(b)}$ by (2.2.1), (2.2.2); $(c^q)^{rp}_{(b)} = (c^q)^r = c^{qr} = (c^{qr})^p_{(b)} = c^{qrp}_{(b)}$, and $(cd)^{qp}_{(b)} = (cd)^q = c^q \cdot d^q = c^{qp}_{(b)} \cdot d^{qp}_{(b)}$ by (2.2.2) and elementary algebra.

2.3. Base zero. Since $N_j^k(b)$ reduces to a simple formula when b is a root of unity and when $b = 0$, and since $c^k_{(b)}$ has some interesting properties when b is a root of unity, we are now prompted to investigate $c^k_{(0)}$. The following theorem establishes a formula for computing

* Theorem 3.1.3 validates this theorem when c and d are complex variables.

generalized powers with base zero.

THEOREM 2.3.1.* $(c+1)^{k}_{(0)} = \binom{c+k}{k}$.

PROOF. If $c = 0$, the theorem follows by Theorem 2.1.1. If we assume inductively that the theorem holds for c, then

$$(c+2)^{k}_{(0)} = \sum_{h=0}^{k} (c+1)^{h}_{(0)} = \sum_{h=0}^{k} \binom{c+h}{h} = \binom{c+k+1}{k}$$

by (2.1.3), Theorem 1.3.8, and elementary algebra. The theorem follows by finite induction.

The above theorem yields some interesting consequences. For example,

(2.3.1) $$(c+1)^{k}_{(0)} = (k+1)^{c}_{(0)},$$

(2.3.2) $$(-c)^{c}_{(0)} = (-1)^{c},$$

and

(2.3.3) $$(-c)^{k}_{(0)} = 0 \quad \text{when } k > c;$$

where the last two formulas require the use of Theorem 3.1.3.

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* Theorem 3.1.3 validates this theorem when c is a complex variable.

CHAPTER 3 GENERALIZED POWERS OF THE FORM $x^k_{(b)}$

3.1. Arbitrary complex medials. The following two theorems lead us to a natural definition for $x^k_{(b)}$.

THEOREM 3.1.1. If c and k are positive integers,

$$c^k_{(b)} = \sum_{j=o}^{c-1} \sum_{h=o}^{k-1} N^k_h(b)\ j^h_{(b)}. \tag{3.1.1}$$

PROOF. Since $(j+1)^k_{(b)} - j^k_{(b)} = \sum_{h=o}^{k-1} N^k_h(b)\ j^h_{(b)}$ [by (2.1.3) and Theorem 1.3.4], and $c^k_{(b)} = \sum_{j=o}^{c-1} [(j+1)^k_{(b)} - j^k_{(b)}]$, (3.1.1) follows.

THEOREM 3.1.2. If c and k are non-negative integers, then

$$c^k_{(b)} = \sum_{j=o}^{k} P^k_j(b)\ c^j, \tag{3.1.2}$$

where $P^k_j(b)$ is a polynomial in b with rational coefficients when $0 < j \le k$. $P^o_o(b) = 1$ and $P^n_o(b) = 0$.

PROOF. Since $c^o_{(b)} = 1$ [by Theorem 2.1.6] and $0^n_{(b)} = 0$ [by (2.1.2)], $P^o_o(b) = 1$, $P^n_o(b) = 0$, and the theorem holds for $k = 0$ and $c = 0$ [by (1.2.5)]. To complete the proof we must show that the theorem holds when c is a positive integer. To accomplish this by finite induction we will establish that the theorem holds for $k+1$ whenever it holds for $0, 1, \cdot\cdot\cdot, k$.

By use of Theorem 3.1.1, the inductive hypothesis, and some rearrangement of terms [4, p. 245], we have

$$c^{k+1}_{(b)} = \sum_{h=o}^{k} N^{k+1}_h(b) \sum_{j=o}^{c-1} \sum_{i=o}^{h} P^h_i(b)\, j^i = \sum_{i=o}^{k} \sum_{h=i}^{k} N^{k+1}_h(b)\, P^h_i(b) \left[\sum_{j=o}^{c-1} j^i \right]. \tag{3.1.3}$$

We next let $\underline{f^i} = \sum_{j=o}^{c-1} j^i$. It follows from elementary algebra that

f^i is a polynomial in c of degree $i+1$ with rational coefficients f^i_j when $0 < j \leq i+1$, and $f^i_o = 0$. Thus we have

(3.1.4) $$f^i = \sum_{j=1}^{i+1} f^i_j \, c^j.$$

After applying (3.1.4) and rearranging the terms [4, p. 245] again, (3.1.3) becomes

(3.1.5) $$c^{k+1}_{(b)} = \sum_{j=1}^{k+1} \left[\sum_{i=j-1}^{k} \sum_{h=i}^{k} N^{k+1}_h(b) \, P^h_i(b) \, f^i_j \right] c^j.$$

We can now observe that the expression within the brackets of (3.1.5) is a polynomial in b with rational coefficients when $0 < j \leq k+1$ since each $N^{k+1}_h(b) \, P^h_i(b)$ is a polynomial in b with rational coefficients (by Theorem 1.3.18 and the inductive assumption), and the f^i_j are rational. Since $P^n_o(b) = 0$, we have $P^{k+1}_o(b) = 0$. Thus (3.1.2) follows from (3.1.5) and finite induction if we let $P^{k+1}_j(b)$ equal the expression within the brackets of (3.1.5) when $0 < j \leq k+1$. The proof is now complete.

