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S.A. Stone

The detection of limit cycles may be accomplished in several ways. This thesis presents a numerical method for detection when the system is described by the Van der Pol equation.

Two numerical integration methods are used to increase the accuracy of the detection. The concept of a transversal is utilized to establish a simple technique for stopping the numerical solution. Detection is accomplished by comparing the closure distance for the solution as a trajectory in the phase-plane against an arbitrary (but justifiable) small number.

Experimental results and computation of optimal interval values for the numerical integrations were obtained using a CDC 3300.

Limit Cycle Detection of the Van der Pol Equation

by

Yoon Goon Bae

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Redacted for Privacy

Head of Department of Electrical and Electronics Engineering

Redacted for Privacy

Dean of Graduate School

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I. INTRODUCTION

Nonlinear systems may have a limit cycle, which is an isolated closed trajectory in the phase-plane corresponding to periodic motions. Engineers need to know how to determine the limit cycle in order to improve system designs. With the increased sophistication in the use of computers, it is possible to write programs which will determine the limit cycles of systems.

It is the purpose of this paper to examine the Van der Pol equation and develop a numerical method for determining the existence of a limit cycle within its parameters.

Definition of Terms Used

Limit Cycle. A disturbed dynamical unforced second-order system can be described by means of a phase portrait in a phase-plane. The phase-plane method is a graphic procedure for [illegible] ing the transient response of a second-o[illegible] solution is not available, the equations m[illegible] equations and time derivatives to obtain th[illegible]

In numerical methods, the nonlinear [illegible] equations are solved by simultaneously usin[illegible] differential equations. The numerical solutions then can be plotted in a phase-plane. In phase-plane terminology, a nonlinear system with a fixed periodic output (an isolated closed trajectory in a phase-plane) is said to have a limit cycle.

A closed trajectory is called stable if it is approached by the trajectories from the inside and the outside, as $t \to +\infty$. If a closed trajectory is approached by the trajectories from the inside and the outside as $t \to -\infty$, it is called unstable. It is semi-stable if the trajectories approach the limit cycle from the outside as $t \to +\infty$, and from the inside, as $t \to -\infty$, or vice versa. A common feature of the limit cycles is that they do not depend on the initial conditions but uniquely depend on the parameters of a system.

Detecting Method for a Limit Cycle. Throughout this paper the term "detecting method for a limit cycle" is interpreted as the steps necessary in finding the limit cycle of the Van der Pol equation. The necessary steps are described in the order of computer programming. Additional details on the detecting method for a limit cycle are provided in Chapter III.

Point. Since a numerical solution may be considered as a point on a phase-plane trajectory, the term "point" is used to indicate a numerical solution of a differential equation.

Optimal Interval Value. The term "optimal interval value" indicates the interval value which makes the truncation error term per iteration equal to the quotient of the average round-off error per iteration and the order of the numerical method.

The final error in the numerical integration is governed by the interval value and is the sum of the final truncation error and the final round-off error. The optimal interval value is found by differentiating the equation (Equation 20 on page 42) that expresses the final error and setting it equal to zero. The study of numerical errors in the numerical integration method is summarized in the Appendix.

Transversal. The term "transversal" is defined as a line on a phase-plane that traverses or intersects a closed trajectory of a system of equations. In this paper, the transversal is a straight line that is parallel to one of the coordinate axes in a phase-plane, is confined to one of the quadrants, and intersects a closed trajectory.

Value of Tolerance. From the given initial conditions, a numerical integration method is started, and the numerical integration gives a trajectory of the numerical solutions. Let $\ldots, P(t_1), P(t_2), \ldots, P(t_{n-1}), P(t_n), \ldots$ be a sequence of numerical solutions corresponding to $t = \ldots, t_1, t_2, \ldots, t_{n-1}, t_n, \ldots$. Also, let the point $P(t_1)$ on the trajectory be denoted by a crossing of a give? transversal for $t = t_1$, such that the trajectory for $t = t_1$ c?

over the transversal while the trajectory for t_1 - integration interval does not cross over the transversal. Point $P(t_n)$ corresponding to $t = t_n$ will be a successive crossing of the transversal for $t = t_n$ (see Figure 1). The trajectory of numerical solutions may be a limit cycle or an approaching trajectory of the limit cycle.

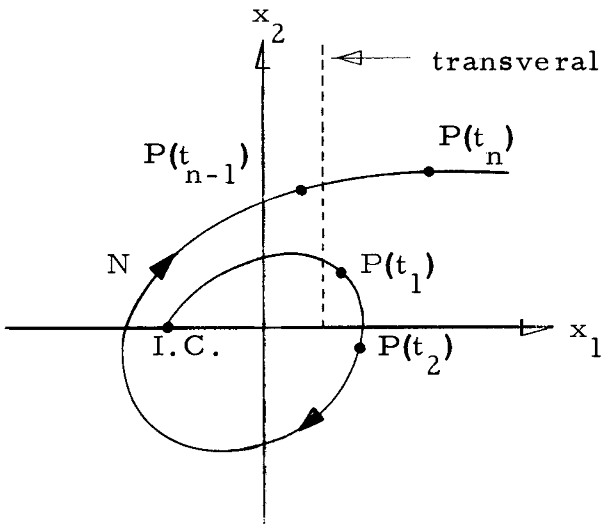

Figure 1. Illustration of the "value of tolerance". Symbol N stands for the trajectory of the system $(\frac{dx_1}{dt} = R(x_1, x_2), \frac{dx_2}{dt} = Q(x_1, x_2))$ in a phase-plane.

The trajectory N represents a trajectory obtained by numerical solutions of a system of equations. It is possible that this solution could correspond to a closed analytic solution. Since numerical solutions are a sequence of discrete values of the independent variable t and have numerical errors, points $P(t_1)$ and $P(t_n)$ are not numerically equal and not on the transversal.

The distance between $P(t_1)$ and points on the trajectory corresponding to $t_{n-1} \leq t \leq t_{n+1}$ are examined to find offset distances (offset distance is defined on page 20) among them. The offset distance thus found, which is small enough to meet the condition (Equation 14 on page 23) for accepting the trajectory as a limit cycle, is defined as a value of tolerance.

Procedures Followed

Numerical solutions for successive crossings over a given transversal are obtained, and a series of tests for the limit cycle are made only in the neighborhood of the successive crossings (successive crossings are defined on page 4). To minimize numerical errors, an optimal interval value, as shown in the Appendix, has been developed and used throughout this thesis to obtain an accurate numerical solution.

The procedure itself involves two distinct steps:

Step I. Calculation of the Optimal Interval Value

To obtain an accurate numerical solution, it is necessary to choose a suitable interval value in employing the numerical integration methods. The optimal interval value was chosen for this thesis. By equating the truncation error term per iteration to the quotient of the average round-off error per iteration and the order of the numerical method, the optimal interval value is calculated by solving the equation (Equation 24 on page 44). In the Appendix, the problems of error and the choice of an interval value are described further.

Step II. Detection of a Limit Cycle.

A limit cycle is detected by checking the offset distance on the trajectory in the neighborhood of $P(t_1)$. The offset distance (defined on page 20) will be tested against the arbitrary standard of two times the optimal interval value in use to decide whether or not the offset distance is small enough to accept the trajectory as a limit cycle.

A more detailed account of the scheme for the detection of a limit cycle appears in Chapter III. Chapter II contains the materials pertaining to this study.

II. SURVEY OF MATERIALS STUDIED

Much has been written in regard to the oscillation of the Van der Pol equation, the limit cycle in nonlinear systems, and numerical integrations in numerical analysis. But only a brief summary of the work done on the problems of limit cycles and numerical methods related to the paper at hand will be given here.

Literature on the Detection of Limit Cycles and the Van der Pol Equation

Van der Pol showed that the behavior of electric circuits of a triode oscillator could be described by a nonlinear differential equation

$$\frac{d^2x}{dt^2} - \epsilon\,(1-x^2)\,\frac{dx}{dt} + x = 0 \qquad (1)$$

The Van der Pol equation exhibits oscillations, and the characteristics of the equation and the graphical solutions are given in his paper (1, 41). Also, the Van der Pol equation is presented as an example of showing the method of isoclines (29, p. 20-21), of applying the First Theorem of Bendixon and the Second Theorem of Bendixon (15, 29, 30, 38, 40), of investigating the stability of periodic motions (29, p. 82-86), and of the geometrical analysis of the existence of periodic motion (24, 29). Furthermore, to develop a new graphical method for constructing velocity—displacement curves for different initial or boundary conditions from the acceleration-velocity curves, Ku (23) used the Van der Pol equation as an example and showed the limit cycle trajectory when $\epsilon = 1$. Also, a difference equation analog to the Van der Pol equation was used by Aseltine and Nesbit (2, 26) to analyze a technique for nonlinear sampled data systems, using the incremental phase-plane method.

In determining limit cycle conditions and the existence of limit cycles in nonlinear control systems, Gronner (16) and Jud (20) employed the describing function technique. The basic principle of the describing function analysis is to linearize all nonlinear components by assuming that harmonic motion exists and by assigning a complex nonlinear gain $J(A, w)$ to each nonlinearity. Then the Nyquist stability criterion is applied (15, 32, 40). The Nyquist stability criterion states that if $J(A, w) = -1$ for one or several parameter pairs A, w, then there exists one, or several, stationary oscillations, which may be stable or unstable. For the study of limit cycles in multiple nonlinear control systems, transformation of a symmetric nonlinear system to a single nonlinearity system was used by Gelb (14).

The method of piecewise linear analysis was developed by Langill (24) for the study of limit cycles in a third-order control system and in a bang-bang controller. In a piecewise linear approach, modes of operation are determined as a function of the specific discontinuity under consideration, and open- or closed-loop system transfer functions throughout each mode are developed. Applying standard inverse Laplace transform techniques, the corresponding time domain relationships are obtained. The resulting numerical equations are solved in steps to yield both a highly accurate system transient response and the limit cycle characteristics.

In analytical ways, several theorems are available in detecting a periodic motion or limit cycles. The Poincaré index (13, 15, 29, 30, 40) represents the net rotation of the direction field along a simple closed curve enclosing the singular points (focus, center, node, and saddle point). The index of a center, a node, or a focus is +1, and the index of saddle point is -1. The Poincaré index of a closed curve, including the individual singularities, is the sum of the indices of the individual singularities. If its index is + 1, it gives a necessary condition for the existence of a limit cycle.

