AN ABSTRACT OF THE THESIS OF

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Title A STUDY OF THE DIFFERENTIAL EQUATION

$$(1)\ \frac{d^2w}{dz^2} + (m+z^2)w = 0$$

Abstract Approved:
(Major Professor)

ABSTRACT

The first part of the paper is devoted to the obtaining of convergent power series solutions and the determination of certain properties of these solutions. In particular, properties are discussed relative to the number of zeros of a solution and the distances between successive zeros. The relationships of orthogonality of two different solutions are shown. Since it is difficult to write down a general term for the power series, the equation is transformed into one which gives rise to a series with simpler coefficients. In addition, the relation of the Weber equation to (1) is shown.

Beginning with Chapter VII the discussion centers on solutions of (1) asymptotic in z. The theorem of Trjitzinsky is applied to show the asymptotic character of the formal solutions. In Chapter VIII the solutions asymptotic in the parameter are obtained by applying the method and existence theorem given by Birkhoff. In both cases, solutions are obtained which are best suited for calculation.

The last Chapter deals with the behavior of the coefficients of a solution due to Stokes' phenomenon. In the case of solutions asymptotic in m, explicit formulas are given for the calculation of the coefficients.

A STUDY OF THE DIFFERENTIAL EQUATION

$$\frac{d^2w}{dz^2} + (m + z^2)w = 0$$

by

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A STUDY OF THE DIFFERENTIAL EQUATION

$$\frac{d^2w}{dz^2} + (m+z^2)w = 0$$

I. INTRODUCTION

This paper is a study of the differential equation

(1) $$\frac{d^2w}{dz^2} + (m+z^2)w = 0$$

The first part of the paper is devoted to the obtaining of convergent power series solutions and the determination of certain properties of these solutions. In particular, properties are discussed relative to the number of zeros of a solution and the distances between successive zeros. The relationships of orthogonality of two different solutions are shown. Since it is difficult to write down a general term for the power series, the equation is transformed into one which gives rise to a series with simpler coefficients. In addition, the relation of the Weber equation to (1) is shown.

Beginning with Chapter VII the discussion centers on solutions of (1) asymptotic in z , and particular stress is given to obtaining real solutions for practical computation. The theorem of Trjitzinsky is applied to show the asymptotic character of the formal solutions.

In Chapter VIII the solutions asymptotic in the parameter m are obtained by applying the method and existence theorem given by Birkhoff. Here again solutions are obtained which are best suited for calculation.

In Chapter IX the behavior of the coefficients of a solution due to Stokes' phenomena is discussed. In the case of solutions asymptotic in m, explicit formulas are given for calculation of the coefficients.

II. EXISTENCE OF A SOLUTION OF

(1) $$w'' + (m + z^2) w = 0$$

Consider the general homogeneous linear differential equation of the second order

(1-a) $$w'' = p_1 w' + p_2 w$$

in which the coefficients p_1 and p_2 are regular throughout the finite plane. Here w is a function of the complex variable z. It is shown by a general existence theorem* that (1-a) admits one and only one analytic solution of form

$$w = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots\ldots ,$$

which with its first derivative takes on assigned values at $z = 0$. This solution is valid over the entire finite plane.

Equation (1) satisfies the conditions necessary in the type form (1-a). Therefore we may assert (1) has a solution of the form

$$w = C_0 + C_1 z + C_2 z^2 + \ldots\ldots .$$

We shall next determine the coefficients of this series.

*Pierpont, James, Functions of a Complex Variable, p. 459, Ginn and Co., 1914

III. THE SOLUTION OF (1) BY MEANS OF A POWER SERIES.

Assume the solution

$$w = C_0 + C_1 z + C_2 z^2 + \ldots\ldots C_i z^i + \ldots\ldots,$$

Then

$$\frac{dw}{dz} = C_1 + 2C_2 z + 3C_3 z^2 + \ldots\ldots + iC_i z^{i-1} + \ldots,$$

$$\frac{d^2w}{dz^2} = 2C_2 + 6C_3 z + 12C_4 z^2 + \ldots\ i(i-1)C_i z^{i-2} + \ldots.$$

If we substitute the foregoing expressions in (1) we have

$$(C_0 m + 2C_2) + (6C_3 + mC_1)z + (12C_4 + mC_2 + C_0)z^2 + \ldots\ldots$$

$$+ (i+2)(i+1)C_{i+2}z^i + mC_i z^i + C_{i-2}z^i + \ldots\ldots = 0$$

Next, the coefficients are equated to zero, and this gives

$$C_2 = -\frac{C_0 m}{2!}$$

$$C_3 = -\frac{C_1 m}{3!}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C_{i+2} = \frac{-mC_i - C_{i-2}}{(i+2)(i+1)}$$

The last formula gives each coefficient in terms of the two preceding ones where C_0 and C_1 are two arbitrary constants. Several of these coefficients have been computed and are given below.

$$C_4 = \left(\frac{m^2-2}{4!}\right)C_0$$

$$C_5 = \left(\frac{m^2-5}{5!}\right)C_1$$

$$C_6 = -\left(\frac{m^3-14m}{6!}\right)C_0$$

$$C_7 = -\left(\frac{m^3 - 26m}{7!}\right) C_1 ,$$

$$C_8 = \frac{m^4 - 44m^2 + 60}{8!} C_0 ,$$

$$C_9 = \frac{m^4 - 68m^2 + 252}{9!} C_1 ,$$

$$C_{10} = -\left(\frac{m^5 - 100m^3 + 844m}{10!}\right) C_0 ,$$

$$C_{11} = -\left(\frac{m^5 - 140m^3 + 2124m}{11!}\right) C_1 ,$$

$$C_{12} = \frac{m^6 - 190m^4 + 4804m^2 - 5400}{12!} C_0 ,$$

$$C_{13} = \frac{m^6 - 250m^4 + 9604m^2 - 27720}{13!} C_1 .$$

When these values for the coefficients are substituted in the assumed power series solution, we have as the most general solution for finite values of

$$w = C_0 w_1 + C_1 w_2$$

where

$$w_1 = 1 + \frac{m}{2} z^2 + \left(\frac{m^2 - 2}{24}\right) z^4 - \left(\frac{m^3 - 14m}{6!}\right) z^6 + \left(\frac{m^4 - 44m^2 + 60}{8!}\right) z^8 - \left(\frac{m^5 - 100m^3 + 844m}{10!}\right) z^{10} + \cdots\cdots ,$$

$$w_2 = z - \frac{m}{3!} z^3 + \left(\frac{m^2 - 6}{5!}\right) z^5 - \left(\frac{m^3 - 26m}{7!}\right) z^7 + \left(\frac{m^4 - 68m^2 + 252}{9!}\right) z^9 - \left(\frac{m^5 - 140m^3 + 2124m}{11!}\right) z^{11} + \cdots\cdots$$

IV. PROPERTIES OF SOLUTIONS OF (1) FOR REAL $z=x$

1. A solution of (1) has an infinite number of zeros.

Proof:

We shall consider the equation

(2) $$\bar{y}''+n^2y=0$$

which has a solution $\bar{y}=C\sin n(x-a)$, where a may be given any value. Since (1) has a solution which has a zero, let us choose a at this value of x.

Now we multiply (1) by $\bar{y}$ and (2) by y and subtract (2) from (1). The result gives us

$$\bar{y}y''-y\bar{y}''+[(m+x^2)-n^2]y\bar{y}=0$$

Choose n so that the expression in brackets is positive. Furthermore, pick the solution $\bar{y}$ such that $\bar{y}'(a)=y(a)$. When the last equation is integrated between the limits a and x, there results the expression

$$\bar{y}y'-y\bar{y}'\Big]_a^x+\int_a^x[(m+x^2)-n^2]y\bar{y}\,dx=0.$$

Assume $\bar{y}$ vanishes again first after a at b; then this equation will be

$$-y\bar{y}'\Big]_a^b+\int_a^b[(m+x^2)-n^2]y\bar{y}\,dx=0.$$

The signs of all the terms in the equation will be positive. The situation is not possible. If y vanishes first, we have

$$\bar{y}y'\Big]_a^b+\int_a^b[(m+x^2)-n^2]y\bar{y}\,dx=0.$$

Here the signs are opposite, and the relation could be satisfied. Since (2) has an infinite number of zeros, the process might be continued an infinite number of times, thus showing (1) has an infinite number of zeros.

2. The squares of the amplitudes between successive zeros of a solution of (1) are always decreasing.

Proof:

Let y be a solution of (1) and form the equation

$$w = \frac{y^2 - y'^2}{m + x^2} .$$

Then

$$w' = 2yy' + \frac{2y'y''}{m+x^2} - \frac{2xy'}{(m+x^2)^2}$$

$$= \frac{2y'}{m+x^2}\left[y'' + (m+x^2)y\right] - \frac{2xy'^2}{(m+x^2)^2}$$

$$= \frac{2xy'^2}{(m+x^2)^2} .$$

When x is positive w' is always negative, and w is a decreasing function. Let y_i , $i = 1,2,3,$-----, represent the amplitudes of the intervals, and let y_i represent slopes at x_i. The slopes at x_i will be zero, and we have

$$w_1 = y_1^2 , \quad w_2 = y_2^2 \quad \text{etc.}$$

Since w decreases

$$y_1^2 > y_2^2 > y_3^2 \quad \text{etc.,}$$

and the proof is complete.

3. The intervals between successive zeros of a solution of (1) are continually decreasing.

Proof:

Assume a solution of equation (1) has successive zeros at a and b. In (1) make the substitution $x = z - a + b$. Now this new equation will become

$$\frac{d^2y}{dz^2} + [m + (z+b-a)^2]\, y = 0$$

which, since $dx = dz$, may be written

$$\text{(1-b)} \qquad \frac{d^2y}{dx^2} + [m + (z+b-a)^2]\, \hat{y} = 0 \quad .$$

Equation (1-b) has a solution $\hat{y}$ which will have a zero at a when the corresponding solution y of (1) has a zero at b. We seek to show that y will vanish at b before $\hat{y}$.