If $0 \leq j \leq k$, $\underline{P^k_j(b)}$ is the polynomial in b which has been established in the above theorem. We thus have

(3.1.6) $$\underline{P^o_o(b)} = 1,$$

(3.1.7) $$\underline{P^n_o(b)} = 0,$$

and

(3.1.8) $$\underline{P^k_j(b)} = \sum_{i=j-1}^{k-1} \sum_{h=i}^{k-1} N^k_h(b) \, P^h_i(b) \, f^i_j \quad \text{when } 0 < j \leq k.$$

By virtue of (3.1.2) we can now let

(3.1.9) $$\underline{x^k_{(b)}} = \sum_{j=o}^{k} P^k_j(b) \, x^j.$$

The following two identities are direct consequences of (3.1.9)

(3.1.10) $$[(x+y)+t]^k_{(b)} = [x+(y+t)]^k_{(b)}$$

and

$$(x+y)^k_{(b)} = (y+x)^k_{(b)}. \tag{3.1.11}$$

Since most of our formulas for generalized powers have been established by finite induction on the medial, the following theorem serves well in its role of validating these formulas when the integral medials are replaced by complex medials.

THEOREM 3.1.3. Let E be an equation in which each member is a polynomial (with complex coefficients) in generalized powers with non-negative integral exponents, and such that each variable medial is a polynomial (with complex coefficients) in its "component" variables. If E contains a fixed number of variable medials, and if E holds when each variable medial component is restricted to an infinite subset of the integers, then E is an identity when each variable medial component is converted to a complex variable.

PROOF. If in E we express each generalized power with a variable medial in polynomial form [by use of (3.1.9)], we obtain an identity by [5, p.66]. Since this identity holds when each variable medial component is converted to a complex variable, the theorem follows by (3.1.9).

3.2. Negative medials. The next theorem serves as a lemma for its successor, which in turn serves as a lemma in establishing a formula for $(-x)^k_{(b)}$.

THEOREM 3.2.1. $\sum_{j=0}^{n} (-1)^j \, b^{\binom{j}{2}} \, N^n_j(b) = 0.$

PROOF. If $n=1$, the theorem is immediate. If we assume inductively that the theorem holds for n, then by some rearrangement of terms and

Theorems 1.3.15, 1.3.5, 1.3.3 we have

$$\sum_{j=0}^{n+1} (-1)^j\, b^{\binom{j}{2}}\, N_j^{n+1}(b)$$

$$= \sum_{j=0}^{n} (-1)^j\, b^{\binom{j}{2}}\, N_j^n(b) + \sum_{j=1}^{n+1} (-1)^j\, b^{\binom{j}{2}}\, b^{n+1-j}\, N_{j-1}^n(b)$$

$$= -b^n \sum_{j=1}^{n+1} (-1)^{j-1}\, b^{\binom{j-1}{2}}\, N_{j-1}^n(b).$$

If we now let $h = j-1$, the last expression reduces to zero by the inductive assumption. Thus the theorem holds for $n+1$ whenever it holds for n, and the proof by finite induction is complete.

THEOREM 3.2.2. $(-1)_{(b)}^k = (-1)^k\, b^{\binom{k}{2}}$.

PROOF. If $k = 0$, the theorem follows by Theorem 2.1.6. If we assume inductively that the theorem holds for $0, 1, \cdots, k$ we have

$$(3.2.1) \qquad 0 = (1-1)_{(b)}^{k+1} = \sum_{j=0}^{k} N_j^{k+1}(b)\, (-1)^j\, b^{\binom{j}{2}} + (-1)_{(b)}^{k+1}$$

$$= \sum_{j=0}^{k+1} N_j^{k+1}(b)\, (-1)^j\, b^{\binom{j}{2}} + (-1)_{(b)}^{k+1} - (-1)^{k+1}\, b^{\binom{k+1}{2}}$$

by (2.1.2), the extended binomial theorem, and Theorems 2.1.1, 1.3.4. If we now apply Theorem 3.2.1, (3.2.1) reduces to

$$(3.2.2) \qquad 0 = 0 + (-1)_{(b)}^{k+1} - (-1)^{k+1}\, b^{\binom{k+1}{2}}.$$

Since (3.2.2) now validates the theorem for $k+1$, the proof by finite induction is complete.

THEOREM 3.2.3. $(-x)_{(b)}^k = (-1)_{(b)}^k\, x_{\left(\frac{1}{b}\right)}^k$ if $b \neq 0$.

PROOF. We first establish the theorem for $x = c$. If $c = 0$, the theorem follows directly from (2.1.1), (2.1.2), and Theorem 2.1.6. If we assume inductively that the theorem holds for c, we have

$$(-c-1)^k_{(b)} = \sum_{h=o}^{k} N^k_h(b)\ (-1)^{k-h}_{(b)}\ (-c)^h_{(b)} = (-1)^k_{(b)} \sum_{h=o}^{k} N^k_h\left(\frac{1}{b}\right) c^h_{\left(\frac{1}{b}\right)}$$

$$= (-1)^k_{(b)}\ (c+1)^k_{\left(\frac{1}{b}\right)} \quad \text{if } b \neq 0,$$

by (3.1.11), the extended binomial theorem, Theorem 3.2.2, (1.3.4), (2.1.3). The theorem now follows for $x = c$ by finite induction, and an application of Theorem 3.1.3 completes the proof.

