The development of modern industries calls for the robotic manipulators with high speed and accurate tracking performance. Many authors have paid attention to robust control of robotic manipulators; however, only few authors have also considered the control problem of manipulators with power limitation.

In this dissertation, the robotic manipulator is modeled as an uncertain system, with such uncertainties as varying moments of inertia, damping and payloads during tracking. The resulting uncertain part of the system is norm-bounded by a known constant.

The total control consists of a linear part with gain matrix $K$, and a nonlinear part $\Delta v$, typically used for control of uncertain dynamical systems. Saturation of the resulting controller is assumed, with bounds imposed by the power limitation of actuators. It is proved at the
dissertation that such a system is globally uniformly practically stable. The distribution of the control power between two controllers is discussed. It is found that when small gain matrix K is used and Av dominates the controller, the solution to the system can approach a smaller region with faster response; that is, higher tracking accuracy is obtained.

Theoretical analysis is provided to support the proposed control scheme. A two-link robotic manipulator is simulated with the results confirming the prediction.
Robust Controller Design
for Robotic Manipulators with Saturation
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Typed by researcher for Zuyang Liang
To my dear family
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NOMENCLATURE

A  \( m \times m \) constant error dynamic system matrix
A  \( m \times m \) error dynamic closed loop system matrix (\( A = A - BK \))
B  \( m \times n \) constant input matrix
C  \( m \times m \) constraint matrix
C(q,\( \dot{q} \)) \( n \times 1 \) vector of centrifugal, Coriolis and viscous friction moments
D  \( m \times p \) constant matrix (Section 3.1)
E  \( n \times n \) matrix (Eq. (3.2.4))
e  \( m \times 1 \) error vector
F  \( n \times p \) constant matrix (Section 3.1)
G(q) \( n \times 1 \) vector of gravity moments
h(q,\( \dot{q} \)) \( n \times 1 \) vector (= \( C(q,\dot{q}) + G(q) \))
\( \hat{h}(q,\dot{q}) \) a nominal or computed version of \( h(q,\dot{q}) \)
I  \( m \times m \) identity matrix
K  \( n \times m \) linear control gain matrix
M(q) \( n \times n \) generalized inertia matrix
\( \hat{M}(q) \) a nominal or computed version of \( M(q) \)
m  \( m = 2 \times n \)
n  the number of degree-of-freedom of a robotic manipulator
N  \( n \times n \) matrix (Eq. (2.2.1))
P  \( m \times m \) matrix, the unique solution to the Lyapunov equation
Q  \( m \times m \) positive definite matrix
\( q, \dot{q}, \ddot{q} \)  
\( n \times 1 \) vector representing joint positions, velocities and accelerations

\( u(t) \)  
\( n \times 1 \) vector of control inputs

\( U_m \)  
\( m \times 1 \) vector of control bounds

\( V(x) \)  
Lyapunov function

\( v \)  
\( n \times 1 \) vector of the total control

\( v_1 \)  
\( n \times 1 \) vector of linear control

\( w \)  
\( p \times 1 \) vector of system uncertainties

\( x \)  
\( m \times 1 \) vector of state variables

\( \beta_\delta \)  
Lyapunov ellipsoid (Section 4.3)

\( \Delta v \)  
\( n \times 1 \) vector of nonlinear control

\( \varepsilon \)  
a small positive number

\( \lambda_{\text{max}} \)  
the maximum eigenvalue

\( \lambda_{\text{min}} \)  
the minimum eigenvalue

\( \rho \)  
a bound of the norm of system uncertainties

\( \tau(t) \)  
\( n \times 1 \) vector of forces or torques applied to robotic links (Section 2.1)
CHAPTER 1

INTRODUCTION

1.1 About Robotic Manipulators

Robotic manipulators have a major impact on the nature of manufacturing systems. Robots are not only replacing workers in dull, repetitive, or hazardous environments, but also increasing productivity, quality and safety of the manufacturing processes. Today more and more complex tasks are assigned to robotics in industries. The automation through the introduction of robots will undoubtedly accelerate in the future.

Robotic systems are essentially dynamical systems. In the case of fast motion and mechanical configurations with strongly coupled subsystems, the control task to be solved is essentially dynamic. Moreover, robotic systems are highly nonlinear systems. The motion of robotic systems as active spatial mechanisms is described via time-varying, coupled, nonlinear second-order differential equations. Linearized or linear models of such motion are inaccurate
and non-effective in practical application. In any physical system there is a degree of uncertainty regarding the values of various parameters. In the case of a system as complicated as a robot, this is particularly true, especially if the robot is carrying unknown loads. In addition, the burden of computing the complete model of manipulator may be prohibitively expensive or impossible with the bounds imposed by the available computer architecture. In such cases it is desirable to simplify the equations of motion as much as possible by ignoring certain terms in the equations in order to speed up the computation of the control law. Thus, practical implementation of nonlinear control for robotic manipulators requires consideration of issues of robustness to parameter uncertainty, external disturbances, sensor noise, computational complexity, input disturbances and actuator saturation.

1.2 Robust Control

The robust control problem is classical; however, the term robust control for this classical problem is only of recent vintage. The term robust control here is confined to the nonadaptive or nonself-tuning solution to the problem of controlling uncertain systems.
Table 1.2.1 shows the outline of nonadaptive approaches to the robust control problem. A historical review of robust control can be found in the paper by Peter Dorato (1987) and the paper by Dragoslav D. Šiljak (1989).

Classical Sensitivity Design:
. Feedback & large loop gain (Black 1927)
. Nyquist frequency domain stability Criterion (1932)
. Differential sensitivity function (Bode 1945)

State-variable:
. Sensitivity comparison matrix
. Trajectory insensitivity
. Performance insensitivity
. Eigenvalue/eigenvector insensitivity

Modern Robust Control
. Frequency domain
  . parameter space
  . H∞ optimal sensitivity design
  . H2 optimal sensitivity design
. Model parameter uncertainty stochastically
. Game-theory
. Guaranteed-cost-control
. Lyapunov-function
. Qualitative-feedback-theory
. Hurwitz-condition
. norm-uncertainty

Table 1.2.1 Non-adaptive Robust Control
The increase in application of manipulators in industry and automated manufacturing systems calls for more robust manipulator controllers. In recent years, much effort has been devoted to the problem of obtaining stabilizing controllers for uncertain systems, especially in robotics. In much of this research, the uncertainties are modelled deterministically, rather than stochastically, and they are characterized by certain structural conditions and known bounds, such uncertainties could be due to uncertain disturbance input, uncertain parameters, or model simplification. Table 1.2.2 shows the methods for robust control of robotic manipulators.

- Robust servomechanism
- PID control
- Pole placement
- Two-stage synthesis
- Uncertain dynamical system theory
- Variable structure systems (VSS)
- other

Table 1.2.2 Robust Control of Robotic Manipulators

Desa et al. (1985) described a framework for manipulators based on robust servomechanism theory for multivariable linear systems. Figure 1.2.1 shows the control scheme. The nonlinear, dynamical system is
split into a nominal (or global) part and a linear part. Then the linear time-varying system is converted into a linear time-invariant system. A control law for the linear system is then derived on the basis of linear quadratic regulator theory. The implicit model-following technique is used to choose the weights in the resulting performance index. The uncertainties were modeled as input disturbances. The theory is then applied to design a control law for a two degree-of-freedom spatial manipulator following a prescribed trajectory. The weakness of this approach follows from the fact that the investigation to the

Figure 1.2.1 Servomechanism Control of Robotic Manipulators
linear time-invariant system would not reflect the original system, since the conversion of the linear time-varying system into a linear time-invariant system is not reliable.

PID control is popular in practice, since it is easier to implement. Tarokh and Seraji (1988) proposed a scheme for multivariable control of robot manipulators. The scheme is shown in Figure 1.2.2.

![PID Control of Robotic Manipulators](image)

Figure 1.2.2 PID Control of Robotic Manipulators

It is composed of an inner loop stabilizing controller and an outer loop tracking controller. The inner loop utilizes a multivariable PD controller to stabilize the robot by placing the poles at some desired locations. The
outer loop employs a multivariable PID controller to achieve input-output decoupling and trajectory tracking. The gains of the PD and PID controllers are related directly to the linearized robot model by simple closed-form expressions. The controller gains are updated on line to cope with variations in the robot model for gross motion and payload change. No example was given. Since the control scheme proposed is based on linear multivariable control theory, the linearized robot dynamic model has to be constructed. Seraji also presented a control scheme which consists of two independent multivariable feedforward and feedback controllers (Seraji, 1987). Figure 1.2.3 shows the control scheme. Later the feedback controller was updated by an adaptation law which contains both proportional and integral adaptation terms (Seraji, 1989). PID controller design for robotics also can be found in the paper by Kawamura et al (1988).