Let us multiply (1) by $\hat{y}$ and (1-b) by y and subtract. The result gives us

$$\hat{y}y'' - y\hat{y}'' + [x^2 - (x+b-a)^2]\, y\hat{y} = 0 .$$

When we integrate this equation between the limits a and b we obtain the equation

$$\hat{y}y\Big]_a^b + \int_a^b [x^2 - (x+b-a)^2]\, y\hat{y}\, dx = 0.$$

As before in this process, if $\hat{y}$ does not vanish before b at c the signs of the terms do not make the equation possible. If $\hat{y}$ does vanish before b, the equation could be satisfied. Thus the interval ac is shorter than ab which indicates that the next interval between the zeros of a solution y is shorter than the one before it. The process may be continued throughout all the intervals to show that they are continually decreasing.

4. If corresponding to two different values of there are two solutions y_1 and y_2 of (1) both equal to zero at a and b, then

$$\int_a^b y_1 y_2 \, dx = 0 \quad .$$

Consider the equations

$$(3) \qquad y_1'' + (m_1^2 + x^2) y_1 = 0 ,$$

$$(4) \qquad y_2'' + (m_2^2 + x^2) y_2 = 0 .$$

If we multiply (3) by y_2 and (4) by y_1 and subtract there results the eqaution

$$y_2 y_1'' - y_1 y_2'' + (m_1^2 + x^2) y_1 y_2 - (m_2^2 + x^2) y_1 y_2 = 0.$$

When we integrate this expression between the limits a and b we have

$$y_2 y_1' - y_1 y_2' \Big]_a^b + (m_1^2 - m_2^2) \int_a^b y_1 y_2 = 0.$$

Now if y_1 and y_2 are different solutions with zeros at a and b there remains only the eqaution

$$\int_a^b y_1 y_2 \, dx = 0 \; .$$

5. In (1) let $x = at$. We then have

$$(1\text{-c}) \qquad \frac{1}{a^2} \frac{d^2 y}{dt^2} + (m + a^2 t^2) = 0 .$$

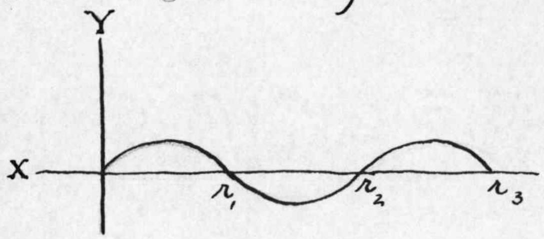
Let y (x) be a solution of (1). Then y (Kt) is a solution of (1-c). Now if

$$y_i (a_i K) = y_j (a_j K) = 0 \qquad i \neq j ,$$

then

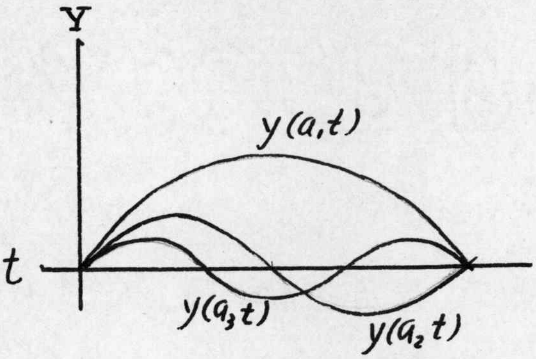
$$\int_0^K \left\{ t^2 (a_i^2 + a_j^2) - m \right\} y_i y_j \, dt = 0.$$

Consider the graph of y (x):



From this it is seen that $y\ (at\ ) = 0$ when $at = r_i$, $i = 1,2,3,$-----. When $t = K$, $aK = r_i$, and $a_i = \frac{r_i}{K}$.

From these results we sketch the graph



We may now write

(5) $$\frac{y_i''}{a_i^2} + (m + a_i^2 t^2)\ y_i = 0,$$

(6) $$\frac{y_J''}{a_J^2} + (m + a_J^2 t^2)\ y_J = 0 .$$

Let us now multiply (5) by y_J and (6) by y_i and subtract (6) from (5). If we integrate the resulting equation between the limits 0 and K we obtain

$$y_J y_i' - y_i y_J' \Big]_0^K + \int_0^K \left[a_i^2(m + a_J^2 t^2) - a_J^2(m + a_J^2 t^2)\right] y_1 y_2\, dt = 0 .$$

The first term of the last equation will be zero at these limits, and we have,

$$\int_0^K \left\{ t^2(a_i^2 + a_J^2) - m \right\} y_i y_J\, dt = 0 ,$$

where

$$y\,(a_i K) = y(a_J K) = 0, \quad i \neq J .$$

V. SOLUTIONS OF TRANSFORMED EQUATION

It is important for some purposes to obtain convergent solutions of (1) for which the general term can explicitly be written. Since it is difficult to write a general term for the power series solutions obtained in the previous section it is desirable to transform the equation into one which will give rise to simpler series coefficients.

Let

$$w = e^{Kz^2} v$$

$$w' = e^{Kz^2}(v' + 2Kzv)$$

$$w'' = e^{Kz^2}[v'' + 2Kzv' + v(2K + 4K^2z^2)].$$

With this transformation (1) becomes

$$v'' + 4Kzv' + (2K + 4K^2z^2 + m + z^2)v = 0. \tag{7}$$

Now to remove the term $[4K^2+1]z^2$ we take

$$4K^2 + 1 = 0$$
$$K = \pm\frac{i}{2}.$$

For $K = \frac{i}{2}$ the equation (7) now becomes

$$v'' + 2izv' + (i+m)v = 0.$$

We now assume a power series solution as before and have

$$v = \sum_{J=0}^{\infty} a_J z^J$$

$$v' = \sum_{J=1}^{\infty} J a_J z^{J-1}$$

$$v'' = \sum_{J=2}^{\infty} J(J-1) a_J z^{J-2}$$

When these values are substituted in (3) there results

$$\sum_{J=2}^{\infty} J(J-1) a_J z^{J-2} + 2i \sum_{J=1}^{\infty} J a_J z^J + (i+m) \sum_{J=0}^{\infty} a_J z^J = 0 .$$

By collecting the coefficients of z^{r-2} and equating to zero we have the recurrence formula

$$a_r = \frac{-m - i(2r-3)}{r(r-1)} a_{r-2} ,$$

Since a_0 and a_1 are arbitrary constants we form now the two power series solutions which converge for all z .

$$v_1 = a_0 \left\{ 1 - \frac{m+i}{2!} z^2 + \frac{(m+i)(m+5i)}{4!} z^4 + \ldots\ldots \right.$$

$$\left. \ldots\ldots \frac{(-1)^n (m+i)(m+5i)\ldots\ldots(m+[4n-3]i)}{2n!} z^{2n} + \ldots\ldots \right\}$$

$$v_2 = a_1 \left\{ z - \frac{m+3i}{3!} z^3 + \frac{(m+3i)(m+7i)}{5!} z^5 + \ldots\ldots \right.$$

$$\left. \ldots\ldots \frac{(-1)^n (m+3i)(m+7i)\ldots\ldots(m+[4n-1]i)}{(2n+1)!} z^{2n+1} + \ldots\ldots \right\}$$

The most general solution is now written

$$v = a_0 v_1 + a_1 v_2 .$$

Two independent solutions of (1) may now be written

$$u_1 = e^{\frac{iz^2}{2}} (1 + a_2 z^2 + \ldots\ldots)$$

(8)

$$u_2 = e^{\frac{iz^2}{2}} (z + b_2 z^3 + \ldots\ldots) .$$

By a well known theorem on homogeneous linear differential equations there exist solutions of (1) $w_1(z)$ and $w_2(z)$, real for real z such that

$$w_1(0) = 1 \qquad w_2(0) = 0$$

$$w_1'(0) = 0 \qquad w_2'(0) = 1$$

Upon differentiation of equations (8) we find that

$$u_1(0) = 1 \qquad . \; u_2(0) = 0$$

$$u_1'(0) = 0 \qquad u_2'(0) = 1.$$

Thus it is evident that u_1 and u_2 are identical with w_1 and w_2 , and, in spite of the complex appearance, they give real values for real values of z . The general solution of (1) is

$$w = a_0 u_1 + a_1 u_2$$

VI. THE RELATION BETWEEN OUR EQUATION AND THE WEBER EQUATION

In (1) make the substitution

$$z = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} \, v = \frac{v}{i-1} .$$

We have now the equation

$$w'' + \left(n + \tfrac{1}{2} - \frac{v^2}{4}\right) w = 0$$

where $n = \frac{mi}{2} - \frac{1}{2}$. This is the well known Weber equation whose solutions are the Weber-Hermite parabolic cylinder functions.* It is obvious that real values for parameter (except $m = 0$) and independent variable in our equation correspond to imaginary values in the Weber equation.

---

*Whittaker and Watson, Modern Analysis, Cambridge University Press, 1933, p.347 § 16.5

VII. THE SOLUTIONS OF (1) ASYMPTOTIC IN THE COMPLEX VARIABLE

A. Introduction

It is often desirable to have solutions of equations such as (1) which are practical to use for large numerical values of the independent variable. In order to emphasize that the independent variable is now complex we denote it by the letter z . It is shown by Fabry* that for (1) there exists a full set** of formal solutions. These solutions have been proved to be asymptotic under certain restrictions.*** The formal solutions are of the form

$$S_i = e^{Q_i(z)} z^{r_i} v_i(z), \qquad i = 1,2$$

where $Q_i(z)$ is a polynomial of form

$$Q_i(z) = \alpha_i z^2 + \beta_i z$$

$$v_i(z) = A_{i0} + \frac{A_{i1}}{z} + \frac{A_{i2}}{z^2} + \cdots,$$

and r_i is a constant.