When a system

$$\frac{dx_1}{dt} = R(x_1, x_2) \tag{2}$$

$$\frac{dx_2}{dt} = Q(x_1, x_2)$$

is considered, in which the righthand members are continuously differentiable in each variable, the Bendixon criterion (13, 15, 29, 30, 40) for limit cycles (or Bendixon's negative criterion) states that if the expression $\frac{\partial R(x_1, x_2)}{\partial x_1} + \frac{\partial Q(x_1, x_2)}{\partial x_2}$ does not change its sign within a region D of the phase-plane, no closed trajectory can exist. This criterion does not give sufficient conditions for the existence of limit cycles, and this criterion is somewhat weak for detecting the existence of limit cycles.

The existence of closed trajectories in a phase-plane by the topological and analytical methods was investigated by Levinson and Smith (25). They considered an equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \tag{3}$$

and introduced the functions $F(x) = \int_0^x f(x)dx$ and $G(x) = \int_0^x g(x)dx$. Equation 3 possesses a unique periodic solution. The conditions are:

First, all functions are continuous, $f(x)$ is an even function of x, $g(x)$ is an odd function of x.

Second, $F(x)$ has a single positive zero x_0; it is negative for $0 < x < x_0$; and for $x > x_0$ it is positive.

Third, $F(x) \to \infty$ with x.

The theorems of Liapunov turn out to be very useful in those instances where it is possible to find a positive definite function V. Szegő (33, 39) has applied this idea to locate limit cycles of second-order systems by finding a closed contour, so that the total derivative

$\frac{dV}{dt}$ is negative definite on the closed contour N_1, and positive definite on the closed contour N_2.

The theorem that gives sufficient conditions for the existence of a closed trajectory is the Poincaré-Bendixon theorem (13, 15, 29, 30, 40). It states that if a half trajectory C remains in a finite domain D without approaching any singularities, then the half trajectory C is either a closed trajectory or approaches such a trajectory. Instead of determining the Poincaré-Bendixon domain, the following theorem related to limit cycles is used for this study.

The Theorem (34, 38). If a trajectory of Equation 2 intersects a transversal (see the definition of the term "transversal") at two points, then the trajectory is not contained in any limit cycles.

The primary object is to determine how numerical methods can be applied to the theorem stated above. More details concerning this theorem are in Chapter III.

Literature on Numerical Methods

In selecting a numerical method for the solution of the Van der Pol equation, it is necessary to examine numerical methods, since the choice of any numerical method and interval value is somewhat arbitrary. The numerical solutions are different for each chosen numerical method and interval value. Thus, it is difficult to decide the value of tolerance to detect the existence of a closed trajectory.

Numerical methods for the solution of an ordinary differential equation may be put in two categories; numerical integration method (i.e., predictor-corrector method) and Runge-Kutta method.[1] One of the key factors to be considered in the selection of a particular predictor-corrector method is the stability of the numerical algorithm. This is particularly crucial in differential equations with a forcing function whose time duration time is long, or where the total integration time is long. Considerable effort has been directed toward the development of algorithms having improved stability characteristics as the interval approaches zero (9, 11), and Hamming (18) and Chase (8) have synthesized corrector algorithms that are stable over an increased range of interval values. Also, Milne and Reynolds (27, 28) have studied the stability in Milne's method.

The principle criticism of the Runge-Kutta method has been the nonexistence of a satisfactory method for the estimation of the

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[1] The numerical integration methods require the evaluation of $f(x, y)$ in $\frac{dy}{dx} = f(x, y)$ only at the points x_i (a slight exception to this rule is considered in which $f(x, y)$ is evaluated at values of $x = x_i + \frac{h}{2}$ in order to simplify the computational procedure where $h = x_{i+1} - x_i$) where $x_i > x_0$, $i = 1, 2, 3, \ldots$. This is a basic difference between the numerical integration methods and the Runge-Kutta methods in which evaluation of $f(x, y)$ are made both at the points x_i and at values of x in between the successive x_i.

truncation error. However, Runge-Kutta methods have adapted well to automatic computation because of their self-starting and stability features. Methods of error estimation in Runge-Kutta methods are known (5, 7, 21, 35) and optimum Runge-Kutta methods of order two, three, and four have been developed.

The stability region of a given differential equation is a function of the interval value. Higher order methods usually have very small stability regions and thus require very small interval values. A method for increasing the interval value without violating the stability region has been obtained (12). The study includes the choice of interval values in an optimal way without affecting the local truncation error (31), and the convergence of the numerical solution to the exact solution as the interval value tends to zero (4).

But none of these studies allows for the following two sources of error; namely, the truncation error (6, 37, 42) and the round-off error (3, 19). Only the optimal interval value for the accuracy of numerical value of various derivatives has been obtained (22).

In this paper, instead of examining the stability of a chosen numerical method (17, 36), an extensive study is made to find the optimal interval value for the numerical integration methods by following similar steps in finding the optimal value for derivatives. The optimal interval value for the numerical integrations is tested for a chosen differential equation. More details concerning the optimal interval value are in the Appendix.

III. DETECTING METHOD FOR A LIMIT CYCLE

In the introduction, the detecting method for a limit cycle of the Van der Pol equation and the calculation of the optimal interval value for the numerical integration methods are briefly mentioned. This chapter is divided into two main parts; a broad discussion of the detecting method for a limit cycle and the calculation of the optimal interval value. The first part includes (1) a brief explanation about the Van der Pol equation, and (2) the calculation of the optimal interval value of Euler's method and that of the fourth-order predictor-corrector method for numerical solutions of the Van der Pol equation. The second part includes (1) a discussion of the theorem for the detection of a limit cycle, (2) a definition of an offset distance in terms of the Euclidean measure, and (3) a definition of a closed trajectory.

Optimal Interval Value

The selection of the interval value for the numerical integration methods is based on the size of numerical errors in numerical analysis. A suitable interval value for numerical integrations is necessary for detecting a limit cycle by numerical check, since for different interval values, different sequences of numerical solutions and different values of tolerance are obtained. The truncation error comes from the use of a finite number of terms in the integration algorithm.

Round-off errors arise from using a finite number to approximate real numbers and from the rounding off procedure of a machine. A final numerical error in numerical analysis depends upon two major factors; the first factor is an interval value used in numerical integrations, and the second factor is the stability in numerical analysis.

The concept of the optimal interval value for numerical integrations is developed in the Appendix and the term "optimal interval value" is defined as an interval value for numerical integrations which gives minimum errors. An optimal interval value is calculated by using Equation 24 (see page 44 in the Appendix).

Van der Pol Equation

The Van der Pol equation is a nonlinear second-order differential equation (1, 10, 41):

$$\frac{d^2x}{dt^2} - \epsilon\,(1-x^2)\,\frac{dx}{dt} + x = 0. \tag{4}$$

The Van der Pol equation describes the behavior of an electronic oscillator with a nonlinear resistance. If ϵ is smaller than unity, one type of solution results; if ϵ is larger than a unit, there is a different type of solution. A phase-plane solution for the Van der Pol equation with $\epsilon = 1$ is obtained by solving Equation 5 (by changing notations; $x_1 = x$ and $x_2 = \frac{dx_1}{dt}$).

$$\frac{dx_1}{dt} = x_2 \tag{5}$$

$$\frac{dx_2}{dt} = \epsilon(1-x_1^2)x_2 - x_1$$

Optimal Interval Value

The optimal interval value of the fourth-order predictor-corrector method and Euler's method for the Van der Pol equation are calculated by using Equation 24 on page 44. For the integrations for the fourth-order predictor-corrector method, Equation 6 is used by putting $x = x_1$ and $y = x_2$.

$$x'_n = y_n \tag{6}$$

$$y'_n = y_n \times (1 - x_n \times x_n) - x_n$$

$$\overline{x}_{n+1} = x_{n-3} + 4 \times h \times (2 \times x'_n - x'_{n-1} + 2 \times x'_{n-2})/3$$

$$\overline{y}_{n+1} = y_{n-3} + 4 \times h \times (2 \times y'_n - y'_{n-1} + 2 \times y'_{n-2})/3$$

$$\overline{x}'_{n+1} = \overline{y}_{n+1}$$

$$\overline{y}'_{n+1} = \overline{y}_{n+1} \times (1 - \overline{x}_{n+1} \times \overline{x}_{n+1}) - \overline{x}_{n+1}$$

$$x_{n+1} = x_{n-1} + h \times (\overline{x}'_{n+1} + 4 \times x'_n + x'_{n-1})/3$$

$$y_{n+1} = y_{n-1} + h \times (\overline{y}'_{n+1} + 4 \times y'_n + y'_{n-1})/3$$

For the integrations of Euler's method, Equation 7 is used.

$$x_n' = y_n \tag{7}$$

$$y_n' = y_n \times (1 - x_n \times x_n) - x_n$$

$$x_{n+1} = x_n + h \times x_n'$$

$$y_{n+1} = y_n + h \times y_n'$$

Numerical calculation of derivatives by a digital computer (CDC 3300) gives

$$|y^{(5)}|_{max} \doteq 3760 \qquad |y^{(2)}|_{max} \doteq 9.7$$

$$|x^{(5)}|_{max} \doteq 550 \qquad |x^{(2)}|_{max} \doteq 4.76$$

The fourth-order predictor-corrector method for the Van der Pol equation gives 36 operations in each iteration and the Euler's method for the Van der Pol equation gives eight operations in each iteration.

The optimal interval values listed below are found by using Equation 24 (see page 44). By equating the truncation error term per iteration with the quotient of the average round-off error per iteration and the order of the numerical method, the optimal interval value for numerical integrations is calculated by solving the equation for the interval. The round-off error per operation in the CDC 3300 computer is found experimentally by examining the numerical errors of a known second-order differential equation (see page 45).

The optimal interval value of the fourth-order predictor-corrector method for the Van der Pol equation is given by Equation 8.

$$h_o \doteq 2 \times 10^{-3} \quad (8)$$

The optimal interval value of Euler's method for the numerical solutions of the Van der Pol equation during short duration time is

$$h_o \doteq 2 \times 10^{-6} \quad (9)$$

Detection of a Limit Cycle of the Van der Pol Equation

The procedures for detecting the limit cycle by examining the Van der Pol equation through a numerical method will be discussed in this section. Prior to the detailed procedures, an introduction of the related theorem is presented.

