AN ABSTRACT OF THE DISSERTATION OF

Branwen Schaub for the degree of Doctor of Philosophy in Mathematics presented on June 8, 2021.

Title: Creating Community: A Case Study of Students' Experiences in Inquiry-Based Learning.

Abstract approved: ____
Elise N. Lockwood

We now have broad consensus in the mathematics education research community that active, inquiry-based classrooms provide a wealth of learning benefits for students (Freeman et al., 2014; Laursen et al., 2014; Theobald et al., 2020). Classrooms that utilize inquiry throughout the entire structure of the course, as opposed to the occasional group activity or unit, fall under the category *inquiry-based mathematics education* (IBME) (Laursen & Rasmussen, 2019). These classrooms are broadly characterized as consisting of four pillars: (1) students engage deeply with coherent and meaningful mathematical tasks, (2) students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, and (4) instructors foster equity in their design and facilitation choices (p. 138). Given the wide acceptance of inquiry-based methods in undergraduate mathematics, a number of studies have addressed student experiences in these classrooms to provide further insight into the benefits of IBME, the potential downsides of IBME (e.g., Brown, 2018; Johnson et al., 2020; Stone-Johnstone et al., 2019), and to share the "what/when/how/why" of mathematical content as it is taught in these spaces (e.g., Dawkins, 2014a; Kuster et al., 2018; Rasmussen & Kwon, 2007; Wawro et al., 2012). Moreover, there are few studies that combine classroom level observation data with student and professor interview

data across the span of an entire term (Dawkins et al. (2019)is a notable exception) in order to capture the interplay between students' social and mathematical experiences.

For my dissertation study, I observed an *inquiry-based learning* (IBL, which is a particular strand of IBME) undergraduate advanced calculus course. Thus, in addition to the context of being an IBME classroom, my study contributes to research on the teaching and learning of advanced calculus (also known as real analysis), which in turn heavily involves proof-based arguments and reasoning. Real analysis, often introduced as advanced calculus at the undergraduate level is a required course for most mathematics major degrees across the country (Blair et al., 2018). Existing literature shows that proof is not a trivial activity for students to engage in (Stylianides et al., 2016; Stylianides et al., 2017; Stylianou et al., 2015; Weber, 2010) and that advanced calculus is a useful setting in which to study students' proof activity (Alcock & Simpson, 2002; Alcock & Weber, 2005; Dawkins & Roh, 2016; Weber & Alcock, 2004; Zazkis et al., 2016). In particular, the IBME style of this advanced calculus classroom meant that students were engaging in authentic, student-centered proof activity for the majority of class time. Furthermore, the classroom I observed was run by a highly experienced instructor, who had over 12 years of experience teaching with IBL materials and had spent several years developing this course. Thus, this was an ideal case study for me to observe the full potential of the possibilities of what IBL can offer advanced calculus students, while still emphasizing the difficulties of IBL teaching no matter an instructor's experience level.

My broad research goal for the dissertation was to capture students' social and mathematical experiences in this classroom setting, and to explore the relation between these experiences and the IBL structure. Additionally, the instructor I observed added several activities to her course that were not prescribed by the IBL structure (such as a reflective essay on personal axioms), and so I also wanted to explore the relation between these activities and students' experiences. Notably, this class occurred over the Spring 2020 semester, which meant that I inadvertently captured the class' transition to remote learning and the instructor's expert facilitation of a safe classroom space throughout that difficult period. My data collection consisted of classroom observations for the entire term, along with a series of individual interviews across the term with the professor and five selected volunteer students.

I am broadly motivated to explore the following questions regarding student experiences in inquiry-based learning and how a sense of classroom community was created over the course of a term:

1) *In what ways did the IBL structure of the classroom influence and support interplay between the combined social and mathematical experiences of students in this classroom?*

2) *In what ways did the instructor influence and support interplay between the combined social and mathematical experiences of students in this classroom?*

3) *How, if at all, did the interplay between these combined social and mathematical experiences work to create an overall sense of classroom community?*

To answer these questions, I analyzed both classroom and interview data, which enabled me to write three papers that address different aspects of the term. Each paper combines the social and mathematical experiences of the students in a different way (a narrative analysis of the first three days of the term, a thematic analysis of how rehumanizing mathematics occurred and benefited students during the remote transition to online learning due to COVID-19, and an application of a theoretical framework for understanding students' developing values and norms around proof across the term). In order to address my first two research questions, in each paper I emphasize ways in which the IBL structure and the instructor's additional activities influenced, or provided opportunity for, these experiences. My third research question is addressed primarily in the second paper in my data regarding classroom community at the end of the term.

Creating Community:
A Case Study of Students’ Experiences in Inquiry-Based Learning

by
Branwen Schaub

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented June 8, 2021
Commencement June 2022

Doctor of Philosophy dissertation of Branwen Schaub presented on June 8, 2021

APPROVED:

Major Professor, representing Mathematics

Head of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Branwen Schaub, Author

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my adviser, the incredible Dr. Elise Lockwood. She took a chance on me six years ago as an undergraduate looking to gain experience in mathematics education research and has blessed me with so many opportunities along this journey. I would not be in graduate school without her, her grant work provided me an invaluable amount of research experience and funding, and I would not have been able to write this dissertation without her continued guidance. I thank her for introducing me to the RUME community and for trusting me to do this dissertation that was so tangential to her own research projects. In the words of Hamilton, thank you for not letting me throw away my shot.

I would like to thank the fabulous OSU Mathematics Graduate Students cohort of 2016 and the many students above and below me for their support and friendship. I wouldn't have wanted to do graduate school without them, and I will carry many fond memories of wandering Kidder Hall with a cup of coffee looking for a good distracting conversation among friends. In particular, I need to thank Ayşe Yiltekin. Ayşe went from stranger to sister almost overnight, and I would not have survived the long-distance commutes to graduate school without her hospitality. There's no one I would rather have taken quals with, eaten so many meals with, and had as my officemate and roommate. Teşekkür ederim arkadaşım.

Along my graduate school journey, I have had several important mentors. I would like to thank my Masters adviser, Dr. Christine Escher, for inspiring my continued love of topology, my teaching mentor, Dr. Daniel Rockwell, for showing me what it means to put students first, and Dr. Mary Beisiegel for her unlimited advice. Additionally, I would like to thank Dr. David Pengelley, who willingly read every word of my dissertation and always provided me with thoughtful conversation on all topics mathematics and education related over a nice hot drink.

I would also like to thank my undergraduate alma mater, the University of Portland, and the mentors from the mathematics department that helped me succeed there: Dr. Christopher Lee, Dr. Valerie Peterson, Dr. Stephanie Salomone, and Dr. Craig Swinyard. You all helped point me towards mathematics and provided an innumerable number of positive experiences that made me fall in love with the subject. I am blessed to have such powerful models of women in mathematics, and I am extremely thankful to Craig for introducing me to Elise.

To my family members: Angela, Dan, Rhiannon, and Alan, thank you for all the ways you raised and guided me on this journey. My dance family, the Muellers, thank you for providing unconditional love and a much-needed weekly respite from academia. Brendan, thank you for keeping the light on at the end of the tunnel this past year. And of course, Brody, for teaching me the true meaning of patience and progress over perfection.

I would like to pay tribute to the land which has raised me and graciously given me the space to complete my schooling from elementary, to undergraduate, and finally graduate education. To honor the past and present of these indigenous lands means to recognize the devastating effects of colonialism on indigenous people, and the land itself. My schooling was completed within the states of Oregon and Washington, and in particular the outdoor offerings of the Wenatchee Valley were essential to my mental health during the dissertation writing process.

The University of Portland, and the surrounding Portland Metro area, rests on traditional village sites of the Multnomah, Wasco, Cowlitz, Kathlamet, Clackamas, Bands of Chinook, Tualatin, Kalapuya, Molalla, and many other tribes who made their homes along the Columbia River creating communities and summer encampments to harvest and use the plentiful natural resources of the area (Portland Indian Leaders Roundtable, 2018).

Oregon State University in Corvallis, Oregon, is located within the traditional homelands of the Mary's River or Ampinefu Band of Kalapuya. Following the Willamette Valley Treaty of 1855, Kalapuya people were forcibly removed to reservations in Western Oregon. Today, living descendants of these people are a part of the Confederated Tribes of Grand Ronde Community of Oregon (grandronde.org) and the Confederated Tribes of the Siletz Indians (ctsi.nsn.us).

The Wenatchee Valley is located on the traditional homelands of the p'squosa (wenatchi), at the place the p'squosa call Skwikwiast, which we call Wenatchee. The traditional territories of the Colville Tribes extend across eastern Washington and into portions of the British Columbia, Oregon and Idaho. The Confederate Tribes of the Colville Tribes include the Lakes, Colville, Okanogan, Moses-Columbia, Wenatchi, Entiat, Chelan, Methow, Nespelem, San Poli, Chief Joseph Band of Nez Perce and Palus Indians.

Thank you.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

LIST OF FIGURES (Continued)

LIST OF TABLES

DEDICATION

This dissertation is dedicated to Dr. Stephanie Anne Salomone. Thank you for wearing so many hats in my life, including but not limited to the following: an inspiration, a teacher, a mentor, a mother figure, a shoulder to cry on, a motivator, a participant, a research partner, a boss, an ally in the fight for racial and social justice in mathematics education, and most importantly a friend.

1 Introduction

We now have broad consensus in the mathematics education research community that active, inquiry-based classrooms provide a wealth of learning benefits for students (Freeman et al., 2014; Laursen et al., 2014; Theobald et al., 2020). Classrooms that utilize inquiry throughout the entire structure of the course, as opposed to the occasional group activity or unit, fall under the category *inquiry-based mathematics education* (IBME) (Laursen & Rasmussen, 2019). These classrooms are broadly characterized as consisting of four pillars: (1) students engage deeply with coherent and meaningful mathematical tasks, (2) students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, and (4) instructors foster equity in their design and facilitation choices (p. 138). Given the wide acceptance of inquiry-based methods in undergraduate mathematics, a number of studies have addressed student experiences in these classrooms to provide further insight into the benefits of IBME, the potential downsides of IBME (Brown, 2018; Johnson et al., 2020; Stone-Johnstone et al., 2019), and to share the "what/when/how/why" of mathematical content as it is taught in these spaces (Dawkins, 2014a; Kuster et al., 2018; Rasmussen & Kwon, 2007; Wawro et al., 2012). Moreover, there are few studies that combine classroom level observation data with student and professor interview data across the span of an entire term (Dawkins (2019) is a notable exception) in order to capture the interplay between students' social and mathematical experiences.

For my dissertation study, I observed an *inquiry-based learning* (IBL, which is a particular strand of IBME) undergraduate advanced calculus course. Real analysis, often introduced as advanced calculus at the undergraduate level, is a required course for most mathematics major degrees across the country (Blair et al., 2018). It is one of the core courses of undergraduate mathematics majors, both because it requires students to justify and explore foundations for calculus, and because it exposes students to abstract mathematical topics and rigorous mathematical proof. In this way, it stands out as an important proof-based course that is known to be challenging for students (Reed, 2018, p. 26). Existing literature shows that proof is not a trivial activity for students to engage in (Stylianou et al. 2015; Stylianides et al., 2016; Stylianides et al., 2017; Weber, 2010) and that advanced calculus is a useful setting in which to study students' proof activity (Alcock & Simpson, 2002; Alcock & Weber, 2005; Dawkins & Roh, 2016; Weber & Alcock, 2004; Zazkis et al. 2016). In particular, the IBME style of this advanced calculus classroom meant that students were engaging in authentic, student-centered proof activity for the majority of class time. Thus, in addition to the context of being an IBME classroom, my study contributes to research on the teaching and learning of advanced calculus (also known as real analysis), which in turn heavily involves proof-based arguments and reasoning. Furthermore, the classroom I observed was run by a highly experienced instructor, who had over 12 years of experience teaching with IBL materials and had spent several years developing this course. Thus, this was an ideal case study for me to observe the full potential of the possibilities of what IBL can offer advanced calculus students, while still emphasizing the difficulties of IBL teaching no matter an instructor's experience level.

My broad research goal for the dissertation was to capture students' social and mathematical experiences in this classroom setting, and to explore the relation between these experiences and the IBL structure. Additionally, the instructor I observed added several activities to her course that were not prescribed by the IBL structure (such as a reflective essay on personal axioms), and so I also wanted to explore the relation between these activities and students' experiences. Notably, this class occurred over the Spring 2020 semester, which meant that I inadvertently captured the class' transition to remote learning and the instructor's expert facilitation of a safe classroom space throughout that difficult period. My data collection consisted of classroom observations for the entire term, along with a series of individual interviews across the term with the professor and five selected volunteer students.

1.1 Research Questions

I am broadly motivated to explore the following questions regarding student experiences in inquiry-based learning and how a sense of classroom community was created over the course of a term:

1) *In what ways did the IBL structure of the classroom influence and support interplay between the combined social and mathematical experiences of students in this classroom?*

2) *In what ways did the instructor influence and support interplay between the combined social and mathematical experiences of students in this classroom?*

3) *How, if at all, did the interplay between these combined social and mathematical experiences work to create an overall sense of classroom community?*

To answer these questions, I wrote three papers (Chapters 5, 6, and 7) that address different timelines during the term. Each paper combines the social and mathematical experiences of the students in a different way and emphasizes various ways in which the IBL structure and the instructor's additional activities influenced, or provided opportunity for, these experiences.

1.2 Overview of Papers

In Paper 1, *Are We Allowed to Do That? A Narrative Analysis of Student Experiences in an IBL Class*, I analyze a narrative of the class' discussions on the use of algebra in their first proof. The students' work proving a statement on the uniqueness of the additive inverse for every real number took an interesting turn when one student cited what she called "the Law of Cancellation" to cancel like terms on both sides of an equation; leading the students to question whether they could assume the use of cancellation, or alternatively whether their proof of the statement was justification for using cancellation in future proofs. My research goal in this paper is to share a nuanced picture of how students' spontaneous mathematical ideas can occur within and impact the course of an IBL classroom. In particular, I highlight how these conversations, and the instructor's role in these conversations, afforded opportunities for the class to engage in the creation of social and sociomathematical norms (Cobb & Yackel, 1996) that promoted more equitable (Gutiérrez, 2009) learning experiences.

In Paper 2, *Inquiry-Based Learning and Beyond: A Case Study of Rehumanizing Mathematics in Action*, I present an empirically grounded case study that considers how the dimensions of rehumanizing mathematics (Gutiérrez, 2018) occurred and developed, both through the structure of the course and through course elements that an instructor incorporated (such as a writing assignment that asked students to articulate a personal axiom). My evidence that the course engaged in rehumanization comes from student data at the end of the Spring 2020 term emphasizing how important this class was to them during the transition to remote learning due to COVID-19. I also employ the four pillars of inquiry-based mathematics education (IBME) (Laursen & Rasmussen, 2019) to frame my understanding of the IBL structure I observed and contribute to larger conversations on equity in undergraduate mathematics education.

In Paper 3, *Students' Shifting Values and Norms on Proof in an IBL Real Analysis Course*, I provide three examples of students developing proof values and norms over the term. I rely on Dawkins & Weber's (2017) theoretical framework for conceptualizing proof in terms of values and norms that uphold hose values. My work provides data to shed light on their framework as well as extends the notions of why these values are hard to create with students, and how an inquiry-based learning classroom provides opportunities to practice proof that are valuable in creating these norms. I also share three key activities from the instructor that were instrumental in helping students' recognize these values and recall them in final individual interviews where they reflected on what they had learned about themselves as mathematicians and their proof writing abilities over the term.

Together, these three papers address my overall research goals by taking different approaches (via separate theoretical frameworks and portions of the data set) to analyze the social and mathematical experiences of the students and emphasize ways in which the IBL structure and the instructor's additional activities influenced, or provided opportunity for, these experiences. As a small caveat, I want to draw attention to the fact that these three papers by no means encompass the full range of experiences that occurred in the classroom over the course of the term. My three papers were written primarily from observational classroom data and the interview data of five voluntary students who had overall positive experiences in the classroom. In an IBME classroom, a large majority of the instructor's work lies "behind the scenes". During class time, the instructor acts as a facilitator, and is continually making small, unseen, choices to guide the class in ways that maintain the student-centered nature of IBME. My data collection was focused on students and did not capture this difficult and nuanced work of the instructor. I therefore encourage the reader to keep in mind the untold story of the instructor behind each of these papers; there is simply more to this story than I had space to tell. In addition, there were some students in class who did not participate as actively in class, and my data collection did not permit me to investigate or analyze their experiences deeply. Overall, then, my three papers should be read as proofs of existence of what's possible with IBME, and I offer specific examples of what happened in this classroom, using certain theoretical frameworks to understand student experiences. The dissertation as a whole should not be considered an all-encompassing picture of the class or the instructor, and I acknowledge that there are more layers and nuance to all of these stories than three papers can accurately portray.

2 Literature Review

The classroom I observed for my dissertation was described as an inquiry-based learning classroom by the instructor. This classroom structure defined several aspects of the instruction, and I emphasize the impact of the structure of the course throughout the results of all three papers. Thus, in this literature review, I provide an overview of what I mean by *inquiry-based learning* (IBL) and its history, the broader term *inquiry-based mathematics education* (IBME), and a variety of studies that have been conducted in IBME classrooms. In particular, I consider ways in which studies have looked at students' mathematical experiences, affective experiences, equity related experiences, and experiences in classrooms that specifically used proofs or dealt with advanced mathematics material such as real analysis. I then consider both proof and real analysis more broadly, showing that proof is a fundamental practice at the undergraduate level that warrants further investigation and study, and that real analysis is a relevant and important course at the undergraduate level. In total, these studies all help to situate my own work in the wider context of mathematics education research describe the contributions of my own study to the existing literature base.

2.1 Inquiry Based Mathematics Education

First, I examine the broadest term possible for classrooms that use inquiry teaching methods: *active learning*. The benefits of active learning for mathematics have been well established by previous researchers (Freeman et al. 2014; Theobald et al. 2020). However, a definition of active learning that is both all-encompassing and non-trivial is difficult to write. In their article for the *Notices of the American Mathematical Society*, Braun et al. defined active learning as "classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving, that promote higher-order thinking" (Braun et al. 2017, p. 124). Thus, the overall goal of active learning is to engage students in their own learning process in a variety of ways, which can occur at a variety of different levels in a classroom. For example, a lecture-based classroom could instill a problem-solving session once a week for students, or provide small active-learning opportunities such as clicker questions throughout every class period.

The term *inquiry-based mathematics education* (IBME) was jointly constructed by Laursen and Rasmussen (Laursen & Rasmussen, 2019), to unite various lines of inquiry-based teaching and research. IBME classrooms use active learning with “a longer-term trajectory that sequences daily tasks to build toward big ideas,” to “reinvent or create mathematics that is new to [students],” and to “offer students and instructors greater opportunities to develop a critical stance toward previous, perhaps unquestioned, learning and teaching routines” (Laursen & Rasmussen 2019, p. 133-134). This characterization of IBME helps to shed light on Braun et al.’s definition and show how active learning can be systematically used as a classroom structure.

Broadly speaking, Laursen and Rasmussen (2019) characterize IBME classrooms as encompassing four “pillars” (see Figure 1) that prescribe expectations for instructor and student involvement both mathematically and socially in the classroom. The language of the top row of pillars concerning student needs were established in Laursen et al. (2014), the third pillar on inquiring into student thinking was developed in Rasmussen and Kwon (2007), the fourth pillar on equity was established in Laursen and Rasmussen (2019), and the graphic organization is shown as presented in the observed instructor’s syllabus[1].

[1] The instructor credited Dr. Nina White with the graphic.

	Mathematical Space	Social Space
Students Need	Deep engagement in rich mathematics	Opportunities to collaborate
Instructors Must	Inquire into student thinking	Foster an equitable environment

Figure 2-1: The Four Pillars of Inquiry-Based Mathematics Education

2.2 Inquiry-Based Learning and Inquiry-Oriented Instruction

One of the main goals of the term IBME is to unite two branches of inquiry, *inquiry-based learning* (IBL) and *inquiry-oriented instruction* (IOI), through their common goals and themes. However, since the classroom I observed had features that drew heavily on the IBL tradition I find it useful to explicate the differences between these two approaches to inquiry. These distinctions also help to make sense of the research that has risen out of both lines of work, and position my own contributions to the IBL community in particular. We will see that the main distinction between these two approaches comes from their historical paths; the term IOI stemmed out of undergraduate mathematics education research tradition in the early 2000s, whereas IBL was developed along multiple concurrent paths as early as the 1960s through the broader mathematical community and practitioners of various education levels.

IOI comprises a body of curriculum and research literature that is centered on design-based research (Cobb, 2000; Gravemeijer, 1994), Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1999), and is inspired by work on sociomathematical norms (Cobb & Yackel, 1996). Laursen and Rasmussen (2019) describe IOI classrooms as follows,

> Visitors to IO classrooms would see students working in small groups on unfamiliar and challenging problems, students presenting and sharing their work, even if tentative, and whole-class discussions where students question and refine their classmates’ reasoning. The students’ intellectual work lies in creating and

> revising definitions, making and justifying conjectures and justifying them, developing their own representations, and creating their own algorithms and methods for solving problems (p. 134).

Due to the origins of this work in undergraduate mathematics education research, the majority of qualitative studies in IBME classrooms currently comes from IOI researchers and their curricula. For example, the National Science Foundation supported project *Teaching Inquiry-Oriented Mathematics: Establishing Supports*[2] (TIMES) has reported on work related to the development and implementation of inquiry-oriented curricula within several mathematical domains (including abstract algebra, linear algebra, and differential equations). Within each domain, researchers have investigated work on student thinking, development and refinement of tasks and materials, and issues related to the effective implementation of such curricula in classrooms. In general, IOI can be seen as a particular characterization of IBL that has established curricula and ways of teaching (Kuster et al., 2018).

IBL comes from a long history with both mathematicians (see Section 2.3) and practitioners. Laursen and Rasmussen describe the IBL community as "a lively place for practitioners to exchange ideas and deepen their practice – a network of people and events" (Laursen & Rasmussen 2019, p. 135). The large variety of inquiry-based learning classrooms and the focus on practitioner viewpoints has produced quantitative literature such as large-scale studies about student learning outcomes in IBL classrooms (Laursen et al., 2014), and personal accounts of teacher experiences in practitioner journals such as PRIMUS (Shannon, 2016). The description below from Laursen and Rasmussen aligns well with the classroom I observed.

> Visitors to IBL courses would see class work that is highly interactive, emphasizing student communication and critique of these ideas, whether through student presentations at the board or small group discussions. Whole-class discussions and debriefs are used to aid collective sense-making, and instructors

[2] https://times.math.vt.edu/

> may provide mini-lectures to provide closer and signposting. Instructors' classroom role is thus shifted from telling and demonstrating to guiding, managing, coaching, and monitoring student inquiry (Laursen & Rasmussen 2019, p. 136).

In particular, the classroom I observed utilized whole class presentations at the board and a high amount of student interaction as they provided feedback and advice on each other's presentations. The instructor's role was mainly that of a guide or facilitator of these student conversations. Since the publication of Laursen and Rasmussen (2019), the *Academy for Inquiry Based Learning* now defines IBL using the four pillars discussed in Section 2.1, which also draw on these components of students engaging in deep mathematics while the instructor inquires into student thinking. Thus, there is a fair amount of overlap between inquiry-based learning and inquiry-based mathematics education. One way in which my study adds to the existing literature then is by providing detailed examples of how an IBL classroom can attend to the four pillars of IBME with lens' and frameworks from the undergraduate mathematics education community more similar to the existing work in the IOI community.

2.3 History of Inquiry-Based Learning and the Moore Method

The classroom I observed employed a version of inquiry-based learning that was historically developed among mathematicians and called the Moore Method (or Modified Moore Method). However, the instructor recognized that the Moore Method had a highly problematic history and preferred to not label her classroom in a way that associated her with the method. In this section, I review the history of the Moore Method, and explore the reasons behind its controversy. I find it important to include this history in order to provide a full picture of an important aspect to the foundation of inquiry-based learning and the historical development of the classroom style that I observed. I begin by describing Moore's teaching method, the ethical considerations of his method, and how the method evolved over time to present day classrooms.

Robert Lee Moore (1882-1974) was a prominent mathematician who had a significant impact on mathematics and mathematics education. In the 1960s, Moore's classroom was considered highly radical because he did not lecture. Instead, students were responsible for creating mathematical proofs of theorems that Moore gave them, and they spent class time presenting their original work to their peers. Moore disallowed his students from talking with each other, other professors, or consulting any outside materials when writing their proofs. He even went so far as to change the definitions of terms so they could not be looked up in books. While there are several more details to his teaching, I emphasize two additional factors: the way he picked students to present, and his students' transition from classwork to research. Moore made a point of asking whom he believed to be the weakest students in class for their problem solution first, and then letting one of the stronger students finish the proof. He was also known for giving the top students additional homework that were, unbeknownst to them, actually unsolved research questions that led to graduate school level work and publications. One of Moore's students, Lucille Whyburn, wrote an article describing his methods. She describes how "the Moore Method develops rugged individualism" and comments on how personal responsibility for class material shows the "question of whether or not you have the ability" (Whyburn, 1970). One can imagine how attractive this style of teaching was for mathematicians at the time, particularly the idea that one can pull themselves up by their own mathematical bootstraps if they work hard enough. While Moore pioneered this way of teaching, his methods are not considered without controversy today. Moore handpicked the members of his class, and his qualifications were white students, typically but not always male, who had what he considered to be a great aptitude for logic and problem solving, and very little formal mathematical background (Starbird, 2015).

From these descriptions, it is understandable why individuals have shied away from the Moore method or felt victimized by its tenets (Hersh, 2010).

Nevertheless, the Moore method continued to gain slow popularity over the next few decades, in part because several of Moore's students went on to become key figures in mathematical organizations such as the *Educational Advancement Foundation* (EAF) and the *Mathematical Association of America* (MAA). These students had positive experiences with the Moore Method and hoped to spread their vision by attaching the name Moore Method to the growing concept of inquiry-based learning. However, the results of a historical sociological research project on the Moore Method showed that "the ways in which this IBL movement's founders framed and labeled their movement, and consequently the pedagogical innovation they were hoping to disseminate, constrained its growth for several years" (Haberler et al., 2018). In their research, Haberler et al. found a difference in how older and younger generations of IBL educators framed the IBL movement: in general, older members desired to focus on preserving the legacy of Moore and younger members wanted to include more sociocultural viewpoints such as group work, collaboration, and social justice. As a result of such attempts by younger generations, multiple variations of Moore's method have been introduced over the years. For instance, Mahavier describes his class as a "gentle discovery method" (Mahavier, 1997). He allows for group work, office hours, and the usage of outside reference materials on the grounds that students will have to decipher book notation from class notation. Mahavier cites an unhealthy atmosphere as his reason for modifying the original method; a personal experience with a traditional Moore method class had led him away from mathematics for several years. As mathematicians have drifted further from Moore's original design, and educators outside of the Moore lineage have become interested in IBL type teaching, it has become unclear what should count as being called the Moore method, and what barriers the name may place on the movement's progress. For these reasons, the professor whose classroom I observed chose to associate her classroom within the *Academy of Inquiry Based Learning*, a much broader community of practitioners who use inquiry in the classroom, and not the Moore Method. Even so, her classroom style did carry several traits of the modified Moore Method such as students presenting proofs to their peers and using no resources besides each other and the instructor, and I find it important to acknowledge the historical development of this classroom style and its potentially inequity implications.

2.4 Studies in Inquiry Based Mathematics Education

My three papers address both social and mathematical experiences of students in an IBL classroom. In this section, I survey the wide variety of IBME literature in both of these areas. The purpose of reviewing these studies is to give an over-arching view of the ways in which student experiences in IBME classrooms have been studied previously, and to situate my dissertation within this existing body of work. In each paper, I offer additional details about IBME studies that are relevant to that respective paper.

2.4.1 IBME Studies on Students' Social Experiences

Due to the student-centered nature of IBME classrooms, there are a number of studies that have looked primarily at students' social experiences in the classroom. By this I mean the data presented in the papers are centered on student data that is not directly related to any mathematical content. On the individual level this comes across through studies that look at affect, the collection of one's attitudes, beliefs, and emotions (McLeod, 1994). At the classroom level this comes across most frequently as studies that consider aspects of equity.