In general the series $v_i(z)$ do not terminate and do not converge. Trjitzinsky**** has shown that these formal solutions are asymptotic expansions of true solutions and has determined their validity.

*M.E. Fabry, Thèse, University of Paris, 1885

**Here "full set" means two formal solutions such that the Wronskian formed from their formal derivatives does not vanish.

***W.J. Trjitzinsky, _Acta Mathematica_, vol.62: 1-2 1933, pp.167-226

****W.J. Trjitzinsky, loc. cit..

We shall proceed by assuming the above solutions and determining their unknown coefficients. Then the existence of true solutions asymptotic to the formal solutions and the regions in which the asymptotic developments are valid will be considered.

B. The formal series solutions of (1) in descending powers of z.

Assume a solution

$$
\begin{aligned}
S(z) &= z^{r} e^{P(z)}\, v(z) \\
&= e^{\beta z^{2}+\nu z} z^{r}\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^{2}} + \cdots\right)
\end{aligned}
\tag{9}
$$

where β, γ, r, a_0, a_1, a_2, ----, a_n, ----- are constants to be determined. We shall find first the particular solutions for which $a_0 = 1$.

Write

$$S(z) = e^{\beta z^{2}+\gamma z}\sum_{J=0}^{\infty} a_J z^{r-J}$$

Differentiating, we get

$$
\begin{aligned}
S(z) &= (2\beta z+\gamma)e^{\beta z+\gamma z}\sum_{J=0}^{\infty} a_J z^{r-J} + e^{\beta z+\gamma z}\sum_{J=0}^{\infty}(r-J)a_J z^{r-J-1} \\
&= e^{\beta z^{2}+\gamma z}\sum_{J=0}^{\infty}\left(2\beta a_J z^{r-J+1} + a_J\gamma z^{r-J}\right) + e^{\beta z+\gamma z}\sum_{J=0}^{\infty}(r-J)a_J z^{r-J-1}
\end{aligned}
$$

$$
\begin{aligned}
S''(z) &= (2\beta z+\gamma)e^{\beta z+\gamma z}\sum_{J=0}^{\infty}\left(2\beta a_J z^{r-J+1} + a_J\gamma z^{r-J}\right) \\
&\quad + e^{\beta z^{2}+\gamma z}\sum_{J=0}^{\infty}\left[2\beta a_J(r-J+1)z^{r-J} + a_J\gamma(r-J)z^{r-J-1}\right]
\end{aligned}
$$

$$+e^{\beta z^2+\gamma z}\sum_{J=0}^{\infty}\left[(r-J)(r-J-1)a_J z^{r-J-2}\right]$$

$$+(2\beta z+\gamma)e^{\beta z^2+\gamma z}\sum_{J=0}^{\infty}\left[(r-J)a_J z^{r-J-1}\right]$$

or

$$S''(z)=e^{\beta z^2+\gamma z}\Big[\sum_{J=0}^{\infty}\left(4\beta^2 a_J z^{r-J+2}+4\beta\gamma a_J z^{r-J+1}+\gamma^2 a_J z^{r-J}\right)\Big]$$

$$+\sum_{J=0}^{\infty}(r-J)(r-J-1)a_J z^{r-J-2}$$

$$+\sum_{J=0}^{\infty}\left[2\beta a_J(r-J+1)z^{r-J}+a_J\gamma(r-J)z^{r-J-1}\right]$$

$$+\sum_{J=0}^{\infty}\left[2a_J\beta(r-J)z^{r-J}+a_J\gamma(r-J)z^{r-J-1}\right]$$

$$(m+z^2)S(z)=e^{\beta z^2+\gamma z}\Big[\sum_{J=0}^{\infty}(ma_J z^{r-J})+\sum_{J=0}^{\infty}a_J z^{r-J+2}\Big]$$

The above values may be substituted in (1) and we have

$$e^{P(z)}\Big\{(4\beta^2+1)z^{r+2}+(4\beta^2 a_1+4\beta\gamma a_1)z^{r+1}+\big[4\beta^2 a_2 \tag{10}$$

$$+4\beta\gamma a_1+2\beta(r+1)+2\beta r+m+a_2\big]z^r+\ldots\ldots\Big\}=0.$$

Equating the coefficients of $e^{P(z)}z^K$ to zero we find

$K=r+2$: $\beta=\pm^{*}\frac{i}{2}$

$K=r+1$: $\gamma=0$

$K=r$: $r=\pm^{*}\frac{mi}{2}-\frac{1}{2}$

$K=r-1$: $a_1=0$

---

*These are corresponding notations, that is, when $\beta=+\frac{i}{2}$, $r=+\frac{mi}{2}-\frac{1}{2}$ etc..

To determine the coefficients a_1, a_2, a_3, -----, the most efficient procedure is to derive a recurrence formula. This is done below. If the coefficient of $e^{P(z)} z^{r-K}$ in (10) is equated to zero, we have

$$2\beta a_K (r-K+1) + (r-K+1)(r-K+2) a_{K-2}$$
$$+ 2 a_K \beta (r-K) + m a_K = 0.$$

Solving this equation for a_K we have

$$a_K = \frac{-(r-K+1)(r-K+2)}{2\beta(2r-2K+1)+m} a_{K-2}, \quad K>1$$

which gives an expression for the value of each coefficient in terms of a preceding one.

When the values of r and β are substituted in (9) the equation may further be reduced to

(11) $$a_K = \frac{\left(\pm \frac{mi}{2} - K + \frac{1}{2}\right)\left(\pm \frac{mi}{2} - K + \frac{3}{2}\right)}{\pm 2Ki} a_{K-2}$$

Since we have shown $a_1 = 0$ it follows that all a's with odd subscripts will be zero, since for these by (11) a_1 will always appear as a factor in the recurrence formula. The coefficients a_2, a_4, a_6, ----, may now be calculated. A few of these are given below:

$$a_2 = \frac{\frac{1}{4}(\pm mi-3)(\pm mi-1)}{\pm 4i}$$

$$a_4 = \frac{\frac{1}{4^2}(\pm mi-7)(\pm mi-5)(\pm mi-3)(\pm mi-1)}{-32}$$

$$a_6 = \frac{\frac{1}{4^3}(\pm mi-11)(\pm mi-9)(\pm mi-7)(\pm mi-5)(\pm mi-3)(\pm mi-1)}{\pm 384 i}$$

(12)

....

$$a_n = \frac{\frac{1}{4^{\frac{n}{2}}}(\pm mi-[2n-1])(\pm mi-[2n-3])\cdots\cdots(\pm mi-1)}{\pm 2ni\,[\pm(2n-4)i]\,[\pm(2n-8)i]\cdots\cdots\pm 4i}$$

When the values of these coefficients are substituted back in (9) there results the following expression for $S\,(z)$:

$$\text{(13)}\qquad S_J(z) = e^{\frac{\pm iz^2}{2}}\, z^{\frac{\pm mi}{2}-\frac{1}{2}}\left(1+\frac{a_{J2}}{z^2}+\frac{a_{J4}}{z^4}+\cdots\cdots\right), \qquad J=1,2$$

where the value of a_{JK} is given by the formula (12) for a_K. For $J=1$, the upper sign of the symbol ($\pm$) is to be used in both (12) and (13), and for $J=2$, the lower sign. Since the above gives two distinct formal solutions we may write them as

$$\text{(14)}\qquad \begin{aligned} S_1 &\equiv e^{\frac{iz^2}{2}}\, z^{\frac{mi}{2}-\frac{1}{2}}\left(1+\frac{a_{12}}{z^2}+\frac{a_{14}}{z^4}+\cdots\cdots\right) \\ S_2 &\equiv e^{\frac{-iz^2}{2}}\, z^{\frac{-mi}{2}-\frac{1}{2}}\left(1+\frac{a_{22}}{z^2}+\frac{a_{24}}{z^4}+\cdots\cdot\right). \end{aligned}$$

If z and m are real it is desirable to change the form of these expressions as follows: For z real

$$\arg z = K\pi, \qquad K = 0, \pm 1, \pm 2, \cdots\cdots$$

Now

$$\begin{aligned} e^{\pm iz^2}\, z^{\pm\frac{mi}{2}-\frac{1}{2}} &= e^{\frac{\pm iz^2}{2}}\, e^{\pm\frac{mi}{2}\log z}\, z^{\frac{1}{2}} \\ &= e^{i(\pm\frac{z^2}{2}\pm m\log|z|\pm\frac{m}{2}i\arg z)}\left(|z|e^{i\arg z}\right)^{\frac{1}{2}} \\ &= e^{\pm\frac{m}{2}\arg z}\, e^{\pm i(\frac{z^2}{2}+\frac{m}{2}\log|z|)}\,|z|^{\frac{1}{2}} e^{\frac{1}{2}\arg z} \\ &= Ce^{\pm i(\frac{z^2}{2}+\frac{m}{2}\log|z|)}\,|z|^{\frac{1}{2}} \end{aligned}$$

It is possible to separate the real and imaginary parts of the last expression by writing

$$e^{\pm iz^2} z^{\pm\frac{mi}{2}-\frac{1}{2}} = C|z|^{\frac{1}{2}}\left\{\cos\left[\pm\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)\right]+i\sin\left[\pm\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)\right]\right\}$$

$$= C|z|^{\frac{1}{2}}\left\{\cos\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)\pm i\sin\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)\right\}.$$

The equations (14) may now be written for real m and z as

$$S_1 = |z|^{\frac{1}{2}}\left[\cos\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)+i\sin\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)\right]\left[v_1(z)\right],$$

$$S_2 = |z|^{\frac{1}{2}}\left[\cos\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)-i\sin\left(\frac{z^2}{2}+\frac{m}{2}\log|z|\right)\right]\left[v_2(z)\right].$$

where the constant factor introduced by the multivaluedness of log z has been dropped.

