AN ABSTRACT OF THE THESIS OF

VISUTDHI UPATISRINGA for the Master of Arts
(Name) (Degree)
in Mathematics presented on March 17, 1967
(Major) (Date)

Title A PROGRAM FOR THE FUNDAMENTAL THEOREM ON
SYMMETRIC FUNCTIONS

Abstract approved Redacted for Privacy

Harry E. Goheen

The computation of symmetric functions can be tedious. For this reason, the object of this paper is to devise a computer program so that these symmetric functions can be handled automatically. The contribution of this investigation is a heuristic for finding the polynomials proved to exist by the Fundamental Theorem on Symmetric Functions. This program is heuristic, since a proof of the necessary existence theorem is lacking. Examples and Alcom program are given in the appendix.
A Program for the Fundamental Theorem on Symmetric Functions

by

Visutdhi Upatisringa

A THESIS

submitted to

Oregon State University

in partial fulfillment of the requirements for the degree of

Master of Arts

June 1967
APPROVED:

Redacted for Privacy
Professor of Mathematics
In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy
Dean of Graduate School

Date thesis is presented ________________

Typed by Carol Baker for ________________
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. FUNDAMENTAL CONCEPTS ON SYMMETRIC FUNCTIONS</td>
<td>3</td>
</tr>
<tr>
<td>III. DESCRIPTION OF A PROGRAM FOR THE FUNDAMENTAL THEOREM ON SYMMETRIC FUNCTIONS</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>25</td>
</tr>
<tr>
<td>IV. AN EXAMPLE</td>
<td>27</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>34</td>
</tr>
</tbody>
</table>

Appendix

| I. ALCOM PROGRAMMING | 35 |
| II. SYMMETRIC FUNCTIONS OF SIXTH AND SEVENTH DEGREES | 41 |
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1.</td>
<td>System of Linear Equations.</td>
<td>24</td>
</tr>
<tr>
<td>3.2.</td>
<td>Triangular Form of the System of Equations.</td>
<td>25</td>
</tr>
</tbody>
</table>
A PROGRAM FOR THE FUNDAMENTAL THEOREM ON SYMMETRIC FUNCTIONS

I. INTRODUCTION

The proofs of existence and uniqueness of the Fundamental Theorem on Symmetric Functions (Theorem 2.1) are well known. The symmetric functions or the sigma functions (Definition 2.2) of different totality in degrees have also been discussed and computed. Symmetric functions are related to some problems of combinatorial analysis and may be used as a tool for the study of partitions and other problems of combinations.

The purpose of this paper is to devise a program for the Fundamental Theorem on Symmetric Functions so that solutions may be found by automatic processes. The first section of this paper illustrates some of the elementary properties of symmetric functions. The second portion of this paper describes a computational program which actually consists of three main parts. First, we generate all possible partitions (Definition 2.6) corresponding to the given type of the sigma function (Definition 2.2). Secondly, we generate a system of linear equations from these partitions by assigning values to the variables \( x_1, x_2, \ldots, x_q \). Thirdly, we apply Gauss's elimination method for solving the generated system of equations. Hence, the desired coefficients are determined. In order to assist the machine to print out the results, we shall represent symmetric functions and
the elementary symmetric functions by means of numbers. In the third section we shall simply illustrate an example.

The heuristic has been tested on the ALWAC III-E computer and the program was written in Alcom language. The results have been shown to be correct in all the instances tested but the proof that the system of simultaneous equations can always be obtained by the method here described is still incomplete.
II. FUNDAMENTAL CONCEPTS ON SYMMETRIC FUNCTIONS

It is not our desire to present the complete theory on symmetric functions which can be found in texts on Higher Algebra and Theory of Equations. Instead we present only those portions necessary for our discussion. We shall find methods for expanding by automatic processes any symmetric polynomial as a polynomial in the elementary symmetric functions. Before we state the Fundamental Theorem on Symmetric Functions, we present a few basic definitions.

**Definition 2.1**: A polynomial \( f(x_1, x_2, \ldots, x_q) \) is called a symmetric polynomial or symmetric function if it is unchanged by arbitrary interchanges of the variables \( (x_1, x_2, \ldots, x_q) \).

For examples, the following polynomials

\[
(1) \quad f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2
\]

\[
(2) \quad f(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_2^2 x_1 + x_3^2 x_1 + x_3^2 x_2
\]

\[
(3) \quad f(x_1, x_2, x_3) = (x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2
\]

are symmetric polynomials or symmetric functions of the variables \( (x_1, x_2, x_3) \). However, the polynomial
(4) \[ f(x_1, x_2, x_3) = x_1^2 + x_2 + x_3^2 \]

is not a symmetric polynomial, since it is changed to a different polynomial if \( x_1 \) and \( x_2 \) are interchanged.

**Definition 2.2:** The sum of the monomial

\[
\sum_{\ell_1, \ell_2, \ldots, \ell_p} x_1^{\ell_1} x_2^{\ell_2} \cdots x_p^{\ell_p}
\]

and all other similar monomials obtained from it by interchanging their subscripts is called a sigma function or \( \sum \)-function of the type \( (\ell_1, \ell_2, \ldots, \ell_p) \) in the variables \( (x_1, x_2, \ldots, x_q) \) and is denoted by

\[
\sum_{\ell_1, \ell_2, \ldots, \ell_p} x_1^{\ell_1} x_2^{\ell_2} \cdots x_p^{\ell_p}.
\]

For example, the above symmetric polynomial (2) is a sigma function of the type \( (2, 1) \) in the variables \( (x_1, x_2, x_3) \).

For convenience in specifying the type, we may assume that the exponents \( \ell_1 \) form a monotone, non-increasing sequence.

**Definition 2.3:** The symmetric functions
\[ a_1 = \sum_{i} x_i \]
\[ a_2 = \sum_{i < j} x_i x_j \]
\[ a_3 = \sum_{i < j < k} x_i x_j x_k \]
\[ \vdots \]
\[ a_n = x_1 x_2 x_3 \cdots x_n \]

of the variables \((x_1, x_2, \ldots, x_n)\) are called the elementary symmetric functions of the variables \((x_1, x_2, \ldots, x_n)\).

The elementary symmetric functions are sigma functions; \(a_1\) is of type \((1)\), \(a_2\) is of type \((1, 1)\), and the type of \(a_i\) is a string of \(i\) 1's.

Hence, the elementary symmetric functions \(a_1, a_2, \ldots, a_n\) are clearly among the sigma functions. We shall now state the

**Fundamental Theorem on Symmetric Functions.**

**Theorem 2.1:** Every symmetric polynomial

\[ F(x_1, x_2, \ldots, x_q) \]

is equal to a polynomial, with integral coefficients, in the coefficients of \(F\) and the elementary symmetric functions \(a_1, a_2, \ldots, a_q\).
There are several proofs of this theorem. We shall refer in particular to the proof given by Dickson [2].

The ideas of the proof which are useful in our discussion are described briefly in the following definitions, lemmas, and Theorem 2.2.

**Definition 2.4:** A polynomial $F$ is homogeneous if it is a sum of monomials

$$c x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p}$$

each having the same total degree

$$n = l_1 + l_2 + \cdots + l_p.$$ 

**Definition 2.5:** In a homogeneous polynomial $F$, the monomial $f = c x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p}$ is said to be higher than the monomial $f' = c' x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ if $l_1 > k_1$, or if $l_1 = k_1$, $l_2 > k_2$, or if $l_1 = k_1$, $l_2 = k_2$, $l_3 > k_3$ and so forth, such that the first one of the differences $l_1 - k_1$, $l_2 - k_2$, $l_3 - k_3$, etc., which is not zero, is positive.

**Lemma 2.1:** If $f = c x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p}$ is the highest monomial of the symmetric polynomial $F$, then $l_1 \geq l_2 \geq \cdots \geq l_p$. (i.e. the exponents $l_i$ form a monotone non-increasing sequence.)
The proof is well known and sheds no light on our discussion.