Multiple studies have explored affect in IBME courses (Hassi, 2009; Love et al., 2014; Zwanch et al., 2019) through surveys and individual student interviews. For example, in Zwanch et al.'s study on the relation between IOI classes and motivation, students from 16 inquiry oriented differential equations classes were given a post-class survey question that asked them to "please comment on how the way this class was taught affects your ability to remember key ideas" (Zwanch et al. 2019, p. 3). Open coding of these answers led to four themes: (1) Engaging with Math and Each Other, (2) Less to "Know", (3) Feelings of Helplessness, and (4) Resistance to Change. Additionally, I note Dawkins et. al (2019) which detailed the relationship between an IBL teacher's goals (such as creating proofs and overcoming challenges), their practices (such as differentiated feedback) and student experiences (such as comparing students with low and high buy-in) in an IBL Real Analysis course with a similar structure as the one I observed. However, to my knowledge, no previous studies have connected this type of student or teacher data with classroom observations in order to draw connections between students' classroom experiences and shifts in affect. Thus, one way in which my study adds to the literature is by providing a case study that examines the affective experiences students have in an IBL classroom.

With regards to equity, studies such as (Laursen et al., 2014) have produced compelling results about the positive impact of IBL classrooms on the course performance by women students. In particular, they found that "women in non-IBL courses reported substantially lower cognitive gains than did their male classmates" while in IBL classes "women's cognitive and affective gains were statistically identical to those of men, and their collaborative gains were higher" (Laursen et al. 2014, p. 411-412). However, more recent studies have questioned the assumption of a necessary and sufficient relationship between equity and IBL classrooms (Brown, 2018; Johnson et al., 2020; Stone-Johnstone et al., 2019). For example, Stone-Johnstone et al. (2019) claim, "even though women may have improved outcomes, it is possible that they could be marginalized at the level of classroom participation" (p. 1). Such recent work suggests that more research is warranted to understand how IBME might affect some students.

While my study as a whole does not have research goals or questions that contribute directly to the affect or equity literature, each of my three papers contains data of students' affective experiences in the classroom and provides windows of insight into how the classroom structure contributed to the four components of equity – access, achievement, identity, and power – as characterized by Gutiérrez (2009).

2.4.2 IBME Studies on Students' Mathematical Experiences

In addition to studies on social experiences, there have been a number of studies that look specifically at the mathematical experiences, or the interrelated sociomathematical experiences, of students in IBME classrooms.

The inquiry-oriented instruction community has produced a number of results regarding how students' experiences with the various curriculum materials they have produced (e.g., Larsen, 2013; Rasmussen et al., 2006; Strand, 2016; Wawro et al., 2012). For example, Wawro et al. (2012) describe group board work and classroom conversations as students engage in an activity called *The Magic Carpet Ride*, which guided students towards reinventing the linear algebra concepts of span, linear dependence, and linear independence. There are less studies that look at IBL curriculum, in particular because it does not have the same research-design base as inquiry-oriented instruction. Thus, one way in which my dissertation contributes to the IBME literature is by providing an in-depth look at students' experiences of an IBL Real Analysis curriculum and their proof presentations of the material.

In addition, I highlight two studies that have considered the relationship between the social and mathematical experiences of students, through identifying norms and practices in IBME classrooms. Dawkins (2009) addressed sociomathematical norms in a non-traditional (i.e., not lecture-based with elements of IBME) advanced calculus classroom. He found three clusters of sociomathematical norms around valuing visualization, mathematical communication, and developing mathematics (i.e., creating definitions). In his conclusion he writes,

> The establishment of these three clusters of non-traditional sociomathematical norms transformed the students' classroom experience from one of acquisition of externally imposed mathematical abstractions into one of construction, participation, and advanced meaning making. The students learned about and participated in many cognitive and social activities that characterize the greater mathematical community (p. 181-182).

Similarly, Fukawa-Connelley (2012) categorized several social and sociomathematical norms from an Abstract Algebra class around student participation in class proof presentations. He categorized norms around presenter responsibilities (explaining and defending your work, responding to questions), audience responsibilities (reading carefully, convincing yourself, asking questions), and norms such as only using peer-validated knowledge, and working with others (p. 413). My dissertation contributes to existing literature by providing multiple examples of how the creation and negotiation of norms occur over a course term through both social and mathematical activities among the students and with the professor.

2.5 Literature on Proof and Motivating Connections to IBME

The classroom I observed was structured so that students spent most of class time presenting proofs they had written at the board, and giving feedback to each other. Proof is a well-studied practice at the K-12 (Ball & Bass, 2000, 2003), undergraduate (Raman, 2003; Weber, 2001; Weber & Alcock, 2009; Zazkis et al., 2016) and graduate (Reed, 2018) levels of mathematics. Furthermore, agendas for future research on proof, such as those described in Stylianides et al.

(2017), show that there is much left to be uncovered and understood about the ways in which students engage in proof. In this section, I define proof and explore ways in which it has been studied from the existing literature, showing that proof is a valuable practice worthy of continued study. I then consider some motivation as to why studying proof in the context of an IBL classroom is relevant to researchers today.

2.5.1 Defining Proof

Proof has been defined from a number of perspectives such as a rigorous mathematical process of logic, a cognitive process of an individual, and as a social phenomenon (Stylianides et al., 2017). Here I highlight a few ways that proof has been defined in the literature to justify my claims that proof is an area of great interest to the mathematics education community.

From the mathematical perspective, one can think of proof as “logical deductions that link premises with conclusions (e.g., Healy & Hoyles, 2001; Knuth, 2002b; Mariotti, 2000a)” (Stylianides et al., 2017, p. 238). This perspective focuses on describing proof through its function, which is typically thought of as explanation of why a statement is true using a series logical steps. An example of this then, are the two-column proofs used in high school geometry classrooms where students begin with a premise and then write a list of claims in one column and the definition, axiom, or theorem that justifies that claim in the second column until a specific conclusion is reached. However, this perspective does not capture the entire experience of proof, as a formal logic proof that rigorously explains why some mathematical fact is true might not be convincing to a reader or bring them to new understanding of mathematics.

The cognitive perspective addresses this concern by considering how individuals interact with proof and why they do so. One main concept from proof researchers under this cognitive perspective is distinguishing between proofs that *convince* and proofs that *explain* (Hanna, 2000;

Hersh, 1993; Weber, 2002, 2010). A proof that convinces leans on the use of logic and accepted definitions or axioms to conclude why the proposition is true (Weber, 2002) as in the mathematical perspective. In contrast, a proof may rely more on intuition or less formal mathematical reasoning to help the reader understand why the proof is true and thus may not be as rigorous (Weber, 2002). Another example is Harel and Sowder's (1998) framework of *proof schemes*, which demonstrates a variety of ways in which students can engage in proof (namely external conviction, empirical, and analytical proofs). Finally, Selden and Selden (2013) developed the notion of *proof frameworks* to distinguish between the formal proof writing process and the problem-solving process to better understand the way that an individual engages in creating a proof.

Finally, the social perspective addresses not just the mathematical nature of proof and how individuals engage in proof, but also how proof is socially constructed within the larger mathematical community. Combining all of these viewpoints, Czocher and Weber (2020) developed a descriptive definition of proof as a *cluster category*, which they describe as "a collection of properties that an object can satisfy to 'count toward' category membership, but no single property is necessary or sufficient for category membership" (p. 59-60). Czocher and Weber's categories include a proof as a convincing justification, a perspicuous justification, an a priori justification, a transparent justification, or a justification that has been sanctioned by the mathematical community. This definition fits in best with my study because I am both observing students that are learning at the individual level how to write proofs that fit a number of these categories, as well as considering their proof activity at the classroom level and how they come to justify their mathematical activity as a group.

Stylianides et al. (2017) write the following about the socially-embedded perspective of proof,

> … one goal of instruction within this perspective is for students to engage in an authentic way with proving as this activity is practiced in the mathematical community, including meaningfully using proof as a tool for settling debates about the truth of contentious mathematical assertions (Alibert & Thomas, 1991; Zack, 1997) and for generating and communicating mathematical knowledge… A second goal is for classroom communities to use proof for the same reasons as mathematicians, including providing explanations (e.g., Hanna, 1990), illustrating new methods to solve problems (e.g., Hanna & Barbeau, 2010), and deepening one's understanding of concepts (e.g., Larsen & Zandieh, 2008) (p. 247).

In particular, they highlight a number of critical issues in this perspective: 1) understanding mathematical practice with respect to proof, 2) identifying what proving is for students and teachers, 3) designing classroom environments where proof can be seen as a tool for generating and communicating mathematical knowledge, and 4) creating social norms with respect to proof that invite students to prove and provide learning opportunities for students when engage in proving activities (p. 247-248). This view of proof is relatively new when compared to the perspectives of proving as problem-solving or convincing and so the past literature is relatively thin. Fukawa-Connelley's (2012) study discussed in Section 2.4.2 is a good example of studying proof from the socially-embedded perspective. Overall then, I have shown clear development and interest on the part of researchers' understanding of proof as a phenomenon, and new directions in which to study proof, such as the critical issues remaining in the socially-embedded perspective of proof.

2.5.2 Student Experiences with Proof

In this section, I consider a few ways in which proof has been studied among undergraduate students, showcasing different ways in which students tend to have difficulty with proof, thus further justifying the mathematics education community's interest in studying proof.

One source of difficulty for students comes from distinguishing between informal arguments and formal proofs. Pedemonte and colleagues have distinguished ways in which students might have difficulty constructing proofs from informal arguments (Pedemonte, 2007, 2008; Pedemonte & Reid, 2011). In a similar vein, Alcock and Weber (2005) distinguished between students that take a referential (based on examples) or syntactic (based on manipulation of formal facts) approach to proof and the different ways that these two approaches created barriers for students. This line of research has continued with work such as Zazkis, Weber, and Mejia-Ramos (2014) in which researchers investigated ways to move students from referential proof writing to a successful proof (namely through three actions of syntactifying, re-warranting, and elaborating). Researchers have also pinpointed more specific areas of distinguishing between argument and proof, such as Zazkis et al. (2016), which considered how to assist students in forming connections between graphical arguments and verbal-symbolic proofs.

Another source of difficulty for students comes from students drawing distinctions between what they feel is convincing on a personal level and what they think valid mathematics looks like. Stylianou et al. (2015) conducted a study of 535 early undergraduate students with no formal proof-based mathematics classes, and found that “even though the majority of those students who were asked to construct their own arguments did not construct valid deductive proofs, the same students still indicated a preference for general arguments [when reading proofs]” (Stylianou et al., 2015, p. 116). This corresponds to a discrepancy between students seeing proof as a verification tool (Schoenfeld, 1994) used by instructors, versus a tool for students to engage more deeply in mathematical thinking (Alibert, 1988). In addition, research shows that undergraduate students draw distinctions between “deciding if an argument is personally convincing or would be sanctioned as a mathematical proof” (Weber, 2010).

Similarly, Erickson (2020) employed Harel and Sowder's (1998) framework of proof schemes to specifically investigate students' activity regarding combinatorial proof, and found that students' often dismiss these proofs as being rigorous mathematical proofs because they are often more intuitive and do not involve symbolic manipulation.

I have included this literature to emphasize that proof is a highly developed skill for students that takes time to develop as they are slowly enculturated into mathematicians' values and norms of proof (Dawkins & Weber, 2017), and to provide justification that proof is indeed a widely studied phenomenon in mathematics education that is of interest to the wider community.

2.5.3 Motivating Connections Between Proof and IBME

One of the large motivating factors for observing the particular IBL classroom I did, is that the curriculum was entirely proof-based. In other words, students were engaging in writing, reading, interpreting, and critiquing proofs on a daily basis in the classroom. Since IBME classrooms are student-centered and provide opportunities for deep, rich mathematical thinking, I was especially interested to see how students' proof activity developed over time in the classroom. One major contribution of my study then, is that it contributes to the socially-embedded view of proof. The collaborative nature of students' proof presentations addresses the third and fourth critical issues from Stylianides et al. (2017) (designing classroom environments where proof can be seen as a tool for generating and communicating mathematical knowledge and creating social norms with respect to proof that invite students to prove and provide learning opportunities for students when engage in proving activities). I discuss these topics across all three papers in regard to students' social and mathematical experiences in the classroom. While none of my overarching research questions specifically address proof, it does come through in each paper, for example as the students work through the use of axioms and justifications with the Law of Cancellation in Paper 1. In addition, Paper 3 focuses on students' shifting values and norms of proof over the term and draws heavily on the socially-embedded view.

2.6 Literature on Advanced Calculus and Motivating Connections to IBME

The proofs that students completed in the classroom I observed were all situated in the context of advanced calculus (called Real Analysis at the university where I collected data). In this section I explain the particulars of the course content and address studies that have looked at students' experiences working with advanced calculus material, and their experiences in advanced calculus classrooms. I then situate my own study in relation to this literature and justify why the uniqueness of an IBL advanced calculus classroom should be of interest to the mathematics education community.

2.6.1 Characterizing Advanced Calculus

Advanced calculus can broadly be described as the study of the real numbers and a formal proof-based undertaking of one-dimensional calculus, covering topics such as the real number field axioms, limits, convergence, continuity, and derivatives. To emphasize the importance of this course, I note that a course in advanced calculus is required of most undergraduate mathematics majors and as a prerequisite for mathematics graduate school programs. Additionally, the topics of advanced calculus lead to other mathematical areas such as statistics and applied mathematics, and provide a rigorous foundation for secondary teachers' knowledge of the real numbers (e.g., Fukawa-Connelly et al., 2020). Furthermore, as one of the first rigorous, proof-based courses that undergraduate students' encounter, it has been a prime area in which to study students' advanced mathematical thinking and proof activity (e.g., Alcock & Weber, 2005; Lew et al., 2016; Zazkis et al., 2016). For these reasons, it is of natural importance to understand and study students' understandings of advanced calculus content. At the graduate level, advanced calculus continues into the more general studies of metric spaces and calculus concepts that go beyond the undergraduate level, and the course is often referred to as Real Analysis. The university I completed my study at was an undergraduate only institution, and titled their advanced calculus course as Real Analysis, and so I use the two terms somewhat interchangeably, noting that Real Analysis might mean a more advanced content than covered in the classroom I observed.

2.6.2 Research on Student Experiences in Advanced Calculus and Real Analysis

Although advanced calculus is an incredibly important subject in mathematics and notoriously difficult for students, there remains a wide gap in the literature on real analysis content when compared to other mathematical domains, with the exception of formal definitions of limits (e.g., Oehrtman et al., 2014; Swinyard, 2011). One reason for this may be that advanced calculus is a fairly high-level course and not as widely taken by students such as a course like linear algebra or differential equations. However, there has been a recent shift in attention towards research on advanced calculus content and I now address a few of these studies.

Reed (2018) pointed to ways in which students can leverage ideas in advanced calculus for understanding more advanced topics (like metric spaces) that are important and necessary for explore in higher-level mathematics (such as what they might see in graduate school). Strand (2016) used local instructional theory to assist students in reinventing several advanced calculus concepts by developing a local instructional theory for “supporting the reinvention of formal conceptions of sequence convergence, the completeness property of the real numbers, and continuity of real functions” through the use of Cauchy’s proof of the Intermediate Value Theorem. In particular, Strand considers “two students' reinventions of formal conceptions of sequence convergence and the completeness property of the real numbers in the context of developing a proof of the Intermediate Value Theorem (IVT)" (p. 1). In a similar style, Vroom (2020) considers guided reinvention as a way to develop students' mathematical language fluency, and in particular the use of multiple quantifier statements in advanced calculus. While all of these studies offer new and exciting insights into students' experiences with advanced calculus, there is still much more to be learned about how students develop their formal, proof-based understandings of advanced calculus topics.

Researchers are also considering ways in which to make the advanced calculus curriculum of interest and important to students that are not going on to higher mathematics. Notably, the Upgrading Learning for Teachers in Real Analysis (ULTRA) project has developed modules specifically targeting teaching advanced calculus to pre-service teachers (Fukawa-Connelley et al., 2020). These modules connect advanced calculus content to secondary mathematics by beginning with a teaching episode of secondary mathematics, building up to how advanced calculus can address the episode, and then stepping down again to draw some conclusions on how to connect their advanced mathematical knowledge to the practices of teaching secondary content. In general, the results of ULTRA so far have shown promise in both teaching advanced calculus content to students and making future teachers feel that a course in advanced calculus is beneficial and useful for their future career.

I now consider ways in which studies have used an advanced calculus classroom for studies with other separate focuses, namely as a place to study proof and sociomathematical norms in advanced mathematics. First, advanced calculus is a common context for studying students' proving activity (e.g., Alcock & Weber, 2005; Lew et al., 2016; Weber & Alcock, 2009). This is because it tends to be one of the first proof-based mathematics classes that students encounter, and it poses challenges for students as they must reason abstractly about novel concepts. For example, in Lew et al. (2016) the authors use an advanced calculus classroom to investigate what students take away from proofs presented in advanced mathematics classes and whether or not these takeaways matched the instructor's intentions. The authors video recorded a lecture on several proofs by a highly experience instructor and then interviewed the professor to determine what the main ideas they tried to convey during the lecture were. They then interviewed six students to see what they took away from the lecture and found several discrepancies between the instructor's goals and the students' experiences, mainly that the students' did not internalize what the instructor had expected them to learn. Weber (2008) used undergraduate real analysis students as the subjects for his study on the role of affect in students' proof writing experiences, asking students to think aloud as they worked on proofs in order to document moments of frustration, anxiety/despair, encouragement, and pleasure. He found strong that affect can have a strong influence on a students' proving experience, i.e., "frustration and anxiety led the observed student to place more emphasis on rote learning strategies and avoid engaging in the course material" and "small gains in [a student's] understanding provided her with feelings of pleasure and encouragement, which in turn motivated her to seek out opportunities to study the material further" (Weber, 2008, p. 71).

Advanced calculus is also a useful place to study students' socio-cultural experiences with advanced mathematics because it is often one of the most advanced courses undergraduate mathematics majors are required take (perhaps alongside a course in abstract algebra). Dawkins (2009) and subsequent works (2013, 2014b) consider the role of social and sociomathematical norms in advanced calculus classrooms. For instance, Dawkins (2013) considers "students' individual patterns of adherence to a norm for creating and assessing definitions in an undergraduate real analysis classroom" (p. 237). However, the use of formal definitions occurs in many classes besides real analysis, and the results presented in the paper identified patterns of non-adherence, peripheral adherence, and authoritative adoption of the norm on creating and assessing definitions. While a real analysis classroom was the setting for the study, there was no inherent connection to students' understanding of real analysis content. I do not view this as a drawback of their study; I am merely reasserting the fact that many studies in real analysis classrooms do not address or expand on students' understanding of real analysis content.

2.6.3 Motivating Connections Between Advanced Calculus and IBME

Not only did I observe a proof-based IBL classroom, but the particular proofs and material that students worked through was advanced calculus content. Thus, one major way in which my study contributes to the existing literature is by providing a detailed case study of student experiences in an IBL advanced calculus classroom and showcasing a number of aspects of the curriculum that are unique when compared to the existing IBME literature. In particular, I identify several connections between the real analysis content and the students' social and mathematical experiences, such as the field axioms' use in the Law of Cancellation in Paper 1, and the use of multiple quantifier statements in shifting students' proof norms around understanding and interpreting others' proofs in Paper 3. Additionally, the humanistic activities that the instructor added were centered around connecting the concept of axiomatic formal mathematics with students' personal identities, as in the This I Believe essay on a core belief of an individual that they take without proof. While it is not heavily discussed in my papers, I do see this connection as more relevant in real analysis as opposed to a class like abstract algebra because the students are already familiar with most of the concepts of advanced calculus from their earlier classes. Thus, addressing their own identities and personal belief systems is an easier parallel to draw as they identify the axioms and formal definitions behind the calculus that they already know. Thus, my dissertation as a whole brings out some of the unique features of how the proof-based structure of advanced calculus content is well-suited for an IBME classroom and ways in which humanistic activities can correlate to the advanced calculus content.

2.7 Summary of Literature Review

In this literature review, I have characterized the term *inquiry-based mathematics education*, as well as the history and differences between *inquiry-based learning* and *inquiry-oriented instruction*. Based on these definitions I have surveyed literature on students' social and mathematical experiences in IBME classrooms, supporting my overall research goals by showing how past literature has collected data on, and analyzed student experiences in IBME. I have also argued for why the particular case of a proof-based advanced calculus IBME classroom should be of interest to the research community, based on the past literature in both of these areas of study. My work thus extends the mathematics education research community's understandings of IBME classrooms by combining classroom observation data and individual interview data to paint a fuller picture of the interconnected nature of students' social and mathematical experiences in an IBL undergraduate Real Analysis classroom.

3 Theoretical Perspectives

In each of my three papers, I employ a different theoretical framework that targets the research questions of that particular paper. While each framework will be discussed in detail within the relevant paper, I provide a broad overview of the three frameworks, how they relate to my overall theoretical perspective as a social constructivist, and how they help me address my broader research questions.

3.1 Social Constructivism

Social constructivism is a theory of learning that considers both students' individual cognitive efforts and the context of their environment to create mathematical meaning for themselves. Glasersfeld (1989) states the first two principles of constructivism as: (1) that knowledge is not passively received but actively built up the cognizing subject, and (2) that the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality (p. 182). Taking (1) by itself gives *trivial constructivism,* named so by von Glasersfeld because it failed to consider the purpose of cognition, which is stated in (2). (1) and (2) together are called *radical constructivism*; the radical piece involves suspending one's belief in an ontological (external) reality and shifting focus to how individuals build up their own experiential worlds. However, these principles focused on the cognition of an individual, and do not take their environments into account. Thus, *social constructivism* (Ernest, 1994) extended the work of von Glasersfeld to include the physical and social worlds that an individual engages in. This idea is nicely summarized in a third constructivism principle added by Taylor and Campbell-Williams (1993); cited in Jaworski (1990). She writes,

> The third principle derives from the sociology of knowledge, and acknowledges that reality is constructed intersubjectively, that is it is socially negotiated between significant others who are able to share meanings and social perspectives of a common *lifeworld* (Berger and Luckmann, 1966) (Jaworski 1990, p. 24).

From this third principle, I highlight the concepts of intersubjectivity and shared meaning. Intersubjectivity considers how meaning is co-created between individuals as they negotiate mutual understanding. This perspective aligns with my research questions because I am focused on collecting data at the classroom level and understanding students' shared social and mathematical experiences due to the structure of the classroom and the instructor's activities. While I do not explicitly discuss social constructivism in any of my three papers, it is implicit in the nature of my data collection, my research questions, and my choice of theoretical frameworks.

3.2 Specific Theoretical Frameworks for Each Paper

3.2.1 Interpretive Framework

In my first paper, I employ Cobb & Yackel's (1996) *interpretive framework* for relating the social and psychological perspectives of classroom activity. This framework put forth three constructs (classroom social norms, sociomathematical norms, and classroom mathematical practices) – see Table 1, designed to characterize social analogues of the psychological (beliefs about roles and nature of mathematical activity, mathematical beliefs and values, and mathematical conceptions and activity) aspects of students' individual activity.

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Table 3-1: Cobb and Yackel's interpretive framework (1996, p. 177)

In Paper 1 (Chapter 5), I focus primarily on the social aspects of Cobb and Yackel's framework, and use their language to identify and explain the classroom activity that I observed. Notably, this social perspective ties into my overall positioning as a social constructivist since I am viewing the social, mathematical, and sociomathematical norms as being co-created in the classroom environment by the students and instructor. This framework also assists me in addressing my overall research questions by providing language with which to view activity at the classroom level and make claims about the overall classroom environment.

3.2.2 Rehumanizing Mathematics Framework

In Paper 2 (Chapter 6), I use Gutierrez's (2018) framework for *rehumanizing mathematics* as a lens to understand the instructor's actions in the classroom, and to consider both how the instructor's teaching actions were informed by the IBL class structure and how she took additional actions of her own to create an equitable experience for her students. I specifically chose the rehumanizing mathematics framework over other frameworks that focus on equity (Gutiérrez, 2009; Tang et al., 2017) in order to create a more detailed picture of how an instructor might foster equitable actions in a mathematics classroom. From Gutiérrez, I interpret that rehumanization, as opposed to equity, focuses on actionable, evidence-based, ways to encourage equitable experiences in the classroom. Gutiérrez (2018) lists eight dimensions of rehumanization: "(1) participation/positioning, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) body/emotions, and (8) ownership" (Gutiérrez, 2018, p. 4). This lens of rehumanization ties into my focus as a social constructivist by providing ways in which I can explain the purpose and impact of various social interactions in the classroom that I observed and how these social aspects impacted the overall mathematical learning. This in turn helps me address my overall research questions by showing ways in which the social aspects of the classroom interacted with the mathematical learning space.

3.2.3 Proof Values and Norms Framework

In Paper 3 (Chapter 7), I use Dawkins and Weber's (2017) theoretical framework on proof values and norms to understand how students' classroom proof practice developed over the term. Dawkins and Weber developed this framework as one way in which to help make sense of why numerous research studies have shown that it is difficult to "apprentice students into the mathematical practices associated with proof" (Dawkins & Weber, 2017, p. 123). Many students find proof problematic or confusing, and in their paper, Dawkins and Weber argue that these struggles stem from students not having adopted the proof values and norms of mathematicians.

In Table 1, I provide the values and norms around proof as listed in the section headings of Dawkins & Weber (2017); the numbering of norms and values is specific to my dissertation for ease of discussion. I note that there are no norms associated with the fourth value. Please see the Results of Paper 3 for more detail on these values and norms.

Value	Norm	Description
Value 1		Mathematical knowledge is justified by a priori arguments.
	Norm 1.1	Justification in proof must be based on stipulated definitions.
	Norm 1.2	Justification in a proof should be deductive and not admit rebuttals.
Value 2		Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
	Norm 2.1	Mathematical proof is written without reference to author or reader's agency.
	Norm 2.2	A proof is an autonomous object, not a description of a problem-solving process.
Value 3		Mathematicians desire to increase their understanding of mathematics.
	Norm 3.1	Routine calculations and obvious justifications are omitted from a proof.
	Norm 3.2	Irrelevant statements are not presented in a proof.
	Norm 3.3	Published proofs are typeset to reveal their mathematical structure.
	Norm 3.4	Symbol choice follows conventions.
Value 4		Mathematicians desire a set of consistent proof standards.

Table 3-2: Proof Values and Norms, adapted from Dawkins & Weber (2017)

This framework ties into my overall theoretical perspective as a social constructivist because it takes a social view of proof. This in turn helps me address my overall research questions by showing ways in which mathematics is socially constructed and impacts students' social interactions.

4 Methods

In this section I outline my overall methods for the whole dissertation – the data I gathered, the context of the research site, and the overall analytical approach I took with the data. I offer more methodological details that are relevant to each respective paper in Chapters 5, 6, and 7.

4.1 Data Site

The data collection for my dissertation took place at a small liberal arts university in the western United States. I observed an upper-division mathematics course which was entitled "Real Analysis" at this university. The topics, described in the course syllabus as follows, often fall under the purview of Advanced Calculus at the undergraduate level, but the course was called Real Analysis. The course syllabus described this class in the following way: "A rigorous treatment of properties of the real numbers and functions of a single real variable. Topics include completeness, limits, continuity, differentiation, integration, and sequences. Additional topics may include series, an introduction to Euclidean or metric spaces". I chose to study this classroom for its uniqueness of being an upper-division mathematics course using IBL teaching methods that also engaged in humanistic activities (such as an essay on personal axioms called *This I Believe*). Furthermore, this classroom was run by a highly experienced instructor, who had over 12 years of experience teaching with IBL materials and had spent several years developing this course. Thus, this was an ideal case study for me to observe the full potential of the possibilities of what IBL can offer students, while still emphasizing the difficulties of IBL teaching regardless of an instructor's experience level.