By a linear combination of S_1 and S_2 we may obtain formal solutions free from i. Thus if we denote by $\hat{S}_J$ the leading term of S_J, (set $v_J \equiv 1$),

$$\hat{G}_1 = \frac{\hat{S}_1+\hat{S}_2}{2} = |z|^{-\frac{1}{2}}\left\{\cos\left(\frac{z^2}{2}+\log|z|^{\frac{m}{2}}\right)\right\},$$

(15)

$$\hat{G}_2 = \frac{\hat{S}_1+\hat{S}_2}{2} = |z|^{-\frac{1}{2}}\left\{\sin\left(\frac{z^2}{2}+\log|z|^{\frac{m}{2}}\right)\right\}.$$

When all the terms of $v_i(z)$ are considered these combinations will still be free from i. We then call G_1 and G_2 the complete formal solutions defined analogously to (15).

Let

$$p = \left[\frac{z^2}{2}+\log|z|^{\frac{m}{2}}\right].$$

We then get the following:

$$G_1 = |z|^{-\frac{1}{2}} \left\{ \cos p + \left[\frac{m^2 \sin p - 4m \cos p + 3 \sin p}{4z} \right] + \cdots \right\}$$

$$G_2 = |z|^{-\frac{1}{2}} \left\{ \sin p + \left[\frac{m^2 \cos p - 4m \sin p - 3 \cos p}{4z} \right] + \cdots \right\}.$$

For most practical purposes when z and m are real these last formulas are better suited for computation since they involve only real quantities. The most general formal solution for real z and m can hence be written

$$G(z) = C_1 G_1(z) + C_2 G_2(z).$$

C. Existence and validity of asymptotic solutions.

Consider the formal solutions just derived

$$S_1 = e^{Q_1(z)} z^{r_1} (1 + \cdots\cdots),$$

$$S_2 = e^{Q_2(z)} z^{r_2} (1 + \cdots\cdot),$$

where $Q = \frac{iz^2}{2}$, $Q_2 = \frac{-iz^2}{2}$, and $r_1 = \frac{mi}{2} - \frac{1}{2}$.

In order to apply the theory of Trjitzinsky it is necessary to consider regions R_i * bounded by Q-curves which are defined as follows:

A Q-curve is one on which

$$\mathcal{R}\,[Q_{iJ}(z)] = 0$$

where

$$Q_{iJ} \equiv Q_i - Q_J\,, \quad i, J = 1, 2, \quad i \neq J$$

and $\mathcal{R}[f(z)$ represents the real part of $f(z)$.

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* Trjitzinsky, loc. cit., p.171 § 2

Now $$\mathcal{R}[Q_{12}] = \mathcal{R}[iz^2]$$

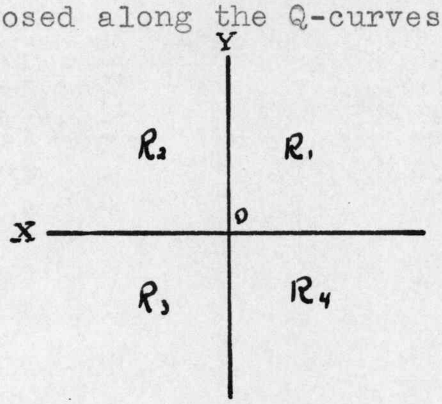
Let $$z = x + iy^{*} \;;$$

then $$z^2 = x^2 - y^2 + 2i\,xy$$

$$iz^2 = i\,(x^2 + y^2) - 2\,xy$$

Finally $$\mathcal{R}[Q_{12}] = -\mathcal{R}[Q_{21}] = \mathcal{R}[iz^2] = -2xy.$$

We must now consider along what curves the equation $2xy = 0$ is satisfied. These are obviously the axes $x = 0$ and $y = 0$. These are hence our Q-curves, and they divide the z-plane into four regions R_i as shown in the diagram. They are closed along the Q-curves but open at ∞.



Trjitzinsky states further** that we must take into account the possibility of a region R_i where one of the differences $Q_i(z) - Q_J(z)$ has a non negative real part and is such that for some $\beta > 0$

(16) $$|z|^{-\beta} e^{\mathcal{R}[Q_{iJ}(z)]} \longrightarrow 0$$

as $z \longrightarrow \infty$ along both boundaries of R_i.

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*The x here is, of course, not to be confused with the x of equation (1).

**Trjitzinsky, loc. cit., p.180

We note that in every one of the four regions R_i bounded by our Q-curves there will be at least one difference $Q_i - Q_j$ which has a non negative real part. In fact, for the entire plane we have the following: In both R_1 and R_3, $\mathcal{R}[Q_{12}]$ is negative and $\mathcal{R}[Q_{21}]$ is positive. In both R_2 and R_4, $\mathcal{R}[Q_{12}]$ is positive and $\mathcal{R}[Q]$ is negative.

Specifically then, we must consider the possibility of relation (16) for

$$\mathcal{R}[Q_{21}] \equiv \quad 2xy \quad \text{in } R_1 \text{ and } R_3,$$

$$\mathcal{R}[Q_{12}] \equiv \quad -2xy \quad \text{in } R_2 \text{ and } R_4,$$

and henceforth in referring back to the relation (16) it will be understood that in any region we are considering it <u>only for the Q_{ij} mentioned above</u>.

In the regions R_1, R_3 the left member of (16) is

(16-a) $$|z|^{-\beta} e^{2xy},$$

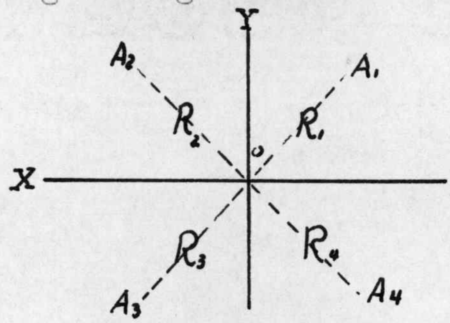
and in the regions R_2, R_4 it is

(16-b) $$|z|^{\beta} e^{-2xy}.$$

Evidently these expressions approach zero along both boundaries of their respective regions, $x = 0$ or $y = 0$.

When this happens along both boundaries of R_i it is necessary to subdivide R_i into two subregions R_i', R_i'' each with one boundary in common with R_i and another boundary dividing R_i' from R_i'' and interior to R_i such that along it all the left members of (16) increase indefinitely for every $\beta > 0$.

Take the 45° lines for such boundaries interior to R_i We have the following configuration:



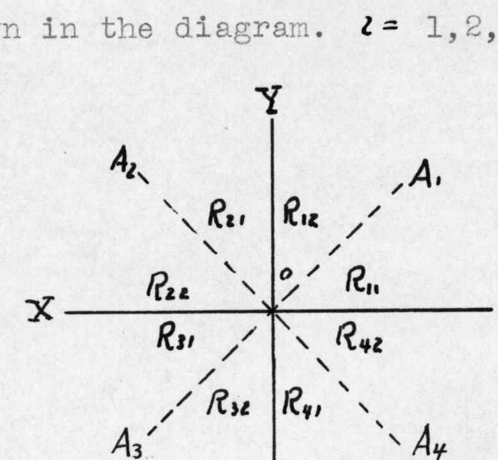
Along the new boundaries OA_i, consider the appropriate left members of (16). From (16-a) and (16-b) we see these are in every case

$$|z|^{-\beta} e^{2|xy|}.$$

Along the new boundaries this becomes

$$|z|^{-\beta} e^{|z|^2}.$$

Now along any of these lines $|z|^{-\beta} e^{|z|^2}$ increases indefinitely since in the limit $|z|^{-\beta}$ is negligible compared to $e^{|z|^2}$. Since the conditions are satisfied along these boundaries we now have the regions R_i divided into subregions R_{ij} as shown in the diagram. $i = 1,2,3,4,\ \ j = 1,2,$



From the fundamental existence theorem given by Trjitzinsky* we may now state the following:

In any region R_{ij}, $i = 1,2,3,4$, $j = 1,2$, true solutions ${}_{ij}w_1$, ${}_{ij}w_2$ of the equation (1) exist such that

$$ {}_{ij}w_1 \sim S_1 , \qquad {}_{ij}w_1' \sim S_1' $$

$$ {}_{ij}w_2 \sim S_2 , \qquad {}_{ij}w_2' \sim S_2' $$

for all values of z in R_{ij}. Here S_i' indicates the series obtained by formally differentiating the series S_i.

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*Trjitzinsky, loc. cit., p.208 § 7

VIII. THE SOLUTIONS OF (1) ASYMPTOTIC IN THE PARAMETER

A. Introduction

In a paper by Birkhoff* the asymptotic character of the solutions of

$$\frac{d^n z}{dx^n} + \rho a_{n-1}(x,\rho)\frac{d^{n-1} z}{dx^{n-1}} + \rho^2 a_{n-2}(x,\rho)\frac{d^{n-2} z}{dx^{n-2}} + \cdots + \rho^n a_0(x,\rho) z = 0$$

is discussed for large values of ρ. The coefficients $a_i(x,\rho)$ are assumed to be analytic in the complex parameter ρ at $\rho = \infty$ and to have derivatives of all orders in the real variable x in some interval $a \leq x \leq b$.