A nonlinear control law and arbitrary placing of poles was proposed by Freund (1977). In his investigation, however, moments of inertia were neglected. Application of such nonlinear control to complex configurations leads to complex control laws. Dib (1987) proposed a design method for optimally placing the closed-loop poles of a discretized robotic control system at exact prescribed locations in the unit circle of a complex z-plane. The system should be
linearized. In practice, however, these linearized systems are only approximation of nonlinear models at various operating conditions. Thus the exact location of poles for each optimal condition is difficult to obtain. The system uncertainties were not considered. Fadali (1990) proposed a robust pole assignment for computed torque robotic manipulator control. The computed torque method is used to reduce the manipulator controller design to a linear problem. A robust pole assignment approach is used to select a suitable linear state feedback for the nominal computed torque method. The effect of modeling errors is accounted for by a state-dependent acceleration disturbance.
vector. A two degree-of-freedom manipulator design was discussed.

The two-stage approach was introduced by Vukobratović and Stokić (1981, 1982, 1983). The first stage of control synthesis consists of synthesizing nominal programmed control and implementing the desired system motion for some chosen initial state. The second stage of control synthesis consists of synthesizing control for the tracking of nominal trajectories when the actual initial state deviates from the nominal initial state (but belongs within a bounded region of initial states). The two-stage approach has been widely used along with other approaches.

The theory of variable structure systems (VSS) was developed in the USSR and has found applications in control of a wide range of processes in steel, power, chemical, and aerospace industries (Young, 1978). The salient feature of VSS is that the so-called sliding mode occurs on a switching surface. While in sliding mode, the system remains insensitive to parameter variations and disturbances. It is this insensitivity property of VSS that enables the elimination of interaction among the various joints of the manipulator. Young (1978) and Hached et al (1988) utilized a transformation for decoupling the "fast" and "slow" states. An estimate of the region is found where the
Lyapunov derivative of the "slow" subsystem is negative. The results found in the investigation of both the "slow" and "fast" subsystems are then used to obtain an estimate of the region of attraction of the overall system. A two degree-of-freedom manipulator was considered with and without the Coriolis and centripetal terms (Hached et al, 1988).

Uncertain dynamical system theory based on Lyapunov function for stability analysis provides another approach to robust stabilization. It is assumed that the uncertain quantities are Lebesgue measurable function whose values may range in prescribed sets; that is, loosely speaking, only the possible magnitudes of uncertainties are presumed known (Leitmann, 1981). The early investigations on the theory can be found in the papers by Gutman (1979, 1983), Leitmann (1979, 1981) and Barmish et al (1983). An advantage of the approach is that time-varying and nonlinear systems can also be treated. Spong and Vidyasagar et al. found the application in robust control of robotic manipulators (1985, 1987, 1989). Figure 1.2.4 illustrates the control scheme which consists of a feedback linearizing control (inner loop) based on a nominal system, followed by a robust linear feedback control (outerloop) based on the uncertain dynamical system theory. Note that the outer loop control is more in line with the notion of a feedback control in the
usual sense of being error-driven. Similar applications can be found in the paper by Dolphus et al (1990). The uncertain dynamical system theory application on robotics is also discussed in the paper by Shoureshi et al (1990).

Figure 1.2.4 Uncertain Dynamic System Control of Robots

A number of other approaches to the control of uncertain systems were developed in parallel with the approaches outlined above. Ha et al (1987) presented a nonlinear feedback multivariable controller which requires restrictive assumptions on the structure of the model and modeling errors. A two degree-of-freedom manipulator was discussed. For tracking dynamic signals in time-varying uncertain systems, Hopp et al (1990) proposed a linear
controller instead of nonlinear. With this linear control, similar results to those obtained with the nonlinear controller would be reached; however, no consideration was given to robotics.

1.3 Input Saturation

One of the common problems encountered in control of dynamical systems is that the control actions calculated by a controller cannot be implemented in full; that is, the control input saturates.

Input saturation also refers to control constraints or control bounds. Most physical systems have a limited range of available control effort. If a calculated control input to the system violates a saturation limit then the subsequent control will not in general be satisfactory.

However, only few authors considered the control problem of robotic manipulators with input saturation (A. Weinreb and A. E. Bryson, 1985; Mehrez Hached and Mehdi Madani-Esfahani et al, 1988; Fadali et al, 1990). Weinreb (1985) proposed a method for optimal control of systems with hard control bounds. It incorporates control bounds into the gradient algorithm formulation and uses the control-variation step-size weight to satisfy the control bounds.
The method is applied to a two-link robot arm without uncertainties. Hached et al (1988) proposed a method based on the theory of variable structure systems (VSS). A transformation for decoupling the "fast" and "slow" states was used to investigate stability domain estimates of the system. The results were then applied to a two-joint planar manipulator. This controller is for a class of linear time-invariant systems subject to uncertainties. Fadali et al. (1990) considered a random disturbance which could result from the variation of system parameters or from random clipping of the nominal actuator torques in the presence of actuator torque bounds. The resulting disturbance is a Gaussian white noise acceleration error with covariance matrix. A two degree-of-freedom cylindric manipulator with generalized coordinators was discussed.

Recently, Soldatos, Corless and Leitmann (1990) proposed a method for stabilizing uncertain systems when the norm of the control is bounded by a prespecified constant. The method treats continuous-time dynamical systems whose nominal part is linear and whose uncertain part is norm-bounded by a known constant. Given a ball of initial states, the controller yields "practical stability" with a region of attraction which includes the given ball. The proposed method is applied to a single scalar example and an inverted pendulum subject to a bounded control torque. No
consideration is given to improving stability, tracking accuracy and robotics.

A one-step optimal method for compensating for any form of input saturation in discrete linear controllers was presented by Segall et al (1991). The correction for multivariable controllers is to simultaneously adjust the remaining control inputs if some inputs saturate. The algorithm was applied to linear process systems.

1.4 The Organization of the Contents

In this research, we considered not only the uncertainties of the systems, but also the input saturation. Using the feedback linearizing control scheme the nonlinear time-varying dynamical systems are simplified to continuous-time dynamical systems whose nominal part is linear and whose uncertain part is norm-bounded by a known constant. The systems are stabilized in the sense of "practical stability". Figure 1.4.1 shows the control scheme.

After the introduction, robot dynamics are discussed in Chapter 2. Since the requirements imposed on robots concerning the speed and quality of operations (precision of tracking) are increasing, the control at the executive level must take into account the dynamics of the robot. Several
analytical methods for the dynamical systems will be discussed in this chapter. Inverse dynamics is used in the design of controllers since it is a very attractive dynamics control method.

In Chapter 3, the uncertain systems theory is discussed, then the applications to robotic manipulators are explored.

In Chapter 4, a design procedure is presented for
designing a robust controller of robotic manipulators with input saturation. The issue of improving the stability and tracking accuracy is also discussed.

Conclusions are summarized in Chapter 5 and numerical examples are given at the end of Chapter 4 to illustrate the concepts of the material presented in this paper.
CHAPTER 2

ROBOTIC DYNAMICS

Most of the robots on the market today are not capable of ensuring precise tracking of fast trajectories (Vukobratović, 1989), since the dynamic forces are not compensated. Because robotic manipulators are dynamical systems, especially for those concerning the speed and precision of tracking, the dynamics of the robots must be taken into account. In this chapter we investigate the dynamics of the systems and the control methods to the systems.

2.1 Mathematic Modeling of Robotic Manipulators

Dynamical equations governing the behavior of robotic manipulators play an important role in the design and control of the manipulators. The main methods most frequently employed in the mathematical modeling of robotic manipulators are: Lagrange Equations, Newton-Euler Equations and Kane's Dynamical Equations. Based on these methods a lot of algorithms have been proposed. Vukobratović and Kirčanski (1985) present a review of some algorithms based on these methods.
The Lagrange method tends to lead to computational algorithms involving large numbers of unnecessary arithmetic operations. The Newton-Euler approach can force one to perform unnecessary calculations associated with the elimination of certain forces and torques of interaction between elements of a robot. Kane and Levinson (1983) provided methodology for application of Kane’s Equations to robotics. Kane’s method provides an efficient way to generate the mathematical models. The advantage of this approach is its simple treatment of complex manipulators. However, this generality complicates the application to the simple joints.