The Theorem Related to Limit Cycles

The theorem related to limit cycles (34, 38, and page 11) states that if a trajectory of a system (Equation 10)

$$\frac{dx_1}{dt} = R(x_1, x_2) \quad (10)$$

$$\frac{dx_2}{dt} = Q(x_1, x_2)$$

intersects a transversal (a transversal is a straight line that is parallel to one of the coordinate axes in a phase-plane and intersects a limit cycle, and is confined to one of the quadrants) at two points, then the trajectory is not contained in any limit cycle (Figure 2a).

In Figure 2a, N is a trajectory through $P(t_1)$, and ℓ is a transversal at $P(t_1)$. If the trajectory N arrives at $P(t_1)$ for $t = t_1$, then, for $t > t_1$, then trajectory N will intersect a transversal at a point $P(t_n)$ for $t = t_n$. If t_n is the least time required to approach $P(t_n)$ from $P(t_1)$, then that portion of N can be plotted for $t_1 \leq t \leq t_n$. If the end points of this portion of trajectory N are the same point, then the trajectory will be a closed trajectory.

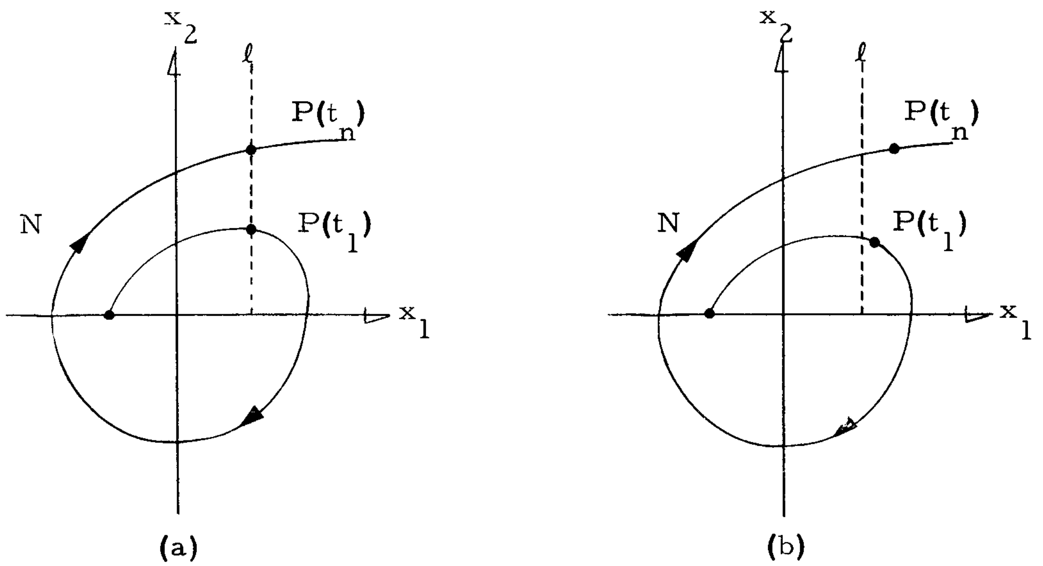

Figure 2. Drawings to illustrate the theorem related to limit cycles. (a) Symbol N stands for the trajectory of analytic solution. (b) Symbol N stands for the trajectory of numerical solution.

Offset Distance

When a numerical integration method is used for a system of equations, the resulting trajectory can not, in general, be expected to close at the transversal (see Figure 2b, $P(t_1) \neq P(t_n)$). The point $P(t_n)$ corresponding to the crossing of the transversal (a crossing of a given transversal is defined on page 3) can be obtained by continuing the numerical integration from $P(t_1)$ (Figure 2b). The offset distance in the neighborhood of $P(t_1)$ is the distance between point $P(t_1)$ and the trajectory N through $P(t_n)$.

If the point $P(t_1)$ and $P(t_n)$ are numerically equal, the sequence of numerical solutions from point $P(t_1)$ to point $P(t_n)$ is closed and it can be considered as a trajectory of a limit cycle. However, the trajectory from $P(t_1)$ to $P(t_n)$, resulting from the numerical solution, does not close ($P(t_1) \neq P(t_n)$; see Experimental Result I in Chapter IV) because of the numerical errors and the discrete nature of the numerical solution.

To study the trajectory at the point where it does not close because of the application of numerical methods, the Euclidean measure of distance is introduced as a measure of the distance between points of offset. If $P(t)$ is a set of numerical solutions of a system:

$$\frac{dx_1}{dt} = R(x_1, x_2) \qquad (11)$$

$$\frac{dx_2}{dt} = Q(x_1, x_2)$$

then a real-valued function d, which is defined on all of $P(t) \times P(t)$ (Cartesian product), is called a distance on $P(t)$ if the following conditions are satisfied:

First, $d(P(t_i), P(t_j)) \geq 0$ for all $P(t_i)$, $P(t_j)$ in $P(t)$ and $d(P(t_i), P(t_j)) = 0$ if and only if $P(t_i) = P(t_j)$.

Second, $d(P(t_i), P(t_j)) = d(P(t_j), P(t_i))$ for all $P(t_i)$, $P(t_j)$ in $P(t)$.

Third, $d(P(t_i), P(t_k)) \leq d(P(t_i), P(t_j)) + d(P(t_j), P(t_k))$ for all $P(t_i)$, $P(t_j)$, $P(t_k)$ in $P(t)$.

$d(P(t_i), P(t_j))$ is referred to as the distance between points $P(t_i)$ and $P(t_j)$, and $d(P(t_i), P(t_j))$ is defined by setting

$$\begin{aligned} d(P(t_i), P(t_j)) &= \| P(t_i) - P(t_j) \| \\ &= \sqrt{\langle P(t_i) - P(t_j), P(t_i) - P(t_j) \rangle} \\ &= \sqrt{(x_1(t_i) - x_1(t_j))^2 + (x_2(t_i) - x_2(t_j))^2} \end{aligned} \qquad (12)$$

where

$$P(t_i) = (x_1(t_i), x_2(t_i)) \quad \text{and} \quad P(t_j) = (x_1(t_j), x_2(t_j)).$$

Closed Trajectory

It may be recalled that the points $P(t_1)$, $P(t_2), \ldots, P(t_{n-1})$, $P(t_n)$ are defined as a sequence of solutions of a system of equations, and points $P(t_1)$ and $P(t_n)$ are the successive crossings of a given transversal for $t_n > t_1$ (Figure 1). A closed trajectory from a trajectory obtained through numerical integrations may be determined by testing the offset distance d for $P(t_i)$ $(t_{n-1} \leq t_i \leq t_{n+1})$. The testing is simply a comparison of d and the value of tolerance. If $d \leq 2 \times$ (the optimal interval vlaue), then the trajectory is considered to be a limit cycle. If d is not smaller than $2 \times$ (the optimal interval value), the whole integration process will be repeated with $P(t_1)$ replaced by the $P(t_n)$.

It seems to be justified to choose the value of tolerance as $\frac{1}{2} \times$ interval distance (the distance corresponding to one interval value), since the maximum offset distance for a closed trajectory would be $\frac{1}{2} \times$ interval distance. The interval distance in the neighborhood of $P(t_i)$ was found to be approximately $4 \times$ the optimal interval value (see the results on page 30). The dimension of the foregoing constant is that of the speed. Thus, the value of tolerance is defined as:

$$\text{Value of tolerance} = 2 \times (\text{the optimal interval value}). \qquad (13)$$

In other words, if $s(P(t_1), \Delta)$ is defined by the set

$$S(P(t_1), \Delta) = \{P(t_i) \in P(t), \ d(P(t_1), P(t_i)) \leq \Delta\} \tag{14}$$

for the real number Δ (value of tolerance) > 0. Then point $P(t_i)$ $(t_{n-1} \leq t_i \leq t_{n+1})$ is within a circle of radius Δ about $P(t_1)$, if the trajectory of numerical solutions is a closed trajectory.

The numerical integration process requires small interval values for the accuracy in general. Experimental results (Figures 6, 7, 8, and 9), however, indicated an optimal interval value beyond which the further decreasing of the interval value does not improve the numerical results. From the experimental caclualtion, it was found that the optimal interval value of Euler's method is smaller than that of the fourth-order predictor-corrector method (see Equations 8 and 9 on page 18). Thus, Euler's method with its optimal interval value is used from point $P(t_{n-1})$ to point $P(t_{n+1})$, while the fourth-order predictor-corrector method with its optimal interval value is used elsewhere. It is shown, theoretically, that the fourth-order predictor-corrector method has better accuracy than Euler's method. The above choice of Euler's method at the specified portion of the trajectory is solely from its smaller optimal interval value; since the $P(t_i)$ may fall in between discrete points, Euler's method with its smaller optimal interval value provides smaller value of tolerance (two times the optimal interval value). In addition to the smaller value of tolerance in using Euler's method, it also has a much simpler iteration process.

IV. RESULTS

A differential equation, whose analytic solutions are a family of circles in a phase-plane, is chosen to examine the successive crossings of a given transversal. It is also shown that for examining the existence of a limit cycle the numerical integration with a small optimal interval value should be employed in the neighborhood of the crossing at the transversal. The Van der Pol equation is chosen for detecting a limit cycle described in Chapter III.

Experimental Result I

The following system is chosen for the examination of the successive crossings of a given transversal.

$$\frac{dx_1}{dt} = x_2 \qquad\qquad \frac{dx_2}{dt} = -x_1 \tag{15}$$

The analytic solutions of the above equations are a family of circles (closed trajectory) in a phase-plane. For the numerical integration, the fourth-order predictor-corrector method with its optimal interval value $(h = 8 \times 10^{-3})$ is used (the calculation of this optimal interval value is shown on page 51 and the system of equations (Equation 15) is also used for examining the numerical errors in the Appendix).

As a starting procedure, the Taylor series expansion up to the eighth-order term is used. The results are shown in Figure 3. The results have shown that the trajectory neither closes at the crossing of a given transversal nor does a numerical solution fall exactly on the transversal (Figure 3). As initial conditions, $x_1 = 2$ and $x_2 = 0$ are used.