3.3. $\underline{P^k_j(b)}$. By means of Theorem 3.1.1 we can find $P^k_j(b)$. We now exhibit the results obtained for $k = 2, 3, 4$.

(3.3.1) $x^2_{(b)} = \frac{x^2}{2!}[N^2_1(b)]\& + \frac{x}{2}(1-b).$

(3.3.2) $x^3_{(b)} = \frac{x^3}{3!}[N^3_1(b)]\& + \frac{x^2}{2}(1-b)N^3_1(b) + \frac{x}{3}(1-b)^2[N^2_1(b)]\&.$

(3.3.3) $x^4_{(b)} = \frac{x^4}{4!}[N^4_1(b)]\& + \frac{x^3}{4}(1-b)N^4_1(b)N^3_1(b)$

$$+ \frac{x^2}{4!}(1-b)^2(1+b^2)(11+19b+11b^2) + \frac{x}{4}(1-b)^3[N^3_1(b)]\&.$$

We will next consider some properties of $P^k_j(b)$ when b is a root of unity.

THEOREM 3.3.1. Let $\underline{P^r_h(b)} = 0$ when $h < 0$ or $h > r$. If b has period p, $0 \leq r < p$, and $0 \leq q + h \leq pq + r$; then $P^{pq+r}_{q+h}(b) = P^r_h(b)$.

PROOF. If we set $k = pq + r$ and apply (2.2.1) we obtain $c^k_{(b)} = c^q \cdot c^r_{(b)}$ which in turn yields $\sum_{j=o}^{k} P^k_j(b)\ c^j = \sum_{h=o}^{r} P^r_h(b)\ c^{q+h}$. If we now let $j = q + h$, the latter equation reduces to

$$\sum_{h=-q}^{k-q} P^k_{q+h}(b)\ c^{q+h} = \sum_{h=o}^{r} P^r_h(b)\ c^{q+h}.$$

Since this equation is an identity in c, the theorem follows.

By use of Theorem 2.1.8 we find

$$(3.3.4)\qquad P_j^k(1) = 0^{k-j} \quad \text{when } 0 \leq j \leq k$$

and by Theorem 3.3.1, we have

$$(3.3.5)\quad P_j^{2q+r}(-1) = 0^{|q+r-j|} \quad \text{when } 0 \leq j \leq 2q+r \text{ and } 0 \leq r < 2.$$

We next consider $P_j^k(0)$.

THEOREM 3.3.2. $P_j^k(0) > 0$ when $0 < j \leq k$.

PROOF. By (3.1.2) and Theorem 2.3.1 we have

$$c_{(0)}^k = \sum_{j=o}^{k} P_j^k(0)\ c^j = \binom{c+k-1}{k}.$$

Thus $k!\,P_j^k(0)$ is the sum of all products with $(k-j)$ different factors which are formed from the first $(k-1)$ natural numbers. Since there are $\binom{k-1}{k-j}$ of these products, the theorem follows.

The above theorem establishes the fact that the constant term in the polynomial expansion of $P_j^k(b)$ is positive when $0 < j \leq k$. This fact, together with the following theorem, will be used to establish the degree of $P_j^k(b)$ when $0 < j \leq k$.

THEOREM 3.3.3. $P_j^k(b) = (-1)^{j+k}\ b^{\binom{k}{2}}\ P_j^k\left(\frac{1}{b}\right)$ if $b \neq 0$.

PROOF. By (3.1.9) and Theorems 3.2.3, 3.2.2 we have

$$\sum_{j=o}^{k} P_j^k(b)(-x)^j = (-x)_{(b)}^k = (-1)^k\ b^{\binom{k}{2}} \sum_{j=o}^{k} P_j^k\left(\frac{1}{b}\right) x^j$$

if $b \neq 0$. Since this is an identity in x, the theorem follows by equating the coefficients of x^j.

The above two theorems together with Theorem 3.1.2 yield

THEOREM 3.3.4. $P_j^k(b)$ is a polynomial in b with rational coefficients and is of degree $\binom{k}{2}$ when $0 < j \leq k$. It is coefficient symmetric if

$j+k$ is even, and is coefficient anti-symmetric if $j+k$ is odd.

We now establish some additional properties of $P_j^k(b)$.

THEOREM 3.3.5. $\sum_{j=0}^{k} P_j^k(b) = 1.$

PROOF. If we let $c = 1$ in (3.1.2), the theorem follows by Theorem 2.1.1.

THEOREM 3.3.6. If $0 \le j \le k$, then

$$\sum_{i=j}^{k} \binom{i}{j} P_i^k(b) = \sum_{i=j}^{k} N_i^k(b) P_j^i(b). \tag{3.3.6}$$

PROOF. By use of (3.1.2) and (2.1.3) we obtain the following two expressions for $(c+1)_{(b)}^k$ which we equate.

$$\sum_{i=0}^{k} P_i^k(b)(c+1)^i = \sum_{i=0}^{k} N_i^k(b) \sum_{j=0}^{i} P_j^i(b) c^j. \tag{3.3.7}$$

By use of the binomial theorem and a rearrangement of terms [4, p. 245], (3.3.7) becomes

$$\sum_{j=0}^{k} \sum_{i=j}^{k} \binom{i}{j} P_i^k(b) c^j = \sum_{j=0}^{k} \sum_{i=j}^{k} N_i^k(b) P_j^i(b) c^j. \tag{3.3.8}$$

Since (3.3.8) is an identity in c, (3.3.6) follows.