By using either the Newton-Euler or Lagrange’s equations, the equation of motion for a rigid manipulator with n degree-of-freedom of body can be obtained and written as

\[ M(q) \ddot{q} + C(q, \dot{q}) + G(q) = \tau(t) \] (2.1.1)

where

\( \tau(t) \) is an \( n \times 1 \) vector of forces or torques applied to links,

\( M(q) \) is an \( n \times n \) generalized inertia matrix,

\( C(q, \dot{q}) \) is an \( n \times 1 \) vector of centrifugal, Coriolis and viscous friction moments

\( G(q) \) is an \( n \times 1 \) vector of gravity moments
\( \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \) are \( n \times 1 \) vectors representing joint positions, velocities and accelerations. An example is shown in Section 4.4.

2.2 Dynamic Analysis of Robotic Systems

The properties of dynamical model matrices play an important role in the dynamical analysis. The inertia matrix \( \mathbf{M}(\mathbf{q}) \) of the dynamical model (2.1.1) is symmetric and positive definite (Vukobratović and Kirćanski, 1985). Therefore, the inverse of \( \mathbf{M}(\mathbf{q}) \) exists and also is positive definite. When several joints are moving simultaneously the moment of inertia of the mechanism is varying during the motion. The performance of a robot can be uneven if the moment of inertia is significantly varied. Gravity moments also vary during the movement, causing errors both in positioning and in tracking of a trajectory. Centrifugal and Coriolis moments are significant if the joints are moving at high speeds, causing errors in tracking of fast trajectories. The centrifugal and Coriolis moments must be taken into account in controller design if the precise tracking is required.

The linearization methods which are used in dynamic analysis and control of robotic manipulators are discussed next.
2.2.1 Linearization

Three methods of linearization of a robotic system are considered in engineering practice. The first method is based on the cancellation of the gravity terms and the piecewise parameterization. The second method is based on the perturbation equations associated with a given nominal trajectory. The third is based on the design of inertia distribution.

1. **Cancellation of Gravity Terms and Parameterization**

The Coriolis and centrifugal term $C(q, \dot{q})$ is a quadratic vector form of $\dot{q}$. Hence this term can be expressed as

$$C(q, \dot{q}) = N(q, \dot{q}) \dot{q}$$  \hspace{1cm} (2.2.1)

where $N(q, \dot{q})$ is defined as an $n \times n$ matrix. The dynamical equation of the robotic system in Eq. (2.1.1) can be rewritten as

$$M(q) \ddot{q} + N(q, \dot{q}) \dot{q} = \tau - G(q)$$  \hspace{1cm} (2.2.2)
or
\[ \ddot{q} = -M^{-1}(q) N(q, \dot{q}) \dot{q} + M^{-1}(q) [\tau - G(q)] \]
which can be written in the state-space representation as
\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \] (2.2.3)

where
\[ A(t) = \begin{bmatrix} 0_n & I_n \\ 0_n & -M^{-1}(q) \end{bmatrix} \] (2.2.4)
\[ B(t) = \begin{bmatrix} 0_n \\ M^{-1}(q) \end{bmatrix} \] (2.2.5)

\[ x^T(t) = [q^T(t), \dot{q}^T(t)] \] (2.2.6)
\[ u(t) = \tau(t) - G[q(t)] \] (2.2.7)

Thus, the control problem can be considered as a linear time-varying control problem. The linearized dynamics equations of a robotic system can be computed at each sampling period where the nominal trajectory is known.

2. Perturbation Equations
Given a desired trajectory $q^d(t)$, $\dot{q}^d(t)$ and $\ddot{q}^d(t)$, the nominal applied torque $\tau^d(t)$ required for motion along the specified trajectory can be precomputed. The dynamical equations in Eq. (2.1.1) can be expressed as a sum of the nominal equation

$$M(q^d)\dddot{q}^d + C(q^d, \dot{q}^d) + G(q^d) = \tau^d(t) \quad (2.2.8)$$

plus a perturbation equation

$$\delta [M(q)\ddot{q}] + \delta C(q, \dot{q}) + \delta G(q) = \delta \tau(t)$$

The variations $\delta [M(q)\ddot{q}]$, $\delta C(q, \dot{q})$ and $\delta G(q)$ can be expressed in terms of the following linear approximations

$$\delta [M(q)\ddot{q}] = A'(t)\delta q + B'(t)\delta \dot{q}$$

$$\delta C(q, \dot{q}) = C'(t)\delta q + D'(t)\delta \dot{q}$$

$$\delta G(q) = E'(t)\delta q$$

where

$$A'(t) = \left. \frac{\partial (M(q)\ddot{q})}{\partial q} \right|_{q^d, \dot{q}^d}$$

$$B'(t) = \left. \frac{\partial (M(q)\ddot{q})}{\partial \dot{q}} \right|_{q^d, \dot{q}^d}$$
Thus the perturbation equations for the manipulator can be approximated as

\[ R(t) \delta \dot{q} + S(t) \delta \dot{q} + T(t) \delta q = \delta \tau \quad (2.2.9) \]

where

\[ R(t) = B'(t) \]
\[ S(t) = D'(t) \]

and

\[ T(t) = A'(t) + C'(t) + E'(t) \]

Therefore, the state-space representation of the manipulator dynamics can be written as

\[ \delta \dot{x}(t) = A(t) \delta x(t) + B(t) \delta \tau(t) \quad (2.2.10) \]

where

\[ x = [\delta q_1, \delta q_2, \ldots, \delta q_n, \delta \dot{q}_1, \delta \dot{q}_2, \ldots, \delta \dot{q}_n]^T \]
Designing a linear state-feedback gain $K$, the control law for the system (2.2.10) is

$$\delta \tau(t) = -K(t) \delta x(t)$$  \hspace{1cm} (2.2.11)$$

The total input torque becomes

$$\tau(t) = \tau^d(t) + \delta \tau(t)$$  \hspace{1cm} (2.2.12)$$

3. **Design of Inertia Distribution**

The design of inertia distribution is based on eliminating coefficients of nonlinear terms in the system’s kinematic and potential energy equations (Yang and Tzeng, 1986). The robot’s structure can be improved by examining the complete expanded Lagrange equations to redesign the link’s inertia property, including inertia, mass, and the location of the mass center. Accordingly, the manipulator
dynamics is simplified and linearized.

4. Comments

An important advantage of the perturbation equation method is that when the desired trajectory is preplanned, the feedback gain matrix $K$ can be computed off-line and stored in a look-up table. But the linearized system is only the approximation of the original system. The parameterization approach does not require any prior knowledge of the path. The system parameters must be either computed on-line or stored in a table based on segmentation of the workspace. However, the piecewise-linearized system would not always reflect the original system. The third method can not be used to analyze an existing manipulator to be controlled. It would be used in the design of a new manipulator. For some configurations of simple robots with three or four links, a completely dynamic linearization is possible, while for complicated robots completely dynamic linearization is impossible.

2.2.2 Inverse Dynamics

In this paper, we use inverse dynamics, or feedback linearizing method instead of the linearization methods discussed above. The idea of inverse dynamics is to
construct a nonlinear feedback control law which cancels the highly nonlinear coupled dynamics of the manipulator and results in a decoupled and linear closed loop system.

For simplicity we rewrite the Eq. (2.1.1) as

$$M(q) \ddot{q} + h(q, \dot{q}) = u$$  \hfill (2.2.13)$$

where

$$h(q, \dot{q}) = C(q, \dot{q}) + G(q)$$

If we choose the control $u(t)$ as follows

$$u(t) = M(q) v + h(q, \dot{q})$$ \hfill (2.2.14)$$

then the system (2.2.13) becomes

$$\ddot{q} = v$$ \hfill (2.2.15)$$

setting

$$v = \dot{q}^d(t) - K_1 [q(t) - \dot{q}^d(t)] - K_2 [\dot{q}(t) - \dot{q}^d(t)]$$ \hfill (2.2.16)$$

and letting the tracking error $e(t) = q(t) - \dot{q}^d(t)$, the nonlinear time-varying control problem now becomes stabilizing the linear time-invariant system (2.2.17)

$$\ddot{e}(t) + K_2 \dot{e}(t) + K_1 e(t) = 0$$ \hfill (2.2.17)$$
which is easier to control. However, exact nonlinear dynamics cancellation is not true in practice, because of the modelling errors, computation errors and uncertainties of the robotic systems. The uncertain systems will be discussed in Chapter 3.

Compared with perturbation linearization method, inverse dynamics is more effective. Inverse dynamics can be viewed as input transformation, which does not change system dynamics, while perturbation linearization is only a local approximation method which could lead to undesired response, owing to deviation of the assumed condition from real condition. Inverse dynamics requires on-line computation; however, perturbation method can be off-line computation.
CHAPTER 3

STABILIZING UNCERTAIN SYSTEMS

Uncertain system theory is discussed in this chapter. The application of the theory to robotic manipulators is also described. By using inverse dynamics or feedback linearization method discussed in Chapter 2, an n-link rigid robot is globally linearized and decoupled. The system is treated as a continuous-time dynamical system whose nominal part is linear and whose uncertain part is norm-bounded by a known constant.