$P(t_n)$ $x_1 = 0.13194084160$
$x_2 = 1.99564315806$

$P(t_{n-1})$ $x_1 = -0.02796235621$
$x_2 = 1.99780451714$

$P(t_1)$ $x_1 = 0.05839904473$
$x_2 = 1.99914720608$

(Pt_n) $x_1 = -1.99864483165$
$x_2 = 0.07361275554$

$P(t_1)$ $x_1 = -2.0$
$x_2 = 0$

$P(t_{n-1})$ $x_1 = -1.99813531927$
$x_2 = -0.08643769724$

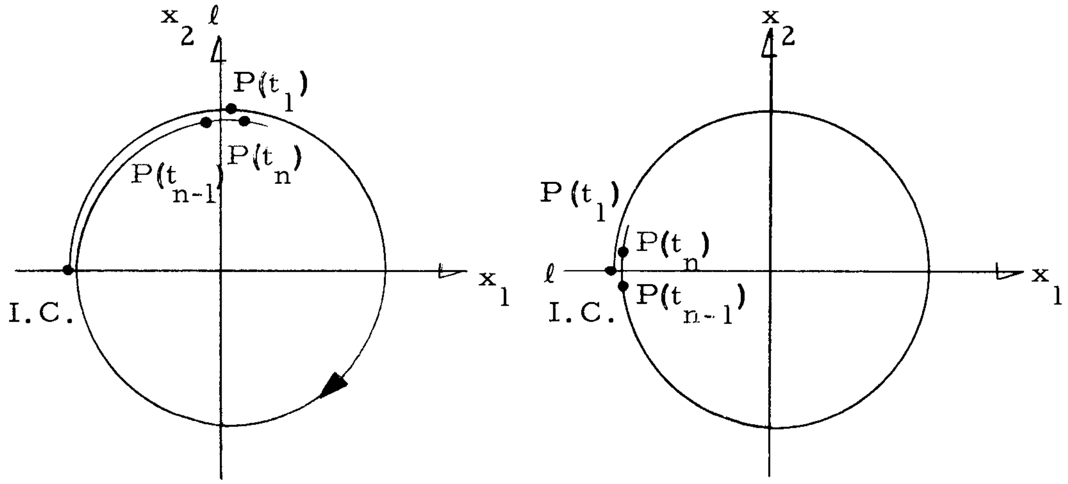

Figure 3. Experimental result I. (a) The positive direction of the vertical axis is used for a transversal. (b) The negative direction of the horizontal axis is used for a transversal.

Experimental Result II

The Van der Pol equation

$$\frac{d^2x}{dt^2} - \epsilon \times (1-x^2)\frac{dx}{dt} + x = 0 \qquad (16)$$

is used to show that a small interval value is required to examine the existence of a closed trajectory, by testing the offset distance between successive trajectories, when $\epsilon = 1$. For the numerical integration, the fourth-order predictor-corrector method with its optimal interval value $(h = 2 \times 10^{-3})$ is used. As a starting procedure, the fourth-order Runge-Kutta method is used.

In Figure 4, the position of numerical solution $P(t_n)$ with respect to point $P(t_1)$ is shown. The results shown in Figure 4 indicate that a small interval vlaue is required to find the offset distance between successive trajectories of numerical solutions.

Experimental Result III

In Chapter III, the detecting method for a limit cycle of the Van der Pol equation is described. Applying the steps mentioned in Chapter III, the limit cycle of the Van der Pol equation is detected. Using the fourth-order predictor-corrector method with its optimal interval value $(h = 2 \times 10^{-3})$, the Van der Pol equation is numerically solved for $t \leq t_n$, $t_{n+1} \leq t \leq t_{2n}$, $t_{2n+1} \leq t \leq t_{3n}$, and $t_{3n+1} \leq t \leq t_{4n}$.

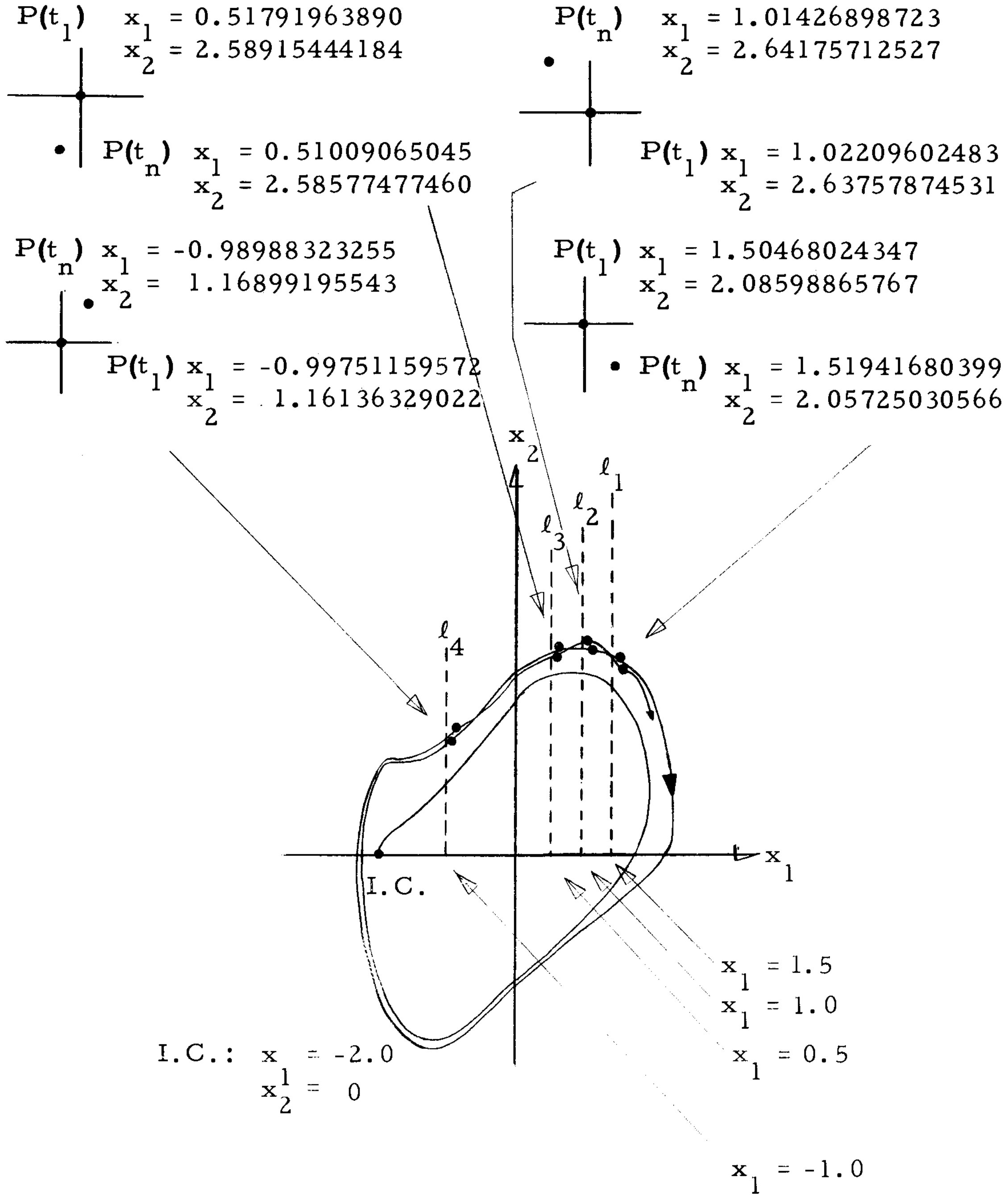

Figure 4. Experimental result II.

For $t_{n-1} \leq t \leq t_{n+1}$, $t_{2n-1} \leq t \leq t_{2n+1}$, $t_{3n-1} \leq t \leq t_{3n+1}$, and $t_{4n-1} \leq t \leq t_{4n+1}$, Euler's method with its optimal interval value ($h = 2 \times 10^{-6}$) is used to find the offset distances. The points necessary in analyzing the results are shown in Figure 5.

The necessary data for this experiment are given as follows:

Optimal interval value of the fourth-order predictor-corrector method . . . . 2×10^{-3}

Optimal interval value of Euler's method . . . . . . 2×10^{-6}

Chosen transversal . . . . $x_1 = 1.4$, $x_2 \geq 0$

Initial conditions . . . . $x_1 = -2.0$, and $x_2 = 0$

2 × (the optimal interval value of Euler's method) . . . . 4×10^{-6}

Through the fourth-order predictor-corrector method with its optimal interval value, the successive crossings of a given transversal are obtained as follows:

$P(t_1)$: $x_1 = 1.41804153446$, $x_2 = 2.24290302774$

$P(t_n)$: $x_1 = 1.40245844488$, $x_2 = 2.26899615792$

$P(t_{2n})$: $x_1 = 1.40407638173$, $x_2 = 2.26642774499$

$P(t_{3n})$: $x_1 = 1.40115674859$, $x_2 = 2.27105702483$

$P(t_{4n})$: $x_1 = 1.40277533795$, $x_2 = 2.26849451417$

for $t_{4n} > t_{3n} > t_{2n} > t_n > t_1$.