4.1.1 Class Structure

The entire Real Analysis course was based around a highly scaffolded packet of materials (a printed and stapled set of mathematical content for the class that was given to each student on the first day); I subsequently refer to these materials as "the packet," which is how the instructor and students referred to them. The packet began with the definition of the real numbers, the field axioms, a definition of subtraction and division, and a list of theorems to prove in numerical order (the packet included additional axioms and definitions that are beyond the scope of the data presented in this paper). Because the theorems were listed in order, the expectation was that the first proof could only be completed using the definitions and axioms given, the second proof could use the definitions, axioms, and first theorem, and so on. The standing homework assignment for students was to “try something” by working through the packet and claiming problems on an Excel spreadsheet when they had a proof to share with the class. Students were expected not to use any resources other than the definitions and axioms in the packet, the instructor’s office hours, and each other. In addition to the mathematical content addressed in class, the instructor included several humanities-based journal assignments and small class activities that focused on addressing students’ personal understandings of axioms, belief, and their identity as mathematicians.

The class was structured so that most of the class time was spent with students at the board giving presentations of proofs from a packet that the instructor gave them, along with questions and corrections from their peers. A typical class day started with an open call for questions or discussion on previous work, and then the instructor would pick a student to present the next proof at the board, alphabetically from the Excel spreadsheet. After a student presented, they would answer questions and potentially engage with peers in a collaborative revision process, sometimes trading off who went to the board to explain new ideas. The instructor continually directed students to lead their own discussions, and she carefully chose when to engage in explanation or mediation of a debate. Time out of class was spent working alone or with peers on these packet problems. Final grades were based on class participation, weekly homework assignments of writing up proofs that were presented in class, two exams, and a portfolio that combined commentary on proofs done throughout the term with a final reflective essay.

In Appendix A, I provide a general overview of the term schedule, including the timeline of mathematical topics, humanistic activities, and protocols for the individual interviews with students and the professor. I note that one unexpected aspect of my data collection was the emergency transition to remote teaching that occurred between Week 9 and 10 of the term due to COVID-19. In Appendix B, I provide the syllabus with any identifying information for the school redacted. I include this syllabus to provide the interested reader with a more detailed look at how the inquiry-based learning structure was introduced and explained to the students. Additionally, in Appendix C, I provide the entire “packet” of mathematical material that was given to students over the course of the term. Again, I have included this material for the interested reader, and to provide further context for the class excerpts presented in the results of the three papers. These materials also highlight the depth of time and work that the instructor put into her course, and again I emphasize that this case study is an exemplary case of what is possible with IBL classroom design.

4.1.2 Researcher Positionality

One unique aspect of qualitative research is that the researcher (in this case, me) is the instrument for data collection as well as the analyst. Thus, every piece of data is filtered during collection through all of the bias, prejudices, and orientations of the researcher before the analysis stage has even begun. Given that this subjectivity is unavoidable, it becomes the researcher’s job to dutifully and ethically report as much bias as possible, to make obvious their own thoughts and to observe themselves just as much as their participants. To this end, I kept a research journal to write in daily and keep track of my own understandings of the study as it evolves over time. Going into this study, I was aware of my friendship with the classroom professor, which both granted me access to the classroom and biased me to interpret her actions in an overly friendly light. Furthermore, my own experiences with Real Analysis and inquiry-based learning endowed me with memories and my own feelings about the class experience that I had to both notice and separate from my dissertation work. I also acknowledge that my appearance as a young cis-female White graduate student undoubtedly positioned students and the instructor to act in certain ways around and towards me, both in my observations and interviews. These unavoidable lens were also a natural part of my own bias and again something I considered in my personal reflections. While none of these concerns were reason not to do my study, it is important that I remained aware of my personal biases throughout the research process.

Throughout my data collection and analysis, I actively searched for alternative explanations to the themes I was generating and to be aware of whether and how I was providing a balanced view of the classroom. This class was extremely unique both in terms of the IBL mathematics materials, the humanistic activities, the instructor's expertise, and comfort with facilitating an IBL classroom and with sharing her own humanity with students, and the transition to remote learning due to COVID-19. In presenting my results and considering the larger picture of what this case study has to offer, I worked diligently to maintain a frame of mind that acknowledged the uniqueness of this case study and how the results may or may not transfer to other IBME classrooms. This dissertation and the results are not meant to portray how every IBME classroom works, or the level of community that IBME on its own produces, but rather to exemplify how the intersectional space of IBME, humanism, and a highly skilled instructor can create beneficial social and mathematical experiences for students that promote the formation of a classroom community.

4.2 Participants

The participants of my study were the students and the professor of the Real Analysis class. In this section I describe the student population and the professor.

4.2.1 The Students

The Real Analysis class had 19 students, mostly junior or senior students, mathematics majors, minors, or double majors ranging from theology, engineering, computer science, and biology. I use pseudonyms throughout all three papers to protect student identities. One important aspect of my student population is that because this was a small university and an upper division course, several of these students knew each other and were friends. Additionally, several of them had taken classes with the professor before, including a Discrete Mathematics (introduction to proof) course that had also utilized inquiry-based learning techniques. These aspects of familiarity with each other, the professor, and some level of inquiry-based learning had an unavoidable impact on the results of my data collection and the level of community that this classroom ultimately created.

4.2.2 The Professor

Dr. Miya (a pseudonym) is a full professor and was the chair of the mathematics department at the time I completed my study. Her interest in IBL teaching methods started in the summer of 2006 when she attended a 4-day intensive IBL through the Educational Advancement Foundation. She has attended several IBL trainings since then and is now a trainer herself, leading workshops for the *Academy of Inquiry Based Learning*. In her Real Analysis syllabus, she states,

> It's entirely reasonable that you would wonder why I'd run a class this way. I mean, technically I am an expert in real analysis with a diploma signed by Arnold Schwarzenegger to prove it. I love talking about mathematics, so why take a back seat? The full answer to "WHY?" will unfold over the course of the semester. There will be times when you are certain this experiment in learning won't work, or that what you're doing won't possibly get you to whatever academic goal you've set for yourself. When that happens, please step back and trust me. Trust that I believe you can do this, and do it well. Trust that I believe that learning using IBL is the best way to learn to think like a mathematician. Trust that the takeaways from learning using IBL are much bigger than you can predict now. You'll see. I promise.

Thus, Dr. Miya was fully engaged in and aware of IBL teaching methods, making her class an exemplary model with which to study student experiences in IBL classrooms.

I note a number of demographic traits, and characteristics of the class, that afforded Dr. Miya the privilege to engage in this type of classroom, especially the additional humanistic activities. Dr. Miya identifies as a White, straight, cis-gender woman, and has tenure along with years of leadership in her department. I hypothesize that these traits gave her a level of social expectation/acceptance to engage in humanistic and vulnerable activities with her students and run an unconventional classroom. Additionally, she had cultivated a number of friendships and mentorships with students in this classroom before the beginning of the Real Analysis class that I am sure played an implicit role in the way community was established in this particular classroom. Finally, Real Analysis is considered a "capstone" course for the mathematics major and has no required topics as a pre-requisite for future courses (at this institution, Real Analysis 2 is an optional course that picks up wherever the last professor ended Real Analysis). Thus, the transition to remote learning was somewhat easier for this professor in that she could pick and choose the pace of the course and provide more space for her students to engage in community building as opposed to necessary mathematical content to proceed to a future course (as might have been the case in a Calculus 1 class for example). I add these caveats to recognize that not every instructor may have the same opportunities or access to engage their students in mathematics and create community as Dr. Miya did. However, I do believe that the overall classroom style and activities completed by students provide several great examples for other instructors and serve as a proof of existence for the benefits of inquiry-based learning methods in advanced undergraduate mathematics classes.

4.3 Data Collection

My data collection was comprised of two components: classroom observations and interviews. I describe both methods of data collection in the following section. Notably, during the 10th week of term (of a 16-week semester), the classroom had to transition to remote learning due to COVID-19. I continued both components of my data collection via video conference technology and updated consent forms.

4.3.1 Classroom Observations

Due to the IBL structure of the classroom, I could not guarantee when interesting episodes might occur, when certain topics would be discussed, or how any notions of community would develop over the term. Thus, I chose to complete in-person, and then virtual, classroom observations over the whole term, approximately 35 hours of observations. The classroom was already being recorded for professional development purposes, and I used these videos in my own data analysis as well. The video recorded observations came from one camera at the back of the classroom, focused the camera on the whiteboard and student presenting at the front of the room. I also took consistent fieldnotes detailing what proofs were done at the board, and noting any interpretations of classroom interactions that I would later triangulate with interview data from the students and/or professor. In particular, I made note of any instances in the class that I thought captured important features for my research questions and the accompanying time stamp. These timestamps allowed me to re-listen/watch the classroom event for further detail.

Thus, while a large amount of audio/video data was collected, I only interacted with this data on a daily basis as necessitated by my field notes.

4.3.2 Individual Interviews

In addition to the classroom observations, I conducted selection interviews (see Appendix D for the selection interview guide) and recruited 5 students for a series of individual interviews with the aim of obtaining a wide variety of student experiences (different backgrounds, career goals, and attitude towards the class). These students were selected based on availability and interest in participating in the research study. They were paid $20 an hour for their time. Student pseudonyms, areas of interest, and number of interviews are listed below in Table 3.

Name	Major	Post-College Plan	# of Interviews
Ash	Mathematics	Graduate School, Mathematics	3
Hayden	Mathematics	Graduate School, Mathematics	4
Parker	Mathematics, Secondary Education	High School Teacher	4
Sloan	Mathematics, Theology	Graduate School, Theology	4
Taylor	Mathematics, Biology	Graduate School, Biology	3

Table 4-1: Individual Student Interview Participants

For the 5 students that agreed to participate, I held three to four individual interviews over the course of the term: a beginning of class interview (Week 4), a pre-COVID transition interview (Week 8 for participants dependent on availability), a post-COVID transition interview (Week 11), and an end of class interview (Week 11) (see Appendix E for the individual student interview guide). These interviews ranged from 30 minutes to 1 hour. For student interviews, I used a standardized interview guide (Patton, 2015 p. 344), so that each student was asked the same questions about their course experience. However, I left myself freedom to combine the guide with a more informal conversation approach (Patton, 2015, p. 342) that allowed to me to probe deeper if I feel that a student has more to say about a particular topic. I made this decision because I believed it likely that treating the interview as a jointly produced conversation would help interviewees give more robust and honest answers about their classroom experiences. I drew my interview questions from Darragh's work on mathematical identity (Darragh, 2014, 2015, 2016) to draw out each individual's experience of the classroom and how they interpreted others' actions in class. Example interview questions include:

-Who stands out to you in class?

-Is there anyone in class who stands out to you as being good at mathematics?

-Can you describe for me what someone who is good at math would be like?

-Do you fit this description?

-What do successful students do in this class?

-Describe your last proof presentation to the class, what was it like?

In addition, I held three interviews with the class professor (pre-class, Week 4, and Week 8). These interviews were more open-ended discussions in which we discussed the overall progress of the class and compared interpretations of classroom events that I had selected for further analysis, as a form of data triangulation (See Appendix F for the professor interview guide). Example questions include:

- How would you describe the classroom environment over the past two weeks?

- What changes have you noticed in the classroom over the past two weeks?

- I noticed this (example) happen, what is your take on why this happened?

- How would you describe Student (name) in class?

- I noticed you did this (example) in class, what was your reasoning? How did you expect students to respond? Was it successful?

All of the student and professor interviews were audio recorded and transcribed for further analysis.

4.4 Data Analysis

I consider my data analysis as consisting of two stages. The first stage was the preliminary analysis that occurred while I was still collecting data in the field. Examples of this analysis are starring classroom episodes as interesting or relevant, planning interview questions based around moments I noticed in the classroom that week, and adjusting my specific areas of focus as the term progressed and I noticed the growing sense of classroom community. The second stage was a series of thematic analyses (Braun & Clark, 2006) across the data set, spanning the classroom observations, interviews with the 5 students, and professor interviews, culminating in my three papers. This level of analysis did not occur until my data collection was complete since I had to step back and consider the entire term and allocate myself three distinct yet cohesive papers that exemplified the creation of community that I witnessed across the term.

4.4.1 Thematic Analysis

Thematic analysis is “a method for identifying, analyzing, and reporting patterns (themes) within data” (Braun & Clark 2006, p. 6). The flexibility in the definition of thematic analysis allows it to be considered as a methodology for a variety of studies. Thus, Braun and Clark are careful to describe three choices that researchers must explicitly make in order for their study to remain rigorous and make sound use of the definition.

First, a thematic analysis may focus on an *inductive* (bottom-up) or *deductive* (top-down) approach. Typically, the deductive approach stems from a researcher using a particular frame, such as an existing theory as a way to generate codes for their data set. In contrast, an inductive approach allows codes to emerge and evolve from the data in whatever shape fits best. Due to the broadness of my overarching research questions, and the student-centered nature of the classroom, while I always had a general idea of what to expect in the classroom, I allowed myself to stay curious and open to interesting turns the class was taking or unexpected ways in which students engaged in the content or IBME class structure. Thus, I chose an *inductive* approach to my data analysis, where my questions and themes rose out of the novel data I collected. One particular example of this is Paper 1 on the Law of Cancellation, which was a highly unexpected experience (both for the students and the instructor) that I had not planned to see or have an existing framework in mind ready to capture data on.

Second, a thematic analysis can identify themes at either the semantic or latent level. A *semantic* approach focuses on the "surface meanings of the data and the analyst is not looking for anything *beyond* what a participant has said or what has been written" (Braun & Clark 2006, p. 13). In this approach, a researcher will focus on describing, organizing, and interpreting patterns found in their data. Constructionists tend to work under the *latent* approach, which "starts to identify or examine the underlying ideas, assumptions, and conceptualizations – and ideologies – that are theorized as shaping or informing the semantic content of the data" (Braun & Clark 2006, p. 13). As a social constructionist that was trying to infer meanings behind activity at the classroom level, I employed the latent approach in my data analysis. An example of this is my work in Paper 3 on students' shifting proof values and norms in which I coded beyond what participants had said in interviews, or I had observed in the classroom, to identify the underlying norms and values of the conversations. For example, here is an excerpt from an interviewee recalling a particular classroom episode of a student receiving feedback on their proof.

Ash: *Um I remember a moment in class when like Easton presented a proof and we talked about it the entire class. And we talked about the quantifiers and like what position they needed to be in. And um we came to the conclusion that Easton's last line needed to be written differently for the quantifiers to be in the right spot.* ***And then I remember him*** *saying, I remember him saying "oh that was just my preference, I know that it's supposed to be written that way but I just put it on the board like that".*

Int.: *Mmm.*

Ash: *And like that definitely happened. Um I feel like those are really weird moments for me. I don't really know how to respond. Cause part of me is like "did you really know? or like are you just saying that was your preference and you know what they're talking about? Or like, you know?* ***Sometimes I feel like it's easier to like use "yeah it was my preference" instead of saying like "oh yeah you're right, the logic of my proof is confused".***

In this quote, Ash does not explicitly say that she had a shift in how she viewed others' proofs, but when compared to data at the beginning of the term and other interviewee responses as well, I was able to use the *latent* approach to help me in constructing my best interpretation of Ash's underlying assumptions and views of proof that had shifted over the course of the term.

Finally, Braun and Clark distinguish between researchers who use an *essentialist/realist* or a *constructionist* paradigm in their work. An essentialist thematic analysis would assume a one-directional relationship between meaning and experience at the level of the individual, whereas a constructionist thematic analysis "seeks to theorize the socio-cultural contexts, and structural conditions that enable the individual accounts that are provided" (Braun & Clark 2006, p. 14). Again, my social constructivist perspective and the latent nature of my data analysis tended me towards a *constructionist* paradigm in my work. In particular, in my data analysis I was consistently aiming to understand relationships between the structure of the IBME classroom and the instructor's particular facilitation of the IBME classroom on students' social and mathematical experiences. An example of this is Paper 2, in which I provide a number of examples that contributed to the students' "rehumanizing mathematics" and creating a sense of classroom community.

There are six phases of thematic analysis: (1) familiarizing yourself with your data, (2) generating initial codes, (3) searching for themes, (4) reviewing themes, (5) defining and naming themes, and (6) producing the report (Braun and Clark 2006, p. 17). These phases should not be considered as a strictly linear process, for example, when reviewing themes, one might realize that it would be helpful to add an additional code or dissolve a code into other existing ones. With this broad framework of thematic analysis in mind I was able to identify my three papers through multiple, iterative processes of familiarizing myself with the data, generating codes, and searching for themes.

Paper 1 arose from starred episodes at the beginning of the term that ultimately had a significant impact on how the classroom developed sociomathematical norms (cite) and how students experienced that development over time. I found that these themes were best organized and described through a narrative analysis approach, which is detailed in the Methods of Paper 1. Paper 2 arose from a desire to capture the broader social experience of students across the term, and in particular the shift to remote learning due to COVID-19. In this paper, I especially employed the inductive lens of thematic analysis to work from the ground up and produce themes that ultimately aligned with Gutierrez (2018)'s eight dimensions of rehumanizing mathematics. Paper 3 arose from a targeted thematic analysis of students' final interviews in which I coded for shifts in students' perceptions of their proof activity. In this paper, I utilize Dawkins and Weber's (2017) framework for mathematicians' proof values and norms to inform my analysis.

Together, all three papers address my research questions for the entire dissertation by providing different lens with which to view students' social and mathematical experiences in the classroom and to pinpoint which of these experiences arose from the IBL structure and which arose from added elements of the particular instructor I observed. In addition, Gutiérrez's (2018) framework for rehumanizing mathematics provided a particularly potent lens with which to view my third research goal on establishing classroom community.

Finally, I want to repeat that these three papers by no means encompass the full range of experiences that occurred in the classroom over the course of the term. and My three papers should be read as proofs of existence of what's possible with IBME and specific examples of what happened in this classroom, using certain theoretical frameworks to understand student experiences. The dissertation as a whole should not be considered an all-encompassing picture of the class or the instructor, and I acknowledge that there are more layers and nuance to all of these stories than three papers can accurately portray.

5 (Paper 1) Are We Allowed to Do That? A narrative analysis of student experiences in an IBL class

(Submitted to the Journal of Mathematical Behavior in March of 2021)

Abstract: In this paper, I analyze a narrative of an inquiry-based learning undergraduate Real Analysis class' discussions on the use of algebra in their first proof. The students' work proving a statement on the uniqueness of the additive inverse for every real number took an interesting turn when one student cited what she called "the Law of Cancellation" to cancel like terms on both sides of an equation; leading the students to question whether they could assume the use of cancellation, or alternatively whether their proof of the statement was justification for using cancellation in future proofs. My research goal is to share a nuanced picture of how students' spontaneous mathematical ideas can occur within and impact the course of an IBL classroom. In particular, I highlight how these conversations, and the instructor's role in these convesations, afforded opportunities for the class to engage in the creation of social and sociomathematical norms (Cobb & Yackel, 1996) that promoted more equitable (Gutiérrez, 2009) learning experiences.

Keywords: *inquiry-based learning, social and sociomathematical norms, equity, axioms, proof*

5.1 Introduction and Motivation

Real Analysis, often introduced as advanced calculus at the undergraduate level, is a milestone course for mathematics students. Over 78% of mathematics major programs require a course in real analysis (Blair et al., 2018), and most graduate programs require students to demonstrate competency with the subject. This class, along with abstract algebra, often serves as the foundation for students developing their ability to rigorously read, interpret, and participate in advanced mathematics (Tall, 1991) and advancing mathematical thinking (Rasmussen et al., 2005). In this paper, I examine an undergraduate, inquiry-based learning (IBL) introductory Real Analysis class. I focus on the first three days of class, in which students proved that the additive inverse of a real number is unique. The narrative begins when a student cited something she called "the Law of Cancellation" as justification for cancelling like terms on both sides of an equation in the proof. This unexpected episode led to exchanges that afforded opportunities for students to take part in the inception and development of social and sociomathematical norms within their classroom, and shed light on interesting mathematical distinctions that the students made. My research goal is to share a nuanced picture of how students' spontaneous mathematical contributions, such as the Law of Cancellation, can occur within and impact the course of an IBL classroom. In presenting this case, I address the following research question:

1) *How did one student's use of the Law of Cancellation afford opportunities for the students and instructor to engage in the development of social and sociomathematical norms (Cobb & Yackel, 1996) that promoted more equitable (Gutiérrez, 2009) experiences for students?*

By answering this research question, I provide insights about ways in which specific mathematical conversations can be leveraged for establishing and negotiating norms in an IBL classroom at the beginning of the term.

5.2 Literature Review

5.2.1 Inquiry-Based Learning

The instructor observed for the study described in this paper utilized an *inquiry-based learning* (IBL) teaching style in her classroom. A single encompassing definition of IBL is elusive due to its long history in mathematics as a pedagogical style[3], and varied use of the term "inquiry" when describing classroom styles. Broadly speaking, an IBL classroom encompass four "pillars" (see Figure 2) that prescribe expectations for instructor and student involvement both mathematically and socially. The language of the top row of pillars concerning student needs were established in Laursen et al. (2014), the third pillar on inquiring into student thinking was developed in Rasmussen and Kwon (2007), the fourth pillar on equity was established in Laursen and Rasmussen (2019), and the graphic organization in Figure 5-1 is shown as it was presented in the observed instructor's syllabus[4]. Together, these four pillars form the basis of a

3 The presentation focus of this classroom is reminiscent of the Moore Method, however the professor observed in this study explicitly chose to not affiliate her classroom with his work due to Moore's history of racial intolerance and competitive teaching style (see Haberler et al., 2018).
4 The instructor credits Dr. Nina White with the graphic.

broader framework called *inquiry-based mathematics education* (IBME), as developed and articulated in Larsen and Rasmussen (2019). The pillars have also been adopted by the IBL community at large (for example they can be found on the *Academy of Inquiry Based Leaning*'s website homepage[5]), and thus I use them in my own characterization of IBL. I offer a description of the observed instructor's specific classroom structure and participation expectations in the Methods, Section 5.4.1.

	Mathematical Space	Social Space
Students Need	Deep engagement in rich mathematics	Opportunities to collaborate
Instructors Must	Inquire into student thinking	Foster an equitable environment

Figure 5-1: The Four Pillars of Inquiry-Based Mathematics Education

Most papers written on IBL courses are focused on the practitioner viewpoint (e.g., Capaldi, 2015; Kinsey & Moore, 2015; von Renesse & Ecke, 2015), with a few studies that report on student experiences (e.g., Hassi & Laursen, 2015), teacher experiences (e.g., Hayward et al., 2016; Mesa et al., 2020), and learning outcomes (e.g., Laursen et al., 2014, 2016). Thus, for the remainder of this section, I widen my review to studies of IBME classrooms. While these classrooms may not have the exact IBL setup as the classroom studied in this paper, the studies are relevant to my work because they highlight important aspects of the student experience in classrooms that prioritize inquiry.

5 www. inquirybasedlearning.org

A wide variety of mathematics topics have been studied through the implementation of *inquiry-oriented instruction* (IOI) materials (Kuster et al., 2018). The use of IOI materials in classrooms arose from a group of undergraduate mathematics researchers' and educators' work in creating inquiry-oriented instructional materials based in Realistic Mathematics Education (Gravemeijer, 1999). Notable examples include IOI materials developed for differential equations (Rasmussen et al., 2006), linear algebra (Wawro et al., 2012), abstract algebra (Larsen, 2013), and most recently real analysis (Strand, 2016). These studies demonstrate ways in which mathematics education researchers have sought to incorporate active, inquiry-oriented instruction into undergraduate mathematics classrooms. These papers have tended to focus on examining the content side of IBME classrooms, understandably because the goal of IOI has been to create inquiry-oriented curriculum. In this study, I hope to share other important aspects IBME classrooms, in particular the types of interactions and dynamics involved in student proof presentations to the class.

I highlight two studies that have considered the relationship between the social and mathematical experiences of students, through identifying norms and practices in IBME classrooms. Dawkins (2009) addressed sociomathematical norms in a non-traditional (i.e., not lecture-based, with IBME elements) advanced calculus classroom. He found three clusters of sociomathematical norms around valuing visualization, mathematical communication, and developing mathematics (i.e., creating definitions). In his conclusion he writes,

> The establishment of these three clusters of non-traditional sociomathematical norms transformed the students' classroom experience from one of acquisition of externally imposed mathematical abstractions into one of construction, participation, and advanced meaning making. The students learned about and participated in many cognitive and social activities that characterize the greater mathematical community (p. 181-182).

Similarly, Fukawa-Connelley (2012) categorized several social and sociomathematical norms from an Abstract Algebra class around student participation in class proof presentations. He categorized norms around presenter responsibilities (explaining and defending your work, responding to questions), audience responsibilities (reading carefully, convincing yourself, asking questions), and norms such as only using peer-validated knowledge, and working with others (p. 413). While defining established norms and practices for the class I observed is outside the scope of this paper, which focuses on the first three days of class, I hope to add to the literature by providing further insights into what the creation and negotiation of norms and practices in an IBME class might look like.

Finally, I consider the current relationship in the literature between IBME classrooms and equity. Several studies have looked at IBME classrooms with an equity lens, and there is even an existing theoretical framework (Tang et al., 2017) that identifies several ways in which themes of inquiry learning align with equity as defined in Gutiérrez (2009). While some studies have shown inquiry-based learning to have promising outcomes for women and minority students (e.g., Laursen et al., 2014), this does not imply that IBME classrooms automatically create equitable environments for students. In Johnson et al. (2020), the authors provided contradictory findings that their IOI abstract algebra courses benefited men but not women. Other studies have also shown that inequities can still occur on a day to day basis in IBME classes and that we need more awareness around the actions that students must take during class to fully participate (e.g., Brown, 2018; Stone-Johnstone et al., 2019). In this paper, I hope to further our understanding of the relationship between equity and IBME by providing a window into the dynamics of an IBL classroom, highlighting how it could promote both equitable and inequitable outcomes.

5.2.2 Axioms, Proof, and Advanced Calculus

Because the students' negotiation of mathematical ideas is central to the narrative of this paper, I now briefly survey literature on the relevant mathematics. First, axioms are statements that are taken as true without proof and, as such, they are used as the foundational building blocks of mathematical systems. Introducing students to axiomatic thinking is a nontrivial endeavor and works such as Dawkins (2014b, 2018) provide evidence of the explicit attention needed from the instructor and difficulties of the student to "develop conscious meta-mathematical understandings of formal mathematics games like systematizing" (Dawkins, 2014b, p. 22). In this paper, I focus on students' reasoning about and proof of the uniqueness of the additive inverse for every real number, using the field axioms for justification, and in particular their discussion over the act of cancelling like terms on both sides of an equation. There are studies that have considered various aspects of post-secondary work in elementary algebra, such as Wasserman's (2014) study on teachers solving one-step equations with the field axioms, and Cook's (2014) study on students' reinvention of the zero-divisor and unit for a ring. However, to my knowledge, there are no studies that examine algebra as the basis from which to begin an advanced calculus course or student difficulties in proving basic algebraic properties through justifications with the real number field axioms.

There are several examples in the research literature of how students, mathematics majors included, struggle with proof (e.g., Harel & Sowder, 2007; Stylianides et al., 2017). Due to the proof-based nature of the curriculum, many studies have looked at proof in the context of real analysis classrooms. In Alcock and Weber (2005), the authors focused on the ways in which students validate real analysis proofs. Through student interviews that involved reading an illogical analysis proof, they found that "failure to consider the warrants used in a proof will not only cause students to be unable to validate proofs reliably but... can also prevent them from gaining conviction and understanding from proofs presented in their classrooms" (p. 133). In Lew et al. (2016), the authors explored the relationship between the key points of a professor's lecture, in which he gave several proofs, and what students recalled afterwards, especially after re-watching certain video clips. Their results showed a large disparity between what the professor expected students to take away from the lecture, and what students tend to write/remember. To my knowledge there are no studies whose data collection consisted of observing students present and discuss proofs with their peers at the advanced calculus level. I hope to offer new insights by observing this novel class environment.