If we let j take on the values $k-1, k-2, \cdots, 1, 0$ successively, we can use (3.3.6) to find $P_i^k(b)$ after we have $P_j^i(b)$ for $i \le k-1$. These $P_j^i(b)$ can be found by a similar process since we know $P_0^0(b) = 1$ [by (3.1.6)]. We now use this procedure to establish

THEOREM 3.3.7. $P_k^k(b) = \frac{1}{k!}[N_1^k(b)]\&.$

PROOF. If $k = 0$, the theorem follows by (3.1.6). If $k > 0$ and $j = k-1$, then (3.3.6) reduces to $P_k^k(b) = \frac{1}{k} N_1^k(b) P_{k-1}^{k-1}(b)$ by Theorem 1.3.13. The theorem now follows by recursion and the fact that $P_0^0(b) = 1$.

THEOREM 3.3.8. If $k \ge 1$, then

$$P_{k-1}^k(b) = \lim_{a \to b} \binom{k}{2} (1-a) [N_1^2(a)]^{-1} P_k^k(a). \tag{3.3.9}$$

PROOF. If $k = 1$, (3.3.9) follows from (3.1.7). By use of (3.3.6) with $j = k - 2$, and Theorem 3.3.7, we find that (3.3.9) holds for $P^k_{k-1}(b)$ whenever it holds for $P^{k-1}_{k-2}(b)$. The theorem now follows by finite induction.

We conclude this section with some conjectures.

CONJECTURE 3.3.1. $P^k_1(b) = \frac{(1-b)^{k-1}}{k}[N^{k-1}_1(b)]\&$ when $k \geq 1$.

CONJECTURE 3.3.2. If $0 \leq j \leq k$, $P^k_j(b)$ contains $(1-b)^{k-j}$ as a factor.

CONJECTURE 3.3.3. $P^k_j(b) = 0$ implies $|b| = 1$ when $0 < j \leq k$.

CONJECTURE 3.3.4. $P^k_j(b) > 0$ when $-1 < b < 1$ and $0 < j \leq k$.

CONJECTURE 3.3.5. If $x > 0$, then $x^k_{(b)} > 0$ when $-1 \leq b \leq 1$.

We note that the last conjecture follows from (3.1.9), (3.3.4), (3.3.5), (3.1.6), (3.1.7), and Conjecture 3.3.4; and that Conjecture 3.3.4 follows from Theorem 3.3.2, continuity (by Theorem 3.1.2) and Conjecture 3.3.3. Conjecture 3.3.3 has been verified for $k \leq 5$.

3.4. $Q^k_i(x)$. In this section we investigate another approach for finding a simple formula for $x^k_{(b)}$, and again we are unsuccessful. By use of (3.3.1), (3.3.2), (3.3.3) we can express $x^k_{(b)}$ in powers of b when $k = 2, 3, 4$. We now exhibit these results.

(3.4.1) $$x^2_{(b)} = b\binom{x}{2} + \binom{x+1}{2}.$$

(3.4.2) $$x^3_{(b)} = b^3\binom{x}{3} + 2b^2\binom{x+1}{3} + 2b\binom{x+1}{3} + \binom{x+2}{3}.$$

(3.4.3) $$x^4_{(b)} = b^6\binom{x}{4} + 3b^5\binom{x+1}{4} + \frac{b^4}{4}\binom{x+1}{3}(5x-6) + \frac{3}{2}b^3\binom{x+1}{3}x + \frac{b^2}{4}\binom{x+1}{3}(5x+6) + 3b\binom{x+2}{4} + \binom{x+3}{4}.$$

The next theorem follows directly from (3.1.9) and Theorems 3.1.2, 3.3.7.

THEOREM 3.4.1. $x^k_{(b)} = \sum_{i=0}^{\binom{k}{2}} Q_i^k(x)b^i$ where $Q_i^k(x)$ is a polynomial in x of degree k with rational coefficients.

If $0 \leq i \leq \binom{k}{2}$, $Q_i^k(x)$ is the polynomial in x which has been established in the above theorem. The following theorem is now an immediate consequence of Theorem 3.2.2.

THEOREM 3.4.2. $Q_i^k(x)$ has $x+1$ as a factor when $i \neq \binom{k}{2}$.

Since $0^k_{(b)}$ and $1^k_{(b)}$ are both independent of b [by (2.1.1), (2.1.2), and Theorem 2.1.1] we have

THEOREM 3.4.3. $Q_i^k(x)$ has $x(x-1)$ as a factor when $i \neq 0$, and x as a factor when $k \neq 0$.

By virtue of Theorem 2.3.1 we have

THEOREM 3.4.4. $Q_0^k(x) = \binom{x+k-1}{k}$.

If we let $b = 1$ in the equation of Theorem 3.4.1 and apply Theorem 2.1.8 we obtain

THEOREM 3.4.5. $\sum_{i=0}^{\binom{k}{2}} Q_i^k(x) = x^k$.

We now establish two additional properties of $Q_i^k(x)$.

THEOREM 3.4.6. $Q^k_{\binom{k}{2}-i}(x) = (-1)^k Q_i^k(-x)$ when $0 \leq i \leq \binom{k}{2}$.