The offset distance in the neighborhood of $P(t_1)$ when Euler's method with its optimal interval value is used during $t_{n+1} \sim t_{n-1}$

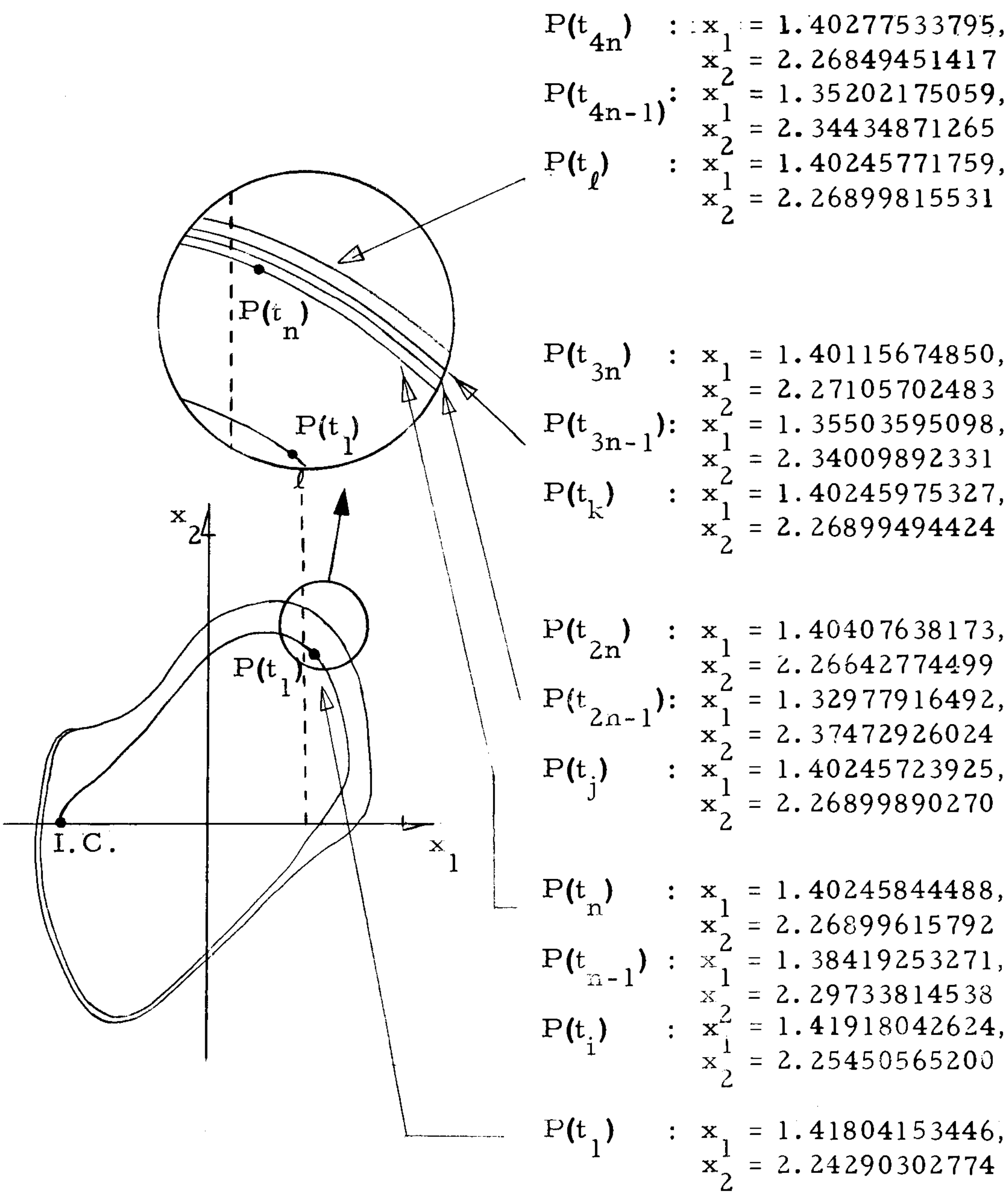

Figure 5. Experimental result III.

seconds is:

$$d(P(t_1), P(t_i)) = d_1 = 1.16591723355 \times 10^{-2} \text{ for } t_{n-1} < t_i < t_{n+1}$$

Since d_1 is not smaller than the value of tolerance, the integration is repeated to get d_2:

$$d(P(t_n), P(t_j)) = d_2 = 2.99789327354 \times 10^{-6} \text{ for } t_{2n-1} < t_j < t_{2n+1}.$$

Since d_2 is less than the value of tolerance, the trajectory examined here satisfies as a limit cycle.

The value of tolerance is:

$$\text{Value of tolerance} = 2 \times (\text{optimal interval value of Euler's method}). \tag{17}$$

The interval distance in the neighborhood of point $P(t_j)$ is computed to determine the value of tolerance as follows:

$P(t_j)$: $x_1 = 1.40245723925$, $x_2 = 2.26899890270$

$P(t_{j+1})$: $x_1 = 1.40246177724$, $x_2 = 2.26899171004$

$d(P(t_{j+1}), P(t_j)) = 8.49810044737 \times 10^{-6}$

Thus it is approximated that the interval distance is four times the optimal interval value of Euler's method.

It is interesting to note that the offset distance is not stable at the successive crossings of the transversal. The offset distances for the third and fifth crossings of the transversal are computed as follows:

$$d(P(t_n), P(t_k)) = 1.78459693784 \times 10^{-6} \text{ for } t_{3n-1} < t_k < t_{3n+1}$$

$$d(P(t_n), P(t_\ell)) = 2.21556163641 \times 10^{-6} \text{ for } t_{4n-1} < t_\ell < t_{4n+1} ;$$

V. SUMMARY

The experimentation and research of this study verifies the postulate that a numerical method could be developed for determining the existence of a limit cycle within the parameters of the Van der Pol equation.

Experiment I was conducted for a system of equations whose analytic solutions represent a family of closed trajectories in a phase-plane. The experimental results obtained by the fourth-order predictor-corrector method with its optimal interval value do not give a family of closed trajectories. This is due to the method employed and to the discrete nature of numerical methods in general.

In Experiment II the position of numerical solution $P(t_n)$ with respect to point $P(t_1)$ was shown, and it was justified that a small interval value was required to find the offset distance between successive trajectories of numerical solutions.

The main study was Experiment III. In this experiment it was shown that it was possible to establish a numerical method for determining the existence of the limit cycle of the Van der Pol equation. The numerically solved trajectory with an offset distance is approximated as a closed trajectory when the offset distance in terms of Euclidean measure is less than the value of tolerance. The value of tolerance is determined experimentally as two times the optimal interval value. The offset distance may be checked at any point on the trajectory. A transversal is arbitrarily chosen so that the testing is simplified at a point near the transversal. The optimal interval value obtained emperically showed that Euler's method has a smaller optimal interval value than that of the fourth-order predictor-corrector method, and that accuracy of the offset distance depends on the interval value. Euler's method replaces the fourth-order predictor-corrector method for the portion of the trajectory where the offset distance is evaluated.

The numerical method of detecting the limit cycle of the Van der Pol equation may be a useful tool for future numerical studies of nonlinear second-order differential equations.

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APPENDIX

APPENDIX

Interval for Accuracy in Numerical Integration Methods

I. Optimal Interval Value

If a computing machine is used, a numerical integration method is required for the solution of the differential equations. To choose the best method which yields satisfactory numerical solutions, several factors must be considered; the accuracy required, the ease of the estimation of the truncation error at each step, the computing time, and the computer program. If a high degree of accuracy is needed, higher-order methods are required, such as Adam's method Milne's method. If accuracy is not important, a low-order method, such as Euler's method or the second-order method, will suffice.

Numerical errors in numerical analysis are truncation and round-off error. Truncation error comes from the use of a finite number of terms in the integration algorithm. Round-off errors arise from using a finite number to approximate real numbers and from the rounding off procedure of a machine.

When two n-digit numbers are multiplied, the product produces more than n-digits (for example: 2.3 (2 digits) × 6.8 (2 digits) = 15.68 (4 digits)). Similarly, division or square root may produce a quotient having an infinite number of digits. If only n digits are needed after an operation (division, multiplication, addition, or subtraction), the product, the quotient or the final values are gotten by a round-off procedure.

When a differential equation is to be solved numerically, a value of total integration time should be specified, and the final error of numerical solutions is a sum of the final truncation error and the final round-off error. The final error is governed by the value of the integration interval. When the interval value becomes large, then the per-iteration truncation error, which determines the final error, increases. If the interval value becomes small, then the per-iteration round-off error is greater than the per-iteration truncation error, and the per-iteration round-off error determines the final error.

Various symbols are here defined such as

T = total integration time.

R = average round-off error per iteration.

h = interval value.

h_o = optimal interval value.

Z_1 = final error in Euler's method.

Z_2 = final error in the second-order method.

Z_3 = final error in the second-order predictor-corrector method.

Z_4 = final error in the fourth-order predictor-corrector method.

The optimal interval value for the numerical iterations are formulated here for a possible maximum error, such as errors accumulated with each iteration and having the same sign. If the interval value is equal to the value of the total integration time, just one iteration is necessary to finish the numerical iteration, and the value of the round-off error becomes the value of R (average round-off error per iteration). If the interval value decreases by one half, then the number of iterations is two, and the value of the round-off error becomes $2 \times R$, and so on. The final round-off error as a function of interval is given as,

$$\text{Final round-off error} = R \times \frac{T}{h} \qquad (18)$$

If the interval value equals the value of total integration time, one iteration is enough to finish the numerical iteration, and the final truncation error is given by the truncation error term per iteration. If the interval value decreases by one half, the truncation error term per iteration becomes small, but the number of iterations is two, and so on. If the truncation error term per iteration is expressed in the following equations,

$$\frac{x''(s) + y''(s)}{2} h^2 \quad \ldots \text{ in Euler's method,} \tag{19a}$$

$$\frac{x'''(s) + y'''(s)}{3} h^3 \quad \ldots \text{ in the second-order method,}$$

$$\frac{5(x'''(s) + y'''(s))}{12} h^3 \quad \ldots \text{ in the second-order predictor-corrector method,}$$

$$\frac{29(x^{(5)}(s) + y^{(5)}(s))}{90} h^5 \quad \ldots \text{ in the fourth-order predictor-corrector method.}$$

then the final truncation errors as a function of interval are;

$$\frac{x''(s) + y''(s)}{2} hT \quad \ldots \text{ in Euler's method,} \tag{19b}$$

$$\frac{x'''(s) + y'''(s)}{3} h^2 T \quad \ldots \text{ in the second-order method,}$$

$$\frac{5(x'''(s) + y'''(s))}{12} h^2 T \quad \ldots \text{ in the second-order predictor-corrector method,}$$

$$\frac{29(x^{(5)}(s) + y^{(5)}(s))}{90} h^4 T \quad \ldots \text{ in the fourth-order predictor-corrector method.}$$

Since the final error is a sum of the final round-off error and the final truncation error, the final error in following numerical methods is

$$R \times \frac{T}{h} + \frac{x''(s) + y''(s)}{2} hT \quad \text{... final error in Euler's method,} \tag{20}$$

$$R \times \frac{T}{h} + \frac{x'''(s) + y'''(s)}{3} h^2 T \quad \text{... final error in the second-order method,}$$

$$R \times \frac{T}{h} + \frac{5(x'''(s) + y'''(s))}{12} h^2 T \quad \text{... final error in the second-order predictor-corrector method,}$$

$$R \times \frac{T}{h} + \frac{29(x^{(5)}(s) + y^{(5)}(s))}{90} h^4 T \quad \text{... final error in the fourth-order predictor-corrector method.}$$

The derivatives $x^{(5)}(s)$, $y^{(5)}(s)$, $x'''(s)$, $y'''(s)$, $x''(s)$, $y''(s)$, in the truncation error term are considered as a constant here, by using the median of the maximum absolute values. If the above equations are differentiated with respect to the interval value, set equal to zero and rearranged, then the interval value which satisfies the following equation gives a minimum final error in the numerical methods.

$$\text{Truncation error per iteration} = \frac{\text{average round-off error per iteration}}{\text{the order of the numerical method}} \tag{21}$$

<u>For example</u>: In the fourth-order predictor-corrector method, the final error is

$$Z_4 = R \times \frac{T}{h} + \frac{29(x^{(5)} + y^{(5)})}{90} h^4 T$$

By differentiating with respect to the interval

$$\frac{dZ_4}{dh} = -R \times \frac{T}{h^2} + \frac{4 \times 29(x^{(5)} + y^{(5)})}{90} h^3 T = 0$$

$$4 \times \frac{29(x^{(5)} + y^{(5)})}{90} h^3 T = R \times \frac{T}{h^2}$$

and by rearranging the equation,

$$\frac{29(x^{(5)} + y^{(5)})}{90} h^5 = \frac{R}{4} \qquad (22)$$

The left part of Equation 22 is the truncation error term per iteration, and the right part of Equation 22 is the average round-off error per iteration over the order of the numerical method. Hence, the optimal interval value of the fourth-order predictor-corrector method is the interval value that satisfies Equation 21.