There have also been studies more broadly on student and instructor experiences in advanced calculus classrooms. Weber (2008) reports on the affective experiences of a single real analysis student, using weekly interviews that asked them to describe their class experiences and complete mathematical tasks while explaining their reasoning aloud. Dawkins (2014a) considered how the act of mathematical defining in an inquiry-oriented real analysis classroom contributed to students' acculturation into advanced mathematical practices. Dawkins et al. (2019) detailed one Real Analysis IBL instructor's goals for student development and provided supporting data from instructor and student interviews. Several studies have looked at various aspects of teaching in IBME advanced calculus classrooms (e.g., Mullen, 2012; Reinholz, 2020; Roh & Lee, 2017; C. A. Shannon, 2016; K. Shannon, 2018; Zazkis et al., 2016). Finally, the ULTRA (Upgrading Learning for Teachers in Real Analysis) project has created a curriculum of modules that each build up from a situation in teaching secondary mathematics, to secondary mathematics, to real analysis, and then back down the ladder again, leaving students with an understanding of how real analysis makes sense of underlying mathematical issues that pre-service/in-service teachers might face in the classroom (see Fukawa-Connelly et al., 2020).

5.3 Theoretical Perspectives

5.3.1 Classroom Norms and Practices

In my analysis and interpretation of the classroom data I collected, I used the interpretive framework for relating the social and psychological perspectives of classroom activity developed by Cobb and Yackel (1996), which provided language by which to articulate classroom phenomena I observed. This framework put forth three constructs (classroom social norms, sociomathematical norms, and classroom mathematical practices) – see Table 5-1, designed to characterize the social analogues of individual, psychological aspects of students' mathematical beliefs and activity. Using a social constructivist perspective (Cobb et al., 1992), I assume that all norms and practices were co-created by students and the teacher. Furthermore, I see the relationship between the social and psychological perspectives as reflexive – neither one "existed" first or is more important, although I focus on the social classroom level of interactions in this paper. Next, I offer a description of each of the social categories, along with the psychological counterparts to further clarify each social category.

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Table 5-1: Cobb and Yackel's interpretive framework (1996, p. 177)

Classroom social norms are characterized by "regularities in communal or collective classroom activity and are considered to be jointly established by the teacher and students as members of the classroom community" (Cobb & Yackel, 1996, p. 178). These norms are not discipline specific – for example, a classroom social norm could be that students raise their hand before answering a question, which is applicable to many subjects and classrooms outside of mathematics. Furthermore, this social norm could imply that students' raise their hands to offer correct answers that are in line with the teacher's procedures or explanations of content.

From the individual perspective, a student might have the *belief about their role* that in class they "follow instructions and to solve problems in the way the instructor and/or textbook demonstrate" (Yackel & Rasmussen, 2003, p. 318). These personal beliefs may conflict with an inquiry-based teacher's expectations to have a classroom where they do not demonstrate problems or verify for students, creating a need for students to shift their individual beliefs to be successful in this new classroom environment (Yackel & Rasmussen, 2003). Thus, the interpretive framework makes sense of the connection between students' personal beliefs about their role in a mathematics classroom and the way they participate socially in the classroom.

Sociomathematical norms are norms specific to the discipline of mathematics; they have to do with shared expectations among a community, particularly related to doing mathematics. These norms can be thought of as delineating what counts as a different, sophisticated, efficient, or acceptable mathematical explanation (Cobb and Yackel, 1996, p. 179). An example of a sociomathematical norm is a class agreeing that "Visualization is an acceptable and helpful tool for sense-making, defining, and proof production" (Dawkins 2009, p. 162). The fact that visualization and graphical representations of proofs are not always accepted or emphasized in mathematics (Eisenberg & Dreyfus, 1991) shows how sociomathematical norms are specific to each classroom; in this case it took explicit work on the part of the instructor to guide and promote students' visual thinking as an acceptable form of mathematical explanation. I note that at the beginning of a class term, the term *expectations* may be used instead of sociomathematical norms (Dawkins, 2009), as the teacher sets the expectation for certain mathematical activity to occur before it has been taken up and normalized by the students. In addition, Cobb and Yackel pinpoint sociomathematical norms as a means by which teachers can develop intellectual autonomy for their students, by supporting a community that can validate mathematical work and spread the responsibility for judging mathematical solutions among each other.

I now provide two examples to relate *sociomathematical norms* to the individual perspective of *mathematical beliefs and values*. In Hassi and Laursen (2015), they considered "the opportunity for empowering changes in students' perceptions and activities, as they study mathematics and solve problems in classes that use inquiry, peer interaction, and communication" (p. 317). Through semi-structured individual interviews, they found aspects of self, cognitive, and social empowerment for students in an IBL class that promoted transformative learning experiences. In another study, Dawkins (2014a) interviewed students over the course of a term and looked at their individual adherence to a classroom sociomathematical norm on defining by classifying for each student whether the norm was taken-as-expected, taken-as-beneficial, or taken-as-meaningful. Thus, interview data can be used to both explicate student beliefs at the individual level and to verify developing sociomathematical norms at the classroom level through changes in student beliefs.

Classroom mathematical practices are defined as "the mathematical practices established by the classroom community [that] can be seen to constitute the immediate, local situations of the students' development" (Cobb & Yackel, 1996, p. 180). These practices are one way to think describe how a classroom community decides what sorts of mathematics need explanation and what mathematics is implicitly known by the community. Due to their local nature, classroom mathematical practices are constantly shifting and evolving over the course of a term. For example, Rasmussen et al. (2015) provided data of students working in class to prove that for a set of n vectors in R^m, if $m<n$, the set of vectors will be linearly dependent. One student used

geometrical reasoning to describe how such a set of vectors would always allow you to "get back home". However, the connection between a linearly dependent set of vectors and being able to the origin was itself a topic of debate for the students earlier in the term. This is an example of how a topic of debate, linearly dependent vectors have a nontrivial linear combination that equal the zero vector, can become a local classroom mathematical practice over time that is then used to assist in a new claim, in this case about linear independence and dimension.

An individual student contribution such as the student describing how to "get back home" with a set of linearly dependent vectors is an example of a student participating in *mathematical activity* within their classroom. In addition, the student had their own mathematical conceptions of what linear dependence and linear independence meant based on the reasoning they described to their classmates. Thus, using the individual perspective, one can ascertain a student's experience, participation in, and understanding of norms and practices at the classroom level.

5.3.2 Equity in the Classroom

I view equity through the perspective of Gutiérrez (2009), in which she characterizes equity as working towards the re-distribution of power in the classroom (and ultimately one's every day life and the global society). She defines equity as being framed as a dominant axis (*access* and *achievement*), which focuses on helping students succeed in the currently inequitable system (i.e., "playing the game"), and a critical axis (*identity* and *power*), which focuses on helping students "change the game". *Access* refers to the resources that students have to participate in and learn mathematics. *Acheievement* then, is the outcome affected by students' access, and is measured in results such as participation, grades, and college/career choices. *Identity* refers to attending to the history of marginalization and discrimination in mathematics, helping students find ways to "balance between opportunities to reflect on oneself and others as part of the mathematics learning experience" (p. 5), along with more global acknowledgements of the relations between their personal identity and the field of mathematics. *Power* is the outcome of this focus on identity, where students become critical and capable of making social transformation through mathematics. Tang et al. (2017) proposed a theoretical framework that delineates several connections between IBME classrooms and Gutiérrez's definition of equity. While I do not make use of this exact framework in my study, I highlight their idea that the critical axis can benefit from an IBME class where "students can be involved in decision-making on acceptance or rejection of mathematical knowledge presented during class" (p. 59), thus positioning students as authority figures with power in the classroom. I particularly draw on this language to talk about ways in which the instructor distributed mathematical authority to students in the class.

5.4 Methods

5.4.1 Data Site and Data Collection

Since the term *inquiry-based learning* covers a wide variety of classroom styles, in this section I give context for the classroom I observed. First, the data was collected at a small liberal arts university in a class of 19 upper-division mathematics major/minor students (pseudonyms are used throughout the data excerpts). The title of the course was Real Analysis, and according to the class syllabus, the purpose of the class was to "prov[e] all of those theorems you accepted as true back in calculus." The class was structured so that most of the class time was spent with students at the board giving presentations of proofs from a packet that the instructor gave them, along with questions and corrections from their peers. This packet was a printed and stapled set of mathematical content for the class that was given to each student on the first day (see

Appendix C for a full set of the packet materials). Time out of class was spent working alone or with peers on these packet problems.

The packet began with the definition of the real numbers, the field axioms, a definition of subtraction and division, and a list of theorems to prove in numerical order (the packet included additional axioms and definitions that are beyond the scope of the data presented in this paper). Because the theorems were listed in order, the expectation was that the first proof could only be completed using the definitions and axioms given, the second proof could use the definitions, axioms, and first theorem, and so on. The standing homework assignment for students was to "try something" by working through the packet and claiming problems on an Excel spreadsheet when they had a proof to share with the class. Students were expected to not use any resources other than the definitions and axioms in the packet, the instructor's office hours, and each other. In addition to the mathematical content addressed in class, the instructor included several humanities-based assignments and small class activities that focused on addressing students' personal axioms, beliefs, and their identity as mathematicians.

A typical class day started with an open call for questions or discussion on previous work, and then the instructor would pick a student to present the next proof at the board, alphabetically from the Excel spreadsheet. After a student presented, they would answer questions and potentially engage with peers in a collaborative revision process, sometimes trading off who went to the board to explain new ideas. The instructor continually directed students to lead their own discussions, and she carefully chose when to engage in explanation or mediation of a debate. Final grades were based on class participation, weekly homework assignments of writing up proofs that were presented in class, two exams, and a portfolio that combined commentary on proofs done throughout the term with a final reflective essay.

The data for this paper comes from a larger data set of classroom observations across an entire semester term. Every class day was filmed with a single camera at the back of the room, focused on the front whiteboard where students would present proofs. I sat at the back of the classroom with the camera and did not interact with the students or instructor during class time. The rest of the classroom was set up into small table groups of four to five students and the instructor sat at the back of the classroom. Occasionally the camera would zoom out to capture whole class discussions easier, but any small group discussions or side conversations between a student and the instructor were not captured. I took written field notes to keep track of proofs as they were presented and any inferences I had about what was happening socially in the classroom, and starred any episodes that I found especially interesting for data analysis. Due to the large amount of classroom observation data that I collected, I primarily returned to these starred episodes in the first stage of my inductive thematic analysis and began by questioning whether and how they addressed my overarching research questions on students' social and mathematical experiences in the classroom. In this paper, I report on the first three days of proof presentations, which were the second, third, and fourth days of class (the first day of class did not have any presentations) that I had starred as extremely interesting because they represented an unexpected shift in classroom focus for the students and instructor over the course of the three days on this Law of Cancellation and proving Statement 1.

5.4.2 Data Analysis

The strong story-arc of the data explored in this paper led me to analyze and frame my results as a narrative analysis. In Creswell and Poth's (2018) book on qualitative inquiry and research design, they write "Czarniawska (2004) defines narrative analysis as a specific type of qualitative design in which 'narrative is understood as a spoken or written text giving an account of an event/action or series of events/actions, chronologically connected' (p. 17)" (p. 67). Once I had identified that there was something to explore about the Law of cancellation and what had occurred in these first several class periods, my analysis began by creating enhanced transcripts of the first three class sessions with images of board work. I chose these three class sessions because I knew that the discission about the Law of Cancellation was restricted to these sessions. From the enhanced transcripts, I highlighted every conversation that involved the Law of Cancellation or proving Statement 1, from its first use in class to the final resolution when the class considered a proof of Statement 1 complete. I made note of these, and then I began to articulate a chronological narrative of what transpired related to the Law of Cancellation across these three days. Creswell and Poth (2018) write,

> One aspect of the chronology is that the stories have a beginning, a middle, and an end. Similar to the basic elements found in good novels, these aspects involve a predicament, conflict or struggle; a protagonist, or main character; and a sequence with implied causality (i.e., a plot) during which the predicament is solved in some fashion (Carter, 1993) (p. 71-72).

The act of writing a chronology of the first three days of proof presentation focused solely on the Law of Cancellation served as a main feature of my data analysis. Specifically, I determined the beginning and the end of the narrative based on when the Law of Cancellation was first brought up, addressed, and then ultimately reconciled at the end of the third day. However, the middle of the narrative was much more difficult to discern. There were a number of contributing factors to the rising action: students were developing their social skills of interacting in class with each other, the instructor was establishing her primary role as a facilitator and not a content expert, and the students were developing their understanding of the field axioms and how to communicate their mathematics through proofs. For this reason, I consider my research to be inductive as I made multiple passes through the data to detail this middle part of the narrative and make sense of and articulate the perspectives of the students and instructor through their class conversations. I worked through several versions of proofs of Statement 1 and wrote versions of the narrative, while actively working to balance these developing narratives against the broader dataset and my interview knowledge of students to make educated inferences as to the nature of the classroom activity. This process yielded an initial complete narrative, which was written in a way to emphasize the goals of my research questions regarding the IBL structure and the instructor's facilitation of the class, while also recognizing the subtle mathematical implications of the Law of Cancellation and how this episode fits in to a larger picture of working with the real number system.

I then analyzed the narrative using my theoretical framework of norms and practices (Cobb & Yackel, 1996). I systematically went through the transcript pieces of the narrative and coded for whether the participants were positioning themselves as individuals or members of a classroom through their pronoun choice. My main evidence of sociomathematical norm development came from students' shifting their pronoun from "I", to "We", noting that the instructor always referred to the class as "We" or "Us" to emphasize the collaborative nature of an IBME classroom. Thus, I interpreted students' use of the pronoun "We" during class discussions as attempts to engage their classmates in forming classroom norms and practices. I note that I took care to be aware of not just labeling pronouns as they appeared, but taking a critical lens to each episode and asking how the speaker used pronouns to position themselves in specific ways. For example, the pronoun "we" used in general class discussion often indicated trying to make classroom level decisions that developed sociomathematical norms, whereas "we" used when describing actions of doing mathematics within a proof was more indicative of the "royal we" presenter mode often used in mathematics by lecturers. Coding pronouns helped me further define and describe the narrative's description of the sociomathematical norm development amongst the students as to how mathematical "truth" was determined in their classroom.

Table 5-2 is an example of the type of coding I did to highlight pronoun usage and subsequently sociomathematical norm development in the classroom. This excerpt comes from the third class day, when a student named Jordan gave a voluntary idea of how to prove Statement 1 in a way that does not involved the Law of Cancellation, and her classmates subsequently pointed out that her proof implicitly used the same mechanisms of the Law of Cancellation (i.e., subtracting from both sides of the equation).

Student	**Transcript**	**Coding for Pronouns**	**Narrative Description of Sociomathematical Norm Development**
Jordan	*What I think you could do, to uh escape the add, is if we just added zero to one side. Because technically adding 0 it's just ($a + (-a)$), and if you add that can't you still move one of the a's to the other side with the additive inverse?*	-"I" refers to Jordan's self -"You" refers to the general class	This was a voluntary idea brought up by an individual student of how to work around using the Law of Cancellation in proof of Statement 1.
Emory	*But if you can do that, couldn't you just do that on the first line? Say $a + b = a + c$, then next line, $b = a – a + c$.*	-"You" refers to Jordan	Emory extends Jordan's individual idea to show that her logic is the same as a simpler version of the proof, which was already rejected.
Sloan	*And then by associativity you'd get to $b = c$ more quickly. But can we subtract?*	-"You" refers to Jordan -"We" refers to the class	Sloan questions whether the class as a whole can use subtraction, and May agrees. There is confusion here as to what they have defined and decided upon as a class. Since the students are questioning what mathematics the accept as a class, this is a moment of where socio-mathematical norm development occurs.
May	*Are we allowed to do that? Is that defined?*	-"We" refers to the class	
Sloan	*I feel like we decided that we weren't allowed to do that on Wednesday.*	-"We" refers to the class	
Jordan			

	But wasn't the axiom just division is doing the inverse, or subtraction was just like an inverse?		Jordan reiterates that subtraction is defined in their packet materials as adding the inverse element. Jo agrees and restates Jordan's point.
Jo	*Yeah.*		
Jordan	*So technically we're just adding an inverse.*	- Royal "We" refers to an individual in the class doing the mathematics of the proof	
Jo	*You're just adding an inverse to the other side.*	-"You're" refers to an individual (in particular, Sloan) doing the mathematics of the proof	
Sloan	*But you're taking it from the left side…*	-"You're" refers to an individual (in particular, Jo) doing the mathematics of the proof, similar to royal "we"	Sloan emphasizes that the the inverse is coming from (the other side of the equation) which aligns with the class' concern about the Law of Cancellation operating on both sides of the equation.
Jo	*Yeah you're right…*	-"You're" refers to Sloan	Jo agrees with Sloan.

Table 5-2: Example of Coding and Analysis of Narrative

Next, I read through the narrative, specifically flagging for instances where the instructor positioned the students as sources of mathematical authority, and whether and how the students began to position each other in these ways as well. This helped me to develop a more descriptive picture of the class' experience, by drawing attention to moments that afforded equitable (Gutiérrez, 2009), and at times potentially inequitable, experiences for students.

Finally, I enlisted an mathematics education researcher and a mathematician to read through and comment on the narrative, in order to help refine and make sense of what I had written. Through continued discussions with these researchers, I gained insights into both the mathematics behind the Law of Cancellation and the sociomathematical norm development of students that helped me write a clearer narrative of what had occurred in the classroom. These readers were especially helpful in coming up with possible counter-narratives for me to explore in the data, such as whether the instructor herself realized that the Law of Cancellation was an appropriate fact to use given the packet materials or if she was trying to guide students towards a particular proof of Statement 1 that used the full strength of the hypotheses. I continued to edit the narrative until it satisfied both myself and my two readers, to a point where I felt confident that I had articulated the narrative as faithfully as possible.

5.4.3 Mathematical Discussion

In this section, I explain the mathematics that will be discussed in the Results. I start with information that students received in their packet prior to working on problems, including the field axioms for the real numbers. Then, I consider the statement that students' worked on along with explanations for proofs of the statement that will aid in the reader's understanding of the Results. I refer to this as Statement 1, since it was the first statement in the packet, and the purpose of the statement is to justify the uniqueness of the additive inverse for every real number. Finally, I explore a mathematical concept that became increasingly important to the students regarding the proof of Statement 1, which they called the "Law of Cancellation".

5.4.3.1 Introductory Material

First, I describe the introductory packet material that came before the statements to be proven. This material consisted of the field axioms and definitions of division and subtraction. The field axioms for the real numbers (see Figure 5-2) are familiar properties of numbers which students gain exposure to as early as elementary school. These axioms are reinforced through several years of basic mathematics, algebra, calculus, and advanced coursework for undergraduate mathematics majors. One reason why these axioms become so innate in students is that the K-12 mathematics education system primarily focuses on the real numbers (or subsets of the real numbers), so that students become familiar with these axioms and take them, and their consequences, as givens. Thus, it can be a surprise to students to be required to justify properties of the real numbers that have been treated as givens for much of their mathematical careers.

Field Axioms: There exists a nonempty set R (which we call the real numbers) with two binary operations on R called addition (+) and multiplication (×) that have the following properties.

(i) Commutative Property: + and × are commutative operations, i.e., if a and b are any two real numbers then $a + b = b + a$ and $a \times b = b \times a$.
(ii) Associative Property: + and × are associative operations, i.e., if a and b are any two real numbers then $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.
(iii) Additive and Multiplicative Identities" There exists a real number, which we call 0 (zero) such that if x is any real number then $x + 0 = x$. There exists a real number distinct from zero, which we call 1 (one), such that if x is any real number then $x \times 1 = x$.
(iv) Additive and Multiplicative Inverses: For each real number x, there is a real number which we call $-x$ (minus x) such that $x + (-x) = 0$. For each non-zero real number x there exists a real number which we call $\frac{1}{x}$ such that $x \times \frac{1}{x} = 1$. We also denote the multiplicative inverse by x^{-1}.
(v) Distributive Property: Multiplication distributes over addition, i.e. if a, b, and c are real numbers then $a \times (b + c) = a \times b + a \times c$. We adopt the usual order of operations rules so that the expression $a \times b + a \times c$ is equivalent to $(a \times b) + (a \times c)$.

Figure 5-2: Field Axioms in Class Packet

In addition to the field axioms, the packet also included a definition of division and subtraction which utilized the definitions of the additive and multiplicative inverses (Figure 5-3).

Division and subtraction are defined in terms of multiplicative and additive inverses, so that $\frac{x}{y}$ means x * $\frac{1}{y}$ and x – y means x + (-y). Furthermore, we adopt the usual convention of writing xy to mean x*y.

Figure 5-3: Definition of Division and Subtraction in Class Packet

5.4.3.2 Statement 1

I now discuss the statement that will be focused on in the Results section, and I provide three correct proofs of the statement. The first four statements are shown below (Figure 5-4) for context as they appeared in the packet, although this paper focuses on Statement 1.

1. Suppose a, b, and c are real numbers. Show that if $a + b = 0$ and $a + c = 0$, then $b = c$. In other words, the additive inverse of a is unique.
2. Prove that the multiplicative inverse is unique.
3. Prove that if x is a real number then $x \times 0 = 0$.
4. Prove that if x is any real number $(-1)x = -x$.

Figure 5-4: The first four packet statements

One correct proof of Statement 1 starts with the hypothesis that $a+b = a+c$, adds an additive inverse of a to both sides of the equation, employs the additive inverse axiom to write $(-a)+(a)$ as 0, and uses the additive identity axiom to write $0+b=0+c$ as $b=c$ (see Figure 5-5).

Proof of Statement 1 (a)

$$a + b = a + c \text{ , by hypothesis} \quad (1)$$
$$(-a) + a + b = (-a) + a + c \text{ , (-a) is an additive inverse element of a} \quad (2)$$
$$0 + b = 0 + c \text{ , by Axiom IV, property of additive inverse} \quad (3)$$
$$b = c \text{ , by Axiom III, property of additive identity} \quad (4)$$

Figure 5-5: Proof of Statement 1 (a)

While not explicitly stated as a field axiom, the ability to add equal quantities to both sides of an equation, which by the definition of subtraction also means the ability to subtract equal quantities from both sides of an equation, is embedded in the fact that (+) is a binary operation (which is stated at the beginning of the field axioms in Figure 5-2). In particular, if $a=b$, then $a+b=a+c$ because on each side of the equation, the binary operation + is being applied to the same element of R. Since a binary operation is a function, its value is uniquely determined on the ordered pair of elements of the set R. However, this is a subtle point about the definition of a binary operation, and students may not recognize this without explicit explanation from an instructor[6]. A student who is concerned as to whether the field axioms allow for addition or subtraction on both sides of the equation may prefer a proof that uses the same logic as the proof in Figure 5-5, but that only manipulates one side of the equation (see Figure 5-6).

6 I credit and thank Dr. David Pengelley for his insightful conversations on algebra and the field axioms that informed this section of the mathematical discussion.

Proof of Statement 1 (b)

$$
\begin{aligned}
b &= 0 + b \text{ , by Axiom III, property of additive identity} && (1)\\
&= ((-a) + a) + b \text{ , (-a) is an additive inverse element of a} && (2)\\
&= (-a) + (a + b) \text{ , by Axiom II, property of associativity} && (3)\\
&= (-a) + (a + c) \text{ , by hypothesis} && (4)\\
&= (-a + a) + c \text{ , by Axiom II, property of associativity} && (5)\\
&= 0 + c \text{ , by Axiom IV, property of additive inverse} && (6)\\
&= c \text{ , by Axiom III, property of additive identity} && (7)
\end{aligned}
$$

Figure 5-6: Proof of Statement 1 (b)

Finally, I provide a third proof of Statement 1 that uses one side of the equation and no inverse element of *a* (see Figure 5-7). This proof begins with rewriting *b* as *0+b* by the additive identity axiom, substituting *0* with the hypothesis that *0=a+c*, uses the associative and commutative axioms to rearrange elements, uses the hypothesis that *a+b=0*, and finishes by using the additive identity axiom to write *c+0* as *c*.

Proof of Statement 1 (c)

$$
\begin{aligned}
b &= 0 + b \text{ , by Axiom III, property of additive identity} && (1)\\
&= (a + c) + b \text{ , by hypothesis} && (2)\\
&= a + (c + b) \text{ , by Axiom II, property of associativity} && (3)\\
&= a + (b + c) \text{ , by Axiom V, property of commutativity} && (4)\\
&= (a + b) + c \text{ , by Axiom II, property of associativity} && (5)\\
&= 0 + c \text{ , by hypothesis} && (6)\\
&= c \text{ , by Axiom III, property of additive identity} && (7)
\end{aligned}
$$

Figure 5-7: Proof of Statement 1 (c)

5.4.3.3 The Law of Cancellation

In the Results section, we will see students justify a proof that uses subtraction on both sides of the equation with a rule that one student introduces as the “Law of Cancellation”. I describe the rule briefly here and how it relates to Statement 1 to provide context for the reader. My understanding and description of this rule comes from classroom data that I will elaborate in Section 5.2.2.

The Law of Cancellation says the following for any real numbers *a*, *b*, and *c*: "If $a+b=a+c$, then $b=c$". Notably, the proofs of Statement 1 in Figures 5-5 and 5-6 only use the hypothesis that $a+b=a+c$, without referencing that they both equal 0, and thus they are proofs of the Law of Cancellation. In fact, one could prove the Law of Cancellation with either of the proofs in Figures 5-5 or 5-6, and then claim Statement 1 as a corollary or result of the proof by adding the additional hypothesis that $a+b=a+c=0$. The proof in Figure 5-7 proves the narrower result of Statement 1 only, by making use of the additional hypothesis. While the name "Law of Cancellation" may not be listed in the field axioms, it is a direct result of the binary operation (+) which guarantees that one can add, and thus subtract, equal quantities from both sides of an equation, and it can be justified using the more general versions of the proof of Statement 1 (Figures 5-5 and 5-6). However, the IBL packet did not explicitly state the definition of a binary operator or this quality of the real numbers. Thus, in the Results we will see how one student's specific naming of the concept as the "Law of Cancellation", which was not a phrase from the packet, created concern among the class that adding or subtracting equal quantities from both sides of an equation was prior knowledge that was not allowed in their class and that only a proof that operates on one side of the equation, such as in Figures 5-6 and 5-7, was acceptable.

5.5 Results

The results for this paper focus on student work during the first three days of proof presentations (the first day of class did not include any presentations), which I call Day One, Day Two, and Day Three. I have split the results sections by class day to help the reader visualize the chronological progression of the students' and instructor's reasoning. I abbreviate the Law of Cancellation as LoC.

In Section 5.5.1, I set the stage for the narrative by sharing how the LoC arose during the class' feedback after a student presented their proof of Statement 1. In Section 5.2, we see how the LoC was retracted by the same student that proposed it, and how the instructor responded by turning the mistake into an important learning moment for the class. I also share two key moments of students recognizing the LoC in subsequent proof attempts, thus highlighting the level of caution the students had developed and their growing ability to debate mathematical truth with each other. In Section 5.3, I provide comments from two students that tried to convince the class of the LoC. Finally, I show how the class accepted a proof of Statement 1 that did not use the LoC, and I end with the instructor's summary of what the importance of the whole experience was for them as a class.

5.5.1 Day One

5.5.1.1 The Law of Cancellation arises

The first day of proof presentations began with the instructor asking a student, Rose, to present her proof of Statement 1 on the board (see Figure 5-8). Note that the proof does not complete all steps equally on the left and right sides of the equations, but it is similar in form to the one given in Section 4.3.2, Figure 5-5.

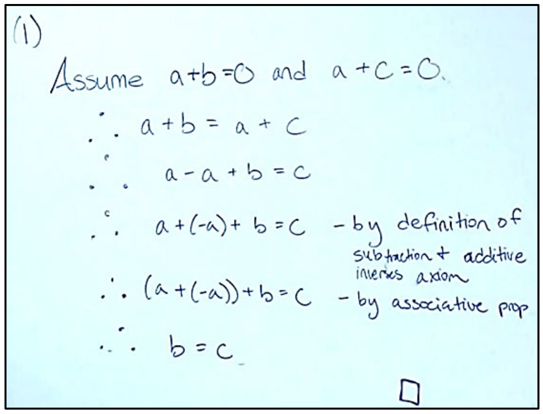


Figure 5-8: Rose's First Proof of Statement 1

After Rose finished her presentation, the instructor prompted the class for questions and comments on the proof. In Excerpt 1 below, a student named Josh asked whether subtraction was allowed in their proofs. The class decided that while subtraction was allowed via the packet definition; they could instead cite what one student, Sloan, called the "Law of Cancellation".