**Lemma 2.2:** If $f = c_1 x_1^1 x_2^2 \cdots x_p^p$ is the highest monomial of the homogeneous symmetric polynomial $F$ and if $f' = c'_1 x_1^{l'_1} x_2^{l'_2} \cdots x_p^{l'_p}$ is that of $F'$, then the highest monomial in their product $FF'$ is

$$ff' = cc' x_1^{l_1+l'_1} x_2^{l_2+l'_2} \cdots x_p^{l_p+l'_p}.$$

**Proof:** Suppose $t = \beta x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ is a monomial of $FF'$, higher than $ff'$. Then $t$ is a product of the monomials

$$g = dx_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$$

and

$$g' = d'x_1^{k'_1} x_2^{k'_2} \cdots x_p^{k'_p}$$

Since $t$ is higher than $ff'$, then by Definition 2.5, the first one of the differences

$$k_1+k'_1 - \ell_1 - \ell'_1, k_2+k'_2 - \ell_2 - \ell'_2, \cdots, k_p+k'_p - \ell_p - \ell'_p$$

which is not zero is positive. However, either all the differences

$$k_1-\ell_1, k_2-\ell_2, \cdots, k_p-\ell_p$$

are zero or the first one which is not zero is negative, since $f$ is either identical to $g$ or is higher than $g$. Similarly, for the differences

$$k'_1-\ell'_1, k'_2-\ell'_2, \cdots, k'_p-\ell'_p.$$
and for \( f' \) and \( g' \). Thus, in both cases we have contradictions and hence the assumption is false. There is no monomial of \( FF' \) higher than \( ff' \).

**Lemma 2.3:** If \( f = x_1 x_2 \cdots x_p \) is the highest monomial of the homogeneous symmetric polynomial \( F \), then the highest monomial in the expansion of \( \phi = a_1 a_2 \cdots a_{p-1} a_p \) is \( f \).

**Proof:** Since the highest monomial in

\[
a_1 = \sum_{i} x_i
\]

\[
a_2 = \sum_{i<j} x_i x_j
\]

\[\vdots\]

\[
a_p = x_1 x_2 \cdots x_p
\]

are \( x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_p \) respectively, then by repeated application of Lemma 2.2, the highest monomial in

\[
\phi = a_1 a_2 \cdots a_{p-1} a_p
\]

is
Theorem 2.2: Any homogeneous symmetric polynomial with integral coefficients is a polynomial with integer coefficients in the elementary symmetric polynomials.

The proof is based on Lemma 2.3 and the idea of complete induction, for there can be only lower symmetric polynomials in the difference

$$\sum x_1 x_2 \cdots x_p - a_1 - a_2 - a_3 \cdots a_{p-1} - a_p$$

than in $$\sum x_1 x_2 \cdots x_p$$, if $$\ell_i > \ell_j, \ i < j$$. For details, see [2].

Theorem 2.2 leads us to consider the partition functions.

Definition 2.6: The partition function $$p(n)$$ is defined as the number of different ways by which the positive integer $$n$$ may be expressed as a sum of positive integers.

If we consider the sigma function

$$\sum x_1 x_2 \cdots x_p$$

of the type $$(\ell_1, \ell_2, \cdots, \ell_p)$$ whose total degree is $$\ell_1 + \ell_2 + \cdots + \ell_p = n$$,
then we may regard the type \((l_1, l_2, \ldots, l_p)\) as the components of a partition of \(n\). Theorem 2.2 relates the finite number of such partitions of \(n\) to the possible monomials in the elementary symmetric functions. However, Theorem 2.2 only guarantees that there are \(m\) monomials \(\phi_i(a_1, a_2, \ldots, a_q)\)'s where

\[
c_i \phi_i(a_1, a_2, \ldots, a_q) = c_1 a_1^{k_1} a_2^{k_2} \cdots a_q^{k_q} \quad \text{for } i = 1, 2, \ldots, m
\]

and \(c_i\)'s are the coefficients. It does not guarantee that these \(c_i\)'s may be found by an automatic process. Since there are \(m\) unknowns \(c_i\), it seems logical to generate \(m\) equations.

We have not proved that these \(m\) linear equations in the \(m\) unknown coefficients are linearly independent. Therefore, the program which is discussed in Chapter III must be considered as a heuristic program, although it has worked in every case for which the author has tested it.
III. DESCRIPTION OF A PROGRAM FOR THE FUNDAMENTAL THEOREM ON SYMMETRIC FUNCTIONS

In this section we describe a computational method for finding the polynomials in the elementary symmetric functions discussed in the Fundamental Theorem. Since there is no loss in generality by assuming the discussion restricted to homogeneous symmetric polynomials of type \( (l_1, l_2, \ldots, l_p) \), we shall make this restriction.

Generating Restricted Partitions

From the sigma function \( \sum x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p} \) of the type \( (l_1, l_2, \ldots, l_p) \) with the total degree \( n = l_1 + l_2 + \cdots + l_p \), we consider the type \( (l_1, l_2, \ldots, l_p) \) as a particular partition of the number \( n \) with \( p \) components. It follows from Lemma 2.1 that \( l_1 \geq l_2 \geq \cdots \geq l_p \) and Theorem 2.2 has claimed that there is a connection between the finitely many such partitions dominated by \( (l_1, l_2, \ldots, l_p) \) and the sought polynomials. We therefore order the sought polynomials according to their heights and group them into sequences. Each sequence is determined by a restricted partition of length greater than or equal to \( p \).

For example, in the calculation of \( \sum x_1^3 x_2^2 x_3^2 \) we concern ourselves with the following sequences of partitions:
(a) 3, 2, 1, 1; 2, 2, 2, 1;
(b) 3, 1, 1, 1, 1; 2, 2, 1, 1, 1;
(c) 2, 1, 1, 1, 1;
(d) 1, 1, 1, 1, 1, 1.

We first compute \( n = \sum_{i=1}^{p} l_i \). If \( p \leq 2 \), let \( k_i = l_i \) for \( i = 1, 2 \), and store these \( k_i \) in arrays \( c_{ji} \) where \( j \) denotes the \( j^{th} \) partition and \( i \) denotes the \( i^{th} \) component of the \( j^{th} \) partition. Thus, \( j \) begins with one and with the increment of one for each new partition generated. To keep track of the number of the components for each partition, we set

\[
F_j = s
\]

number of components of the partition.

If \( k_1 - k_2 > 1 \), then we take for the new \( k_2 \) the successor of the old \( k_2 \) and take for the new \( k_1 \), \( n \) minus the new \( k_2 \), and store these \( k_i \) components in arrays \( c_{ji} \). If \( k_1 - k_2 = 1 \) or 0, then we increase \( s \) by one and also if we begin with \( p > 2 \), then we take for the new \( k_i \), \( i = 2, 3, \ldots, s \), the value 1, and take for the new \( k_1 \), \( n \) minus the sum of these \( k_i \), \( i = 2, 3, \ldots, s \). Again we compare as before for \( k_1 \) and \( k_2 \). If \( k_1 - k_2 = 1 \) or 0, but \( k_1 - k_3 > 1 \), then we take for the new \( k_2 \) and \( k_3 \) the successor of the old \( k_3 \), and take for the new \( k_1 \), \( n \) minus the sum of these \( k_i \), \( i = 2, 3, \ldots, s \). We begin with comparison between
and $k_3$, then with $k_1$ and $k_3$. If $k_1 - k_2 = 1$ or 0, and $k_1 - k_3 = 1$ or 0, but $k_1 - k_4 > 1$, then we take for the new $k_2$, $k_3$ and $k_4$ the successor of the old $k_4$ and take for the new $k_1$, $n$ minus the sum of these new $k_i$, $i = 2, 3, \cdots, s$. Finally, we get to $k_1 - k_i = 1$ or 0 for $i = 2, 3, \cdots, s$, and come to the end of the first sequence.