If the average round-off error per iteration can be assumed known at each iteration, it is easy to find an interval value which makes the truncation error term per iteration equal to the quotient of the average round-off error per iteration and the order of the numerical method, since the truncation error term per iteration is known.

Because of the rounding off, the last digit values are not significant at each operation, and the average round-off error per iteration is proportional to the number of operations and the average round-off error per operation. The value 0.3×10^{-11} is chosen for the value of the average round-off error per operation after several trials were made using a selected differential equation in the next section.

It is assumed that the average round-off error per iteration in a CDC 3300 Computer is

$$\text{Average round-off error per iteration} = 0.3 \times 10^{-11} \times 0p \qquad (23)$$

where 0p = number of operations per iteration. The power of 10 in Equation 23 stands for the maximum number of digits that can be used in the CDC 3300 (mantissa).

The optimal interval value for the numerical iterations can be found by using the following equations.

$$\frac{\underset{\text{of max.}}{\overset{\text{median}}{|x''|}} + \underset{\text{of max.}}{\overset{\text{median}}{|y''|}}}{2} h^2 = 0.3 \times 10^{-11} \times 0p \quad \text{in Euler's method.} \qquad (24)$$

$$\frac{\underset{\text{of max.}}{\overset{\text{median}}{|x'''|}} + \underset{\text{of max.}}{\overset{\text{median}}{|y'''|}}}{3} h^3 = \frac{0.3 \times 10^{-11} \times 0p}{2} \quad \text{in the second-order method.}$$

$$\frac{5\left(\underset{\text{of max.}}{\overset{\text{median}}{|x'''|}} + \underset{\text{of max.}}{\overset{\text{median}}{|y'''|}}\right)}{12} h^3 = \frac{0.3 \times 10^{-11} \times 0p}{2} \quad \text{in the second-order predictor-corrector method.}$$

$$\frac{29\left(\underset{\text{of max.}}{\overset{\text{median}}{|x^{(5)}|}} + \underset{\text{of max.}}{\overset{\text{median}}{|y^{(5)}|}}\right)}{90} h^5 = \frac{0.3 \times 10^{-11} \times 0p}{4} \quad \text{in the fourth-order predictor-corrector method.}$$

The number of operations are determined by the number of operations in the iteration algorithm. The number of operations depends on the numerical method and the differential equation chosen.

In the next section, a differential equation (second-order linear differential equation) is chosen, and by using Equation 24, optimal interval values of each method (Euler's method, the second-order method, the second-order predictor-corrector method, and the fourth-order predictor-corrector method) are obtained. The results with an optimal interval value (h_o) derived from Equation 24 are compared with the results with the interval values $h_o \times 10^{-1}$ and $h_o \times 10^{+1}$. The results are shown in Figures 6, 7, 8, 9, and 10.

II. Experimental Results

For this experiment, a second-order differential equation is chosen. Its solutions are a family of circles in the phase-plane.

$$\frac{d^2x}{dt^2} + x = 0 \tag{25}$$

The numerical integration methods for the solution of the differential equation can be found from the Taylor series expansion, and the truncation error term per iteration can be approximated by the Taylor series term or modified form of it. Here the truncation error term per iteration is used in a general sense of the definition where the series converges. Since any higher order differential equation can be reduced to the first-order differential equations by a simple change in notation $(x_1 = x,\ x_2 = \frac{dx_1}{dt})$, the following equations are obtained.

$$\frac{dx_1}{dt} = x_2 \qquad (26)$$

$$\frac{dx_2}{dt} = -x_1$$

Throughout the experiments the following initial conditions are chosen:

$$x_1 = -2 \qquad (27)$$

$$x_2 = 0$$

The solution is a circle of radius 2 in a phase-plane, and any error of the numerical solution is measured by Equation 28, to avoid an error due to the process of computing the square root.

$$\text{Error} = (x_1^2 + x_2^2) - 4 \qquad (28)$$

The required starting values of x_1 and x_2 are determined by the use of the Taylor series expansions.

Whether or not the optimal interval value is used, an accurate starting value is required when the over-all numerical results are going to be compared and checked. For the second-order methods, therefore, the Taylor series expansion up to a sixth order term is used, and for the fourth-order predictor-corrector method, the Taylor series expansion up to the eighth-order term is used. In this chosen differential equation, the maximum value of derivatives (first, second, third, fourth, and fifth, etc.) is 2. Hence, the median of maximum absolute value of derivatives is 1. The derivatives in the truncation error term per iteration are considered as a constant here, by using the median of maximum absolute value.

Throughout this experiment the optimal interval value (h_o) derived from Equation 24 is compared with the interval value $h_o \times 10^{-1}$ and $h_o \times 10^{+1}$. Errors are shown on page 52-56 of this paper.

Euler's method. Euler's method takes only two terms of the Taylor series (by putting $x = x_1$ and $y = x_2$):

$$x_{n+1} = x_n + h \times x'_n \tag{29}$$

$$y_{n+1} = y_n + h \times y'_n$$

and the truncation error term per iteration is

$$T_r = \frac{x''(s) + y''(s)}{2} h^2 \tag{30}$$

where $t_n < s < t_{n+1}$. In the calculation of the truncation error term per iteration, median values of $|x''|_{max}$ and $|y''|_{max}$ are used. The optimal interval value of Euler's method for a given differential equation is obtained by Equation 24. Substituting specific values in Equation 24,

$$\frac{2 \times h^2}{2} = 0.3 \times 10^{-11} \times 4$$

Since $|x''|^{median}_{of\ max} = 1$ and $|y''|^{median}_{of\ max} = 1$,

$$h^2 = 12 \times 10^{-12}$$

$$h_o \doteq 3 \times 10^{-6} \qquad (31)$$

The results of Euler's method are shown in Figure 7 on page 53 of this paper.

Second-order method. The set of equations for the numerical iteration takes the form of

$$x_{n+1} = x_{n-1} + 2 \times h \times x'_n \qquad (32)$$

$$y_{n+1} = y_{n-1} + 2 \times h \times y'_n$$

and the truncation error term per iteration is

$$T_r = \frac{x'''(s) + y'''(s)}{3} h^3 \qquad (33)$$

If median values of $|x'''|_{max}$ and $|y'''|_{max}$ are used for the values of $x'''(s)$ and $y'''(s)$, then

$$\frac{2}{3}h^3 = \frac{0.3 \times 10^{-11} \times 6}{2}$$

$$h^3 = 13.5 \times 10^{-12}$$

$$h_o \doteq 2 \times 10^{-4} \qquad (34)$$

The numerical results of the value of h_o, $h_o \times 10^{-1}$, and $h_o \times 10^{+1}$ are shown in Figure 8 on page 54 of this paper.

Second-order predictor-corrector method. The complete set of equation for the numerical iterations and the truncation error term per iteration are given by Equation 35 and 36.

$$\bar{x}_{n+1} = x_{n-1} + 2 \times h \times x'_n \qquad (35)$$

$$\bar{y}_{n+1} = y_{n-1} + 2 \times h \times y'_n$$

$$\bar{x}'_{n+1} = \bar{y}_{n+1}$$

$$\bar{y}'_{n+1} = -\bar{x}_{n+1}$$

$$x_{n+1} = x_n + h \times (\bar{x}'_{n+1} + x'_n)/2$$

$$y_{n+1} = y_n + h \times (\bar{y}'_{n+1} + y'_n)/2$$

$$T_r = \frac{5(x'''(s) + y'''(s))}{12}h^3 \qquad (36)$$

If the median values of $|x'''|_{max}$ and $|y'''|_{max}$ are used for the values of $x'''(s)$ and $y'''(s)$, then

$$\frac{5 \times 2}{12} h^3 = \frac{0.3 \times 10^{-11} \times 14}{2}$$

$$h^3 = 25.2 \times 10^{-12}$$

$$h_o \doteq 3 \times 10^{-4} \tag{37}$$

The results are shown in Figure 9 on page 55 of this paper.

Fourth-order predictor-corrector method. The set of equations for the iterations is given in Equation 38, and the truncation error term per iteration is given in Equation 39.

$$\bar{x}_{n+1} = x_{n-3} + 4 \times h \times (2 \times x'_n - x'_{n-1} + 2 \times x'_{n-2})/3 \tag{38}$$

$$\bar{y}_{n+1} = y_{n-3} + 4 \times h \times (2 \times y'_n - y'_{n-1} + 2 \times y'_{n-2})/3$$

$$\bar{x}'_{n+1} = \bar{y}_{n+1}$$

$$\bar{y}'_{n+1} = -\bar{x}_{n+1}$$

$$x_{n+1} = x_{n-1} + h \times (\bar{x}'_{n+1} + 4 \times x'_n + x'_{n-1})/3$$

$$y_{n+1} = y_{n-1} + h \times (\bar{y}'_{n+1} + 4 \times y'_n + y'_{n-1})/3$$

$$T_r = \frac{29(x^{(5)}(s) + y^{(5)}(s))}{90} h^5 \tag{39}$$

By using the median values of $|x^{(5)}|_{max}$ and $|y^{(5)}|_{max}$ for the values of $x^{(5)}(s)$ and $y^{(5)}(s)$, the following optimal interval value is obtained:

$$\frac{29 \times 2}{90} h^5 = \frac{0.3 \times 10^{-11} \times 28}{4}$$

$$h^5 \doteq 30 \times 10^{-12}$$

$$h_o \doteq 8 \times 10^{-3} \qquad (40)$$

The results are shown in Figure 6 on page 52 of this paper.