Transcript Excerpt 1

1	*Josh:*	*Um just like open question to the class. I was under the impression that we had two binary operations on the real numbers, that were addition and multiplication. And so that we weren't allowed to use subtraction or division?*
2	*Sloan:*	*You can use the Law of Cancellation. You have a on both sides, you could just cancel them.*
3	*Rose:*	*So like add it here? (points to third line)*
4	*Sloan:*	*Yeah*
5	*Inst.:*	*Josh, could you say what you said again?*
6	*Josh:*	*Yeah I said that we have the two binary operations on the set of real numbers that were uh addition and multiplication as defined on the front page. Uh and so that, those would be like the only operations we could do with real numbers.*
7	*Taylor:*	*The definition for division and subtraction comes before the first problem, so wouldn't you be able to use that one as well?*
8	*Josh:*	*That was my question.*
9	*Taylor:*	*I would assume so.*
10	*Rose:*	*That's why I put it here. (points to fourth line)*
11	*Taylor:*	*I would assume that's also, I think the Law of Cancellation makes the proof shorter but I think that's good by itself as well.*
12	*Easton:*	*Also um, existence of (-a) is stated in the additive inverse theorem*
13	*Inst.:*	*Say that one more time?*
14	*Easton:*	*It states um the axiom of additive inverses says that for all (a) there exists a (-a) and a+(-a) = 0.*

Josh made two bids in lines (1) and (6) for people to join him in assessing the validity of the proof through whether subtraction was allowed, using the pronoun “we”. Josh’s question was answered by Taylor in line (7), he showed that using subtraction was allowed since the definition of subtraction came before Statement 1. However, his suggestion was put aside when the class turned to Sloan’s suggestion in line (2) of using the LoC. In line (11), Taylor also confirmed that the LoC was a valid way to make the proof shorter. This was especially noteworthy since the class had never acknowledged anything called the Law of Cancellation beforehand, and there was no statement by this name in the packet. I interpret the students Sloan, Taylor, and Rose as individuals participating in the classroom discussion, noting their use of the pronouns “I” and “you”. I view these interactions as the students validating each other’s individual mathematical conceptions of algebra regarding cancelling on both sides of an equation, which they understood so innately that they were able to make sense of Sloan’s phrase, “Law of Cancellation”. Overall, this first excerpt shows us students’ first attempts at creating sociomathematical norms (Cobb & Yackel, 1996) as to what counted as an acceptable proof, through trying to interpret what their packet did and did not allow in terms of proof justifications.

Rose incorporated everyone’s suggestions to her proof on the board (see Figure 5-9). Notably, Rose accepted the LoC by adding it to her proof. She also changed her notation in line 3 on the left side of the proof from subtracting *a*, to adding (-*a*), which indicates that she was aware of the conversation around subtraction and additive inverses. However, I note that Rose’s proof was not logically consistent line by line. For example, she writes *+(-a)* on the right side of the equation in line 3 and it did not appear on the right until line 4. Furthermore, it is not clear what the role of the LoC was in the proof. A careful reader may be confused as to whether the LoC was used to justify the addition of (-*a*) on both sides of the equation, or whether the law allowed for "cancellation" of the term on the right side, or on both sides of the equation, and ultimately whether the law was necessary given that the students had asserted the validity of both subtraction and using an additive inverse element. This excerpt reveals an early baseline for the level of proof students expected of each other here on the first day of presentations. I argue that Rose's proof was not polished or rigorous, and yet students allowed Rose's proof to stand as complete. I hypothesize that at this point the students had not yet established social norms that would allow them to push back on corrections, or push for more detailed, accurate board work, and the establishment of such norms is outside the scope of this paper. These students had all taken Discrete Math, and many had taken Abstract Algebra, so they were aware of what a logically consistent proof should look like. However, a community that can engage in vulnerable communication (such as critiquing a community member's work) takes time to build.

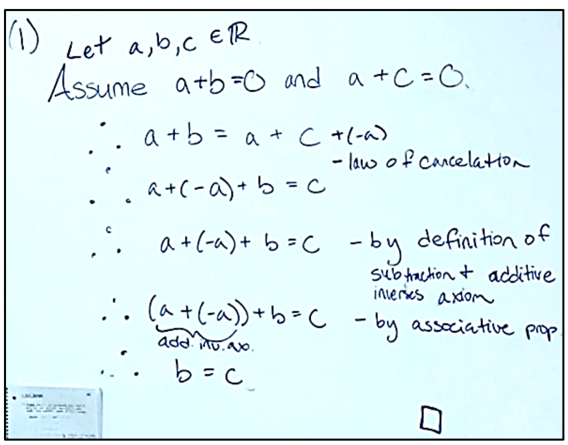


Figure 5-9: Rose's Edited Proof of Statement 1

After Rose edited her proof, the students moved on to presentations of Statements 2 and 3, and it seemed that everyone assumed the proof of Statement 1 was complete and correct. If there were alternative proofs, they were not shared with the class, and to my knowledge no one questioned the validity of the proof of Statement 1 at the time on the first day of presentations.

To summarize this episode, the LoC was proposed by Sloan and then accepted by students in the classroom, both verbally and through Rose's additions on the board. The students understood that their proof justifications should come from the packet, as evidenced by their debate around whether subtraction was allowed, but they did not question where the name "Law of Cancellation" had come from or its relation to their packet. Thus, we see that the students were starting to establish some sense of a norm about what constituted acceptable justifications for proofs in their class, but it had not reached a level of rigor that allowed students to correct each other's feedback or consider the full impact of their suggestions during class.

5.5.2 Day Two

5.5.2.1 The Law of Cancellation is retracted

At the beginning of the next class period, the instructor asked for any questions or outstanding business. As we see in the following excerpt, Sloan raised the issue to her peers of how the LoC should not have been allowed. It is noteworthy that Sloan was reflective and willing to share what she had realized, especially because it cast her previous contribution in a negative light, and that the instructor let her introduce the error on her own. Excerpt 2 below is a class conversation in which Sloan and the instructor reflect on an email exchange they had after class on Wednesday on using the LoC.

Transcript Excerpt 2

15 *Inst.:* *Um anything for the good of the order? Sloan?*

16 *Sloan:* *When I talked about the cancellation law, I don't know what I was talking about Wednesday. But y'all believed me and that was nice [students laugh]. But you can't use it!*

17 *Inst.:* *Right? Right, like we can't do that. We could, we could establish a law of cancellation. And it can be whatever we want it to be. We could make up a law and call it Sloan's Law or call it the Law of Cancellation. And then we could use it, that's legit. But we can't just write it down because the words came out of Sloan's mouth. Like "oh it's the law!" You said it with such authority that I was like "that must be true" (people laugh) Right?*

18 *Sloan:* *No one said anything against it*

19 *Inst.:* *Yeah I was out on a run yesterday and I was like "did Sloan say law of cancellation?" Then I realized, and like I wrote it in my notes "by the law of cancellation", I bet you did too!... I wrote it down and then I was thinking about it*

a day and a half later and was like, I have no idea what she was talking about. I accepted that as fact, right? Talk about accepting as true. I accepted it as true without actually looking like is that something we're accepting as true? And it turns out, when I emailed Sloan she's like "yeah we don't have that thing". So I'm like "ok then we don't have that thing".

20 *Inst:* *But how often, think about that. How often have you done that? Just like "oh somebody said that was true", "I wrote those words down". Every day! Let's not do that. Like and as I responded to Sloan in my second or third email in the back and forth, "yeah that was awesome modeling for me, just write it down oh law of cancellation yeah that seems good" and just write it law of cancellation! So let's try, let's not do that. We have exactly one axiom right now. One axiom and two definitions, yes? That's all we have to work with. That's all we can do. That's all we can assume. We will continue to add axioms, there's another one at the bottom, the mid-bottom of page 2. Ok so yeah, just this axiom. And you're right, axioms are things we do not prove.*

It is unclear from line (17) whether or not the instructor believed that the students could not add or subtract like terms from both sides of the equation (which is allowed by the binary operation (+) defined in the field axioms as explained in Section 4.2.2). Based on this exchange, and the lack of counter arguments given, I infer that the class as a whole did not recognize the use of the binary operation or believe that cancellation was allowed via their IBL packet after this conversation. This inference is further established by the instructor's emphasis that they could talk through and create a Law of Cancellation as a class, but they could not assume its truth based on Sloan's assumptions. Potentially, the instructor was focused primarily on the opportunity to reinforce the IBL packet expectations with her students, and was just as taken aback with how she herself had accepted the LoC as in line (19), that she did not question the validity of the content of the statement. I interpret her use of the conjunction "let's", as in "let us", as an invitation for the students to join her in interrogating what they took for granted and believed as true mathematically in their classroom, and more broadly in their everyday lives. This was an important moment in norm development, as the instructor was explicit in her desire in line (20) for students to consider their mathematical beliefs and assumptions, and suspend those beliefs that had not been justified using the IBL packet materials.

5.5.2.2 The Law of Cancellation is found implicitly in another proof

Following Transcript Excerpt 2, the instructor asked students to re-evaluate their proofs of Statement 1 in their small groups. After about ten minutes of group work, she asked Rose to put up her group's edited proof of Statement 1 and see what changes the class wanted to make (see Figure 5-10). Excerpt 3 below starts after Rose's presentation, when a student raised the question of whether this proof used the same logic as the LoC; suggesting that if they could not subtract equal elements from both sides of the equation, they also should not add equal elements to both sides.

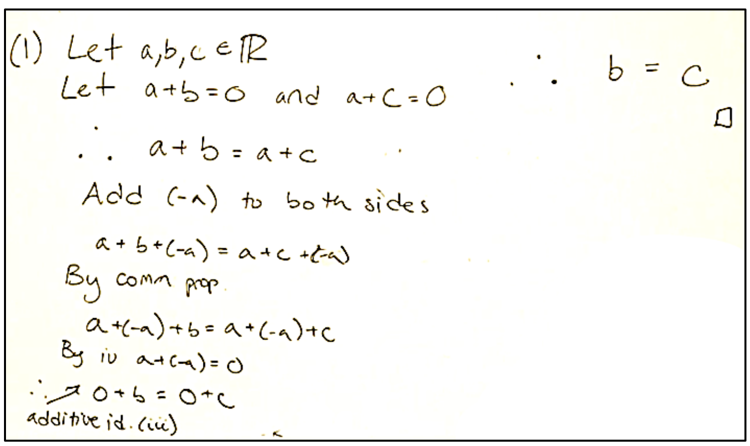


Figure 5-10: Rose's New Proof of Statement 1

Transcript Excerpt 3

20 *Connor:* *So after thinking about it, if we look at 1 in another way, a+b=a+c and then you basically use the Law of Cancellation. But if we think about the Law of Cancellation conversely, so we are cancelling an equal number of things on both sides, can we also add an equal number of things on both sides? Do we also need to prove that?*

21 *Inst.:* *Can you come to the board and write down what we would need to prove?*

22 *Connor:* *So basically, we would need to prove this line. (underlines line four of Rose's proof)*

23 *Jo:* *Like do we need to prove if we're able to do that*

24 *Sloan:* *That we can add to both sides*
25 *Connor:* *Because we said we needed to prove that we can cancel on both sides, for this problem. So do we also need to prove if we can add things on both sides?*
26 *Sloan:* *Oh no.*

In line (20), Connor questioned whether the LoC stated that one can subtract (a) from both sides of an equation or add ($-a$) to both sides of an equation, and his concern was quickly taken up as problematic by other students in class. The students did not consider the difference or relationship between these two operations (recall, subtraction was defined in their packet as adding the additive inverse). Instead, the students drew on their understandings of addition and subtraction as separate operations, as opposed to relying strictly on the packet and using the definition of subtraction given to them. Every participant in the conversation used the pronoun "we", and I interpret that this shift in language suggests that the students were beginning to internalize their contributions as occurring at the classroom level. Thus, we see the class beginning to engage in the instructor's expected sociomathematical norm by asking whether adding on both sides of the equation was something they could assume or whether it needed to be proven.

The instructor turned to Sloan to clarify what she meant by the LoC and decide whether they could add on both sides of the equation. Sloan articulated that the LoC is "If $a+b=a+c$, then $b=c$", and essentially cancelled an (a) from both sides of the equation, but that the explicit formulation of using the LoC involved adding an inverse element ($-a$) to both sides, thus addition on both sides was also not allowed. I note that the instructor did not present any sort of authority over the mathematics; instead she focused on presenting the class' information back to them and asked Sloan what the LoC said. In this way she was offering Sloan an equitable (Gutiérrez, 2009) opportunity to grow her *identity* in class through being positioned as a mathematical authority figure, which in turn gave her more *power* in the classroom. While cancellation may be a true and useful feature of algebra for students, here Sloan articulated that it was not an acceptable truth to use without proof in their classroom community, implying that cancellation did not strictly follow from their IBL packet, and her ideas were accepted by the overall class.

I pause to comment on Rose's multiple times at the board in front of her classmates in regards to potentially inequitable (Gutiérrez, 2009) student experiences. In the first section, we saw her add everyone's ideas to her proof and that the final result was less than satisfactory. While she had *access* in terms of an opportunity to present her mathematical work and get feedback, it did not necessarily lead to the outcome of *achievement* of doing a proof that rendered the problem complete. Furthermore, I do not know what that experience was like for her personally or what effects on her *identity* she might have felt when her second proof was also rejected by the class and whether it led her to feel a lessening sense of *power* in the classroom. We do have evidence of students being comfortable rejecting their own work, as in Sloan's retraction of the Law of Cancellation, but this was not necessarily the experience of every student in class. Thus, one difficulty that instructors must be aware of with an IBL class is how to promote a safe environment for students to present mathematical work in front of their peers as well as be corrected by them.

5.5.2.3 Moving elements from one side of the equation to the other

In an attempt to avoid invoking the LoC by adding to both sides, a student named Jordan proposed adding zero,

($a+(-a)$), to one side of the equation and then "moving" an a to the other side (see Figure 5-11).

$$a+b=a+c$$

$$a+(a+(-a))+b=a+c$$

$$a+(-a)+a+b=a+c$$

$$a+(-a)+b=a+(-a)+c$$

Figure 5-11: Jordan's Idea for Statement 1

Transcript Excerpt 4

28 *Jordan:* *What I think you could do, to uh escape the add, is if we just added zero to one side. Because technically adding 0 it's just (a + (-a)), and if you add that can't you still move one of the a's to the other side with the additive inverse?*
29 *Emory:* *But if you can do that, couldn't you just do that on the first line? Say a+b=a+c, then next line b=a – a +c.*
30 *Sloan:* *And then by associativity you'd get to b=c more quickly.*
31 *Sloan:* *But can we subtract?*
32 *May:* *Are we allowed to do that? Is that defined?*
33 *Sloan:* *I feel like we decided that we weren't allowed to do that on Wednesday.*
34 *Jordan:* *But wasn't the axiom just division is doing the inverse, or subtraction was just like an inverse?*
35 *Jo:* *Yeah*
36 *Jordan:* *So technically we're just adding an inverse*
37 *Jo:* *You're just adding an inverse to the other side*
38 *Sloan:* *But you're taking it from the left side…*
39 *Jo:* *Yeah you're right…*

In line (29), Emory pointed out that moving a term from one side of the equation to the other still implicitly involved subtracting equal elements from both sides, which was verified by Sloan in line (38). This conversation was the first mathematical debate among students that seemed to formulate a bridge between their individual conception of algebraic operations (involving "moving/taking" elements from one side of the equation to another, highlighted with the pronoun use "you" as in lines 28, 29, and 37-39), and the class' agreement to not allow subtraction in their proofs (with the pronoun use "we" as in lines 31-36). Although subtraction was a defined operation in the packet, the use of subtraction to move elements from one side of the equal sign implicitly used subtracting the same element from both sides of the equation. I interpret the intermixing of these two levels of thought and experience in the classroom as evidence that the students were growing both in their social abilities to debate each other productively in class and in their sociomathematical understandings of what it meant to interrogate assumptions in their proofs and justify their work using only the IBL packet materials.

After Jordan sat down, a student named Easton posed his own solution to Statement 1 that only manipulated one side of the equals sign (see Figure 5-12). In his explanation of his proof, Easton emphasized that his proof only used arithmetic on one side of the equation and no algebra[7], thus eliminating the class concerns about using the LoC.

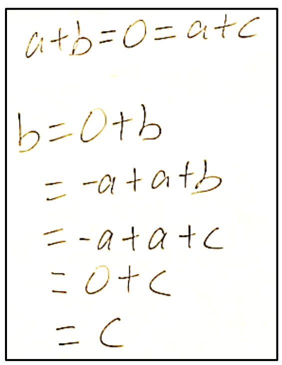


Figure 5-12: Easton's Proof of Statement 1

I end this section by providing some of the instructor's comments before and after Easton presented his proof. These excerpts are especially intriguing in that I interpret the instructor as highlighting that the proofs presented so far, included Easton's, do not make use of the hypothesis that *a+b=0* and *a+c=0*.

Transcript Excerpt 5

40 *Inst.:* *I guess I would pass back to all of you... Rarely do we have extra hypotheses. This isn't a word problem where you throw in a bunch of stuff that is irrelevant to the problem. That doesn't happen that often... This and this are different. (she points*

7 Easton's distinction here was quite notable. The debate over what counts as "algebra" goes as far back as to the Arabic origins of algebra, in which al-jabar literally meant moving a term from one side of an equation to the other.

to Statement 1 and the Law of Cancellation written on the board). Right? And I get one of the issues is that you already believe this is true. That makes it harder to prove. Every time. So. What about that first line have we not used? Right? What makes this and this different? (points again to Statement 1 and the Law of Cancellation)

(Easton presents his proof)

41 *Inst.:* *Can everyone see what he (Easton) did? So, what. What, look at the proof here, Rose's proof. You call out a+b equals 0 and a+c equals zero. What in that never gets used in the proof?*

42 *Jo:* *Oh the fact that they both equal zero?*

43 *Inst.:* *The fact that they're zero! Right? Look at the difference between Sloan's statement (Law of Cancellation) and this (Statement 1). So maybe that has to come into play. Alright we're out of time, see you next time.*

It is possible that the instructor's comments about extra hypotheses and noting "what doesn't get used" in Easton's proof were coming from an expectation that the students would create a proof more similar to Figure 8 in Section 5.4.3.2, which used the full strength of the hypotheses given in Statement 1. This excerpt provides evidence that the instructor may not have realized the connection between the binary operation (+) allowing addition on both sides of an equation, and that the issue with Sloan's statement about the Law of Cancellation was not necessarily the content of the law, but the way in which Sloan had brought it up in class. This inference is strengthened by the fact that the instructor continued to let students claim Statement 1 on the Excel sheet, meaning that she still sensed disagreement in the class, or did not personally view the proof as complete, and would allow another student to present a proof of Statement 1 for presentation points. Thus, overall we see some ways in which the spontaneous work of students in an IBL class can lead away from an instructor's mathematical expectations in favor of maintaining a student-centered classroom experience.

5.5.3 Day Three

5.5.3.1 The students try to justify cancellation

The third day of presentations started with the instructor recalling that not everyone seemed satisfied with the proof of Statement 1 on Day Two, and she asked for new students to go to the board and write proofs of Statements 1-4. The students collected presentation feedback in groups, and then discussed all four proofs as a class. The student who presented Statement 1 gave a proof very similar to Rose, using subtraction on both sides of the equation. In general, the class was satisfied with the proofs of Statements 2-4, rightfully so as the proofs were correct, and they continued to return to the proof of Statement 1, questioning whether subtraction or addition were allowed on both sides of the equation. In Excerpt 6, Josh questioned whether the class had come to any conclusions about operating on both sides of the equals sign. Jo and Emory provided counterarguments in support of operating on both sides of the equation.

Transcript Excerpt 6

44 *Josh:* *Um well we had two people write down the same question for number one, but it's more of a general question not just for number one. Uh just raising the issue of adding/ subtracting/multiplying to both sides of the equal sign. To say like, the statement is still true, like for the third one. Like is that ok, like we were talking about on Friday. Cause I think we left on Friday not sure if we were going to accept that in our groups or not.*

45 *Inst.:* *Community? Jo, you have something to add to that?*

46 *Jo:* *No, yeah on Friday we asked should we be allowed to do that? My opinion is that if we don't allow ourselves to do that we aren't going to get very far. But we can be creative about it, I don't know. I don't really have a problem with it.*

47 *Inst.:* *Ok, Emory?*

48 *Emory:* *Um after thinking about it for a long long while uh I came to the conclusion that in group theory, which is a different class, we weren't allowed to do most of these axioms but we were allowed to do left multiplication and right multiplication and etc. So in my mind, it's more fundamental than the axioms that we're assuming, just because otherwise most of the mathematics, all the fields mathematics still carries over to group theory. Um but the axioms that we accepted here, there are more of them then we're allowed to start with in group theory. So my thinking, those operations, if those don't hold true on both sides of the equals sign to start, then these axioms don't really work either. So that's my thought.*

49 *Inst.:* *And, go ahead Jo.*

50 *Jo:* *Every time I look at inverse or anything like that I make sure to do it on the same side, like on the left like you did up there, um and then like commuting them, like taking the extra step to do that. Just so that I wasn't like adding things willy nilly on like either side of each thing on the equals side. I guess, like adding that (-a) to the other side where it'd be a+c+(-a). Um, but. So I don't know, I guess I did that just to be like ok I'm not totally stretching it.*

(more conversation among students)

51 *Emory:* *... with addition and subtraction, it's like we've accepted that these operators exist. Before we came up with these axioms, the operators addition and subtraction had to exist first. Like you wouldn't be able to do any of these axioms if you didn't have addition or multiplication. And so addition means a+b=a+b, and if I just add another c to the end of both sides it has to be true, right? And so if you don't accept that one, then I feel like you can't even accept these axioms, because these axioms are about these operators. In the real numbers specifically, right? If you can't accept that like why does a+b=b+a? How do you use that step?*

Both Jo and Emory shared their individual conceptions of how necessary and fundamental the operation of cancellation is for working with equations. In line (48), Emory justified his idea by comparing the beginnings of their class to that of group theory and that if group theory allowed cancellation with less than the field axioms, then they should be able to as well. Jo pointed out in line (50) that he was careful to invoke commutativity and stay mindful of how he was adding on both sides of the equation to make sure his argument is rigorous. In line (51), Emory comments on the relationship between the operators of addition and multiplication, and the field axioms. He claimed that these operators existed before the field axioms and that addition on both sides is a necessary aspect of those operators if one even wants to talk about the field axioms. This is a similar argument as is made in Section 4.3.2, and I note that Emory was correct, although his explanation did not convince his classmates.

5.5.3.2 The students agree on a path forward

The students continued discussing the validity of cancellation for the remainder of the class time and several students concluded that they should not use cancellation in the proof of Statement 1. Excerpt 7 below summarizes a class conversation in which Taylor and Ash agreed that if they could prove Statement 1 without cancellation, then they could cite Statement 1 as justification for cancellation in future proofs. Furthermore, they believed that they could prove Statement 1 this way because they had completed other proofs already that only manipulated one side of the equation.

Transcript Excerpt 7

52	*Taylor:*	*… My sort of point was like we learn kind of, I'm just thinking back to whenever I learned what left and right like addition were on both sides, I think I learned that in algebra. It wasn't like a thing that I knew innately... And I think that my little kid self had to learn like hey this is something that isn't, like it doesn't make sense intuitively.*
53	*Ash:*	*Yeah, and I think these first four problems are what we do to then say yes we can do left and right cancellations.*
54	*Taylor:*	*Yeah*
55	*Ash:*	*It's because of how we solved them without doing that so we can prove that it actually does work.*
56	*Taylor:*	*And then I also think that there's proofs that we can do where, like where we only need to use one side of the equation sign to finish it. Like number 2 we did the proof and there's only one side of the equation. Um number 3, even though there is an equals sign, like you don't change the zero on the right-hand side. Like you used all the axioms and kept everything on one side of the equation. So I think it's definitely possible to keep proofs to trying to minimize or probably like avoid adding on both sides of the equations.*

The class continued to reiterate some of these ideas when Taylor said he had come up with a proof of Statement 1 that fit their class requirements, and was invited to the board to present the proof (see Figure 5-13). The class had a very positive response to Taylor's proof and agreed that it sufficed as a correct proof of Statement 1 because it did not use any sort of addition or subtraction on both sides of the equation. I take the class' formulation of what it meant to prove Statement 1 without cancellation, and that Statement 1 then provided the justification for cancellation in future proofs, as evidence that they had internalized the instructor's expected sociomathematical norm for proof justification and choosing to detail exactly when they could use particular features of algebra in their proofs regardless of their personal knowledge of the operations.

Notably, Taylor's proof is very similar to the one Easton had given at the end of the previous class session. I do not know how Easton felt about this experience, whether he recognized the lack of ownership that was attributed to him, and whether it caused him to feel like his voice was under-heard in the classroom. This speaks to another potential inequitable experience that can occur IBL classrooms; students may need multiple times of seeing a proof before they recognize its value, and thus it may not be the first author of a proof who receives the most credit or ownership for their original thoughts. This concern attends to the critical axis of *identity* and *power* from Gutiérrez (2009).

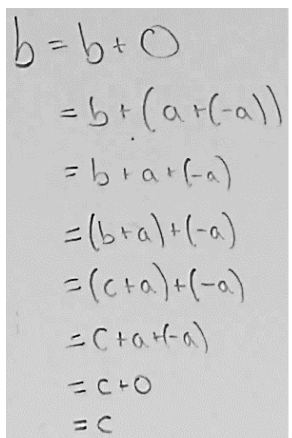


Figure 5-13: Taylor's Proof of Statement 1

5.5.3.3 The instructor summarizes the class' experiences

The instructor ended Day 3 with a summary of the work students had engaged in, and she drew attention to the discomfort that comes with rigorously investigating one's beliefs, inside and outside of mathematics.

Transcript Excerpt 8

57 *Inst.:* *Um. I feel like we've settled this but I'm not sure everyone feels settled on it. Like I feel like between these two, we're at a point where we can add something to both sides of an equation. But my guess is what's getting in the way, that the dissonance in your head, the cognitive dissonance, is that you have this model from algebra of an equals sign being a balance, when you learned algebra. And if you put something on one side, you put something on the other side to balance it out. We're not challenging that model at all. Right? We're still living in the space where that model works. But we are asking the question, why does that model work? Right? That's a much harder question. And so my guess is that those of you who said this is making me uncomfortable, that this feels so detailed, um that the discomfort comes from having believed these "facts" for a very long time. Without proof. Right? My guess is that many of you have taken all of these as axiom. And that's hard, to say maybe I don't need to believe that without proof.*

I interpret this as the instructor summarizing why Statement 1 had been difficult to prove, namely because it went against students' prior experiences and expectations of mathematics. The students were not suddenly believing that algebra was untrue or did not apply in their work, but rather they were asserting that their classroom needed to decide at every moment what they wanted to believe as true without proof, and what they needed to prove. This excerpt goes to show how intent the instructor was on leading a student-centered class and taking their concerns seriously. She did not step in to lead the class through whether cancellation was actually allowed by the field axioms, and let the students decide what mattered to them. This is further stressed by her use of the pronoun "we", as she positioned herself as a member of the class community as opposed to an authority figure who could tell them the "right answer".

5.6 Conclusion, Limitations, and Avenues for Future Research

5.6.1 Conclusion

As noted in the literature review, previous studies have looked at norms (Cobb & Yackel, 1996) and equity (Gutiérrez, 2009) in IBME classes. My research goal for this paper was to provide a detailed narrative of an IBL class' experience that provides insight and raises new questions at the intersection of these two areas. The point of interest of this narrative was how cancelling terms on both sides of the equals sign, which is valid by the definition of a binary operator and embedded in the definition of the field axioms, was not explicitly stated in the packet, and was thus rejected by the students as "true" in their classroom. Furthermore, the students negotiated amongst themselves that proving Statement 1 in a way that did not involve the LoC made it so that they could cite Statement 1 as justification for cancellation in future proofs. By choosing to not accept cancellation as true without justification, these students began to use mathematics as a system that they had control over. The class conversations in this narrative were important experiences because they forced students to come up against prior beliefs of mathematics being a pre-existing structure, and provided opportunities for students to take ownership over their mathematical work. I argue that we witnessed the inception and development of a sociomathematical norm (Cobb & Yackel, 1996) where students saw value in explicitly choosing together what to accept as mathematically "true" in their classroom. Furthermore, we saw growth in the types of social interactions that students were able to engage each other in (giving feedback on proofs, offering new ideas at the board, and debating the justifications used in class). I interpret all of this as being afforded by work on the instructor's part to guide students towards seeing mathematics as a system full of choices that they have power and authority over, which was rooted in her choice to run a student-centered IBL classroom.