Then we increase $s$ by one, and we repeat the same process over again. We continue this process until $s = n$, the degree of the original, that is, each component $k_i$ for $i = 1, 2, \cdots, n$ has the value one.

However, each of these partitions generated will be compared with the original partition $(\ell_1, \ell_2, \cdots, \ell_p)$ and they will be stored in the $c_{ji}$ arrays if they satisfy the following conditions:

if $\ell_1 > k_1$, or if $\ell_1 = k_1$, $\ell_2 > k_2$, or if $\ell_1 = k_1$, $\ell_2 = k_2$, $\ell_3 > k_3$, and so on (Definition 2.5). We reject any partition when any of its components satisfying the relation $k_i < \ell_i$ for $i = 1, 2, \cdots, p$.

In this way, we have generated $m$ partitions as $c_{i1}, c_{i2}, c_{i3}, \cdots, c_{IF_i}$ for $i = 1, 2, \cdots, m$ denoting the order of the partitions which have been generated. They all satisfy the relations $c_{i1} \geq c_{i2} \geq \cdots \geq c_{IF_i}$ and $c_{i1} + c_{i2} + \cdots + c_{IF_i} = n$ for $i = 1, 2, \cdots, m$.

The scheme discussed above is a modification of one presented in reference [1] in another context.
Computation of $M_i$

Since the type $(\ell_1, \ell_2, \cdots, \ell_p)$ of the sigma function satisfies the relation $\ell_1 > \ell_2 > \cdots > \ell_p$, these $\ell_i$'s may be all equal or all distinct, or there may be some equal and some distinct. Thus, we define $M_i$, $i = 1, 2, \cdots, r$ to be the number of $\ell_j$'s which have the same value. For example, if $\ell_1 = \ell_2 > \ell_3 > \ell_4 = \ell_5$, then $M_1 = 2$, $M_2 = 1$, and $M_3 = 2$. It is then clear that

$$\sum_{i=1}^{r} M_i = p,$$

the number of components.

Since we have generated $m$ distinct partitions, thus from Lemma 2.3 and Theorem 2.2, we have the homogenous symmetric polynomial expressed as a sum of products of $m$ distinct monomials $\phi_i(a_1, a_2, \cdots, a_q)$ by integers, i.e.,

$$F = c_1 \phi_1 + c_2 \phi_2 + \cdots + c_m \phi_m.$$

In order to compute the coefficients of these monomials, which we know must exist in the expansion of the homogeneous symmetric polynomial, we generate $m$ linear equations. To do so, we assign values to the variables $x_1, x_2, \cdots, x_q$ and evaluate $\phi_i$'s for these values of $x_j$'s.

Generating Values of $a_i$'s

First, if we assign only 1's and 0's to the variables $(x_1, x_2, \cdots, x_q)$, then we can compute the values of the elementary
symmetric functions $a_i$'s by the formula \( \binom{k}{i} \) for \( i=1,2,\ldots,k \)
where \( k \) is the number of nonzero \( x_i$'s which have the value 1
and whenever we have \( i > k \), we set \( a_i = 0 \). We can always begin
with \( k = p \), since each monomial \( \phi_i$'s will consist of at least one
\( a_i $ as a factor when \( i \geq p \).

We can either increase \( k \) or increase the value of \( x_1 $ and keep \( k \) the same; in either case, we obtain a different set of
values of \( a_i$'s. If \( x_1 = b > 1 \), and all other \( x$'s are 1, then
we again denote \( k \) as the number of nonzero \( x$'s which have the
values 1 except \( x_1 $ which has the value greater than 1. To
compute the values of these elementary symmetric functions,
\( a_i$, \( i=1,2,\ldots,q $, under the assignment of values above, we use
the following formula

\[
a_i = b^{k-1} \binom{k-1}{i-1} + \binom{k-1}{i}
\]

where \( \binom{k-1}{i-1} \) is the number of non-vanishing terms containing \( x_1 $ as a factor, and \( \binom{k-1}{i} \) denotes the number of non-vanishing terms
not containing \( x_1 $ as a factor, which also represents the value of these
terms since all other \( x$'s have the value 1. Whenever we have
\( i \geq k \), we set \( a_i = 0 \).
Generating Values of $\phi_i (a_1, a_2, \ldots, a_q)$

Each of the monomials $\phi_i$ is expressed in terms of the elementary symmetric function $a_i$ and these $a_i$'s have already been computed. Thus, these $\phi_i$'s may be computed from the partition $c_{ij}$, $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, F_i$, (see equation (5) above) which we have generated previously. From Lemma 2.3 and Theorem 2.2 we have

$$
\sum_{i=1}^{m} \xi_1 \xi_2 \cdots \xi_p = \sum_{i=1}^{m} \phi_i (a_1, a_2, \ldots, a_q) c_i,
$$

in which $\phi_i (a_1, a_2, \ldots, a_q) = a_1^{c_{i1}} a_2^{c_{i2}} \cdots a_q^{c_{iF_i}}$. The $c_i$'s are the unknowns to be determined and their coefficients can be computed by finding the products of these $a_i$'s. We denote this value by $A_{d_i}$. Whenever we have $k < F_i$, we have $\phi_i = 0$ since the monomial $\phi_i$ must contain an elementary symmetric function $a_i = 0$. Where $i > k$, therefore, we set $A_{d_i} = 0$.

We assign the next set of values for the variables based on the following decisions. If $A_{d, d+1} = 0$, then we increase, $k$, the number of nonzero variables, by one. If $A_{d, d+1} \neq 0$, then the number of nonzero variables will remain the same, but the value of $x_1$ will be increased by one.
Generating Values of \( \sum x_1^1 x_2^2 \cdots x_p^p (Y_d) \)

Below are some simple examples. Consider \( \sum x_1^1 x_2^2 x_3^3 \).

If the \( l_i \)'s are all equal, we have \( \mathcal{M}_1 = 3 \). Consider the case where we have \( k = 3 \) nonzero variables. We have only a single monomial \( x_1^1 x_2^2 x_3^3 \) which has the value of 1 if we assign \( x_1 = x_2 = x_3 = 1 \), and the value of 2 if we assign \( x_1 = 2, x_2 = x_3 = 1 \). Now consider the case where there are \( k = 4 \) nonzero variables. We may write out the four possible monomials

\[
\begin{align*}
&x_1^1 x_2^2 x_3^3, \quad x_1^1 x_2^2 x_4^4, \quad x_1^1 x_3^3 x_4^4, \quad x_2^2 x_3^3 x_4^4.
\end{align*}
\]

Thus, we are choosing 3 things from 4 things. This can be done in \( \binom{k}{M_1} = \binom{4}{3} = 4 \) ways. If we have \( x_1 = x_2 = x_3 = x_4 = 1 \), then the value of the symmetric function is \( Y_d = 4 \), since each of these monomials has the value 1. However, if we assign \( x_1 = 2, x_2 = x_3 = x_4 = 1 \), it is clear that all the monomials which have \( x_1 \) as factor will have values different from one. Thus we want to find the number of monomials containing \( x_1 \). We see that there are \( \binom{k-1}{M_1-1} = \binom{3}{2} = 3 \) monomials, each of which has the value 2.

There are also monomials which do not contain \( x_1 \). The above example shows that there is only \( \binom{k-1}{M_1} = \binom{3}{3} = 1 \) such monomial with the value 1. Hence, the value of the symmetric function would
be \( Y_d = 3 \cdot 2^{1 + 1} \) under this assignment of values.