3.1 Uncertain System and Robust Control

The control of dynamical systems which contain uncertain elements, input, as well as states, or more generally output, can be treated by a deterministic approach (Leitmann, 1981). The uncertainty is bounded with its bound known. No statistical information on the uncertainty is assumed or utilized.

We consider the following uncertain system

\[ \dot{x}(t) = Ax(t) + Bu(t) + Dw(x(t), t) \] (3.1.1)
where \( x(t) \in \mathbb{R}^m \) is state vector, \( u(t) \in \mathbb{R}^n \) is the control input, \( t \in \mathbb{R} \) is time, and the function \( w: \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p \) represents the uncertainty acting on the nominal system, which is unknown and bounded. The \( w(\cdot) \) may be nonlinear. \( A \in \mathbb{R}^{mxm} \) and \( B \in \mathbb{R}^{mxn} \) are system and input matrix, respectively, which are constant and known. The constant matrix \( D \in \mathbb{R}^{mxp} \) is also known. The following structural assumptions of the system (3.1.1) are proposed.

Assumption 1. The pair \((A,B)\) is stabilizable. Hence there exists a constant matrix \( K \) such that all eigenvalues of \( \bar{A} = A - BK \) have negative real parts.

Assumption 2. There exists a constant matrix \( F \) such that \( D = BF \). This assumption restricts the structure of the uncertainty.

Assumption 3. \( W \subseteq \mathbb{R}^p \) is a known, non-empty compact set. The function \( w(\cdot): \mathbb{R}^m \times \mathbb{R} \to W \) is continuous. Thus we can impose a norm constraint for the uncertainty.

\[
\rho \triangleq \max\{\|Fw\|: w \in W\} \quad (3.1.2)
\]

Based on the above descriptions, the robust control design procedures can be presented as follows.

**Step 1.** Choose a constant matrix \( K \) such that \( \lambda(\bar{A}) \) have
negative real parts, and find the unique solution to the Lyapunov equation

\[ \bar{A}^T P + P \bar{A} = -Q \]  \hspace{1cm} (3.1.3)

for a given \( Q > 0 \).

**Step 2.** Estimate the bound of uncertainty by applying Eq. (3.1.2).

**Step 3.** For a given \( \varepsilon > 0 \), consider the control

\[ u(t) = Kx(t) + p(x(t), t) \]  \hspace{1cm} (3.1.4)

where

\[
p(x, t) = \begin{cases} 
-\rho \frac{B^T P x}{\|B^T P x\|} & \text{if } \|B^T P x\| > \varepsilon \\
-\rho \frac{B^T P x}{\varepsilon} & \text{if } \|B^T P x\| \leq \varepsilon 
\end{cases} \]  \hspace{1cm} (3.1.5)

With the control (3.1.4), the solution \( x(t) \) of the system (3.1.1) which satisfies the three assumptions above will in finite time enter a small region of state space \( \mathbb{R}^n \), containing the equilibrium state \( x \equiv 0 \) and remain there for future time. The size of the region is dependent on the choice of \( \varepsilon \). This region can be made arbitrarily small by appropriately choosing \( \varepsilon \). Detail proof can be found in

3.2 Control of Robotic Manipulators with Uncertainties

After the robotic dynamics model is derived, uncertain dynamical system theory is applied to the manipulator controller design.

3.2.1 Modeling

In section 2.2.2, we discussed inverse dynamics method which utilizes cancellation of the highly nonlinear coupled dynamics of the manipulator. However, there will always be inexact cancellation of the nonlinearities due to uncertainties and also due to computational round-off. A controller whose design is based only on the nominal system may perform poorly due to the inexact cancellation. An example may be found in the book by Spong and Vidyasagar (1989).

To assure satisfactory performance, uncertain system theory is applied to robust controller design for manipulators with bounded uncertainty.

The dynamical equations of an n-link robot is rewritten here (Dolphus and Schmitendorf, 1990).
\[ M(q, r) \ddot{q} + h(q, \dot{q}, r) = u \]  

(3.2.1)

where \( r(t) \in \mathbb{R} \subseteq \mathbb{R}^p \) is continuous, \( R \) is a compact subset of \( \mathbb{R}^p \), \( M(q, \cdot) \) and \( h(q, \dot{q}, \cdot) \) are assumed to be continuous. Since the inertia matrix \( M \) is uniformly positive definite for all \( q \), there exist positive constants \( M \) and \( \bar{M} \) such that

\[ M \leq \|M^{-1}(q, r)\| \leq \bar{M} < \infty \quad \forall \ q \in \mathbb{R}^n \text{ and } r \in \mathbb{R} \]  

(3.2.2)

We consider a fixed nominal set of uncertainties \( \bar{r} \in \mathbb{R} \) and define \( \bar{M}(q) \triangleq M(q, \bar{r}) \) and \( \bar{h}(q, \dot{q}) \triangleq h(q, \dot{q}, \bar{r}) \) and choose a control in the form of (2.2.14)

\[ u = \bar{M}(q) v + \bar{h}(q, \dot{q}) \]  

(3.2.3)

where \( \bar{r} \), \( \bar{M} \) and \( \bar{h} \) represent nominal or computed version of \( r \), \( M \) and \( h \), respectively.

Letting

\[ E \triangleq M^{-1}\bar{M} - I \]  

(3.2.4)

\[ \Delta h \triangleq \hat{h} - h \]  

(3.2.5)

\[ w \triangleq Ev + M^{-1}\Delta h \]  

(3.2.6)
where
\[ \|E\| = \|M^{-1}M_0 - I\| < 1 \]

which results in
\[ \dot{q} = \nu + w \quad (3.2.7) \]

For a given desired trajectory \( q^d(t) \) which satisfies
\[ q^d \in Q^d_1 \subset \mathbb{R}^n, \quad \dot{q}^d \in Q^d_2 \subset \mathbb{R}^n \quad \text{and} \quad q^d \in Q^d_3 \subset \mathbb{R}^n \]
for all \( t \geq 0 \), where \( Q^d_1 \), \( Q^d_2 \) and \( Q^d_3 \) are compact subsets of \( \mathbb{R}^n \). We introduce the error vectors
\[ e_1 = q - q^d \quad \text{and} \quad e_2 = \dot{q} - \dot{q}^d. \]

Then the error dynamics may be written as
\[ \dot{e} = Ae + B(v+w-cid) \quad (3.2.8) \]

where
\[ A = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix}, \quad B = \begin{bmatrix} 0_n \\ I_n \end{bmatrix} \]

3.2.2 Stabilizing

Since the system (3.2.8) satisfies the three assumptions in section 3.1, uncertainty theory can be applied to stabilize the system.
Choosing
\[ v = q^{d-K}e + \Delta v \] \hfill (3.2.9)

Eq. (3.2.8) becomes
\[ \dot{e} = \overline{A}e + B\Delta v + Bw \] \hfill (3.2.10)

where \( \Delta v \in \mathbb{R}^n \) is the control,
\[ \Delta v = \begin{cases} -\rho \frac{B^TPx}{\|B^TPx\|} & \text{if } \|B^TPx\| > \varepsilon \\ -\rho \frac{B^TPx}{\varepsilon} & \text{if } \|B^TPx\| \leq \varepsilon \end{cases} \] \hfill (3.2.11)

\( K \in \mathbb{R}^{nxn} \) is feedback gain matrix such that \( \bar{A} = A - BK \) is stable, and
\[ w = E\Delta v + E(q^{d-K}e) + M^{-1}\Delta h \]
\[ \|w\| \leq \rho \]

where \( \rho \) is a known constant. From Eq. (3.1.2), \( \rho \) can be found as
\[ \rho \geq \max \{ \|w\| : w \in \mathcal{W} \} \] \hfill (3.2.12)
or

\[ \rho \geq \max \{ \| E(q, r) (\dot{q}^d - K e + \Delta v) + M^{-1}(q, r) \Delta h(q, \dot{q}, r) \| : \]

\[ q \in Q_1, \ q \in Q_2, \ \dot{q}^d \in Q^d_3 \text{ and } r \in R \} \]

(3.2.13)

where \( Q_1, Q_2 \) and \( Q^d_3 \) are compact subsets of \( \mathbb{R}^n \). In the absence of uncertainties, that is, when \( w = 0, \dot{M} = M, \dot{h} = h, E = 0, \) and \( \Delta h = 0, \) the control law reduces to inverse dynamics control by letting \( \Delta v = 0. \)

Therefore, for the robotic manipulator with bounded uncertainty, controller design procedures can be stated as the following:

**Step 1.** Choose a gain matrix \( K \) such that \( A = A - BK \) is stable.

**Step 2.** For \( Q = I, \) find unique solution \( P \) to the Lyapunov equation.

**Step 3.** For the given system, choose a scalar \( \rho \) to satisfy Eq. (3.2.13).

**Step 4.** The controller for stabilizing the uncertainties is constructed by using Eq. (3.2.11).
Step 5. The total control is

\[ v = \dot{\theta}^d - Ke + \Delta \nu \]

Substitution of \( v \) for Eq. (3.2.3) defines the torques or forces which control the robotic system to track the given trajectory.
CHAPTER 4

STABILIZING UNCERTAIN SYSTEMS WITH SATURATION

In this chapter, a sequential design procedure is presented for designing a controller of robotic manipulator with bounded uncertainties and input torque saturation. After introduction, the mathematical model of the input saturation and the concepts of practical stability are discussed. The design procedure is presented in Section 4.2.2. The stability of the system with saturation is investigated in Section 4.3. The simulation results with detail discussions are presented in Section 4.4.