Throughout the narrative, the instructor provided Sloan with multiple opportunities to engage in the critical axis of equity (Gutiérrez, 2009) by positioning her as a source of mathematical authority in the classroom. Sloan and her classmates were explicitly "involved in decision-making on acceptance or rejection of mathematical knowledge presented during class" (Tang et al., 2017, p. 59). In Excerpt 2, the instructor let Sloan speak first about the issues regarding the

LoC, and only then did she back Sloan up with her personal story of making the same mistake, thus encouraging the class to join her in not assuming mathematical ideas without proof. She elaborated on Sloan's ideas, and used herself as an example to equalize authority between herself and the students, emphasizing that they were in this learning process together through her communal word choice "let's not do that". In Transcript Excerpt 3, when Connor brought up the question of whether they can add on both sides of an equation, the instructor again refocused the conversation towards Sloan by asking her what the Law of Cancellation explicitly said. The instructor guided the class to focus on Sloan's creation of the law and ensure that the use of the law was consistent with Sloan's values and ideas. Throughout the data, we also saw evidence of students taking ownership over their ideas with voluntary presenters like Jordan, Easton, and Taylor. Furthermore, we saw students position each other as sources of mathematical authority in the classroom through their joint participation in conversations that did not include the instructor's voice or guidance. All of these conversations between the students and instructor occurred at the level of whole class social interactions, and were thus opportunities for the students to establish and negotiate social norms regarding classroom participation.

Overall, the most important takeaway I highlight from this narrative was that the instructor utilized Sloan's idea of the Law of Cancellation and the class' ensuing discussions about cancellation in their proofs as an opportunity to develop their sociomathematical norm (Cobb & Yackel, 1996) regarding the relationship between assumptions, truth, and justification in their classroom. These discussions were enriched by the fact that they were occurring in an IBL classroom and thus they provided students with opportunities to practice growing sociomathematical norms around how they positioned themselves and others as sources of mathematical authority – a major component of creating equity in the classroom through the use of the critical axis (Gutiérrez, 2009). While I did comment on potentially inequitable occurrences in class as well, I see this narrative as offering a nuanced picture of how the development of social and sociomathematical norms at the beginning of the class term can be leveraged as opportunities to also focus on equity in IBME classrooms more broadly.

5.6.2 Limitations and Future Directions

One limitation of this study was that the transcript excerpts did not represent the full range of 19 students in the classroom. IBME instructors must consider how to best encourage balance in student voices, helping quieter students speak up, while also recognizing that some students will feel less comfortable in a public discussion environment for a variety of personality reasons, not to mention the heightened discomfort that a student with a minoritized identity might feel (i.e., they may feel a stereotype threat more strongly while presenting and being corrected in front of their peers). As an observer, I saw a variety of engagement levels in the classroom and that the instructor made every attempt to allow students to participate in a way that was most comfortable for them, but I cannot be sure of every student's experience over the term. An interesting future study would take a closer look at individual student experiences in conjunction with classroom observations throughout the first few weeks of class in order to better make use of the individual side of the interpretive framework and draw deeper connections between the individual and social perspectives of the framework, and equity, in the classroom.

What happened in this class might have never occurred if the instructor had stepped in and verified whether or not the students could use cancellation in their proofs, and I am unsure from the classroom observation data as to whether the instructor caught on to the nuances of the mathematical issues at hand. IBME classes require on the spot navigation on the part of the instructor, and it can be difficult to respond to spontaneous student ideas. If Sloan had not made her comment about the LoC, it's entirely possible that the class would have been happy with the justification of adding (-*a*) to both sides by the additive inverse axiom and moved on from there. Furthermore, the students did not end this narrative with a precise understanding of binary operators, or the relationship between the LoC and Statement 1 as I discussed in Section 5.4.2 and 5.4.3. As such, there was some content knowledge lost at the expense of running the student-centered class. However, these statements were not particularly related to the advanced calculus content, and it is possible that the instructor chose to move on for the sake of class time, and the students were more than ready to move on as well after three days of discussion. However, the mathematics behind students' conversations is still interesting in and of itself, and this study opens opportunities to continue studying students' understanding of the field axioms, for example a teaching experiment that better leads students to understand the Law of Cancellation.

The conversations in this class were possible due to students' prior knowledge of algebra and the cognitive dissonance they encountered when they realized they could not just assume that algebra was true in their class. In this way, the first few statements in the packet provided an intellectual need (Harel, 2008) for students to both understand their axiomatic system fully and read their proofs carefully, habits which continued to serve them throughout the term. The instructor likened these first problems to the Karate Kid practicing tedious chores that seemed inconsequential at the time but were intentionally building foundational skills that he would draw on in his future tournaments. These statements were low-stakes and provided opportunities for students to negotiate and build things like social and sociomathematical norms with each other before engaging in more difficult advanced calculus content. An interesting future study could look at how the introductory curriculum in an IBME class could be used to leverage the sort of social and sociomathematical norm development seen in this paper.

Finally, IBME instructors may feel a tension between acclimating students into the broader established mathematical community and giving students the opportunity to develop their own mathematics. The instructor in this class let students spend almost three periods on one proof (they also completed proofs of Statements 2-4 with much less disagreement during this time). She felt strongly that any time she made a choice to step in she would be upsetting the delicate balance of growing student authority and confidence in the classroom. She wholeheartedly believed in running a "student-centered" classroom, and did not want students to see her as capable (or desiring) of stepping in to fix their mistakes. While it was beyond the scope of this paper to study the details or outcomes of this work on the part of the instructor, there are opportunities for future studies that track the progress of student's experiences of mathematical authority throughout the term and the impact it has on the overall IBME classroom.

References

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior*, *24*(2), 125–134. *https://doi.org/10.1016/j.jmathb.2005.03.003*

Blair, R., Kirkman, E., & Maxwell, J. (2018). *Statistical abstract of undergraduate programs in the mathematical sciences in the united states: Fall 2015 cbms survey*. American Mathematical Society. *https://doi.org/10.1090/cbmssurvey/2015*

Brown, S. (2018). E-IBL: an exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 1–15.

Capaldi, M. (2015). Including inquiry-based learning in a flipped class. *PRIMUS*, *25*(8), 736–744. *https://doi.org/10.1080/10511970.2015.1031303*

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, *23*(1), 2–33. *https://doi.org/10.2307/749161*

Cook, J. P. (2014). The emergence of algebraic structure: Students come to understand units and zero-divisors. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 349–359. *https://doi.org/10.1080/0020739X.2013.837520*

Creswell, J., & Poth, C. (2018). *Qualitative inquiry and research design: Choosing among five approaches* (Fourth Edition). SAGE.

Dawkins, P. C. (2009). *Non-traditional socio-mathematical norms in undergraduate real analysis*. ProQuest LLC.

Dawkins, P. C. (2014a). How students interpret and enact inquiry-oriented defining practices in undergraduate real analysis. *The Journal of Mathematical Behavior*, *33*, 88–105. *https://doi.org/10.1016/j.jmathb.2013.10.002*

Dawkins, P. C. (2014b). When proofs reflect more on assumptions than conclusions. *For the Learning of Mathematics*, *34*(2), 17–23.

Dawkins, P. C. (2018). Student interpretations of axioms in planar geometry. *Investigations in Mathematics Learning*, *10*(4), 227–239. *https://doi.org/10.1080/19477503.2017.1414981*

Dawkins, P. C., Oehrtman, M., & Mahavier, W. T. (2019). Professor goals and student experiences in traditional ibl real analysis: A case study. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 315–336. *https://doi.org/10.1007/s40753-019-00095-4*

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In *W. Zimmermann & S. Cunningham (Eds.), Visualization in teaching and learning mathematics.* (pp. 25–37). Mathematical Association of America.

Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *The Journal of Mathematical Behavior*, *31*(3), 401–416. *https://doi.org/10.1016/j.jmathb.2012.04.002*

Fukawa-Connelly, T., Mejía-Ramos, J. P., Wasserman, N. H., & Weber, K. (2020). An evaluation of ULTRA; an experimental real analysis course built on a transformative theoretical model. *International Journal of Research in Undergraduate Mathematics Education*, *6*(2), 159–185. *https://doi.org/10.1007/s40753-019-00102-8*

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*. *https://doi.org/10.1207/s15327833mtl0102_4*

Gutiérrez, R. (2009). Framing equity" helping students “play the game” and “change the game”. *Teach Excellence Equity Math*, *1*(1), 5–7.

Haberler, Z., Laursen, S. L., & Hayward, C. N. (2018). What’s in a name? Framing struggles of a mathematics education reform community. International Journal of Research in Undergraduate Mathematics Education, 4(3), 415–441. https://doi.org/10.1007/s40753-018-0079-4

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part 1: Focus on proving. *ZDM Mathematics Education*, *40*, 487–500.

Harel, G., & Sowder, L. (2007). Toward comprehensive pserspectives on the learning and teaching of proof. In *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 805–842). *https://math.ucsd.edu/~harel/TowardComprehensivePerspective.pdf*

Hassi, M.-L., & Laursen, S. L. (2015). Transformative learning: Personal empowerment in learning mathematics. *Journal of Transformative Education*, *13*(4), 316–340. *https://doi.org/10.1177/1541344615587111*

Hayward, C. N., Kogan, M., & Laursen, S. L. (2016). Facilitating instructor adoption of inquiry-based learning in college mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 59–82. *https://doi.org/10.1007/s40753-015-0021-y*

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and Gender Inequity in the Undergraduate Mathematics Classroom. *Journal for Research in Mathematics Education*, *51*(4), 504–516. *https://doi.org/10.5951/jresematheduc-2020-0043*

Kinsey, L. C., & Moore, T. E. (2015). Narrative structure in inquiry-based learning. *PRIMUS*, *25*(3), 212–220. *https://doi.org/10.1080/10511970.2014.921650*

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *PRIMUS*, *28*(1), 13–30. *https://doi.org/10.1080/10511970.2017.1338807*

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725. *https://doi.org/10.1016/j.jmathb.2013.04.006*

Laursen, S. L., Hassi, M.-L., & Hough, S. (2016). Implementation and outcomes of inquiry-based learning in mathematics content courses for pre-service teachers. *International Journal of Mathematical Education in Science and Technology*, *47*(2), 256–275. *https://doi.org/10.1080/0020739X.2015.1068390*

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406. *https://doi.org/10.5951/jresematheduc.45.4.0406*

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–146. *https://doi.org/10.1007/s40753-019-00085-6*

Lew, K., Timothy Patrick Fukawa-Connelly, Juan Pablo Mejía-Ramos, & Keith Weber. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162. *https://doi.org/10.5951/jresematheduc.47.2.0162*

Mesa, V., Shultz, M., & Jackson, A. (2020). Moving away from lecture in undergraduate mathematics: Managing tensions within a coordinated inquiry-based linear algebra course. *International Journal of Research in Undergraduate Mathematics Education*, *6*(2), 245–278. *https://doi.org/10.1007/s40753-019-00109-1*

Mullen, E. T. (2012). Teaching an engaged analysis class through active learning. *PRIMUS*, *22*(3), 186–200. *https://doi.org/10.1080/10511970.2010.497957*

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*, *26*(3), 189–194. *https://doi.org/10.1016/j.jmathb.2007.10.001*

Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006). Capitalizing on advances in mathematics and k-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85–93. *https://doi.org/10.1007/BF03036787*

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259–281. *https://doi.org/10.1007/s10649-014-9583-x*

Rasmussen, C., Zandieh, M., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73.

Reinholz, D. L. (2020). Five practices for supporting inquiry in analysis. *PRIMUS*, *30*(1), 19–35. *https://doi.org/10.1080/10511970.2018.1500955*

Roh, K. H., & Lee, Y. H. (2017). Designing Tasks of Introductory Real Analysis to Bridge a Gap between Students' Intuition and Mathematical Rigor: The Case of The Convergence of a Sequence. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 34–68. *https://doi.org/10.1007/s40753-016-0039-9*

Shannon, C. A. (2016). Flipping the analysis classroom. *PRIMUS*, *26*(8), 727–735. *https://doi.org/10.1080/10511970.2016.1162889*

Shannon, K. (2018). It's not the moore method, but… a student-driven, textbook-supported, approach to teaching upper-division mathematics. *PRIMUS*, *28*(2), 118–127. *https://doi.org/10.1080/10511970.2017.1348416*

Stone-Johnstone, A., Reinholz, D., Brooke Mullins, S., Smith, J., & Andrews-Larson, C. (2019). Inquiry without equity: A case study of two undergraduate math classes. *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education*.

Strand, S. R. (2016). The intermediate value theorem as a starting point for inquiry-oriented advanced calculus. In *ProQuest LLC*. ProQuest LLC.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In *Compendium for Research in Mathematics Education* (pp. 237–266). The National Council of Teachers of Mathematics, Inc.

Tall, D. (1991). Preface. In *Advanced mathematical thinking* (pp. xii–xv). Kluwer Academic Publishers.

Tang, G., El Turkey, H., Cilli-Turner, E., Savic, M., Karakok, G., & Plaxco, D. (2017). Inquiry as an entry point to equity in the classroom. *International Journal of Mathematical Education in Science and Technology*, *48*. *https://doi.org/10.1080/0020739X.2017.1352045*

von Renesse, C., & Ecke, V. (2015). Inquiry-based learning and the art of mathematical discourse. *PRIMUS*, *25*(3), 221–237. *https://doi.org/10.1080/10511970.2014.921799*

Wasserman, N. H. (2014). Introducing algebraic structures through solving equations: Vertical content knowledge for K-12 mathematics teachers. *PRIMUS*, *24*(3), 191–214. *https://doi.org/10.1080/10511970.2013.857374*

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577–599. *https://doi.org/10.1080/10511970.2012.667516*

Weber, K. (2008). The role of affect in learning real analysis: A case study. *Research in Mathematics Education*, *10*(1), 71–85.

Yackel, E., & Rasmussen, C. (2003). Beliefs and norms in the mathematics classroom. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (Vol. 31, pp. 313–330). Kluwer Academic Publishers. *https://doi.org/10.1007/0-306-47958-3_18*

Zazkis, D., Weber, K., & Mejía-Ramos, J. P. (2016). Bridging the gap between graphical arguments and verbal-symbolic proofs in a real analysis context. *Educational Studies in Mathematics*, *93*(2), 155–173. *https://doi.org/10.1007/s10649-016-9698-3*

6 (Paper 2) Inquiry-Based Learning and Beyond: A Case Study of Rehumanizing Mathematics in Action

(Submitted to the Journal of Humanistic Mathematics in April of 2021)

Abstract: In this paper, I present an empirically grounded case study that considers how the dimensions of rehumanizing mathematics (Gutiérrez, 2018) can occur and develop in an inquiry-based learning classroom, both through the structure of the course and through course elements that an instructor incorporated (such as a writing assignment that asked students to articulate a personal axiom). My evidence that the course engaged in rehumanization comes from student data at the end of the Spring 2020 term emphasizing how important this class was to them during the transition to remote learning due to COVID-19. I also employ the four pillars of inquiry-based mathematics education (IBME) (Laursen & Rasmussen, 2019) to frame my understanding of the classroom structure I observed and contribute to larger conversations on equity in undergraduate mathematics education.

Keywords: *inquiry-based learning, rehumanizing mathematics, equity*

6.1 Introduction

Laursen and Rasmussen (2019) have broadly characterized *inquiry-based mathematics education* (IBME) as consisting of four pillars: (1) students engage deeply with coherent and meaningful mathematical tasks, (2 students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, and (4) instructors foster equity in their design and facilitation choices (p. 138). While there remains little doubt in the mathematics education community that IBME is in the best interest of our students (e.g., Freeman et al., 2014; Theobald et al., 2020), researchers are continuing to develop their understanding of the fourth IBME pillar regarding equity. Under the broad umbrella term of IBME, there are studies that claim inquiry-based learning (IBL) classrooms have increased benefits for women and minority students (Laursen et al., 2014) as well as cautionary tales (Johnson et al., 2020) that equity is not a given in inquiry-oriented instruction (IOI) classrooms. These contrasting research findings suggest that despite the fourth pillar of IBME on equity and theoretical connections between IBME and equity (see Tang et al.'s (2017) framework aligning Cook et al.'s (2016) characteristics of inquiry and Gutiérrez's (2009) definition of equity), there is a necessary but not sufficient relationship between IBME and equity. In other words, we as researchers may see the most potentially equitable environments for students as occurring in IBME classes, but just because an instructor runs an IBME class does not by itself guarantee an equitable environment for students. In this paper, I argue that in order to more fully realize the fourth IBME pillar, we need to look beyond definitions of equity, in favor of a more detailed picture, such as understanding how instructors attend to *rehumanizing mathematics* in their classrooms (Gutiérrez, 2018). Gutiérrez writes,

> Unlike "equity", which can seem to represent a destination, "rehumanizing" is a verb; it reflects an ongoing process and requires constant vigilance to maintain and to evolve with contexts. Moreover, rehumanizing is an ongoing performance and requires evidence from those for whom we seek to rehumanize our practices that, in fact, the practices are felt in that way (Gutiérrez, 2018, p. 3).

From Gutiérrez, I interpret that using a lens of rehumanization, as opposed to equity, can give teachers active, evidence based, equitable ways to engage students in the classroom. I provide several examples of how the eight dimensions of rehumanizing mathematics can appear through episodes across a semester-long inquiry-based learning (IBL), a type of IBME, undergraduate introductory Real Analysis class. I highlight how the IBL classroom structure supported rehumanizing mathematics, as well as ways in which the observed instructor went above and beyond IBL expectations to engage students in rehumanizing experiences. I argue that Gutiérrez's language of rehumanizing mathematics provides a framework for investigating whether an IBL class provides equitable experiences for students, and that the dimensions of rehumanizing mathematics encourage teacher actions beyond what a traditional IBL class structure necessitates. My evidence for the overall equitable experience of students in this class is demonstrated by data on how well the class handled the transition to remote learning during the Spring 2020 term due to COVID-19, and how impactful the class was for students based on the final day of class and final interviews with students. In one of several similar comments that students offered in their final individual interviews, Taylor had this to say about the class:

Taylor: This time was really rough with COVID, so I think this class kind of transitioned more towards emotional support and making sure people are there for each other. And this class is a very collaborative, integrated course where we all talked with each other, so I think that even emphasized the point of having emotional support and connection for each other… The other classes I had this term… were very quiet… the energy was very dead… So it kind of kept my sanity knowing that I could still come to this class and talk to people and joke around. It was nice to have that.

In light of the positive experience of these students in this class through the transition to remote learning, I seek to address the following research questions:

1) *In what ways did the instructor use the IBL class structure, and add elements beyond the IBL class structure, to promote dimensions of rehumanizing mathematics with students?*
2) *How was the class' engagement in rehumanizing mathematics reflected in the Spring 2020 remote transition and the end of the term?*

Together, answering these questions will inform specific ways in which rehumanizing mathematics might exist within inquiry-based mathematics courses and promote new ways of how to facilitate an IBL style classroom in ways that align with the fourth pillar of IBME. This in turn addresses broader discussions on the relationship between equity and IBME classrooms.

6.2 Literature Review and Theoretical Perspectives

In this section, I explicate the two lens that informed my analysis of the classroom I observed. First, I consider the four pillars of IBME and their historical development along with examples of studies that emphasize each pillar. In order to better frame the classroom I observed, which used a type of IBME called inquiry-based learning (IBL), I also provide distinctions between IBME and IBL. Then I situate my work in relation to Gutiérrez's construct of rehumanizing mathematics. I describe the relationship between equity and rehumanizing mathematics, and the eight dimensions of rehumanizing mathematics in connection to both the IBME pillars and relevant mathematics education literature.

6.2.1 Inquiry Based Mathematics Education

The term IBME was developed to unite several strands of inquiry education, namely inquiry-based learning (IBL) and inquiry-oriented instruction (IOI), under one definition. The instructor I observed identified herself as an inquiry-based learning instructor and is a member of the *Academy of Inquiry-Based Learning*. While IBL has a unique historical development in the mathematics education community, and this instructor's classroom shared many qualities specific to IBL (see Methods section for a classroom description), the IBL community has more recently shifted to a "big tent" view of inquiry type learning and aligns their definition of IBL with the four pillars of IBME[8]. Thus, I use the four pillars of IBME as a lens to broadly understand the structure of the class and the instructor's motivation for particular actions and activities during class time. In the remainder of Section 2.1, I provide explanations of the four IBME pillars, drawing on relevant literature to show how each pillar has been studied. In drawing on the IBME literature broadly, I look at studies under both the inquiry-oriented perspective and the inquiry-based learning perspective.

6.2.1.1 Characterizing IOI and IBL

Inquiry-oriented instruction (IOI) comprises a body of curriculum and research literature centered on design-based research (Cobb, 2000), Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1999), and draws inspiration from Cobb and Yackel's work on

---

8 For example, the Academy of Inquiry Based Learning defines IBL using the four pillars of IBME on their website: http://www.inquirybasedlearning.org/.

sociomathematical norms (Cobb & Yackel, 1996). Laursen and Rasmussen (2019) describe IOI classrooms as follows,

> Visitors to IO classrooms would see students working in small groups on unfamiliar and challenging problems, students presenting and sharing their work, even if tentative, and whole-class discussions where students question and refine their classmates' reasoning. The students' intellectual work lies in creating and revising definitions, making and justifying conjectures and justifying them, developing their own representations, and creating their own algorithms and methods for solving problems (p. 134).

Due to the origins of IOI in undergraduate mathematics education research, the majority of qualitative studies in IBME classrooms currently comes from IOI researchers and their curricula. For example, the National Science Foundation supported project *Teaching Inquiry-Oriented Mathematics: Establishing Supports*[9] (TIMES) has reported on work related to the development and implementation of inquiry-oriented curricula within several mathematical domains (including abstract algebra, linear algebra, and differential equations). Within each domain, researchers have investigated work on student thinking, development and refinement of tasks and materials, and issues related to the effective implementation of such curricula in classrooms. In general, IOI can be seen as a particular characterization of IBME that has established curricula and ways of teaching (Kuster et al., 2018).

Inquiry-based learning (IBL) developed along multiple parallel tracks among both mathematicians and practitioners who teach with active learning methods in their classrooms. Laursen and Rasmussen (2019) write,

> Visitors to IBL courses would see class work that is highly interactive, emphasizing student communication and critique of these ideas, whether through student presentations at the board or small group discussions. Whole-class discussion and debriefs are used to aid collective sense-making, and instructors may provide mini-lectures to provide closure and signposting. Instructors'

9 https://times.math.vt.edu/

> classroom role is thus shifted from telling and demonstrating to guiding, managing, coaching, and monitoring student inquiry (p. 136).

The large variety of IBL style classes and the practitioner-centered nature of the space has encouraged studies such as large-scale quantitative work on student learning outcomes in IBL classrooms (e.g., Laursen et al., 2014), and personal accounts of teacher experiences in practitioner journals such as PRIMUS (e.g., K. Shannon, 2018). Thus, one goal of this paper is to provide a case study of an IBL classroom from a researcher's perspective, which is somewhat lacking when compared to the substantial IOI literature base (e.g., Larsen, 2013; Rasmussen et al., 2006; Strand, 2016; Wawro et al., 2012).

In uniting the overlapping goals of IOI and IBL, Laursen and Rasmussen (2019) describe IBME as classrooms that use active learning with "a longer-term trajectory that sequences daily tasks to build toward big ideas, to "reinvent or create mathematics that is new to [students]," and to "offer students and instructors greater opportunities to develop a critical stance toward previous, perhaps unquestioned, learning and teaching routines" (Laursen & Rasmussen, 2019, p. 133-134). They describe the four pillars of IBME as: (1) students engage deeply with coherent and meaningful mathematical tasks, (2) students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, and (4) instructors foster equity in their design and facilitation choices (p. 138) (see Figure 6-1). I take these four pillars as my definition and characterization of IBME. I now describe each of these pillars individually and examine what evidence of each pillar can look like in an IBME classroom.

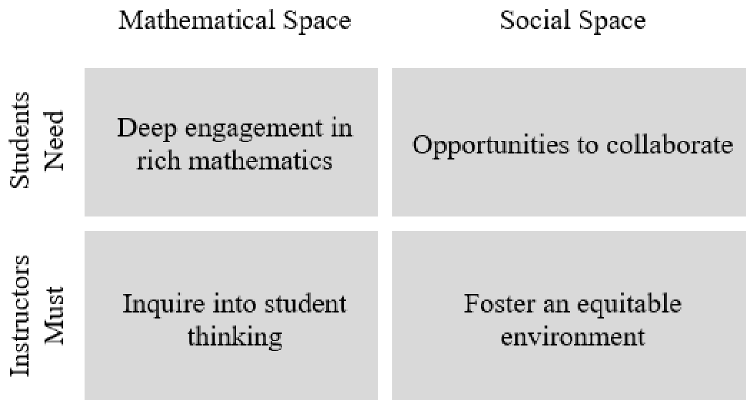


Figure 6-1: The Four Pillars of Inquiry-Based Mathematics Education[10]

6.2.1.2 Pillars 1, 2: Deep Engagement in Rich Mathematics and Opportunities to Collaborate

The first two pillars of IBME are (1) students engage deeply with coherent and meaningful mathematical tasks and (2) students collaboratively process mathematical ideas. These concepts first appeared as "twin pillars" in Laursen et al. (2014). In describing these pillars, Laursen & Rasmussen (2019) state that,

> … deep engagement occurs as students encounter, grapple with, and revisit important ideas over time, in and out of class. And, as students discuss, elaborate and critique these ideas together, they deepen their understanding and build communication skills, collaborative skills, and appreciation for diverse paths to solutions (p. 137).

Thus, evidence of the first two pillars can come from classroom observations of students engaging in mathematics during class, and through reflection in student interviews as to how they do mathematics outside of class. In Ernst et al. (2017), the authors provide expository examples of these two pillars being used in an upper-level proof-based classes, a calculus class, and a pre-service teacher class. In Wawro et al. (2012), the authors examined student group work on an IOI linear algebra instructional sequence covering span and linear independence (the Magic Carpet Ride). The data given in the paper consists of excerpts from classroom transcripts

10 Graphic as it appeared in the instructor's syllabus, credited to Dr. Nina White.

and images of student group work on white boards. Since the focus of this paper was to describe students' understanding of the mathematical content, this data is viewed as a window into students' mathematical experiences in IBME classrooms. My study adds to the literature by addressing these same types of IBME classroom experiences, not with the focus of understanding how students experience the mathematical curriculum, but through a social lens of interpreting how these mathematical experiences contributed to students' overall social, and rehumanizing, experiences in the classroom.

6.2.1.3 Pillar 3: Instructor Inquiry into Student Thinking

The third pillar of IBME considers how instructors inquire into student thinking. This pillar was introduced in Rasmussen and Kwon (2007), and I note that they also describe similar student focused pillars as in Section 6.2.1.1. This pillar focuses on how the instructor can support the first two student centered pillars by highlighting students' mathematical contributions during class and helping to promote a sense of classroom community around creating mathematics together. Thus, evidence of this pillar can come from paying attention to how the instructor interacts with students during class and instructor reflections on the intentions of their in-class actions during interviews. In a follow-up piece to Ernst et al. (2017), Rasmussen et al. (2017), provided a series of instructor questions that target getting students to share their thinking, orient to and engage in others' thinking, deepen their thinking, and build on and extend student ideas (p. 1308). In Dawkins et al. (2019), the authors detail the teaching goals of a professor and compare these to his students' experiences in an IBL classroom. In particular, one of the teaching practices highlighted in this study was the instructor's focus on providing "differentiated feedback". Dawkins et al. write that the instructor "tried to provide minimally sufficient feedback so that students both remained *challenged* and retained *ownership* over their created proofs” (p. 325). The data provided in Dawkins et al. (2019) consists of reflections from both the instructor and students on how these teaching practices were broadly attained over the term. My study adds to the literature by providing examples of this pillar occurring in the day-to-day activities of the classroom, and connections as to how this pillar interacts with the other pillars through the dimensions of rehumanization.