If we consider the case in which \( k_1 = k_2 > k_3 \), then we have \( M_1 = 2, \ M_2 = 1 \). If we have \( k = 4 \) nonzero values of the variables, then we may write down possible monomials with nonzero value as follows:

\[
\begin{align*}
&x_1^1 x_2^1 x_3^3, \quad x_1^1 x_2^1 x_4^3, \quad x_1^1 x_3^1 x_4^3, \\
&x_1^1 x_3^1 x_2^3, \quad x_1^1 x_4^1 x_2^3, \quad x_1^1 x_4^1 x_3^3, \\
&x_2^1 x_3^1 x_1^3, \quad x_2^1 x_4^1 x_1^3, \quad x_3^1 x_4^1 x_1^3, \\
&x_2^1 x_3^1 x_4^3, \quad x_2^1 x_4^1 x_3^3, \quad x_3^1 x_4^1 x_2^3.
\end{align*}
\]

Thus we have \( \binom{k}{M_1} \binom{k-M_1}{M_2} = \binom{4}{2}\binom{2}{1} = 12 \) ways of expressing the monomials. If we have \( x_1 = x_2 = x_3 = x_4 = 1 \), then each of these monomials has the value 1 and the symmetric function has the value 12. If we have \( x_1 > 2 \), say \( x_1 = b \), and \( x_2 = x_3 = x_4 = 1 \), then we would like to determine the number of ways that \( x_a \) or \( x_\beta \) has the value of \( x_1 \) in the determination of the value of

\[
\sum (x_1 x_2)^{k_1} x_3^{k_3} \text{ is } \binom{k-1}{M_{1-1}} (\binom{k-M_1}{M_2}^3) \binom{2}{1} = 6. \quad \text{Thus the polynomial has the value } 6b^1. \quad \text{However, we may consider this separately by finding the number of nonzero monomials in the value of the expansion}
\]
and the number of them different from zero. The number of ways that \(x_1\) is the first factor of \(\frac{\ell_1}{a} \frac{\ell_1}{b} \frac{\ell_3}{\gamma}\) is

\[
\frac{1}{M_1 M_\ell_1 M_\ell_2} \binom{k-1}{M_1_1} \binom{k-M_1}{M_2} = \frac{1}{2} \binom{3}{1} \binom{2}{1} = 3
\]

with value \(3b\) and at \(x_\beta\), we have

\[
\frac{1}{M_1 M_\ell_1 M_\ell_2} \binom{k-1}{M_1_1} \binom{k-M_1}{M_2} = \frac{1}{2} \binom{3}{1} \binom{2}{1} = 3
\]

monomials with common value \(b\). Similarly, the number of ways that \(x_1\) is the third factor of \(\frac{\ell_1}{a} \frac{\ell_1}{b} \frac{\ell_3}{\gamma}\) is

\[
\frac{1}{M_2 M_\ell_2 M_\ell_3} \binom{k-1}{M_1_1} \binom{k-M_1}{M_2} = \frac{1}{2} \binom{3}{1} \binom{2}{1} = 3,
\]

and there is a contribution of \(3b^3\) to the value \(\mathbf{Y}_d\). Finally, the number of monomials which do not contain \(x_1\) as a factor is

\[
\binom{k-1}{M_1_1} \binom{k-M_1}{M_2} = \binom{3}{1} \binom{2}{1} = 3,
\]

and each of these terms has the value 1. Thus, the symmetric function has the total value

\[
\mathbf{Y}_d = 3b + 3b + 3b + 3 = 3.
\]

If we have \(\ell_1 > \ell_2 > \ell_3\), then we have \(M_1 = M_2 = M_3 = 1\). Consider the case in which \(k = 3\) nonzero variables. We have the following 6 monomials:

\[
\begin{align*}
\frac{\ell_1}{x_1} &\frac{\ell_2}{x_2} &\frac{\ell_3}{x_3}, &\frac{\ell_1}{x_1} &\frac{\ell_2}{x_3} &\frac{\ell_3}{x_2}, &\frac{\ell_1}{x_2} &\frac{\ell_2}{x_1} &\frac{\ell_3}{x_3}, \\
\frac{\ell_1}{x_3} &\frac{\ell_2}{x_1} &\frac{\ell_3}{x_2}, &\frac{\ell_1}{x_3} &\frac{\ell_2}{x_2} &\frac{\ell_3}{x_1}, &\frac{\ell_1}{x_3} &\frac{\ell_2}{x_3} &\frac{\ell_3}{x_1}.
\end{align*}
\]

This shows that there are \(\binom{k}{M_1} \binom{k-M_1}{M_2} \binom{k-M_1-M_2}{M_3} = \binom{3}{1} \binom{2}{1} \binom{1}{1} = 6\) ways of expressing the monomials. Hence, if we assign
\(x_1 = x_2 = x_3 = 1\), then each of these monomials has the value 1 and the value of the symmetric function is \(Y_d = 6\). If we assign

\[x_1 = b \geq 2, \quad \text{and} \quad x_2 = x_3 = 1,\]

then we would consider the monomial \(x_1 x_2 x_3^3\) and locate the variable \(x_1\) in these monomials. Thus, the number of ways that \(x_1\) is the first factor of \(x_1 x_2 x_3^3\) is

\[
\binom{k-1}{k-M_1} \binom{k-M_2-M_3}{M_1-1} = \binom{2}{0} \binom{2}{1} \binom{1}{1} = 2,
\]

and the sum of these monomials would have the value \(2b\). Similarly, the number of ways that \(x_1\) is the second factor of \(x_1 x_2 x_3^3\) is

\[
\binom{k-1}{k-M_2} \binom{k-M_1-M_3}{M_2-1} = \binom{2}{0} \binom{2}{1} \binom{1}{1} = 2
\]

and the sum of those monomials would be \(2b^2\). The number of monomials that \(x_1\) is the third factor of \(x_1 x_2 x_3^3\) is

\[
\binom{k-1}{k-M_3} \binom{k-M_1-M_2}{M_3-1} = \binom{2}{0} \binom{2}{1} \binom{1}{1} = 2
\]

and each has the value \(b^3\). However, the number of monomials which do not contain \(x_1\) as a factor would be none, since there are only 3 variables and we would have to have chosen 3 things from 2 things if we wanted to exclude \(x_1\). We use the previous formula to compute the number of ways that there can be monomials which do not contain \(x_1\) as a factor, i.e.

\[
\binom{k-1}{k-M_1} \binom{k-M_1-M_2}{M_1-1} = \binom{2}{0} \binom{1}{1} \binom{0}{1} = 0.
\]

Thus the value of the symmetric function would be \(Y_d = 2b + 2b^2 + 2b^3\).
We may generalize these simple examples to any arbitrary symmetric functions \[ \sum \ell_1 \ell_2 \cdots \ell_p \] of the type \((\ell_1, \ell_2, \cdots, \ell_p)\) and assume that the type has the following form of exponents \(M_1, M_2, \cdots, M_r\). If we assign only 1 to all \(x\)'s, then each monomial has the value 1, and the value of the symmetric functions would simply be

\[
Y_d = \left( \frac{k}{M_1} \right) \left( \frac{k-M_1}{M_2} \right) \cdots \left( \frac{k-M_1-M_2}{M_r} \right)
\]

\[
= \frac{k!}{M_1! M_2! \cdots M_r! (k-M_1-M_2-\cdots-M_r)!}
\]

\[
= \frac{k!}{\prod_{i=1}^{r} (M_i)!} (k-p)!
\]

If \(x_1\) is different from 1 and all other \(x\)'s have the value 1, then we would consider the monomial \(x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p}\), and we would like to determine the number of ways that \(x_1\) is the \(i\)th factor of \(x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p}\) for \(i = 1, 2, \cdots, p\). Hence at \(x_1^{l_1}\), we would have
\[
\frac{1}{M_1} \binom{k-1}{M_1-1} \binom{k-M_1}{M_2} \binom{k-M_1-M_2}{M_3} \ldots \binom{k-M_1-M_2-\ldots-M_{r-1}}{M_r}
\]

\[
= \frac{1}{M_1} \frac{(k-1)!}{(M_1-1)! M_2! M_3! \ldots M_r! (k-M_1-M_2-\ldots-M_r)!}
\]

\[
= \frac{(k-1)!}{(M_1)(M_1-1)! \left( \prod_{i=2}^{r} M_i! \right) (k-p)!}
\]

where \((k-1)\) denotes the number of nonzero \(x_1\)'s not containing \(x_1\). The above formula would denote the number of nonvanishing terms that \(x_1\) is the first factor of \(x_1 \cdot x_2^{p} \ldots x_p^{p}\). Thus it has the value

\[
S_1 = \frac{x_1^{(k-1)!}}{(M_1)(M_1-1)! \left( \prod_{i=2}^{r} M_i! \right) (k-p)!}
\]