PROOF. By Theorems 3.4.1, 3.2.3 we have

$$(3.4.4) \qquad \sum_{i=0}^{\binom{k}{2}} Q_i^k(-x)b^i = (-x)^k_{(b)} = (-1)^k_{(b)} \sum_{j=0}^{\binom{k}{2}} Q_j^k(x)\left(\frac{1}{b}\right)^j$$

when $b \neq 0$. If we now let $i = \binom{k}{2} - j$ in the last expression of (3.4.4), that equation reduces to

$$(3.4.5) \qquad \sum_{i=0}^{\left[\frac{k}{2}\right]} Q_i^k(-x)b^i = (-1)^k \sum_{i=0}^{\left[\frac{k}{2}\right]} Q_{\left[\frac{k}{2}\right]-i}^k(x)b^i$$

by use of Theorem 3.2.2. Since (3.4.5) is an identity in b (when $b \neq 0$), the theorem follows by equating the coefficients of b^i.

By use of Theorems 3.4.4, 3.4.6, and elementary algebra we obtain

THEOREM 3.4.7. $Q_{\left[\frac{k}{2}\right]}^k(x) = \binom{x}{k}$.

We conclude this section with the following conjectures:

CONJECTURE 3.4.1. $Q_1^k(x) = (k-1)\binom{x+k-2}{k}$.

CONJECTURE 3.4.2. $Q_{\left[\frac{k}{2}\right]-1}^k(x) = (k-1)\binom{x+1}{k}$.

We note that Theorem 3.4.6 establishes the equivalence of the above two conjectures.

CONJECTURE 3.4.3. If $k > 1$, then $x_{(b)}^k$ is independent of b only if $x = 0$ or $x = 1$.

This last conjecture follows from Theorem 3.4.7 and Conjecture 3.4.1, since $x(x-1)$ would then be the greatest common factor of $Q_{\left[\frac{k}{2}\right]}^k(x)$ and $Q_1^k(x)$.

All of the above conjectures have been verified for $k \leq 6$.

3.5. $\underline{x_{(b)}^2}$. Our purpose in investigating generalized powers with exponent 2 at this time is to indicate some reasons why any attempt to further extend the properties of ordinary powers may well prove to be fruitless.

All theorems of this section are consequences of (3.3.1) and (3.3.3).

THEOREM 3.5.1. $x_{(b)}^2 = x_{(b)}^1 \cdot x_{(b)}^1$ implies $x(x-1)(b-1) = 0$.

THEOREM 3.5.2. $(xy)^2_{(b)} = x^2_{(b)} \cdot y^2_{(b)}$ implies $xy(x-1)(y-1)(b^2-1) = 0$.

THEOREM 3.5.3. $(x^2)^2_{(b)} = x^4_{(b)}$ implies $x(x-1)(b^2-1) = 0$.

THEOREM 3.5.4. If $b \neq -1$, $\left(\frac{-2}{b+1}\right)^2_{(b)} = 1$.

THEOREM 3.5.5. If $b \neq -1$, $\left(\frac{b-1}{b+1}\right)^2_{(b)} = 0$.

The above theorem, which reveals that $x^2_{(b)} = 0$ does not imply $x = 0$ when $b^2 \neq 1$, can be extended (by use of Theorems 2.1.6, 2.1.7, 2.2.1, 3.3.7, 3.3.8). For if $k \geq 2$, there exists an $x \neq 0$ such that $x^k_{(b)} = 0$ [except when b has period p and $k \equiv 0$ or $1 (\mathrm{mod}\ p)$]. Thus when $b^2 \neq 1$ there exists a $k \geq 2$ and an $x \neq 0$ such that $x^k_{(b)} = 0$.

We conclude this section with two additional properties of $x^2_{(b)}$.

THEOREM 3.5.6. For every real $x \neq 0, 1$ there exists a real b such that $x^2_{(b)} < 0$.

PROOF. $x^2_{(b)} < 0$ when (1) $0 < x < 1$ and $b > \frac{1+x}{1-x}$, or (2) $x(x-1) > 0$ and $b < \frac{1+x}{1-x}$.

THEOREM 3.5.7. For every real $b \neq 1$, there exists a real x such that $x^2_{(b)} < 0$.

PROOF. $x^2_{(b)} < 0$ when (1) $x(b+1) < 0$ and $x > \frac{b-1}{b+1}$, or (2) $x(b+1) > 0$ and $x < \frac{b-1}{b+1}$, or (3) $b = -1$ and $x < 0$.

CHAPTER 4 GENERALIZED POWERS OF THE FORM $x^t_{(b)}$

4.1. Arbitrary complex exponents. Thus far we have considered only generalized powers with non-negative integral exponents. With a view to eliminating this restriction we will let

$$(4.1.1) \qquad \underline{E(b,t,x)} = \sum_{j=0}^{\infty} N_j^t(b) x^j_{(b)}.$$

Since $E(b,t,x) = (1+x)^t_{(b)}$ when t is a non-negative integer (by the extended binomial theorem and Theorems 2.1.1, 1.3.5), it is now reasonable to let

$$(4.1.2) \qquad \underline{(1+x)^t_{(b)}} = E(b,t,x)$$

when $E(b,t,x)$ is convergent.

We have not been able to establish a general theorem on the existence of $(1+x)^t_{(b)}$ when t is not a non-negative integer. Existence theorems for $(1+x)^t_{(b)}$ have been found when $b=1$, $b=0$, b is a root of unity, $x=0$, $x=-1$, $x=1$. We shall now explore these cases.