6.2.1.4 Pillar 4: Instructor Fosters an Equitable Environment

The fourth pillar of IBME tends to how instructors foster equity (see Section 6.2.2 for more on defining equity) in their design and facilitation choices. This pillar was added to the existing three pillars in Laursen and Rasmussen (2019). Several researchers have found positive links between IBME classrooms and equity, such as Laursen et al.’s (2014) work on how IBL classrooms impact course performance by women students. They found that "women in non-IBL courses reported substantially lower cognitive gains than did their male classmates", while in IBL classes, "women’s cognitive and affective gains were statistically identical to those of men, and their collaborative gains were higher" (Laursen et al., 2014, p. 411-412). However, more recent studies have dug deeper into whether there is a necessary and sufficient relationship between equity and IBME classrooms. Brown (2018) argues that we must intentionally pursue Equity Oriented Inquiry Based Learning (EO-IBL). She provides an instance of an IBL classroom in which students were told to form small groups in order to work on an assignment. A Hispanic female was left to the edges of a group of three male students and worked on her own without any recognition from the group. Brown calls this "the illusion of participation" and states

> Even if all IBL students are expected to advocate for their own participation it is not the case that all students are called on to do so (often again and again). More importantly, it is not the case that all will have cultural habitus, disposition, or identity that will support them in doing so (Brown 2018, p. 6).

Brown’s work suggests that even if IBL classrooms provide overall improvement for women and minorities, instructors need to be aware of the burdensome and inequitable actions that these same students might have to take in order to fully participate in class. Stone-Johnstone et al. (2019) voice a similar hypothesis, saying "we must not only be vigilant about increasing student engagement but also conscientious about the ways in which we engage different students" (p. 5). Finally, Johnson et al. found gendered differences in their work regarding an IOI classroom and "suspect[ed] that there are important instructional differences between IOI and IBL that may impact different groups differently [which includes] the routine use of student presentations in IBL classrooms" (Johnson et al. 2020, p. 7). My study addresses this literature by focusing on an IBL classroom that regularly employs student presentations, and I give concrete examples of ways in which the instructor provided different, rehumanizing, ways for students to engage in the classroom that I believe helped to create a more equitable classroom environment.

6.2.2 Rehumanizing Mathematics

I primarily employ Gutiérrez’s (2018) framework for rehumanizing mathematics to understand the instructor’s actions in the classroom. In particular, I consider how the instructor’s teaching actions were informed by the IBL class structure, and how she took additional actions of her own to create an equitable experience for her students, by utilizing the dimensions of rehumanizing mathematics as my theoretical lens. First, I distinguish the relationship I see between equity and rehumanizing mathematics, and I explain why I chose to use rehumanizing over equity as my lens for this work. Then I characterize both equity and the eight dimensions of rehumanization, providing connections between these concepts and the four pillars of IBME.

Gutiérrez (2009) characterizes equity as operating on two axes: a dominant axis (*access* and *achievement*), and a critical axis (*identity* and *power*) (see Table 6-1).

Axis Component	Description
Access	Students' available resources to participate in and learn mathematics.
Achievement	The outcome affected by *access*, measured through class participation, grades, and test scores, etc.
Identity	Attention to the history of marginalization and discrimination in mathematics along with more global acknowledgements of the relations between students' personal identities and the field of mathematics.
Power	The outcome affected by *identity*, where students become critical of systems that use mathematics and are capable of making social transformation with their mathematical knowledge.

Table 6-1: Characterization of Equity, Gutiérrez (2009)

This characterization of equity provides a big picture view of what equity looks like in mathematics, through helping students "play the game" of mathematics as it is, via *access* and *achievement*, and by empowering students to "change the game" of what mathematics could be, via *identity* and *power*. Researchers have drawn connections from Gutiérrez's work to IBME, most notably Tang et al. (2017) created a theoretical framework that put forth connections between each axis component and six qualities of inquiry-based instruction (Cook et al., 2016). Thus, I see this work as a relevant and useful way to frame equity and work around the fourth pillar of IBME.

To this end, I have chosen Gutiérrez's lens of rehumanizing mathematics over equity in order to create a more detailed picture of how an instructor might foster equitable actions in a mathematics classroom. Gutiérrez writes,

> Unlike "equity", which can seem to represent a destination, "rehumanizing" is a verb; it reflects an ongoing process and requires constant vigilance to maintain and to evolve with contexts. Moreover, rehumanizing is an ongoing performance and requires evidence from those for whom we seek to rehumanize our practices that, in fact, the practices are felt in that way (Gutiérrez, 2018, p. 3).

From Gutiérrez, I interpret that rehumanizing mathematics, as opposed to equity, can give teachers actionable, evidence based, ways to encourage equitable experiences in the classroom.. As we will see in the descriptions below, several of these dimensions relate to the axes of equity and I see the dimensions as complementary to Gutiérrez's (2009) characterization of equity. Note, this is not to say that there is any one-to-one correspondence between the dimensions of rehumanization and the axes of equity, but rather that the dimensions of rehumanization give some ideas and ways in which teachers can be actively working in their classrooms towards the overarching goals of equity.

Gutiérrez (2018) lists eight dimensions of rehumanization: "(1) participation/positioning, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) body/emotions, and (8) ownership" (Gutiérrez, 2018, p. 4). Table 6-2 provides a brief overview of these dimensions, and I define each one along with clarifying examples in the remainder of this section. As we will see in the Results, some dimensions of rehumanizing mathematics fit well within the existing structure of an IBL classroom, and others take more explicit work on the part of an instructor beyond what a traditional IBL structure necessitates. Throughout this section I draw connections between these dimensions and the four pillars of IBME. Recall that the *Academy of Inquiry-Based Learning* defines IBL based on these four pillars as well, and I draw on them as the basis for my understanding of the IBL classroom structure I observed. Furthermore, I recognize that many of these dimensions are complementary and can influence each other. Thus, in the following theoretical descriptions and Results section I give my best definitions and categorizations of each episode while recognizing that one example may fall under multiple dimensions.

Dimension	Description
Participation/Positioning	Considers how students are participating and positioned within the classroom, and whether they are given opportunities to act as mathematical authorities.
Cultures/Histories	Considers whether students' cultures and histories are represented in the classroom through attending to the history of mathematics and the mathematical work of their ancestors.
Windows/Mirrors	Considers whether students see mathematics through the different perspectives of their peers (windows) and themselves by reflecting on their own identities in their mathematical work (mirrors).
Living Practice	Considers whether students experience mathematics as a dynamic process full of power dynamics, debate, and rule breaking.
Creation	Considers whether students get to do mathematics in line with their own values, and not simply a reproduction of others' (teacher, textbook, etc.) work.
Broadening Mathematics	Considers whether students experience mathematics beyond the traditional structure of quantitative knowledge, i.e. experiencing mathematics in more qualitative ways through the humanities.
Body/Emotion	Considers whether students are invited to experience mathematics with their full senses intact (intuition, vision, touch, emotion).
Ownership	Considers whether students experience mathematics as something they do from a place of internal motivation, as opposed to external reasons such as grades or teacher approval.

Table 6-2: Dimensions of Rehumanizing Mathematics, adapted from Gutiérrez (2018)

6.2.2.1 Participation/Positioning and Living Practice

The dimension of *participation/positioning* considers how students are participating and positioned within the classroom, and whether they are given opportunities to act as mathematical authorities. In particular, the dimension promotes a shifting of classroom authority from the teacher/text to the students. This language comes from positioning theory (Davies & Harre, 1990) which views personal identity as flexible and capable of changing depending on how that person position themselves and how others position them in various situations. Authority has also been well researched within mathematics education, in particular I note Langer-Osuna's (2017) work on authority and identity in collaborative mathematics. My use and interpretation of this dimension is not necessarily to bring something new to light about the deep constructs that the dimension draws on, but more to recognize their combined benefit and use in an IBME classroom towards the purpose of rehumanizing mathematics, especially in tandem with other dimensions.

A class that addresses the *participation/positioning* dimension of rehumanizing mathematics may create opportunities for students to participate in mathematical activity during class time or afford students authority over the legitimacy and value of the mathematical work being discussed in class. This dimension aligns with the second pillar of IBME where students have opportunities to collaborate (*participation*), and the third pillar in which the instructor focuses their time on inquiring into student thinking as opposed to explaining their own thinking (thus *positioning* students as mathematical authorities). I see this dimension as related to the equity axis component of *access* (student access to participation during class) and *power* (students are positioned as authority figures).

Additionally, the dimension of *living practice* considers whether students experience mathematics as a dynamic process full of power dynamics, debate, and rule breaking (Gutiérrez, 2018, p. 5). The first pillar of IBME focuses on students' deep engagement in rich mathematics and, in an IBL classroom, students debate and create mathematics together, which affords opportunities to experience this dimension. For example, Dawkins (2014a) looked at the practice of defining in an inquiry-oriented undergraduate real analysis class, meaning that definitions were treated as "under construction" and students were "consistently discussing and negotiating these formal statements" (p. 89). Dawkins found differences among students' "perceived frames, roles, expectations, and values" (p. 101) depending on how they understood authority in the classroom and how they acculturated themselves into the mathematical practice of defining. While most, if not all, of the rehumanizing dimensions can be considered in tandem with each other, I see *position/participation* and *living practice* as being deeply valued in an IBME classroom. Furthermore, given students' participation and positioning as mathematical authority figures in an IBL class, I interpret the dimension of *living practice* as related to the critical axis component of *power* (students get to dynamically engage in mathematics and use their mathematical authority to make choices in the classroom).

6.2.2.2 Cultures/Histories and Windows/Mirrors

The dimension of *cultures/histories* considers whether students' cultures and histories are represented in the classroom through attending to the history of mathematics and the mathematical work of their ancestors. This could occur through acknowledgment of students' prior knowledge, mathematics from other countries, ethnomathematics, or making connections between the roots of students' pasts and the history of mathematics. The *Journal of Humanistic Mathematics* provides several examples of instructors attending to this dimension in their classrooms (e.g., Anderton & Wright, 2012; Lipka et al., 2019; Maxwell & Chahine, 2013). One relevant example of this dimension occurring in IBME classes is Matthew & Hodge's (2016) piece on using IBL teaching techniques in a history of mathematics course. However, in general, the IBL class structure requires using carefully scaffolded theorem statements with little context, which creates a number of incompatibilities with the *cultures/histories* dimension (see Section 6.4.1.8 for more details).

While IBL classes may have some difficulty in exemplifying the *cultures/histories* dimension, they do provide ample opportunities in the *windows/mirrors* dimension. This dimension considers whether students see mathematics through the different perspectives of their peers (*windows*) and themselves, by reflecting on their own mathematical work (*mirrors*). An important aspect of this dimension is that students can learn to appreciate, not just critique, the views of others around them. Wawro et al. (2012) provide a nice example of the *windows* aspect of this dimension, through sharing images of the board work of multiple small groups in class on a problem involving linear independence in the *Magic Carpet Ride Problem*. They write, “whole class discussion of the various approaches offered insight into [how] the students were thinking about linear combination[s] of the three vectors” (p .587). As an example of the *mirrors* aspect, Hassi and Laursen (2015) share ways that IBL classes provide students with self, cognitive, and social empowerment opportunities that can lead to transformative learning experiences. Under the self-empowerment section, they considered the role of agency and self-reflection in IBL. They write, “student-centered and collaborative IBL practices fostered students’ need and ability to be reflective in studying mathematics” (p. 326) and they share data excerpts of students’ meta-reflections on what they learned about how they learn and do mathematics in an IBL class. Thus, IBL classrooms provide ways for students to both experience the mathematical thinking of those around them, and to reflect deeply on their own mathematical thinking process.

None of the four IBME pillars address either of these dimensions, however one could consider the *windows* aspect within the second pillar on collaboration and the *mirrors* aspect within the first pillar on deep engagement in rich mathematics. I also view these dimensions as related to the equity axis component of *identity* (students come to see their identity, and those of their classmates, as relevant to the mathematics they are doing in class).

6.2.2.3 Creation and Ownership

Creation considers a student's autonomy and ability to do mathematics in ways that are consistent with their own values, and not simply a reproduction of work that has come before them (i.e., replicating a teacher or textbook). Again, in line with the first pillar of IBME, students' deep engagement in rich mathematics, IBL classes are built around students developing their own solutions to problems without influence from outside resources like a textbook or a teacher. I see the dimension of creation as also relating to creativity, in that students' coming up with ways to do mathematics in line with their values is often creative. Creativity is beginning to be studied in undergraduate mathematics education, notably with the *Vol. 10 Issue 2: Special Issue on Creativity in Mathematics* from the *Journal of Humanistic Mathematics*. In particular, Adiredja and Zandieh's (2020) considered how individual and collective creativity in a linear algebra class speaks to ways that we might recognize creativity in a mathematics class. They write, "In this way, we focus on the potential for students' creative products to reveal insights about their mathematical understanding, and also the way that mathematical analysis of these products might reveal insights into students' creativity" (p. 46). In addition, I see this dimension as related to the critical axis component of *identity* (students have opportunities to create mathematics that reflects their individual identity and views of mathematics).

Similarly, *ownership* promotes student experiences in mathematics as something they do from a place of internal motivation, as opposed for external reasons such as grades or teacher approval. Gutiérrez defines *ownership* as the dimension that recognizes joy and play in mathematics, or mathematics as a form of self-expression, which also aligns with the first pillar of IBME, students' deep engagement in rich mathematics. In fact, several existing studies on IBME classrooms consider ownership an important outcome and facet of the experience. For example, in Dawkins et al. (2019), they describe one of the instructor's overarching learning goals as having students "learn to create their own mathematics" and that the instructor believed "having to create their own proofs gives students *deeper understanding* and *ownership* over what they learn" (p. 323). I see this dimension as related to the outcome ends of both the dominant and critical axes of equity: *achievement* (student success is viewed through an internal lens) and *power* (students have the power to do mathematics for themselves and for their own reasons).

6.2.2.4 Broadening Mathematics and Body/Emotion

The dimension of *broadening mathematics* considers whether students experience mathematics beyond the traditional realm of quantitative knowledge, i.e., experiencing mathematics in more qualitative ways through the humanities. A number of instructors have discussed their humanities-based projects in mathematics classes such as von Renesse and DiGrazia's (2018) expository work on combining mathematics and writing in a first-year inquiry-based learning community, and Gordon's (2019) work on broadening the mathematics curriculum by seeing students as artists and giving them "opportunities to appreciate the aesthetic dimension of mathematics" (p. 192). In particular, one way to broaden mathematics is through the dimension of *body/emotion*. Through this dimension, students are encouraged to use their senses and participate in mathematics as a full body experience, for example through "voice, vision, touch, and intuition" (p. 5). Note, there is no explicit connection between Gutiérrez's use of the term 'body' and current research around embodied cognition (Nunez, 1999), and I do not make any such connections in this work. Due to the nature and focus of my data collection, I tend primarily to focus on the *emotion* aspect of this dimension in my results. Affect (broadly understood as students' emotions, moods, attitudes, motivation) has grown steadily as a research interest (for example the *Educational Studies in Mathematics Vol. 63*

Special Issue on Affect in Mathematics Education, 2006) but there are few examples studying affect in IBME classes. One example, Hassi and Laursen (2015), showed that IBL classes provided students with self, cognitive, and social empowerment opportunities that afforded transformative learning experiences and thus positive affective experiences. However, this is subtly different than considering actions the instructor took to use *emotion* as a way to engage during class in rehumanizing mathematics, as opposed to an outcome from being in an IBL classroom.

Neither *broadening mathematics* nor *body/emotion* are explicitly addressed in the four pillars of IBME. I hypothesize that the dimension of *body/emotion* is related to the fourth pillar on fostering an equitable environment, since recognizing student emotion and drawing on other ways of 'knowing' is an important feature of an inclusive mathematical experience, but it is not explicitly stated or described as such in the defining IBME literature. I also see these two dimensions as related to the dominant axis component of *access* (students are given various access points to participate in class through activities that broaden mathematics).

I find it important to acknowledge that Gutiérrez's work focuses on raising the voices of Indigenous, Black, and Latinx students, which is not the goal of this paper, and it is not my intention to draw attention away from these important discussions. In particular, the dimension of *cultures/history* does not appear in the four pillars of IBME, and IBL classrooms in particular tend to remove all historical context from their materials in favor of students recreating the material themselves with little external input. I recognize that the dimension of *cultures/histories* is deeply important when considering our Black, Indigenous, and Latinx students whose cultures and histories in mathematics have been systematically discriminated against and pushed aside.

Thus, one concern and call for future research in this paper is to consider how this dimension could be more deeply integrated into IBME classrooms (see the Discussion section).

Overall, I use Gutiérrez's work to help reframe what active, evidence-based, equitable actions can look like in an IBL classroom. In addition, I find the lens of rehumanization to be deeply illuminating when interpreting the episodes provided throughout the Results section and understanding the end of term data on students' positive class experiences. Thus, my goal is to extend Gutiérrez's work in ways that explore what these dimensions can look like in practice, and to provide explanation as to the progress of this classroom community over an incredibly difficult term.

6.3 Methods and Data Collection

6.3.1 Classroom Context

The professor that I observed, Dr. Miya[11], is an active member of the *Academy of Inquiry Based Learning*, who has taught IBL courses for over twelve years. According to the syllabus, the purpose of the class is to "prov[e] all of those theorems you accepted as true back in calculus." IBL classes can take on a variety of structures and so I provide some context as to the day-to-day operations of the class I observed. Prior to the remote transition, the class was designed as follows. Students were not given a textbook, instead they received packets containing definitions, axioms, and a list of theorems to prove in a specified order. Students worked through problems at their own pace and used a shared Excel spreadsheet to "claim" problems that they felt comfortable presenting in class. They were not supposed to use resources other than Dr. Miya's office hours and each other. Class time was spent with a single student

11 The pseudonym Miya references "Mr. Miyagi" from the Karate Kid. The instructor shared a clip from the Karate Kid on the first day of class and made references to herself as the Mr. Miyagi of the class multiple times. She saw herself as a guide to the students on a journey that they would not fully understand until they reached their final destination (much like the Karate Kid washing windows and painting houses to prepare for his karate tournament).

presenting their proof on a whiteboard and engaging in a collaborative process of revision with their peers. Dr. Miya primarily stayed silent during these conversations, and she carefully chose when to engage in conversation mediation, small group activities, or extended explanation of a topic. Grades were based on class participation, weekly homework, reflective journal pieces, and a final portfolio that combined mathematics and creative writing to summarize their classroom experience.

During the second week of March 2020, colleges around the world made the decision to transition to remote learning due to the COVID-19 pandemic. The university in which this study took place is on the semester system and gave students and faculty two days off from school in between Week 9 and Week 10 (out of 16 total) to prepare for remote learning. After the emergency remote transition, the class had to develop a new way of maintaining their IBL structure. The professor opted to keep the original MWF class time for optional synchronous class sessions, although every class had full attendance – that is, while there were options for students to participate non-synchronously if needed, all of the students ended up participating synchronously online during the regular meeting time. Proofs were posted on a discussion board the day before class, where students had the option of posting comments on the work. Class time was spent with students explaining their proofs through a shared screen and discussing comments left on the online proof discussion boards.

6.3.2 Data Collection and Analysis

The data for this paper comes from a larger collection of classroom observations, individual student interviews, and interviews with Dr. Miya[12]. The mathematics department considers Real

12 Due to the uniqueness of this class and the Spring 2020 term information, I have chosen not to disclose any other identifying information to protect participant identities.

Analysis a capstone class for their majors and it is considered to be one of the most difficult classes offered. The class had 19 students, mostly junior or senior math majors, and met for three hours a week over the course of a 16-week semester. I observed and video recorded every class, writing detailed fieldnotes and flagging specific episodes that would be worth returning to for video analysis. Five students were recruited for 3-4 individual interviews over the course of the term, during which they reflected upon and discussed their classroom experiences. I interviewed Dr. Miya four times as well, and during those interviews she talked about the challenges and rewards of teaching this class and her perspectives on classroom episodes that I had flagged.

The main goal in analyzing this data was to explore notions of classroom community development over the course of the term. For the results reported in this paper, I drew on data that could illuminate what students remembered as important experiences over the term. I used final interviews with students and the instructor as a starting place, as these interviews were summative of the term. In particular, I analyzed interview questions where I asked students to recall moments in which they had affective experiences (for example, “What is a moment in class where you felt proud/frustrated/challenged?”) or were talking about any of the creative assignments from the term. Specifically, I coded student responses to these questions using the eight dimensions of rehumanizing mathematics as described in Section 6.2.2. I then turned to the classroom observation data for two types of analysis. First, I returned to any coded student interview segments that mentioned a specific classroom episode and transcribed that segment for data triangulation and to add further context to the student’s interview response. I then went through a number of classroom episodes that I had starred, such as creative assignment days or class episodes that had intense affective aspects, and coded these for the dimensions of rehumanizing mathematics as well. In particular, I coded the interview responses and classroom episodes based on student language that fit the name or description of the dimension (i.e., “living”, “emotion”, “create”, “own”) or my best educated interpretation of the situation (i.e., if I had starred a classroom episode as the “participation/positioning” dimension, I explained how I was interpreting students being positioned or participating as mathematical authorities).

I note that the dimensions of rehumanizing mathematics are not mutually exclusive, and in some cases a single episode carried traits of multiple dimensions. I used my best judgement and understanding of the dimensions to code for the strongest dimension present and make note of other contributing factors. In the following quote, the instructor was describing a moment in class where a student had come up with a new theorem and proof based off of a student’s proof presentation.

Miya: *Um Parker, after her thing, she told me "oh yeah I joked to my brother that someday I would have a theorem named after me" and I was like "well did you take a picture of the board?!" cause her name was on the board. And she said no, and like Jo was absent that day or something, and I was like "Rose took a picture of it, I'm gonna get it for you so you can share it with your brother". And like that's meaningful.*

Int.: *Mm-hmm.*

Miya: *That's an accomplishment that people I think, when you're a student you don't know what it means to have something named after you until it happens. And so I think it's a way to honor people's contributions in a way that is authentic. Right? That is something they contributed to the class that we may not have gotten otherwise. Um, you know I probably think too much about these things.*

When coding this segment, I saw three potential dimensions that could apply: participation/ positioning, ownership, and creation. This was an example that considered a student’s volunteer participation in the classroom, and how the instructor was positioning them as a mathematician by helping the student show her brother that she had a theorem named after her in class. However, participation and positioning happened so consistently throughout my dataset, that I had much stronger examples than this one to showcase the dimension. I also saw aspects of the dimensions of ownership, in that a student was motivated to come up with her own theorem and the simple fact of naming it after her implied some level of ownership. Upon close consideration, I decided that while ownership was potentially occurring under the surface in this episode, I didn't have the perspective of the student to know whether it brought her the sense of joy and satisfaction that is a clear indication of ownership and that was noticeable in other data excerpts. Ultimately, I decided that the dimension of creation was the strongest fit for this excerpt, especially when I considered the instructor's words "*… And so I think it's a way to honor people's contributions in a way that is authentic. Right? That is something they contributed to the class that we may not have gotten otherwise".* I understand the dimension of creation as allowing students to engage in mathematics in ways that fit their own values and bring unique contributions to the classroom, and the lens of rehumanizing mathematics helped me to articulate one way in which an instructor was manifesting this dimension within her classroom. Notably then, my process of coding was iterative and I had to go through the dataset multiple times as I cross-compared excerpts to make sure I was producing the clearest possible examples of each dimension of rehumanizing mathematics.

After I completed coding for the dimensions of rehumanizing mathematics, I began to align the themes that arose from my coding with the IBL structure and the instructor's facilitation of the course by connecting back to the four pillars of IBME as discussed in Section 6.1.1. For example, the excerpt above from the instructor interview emphasized a relation between the second pillar (students must have opportunities to collaborate) and how an instructor can pick up on those collaborations and frame them in ways that showcase students' creations and contributions to the class in a worthwhile way. I kept detailed notes of these connections and used them primarily to assist in writing a Results section that used both the language of rehumanizing mathematics and the IBME pillars.

I do not claim that the episodes discussed in the Results provide a comprehensive set of ways in which rehumanizing mathematics appeared in the classroom or could appear in other classrooms, but rather they are a set of illustrative examples. Each example is given as both proof of existence of the dimension and as a concise, potent, best case example from my data collection and analysis.

6.4 Results

My initial interest in using a lens of *rehumanizing mathematics* came from a desire to understand what made this class such an appealing and effective learning environment for students throughout the emergency remote transition due to COVID-19. Overall, I found evidence that the remote transition for this class was successful and carried a significant amount of value to the students. For example, from one student's final interview, consider Taylor's reflection when I asked him to consider what he thought Dr. Miya's overall goals for the class were.

Taylor: To just think deeper, to not always to assume something to be true. I think we more explicitly wanted to become better proof makers, proofreaders. She wanted us to be better mathematicians after this class. I think that's more specifically really pertaining to math, but I guess more philosophically or just as a human person, just have empathy for people. This time was really rough with COVID, so I think this class kind of transitioned more towards emotional support and making sure people are there for each other. And this class is a very collaborative, integrated course where we all talked with each other, so I think that even emphasized the point of having emotional support and connection for each other… The other classes I had this term… were very quiet… the energy was very dead… So it kind of kept my sanity knowing that I could still come to this class and talk to people and joke around. It was nice to have that.

This is just one quote that exemplifies what I saw across the data, namely a class that had developed impressive ways of managing the difficulties of COVID-19 by leveraging their classroom community. In this Results section, I aim to account for this phenomenon, using Gutiérrez's rehumanizing framework to present data. In Section 6.5.1, I consider each dimension of rehumanizing mathematics, and I provide corresponding episodes over the term that exemplify how the instructor went above and beyond the IBL structure to provide a rehumanizing experience for her students. In Section 6.5.2, I consider how these dimensions led overall to a successful remote transition and powerful final day of class for students.

Importantly, I acknowledge that there were specific affordances of Dr. Miya's identity (white, cis-gender, female) and position as a tenured faculty member that enabled her to take on and encourage some of these rehumanizing episodes that are not accessible to every college faculty member. Further, I acknowledge that not all students may have felt equally free to share and engage in these episodes; I recognize that there are complexities involved with inviting emotion, participation, creation, etc. in the classroom and this is certainly something that involves nuance and care on the instructor's part. I hypothesize that some of this intentional instructor work occurred through personal e-mail communication and office hours with students, which was outside the scope of my data collection. However, I report on rehumanizing mathematics as something that was established in this particular classroom as a whole, and believe that this paper offers a useful case as we seek to understand how rehumanizing can be developed and explored in IBL classrooms. From analyzing student interviews, and class interactions, the data suggest that overall students were encouraged and uplifted by the episodes and examples that I share in this paper.

6.4.1 Dimensions of Rehumanizing Mathematics

6.4.1.1 Participation/Positioning

The dimension of *participation/positioning* considers how students participate and position themselves and others within the classroom, and whether they are given opportunities to act as mathematical authorities. One way in which an IBL class can promote *participation/positioning* is through the structure of class, where the majority of time is spent with students presenting their proofs at the board and receiving feedback from their peers. I provide a key example of how this worked in the class I observed, which captures both the high level of student participation in class, and a moment in which the instructor positioned a student, Sloan, as a mathematical authority figure. For context, this was the second day of proof presentations, and during the previous class, Sloan had offered what she called the "Law of Cancellation" to justify cancelling like terms on both sides of the equation in a proof. Classroom Excerpt 1 starts at the beginning of the second day, when Sloan retracts the Law of Cancellation because the use of the law did not match their IBL class expectations.