The equation (1) may be put into this type form by letting $m = \rho^2$. It then appears as

(1') $$w'' + \rho^2\left\{1 + \frac{x^2}{\rho^2}\right\}w = 0.$$

For (1') the characteristic equation** defined in the paper becomes

$$\alpha^2 + 1 = 0$$

This equation has the roots $-i$ and i which will be denoted by α_1 and α_2. It is convenient to postpone the decision as to which is α_1 and which is α_2 until later.***

Formal solutions of the form

(17) $$\sigma_i = e^{\rho\alpha_i x}\left\{\sum_{j=0}^{\infty} u_{ij}(x)\rho^{-j}\right\} \qquad i = 1, 2,$$

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*Transactions American Mathematical Society, vol.9, 1908, pp.219-232

**Trans. Am. Math. Soc., vol.9, 1908, p.220, equation 4

***Considered in part C, § VIII

are then shown to exist.

B. The formal solutions.

To determine the $u_{ij}(x)$ of (17) we assume that σ_i is a solution of (1'). Then

$$\sigma_i' = e^{\rho\alpha_i x}\sum_{j=0}^{\infty} u_{ij}'(x)\rho^{-j} + \rho\alpha_i e^{\rho\alpha_i x}\sum_{j=0}^{\infty} u_{ij}(x)\rho^{-j}.$$

$$\sigma_i'' = e^{\rho\alpha_i x}\Big\{\sum_{j=0}^{\infty} u_{ij}''(x)\rho^{-j} + 2\alpha_i\sum_{j=0}^{\infty} u_{ij}'(x)\rho^{-j+1},$$

$$-\sum_{j=0}^{\infty} u_{ij}(x)\rho^{-j+2}\Big\},$$

$$\rho^2\sigma_i = e^{\rho\alpha_i x}\sum_{j=0}^{\infty} u_{ij}\rho^{-j+2},$$

$$x^2\sigma_i = e^{\rho\alpha_i x}\sum_{j=0}^{\infty} u_{ij}\rho^{-j}.$$

When these values are substituted back in (1') we obtain the equation

$$e^{\rho\alpha_i x}\Big\{\sum_{j=0}^{\infty} u_{ij}''(x)\rho^{-j} + 2\alpha_i\sum_{j=0}^{\infty} u_{ij}'(x)\rho^{-j+1} + x^2\sum_{j=0}^{\infty} u_{ij}\rho^{-j}\Big\} = 0. \tag{18}$$

We now collect the coefficients of $e^{\rho\alpha_i x}\rho^{k}$ in (18) for each k:

$$u_{i0}'(x) = 0.$$

By integration

$$u_{i0}(x) = C.$$

This constant C may be taken as 1 since it is arbitrary.

The next equation becomes

$$u_{i0}'' + 2\alpha_i u_{i1}' + x^2 u_{i0} = 0$$

where it is understood that the u's are functions of x.

Since $u_{io} = 0$ and $u_{io} = 1$ there results

$$2\alpha_i u_{i1}' = -x^2 \quad , \qquad \text{or}$$

$$u_{i1} = \int^x -\frac{x^2}{2\alpha_i}\,dx = \frac{-x^3}{6\alpha_i}\ .$$

We take the constant of integration involved in the determination of u_{iJ} , $J > 0$, so that u_{iJ} (0) = 0. If we used any functions of ρ , f_{2J} (ρ), as constants of integration we could obtain other asymptotic series which are less convenient. Likewise the process may be repeated for other coefficients, but it is simpler to develop a recurrence formula. We have

$$2\alpha_i u_{i,K+1}' + u_{iK}'' + x^2 u_{iK} = 0.$$

From this

$$u_{i,K+1} = \frac{-1}{2\alpha_i}\int^x (u_{iK}'' + x^2 u_{iK})\,dx,$$

or

$$u_{i,K+1} = -\frac{1}{2\alpha_i}\left[u_{iK}' + \int^x x^2 u_{iK}\,dx\right].$$

By use of this formula a number of the coefficients have been obtained explicitly and are given below.

$$u_{io} = 1$$

(19) $$u_{i1} = -\frac{1}{2\alpha_i}\cdot\frac{x^3}{3}$$

$$u_{i2} = \left(\frac{1}{2\alpha_i}\right)^2\left[x^2 + \frac{x^6}{18}\right]$$

$$u_{i3} = -\left(\frac{1}{2\alpha_i}\right)^3\left[2x + \frac{8x^5}{15} + \frac{x^9}{162}\right]$$

$$u_{i4} = \left(\frac{1}{2\alpha_i}\right)^4 \left[\frac{19x^4}{6} + \frac{11x^8}{90} + \frac{x^{12}}{1944} \right]$$

$$u_{i5} = -\left(\frac{1}{2\alpha_i}\right)^5 \left[\frac{40x^3}{3} + \frac{2703x^7}{1855} + \frac{2772x^{11}}{1472580} + \frac{x^{15}}{29168} \right]$$

When these coefficients are substituted back in (17) we have

$$\sigma_i = e^{\rho\alpha_i x}\Big\{1 - \frac{1}{2\alpha_i\rho}\left(\frac{x^3}{3}\right) + \left(\frac{1}{2\alpha_i\rho}\right)^2\left[\frac{x^2}{1} + \frac{x^6}{18}\right] - \left(\frac{1}{2\alpha_i\rho}\right)^3\left[2x + \frac{8x^5}{15} + \frac{x^9}{162} + \cdots\right]\Big\}.$$

The derivative series σ_i' are of importance, and they may be expressed upon differentiation of σ_i as follows:

$$\sigma_i' = e^{\rho\alpha_i x}\Big\{\alpha_i\rho + \left(\frac{1}{2}\cdot\frac{x^3}{3}\right) + \frac{1}{2\alpha_i\rho}\left(-\frac{x^2}{2} + \frac{x^6}{36}\right) + \left(\frac{1}{2\alpha_i\rho}\right)^2\left(x + \frac{x^5}{15} - \frac{x^9}{324}\right) + \left(\frac{1}{2\alpha_i\rho}\right)^3\left(-2 - \frac{13}{12}x^4 + \frac{x^8}{90} - \frac{x^{12}}{3888}\right) + \ldots\Big\}.$$

It is possible to make linear combinations of the formal solutions just obtained such that the new form will be real for real m and x. In the original equation (1) we must consider two cases. When m is real and positive ρ will be real, and we have

$$W_1 = \left[\cos\rho x - i\ \sin\rho x\right]\left[\sum_{J=0}^{\infty} u_{iJ}(x)\rho^{-J}\right],$$

$$W_2 = \left[\cos\rho x + i\ \sin\rho x\right]\left[\sum_{J=0}^{\infty} u_{iJ}(x)\rho^{-J}\right].$$

The following combinations give the desired result.

$$H_1 = W_1 + W_2 = 2\Big[\cos\rho x - \frac{x^3}{6\rho}\sin\rho x - \frac{x^3 + \frac{x^6}{18}}{4\rho^2}\quad \cos\rho x.$$

$$+ \frac{2x + \frac{8x^5}{15} + \frac{x^9}{162}}{2^3\rho^3} \quad \sin\rho x + \dots \Big]$$

(20)

$$H_2 = \frac{W_2 - W_1}{i} = 2\Big[\sin\rho x + \frac{x^3}{6} \quad \cos\rho x - \frac{x^3 + \frac{x^6}{18}}{2^2\rho^2} \quad \sin\rho x$$

$$- \frac{2x + \frac{8x^5}{15} + \frac{x^9}{162}}{2^3\rho^3} \quad \cos\rho x + \dots \Big]$$

It is seen that in the solutions (20) the signs alternate in pairs, thus

(20-a)
$$H_1 = + - - + - - + - -$$
$$H_2 = + + - - + + - - + .$$

The coefficient of $\frac{\sin\rho x}{\rho^J}$ or $\frac{\cos\rho x}{\rho^J}$ is the same as the term u_{iJ}, except for the possible factor of (-1), of (19) with α_i replaced by unity. The ($\pm$) signs can then be inserted from (20-a). Since these coefficients are obtained from the recurrence formula we may write as many terms of (20) as needed.

When m is real and negative ρ may be taken in the form $|\rho|\, i$, and the original formal solutions will be real for real m and x.

In this case we have

$$\hat{\sigma}_1 = e^{|\rho|x}\Big\{1 - \frac{1}{2\rho}\Big[\frac{x^3}{3}\Big] + \frac{1}{2^2\rho^2}\Big[x^2 + \frac{x^6}{18}\Big] - \frac{1}{2^3\rho^3}\Big[2x + \frac{8x^5}{15} + \frac{x^9}{162}\Big] + \dots\Big\},$$

$$\hat{\sigma}_2 = e^{|\rho|x}\Big\{1 + \frac{1}{2\rho}\Big[\frac{x^3}{3}\Big] + \frac{1}{2^2\rho^2}\Big[x^2 + \frac{x^6}{18}\Big] - \frac{1}{2^3\rho^3}\Big[2x + \frac{8x^5}{15} + \frac{x^9}{162}\Big] + \dots\Big\}.$$

C. Existence and validity of asymptotic solutions.

For x confined to the real axis we shall define the region S of the ρ plane as one in which the indices 1 and 2 may be so arranged that

$$\mathcal{R}[\rho\alpha_1 x] \leqq \mathcal{R}[\rho\alpha_2 x]$$

for ρ in S. Here $\mathcal{R}[u]$ represents the real part of the complex number u.

Let $\alpha_1 = -i$ and $\alpha_2 = i$. The corresponding region S will then be one such that

$$\mathcal{R}[\rho(-ix)] \leqq \mathcal{R}[\rho(ix)].$$

Since ρ is a complex parameter let

$$\rho = \rho_x + i\rho_y$$

where ρ_x and ρ_y are real. We have

$$\mathcal{R}[i\rho_x + \rho_y] \leqq \mathcal{R}[i\rho_x - \rho_y],$$

$$\rho_y \leqq -\rho_y, \qquad \text{or}$$

$$\rho_y \leqq 0.$$

Thus the S corresponding to the above values of α is the lower half of the complex ρ plane.