We then have the following general formula

\[
S_i = \frac{x_1^{(k-1)!}}{(M_i)(M_i-1)! (k-p)! \left( \prod_{j=1, j \neq i}^{r} m_j! \right)}
\]

for \(i = 1, 2, \ldots, p, t = 1, 2, \ldots, r, \) and \(t\) will be increased by one whenever \(i\) increases \(M_t\) times. However, there may be some monomials where there is no \(x_1\) at all, and the number of these monomials may be computed as
(k-1) M_{1}^{k-1-M_{1}} M_{2}^{k-1-M_{1}-M_{2}} \cdots M_{r}^{k-1-M_{1}-M_{2}-\cdots-M_{r}} \\
= \frac{(k-1)!}{M_{1}! M_{2}! \cdots M_{r}! (k-1-M_{1}-M_{2}-\cdots-M_{r})!} \\
= \frac{(k-1)!}{\prod_{i=1}^{r} M_{i}!} (k-1-p)! \\

Since each of these monomials has the value 1, we denote its value by Q and Q = 0 whenever k \leq p. Hence the value of the symmetric function would be \\
Y_d = \sum_{i=1}^{p} S_i + Q \\
where d ranges from 1 to m.

Evaluating System of Linear Equations

We have generated the following system of linear equations.

\begin{align*}
A_{11} c_1 + A_{12} c_2 + \cdots + A_{1m} c_m &= Y_1 \\
A_{21} c_1 + A_{22} c_2 + \cdots + A_{2m} c_m &= Y_2 \\
\vdots & \vdots \vdots \vdots \vdots \\
A_{m1} c_1 + A_{m2} c_2 + \cdots + A_{mm} c_m &= Y_m
\end{align*}
Where $A_{di}$ and $Y_d$ for $d, i = 1, 2, \cdots, m$ are the previously generated values, and $c_i$ for $i = 1, 2, \cdots, m$ are to be determined, we may express this system in the table below:

**Table 3.1. System of Linear Equations.**

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$\cdots$</th>
<th>$c_m$</th>
<th>$Y_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11}$</td>
<td>$A_{12}$</td>
<td>$A_{13}$</td>
<td>$\cdots$</td>
<td>$A_{1m}$</td>
<td>$Y_1$</td>
</tr>
<tr>
<td>$A_{21}$</td>
<td>$A_{22}$</td>
<td>$A_{23}$</td>
<td>$\cdots$</td>
<td>$A_{2m}$</td>
<td>$Y_2$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$A_{m1}$</td>
<td>$A_{m2}$</td>
<td>$A_{m3}$</td>
<td>$\cdots$</td>
<td>$A_{mm}$</td>
<td>$Y_m$</td>
</tr>
</tbody>
</table>

To determine $c_i$, $i = 1, 2, \cdots, m$, we apply Gauss's elimination method. We first reduce the above system to the triangular form by eliminating one of the unknowns at a time. We divide the first equation by $A_{11}$, and subtract this equation multiplied by $A_{di}$ ($d = 2, 3, \cdots, m$) from the remaining $m-1$ equations. We then eliminate the unknown $c_2$ from $m-2$ of the $m-1$ equations not containing $c_1$, repeating the same process until the unknown $c_m$ appears only in the last equation, and $c_{m-1}$ and $c_m$ in the next to the last, etc. Thus the system appears in the triangular
form as in Table 3.2.

Table 3.2. Triangular Form of the System of Equations.

<table>
<thead>
<tr>
<th></th>
<th>c₁</th>
<th>c₂</th>
<th>c₃</th>
<th>...</th>
<th>cₘ</th>
<th>Zd</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b₁₂</td>
<td>b₁₃</td>
<td>...</td>
<td>b₁ₘ</td>
<td>Z₁</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>b₂₂</td>
<td>b₂₃</td>
<td>...</td>
<td>b₂ₘ</td>
<td>Z₂</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>1</td>
<td>Zₘ</td>
<td></td>
</tr>
</tbody>
</table>

By backward substitution, the unknowns for \( c_i \) for \( i = 1, 2, \ldots, m \) are then evaluated. Thus \( c_m \) is obtained from the \( m^{th} \) equation, \( c_{m-1} \) is evaluated by substituting the value of \( c_m \) into the \((m-1)^{st}\) equation; \( c_{m-2} \) is solved by substituting \( c_m \) and \( c_{m-1} \) into the \((m-2)^{nd}\) equation, and so on.

Notations for Print-Out

Since the typewriter will not print exponents, we change our notations and symbols to suit the purpose. Thus, our sigma function will be denoted by its type with the symbol \( \sum \) in front of it. For example, the sigma function \( \sum x_1^3 x_2 x_3 x_4 \) will be printed as
\[ \sum (3 \, 1 \, 1 \, 1). \] We represent the \( j^{\text{th}} \) power of the \( i^{\text{th}} \) elementary symmetric functions by the notation \( (i \, j) \). For example, \( a_1^3 a_2^2 a_4 \) would be printed as \((1 \, 3) \, (2 \, 2) \, (4 \, 1)\).
IV. AN EXAMPLE

A program based on the above consideration has been written for the ALWAC III-E computer in the Alcom language. A drawback of the use of the ALWAC III-E computer with this program is that we may be unable to compute fairly simple symmetric functions even though the program works in general. Trouble occurs when the symmetric functions are expressed by a linear combination of more than 32 monomials \( \phi_i(a_1, a_2, \cdots, a_q) \), because the double-subscript arrays of Alcom are limited to 32 in number. Illustrated below is a simple example. Other examples are given in the Appendix.

Consider the following sigma function \( \sum x_1^3 x_2 x_3 x_4 \) of the type \((3, 1, 1, 1)\), the number of variables being arbitrary. We shall first generate all the restricted partitions. We have

\[
3, 1, 1, 1; \quad \text{as } c_{1j}, \quad j = 1, 2, 3, 4.
\]

\[
2, 2, 1, 1; \quad \text{as } c_{2j}, \quad j = 1, 2, 3, 4.
\]

\[
2, 1, 1, 1, 1; \quad \text{as } c_{3j}, \quad j = 1, 2, 3, 4, 5.
\]

\[
1, 1, 1, 1, 1, 1; \quad \text{as } c_{4j}, \quad j = 1, 2, 3, 4, 5, 6.
\]

From these partitions and Lemma 2.3, we have
Thus, we have

\[
\begin{align*}
&\sum_{x=1}^{3} x_1^2 x_2 x_3 x_4 = c_1^2 a_4 + c_2 a_2 a_4 + c_3 a_1 a_5 + c_4 a_6 \\
&\text{where the } c_i's \text{ are the integral coefficients to be determined.}
\end{align*}
\]

Since \( x_1, x_2, \ldots, x_q \) satisfy the following equation

\[
x^q - a_1 x^{q-1} + a_2 x^{q-2} - \cdots + (-1)^q a_q = 0
\]

and only \( a_1, a_2, a_4, a_5 \) and \( a_6 \) appear in the right member of (6), we can assume that \( a_7 = 0, a_8 = 0, \text{ etc.} \), without affecting (6).