We first note that $E(1,t,x)$ is the binomial series for $(1+x)^t$ (by Theorems 1.3.12, 2.1.8). Therefore $E(b,t,x)$ is an extended binomial series, and the theory of the binomial series [3, p. 426] yields

THEOREM 4.1.1. If t is not a non-negative integer, then $(1+x)^t_{(1)} = (1+x)^t$ in the following three cases: (1) for all t when $|x| < 1$, (2) for $R(t) > 0$ when $|x| = 1$, (3) for $-1 < R(t) \leq 0$ when $|x| = 1$ but $x \neq -1$; and otherwise $(1+x)^t_{(1)}$ fails to exist.

We note that (1.2.3) and (1.2.4) validate the above theorem.

By virtue of (4.1.2) and Theorem 1.3.7, generalized powers with base zero are defined only when the exponent is a non-negative integer. By virtue of (4.1.2) and Theorem 1.3.9, a generalized power whose base is a root of unity different from 1 is defined only when the exponent is an integer.

4.2. Some further laws of exponents. We shall now extend the exponent laws of Section 2.2 to generalized powers with integral exponents.

THEOREM 4.2.1. Let b have period p. Then

$$(1+x)_{(b)}^{-np+g} = (1+x)^{-n}\cdot(1+x)_{(b)}^{g} \tag{4.2.1}$$

when $|x| < 1$, and the left member of (4.2.1) does not exist for any other x when the exponent is negative.

PROOF. In order to establish (4.2.1) we will consider two cases: (1) $0 \leq g < p$ and (2) g is any integer. The proof of Case (1) consists of showing that the coefficient of $x_{(b)}^{jp+i}$ in each member of (4.2.1) is equal to $\binom{-n}{j}N_i^g(b)$ when $i < p$. The coefficient in the left member is found by use of (1.3.1). The coefficient in the right member is found by expanding each factor and then using the distributive law and (2.2.1). Thus Case (1) follows.

To establish Case (2) we let $g = ph+r$ with $0 \leq r < p$. We then have

$$(1+x)_{(b)}^{-np+g} = (1+x)_{(b)}^{(-n+h)p+r} = (1+x)_{(b)}^{-n+h}\cdot(1+x)_{(b)}^{r}$$

$$= (1+x)^{-n}(1+x)^{h}(1+x)_{(b)}^{r} = (1+x)^{-n}(1+x)_{(b)}^{g}$$

by Case (1), elementary algebra, and (2.2.1) for $h > 0$. The theorem now follows since the series for $(1+x)^{-n}$ converges if and only if $|x| < 1$.

If we let $n = g = 1$ in (4.2.1), Theorem 2.1.7 yields

THEOREM 4.2.2. Let b have period p. Then $(1+x)_{(b)}^{1-p} = 1$ when $|x| < 1$, and $(1+x)_{(b)}^{1-p}$ does not exist for any other x when $p \neq 1$.

The above theorem reveals that the exponent 1-p behaves like the exponent zero when b has period p and $|x| < 1$.

The following theorem follows directly from (4.2.1) with $g = 0$ and Theorems 2.1.6, 4.1.1.

THEOREM 4.2.3. If b has period p, $(1+x)^{-np}_{(b)} = (1+x)^{-n}$ if and only if $|x| < 1$.

THEOREM 4.2.4. If b has period p, the following laws of exponents hold if and only if $|x|$ and $|y|$ are both less than one.

(4.2.2) $$(1+x)^{-np}_{(b)} \cdot (1+x)^{g}_{(b)} = (1+x)^{-np+g}_{(b)},$$

(4.2.3) $$[(1+x)^{-n}]^{gp}_{(b)} = [(1+x)^{-ng}]^{p}_{(b)} = (1+x)^{-ngp}_{(b)},$$

and

(4.2.4) $$[(1+x)(1+y)]^{-np}_{(b)} = (1+x)^{-np}_{(b)} \cdot (1+y)^{-np}_{(b)}.$$

PROOF. This proof is similar to that of Theorem 2.2.3. Thus (4.2.2) follows from Theorems 4.2.1, 4.2.3; and both (4.2.3) and (4.2.4) follow from Theorem 4.2.3 and elementary algebra.

4.3. $\underline{1^{t}_{(b)}}$. The two theorems of this section completely characterize $1^{t}_{(b)}$.

THEOREM 4.3.1. $1^{t}_{(b)}$ fails to exist if and only if (1) $b = 0$ and t is not a non-negative integer, or (2) b is a root of unity different from 1 and t is not an integer.

PROOF. If (1) or (2) occurs, $1^{t}_{(b)}$ cannot exist [by (4.1.2) and Theorems 1.3.7, 1.3.9]. Conversely, if both (1) and (2) do not occur, then $N^{t}_{j}(b)$ exists for all j [by (1.2.9) and Theorem 1.3.10] and thus $1^{t}_{(b)}$ exists by virtue of (4.1.2), (2.1.1), (2.1.2).

THEOREM 4.3.2. If $E(b,t,0)$ converges, $1^{t}_{(b)} = 1$.

PROOF. The convergence of $E(b,t,0)$ implies the existence of $N^{t}_{j}(b)$ for all j [by (4.1.1)], which in turn yields $1^t_{(b)} = 1$ [by (4.1.2), (2.1.1), (2.1.2), and Theorem 1.3.2].

4.4. Some results on $O^t_{(b)}$. In this section we will establish a formula for $O^{-n}_{(b)}$ when $|b| > 1$.

THEOREM 4.4.1. If $|b| > 1$, then $O^t_{(b)}$ exists for all t.