4.1 Introduction

One of the most common problems encountered in control systems is that the control actions calculated by a controller can not be implemented in full; that is, the control input saturates. A controller designed without considering the control constraint meets the design specifications only when the system is in its operating region; if a calculated control input to the system violates a saturation limit then the subsequent control will not in general be satisfactory (N. L. Segall, 1991).
Due to the high degree of coupling among joints and severe nonlinearities in the robotic systems, input constraints must be considered, especially if one desires large or fast motion of the manipulator. For this reason we consider here the control problem for a robotic manipulator subject to bounds on the allowable input torques. The diagram of the control system is shown in Figure 1.4.1.

4.1.1 Input Saturation

![Manipulated Variable](image)

**Figure 4.1.1 Saturation Nonlinearity**
The saturation nonlinearity shown in Figure 4.1.1 represents the practical behavior of many actuators and final control elements. For example, a motor amplifier combination can produce a torque proportional to the input voltage over a limited range. However, no amplifier can apply an infinite current and no motor can provide an infinite torque, there is a maximum current and thus a maximum torque that the system can produce in either direction (clockwise or counterclockwise).

The input is constrained according to

$$U_{i_{\text{min}}} \leq u_i(t) \leq U_{i_{\text{max}}} \quad i=1,2,\ldots,n, \quad \forall t \geq 0 \quad (4.1.1)$$

where $U_{i_{\text{min}}}$ and $U_{i_{\text{max}}}$ ($i = 1, 2, \ldots, n$) are lower and upper bounds, respectively. Eq. (4.1.1) may be written as

$$C u(t) \leq U_m \quad (4.1.2)$$

where

$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & -1 \end{bmatrix}$$

and

$$U_m = \begin{bmatrix} U_{i_{\text{max}}} \\ U_{i_{\text{min}}} \\ \vdots \\ U_{i_{\text{max}}} \\ U_{i_{\text{min}}} \end{bmatrix}$$
where \( C \in \mathbb{R}^{mxn}, U_m \in \mathbb{R}^{mx1} \) are constant matrix, \( u(t) \in \mathbb{R}^n \) is the input at time \( t \).

4.1.2 Stability

The stability concept employed here differs slightly from the traditional Lyapunov-type stability (Appendix A).

Consider an uncertain dynamical system described by the state equation

\[
\dot{x}(t) = f(t, x(t)) + \Delta f(x(t), w(t), t) + [B(x(t), t) + \Delta B(x(t), w(t), t)]u(t)
\]

(4.1.3)

where \( t \in \mathbb{R}, x(t) \in \mathbb{R}^m \) is the state, \( u(t) \in \mathbb{R}^n \) is the control, \( w(t) \in \mathbb{R}^p \) is the uncertainty and \( f(x,t), \Delta f(x,w,t), B(x,t) \) and \( \Delta B(x,w,t) \) are matrices of appropriate dimensions which depend on the structure of the system.

[Definition 1] The uncertain dynamical system (4.1.3) is said to be practically stabilizable if, given any \( d > 0 \), there is a control law \( p_d(\cdot): \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \) for which, given any admissible uncertainty \( w(\cdot) \) which is bounded, any initial time \( t_0 \in \mathbb{R} \) and any initial state \( x_0 \in \mathbb{R}^m \), the following conditions hold:

(i) Existence of solutions. The closed loop system
\[ \dot{x}(t) = f(t, x(t)) + \Delta f(x(t), w(t), t) + [B(x(t), t) + \Delta B(x(t), w(t), t)] p_d(x(t), t) \]  

(4.1.4)

possesses a solution \( x(\cdot):[t_0, t_1] \rightarrow \mathbb{R}^n \), with \( x(t_0) = x_0 \)

(ii) Extension of solutions. Every solution \( x(\cdot):[t_0, t_1] \rightarrow \mathbb{R}^n \) can be continued over \([t_0, \infty)\).

(iii) Uniform boundedness of solutions. Given any \( \epsilon > 0 \) and any solution \( x(\cdot):[t_0, t_1] \rightarrow \mathbb{R}^n \) of (4.1.3) with \( x(t_0) = x_0 \) and \( \|x_0\| \leq \epsilon \), there is a constant \( \beta(\epsilon) \) such that

\[ \|x(t)\| \leq \beta(\epsilon) \quad \forall \ t \geq t_0. \]  

(4.1.5)

(iv) Uniform attractivity. Given any \( d \geq d \), any \( \epsilon > 0 \) and any solution \( x(\cdot):[t_0, \infty) \rightarrow \mathbb{R}^n \) of (4.1.3) with \( x(t_0) = x_0 \) and \( \|x_0\| \leq \epsilon \), there exists a finite time \( T(\tilde{d}, \epsilon) < \infty \), such that \( \|x(t)\| \leq \tilde{d} \) for all \( t \geq t_0 + T(\tilde{d}, \epsilon) \).

(v) Uniform stability. Given any \( d \geq d \), and any solution \( x(\cdot):[t_0, \infty) \rightarrow \mathbb{R}^n \) of (4.1.3) with \( x(t_0) = x_0 \), there is a constant \( \delta(d) > 0 \) such that \( \|x_0\| \leq \delta(d) \) implies that

\[ \|x(t)\| \leq d \quad \forall \ t \geq t_0. \]
[Definition 2] The set

$$B_d = \{ x \in \mathbb{R}^n : \|x\| < d \}$$ (4.1.6)

is a region of attraction for the uniformly practically stable system.

[Definition 3] System (4.1.3) is globally uniformly practically stable iff it is practically stable with $\mathbb{R}^n$ as a region of attraction.

The $d$ can be regarded as a measure of the distance of the system behavior from that of asymptotic stability. If the system (4.1.3) satisfies the requirements of Definition 1 with $d = 0$, then it is uniformly asymptotically stable with region of attraction $B_d$. Hence, although we cannot guarantee uniform asymptotic stability, we can nevertheless drive the state to an arbitrarily small neighborhood of the origin.

4.2 Robust Controller Design

After discussing the control system, the control procedure will be presented in this section.

4.2.1 System Description
Consider an n degree of freedom robot with bounded inputs

\[
\begin{align*}
M(q) \ddot{q} + h(q, \dot{q}) &= u(t) \\
Cu(t) &\leq U_m
\end{align*}
\]  

(4.2.1)

where \(M(q)\) is the \(n \times n\) inertia matrix of the manipulator, \(h(q, \dot{q}) = C(q, \dot{q}) + G(q)\), \(u(t)\) is the control.

Recall from Chapter 3, that an uncertain dynamical system, whose nominal part is linear and whose uncertain part is norm-bounded, is described by

\[
\dot{e}(t) = \bar{A}e(t) + B(\Delta v(t) + w)
\]

(4.2.2)

where \(t \in \mathbb{R}\) is the time, \(e(t) \in \mathbb{R}^m\) is the state, \(\Delta v(t) \in \mathbb{R}^n\) is the control input, and \(w\) is unknown and bounded by a constant \(p\)

\[
w = E\Delta v + E(\dot{q}^d - Ke) + M^{-1}\Delta h
\]

(4.2.3)

\[
\|w\| \leq \rho
\]

(4.2.4)

The constant matrices \(\bar{A} \in \mathbb{R}^{mxn}\), \(B \in \mathbb{R}^{mxn}\)

\[
\bar{A} = A - BK
\]

(4.2.5)
\[ A = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0_n \\ I_n \end{bmatrix} \quad (4.2.6) \]

\( K \in \mathbb{R}^{n \times m} \) is the feedback control gain matrix. For any \( \varepsilon > 0 \), the nonlinear control \( \Delta v \) can be found by

\[
\Delta v(t) = \begin{cases} 
-\rho \frac{B^T Pe}{\|B^T Pe\|} & \text{if } \|B^T Pe\| \geq \varepsilon \\
-\rho \frac{B^T Pe}{\varepsilon} & \text{if } \|B^T Pe\| < \varepsilon 
\end{cases} \quad (4.2.7)
\]

where \( P \in \mathbb{R}^{m \times m} \) is the unique solution to the Lyapunov equation

\[
P \bar{A} + \bar{A}^T P = -Q \quad (4.2.8)
\]

and \( P \) is symmetric and positive definite.