Thus (7) may be written to be

\[
x^6 - a_1 x^5 + a_2 x^4 - a_3 x^3 + a_4 x^2 - a_5 x + a_6 = 0
\]

Since \( c_1 \) has the coefficient \( a_1^2 a_4 \), we first choose

\[x_1 = x_2 = x_3 = x_4 = 1 \text{ and } x_5 = x_6 = 0.\]

These x's satisfy the equation
We have

\[
\begin{align*}
\frac{2}{(x-1)^4} &= x^6 - \binom{4}{1}x^5 + \binom{4}{2}x^4 - \binom{4}{3}x^3 + \binom{4}{4}x^2.
\end{align*}
\]

Now each term of the left member of (1) has the form \(x^a x^b x^c x^d\) whose subscripts are four of the first four natural numbers, if we neglect terms which vanish. There are \(\binom{4}{4} = 4\) choices for \(x^a\).

After one of these has been selected, there are \(\binom{3}{3} = 1\) choice for \((x^b x^c x^d)\). The left member of (1) consists of \(\binom{4}{1}\binom{3}{3} = 4\) non-vanishing terms, each of which has the value 1, i.e.,

\[
\sum x_1^3 x_2 x_3 x_4 = 4.
\]

Thus, from (6), (9), and (10), we have

\[
16c_1 + 6c_2 = 4
\]

in which \(A_{11} = 16, A_{12} = 6, A_{13} = A_{14} = 0, Y_1 = 4\), since
A_{12} \neq 0. We assign x_1 = 2, x_2 = x_3 = x_4 = 1 and x_5 = x_6 = 0. To compute the \( a_i \)'s, we use the following formula described in Chapter III.

\[ a_i = 2^{(k-1)} + \binom{k-1}{i} \]

Thus

\[
\begin{cases}
    a_1 = 2(1-1) + \binom{4-1}{1-1} = 2(0) + \binom{3}{1} = 5 \\
    a_2 = 2^{(1)} + \binom{3}{2} = 9 \\
    a_3 = 2^{(2)} + \binom{3}{3} = 7 \\
    a_4 = 2^{(3)} + \binom{3}{4} = 2 + 0 = 2 \\
    a_5 = a_6 = 0
\end{cases}
\]  

(12)

However, since \( x_1 = 2 > 1 \), we need a slight modification for computing the left member of (6). The number of nonvanishing terms that \( x_1 \) is the first factor of the monomial \( x^3_\alpha (x_\beta x x_\gamma) \) is \( \binom{3}{3} = 1 \), since there is only one possible way for \( x_2, x_3, \) and \( x_4 \) to be the factor \( (x_\beta x x_\gamma) \). The sum of the values of such monomials is \( 2^3 \binom{3}{3} = 2^3 = 8 \). Also, the number of non-vanishing terms that \( x_1 \) is a factor of \( x_\beta x x_\gamma \) is \( \binom{3}{1}(\binom{2}{2}) = 3 \) and the value for these terms is \( 2^{(3)} \binom{2}{2} = 2 \cdot 3 \cdot 1 = 6 \). Finally, the number of
non-vanishing terms in which $x_1$ is not a factor of $x_1^3(x_\alpha x_\beta x_\gamma x_\delta)$ is none, since there are only three nonzero variables. The sum of all the values of the left member of (6) is $8 + 6 + 0 = 14$, i.e., for this assignment,

\[(13) \quad \sum x_1^3 x_2 x_3 x_4 = 14.\]

Hence, from (6), (12), and (13), we have

\[(14) \quad 50c_1 + 18c_2 = 14\]

in which $A_{21} = 50$, $A_{22} = 18$, $A_{23} = A_{24} = 0$, $Y_2 = 14$. Since $A_{23} = 0$, we thus choose $x_1 = x_2 = x_3 = x_4 = x_5 = 1$ and $x_6 = 0$. Similarly, we obtain the values of the $a_i$'s as in the first case, i.e.,

\[
\begin{align*}
  a_1 &= \binom{5}{1} = 5 \\
  a_2 &= \binom{5}{2} = 10 \\
  a_3 &= \binom{5}{3} = 10 \\
  a_4 &= \binom{5}{4} = 5 \\
  a_5 &= \binom{5}{5} = 1 \\
  a_6 &= 0
\end{align*}
\]

As in the first case, the left member of (1) consists of $\binom{5}{1}\binom{4}{3} = 20$ non-vanishing terms, each of which has the value 1. Thus
Again, (6), (15), and (16) give

\[ 125c_1 + 50c_2 + 5c_3 = 20 \]  

where \( A_{31} = 125, \ A_{32} = 50, \ A_{33} = 5, \ A_{34} = 0, \ Y_3 = 20. \) 

Since \( A_{34} = 0, \) we finally choose \( x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1, \) from which we obtain

\[
\begin{align*}
    a_1 &= \binom{6}{1} = 6 \\
    a_2 &= \binom{6}{2} = 15 \\
    a_3 &= \binom{6}{3} = 20 \\
    a_4 &= \binom{6}{4} = 15 \\
    a_5 &= \binom{6}{5} = 6 \\
    a_6 &= \binom{6}{6} = 1 \\
\end{align*}
\]

The left member of (6) consists of \( \binom{6}{1}(\binom{5}{3}) = 60 \) non-vanishing terms, each of which has the value 1. Thus

\[ \sum^3_{x=1} x_1^3 \cdot x_2^x \cdot x_3^x \cdot x_4 = 60. \]

Therefore, from (6), (18), and (19), we obtain
(20) \[ 540c_1 + 125c_2 + 36c_3 + c_4 = 60 \]

in which \( A_{41} = 540, \ A_{42} = 125, \ A_{43} = 36, \ A_{44} = 1 \) and \( Y_4 = 60 \).

Having generated the above system of linear equations (11), (14), (17) and (20), we apply Gauss' elimination procedure and discover that there is a unique solution and finally conclude that

(21) \[ \sum x_1^3 x_2^2 x_3 x_4 = a_1 a_4 - 2a_2 a_4 - a_1 a_5 + 6a_6 . \]
BIBLIOGRAPHY


APPENDIX I

ALCOM PROGRAMMING

The program described in the body of the thesis is given in the following pages. When the program tape is run, the computer will stop and ask for input, at which time the number of subscripts in the highest monomial of the homogeneous symmetric polynomial

\[ \sum_{i} l_1 x_1^{l_2} \cdots x_p^{l_p} \]

is typed in decimal form. It will stop \( p \) times thereafter. At the first stop \( l_1 \) is typed in; at the second stop \( l_2 \) and so forth up to \( l_p \). Then the program runs freely.
PROGRAM "UPATISRINGA"

PROCEDURE FACTO(G)
  T=1
  if g-1 neg go to 2
  for s=1 step 1 until g
    T*T*S
  end
  2: FACTO=T
END

READ P
FOR I=1 step 1 until p
read 1(I)
END

J=1
M=P
N=0
for i=1 step 1 until p
N=N+L(I)
END
IF 2-p neg go to 11
for i=1 step 1 until p
k(I)=L(I)
END
GO TO 15
11: FOR I=2 step 1 until m
  k(I)=1
end
12: D=0
for i=2 step 1 until m
  d=D+K(I)
END
K(1)=N-D
r=1
13: IF L(R)-K(R) NZ go to 14
  R=R+1
if p-r neg go to 15
go to 13
14: IF L(R)-K(R) NEG go to 16
15: FOR I=1 step 1 until m
  C(J,I)=K(I)
END
F(J)=M
J=J+1
16: T=2
17: IF T-m-1 NZ go to 18
go to 20
18: IF K(T)-k(1)-1 neg go to 19
  T=T+1
go to 17
19: \( K(T) = K(T) + 1 \)
for I=T STEP -1 until 2
k(I)=K(T)
END