PROOF. If t is a non-negative integer, the theorem follows from (2.1.1) and (2.1.2). If t is not a non-negative integer, the theorem follows from Theorem 3.2.2 and the ratio test.

THEOREM 4.4.2. $O^{-n}_{(b)} = \sum_{j=0}^{\infty} b^{-jn} N_j^{n+j-1}(b)$ if $|b| > 1$.

PROOF. Since E(b,-n,-1) reduces to the above series by (4.1.1) and Theorems 1.3.23, 3.2.2; the theorem follows by Theorem 4.4.1 and (4.1.2).

We note that the series in the above theorem diverges when $|b| < 1$.

THEOREM 4.4.3. $O^{-n}_{(b)} = \prod_{j=1}^{n} (1-b^{-j})^{-1}$ if $|b| > 1$.

PROOF. If $n = 1$, the theorem follows from Theorem 4.4.2 and elementary algebra. We assume inductively that the theorem holds for n and wish to show that

$$O^{-(n+1)}_{(b)} = [O^{-n}_{(b)}][1-b^{-(n+1)}]^{-1}. \tag{4.4.1}$$

By Theorems 4.4.2, 1.3.17 (letting $k = n + j - 1$) we have

$$\begin{aligned} O^{-(n+1)}_{(b)} &= \sum_{j=0}^{\infty} b^{-j(n+1)} N_j^{n+j}(b) \\ &= \sum_{j=0}^{\infty} b^{-jn} N_j^{n+j-1}(b) + \sum_{j=0}^{\infty} b^{-j(n+1)} N_{j-1}^{n+j-1}(b). \end{aligned} \tag{4.4.2}$$

By further application of Theorem 4.4.2, and Theorem 1.3.3, (4.4.2) reduces to

$$(4.4.3)\qquad O_{(b)}^{-(n+1)} = O_{(b)}^{-n} + \sum_{j=1}^{\infty} b^{-j(n+1)} N_{j-1}^{n+j-1}(b).$$

If we now let $j = i + 1$ and again apply Theorem 4.4.2, we find that the second expression in the right member of (4.4.3) reduces to

$$(4.4.4)\qquad b^{-(n+1)}\left[\sum_{i=0}^{\infty} b^{-i(n+1)} N_{i}^{n+i}(b)\right] = b^{-(n+1)} \cdot O_{(b)}^{-(n+1)}.$$

Combining (4.4.3) and (4.4.4) we obtain

$$(4.4.5)\qquad O_{(b)}^{-(n+1)} = O_{(b)}^{-n} + b^{-(n+1)} O_{(b)}^{-(n+1)}.$$

The desired result (4.4.1) now follows, and the proof by finite induction is complete.

4.5. Some results on $2_{(b)}^{t}$. By use of Theorem 2.1.1 and the ratio test we have

THEOREM 4.5.1. When t is not a non-negative integer, $2_{(b)}^{t}$ exists if $|b| > 1$ and does not exist if $|b| < 1$.

THEOREM 4.5.2. $2_{(b)}^{-1} = \sum_{j=0}^{\infty} (-1)^{j} b^{-\binom{j+1}{2}}$ if $|b| > 1$.

PROOF. Since $E(b,-1,1)$ reduces to the above series (by Theorems 1.3.23, 1.3.4, 2.1.1), the theorem follows by Theorem 4.5.1 and (4.1.2).

THEOREM 4.5.3. If $b > 1$, then $2_{(b)}^{-1} > \frac{b-1}{b}$; and if $b < -1$, then $2_{(b)}^{-1} < \frac{b-1}{b}$.

PROOF. Since the series of Theorem 4.5.2 is absolutely convergent and alternating, the theorem follows by grouping the terms in pairs.

CHAPTER 5 SOME APPLICATIONS

5.1. Generalized elementary Abelian groups. In this and the following two sections we shall collect the applications indicated above, establish some new ones, and make conjectures regarding others.

In Theorem 1.3.6 we noted that when b is a prime $N_g^k(b)$ is the number of subgroups of order b^g in an elementary Abelian group of order b^k. This fact and our development in Chapter 2 led us to Theorem 2.1.3 which established that $2^k_{(b)}$ represents the number of subgroups in an elementary Abelian group of order b^k when b is a prime.

We will now define a generalized elementary Abelian group, which will be called a G(b,k), as a closed system (i.e., a groupoid [1, p.1] with respect to each defined operation) which contains $N_j^k(b)$ subsystems of type G(b,j) when $0 \leq j \leq k$. Thus $2^k_{(b)}$ is the number of subsystems in a G(b,k) (by Theorem 2.1.2).

The following three theorems are direct consequences of Theorems 1.3.6, 1.3.8, 1.3.12 respectively.

THEOREM 5.1.1. If b is a prime, an elementary Abelian group of order b^k is a G(b,k).

THEOREM 5.1.2. A single element is a G(0,k).

THEOREM 5.1.3. A set of k distinct elements is a G(1,k).

We do not know whether any other G(b,k) exists.