Now if we let $\alpha_1 = i$ and $\alpha_2 = -i$ and proceed similarly to determine the corresponding region S, we find that this is the upper half of the ρ plane.

It is now possible to define the region S_τ of the plane as that in which

$$\tau\pi \leqq \arg\rho \leqq (\tau+1)\pi, \qquad \tau = 0, \pm 1, \pm 2, \ldots .$$

Let x be confined to a finite portion of the real axis including the point $x = 0$ as interior point. Define this by $-b \leqq x \leqq b$. It is now possible to state the

Theorem*- For x on $(-b, b)$ and ρ in S_T there exist true solutions, ${}_T y_i$, $i = 1,2$, of the differential equation (1') such that

$$ {}_T y_i(\rho, x) = u_i(\rho, x) + e^{\alpha_i \rho x} E_{0i}^{\rho - m} , $$

$$ {}_T y_i'(\rho, x) = u_i'(\rho, x) + e^{\alpha_i \rho x} E_{1i}^{\rho - m + 1} , $$

where

$$ u_i(\rho, x) = e^{\alpha_i \rho x} \sum_{J=0}^{m-1} u_{iJ}(x) \rho^{-J} , $$

and the E 's are bounded functions for large values of in S_T and x on the interval $(-b, b)$.

*Birkhoff, loc. cit., p. 225, 22b

IX. ASYMPTOTIC CONNECTIONS

A. Relations of coefficients for solutions asymptotic in z.

The formal solutions of form

$$S_J(z) = e^{\pm\frac{iz^2}{2}} z^{\pm\frac{mi}{2}-\frac{1}{2}} v_J(z)$$

are not in general single valued due to the factor $z^{\pm\frac{mi}{2}-\frac{1}{2}}$. Any true solution w is single valued since it is a linear combination of the single valued functions w_1 and w_2 discussed in Chapter II.

In order to consider the formal solutions, $S_J(z)$, as single valued we study them on the Riemann surface on which arg z is single valued. This will consist of an infinite number of sheets with branch point at the origin. The regions of validity are extended so that a diagram like that of page 24 is now visualized on each sheet of the Riemann surface. There will then be infinitely many regions R_{iJ} numbered as follows:

$$R_{i1}:\ (i-1)\frac{\pi}{2} \leq \arg z \leq (i-\tfrac{1}{2})\frac{\pi}{2}$$

$$i = 0, \pm 1, \pm 2,$$

$$R_{i2}\ \ (i-1)\frac{\pi}{2} \leq \arg z \leq (i-\tfrac{1}{2})\frac{\pi}{2}$$

First let us fix a particular solution w by means of boundary conditions. This is then a linear combination of any pair of true solutions, ${}_{iJ}W_1$, and ${}_{iJ}W_2$:

(21) $$w = {}_{iJ}C_1\ {}_{iJ}W_1 + {}_{iJ}C_2 + {}_{iJ}W_2 .$$

From the fundamental theorem of Trjitzinsky we may also write $$w \sim {}_{iJ}C_1 S_1 + {}_{iJ}C_2 S_2$$

in R_{iJ},

$$, J = 1,2, \quad i = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$$

Now the value of the formal solution S_J in sheet 1 of the Riemann surface will not be the same as the value in sheet 2. Since W is single valued, it is therefore evident that the coefficients, C, must change somewhere in sheet 1, and this fact is known as the Stokes' phenomenon.

To study the changes in the coefficients we shall note what happens as z is taken across the boundaries OA_K: $\arg z = (K - \frac{1}{2})\frac{\pi}{2}$, $K = 0, \pm 1, \pm 2, \ldots\ldots$, and the X and Y axes.

At this point it is convenient to define the following nomenclature: We shall say the formal solution S_α is "dominant" over the formal solution S_β, in the region R_i if

$$\mathcal{R}[Q_\alpha] > \mathcal{R}[Q_\beta], \alpha, \beta = 1, 2, \alpha \neq \beta \text{ in interior of } R_i.$$

We may without any resulting confusion apply the word "dominant" to the corresponding exponential factors in the sequel. From the relationships

$$\mathcal{R}[Q_2] = \frac{|z|^2}{2} \sin 2\theta = -\mathcal{R}[Q_1],$$

where $\theta \equiv \arg z$, we see that in the regions

S_1 is dominant if i is even,

S_2 is dominant if i is odd.

We now proceed to study the behavior of the coefficients, C, as a curve OA is crossed:

On OA_K $\qquad z = |z| e^{(\frac{2K-1}{4})\pi i}$,

$$z^2 = |z|^2 e^{(\frac{2K-1}{4})\pi i},$$

$$= (-1)^{K+1} \, i |z|^2 .$$

Thus

$$Q_1 = \frac{i}{2}(-1)^{K+1}(iz^2) = (-1)^K \frac{|z|^2}{2}$$

$$Q_2 = (-1)^{K+1} \frac{|z|^2}{2}$$

In the limit as $z \to \infty$ on OA_K, $K =$ odd integer, $e^{Q_1} \to 0$, $e^{Q_2} \to \infty$; e^{Q_2} is dominant over e^{Q_1}. Likewise on OA_K, $K =$ even integer, e^{Q_1} is dominant over e^{Q_2}.

Since the regions R_{iJ} are closed along their boundaries, we may assert both the relations

$$w \sim {}_{11}C_1 S_1 + {}_{11}C_2 S_2 ,$$

$$w \sim {}_{12}C_1 S_1 + {}_{12}C_2 S_2 ,$$

along OA_1, the common boundary of R_{11} and R_{12}. From the definition of an asymptotic expansion it follows that

$$\begin{aligned} & {}_{11}C_1 e^{Q_1} z^{r_1}\left\{1 + {}_{11}\eta_1(z)\right\} + {}_{11}C_2 e^{Q_2} z^{r_2}\left\{1 + {}_{11}\eta_2(z)\right\} \\ (23) \qquad & = {}_{12}C_1 e^{Q_1} z^{r_1}\left\{1 + {}_{12}\eta_1(z)\right\} + {}_{12}C_2 e^{Q_2} z^{r_2}\left\{1 + {}_{12}\eta_2(z)\right\} \end{aligned}$$

where all the $\eta(z) \to 0$ as $z \to \infty$ along OA_1. Dividing both sides of (23) by $e^{Q_2} z^{r_2}$ and taking the limit as $z \to \infty$ on OA_1 we have finally

$${}_{11}C_2 = {}_{12}C_2 .$$

Thus the coefficient of the dominant solution S_2 does not change in R_1. This proof may be extended to show that the coefficient of the dominant solution does not change on any curve OA.

If the coefficient of the dominant solution is equal to zero in R_1 equation (23) becomes

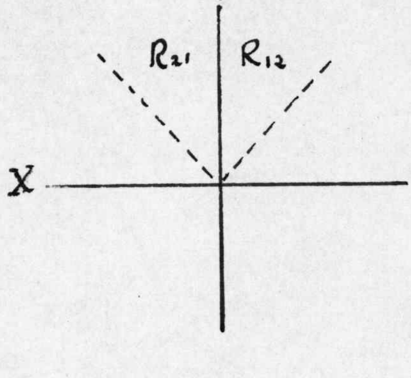
$$ {}_{11}C_1 e^{Q_1} z^{r_1}\{1+{}_{11}\eta(z)\} = {}_{12}C_1 e^{Q_1} z^{r_1}\{1+{}_{12}\eta_1(z)\} $$

where the $\eta(z) \to 0$ as $|z| \to \infty$ along OA_1.
Dividing both sides by $e^{Q_1} z^{r_1}$ and taking the limit as $z \to \infty$ on OA, we have

$$ {}_{11}C_1 = {}_{12}C_1 , $$

which shows that the coefficient of the sub-dominant solution does not change on OA_1. This result may be extended, and we may say that in any region R_i, where the coefficient of the dominant solution is equal to zero, the coefficient of the sub-dominant solution does not change on the curve OA_i.

We shall now determine whether or not the coefficients C_i change on the boundaries of R_i. Consider the diagram



Here it will be necessary to discuss only the case when z is taken across OY since the behavior on the other axes is similar. We write

$$w \sim {}_{12}C_1 S_1 + {}_{12}C_2 S_2 \qquad \text{in } R_{12},$$

$$w \sim {}_{21}C_1 S_1 + {}_{21}C_2 S_2 \qquad \text{in } R_{21}.$$

In other notation we have:

(24) $w = {}_{12}C_1 e^{Q_1} z^{r_1}\left\{1 + {}_{12}\eta_1(z)\right\} + {}_{12}C_2 e^{Q_2} z^{r_2}\left\{1 + {}_{12}\eta_2(z)\right\}$ in R_{12},

(25) $w = {}_{21}C_1 e^{Q_1} z^{r_1}\left\{1 + {}_{21}\eta(z)\right\} + {}_{21}C_2 e^{Q_2} z^{r_2}\left\{1 + {}_{21}\eta_2(z)\right\}$ in R_{21},

where all η's $\to 0$ as $z \to \infty$ on the line $\arg z = \frac{\pi}{2}$.

On the boundary OY we may equate the two solutions (24) and (25) above since it is common to both R_{12} and R_{21}. Also on this boundary $\mathcal{R}[Q_1] = \mathcal{R}[Q_2]$ by the definition of a Q-curve. When we do this the result gives us the equation

(26) $${}_{12}C_1 z^{r_1 - r_2} e^{iz^2}\left\{1 + \eta_1(z)\right\} + {}_{12}C_2\left\{1 + {}_{12}\eta_2(z)\right\}$$

$$= {}_{21}C_1 z^{r_1 - r_2} e^{iz^2}\left\{1 + {}_{21}\eta_1(z)\right\} + {}_{21}C_2\left\{1 + {}_{21}\eta(z)\right\}.$$

Next, let us substitute the values of r_1 and r_2.