GO TO 12

20:M=M+1
if n-m neg go to 21
go to 11

21:J=J-1
I=1
B=1

22:M(B)=1
23:IF F-I neg go to 24
go to 27

24:IF L(I) = L(I+1) NZ GO TO 26
M(B)=M(B)+1
25:I=I+1
go to 23

26:B=B+1
M(B)=1
go to 25
27:V=1

for i=1 step 1 until b
r(I)=FACTO(M(I))
V=V*R(I)
END

31:B=1
K=p
32:X=1
%:=FACTO(K)
I=1

33:IF K-I neg go to 36
?:=FACTO(I)
K=K-I
I:=FACTO(K)
S(I)=S(I)/(?)!
35:I=I+1
if n-I neg go to 37
go to 34
36:S(I)=0
go to 35
37:I=1
38:IF K-F(I) NEG GO TO 40
A:=1
A(d,1)=1
39:IF F(I) neg go to 43
40:IF C(I,2) neg go to 41
A(d,1)=A(A(I),3)
go to 42
41: \( A(d,1) = A(d,i) \times S(g) \) \( \cdot C(i,g) \)  
\[ \text{GO TO 42} \]
42: \( I = I + 1 \)  
\[ \text{if } j-I+1 \text{ nz go to 38} \]
\[ \text{go to 50} \]
43: IF \( C(i,g) - C(i,g+1) \) NZ go to 45  
44: \( G = G + 1 \)  
\[ \text{go to 39} \]
45: IF \( C(I,G) - C(I,G+1) - 1 \) nz go to 46  
\[ A(d,i) = A(d,i) \times S(g) \]  
\[ \text{go to 44} \]
46: \( E = C(i,g) - C(i,g+1) \)  
\[ A(d,i) = A(d,i) \times S(g) \) \( \cdot E \)  
\[ \text{GO TO 44} \]
48: \( K = K + 1 \)  
\[ \text{go to 33} \]
50: \( Z = K - p \)  
\[ \Delta = \text{FACTO}(Z) \]
IF \( X-1 \) nz go to 60  
\[ Y(D) = X/(V+\Delta) \]  
\[ \text{go to 70} \]
60: \( L = K - 1 \)  
\[ t(1) = \text{FACTO}(L) \]
\[ F = 1 \]  
for \( q = 1 \) step 1 until \( b \)  
\[ t(2) = \text{FACTO}(M(Q)-1) \]
\[ H = 1 \]  
for \( i = 1 \) step 1 until \( b \)  
\[ \text{if } q-i \text{ nz go to 61} \]
\[ \text{go to 62} \]
61: \( H = H \times R(I) \)  
62: \( H = H \)  
END  
\[ T(3) = T(1) \times X \times L(F) \]
\[ T(4) = M(Q) \times T(2) \times H \times \Delta \]
63: \( G(F) = T(3)/T(4) \)
\[ R = 1 \]
64: \( F = F + 1 \)  
\[ \text{if } m(q)-r \text{ nz go to 65} \]
\[ \text{go to 66} \]
65: \( R = R + 1 \)  
\[ w = F - 1 \]
\[ g(F) = G(W) \]  
\[ \text{GO TO 64} \]
66: END  
\[ y(D) = 0 \]
for \( i = 1 \) step 1 until \( p \)  
\[ y(D) = y(D) \times G(1) \]
END
IF L-p neg go to 68
T(5)=FACTO(L-p)
T(6)=T(1)/(V*T(5))
go to 69
68:T(6)=0
69:Y(D)=Y(D)+T(6)
70:if j-d nz go to 71
go to 81
71:IF A(d,d+1) nz go to 72
D=D+1
if j-d neg go to 81
go to 48
72:X=X+1
a=X-1
G=K-1
U=FACTO(G)
I=1
73:IF K-i neg go to 76
$=FACTO(I)
$=I-1
w=FACTO($)
C=FACTO(k-i)
S(I)=%/(A+C)+A*U/(W*C)
74:IF N-i nz go to 75
go to 77
75:I=I+1
go to 73
76:S(I)=0
go to 74
77:D=D+1
go to 37
81:M=M-1
for I=1 step 1 until m
J=I+1
for d=G step 1 until J
if A(d,i) nz go to 82
go to 83
82:Q=A(D,i)/a(I,i)
FOR S=G step 1 until J
A(d,s)=A(d,s)-A(I,s)
END
Y(D)=Y(D)-Y(I)
83:END
END
C(J)=Y(J)/A(J,J)
FOR I=1 step 1 until m
K=J-1
V=K:1
for D=V step 1 until J
\[ Y(K) = Y(K) - C(D) \cdot A(k,d) \]
\[ C(K) = \frac{Y(K)}{A(k,k)} \]

PRINT "\( \Sigma \)"
FOR I=1 step 1 until p
PRINT L(I)3.0
end
PRINT ")=" CR
FOR I=1 step 1 until j
PRINT C(I)3.0
K=1
91: IF F(I)-k nz go to 92
PRINT "(" K3.0 C(i,k)3.0 ")"
go to 95
92: IF C(1,k)-C(1,k+1) nz go to 94
93: K=K+1
go to 91
94: Q=C(i,k)-C(i,k+1)
PRINT "(" K3.0 Q 3.0 ")"
go to 93
95: PRINT CR
END
END
APPENDIX II

SYMMETRIC FUNCTIONS OF SIXTH AND SEVENTH DEGREES

In this appendix are included the computer output for the calculation of the homogeneous symmetric polynomials of sixth and seventh degrees. The reader will have to use the conventions for print-out described at the end of Chapter III above. Thus, for example, the first calculation is the calculation of the homogeneous symmetric polynomial $\sum x_1^6$.

$$\sum x_1^6 = a_1^6 - 6a_1^4a_2 + 9a_1^2a_2^2 - 2a_2^3 + 6a_1^3a_3 - 12a_1^2a_3^2 + 3a_3^2 - 6a_1^2a_4 + 6a_2^2a_4 + 6a_1a_5 - 6a_6.$$
### Symmetric Functions of 6th Degree

\[ \Sigma(6) = \]

\[
\begin{array}{ccc}
1 & 1 & 6 \\
-6 & 1 & 4 \\
9 & 1 & 2 \\
-2 & 2 & 3 \\
6 & 1 & 3 \\
-12 & 1 & 1 \\
3 & 3 & 2 \\
-6 & 1 & 2 \\
6 & 2 & 1 \\
6 & 1 & 1 \\
-6 & 6 & 1 \\
\end{array}
\]

\[ \Sigma(5, 1) = \]

\[
\begin{array}{ccc}
1 & 1 & 4 \\
-4 & 1 & 2 \\
2 & 2 & 3 \\
-1 & 1 & 3 \\
7 & 1 & 1 \\
-3 & 3 & 2 \\
1 & 1 & 2 \\
-6 & 2 & 1 \\
-1 & 1 & 1 \\
6 & 6 & 1 \\
\end{array}
\]

\[ \Sigma(4, 2) = \]

\[
\begin{array}{ccc}
1 & 1 & 2 \\
-2 & 2 & 3 \\
-2 & 1 & 3 \\
4 & 1 & 1 \\
-3 & 3 & 2 \\
2 & 1 & 2 \\
2 & 2 & 1 \\
-6 & 1 & 1 \\
6 & 6 & 1 \\
\end{array}
\]

\[ \Sigma(3, 3) = \]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
-3 & 1 & 1 \\
3 & 3 & 2 \\
3 & 1 & 2 \\
-3 & 2 & 1 \\
-3 & 1 & 1 \\
3 & 6 & 1 \\
\end{array}
\]
\[ \Sigma( 4 \ 1 \ 1) = \\
1( 1 \ 3)( 3 \ 1) \\
-3( 1 \ 1)( 2 \ 1)( 3 \ 1) \\
3( 3 \ 2) \\
-1( 1 \ 2)( 4 \ 1) \\
2( 2 \ 1)( 4 \ 1) \\
1( 1 \ 1)( 5 \ 1) \\
-6( 6 \ 1) \]