For simplification we introduce the notation of saturation function \( s(\cdot) \) which satisfies

\[
s(y) = \begin{cases} 
y & \text{if } \|y\| \leq 1 \\
\frac{y}{\|y\|} & \text{if } \|y\| > 1
\end{cases} \quad (4.2.9)
\]
therefore,

$$\Delta v = -ps\left(\frac{B^TPe}{\epsilon}\right)$$  \hspace{1cm} (4.2.10)

The computed input torques to the robot is

$$u(t) = \hat{M}(q)v(t) + \hat{h}(q, \dot{q})$$  \hspace{1cm} (4.2.11)

where \( v = v_1 + \Delta v \). Due to the existence of saturation, the constrained input torques are

$$\bar{u}_i(t) = U_{i\text{max}} s\left(\frac{u_i}{U_{i\text{max}}}\right) \quad i = 1,2,\ldots,n$$  \hspace{1cm} (4.2.12)

4.2.2 Design Procedure

**Step 1:**

Given an \( n \) degree-of-freedom robotic manipulator system as in Eq. (2.1.1). Rewritten as in Eq. (4.2.1), we have \( \hat{M}(q) \) and \( \hat{h}(q, \dot{q}) \), which are a nominal or computed version of \( M, h \), respectively.

**Step 2:**

Choose a small gain matrix \( K \) such that system matrix \( \bar{A} = A - BK \) is stable.
Step 3:
Find matrix $\tilde{A}$ by
$$\tilde{A} = A - B K$$ (4.1.7)

Let $Q \in \mathbb{R}^{n \times n}$ be an identity matrix, and solve the Lyapunov Equation
$$PA + A^TP = -Q$$ (4.2.13)

Then $P$ is a symmetric, positive definite matrix.

Step 4:
Evaluate system uncertainty bound by Eq.(4.2.3) and specify $\rho$ so that Eq. (4.2.4) is satisfied. Given $\varepsilon > 0$, construct nonlinear stabilizing control law $\Delta v$ by Eq. (4.2.7).

Step 5:
The algorithm is completed. The total control $v$ is
$$v = v_1 + \Delta v$$ (4.2.14)
where $v_1 = \dot{q}^d - K e$.

Step 6:
The input torques to the robotic manipulator now are found by Eq. (4.2.11) and Eq. (4.2.12).
4.3 The Stability of the Systems with Saturation

Lemma 4.3.1 Set the Lyapunov function

\[ V(e) = e^T Pe \]

then with the control law (4.2.10), all solutions of (4.2.2) asymptotically approach the Lyapunov ellipsoid \( \beta_{\delta}(e) \).

\[ \beta_{\delta} \Delta \{ e \in \mathbb{R}^n : e^T Pe \leq \delta \} \quad (4.3.1) \]

Proof:

Let \( V(e) = e^T Pe \) then

\[ \dot{V}(e) = e^T Pe + e^T P \dot{e} \]

\[ = e^T (A^T P + PA) e + 2e^T PB (\Delta v + w) \]

\[ = -e^T Q e + 2e^T PB (\Delta v + w) \quad (4.3.2) \]

From the relation

\[ -e^T Q e \leq -\lambda_{\text{min}} (P^{-1} Q) e^T Pe \quad (4.3.3) \]
which is proved in Appendix B, and where $\lambda_{\text{min}}(P^{-1}Q)$ means the minimum eigenvalue of the matrix $(P^{-1}Q)$, and

$$2e^TPB(\Delta v+w) \leq 2\varepsilon \rho$$ (4.3.4)

Eq. (4.3.2) can be rewritten as

$$\dot{V}(e) \leq -\lambda_{\text{min}}(P^{-1}Q) V(e) + 2\varepsilon \rho$$

$$= -\lambda_{\text{min}}(P^{-1}Q) \left[ V(e) - \frac{2\varepsilon \rho}{\lambda_{\text{min}}(P^{-1}Q)} \right]$$ (4.3.5)

Letting

$$\delta(\varepsilon) \triangleq \frac{2\varepsilon \rho}{\lambda_{\text{min}}(P^{-1}Q)}$$ (4.3.6)

Eq. (4.3.5) becomes

$$\dot{V}(e) \leq -\lambda_{\text{min}}(P^{-1}Q) [V(e) - \delta]$$ (4.3.7)

hence, all solutions of (4.2.2) asymptotically approach the Lyapunov ellipsoid $\beta_{\delta(\varepsilon)}$.

Remark 4.3.1 $\beta_{\delta(\varepsilon)}$ can be made arbitrarily small by choosing sufficiently small $\varepsilon$.

Define the maximum and minimum eigenvalues of the matrix $(P^{-1}Q)$ as $\lambda_{\text{max}}(P^{-1}Q)$ and $\lambda_{\text{min}}(P^{-1}Q)$, respectively, we have the following results.
Lemma 4.3.2 Setting $Q = I$, smaller values of ratio $\frac{\lambda_{\max}(P^{-1}Q)}{\lambda_{\min}(P^{-1}Q)}$ correspond to smaller Lyapunov ellipsoid, and larger values of $\lambda_{\min}(P^{-1}Q)$ correspond to faster response. That is, with smaller ratio, $\frac{\lambda_{\max}(P^{-1}Q)}{\lambda_{\min}(P^{-1}Q)}$ and larger values of $\lambda_{\min}(P^{-1}Q)$, the solution of system (4.2.1) will approach a smaller region in shorter time.

Proof:

The solution of Eq. (4.3.7) is

$$V = V_0[\delta_0 + e^{-\mu_{\min}(t-t_0)}] \tag{4.3.8}$$

where $\mu_{\min} = \lambda_{\min}(P^{-1}Q)$, $V_0$ is the initial value of $V$ and $\delta_0$ is a constant. From Eq. (4.3.8), we see that $1/\mu_{\min}$ corresponds to the largest time constant related to changes in the Lyapunov function $V$. Since $V(x)$ may be expanded in a series starting with a quadratic term in $x$, this time constant $1/\mu_{\min}$ is about half the conventional time constant defined for the system (Ogata, 1970). Therefore, the larger values of $\mu_{\min} (= \lambda_{\min}(P^{-1}Q))$ correspond to faster response.

Since

$$\lambda_{\min}(P)\|e\|^2 \leq e^TPe \leq \lambda_{\max}(P)\|e\|^2$$
where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues of matrix $P$, respectively, and

$$e^{T}Pe \leq \delta$$

with $Q = I$ in Eq. (4.3.6), we have

$$\lambda_{\min}(P) \|e\|^2 \leq \frac{2\epsilon \rho}{\lambda_{\min}(P^{-1})}$$

Therefore

$$\|e\| \leq \sqrt{\frac{2\epsilon \rho}{\lambda_{\min}(P) \lambda_{\min}(P^{-1})}}$$

(4.3.9)

Substitute the relation

$$\lambda_{\min}(P) = \frac{1}{\lambda_{\max}(P^{-1})}$$

into Eq. (4.3.9), the result can be written as

$$\|e\| \leq \sqrt{\frac{2\epsilon \rho \lambda_{\max}(P^{-1})}{\lambda_{\min}(P^{-1})}}$$

(4.3.10a)

or

$$\|e\| \leq \sqrt{\frac{2\epsilon \rho \lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

(4.3.10b)
From Eq. (4.3.10a), we see that the smaller the ratio of $\lambda_{\max}(P^{-1}Q)/\lambda_{\min}(P^{-1}Q)$, the smaller the $\|e\|$ is, thus the smaller the Lyapunov ellipsoid $\beta$ is.

Remark 4.3.2 Smaller Lyapunov ellipsoid and faster response imply better tracking performance, therefore, dynamic tracking accuracy can be improved by using a better $P$, the unique solution to Lyapunov equation, such that smaller ratio of $\lambda_{\max}(P^{-1}Q)/\lambda_{\min}(P^{-1}Q)$ and larger $\lambda_{\min}(P^{-1}Q)$ are obtained.