Figures 6, 7, 8, and 9 show that the optimal interval value gives a minimum error: when an interval value is bigger than the optimal interval value, the numerical error increases, while it is smaller than the optimal interval value, the numerical error does not decrease. When an interval value is smaller than the optimal interval value of Euler's method, the numerical error between the numerical integration methods gives the same kind of error curves and the difference between the numerical methods is not greater than 10^{-7} (the numerical error is measured by Equation 28 on page 46 and is shown in Figure 10).

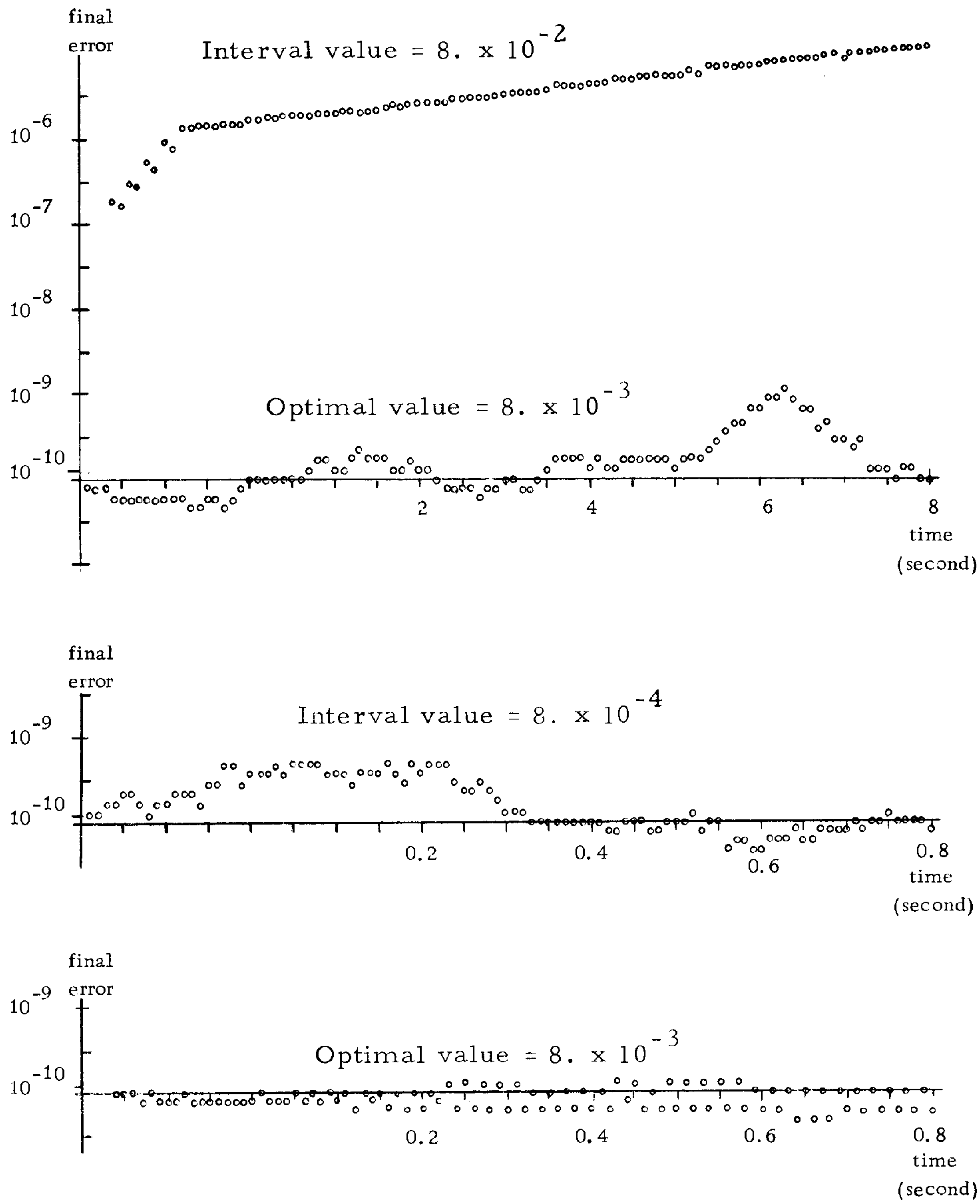
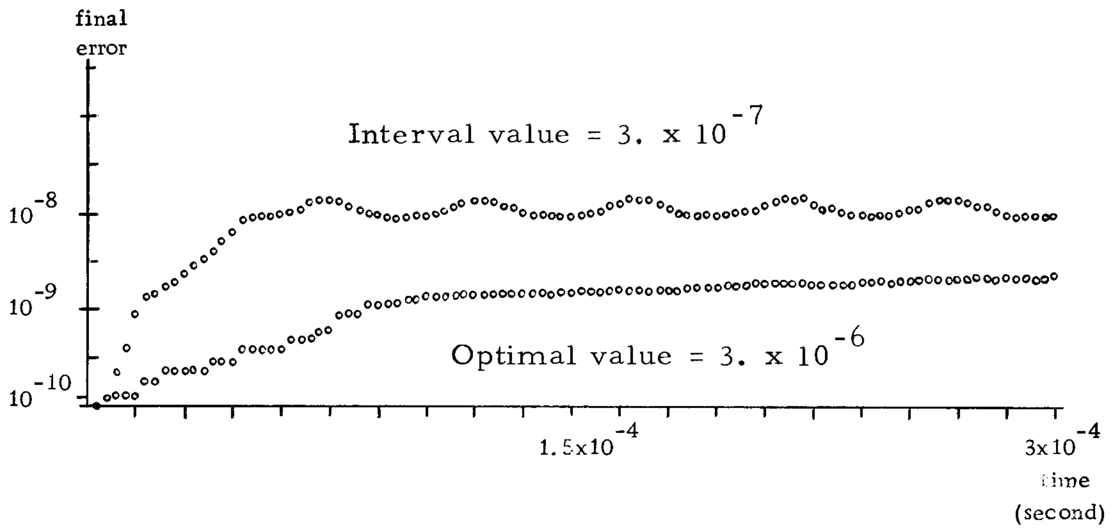


Figure 6. Comparison of errors of the optimal value and another interval value in the fourth-order predictor-corrector method.



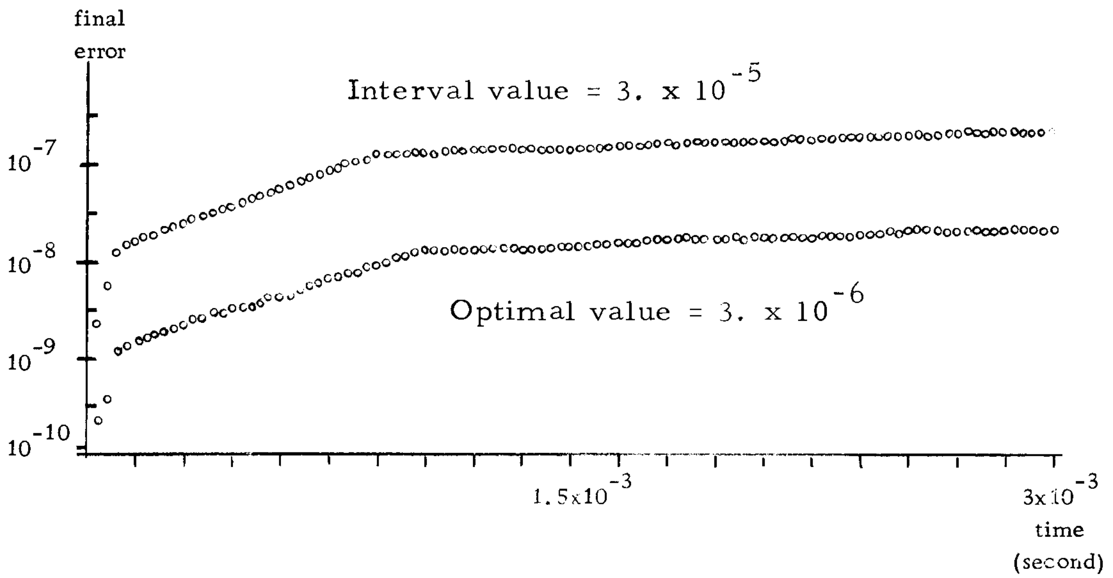


Figure 7. Comparison of errors of the optimal value and another interval value in Euler's method.