\[ \Sigma( 3 \ 2 \ 1) = \\
1( 1 \ 1)( 2 \ 1)( 3 \ 1) \\
-3( 3 \ 2) \\
-3( 1 \ 2)( 4 \ 1) \\
4( 2 \ 1)( 4 \ 1) \\
7( 1 \ 1)( 5 \ 1) \\
-12( 6 \ 1) \]

\[ \Sigma( 2 \ 2 \ 2) = \\
1( 3 \ 2) \\
-2( 2 \ 1)( 4 \ 1) \\
2( 1 \ 1)( 5 \ 1) \\
-2( 6 \ 1) \]

\[ \Sigma( 3 \ 1 \ 1 \ 1) = \\
1( 1 \ 2)( 4 \ 1) \\
-2( 2 \ 1)( 4 \ 1) \\
-1( 1 \ 1)( 5 \ 1) \\
6( 6 \ 1) \]

\[ \Sigma( 2 \ 2 \ 1 \ 1) = \\
1( 2 \ 1)( 4 \ 1) \\
-4( 1 \ 1)( 5 \ 1) \\
9( 6 \ 1) \]

\[ \Sigma( 2 \ 1 \ 1 \ 1 \ 1) = \\
1( 1 \ 1)( 5 \ 1) \\
-6( 6 \ 1) \]
SYMMETRIC FUNCTIONS OF 7th DEGREE

\[ \Sigma (7) = \]
\[
1(1 \ 7)
-7(1 \ 5)(2 \ 1)
14(1 \ 3)(2 \ 2)
-7(1 \ 1)(2 \ 3)
7(1 \ 4)(3 \ 1)
-21(1 \ 2)(2 \ 1)(3 \ 1)
7(2 \ 2)(3 \ 1)
7(1 \ 1)(3 \ 2)
-7(1 \ 3)(4 \ 1)
14(1 \ 1)(2 \ 1)(4 \ 1)
-7(3 \ 1)(4 \ 1)
7(1 \ 2)(5 \ 1)
-7(2 \ 1)(5 \ 1)
-7(1 \ 1)(6 \ 1)
7(7 \ 1)
\]

\[ \Sigma (6 \ 1) = \]
\[
1(1 \ 5)(2 \ 1)
-5(1 \ 3)(2 \ 2)
5(1 \ 1)(2 \ 3)
-1(1 \ 4)(3 \ 1)
9(1 \ 2)(2 \ 1)(3 \ 1)
-7(2 \ 2)(3 \ 1)
-4(1 \ 1)(3 \ 2)
1(1 \ 3)(4 \ 1)
-8(1 \ 1)(2 \ 1)(4 \ 1)
7(3 \ 1)(4 \ 1)
-1(1 \ 2)(5 \ 1)
7(2 \ 1)(5 \ 1)
1(1 \ 1)(6 \ 1)
-7(7 \ 1)
\]

\[ \Sigma (5 \ 2) = \]
\[
1(1 \ 3)(2 \ 2)
-3(1 \ 1)(2 \ 3)
-2(1 \ 4)(3 \ 1)
6(1 \ 2)(2 \ 1)(3 \ 1)
3(2 \ 2)(3 \ 1)
-7(1 \ 1)(3 \ 2)
2(1 \ 3)(4 \ 1)
-4(1 \ 1)(2 \ 1)(4 \ 1)
7(3 \ 1)(4 \ 1)
-2(1 \ 2)(5 \ 1)
-3(2 \ 1)(5 \ 1)
7(1 \ 1)(6 \ 1)
-7(7 \ 1)
\]
\[ \Sigma(4 \ 3) = 
\begin{align*}
1(1 & 1)(2 3) \\
-3(1 & 2)(2 1)(3 1) \\
-1(2 & 2)(3 1) \\
5(1 & 1)(3 2) \\
3(1 & 3)(4 1) \\
-2(1 & 1)(2 1)(4 1) \\
-5(3 & 1)(4 1) \\
-7(1 & 2)(5 1) \\
-7(2 & 1)(5 1) \\
7(1 & 1)(6 1) \\
-7(7 & 1) \\
\end{align*} \]

\[ \Sigma(5 \ 1 \ 1) = 
\begin{align*}
1(1 & 1)(3 1) \\
-4(1 & 2)(2 1)(3 1) \\
2(2 & 2)(3 1) \\
4(1 & 1)(3 2) \\
-1(1 & 3)(4 1) \\
3(1 & 1)(2 1)(4 1) \\
-7(3 & 1)(4 1) \\
1(1 & 2)(5 1) \\
-2(2 & 1)(5 1) \\
-1(1 & 1)(6 1) \\
7(7 & 1) \\
\end{align*} \]

\[ \Sigma(4 \ 2 \ 1) = 
\begin{align*}
1(1 & 2)(2 1)(3 1) \\
-2(2 & 2)(3 1) \\
-1(1 & 1)(3 2) \\
-3(1 & 3)(4 1) \\
8(1 & 1)(2 1)(4 1) \\
-2(3 & 1)(4 1) \\
3(1 & 2)(5 1) \\
-4(2 & 1)(5 1) \\
-8(1 & 1)(6 1) \\
14(7 & 1) \\
\end{align*} \]

\[ \Sigma(3 \ 3 \ 1) = 
\begin{align*}
1(2 & 2)(3 1) \\
-2(1 & 1)(3 2) \\
-1(1 & 1)(2 1)(4 1) \\
5(3 & 1)(4 1) \\
4(1 & 2)(5 1) \\
-7(2 & 1)(5 1) \\
-4(1 & 1)(6 1) \\
7(7 & 1) \\
\end{align*} \]
\[ \Sigma(3\ 2\ 2) = \\
1( \ 1\ 1)( \ 3\ 2) \\
-2( \ 1\ 1)( \ 2\ 1)( \ 4\ 1) \\
-1( \ 3\ 1)( \ 4\ 1) \\
2( \ 1\ 2)( \ 5\ 1) \\
3( \ 2\ 1)( \ 5\ 1) \\
-7( \ 1\ 1)( \ 6\ 1) \\
7( \ 7\ 1) \]

\[ \Sigma(4\ 1\ 1\ 1) = \\
1( \ 1\ 3)( \ 4\ 1) \\
-3( \ 1\ 1)( \ 2\ 1)( \ 4\ 1) \\
3( \ 3\ 1)( \ 4\ 1) \\
-1( \ 1\ 2)( \ 5\ 1) \\
2( \ 2\ 1)( \ 5\ 1) \\
1( \ 1\ 1)( \ 6\ 1) \\
-7( \ 7\ 1) \]

\[ \Sigma(3\ 2\ 1\ 1) = \\
1( \ 1\ 1)( \ 2\ 1)( \ 4\ 1) \\
-3( \ 3\ 1)( \ 4\ 1) \\
-4( \ 1\ 2)( \ 5\ 1) \\
6( \ 2\ 1)( \ 5\ 1) \\
9( \ 1\ 1)( \ 6\ 1) \\
-21( \ 7\ 1) \]

\[ \Sigma(2\ 2\ 2\ 1) = \\
1( \ 3\ 1)( \ 4\ 1) \\
-3( \ 2\ 1)( \ 5\ 1) \\
5( \ 1\ 1)( \ 6\ 1) \\
-7( \ 7\ 1) \]

\[ \Sigma(3\ 1\ 1\ 1\ 1) = \\
1( \ 1\ 2)( \ 5\ 1) \\
-2( \ 2\ 1)( \ 5\ 1) \\
-1( \ 1\ 1)( \ 6\ 1) \\
7( \ 7\ 1) \]

\[ \Sigma(2\ 2\ 1\ 1\ 1) = \\
1( \ 2\ 1)( \ 5\ 1) \\
-5( \ 1\ 1)( \ 6\ 1) \\
14( \ 7\ 1) \]

\[ \Sigma(2\ 1\ 1\ 1\ 1\ 1) = \\
1( \ 1\ 1)( \ 6\ 1) \\
-7( \ 7\ 1) \]