Lemma 4.3.3 The system (4.2.1) is globally uniformly practically stable.

Proof: From Lemma 1, $\forall e(t) \in \mathbb{R}^m$, Eq. (4.3.1) is satisfied. The system (4.2.1) is practically stable with region of attraction $\mathbb{R}^m$. From Definition 3 in Section 4.1.2, Lemma 3 is proved.

4.4 Examples

Consider a two-link robotic manipulator which is shown in Figure 4.4.1. The equations of motion are

$$M(q) \ddot{q} + h(q, \dot{q}) = u$$

(4.4.1)
Figure 4.4.1 Two DOF Robotic Manipulator

\[
M(q) = \begin{bmatrix}
(a_1 + a_2 + a_3 + 2a_4 \cos(q_2) + a_5 + a_6) & (a_3 + a_4 \cos(q_2) + a_6) \\
(a_3 + a_4 \cos(q_2) + a_6) & (a_3 + a_6)
\end{bmatrix}
\]

\[M(q) = \begin{bmatrix}
(a_1 + a_2 + a_3 + 2a_4 \cos(q_2) + a_5 + a_6) & (a_3 + a_4 \cos(q_2) + a_6) \\
(a_3 + a_4 \cos(q_2) + a_6) & (a_3 + a_6)
\end{bmatrix}
\]

(4.4.2)

\[
h(q, \dot{q}) = \begin{bmatrix}
-a_4 \sin(q_2) (2q_1 \dot{q}_2 + \dot{q}_2^2) + (a_7 + a_8) \cos(q_1) + a_9 \cos(q_1 + q_2) \\
(a_4 \sin(q_2) \dot{q}_1^2 + a_9 \cos(q_1 + q_2))
\end{bmatrix}
\]

(4.4.3)
where

\[ a_1 = m_1 L_{c_1}^2 \quad a_2 = m_2 L_1^2 \quad a_3 = m_2 L_{c_2}^2 \]

\[ a_4 = m_2 L_1 L_{c_3} \quad a_5 = J_1 \quad a_6 = J_2 \]

\[ a_7 = m_1 L_{c_1} g \quad a_8 = m_2 L_1 g \quad a_9 = m_2 L_{c_2} g \]

and

\[ m_1 = 20 \ \text{LB} \quad m_2 = (12 + m) \ \text{LB} \]

\[ L_1 = 1.2 \ \text{ft} \quad L_2 = 1.0 \ \text{ft} \]

\[ L_{c_1} = 0.6 \ \text{ft} \quad L_{c_2} = \frac{6 + m}{12 + m} \ \text{ft} \]

\[ J_1 = 2.4 \ \text{LB} \cdot \text{ft}^2 \quad J_2 = \frac{4(m + 3)}{m + 12} \ \text{LB} \cdot \text{ft}^2 \]

\( \ddot{\eta} \) and \( \dot{\eta} \) are obtained by setting \( m = 0 \), i.e., at unloaded state. The torque bounds are [1370, 300] lb-ft. The desired trajectories for both \( q_1 \) and \( q_2 \) are shown in Figure 4.4.2, where

\[ -4 \leq q^d \leq 4 \]

\[ q^d = \begin{cases} 
0 & t \leq 0 \\
2t^2 & 0 < t \leq 0.5 \\
1.0 - 2(1-t)^2 & 0.5 < t \leq 1 \\
1.0 & t > 1 
\end{cases} \]
Figure 4.4.2 Desired Trajectories
Following the procedures described in section 4.2.2, choose the gain matrix $K$

$$K = \begin{bmatrix} 0.0001 & 0 & 0.02 & 0 \\ 0 & 0.0004 & 0 & 0.04 \end{bmatrix} \quad (4.4.4)$$

and let $\epsilon = 0.1$ and $\rho = 40$. We have the results shown in Figure 4.4.3 and Figure 4.4.4. Tracking error is defined as the difference between tracking response and the desired trajectory. The torques are to drive robot arms.

The following discussion will investigate the mechanism and performance of the controller.

1. **About Gain Matrix $K$**

   In our design, a small gain $K$ is recommended. Since $\Delta v$ is related to the solution of the Lyapunov Equation in which the system matrix is related to gain matrix $K$, and

   $$v = q^d - Ke + \Delta v \quad (4.4.5)$$

   choosing a small gain $K$, the system is stable but the values of $Ke$ are much smaller than those of $\Delta v$, thus $\Delta v$ dominates the controller and provides better tracking results. In
Figure 4.4.3  Tracking Response with the Controller 
\( (\varepsilon = 0.1, \rho = 40, \text{torque bounds } = [1370, 300] \text{ lb-ft} \) 
and \( K = [0.0001, 0, 0.02, 0; 0, 0.0004, 0, 0.04 ] \) )
Figure 4.4.4 Tracking Errors
($\varepsilon = 0.1$, $\rho = 40$, torque bounds = [1370, 300] lb-ft, and $K = [0.0001, 0, 0.02, 0; 0, 0.0004, 0, 0.04]$)
fact $\Delta v$ is not only for stabilizing uncertainties, but also for shifting the poles of the system to the left. The $\Delta v$ is derived by applying Lyapunov method which is better than the direct pole assignment method.

Two cases are discussed here. In the first case, the gain matrix is

\[
K = \begin{bmatrix}
100 & 0 & 20 & 0 \\
0 & 144 & 0 & 24
\end{bmatrix}
\] (4.4.6)

In the second case, the values of the gain matrix are much smaller,

\[
K = \begin{bmatrix}
0.0001 & 0 & 0.02 & 0 \\
0 & 0.0004 & 0 & 0.04
\end{bmatrix}
\] (4.4.7)

Both cases have the same bounds on the inputs, and the same $\varepsilon$ and $\rho$. The results below show that for the same power, the tracking accuracy in case two is about 100 times higher than that in case one. The significantly different results indicate that the values of gain matrix need to be insignificant in the controller. The results for case one are shown in Figure 4.4.5 and Figure 4.4.7, and those of case two in Figure 4.4.6 and Figure 4.4.8.
Figure 4.4.5
Tracking Errors (larger K)
($\varepsilon = 0.1, \rho = 40$, torque bounds = [1370, 300] lb-ft)

Figure 4.4.6
Tracking Errors (smaller K)
($\varepsilon = 0.1, \rho = 40$, torque bounds = [1370, 300] lb-ft)
2. About Power

The robot power needs to be discussed since a robot with torque saturation means that the robot has power.
limitation. This is a practical problem, especially when increasing speeds are required on the existing robotic manipulators. As shown above, the robot has significant difference in tracking performance although the same power limitation is imposed. The goal in this paper is to design a robust controller for an uncertain robotic system, even if the system has torque saturation. But it does not mean that the power can be very small. The torque bounds should meet the condition

\[ |U_m|_i \geq |U_{\text{tracking}}|_i + |U_{\text{stabilizing}}|_i \quad (i=1,2,\ldots,n) \]  

(4.4.8)

where \( U_{\text{tracking}} = \ddot{\mathbf{M}} \dot{\mathbf{q}} + \dddot{\mathbf{h}} \) and \( U_{\text{stabilizing}} = \dddot{\mathbf{M}} \Delta \mathbf{v} \). To get accurate tracking, however, it is not necessary to use big power. In Figure 4.4.9 and Figure 4.4.10, the tracking errors are shown for the systems with torque bounds and without torque bounds, respectively. Figure 4.4.11 and Figure 4.4.12 show the torques which are used to drive the systems with torque bounds and without torque bounds. The results indicate that as long as the power is enough for tracking and stabilizing, it is unnecessary to increase motor torques. Moreover, the bigger the motor torque, the larger the inertia momentum is in practice, which would not only waste more energy but also worsen the performance of the robot.
Figure 4.4.9 Tracking Errors (with torque saturation) 
($\varepsilon = 0.1$, $\rho = 400$)

Figure 4.4.10 Tracking Errors (without saturation) 
($\varepsilon = 0.1$, $\rho = 400$)
3. About Stability

In Section 4.3, we have already proved that with the controller, the robotic system is uniformly practically
stable with an arbitrarily small neighborhood of the origin. The bounds of inputs affect the tracking performance. The smaller the input bound, the larger the region of attraction is. Therefore, it reflects the larger tracking error.

4. **About Tracking Accuracy**

With an imposed power limit, a robot system suffers tracking problems. There are many factors which would change tracking accuracy for a robot. Some of them have been discussed above. From the discussions, we have already shown that by using smaller gain $K$ and better $P$, one can have the benefit of the higher tracking accuracy. Increasing power limit also could offer higher tracking accuracy; however, it is probably not practical, since it means that one needs to change design or to buy another robot.