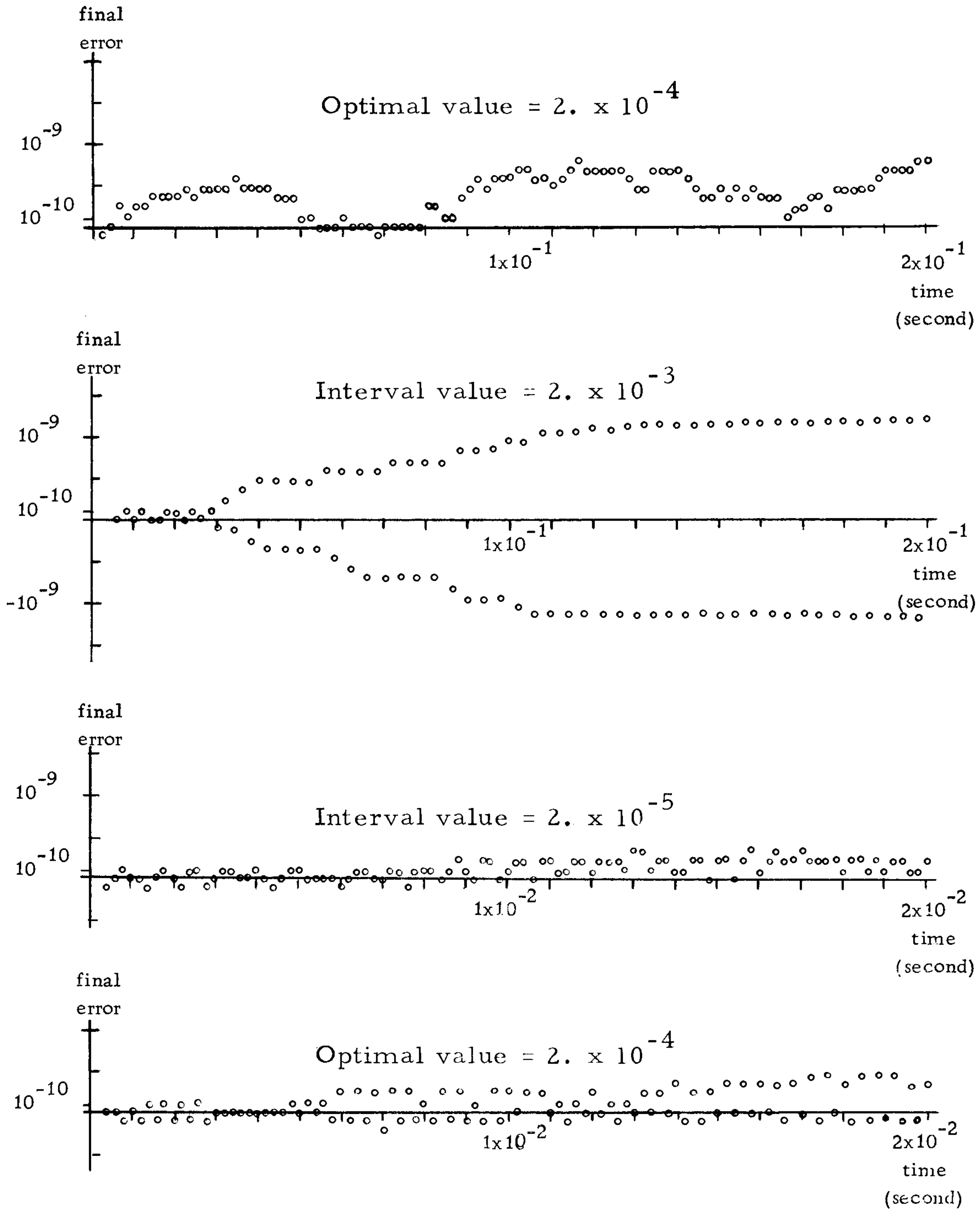


Figure 8. Comparison of errors of the optimal value and another interval value in the second-order numerical method.

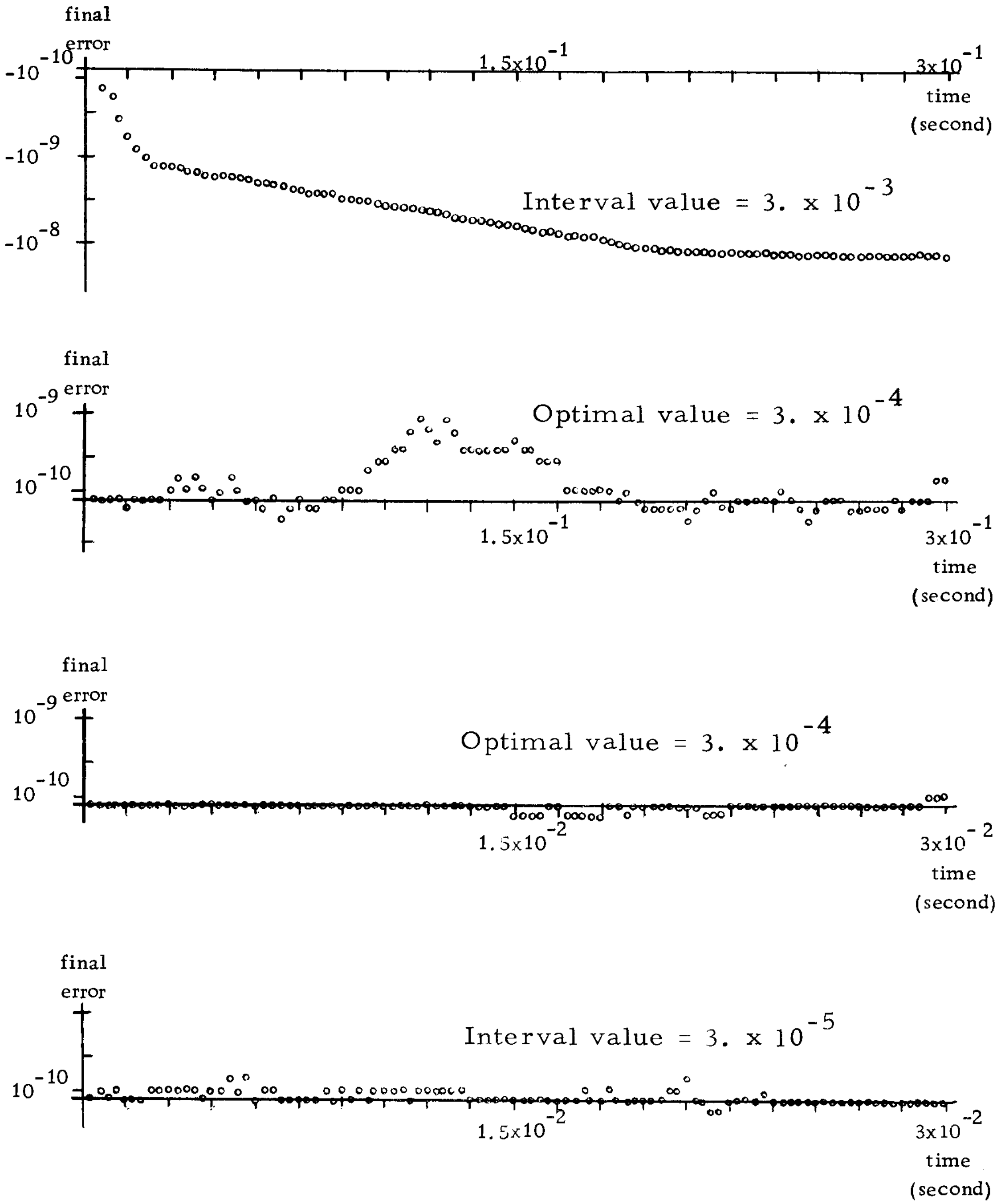


Figure 9. Comparison of errors of the optimal value and another interval value in the second-order predictor-corrector method.

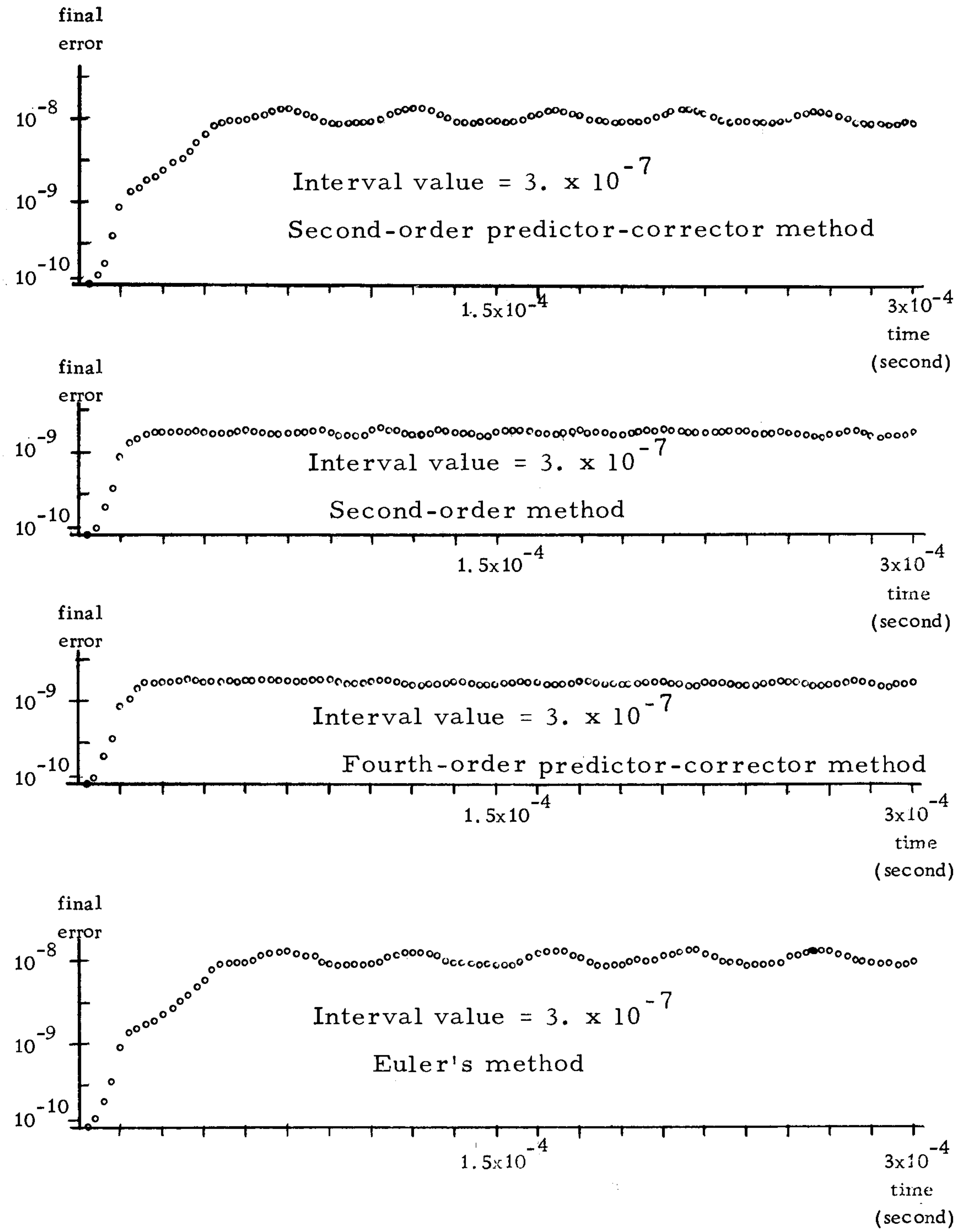


Figure 10. Comparison of errors between numerical methods when an interval value is smaller than the optimal value of Euler's method.