Here
$$r_1 - r_2 = \left(\frac{mi}{2} - \frac{1}{2}\right) - \left(-\frac{mi}{2} - \frac{1}{2}\right),$$

$$= mi.$$

Since m is complex we may write

$$r_1 - r_2 = (m_x + i m_y) i$$

$$= -m_y + i m_x.$$

Now
$$z^{-m_y + i m_x} = e^{(\log z(-m_y + i m_x)} = e^{(\log|z| + i \arg z)(-m_y + i m_x)}$$

$$= e^{-m_y \log|z| - m_x \arg z} \; e^{i(m_x \log|z| - m_y \arg z)}.$$

On or

(27) $$K e^{-m_y \log|z|} \; e^{i m_x \log|z|},$$

since $e^{-m_x \arg z - i m_y \arg z}$ is a constant, K, different from zero.

When the value of $z^{r_1 - r_2}$ as given in (27) is substituted in (26), that equation becomes

(28) $$K e^{-m_y \log|z|} \; e^{i(m_x \log|z| + z^2)} \left[{}_{12}C_1 \left\{ 1 + {}_{12}\eta_1(z) \right\} - {}_{21}C_1 \left\{ 1 + {}_{21}\eta_1(z) \right\} \right] + \left[{}_{12}C_2 \left\{ 1 + {}_{12}\eta_2(z) \right\} - {}_{21}C_2 \left\{ 1 + {}_{21}\eta_2(z) \right\} \right] = 0.$$

The limit of equation (28) as $z \to \infty$ gives us

(29) $$\lim_{z \to \infty} \left[K e^{-m_y \log|z|} \; e^{i(m_x \log|z| + z^2)} A + B \right] = 0$$

where

$$A = {}_{12}C_1 - {}_{21}C_1 ,$$

$$B = {}_{12}C_2 - {}_{21}C_2 .$$

Now we desire to show that $A = B = 0$. Consider Case I in which $m_y \neq 0$. If $m_y < 0$, (29) becomes in effect

$$K \cdot \infty A + B = 0.$$

Therefore $A = 0$ and hence $B = 0$. If $m_y > 0$, we have in effect

$$K \cdot 0 \cdot A + B = 0,$$

or $B = 0$, and substituting this back in (29) we have also $A = 0$.

In Case II, $m_y = 0$. Equation (29) becomes

$$\lim_{z \to \infty} \left[K e^{i(m_x \log|z| + z^2)} A + B \right] = 0.$$

Now let us assume $A = 0$, then $B = 0$. If $A \neq 0$, we have

$$\lim_{z \to \infty} Ke^{i(m_x \log|z| + z^2)} = \frac{-B}{A},$$

but this equation is absurd since the left side represents a vector whose argument is increasing indefinitely.

Thus it is shown that $A = 0$ and $B = 0$ in each case and this gives

$$_{12}C_1 = {}_{21}C_1,$$
$$_{12}C_2 = {}_{21}C_2,$$

on the Q-curve $o\gamma$. This same process may be applied to any Q-curve, and it gives the result that the coefficients do not change when z is taken across these boundaries.

Let us now write the asymptotic solution simply

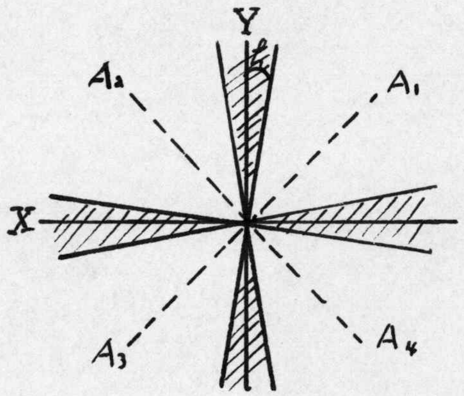
$$w \sim C_1 S_1 + C_2 S_2$$

We have just shown that as z is taken around the origin neither coefficient can change on a Q-curve. Furthermore, the coefficient of the dominant solution will not change on any of the boundaries OA_i. The coefficient of the subdominant solution may change on these boundaries unless the coefficient of the dominant solution is equal to zero, in which case it will not change.

In order to throw more light on the situation we enclose the Q-curves of our Riemann surface in small sectors or D-regions which may be defined as follows:

$$|\arg z - r\tfrac{\pi}{2}| < h > 0, \qquad r = 0, \pm 1, \pm 2, \ldots\ldots$$

Sheet 1 will then appear as in the figure where the D-regions are the shaded areas.



Let that part of the region R_i exclusive of the D-regions be called the restricted region $\bar{R}_i$. Now it can be shown that in any restricted region $\bar{R}_i$, w has the asymptotic development

$$w \sim C_\alpha S_\alpha$$

where S_α is the solution dominant in R_i, and $C_\alpha \neq 0$. Likewise if the coefficient of the dominant solution is equal to zero in R_i we will have

$$w \sim C_\beta S_\beta \quad \text{in } R_i$$

where S_β is the sub-dominant solution.

It will here be noted that the location of curves where the sub-dominant coefficient changes is to a certain extent arbitrary. Any curve $\bar{R}_i$ extending to ∞ would evidently do equally well.

On the Q-curve the two formal solutions become of equal importance and hence in a D-region it is necessary that we retain both terms,

$$w \sim C_1 S_1 + C_2 S_2 \text{ in D}$$

where C_1 is the same as in the adjacent region in which S_1 is dominant and C_2 is the same as in the adjacent region in which S_2 is dominant.

If both coefficients C_1 and C_2 should equal zero in R_i we have the trivial case

$$w \equiv 0.$$

Summarizing this analysis we have found the following situation: If we write simply

$$w \sim C_1 S_1 + C_2 S_2$$

where w is any solution fixed by boundary conditions, then the only possible change of the constants, C, is that C_1 can change on the lines which bisect the second and fourth quadrants, and C_2 can change on the lines which bisect the first and third.* A constant can change on these lines only when the other constant is non-zero. If by some process like contour integration or the saddle point method certain minimum information could be found, this together with our analysis of the change of the coefficients would unravel the complete story of the asymptotic form of the solution w over the whole z-plane. The minimum information just mentioned is a knowledge of the leading term of the asymptotic expansion of the solution w on each of the lines OA_K, $K = 0, \pm 1, \pm 2, \ldots$, and furthermore this is known except for a constant factor.

B. Connections for solutions asymptotic in the parameter m.

*The reader will recall that the location of these lines was somewhat arbitrary when we made the subdivisions of the regions R_i to satisfy the requirements of Trjitzinsky's existence theorem. Any curves extending to ∞ in the respective quadrants would do equally well, providing on the curve $|\arg z - \gamma\frac{\pi}{2}| > h > 0$, that is, the subdividing curve has a different limiting direction from the bounding Q-curve.

From the general existence theorem of page 32 one would suppose that the coefficients of a general solution

$$w = {}_{T}C_1(\rho)\,{}_{T}y_1 + {}_{T}C_2(\rho)\,{}_{T}y_2$$

would depend upon T, and that these coefficients might change due to Stokes' phenomenon when ρ is taken into a different region S_T. It is desired to find the connection of these coefficients in the different regions, and if n terms are needed in the expansion we shall now show how to calculate the coefficients ${}_{T}C(\rho)$ for the solutions $w_K(x, \rho)$, $K = 1,2$, principal* at the origin so that the asymptotic form of these solutions is known explicitly to n terms. We indicate these solutions in the form

$$w_K(x,\rho) = {}_{T}C_{K1}(\rho)e^{\alpha_1\rho x}\left\{\sum_{J=0}^{n-1}\frac{u_{1J}(x)}{\rho^J} + \frac{{}_{T}\varepsilon_{1n}(x,\rho)}{\rho^n}\right\}$$

(30)

$$+ {}_{T}C_{K2}(\rho)e^{\alpha_2\rho x}\left\{\sum_{J=0}^{n-1}\frac{u_{2J}(x)}{\rho^J} + \frac{{}_{T}\varepsilon_{2n}(x,\rho)}{\rho^n}\right\},$$

where the ε's are bounded functions for ρ sufficiently large in S_T and x on $(-b, b)$. The coefficients will be found in the form

$${}_{T}C(\rho) = c_0 + \frac{c_1}{\rho} + \frac{c_2}{\rho^2} + \cdots + \frac{c_{n-1}}{\rho^{n-1}} + \frac{\varepsilon(\rho)}{\rho^n}$$

or $\frac{1}{\rho}$ times this expression, where the c's are determined explicitly and $\varepsilon(\rho)$ is bounded for ρ sufficiently large. This form, however, is sufficient since the $\frac{\varepsilon(\rho)}{\rho^n}$ may be

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*The solutions w_1 and w_2 principal at the origin are such that :

$$w_1(0) = 0 \qquad w_2(0) = 1$$
$$w_1'(0) = 1 \qquad w_2'(0) = 0 .$$

absorbed in the $\frac{\mathcal{E}in(x,\rho)}{\rho^n}$ of (30).