Other factors which would influence tracking accuracy of a robot are $\epsilon$ and $\rho$. The results show that higher tracking accuracy would be derived by using smaller $\epsilon$. Note that Eq. (4.3.10) indicates that the Lyapunov ellipsoid $\beta_0$ can be made arbitrarily small by choosing sufficiently small $\epsilon$. However, it would bring chattering to the control system if $\epsilon$ is too small. Figure 4.4.13 illustrates tracking results. Figure 4.4.14 presents control torques.
Choosing larger $p$ also can improve tracking accuracy since larger $p$ can raise the weight of $\Delta v$ in the total control, but it would cause chattering if $p$ is too large. The tracking results are compared in Figure 4.4.15. The control torques are shown in Figure 4.4.16.

![Graphs of Link One and Link Two showing tracking results with different epsilon values.](image)

Figure 4.4.13 Comparison of Tracking Results of the System with Larger $\epsilon$ and That with Smaller $\epsilon$
Figure 4.4.14 Comparison of Torques for the System with Larger $\varepsilon$ and That with Smaller $\varepsilon$
Figure 4.4.15 Comparison of Tracking Results of the System with Smaller $p$ and that with Larger $p$
Figure 4.4.16 Comparison of Torques for the System with Smaller $\rho$ and That with Larger $\rho$
CHAPTER 5

CONCLUSIONS

1. The development of modern industries calls for high performance robotic systems. Many authors have paid attention to robust control of robotic manipulators in recent years. A number of control algorithms have been proposed to improve the robustness of robotic systems. Servomechanism approach, PID, pole placement and two-stage synthesis have been widely used in industries. Uncertain dynamical system theory and variable structure system theory found applications in robotic control only few years ago. These methods were summarized along with block diagrams for illustration and comparison.

2. The dynamics of robotic manipulators must be taken into account when the robots need to ensure precise tracking of fast trajectories. Dynamic analysis of such dynamical systems were presented with the emphasis on inverse dynamics method, since it is an attractive method for simplifying the control of highly nonlinear and coupled dynamical systems.

3. Uncertain system theory developed by Leitmann, Gutman, Corless and Barmish, etc. is applied to robotic control. The $\rho$ is proposed to be a constant instead of a
function so that the on-line computation time can be saved.

4. The control of robotic manipulators with saturation is a practical problem, but only few authors have considered it. An algorithm for the design of a controller for robotic manipulators with saturation was proposed in this paper. With the control, the robotic system is globally uniformly practically stable. The design procedures were worked out, which are simple and effective, followed by the design of a controller for a two degree-of-freedom robotic manipulator with torque saturation.

5. This research not only found a controller to stabilize the uncertain systems whose final elements would saturate, but also presented the ways to improve the stability and tracking accuracy. These ways are summarized as follows:

(a). Choose a small gain $K$ which is only for ensuring that $\dot{A} = A - BK$ is stable, not for placing poles of the system. In this way, the control $\Delta v$ which is designed by using Lyapunov stability method dominates the total control, so that the solution to the system approaches a smaller region with faster response. That is, higher tracking performance is derived.
(b). Choose a properly small $\varepsilon$, which constrains the region $\beta_0$. The smaller the $\varepsilon$, the higher the tracking accuracy. However, improperly small $\varepsilon$ will cause control chattering.

(c). Choose a proper $\rho$, which is the bound of the uncertainty. Larger value of $\rho$ would raise the weight of $\Delta v$ in the total control, thus, improve the tracking accuracy. Improperly large $\rho$ will cause control chattering.

6. Increasing power is a general way to solve the power limitation problem. But it is not an economical way and sometimes it is impossible. The control presented in this paper can solve the problem without increasing power. Under the same power limitation robotic tracking performance can be significantly improved with the control.

7. Future research will be to investigate more complicated robotic systems and to implement the controller for practical application.
BIBLIOGRAPHY


STABILITY IN THE SENSE OF LYAPUNOV

The three basic concepts of Lyapunov theory are stability, asymptotic stability, and global asymptotic stability.

Consider a dynamical system described by the state equation

\[ \dot{x}(t) = f(x(t), t), \quad t \geq 0 \tag{A.1} \]

where \( x(t) \in \mathbb{R}^m \) and \( f: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \).

[Definition A.1] The equilibrium point \( 0 \) at time \( t_0 \) of system (A.1) is said to be stable at time \( t_0 \) if, for each \( \varepsilon > 0 \), there exists a \( \delta(t_0, \varepsilon) > 0 \) such that

\[ \|x(t_0)\| < \delta(t_0, \varepsilon) \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \tag{A.2} \]

[Definition A.2] The system (A.1) is said to be uniformly stable over \( [t_0, \infty) \) if, for each \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon) > 0 \) such that

\[ \|x(t_1)\| < \delta(\varepsilon) \quad t_1 \geq t_0 \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_1 \tag{A.3} \]
[Definition A.3] The equilibrium point 0 at time \( t_0 \) of system (A.1) is \emph{unstable} at time \( t_0 \) if it is not stable at time \( t_0 \).

[Definition A.4] The equilibrium point 0 at time \( t_0 \) of system (A.1) is \emph{asymptotically stable} at time \( t_0 \) if it is stable at time \( t_0 \) and there exists a number \( \delta_1(t_0) > 0 \) such that

\[
\|x(t_0)\| < \delta_1(t_0) \Rightarrow \|x(t)\| \rightarrow 0 \quad t \rightarrow \infty \quad (A.4)
\]

[Definition A.5] The system (A.1) is \emph{uniformly asymptotically stable} over \([t_0, \infty)\) if it is uniformly stable at over \([t_0, \infty)\) and there exists a number \( \delta_1 > 0 \) such that

\[
\|x(t_1)\| < \delta_1 \quad t_1 \geq t_0 \Rightarrow \|x(t)\| \rightarrow 0 \quad \text{as} \ t \rightarrow \infty \quad (A.5)
\]

Moreover, the convergence is uniform with respect to \( t_1 \).

[Definition A.6] The set

\[
\beta_{\delta_1(t_0)} = \{ x \in \mathbb{R}^n : \|x\| < \delta_1(t_0) \} \quad (A.6)
\]
is a region of attraction for an asymptotically stable equilibrium point 0 at time $t_0$ if all trajectories starting from an initial state $x(t_0)$ sufficiently close to 0 actually approach 0 as $t \to \infty$.

[Definition A.7] The equilibrium point 0 at time $t_0$ is globally asymptotically stable if $x(t) \to 0$ as $t \to \infty$ regardless of what $x(t_0)$ is, or the sphere of attraction is the entire state space $\mathbb{R}^n$. 
APPENDIX B

PROOF OF RELATION (4.3.3)

Given an asymptotically stable system

\[ \dot{x}(t) = Ax(t) \quad (B.1) \]

and matrices P, Q such that

\[ PA + A^T P = -Q \quad (B.3) \]

Then \( \forall \ x \in \mathbb{R}^n \), inequality

\[ x^T Q x \geq \lambda_{\min}(P^{-1}Q) x^T P x \quad (B.2) \]

holds.

Proof:

Letting

\[ V(x) = x^T P x \quad (B.4) \]

the derivative of Lyapunov function is

\[ \dot{V}(x) = x^T P x + x^T P \dot{x} \]

\[ = x^T (A^T P + PA) x \]

\[ = -x^T Q x \quad (B.5) \]
Now define

\[
\mu \Delta - \frac{\dot{V}(x, t)}{V(x, t)} \quad \|x\| \neq 0
\]  

(B.6)

and

\[
\mu_{\text{min}} = \inf_{|x| \in \mathbb{R}^n \setminus 0} \left[ -\frac{\dot{V}(x, t)}{V(x, t)} \right]
\]  

(B.7)

Then

\[
\dot{V}(x, t) \leq -\mu_{\text{min}} V(x, t)
\]  

(B.9)

therefore

\[
-x^TQx \leq -\mu_{\text{min}} x^TPx
\]  

(B.10)

or

\[
x^TQx \geq \mu_{\text{min}} x^TPx
\]  

(B.11)

To find \( \mu_{\text{min}} \), we need to solve the equation

\[
x^TQx - \mu_{\text{min}} x^TPx = 0
\]  

(B.12)
that is

\[ x^T(Q - \mu_{\text{min}} P)x = 0 \quad \text{(B.13)} \]

The solution to the Eq. (B.13) is

\[ \mu_{\text{min}} = \lambda_{\text{min}}(P^{-1}Q) = \lambda_{\text{min}}(QP^{-1}) \quad \text{(B.14)} \]

thus, from (B.11) and (B.14)

\[ x^TQx \geq \lambda_{\text{min}}(P^{-1}Q)x^TPx \]