5.2. Formulas for $\sum_{j=0}^{c-1} j^h_{(b)}$. If $c^n_{(b)}$ is known in closed form for $2 \leq n \leq k$, then $\sum_{j=0}^{c-1} j^h_{(b)}$ can be found for $0 \leq h \leq k-1$ by successive use of (3.1.1). By assuming closed forms for $c^2_{(b)}$ and $c^3_{(b)}$ we now illustrate the procedure for k=3. Since

$$(5.2.1) \qquad c^2_{(b)} = N^2_0(b) \sum_{j=0}^{c-1} j^0_{(b)} + N^2_1(b) \sum_{j=0}^{c-1} j^1_{(b)}$$

by (3.1.1), and $\sum_{j=0}^{c-1} j^0_{(b)} = c$ (by Theorem 2.1.6), we can solve (5.2.1) for $\sum_{j=0}^{c-1} j^1_{(b)}$. If we repeat the process for $c^3_{(b)}$ we have

$$(5.2.2) \qquad c^3_{(b)} = N^3_0(b) \sum_{j=0}^{c-1} j^0_{(b)} + N^3_1(b) \sum_{j=0}^{c-1} j^1_{(b)} + N^3_2(b) \sum_{j=0}^{c-1} j^2_{(b)}$$

by (3.1.1). We can now solve (5.2.2) for $\sum_{j=0}^{c-1} j^2_{(b)}$.

In contrast to the above, the following theorem yields direct methods for finding $c^k_{(b)}$, $P^k_j(b)$, and $\sum_{j=0}^{c-1} j^h_{(b)}$.

THEOREM 5.2.1. $c^k_{(b)} = \sum_{i=0}^{k} R^k_i(b)\binom{c}{i}$ where $\underline{R^k_0(b) = 0^k}$ and $\underline{R^k_n(b)}$ is the sum of all $\binom{k-1}{n-1}$ terms of the form $\prod_{j=1}^{n} N^{[j]}_{[j-1]}(b)$ ($\underline{[j]}$ represents a non-negative integer, $[0]= 0$, $[n]= k$, $[j-1] < [j]$).

PROOF. To establish this theorem by finite induction we will show that it holds for k=0, and for k+1 whenever it holds for $0,1,\cdots,k$.

If k=0, the theorem follows from Theorem 2.1.6. By (3.1.1), the inductive assumption, and elementary algebra we have

$$(5.2.3) \qquad c^{k+1}_{(b)} = \sum_{h=0}^{k} N^{k+1}_h(b) \sum_{j=0}^{c-1} j^h_{(b)} = \sum_{h=0}^{k} N^{k+1}_h(b) \sum_{j=0}^{c-1} \sum_{i=0}^{h} R^h_i(b)\binom{j}{i}$$

$$= \sum_{h=0}^{k} N^{k+1}_h(b) \sum_{i=0}^{h} R^h_i(b) \sum_{j=0}^{c-1} \binom{j}{i} = \sum_{h=0}^{k} N^{k+1}_h(b) \sum_{i=0}^{h} R^h_i(b) \binom{c}{i+1}$$

If we rearrange terms [4, p.245] and let j=i+1, (5.2.3) becomes

$$(5.2.4) \qquad c^{k+1}_{(b)} = \sum_{j=1}^{k+1} \left[\sum_{h=j-1}^{k} R^h_{j-1}(b)\ N^{k+1}_h(b)\right]\binom{c}{j}.$$

We now let $\underline{S(j,k,b)}$ represent the expression within the brackets of (5.2.4). If j=1, $S(j,k,b)= N^{k+1}_0(b)= R^{k+1}_1(b)$. If $j > 1$, $R^h_{j-1}(b)$ is the sum of all $\binom{h-1}{j-2}$ terms of the form $\prod_{i=1}^{j-1} N^{[i]}_{[i-1]}(b)$ (where $[0]=0$, $[j-1]=h$, $[i-1]<[i]$), and the number of terms in $S(j,k,b)$ is $\sum_{h=j-1}^{k}\binom{h-1}{j-2}=\binom{k}{j-1}$. Thus $S(j,k,b)=R_j^{k+1}(b)$ and the proof by finite induction is complete.

By use of the above theorem and (5.2.3) we have

THEOREM 5.2.2. $\sum_{j=0}^{c-1} j^h_{(b)} = \sum_{i=0}^{h} R_i^h(b)\binom{c}{i+1}$.

We note that the above theorem yields a formula for $\sum_{j=0}^{c-1} j^h$ (by Theorem 2.1.8).

5.3. Probability. A real valued function $f(j)$ with domain $\{0,1,\cdots,k\}$ is a probability function if $f(j)\geq 0$ for all j, and $\sum_{j=0}^{k} f(j)=1$.

As a direct consequence of Theorems 1.3.21, 1.3.22, 2.1.2 we have

THEOREM 5.3.1. $f(j)=\left[2^k_{(b)}\right]^{-1} N_j^k(b)$ is a probability function if $b\geq -1$.

We conclude with two conjectures.

CONJECTURE 5.3.1. $f(j)=N_j^k(b)\, x^{k-j}_{(b)}\,(1-x)^j_{(b)}$ is a probability function if $0\leq x\leq 1$ and $-1\leq b\leq 1$.

We note that the above conjecture, which reduces to the binomial distribution if $b=1$ (by Theorems 1.3.12, 2.1.8), follows from the extended binomial theorem, Theorems 2.1.1, 1.3.21, 1.3.22, and Conjecture 3.3.5.

CONJECTURE 5.3.2. $f(j)=P_j^k(b)$ is a probability function if $-1\leq b\leq 1$.

The above conjecture follows from Theorem 3.3.5, (3.3.4), (3.3.5), (3.1.6), (3.1.7), and Conjecture 3.3.4.

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APPENDIX

PASCAL-TYPE TRIANGLE FOR $N_h^k(b)$ WHEN b HAS PERIOD 4

a = 1 + b