Let n be one of the numbers 0,1,2,..... . However, after it is chosen it remains fixed in all further calculations. Now by $u_i(x,\rho)$ we shall denote the n+1 terms

$$u_i(x,\rho) = e^{\alpha_i \rho x}\sum_{J=0}^{n}\frac{u_{iJ}(x)}{\rho^J}.$$

From the general existence theorem of page 32 we may now write the following:

$$(31)\qquad {}_T y_i = u_i(x,\rho) + \frac{e^{\alpha_i \rho x}E_{in}(x,\rho)}{\rho^{n+1}},$$
$$\qquad {}_T y_i' = u_i'(x,\rho) + \frac{e^{\alpha_i \rho x}\hat{E}_{in}(x,\rho)}{\rho^n}.$$

Here the E, $\hat{E}$ are bounded functions for ρ in S_T and x on $(-b, b)$. We may now write the principal solutions and their x derivatives as follows:

$$(32)\qquad W_K = {}_T C_{K1}(\rho)\,{}_T y_1 + {}_T C_{K2}(\rho)\,{}_T y_2,$$
$$W_K' = {}_T C_{K1}(\rho)\,{}_T y_1' + {}_T C_{K2}(\rho)\,{}_T y_2', \qquad K = 1,2$$

where

$$(33)\qquad W_K(0) = \delta_{K1}$$
$$W_K'(0) = \delta_{K2},\ \delta_{KK} \equiv 1,\ \delta_{Ki} \equiv 0 \qquad \text{if } K \neq i.$$

By making use of (32) we write (33) in the following manner, remembering that $u_i(0) = 1$ for all n, and $i = 1,2$:

$$(34)\qquad \delta_{K1} = {}_T C_{K1}(\rho)\left\{1+\frac{E_{1n}(0,\rho)}{\rho^{n+1}}\right\} + {}_T C_{K2}(\rho)\left\{1+\frac{E_{2n}(0,\rho)}{\rho^{n+1}}\right\},$$

$$\delta_{K2} = {}_T C_{K1}(\rho)\left\{u_1'(0,\rho)+\frac{\hat{E}_{1n}(0,\rho)}{\rho^n}\right\} + {}_T C_{K2}(\rho)\left\{u_2'(0,\rho)+\frac{\hat{E}_{2n}(0,\rho)}{\rho^n}\right\}.$$

Equation (34) gives us a system which may be solved for the C 's providing the characteristic determinant Δ does not equal zero. In this case

$$\Delta = u_2'(0,\rho) - u_1'(0,\rho) + \frac{E(\rho)}{\rho^n}.$$

Here $u_i(0,\rho)\ \alpha_i \rho$, and $\alpha_2 - \alpha_1 \neq 0$. Thus $\Delta \neq 0$, and for ρ sufficiently large, we may write the solutions for ${}_T C_{K1}$ and ${}_T C_{K2}$. These become the following:

(35)

$$ {}_T C_{K1}(\rho) = \delta_{K1} \left\{ \frac{u_2'(0,\rho) + \frac{\hat{E}_{2n}}{\rho^n} - \delta_{K-2}\left\{1 + \frac{E_{2n}}{\rho^{n+1}}\right.}{\Delta} \right\},$$

$$ {}_T C_{K2}(\rho) = \delta_{K2} \left\{ \frac{1 + \frac{E_{1n}}{\rho^{n+1}} - \delta_{K1}\left\{u_1'(0,\rho) + \frac{\hat{E}_{1n}}{\rho^n}\right.}{\Delta} \right\}.$$

At this point it will be noted from the series for and σ_i' of Chapter VIII that $u_i(0,\rho)$ is of the form

$$u_i(0,\rho) = A_0 + \frac{A_1}{\rho} + \frac{A_2}{\rho^2} + \cdots + \frac{A_n}{\rho^n},$$

and $u_j(0,\rho)$ is of the form

$$u_i(0,\rho) = \rho\left(B_0 + \frac{B_1}{\rho} + \frac{B_2}{\rho^2} + \cdots + \frac{B_n}{\rho^n} + \frac{\mathcal{E}}{\rho^{n+1}}\right).$$

Thus the C 's of (35) will take the form

(36) $$ {}_T C(\rho) = \frac{A_0 + \frac{A_1}{\rho} + \cdots \frac{A_n}{\rho^n} + \frac{E}{\rho^{n+1}}}{\rho\left(B_0 + \frac{B_1}{\rho} + \cdots + \frac{B_n}{\rho^n} + \frac{E}{\rho^{n+1}}\right)}$$

or ρ times such an expression, where the E 's are bounded for large values of ρ. Now (36) by mere "long division" may be written in the form

$$ {}_T C(\rho) = C_0 + \frac{C_1}{\rho} + \frac{C_2}{\rho^2} + \cdots + \frac{C_n}{\rho^n} + \frac{E}{\rho^{n+1}},$$

or $\frac{1}{\rho}$ times such an expression, where E is bounded, and the C's are known explicitly. This is the procedure for calculation we set out to establish.

From the equation defining a region S_τ on page 31 we may now take $\tau = 0,2,4,\ldots\ldots$ as indicating the regions in which $\alpha_1 = i$ and $\alpha_2 = -i$. Likewise $\tau =$ odd integer when $\alpha_1 = -i$ and $\alpha_2 = i$.

Now if the forms W_K and W_K' of (32) are calculated for ρ in a definite S_τ, say S_0, the resulting expressions are exactly the same for all regions S_τ since we have the following results:

$$\left[{}_\tau C_{K1\,\tau}u_1(x,\rho)\right] = \left[{}_{\tau+1} C_{K2\ \tau+1}u_2(x,\rho)\right],$$

$$\left[{}_\tau C_{K2\,\tau}u_2(x,\rho)\right] = \left[{}_{\tau+1} C_{K1\ \tau+1}u_1(x,\rho)\right],$$

$$\left[{}_\tau C_{K1\,\tau}u_1'(x,\rho)\right] = \left[{}_{\tau+1} C_{K2\ \tau+1}u_2'(x,\rho)\right],$$

$$\left[{}_\tau C_{K2\,\tau}u_2'(x,\rho)\right] = \left[{}_{\tau+1} C_{K1\ \tau+1}u_1'(x,\rho)\right],$$

where the brackets denote that the remainder terms are omitted.

Hence it is no longer necessary to affix the subscript τ in the asymptotic formulas. We shall take $\alpha_1 = i$ and $\alpha_2 = -i$ in all S_τ since it is shown there is no Stokes' phenomenon.

The four coefficients of (35) have been calculated for $n = 4$, and these are given below:

$$C_{11} = \frac{1}{2} + \frac{0}{\rho} + \frac{0}{\rho^2} + \frac{0}{\rho^3} + \frac{0}{\rho^4} + \frac{\varepsilon_1}{\rho^5},$$

$$C_{12} = -\left(-\frac{1}{2} + \frac{0}{\rho} + \frac{0}{\rho^2} + \frac{0}{\rho^3} + \frac{0}{\rho^4} + \frac{\varepsilon_2}{\rho^5}\right),$$

$$C_{21} = -\frac{1}{\rho}\left(-\frac{1}{2i} + \frac{0}{\rho} + \frac{0}{\rho^2} + \frac{0}{\rho^3} + \frac{1}{8i\rho^4} + \frac{\varepsilon_3}{\rho^5}\right),$$

$$C_{22} = \frac{1}{\rho}\left(-\frac{1}{2i} + \frac{0}{\rho} + \frac{0}{\rho^2} + \frac{0}{\rho^3} + \frac{1}{8i\rho^4} + \frac{\varepsilon_4}{\rho^5}\right),$$

where the ε's are bounded functions for x on $(-b, b)$.

The principal solutions (32) may now be expressed for $n = 4$ as follows:

$$w_1 = e^{\rho i x}\left[\frac{1}{2}\left\{1 + \frac{1}{2i\rho}\left[\frac{x^3}{3}\right] + \left(\frac{1}{2i\rho}\right)^2\left[x^2 + \frac{x^6}{18}\right] + \left(\frac{1}{2i\rho}\right)^3\left[2x + \frac{8x^5}{15} + \frac{x^9}{162}\right]\right\}\right.$$

$$\left. + \left(\frac{1}{2i\rho}\right)^4\left[\frac{19x^4}{6} + \frac{11x^8}{90} + \frac{x^{12}}{1944}\right] + \left[\frac{E_1(x,\rho)}{\rho^5}\right]\right]$$

$$+ e^{-\rho i x}\left[\frac{1}{2}\left\{1 - \frac{1}{2i\rho}\left[\frac{x^3}{3}\right] + \left(\frac{1}{2i\rho}\right)^2\left[x^2 + \frac{x^6}{18}\right] + \left(\frac{1}{2i\rho}\right)^3\left[2x + \frac{8x^5}{15} + \frac{x^9}{162}\right]\right\}\right.$$

$$\left. + \left(\frac{1}{2i\rho}\right)^4\left[\frac{19x^4}{6} + \frac{11x^8}{90} + \frac{x^{12}}{1944}\right] + \left[\frac{E_2(x,\rho)}{\rho^5}\right]\right],$$

$$w_2 = \frac{-e^{\rho i x}}{\rho}\left[-\frac{1}{2i}\left\{1 + \left[\frac{x^3}{3}\right]\frac{1}{2\rho i} + \left(\frac{1}{2\rho i}\right)^2\left[x^2 + \frac{x^6}{18}\right] + \left(\frac{1}{2\rho i}\right)^3\left[2x + \frac{8x^5}{15} + \frac{x^9}{162}\right]\right\}\right.$$

$$\left. + \left\{\left(\frac{1}{2i}\right)^4\left[\frac{19x^4}{6} + \frac{11x^8}{90} + \frac{x^{12}}{1944}\right] + \frac{1}{8i}\right\}\frac{1}{\rho^4} + \frac{E_3(x,\rho)}{\rho^5}\right.$$

$$+\frac{e}{\rho}^{\rho i x}\Big[\frac{-1}{2i}\Big\{1-\frac{1}{2i\rho}\Big(\frac{x^3}{3}\Big)+\Big(\frac{1}{2i\rho}\Big)^2\Big[x^2+\frac{x^6}{18}\Big]-\Big(\frac{1}{2i\rho}\Big)^3\Big[2x+\frac{8x^5}{15}+\frac{x^9}{162}\Big]\Big\}$$

$$+\Big\{\Big(\frac{1}{2i}\Big)^4\Big[\frac{19x^4}{6}+\frac{11x^8}{90}+\frac{x^{12}}{1944}\Big]+\frac{1}{8i}\Big\}\frac{1}{\rho^4}+\frac{E_4(x,\rho)}{\rho^5}\Big] \quad .$$

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