Classroom Excerpt 1

Miya: Um anything for the good of the order? Sloan?
Sloan: When I talked about the cancellation law, I don't know what I was talking about Wednesday. But y'all believed me and that was nice [students laugh]. But you can't use it!
Miya: Right? Right, like we can't do that. We could, we could establish a law of cancellation. And it can be whatever we want it to be. ***We could make up a law and call it Sloan's Law*** *or call it the Law of Cancellation. And then we could use it, that's legit. But we can't just write it down because the words came out of Sloan's mouth. Like "oh it's the law!"* ***You said it with such authority that I was like "that must be true"*** *(people laugh) Right?*
Sloan: No one said anything against it!
Miya: Yeah I was out on a run yesterday and I was like "did Sloan say law of cancellation?" Then I realized, and like I wrote it in my notes "by the law of cancellation", I bet you did too!... I wrote it down and then I was thinking about it a day and a half later and was like, I have no idea what she was talking about. I accepted that as fact, right? Talk about accepting as true. I accepted it as true without actually looking like is that something we're accepting as true? And it turns out, ***when I emailed Sloan she's like "yeah we don't have that thing". So, I'm like "ok then we don't have that thing".***

In this excerpt, Dr. Miya positioned Sloan as an authority figure in several ways. First, she let Sloan bring up the mistake to the class herself, and later on referenced how she had emailed Sloan and asked for her opinion, as opposed to just telling her and the class that citing an unknown law was not allowed in their proofs. She also claimed that the class could make up a law and attribute it to Sloan, and that Sloan had such an authoritative voice in the classroom that she, the instructor, had believed her without justification. Thus, while the IBL structure of the class may have facilitated Sloan's initial suggestion of the Law of Cancellation during a proof presentation, the instructor navigated the conversations afterwards in ways that explicitly positioned Sloan and her classmates as mathematical authorities.

As a secondary excerpt, I share part of a class conversation from later in the same class period in which a student named Connor questioned whether another proof was viable because it used addition on both sides of the equation, similar to the Law of Cancellation. I share this excerpt to point out that even on the second day of class, students were positioning each other as sources of authority who could decide whether or not proofs were acceptable in their classroom. In this case, Connor's point was taken up by his classmates and they agreed that the proof was not viable due to its similarity to the Law of Cancellation.

Classroom Excerpt 2

Connor: So, after thinking about it, if we look at 1 in another way, a+b=a+c and then you basically use the Law of Cancellation. But if we think about the Law of Cancellation conversely, so we are cancelling an equal number of things on both sides, can we also add an equal number of things on both sides? Do we also need to prove that?
Miya: Can you come to the board and write down what we would need to prove?
Connor: So basically, we would need to prove this line. (underlines line of proof)
Jo: Like do we need to prove if we're able to do that
Sloan: That we can add to both sides
Connor: Because we said we needed to prove that we can cancel on both sides, for this problem. So, do we also need to prove if we can add things on both sides?
Sloan: Oh no.

This episode describes one example from the beginning of the term of the type of *participation/positioning* that occurred in the classroom. Again, I view this activity as being facilitated by the IBL structure of the class, notably it was already present on only the second day of class, given the requirement of having students present proofs at the board and give feedback to their peers. However, I also interpret the instructor as going above and beyond the IBL structure in the way she explicitly positioned Sloan as a mathematical authority figure in the classroom in Classroom Transcript Excerpt 1. Thus, while the dimension of *participation/ positioning* fits well into the structure of an IBL class, it took explicit work on the part of the instructor to position students as mathematical authorities and make it in some sense a social norm of participation in class. For more detail on the students' work with the Law of Cancellation, see Paper 1 (Chapter 5).

6.4.1.2 Ownership

The dimension of *ownership* focuses on students finding joy in doing mathematics for themselves, as opposed to doing mathematics for the external goals of grades, teacher approval, or to fulfill a major requirement. While the students I observed still had many of these external motivators, I argue that this dimension is inherent to the IBL structure by way of facilitating students to do mathematics on their own, with limited resources. In Interview Excerpt 1, from Ash's first interview of the term, I asked what the most rewarding part of class was so far. In this excerpt I highlight how much enjoyment Ash got from doing math on her own, and that she saw solving problems as a reward in and of itself.

Interview Excerpt 1

Ash: *I love getting a problem. That's SO cool. I remember like two super distinct moments. One was when I was with Sloan, we were doing homework in like the first or second week. And we were both just like spit-balling ideas, and we figured it out! And it was so cool!* ***Like we didn't have to ask anybody else, we didn't like look anything up, we just had an idea and started doing it and it was cool.*** *And then the other day, me and Hayden, this was problem 13, me and Hayden and Jordan were all at Hayden's house and we were doing problems, and it's the night before they were due, so we had to do them -*

Int.: *Mm-hmm. (laugh)*

Ash: *We had to get them. You know, and like Sloan said this idea, she was like "you know I think it's gotta be something with like a+1 because a is a and then plus 1 is strictly greater than a" and I was like "ok ok ok" and then* ***I started like writing stuff down***

> ***and then I was like "oh my god I think I have it" like cool, and then we were all like "yeah that's gotta be it!", so that was cool.*** *Because like I felt like I did it, but like I had, like the intuition was given to me by like communicating about it which was super cool. And I liked that because I feel like, I don't know in the real world you don't really have to do everything by yourself.*

This excerpt shows one way in which students can find *ownership* in an IBL class, namely through doing mathematics themselves and “getting a problem”, an activity that occurs outside of class time. Notably, on the first day of class, a student asked the instructor what her favorite part of this class was, and she expressed a similar sentiment to Ash’s interview response, and continued to encourage this sort of reaction from her students. I argue that this type of experience is a necessary consequence of the IBL class structure because students had to spend time out of class coming up with proofs for their presentation credit in class. Furthermore, the instructor set up the class with the requirement of using no outside resources, other than their peers and the instructor, which meant that students were developing these proofs primarily on their own, and sometimes with the help of their peers or the instructor. I see this decision and explicit encouragement on the instructor’s part as helping in promoting the students to see themselves and others to find joy through ownership of their mathematical content.

6.4.1.3 Creation

The dimension of *creation* considers whether students do mathematics in ways that align with their values and are not simply a reproduction of work from a textbook or a professor’s lecture. This fits into the structure of an IBL class where, again, students are creating proofs on their own without resources such as a textbook or lecture. However, I argue that Dr. Miya went beyond the IBL class structure to intentionally honor student creativity in class, by giving students credit for their unique mathematical ideas. For example, in Week 9, two weeks before moving online, after finishing a proof from the packet that “*if a sequence is increasing and* *bounded above, then it converges*", a student named Parker theorized that "*if a sequence is increasing and bounded above, then it converges to its supremum*". The class worked together on the spot to prove the conjecture, and another student conjectured that a decreasing sequence, bounded below, converged to its infimum. Dr. Miya wrote down the proposed conjectures on the board using the names of the students who proposed them, for example "Parker's Conjecture", and referred to the conjectures by name from that point on. The following excerpt is from an interview with the instructor after this class period had occurred. I asked her about her reasoning behind this moment in class.

Interview Excerpt 2

Miya: *Um Parker, after her thing, she told me "oh yeah I joked to my brother that someday I would have a theorem named after me" and I was like "well did you take a picture of the board?!" cause her name was on the board. And she said no, and like Jo was absent that day or something, and I was like "Rose took a picture of it, I'm gonna get it for you so you can share it with your brother". And like that's meaningful.*
Int.: *Mm-hmm.*
Miya: *That's an accomplishment that people I think, when you're a student you don't know what it means to have something named after you until it happens. And so I think it's a way to honor people's contributions in a way that is authentic. Right? That is something they contributed to the class that we may not have gotten otherwise. Um, you know I probably think too much about these things.*

In the excerpt, Dr. Miya shared her belief that adding students' names to their ideas was meaningful and an important way to show that their creativity was valued in class. While this example also speaks to another way that Dr. Miya *positioned* students as mathematical authorities in class, or gave them *ownership* over the material, I use it primarily to highlight the dimension of *creation*. Not every IBL class using these materials would have the spontaneous moment of a student theorizing the exact claim shared above, and the instructor thought it was important to pick up on and emphasize these unique contributions in real time. In this way, the instructor valued *creation* beyond the IBL structure of the class, by recognizing the importance of giving students opportunities to create and do mathematics of their own design and giving them credit for this work, in class and in front of their peers.

6.4.1.4 Living Practice

The dimension of *living practice* encourages showing mathematics as dynamic, with rules that can break, and dependent upon those interacting in the system. In some ways this can be part of the IBL class structure as students necessarily provide feedback, debate ideas, and experience the growth of their mathematical content knowledge together. However, I highlight an example in which I believe the instructor took the idea of *living practice* a step further by providing students with an opportunity to engage in a novel way with the creation of a definition. "*Friend of a Set*" was an instructor-led whole class activity with the end goal of defining the term limit point[13], on the last day before transitioning to remote learning. This activity was based on, and occurred after, students watched a Youtube video[14] of a similar nature where someone had people guess his rule for creating a sequence of numbers based off of the example "2, 4, 8" (the rule was simply that the numbers were in increasing order). The instructor posed the activity as asking the students to figure out her rule or definition of a "friend of a set" (i.e., a limit point). She started by giving a few examples, such as "*7 is a friend of the open interval (7,9)"* and "*4 is a friend of the closed interval [3,5]".* Students then offered more potential examples and the instructor would verify with a yes/no whether their example satisfied her definition, and she would also verify with a yes/no student guesses of her definition. What students found was that they were more apt to provide examples that fit their belief of what a "friend" was, as opposed to coming up with examples that went against their assumptions to test

13 A point p is a limit point of a set M in the real numbers if every open interval containing p also contains a point of M different from p.
14 Veritasium: Can You Solve This? https://www.youtube.com/watch?v=vKA4w2O61Xo

whether their beliefs were actually true, even after watching a video that demonstrated the same idea. In the excerpt below, Sloan reflects on what she took away from this class period.

Interview Excerpt 3

Sloan: *Um but then I found myself, as we were playing the limit point game, of like coming up with questions that didn't break what we had previously, that didn't attempt to break what we had previously figured out about it. And so that was int-that was an interesting experience of being like* ***"oh I can look in from the outside and be like break the rule, but then when it actually comes up to me creating questions that break the rule, like that's not my go to."***
Int.: *Mm-hmm.*
Sloan: *Um my go to is to like, and that's the, like I mean that's what you do in hypothesis testing like in science. Is you continually do the same thing to see if you get the same result.*
Int.: *Mm-hmm.*
Sloan: *Which is an interesting difference between mathematics and the rest of STEM. But I guess they try to break the rule too, sometimes. Um.* ***Yeah and so it's just interesting like the way that I generate questions is very much so like confirming what I already know.***

Here I interpret that Sloan was coming to view her process of learning new mathematics in a different way, through actively working against her internal confirmation bias and pushing the boundaries of her preconceived mathematical conceptions. Thus, I saw this class period as an example of the instructor working to shift student perspectives on what it means to do mathematics, that their job was to test the foundations of their knowledge, and that it was sometimes more useful to try and break the system than to try and reinforce their beliefs. This was an aspect of treating mathematics as a *living practice* that went beyond the IBL structure of the classroom; not every IBL class contains activities such as the one described above. It also served as an enjoyable break from the usual structure of proof presentations during class and the students seemed to enjoy working as a whole class to try and figure out the definition together, thus it provided a level of social bonding that was valuable towards community creation.

6.4.1.5 Body/Emotion

The dimension of *body/emotion* focuses on valuing the senses and emotions in the classroom, as opposed to treating mathematics as a removed, logical experience. Through this dimension, students are encouraged to attend to these more intuitive measures of understanding to interpret mathematics and their classroom experiences. In my two examples, I focus primarily on the emotional piece of this dimension, and I note that neither body nor emotion are explicitly described in the IBME pillars or any descriptions of an IBL class structure.

One way that Dr. Miya encouraged emotion in class was by emphasizing the importance of vulnerability and acknowledging one's humanity. For example, her syllabus had a paragraph section titled "*When Life Happens*" that normalized recognizing and making space for one's emotions and life events throughout the term. Dr. Miya also worked to model vulnerability by sharing relatable stories from her personal life. In an interview during Week Four, Ash reflected on how much she appreciated Dr. Miya's stories and how it encouraged the students to be more vulnerable themselves. In Interview Excerpt 4, Ash reflects on Dr. Miya's story of being horrified as a student-teacher while listening to a middle school teacher explain that the idea of multiplying two negative numbers making a positive number was justified by the concept that two wrongs make a right.

Interview Excerpt 4

Ash: *That was really cool, 'cause, like, she opened up and was vulnerable about how she was feeling when she was our age sitting in on this other class. That was cool. So, things like that, where you show your vulnerability or she shows her vulnerability are super important to this style of class because it's a super intense, intimidating class, in my opinion.*

Ash drew attention to an important point here, which is that the presentation focus of an IBL classroom requires an immense amount of vulnerability from students. We will see more of this in Section 6.5.2, where Dr. Miya purposefully drew connections between her students' ability to participate in academic vulnerability by presenting proofs at the board and personal vulnerability by sharing their inner-selves with their classmates. These two types of vulnerability were inseparable to Dr. Miya, and she saw promoting one as intrinsically supporting the other.

To address another aspect of *body/emotion*, Dr. Miya emphasized laughter through every class. She encouraged her students to make jokes and told several funny stories herself. Furthermore, she used humor as a way to normalize how difficult the content material was for the students. Classroom Excerpt 3 comes from Week Seven of the term. While dealing with the difficulties of nested quantifiers, Parker commented on how the class was like "academic IBS" (irritable bowel syndrome), which received a large amount of laughter around the class and most of all from the instructor herself.

Classroom Excerpt 3

Parker: This class is like academic IBS.
Emory: IBS?
(students start laughing)
Miya: Did you just say… (doubles over laughing)
Sloan: That was awesome
Ash: That's funny
Jo: What did you say?
Miya: So that's going on Twitter. In about, as soon as I get back to my office. She said this class is like academic IBS. Tell us more about that. Tell us what IBS is for those in the crowd that don't know what you mean by that.
Parker: IBS is irritable bowel syndrome. So you're in constant stomach pain but there's nothing that sets it off that you can pin point, you can kind of just guess and avoid certain things. But like this class is like "well I'm just gonna avoid that one and like claim the others", but then it just (smacks fist to hand) gets ya.
Miya: And then you're doing like the ten-yard dash to the bathroom
Parker: "what's happening?!"
Miya: Yeah, so you don't poop your pants.
Jo: That's why I take bathroom breaks in this class actually.
(laughter)
Jo: Yeah it's a really good way for me to calm down.

In this excerpt, we saw the students and instructor using humor about bodily functions to make light of and normalize the difficulty of the mathematics they did in class. As in the "*Friend of a Set*" example from Section 6.5.1.4, humor was a way in which the class engaged in social bonding with each other. Thus, two ways in which the dimension of *body/emotion* were included in class was through Dr. Miya's emphasis on vulnerability and laughter. Furthermore, Dr. Miya personally chose to spend class time this way; these actions were above and beyond the prescriptions of an IBL class structure.

6.4.1.6 Broadening Mathematics

The dimension of *broadening mathematics* focuses on providing students with opportunities to engage in mathematics beyond the traditional quantitative reasoning associated with mathematics. In other words, students participate in mathematics through qualitative activities such as reading, writing, and creating art. Like *body/emotion*, this dimension does not explicitly come through in the four pillars of IBME or any description of IBL class structures.

Dr. Miya included several elements throughout class that helped students come to understand their individual identities and the mathematics through qualitative activities. For example, she had them make a collage of photos depicting their change "as an individual and as a student and scholar and human" over their college years, spent a day reinventing the "definition of a definition" as an if and only if statement using an activity about how to define a sandwich, and recreated the epsilon-delta definition of continuity using an analogy to making pancakes (Adiredja, 2019). Her largest activity over the term was having students create a *This I Believe* essay. According to the *This I Believe* website[15],

> This I Believe is an international project engaging people in writing and sharing essays describing the core values that guide their daily lives. Over 60,000 of these

15 https://thisibelieve.org/

> essays, written by people from all walks of life, are archived here on our website, heard on public radio, chronicled through our books and television programming, and featured in weekly podcasts. The project is based on the popular 1950s radio series of the same name hosted by Edward R. Murrow.

The original plan after writing these essays was for students to read them aloud to each other and friends/colleagues on campus through an open-mic night. However, COVID-19 interrupted these plans, and the students ended up reading their essays aloud to each other over video conferencing. Two example TIB statements from students are, "*Life is worth living through a constant sense of achievement, being creative in exploring new territories, and generating energy from within*" and "*I believe that you are never too old to have a teddy bear sitting snugly on your bed amidst your pillows*". In terms of broadening mathematics, the students clearly saw Dr. Miya's qualitative assignments as central to the class. When asked in her final interview about what she thought Dr. Miya's goals were for the term, Hayden responded as shown in the following excerpt.

Interview Transcript 5

Hayden: I think she – I would imagine she wants us to teach us how to think critically about math and not just necessarily take things that were given and to just accept them as true. I also think she wants us to realize that math isn't always a way in which we... ***Like, the general idea of it, that there's other ways to look at math, which she did through the sandwich definition, this I believe statement, the curiosity cabinet. So, the creative representation of the semester. So, I think just different ways to look at mathematics.*** *And I think just also how important it is to work in a collaborative environment, especially in this field, because oftentimes it's very hard for one person to get somewhere. You need to work in a collaborative environment, because other people view things differently than you do. That's not like a bad thing, it's just a different way of looking at things.*

I interpret Hayden as recognizing a theme around critically thinking about mathematics and the ways mathematics is interpreted, through more than just the mathematical content of the IBL packet, but also through the many qualitative activities. Thus, I see that Dr. Miya engaged her students in the dimension of *broadening mathematics*, through a variety of hands-on creative activities during the term. I reiterate that this dimension is not integral to the IBL structure of the class, which only dictates the ways in which mathematics is done in class and social expectations in class. Thus, this is an example of a way that Dr. Miya went above and beyond the IBL structure of the class to provide a rehumanizing experience for her students.

6.4.1.7 Windows/Mirrors

The dimension of *windows/mirrors* focuses on students recognizing or being able to reflect on themselves (*mirrors*) in class, and to come to understand different perspectives by watching and learning from their peers (*windows*). As my example for this section, I focus more deeply on student work from the *This I Believe* essays explained in Section 6.4.1.6.

In her second interview of the term, soon after the *This I Believe* assignment had been announced, I asked Hayden how she felt about writing her essay. Her response is below in Interview Excerpt 6.

Interview Excerpt 6

Hayden: I'm excited about it because I think one as a class of all math majors, I think it poses this opportunity to see how each of us thinks because I feel like this will definitely... Regardless of how personal you go into it with your story, I feel like it is a personal project because it shows... Stating something you believe without proof is a personal thing. It is going to be different for every person. People might believe the same things without proof, but everyone's going to have different things. And so I think that it allows us to look into how other math people think. And I think that's fascinating because we are very diverse and different group of people that all are in math and do math and think about math in very different ways. And so, I think the opportunity to do this, one in a math classroom, other than I don't know an English or a philosophy classroom, but two, together instead of just on our own, I think just helps strengthen the bond we have in the classroom and also just see how people work and think, which I think would be cool.

From this excerpt, it is evident that Hayden was excited to write her essay, and that she recognized benefits from doing this qualitative assignment in a mathematics class. She saw this as an opportunity to think about mathematics differently and learn more about her classmates, thus giving them another way to bond as a class (in other words, she anticipated engaging in the *windows* aspect of the *windows/mirrors* dimension). To highlight the *mirrors* aspect of this dimension, I share an excerpt from Parker's final interview of the term. In Interview Excerpt 7, I asked Parker to reflect on the *This I Believe* essay experience and she commented on how one student essay impacted her the most.

Interview Excerpt 7

Parker: I've always been more interested in human communications. And so to have the this I believe papers read. ***Jose's really impacted me because I come from a similar background and I never knew that about him****. And I just, even though we didn't talk,* ***I just felt more connected to the class as a community because I finally knew that somebody understood where I was coming from.*** *And especially because he has such great inputs, he just proved to me that somebody... It doesn't matter what background you come from, you can still do great things.*

In this excerpt, Parker was able to recognize her own experience in that of her classmate's and importantly, it made her feel a greater sense of belonging to this classroom (thus engaging in the *mirrors* aspect of the *windows/mirrors* dimension). Listening to Jose's *This I Believe* essay was impactful for Parker, doubly so given that she considered him to be a successful student in class. Hearing an experience similar to her own from a student she considered successful increased her belief in herself being capable of great mathematical work. Adding the *This I Believe* essay assignment to the class was completely outside of the IBL structure of the class and an intentional move by the instructor. In general, I interpret that it can be difficult for any class to engage in the *windows/mirrors* dimension through an activity such as the *This I Believe* essays because it takes a certain amount of *vulnerability* (see Section 6.4.1.5). Thus, I remind readers that these dimensions are highly interconnected and that by engaging in one dimension of rehumanizing mathematics, the instructor was often re-enforcing or promoting another dimension.

6.4.1.8 Cultures/Histories

The dimension of *cultures/histories* considers whether students are able to witness the work of their ancestors and personal culture in mathematics, namely through history and attention to areas of learning such as ethnomathematics. This dimension did not appear prominently in my data collection, which I argue is a product of the structure of an IBL classroom in which the material is removed from any historical context. This is a major component to take into consideration when running an IBL classroom; the intense focus on student positioning as authorities/ownership/creation of mathematical content from a limited number of resources may promote several other dimensions of rehumanizing mathematics, but potentially at the expense of a lack of the *cultures/histories* dimension. Thus, I see a definite call and need for exploration on more ways in which instructors can balance the student-centered nature of an IBL classroom with influence and appreciation for various histories and cultures in mathematics, or potentially through more humanities-based projects like the *This I Believe* essays that focus on broader aspects of culture beyond one's personal beliefs.

However, a classroom that engages in seven out of eight dimensions in rehumanizing mathematics is still providing students with ample opportunities to reshape their relationship with mathematics and one could supplement a class such as this with one that addresses the cultures/histories dimension more fully, whether through teaching with primary historical sources (Barnett et al., 2016) or a class on the history of mathematics using a book such as *The Crest of the Peacock: Non-European Roots of Mathematics.* In Section 6.4.1 then, I have provided several examples of ways in which the combined IBL course structure and the instructor's intentional actions created opportunities for students to engage in rehumanizing mathematics.

6.4.2 COVID-19 and the End of the Term

In this section, I provide evidence that the students continued to engage in rehumanizing experiences during the remote transition due to COVID-19, and I aim to argue that their overall positive transition to remote learning was possible due to the extensive rehumanizing work they had done as a class throughout the first two-thirds of the term. In particular, I share three examples in which the students engaged in different aspects of rehumanization together. The goal of presenting these examples is to show how the class community developed over time and persisted even in the midst of the sudden changes brought on by COVID-19.

6.4.2.1 Students' Give Emotional Support to Each Other

Approximately one week into remote teaching, Dr. Miya opened class with, "Friends, let's check in. How are you doing?" A few students commented on some lighthearted aspects of their first week of staying home, and then Dr. Miya began to dig deeper by saying "let's talk about peoples' anxiety." Jose, who had not been one to engage much in what I determined to be the *body/emotion* aspects of class, suddenly opened up about how stressed he was and his struggles with the transition to online college amidst the pandemic. His classmates responded with an immense amount of support, sharing their own stories of difficulties, trading phone numbers for homework help, and enjoying some fun conversations about life outside of class. Dr. Miya continued to encourage more and more conversation among the students, saying "What else do we need to talk about? It's ok if we don't get to any math today", and in fact the students spent the entire class period providing emotional support to each other. In the excerpt below, Dr. Miya reflected on this day in an interview, discussing how it related to the overall class goal of becoming more vulnerable as a human through academic vulnerability.

Interview Excerpt 8

Miya: So, Jose actually, um, reached out to me a couple days before that. And just said he was really struggling, and he had actually driven home to San Francisco to just see his family *because he was missing them, and he was just really struggling. And he kind of, he opened up with his frustrations about trying to learn online, which he is really struggling with. And trying to be a part of the class. And um I listened a lot. Um and my suggestion was that he reach out to his classmates, because my feeling was that um, that he -what he was expressing was not, that he was not alone in that. And at the time, he was not super comfortable doing that. Like he didn't, I was like "do you want me to ask a question, bring it up?"... Um, so while it was my suggestion, I didn't think he was actually going to speak. When I left my solo meeting with him, my impression was he didn't want to talk any more about it. So, I was really surprised when he like, bared his soul to his classmates. Like it really does surprise me. Because even though I had this conversation with him, I knew sort of that he was struggling and why he was struggling. Um my impression was that he did not want to let other people in. Like it was ok for him to talk to me about it, but he really did not want to admit to um personal struggles to his classmates. So, when he did it, I was really proud of him. Because for me, um what I heard from him was "I need you to be with me right now, and you can't be".*

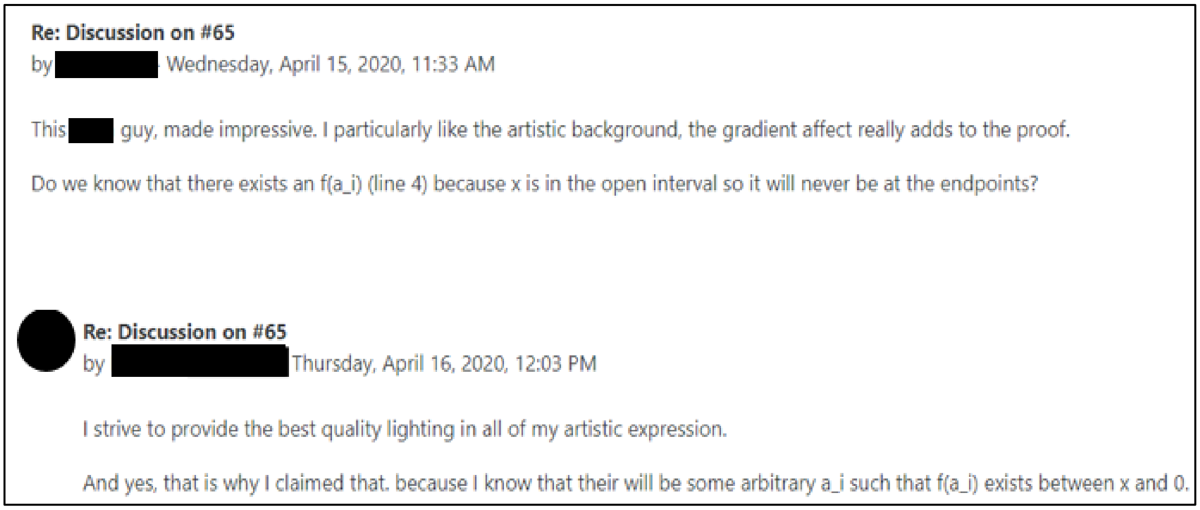
Miya: Um. I think this class, and I think part of what IBL does, is... It allows for students to understand vulnerability. So, when you're at the board presenting something um even if you're sure it's correct, you're in a vulnerable position. And so, the students have, over the course however many now twelve weeks, have gotten to understand because they've all done it, what it feels like to be in an academically vulnerable situation. And my feeling is that the reason that that's important is not for sort of long term academic vulnerability issues, but because my hope is that they can understand that being vulnerable as a human being is a way to grow as a human being. So, it's, and I think that's what Wednesday was about. Was we have created a community of trust around academic issues. I think it's more than that. Let's see. Is it more than that? And the answer was very much yes.

I interpret the moment that Dr. Miya described as her choosing to use the impact of COVID-19 to prioritize and strengthen the emotional well-being of her class (which I characterize as the dimension of *body/emotion*) by encouraging students to express their feelings in class. While many students were familiar with each other at the beginning of the term (as it was a class for juniors and seniors in a moderately sized department), she managed to create a unique space of vulnerability and graciousness that continued to grow over the term. Importantly, Dr. Miya saw this personal vulnerability as an outcome of the academic vulnerability that students had developed with each other through their proof presentations and collaborative work over the term. I hypothesize that all of the class' work on the various dimensions of rehumanization paid off at this moment when a specific student needed the support of their classroom community. I note that this is not to say every instructor should strive or push students to be vulnerable, and that great care must be taken towards a specific student's need and comfort levels. This class had clearly reached a point where such discussions were comfortable, and the students responded extremely positively to Jose's comments, which points to this episode being a powerful example of students engaging in the *emotion* dimension of rehumanization.

6.4.2.2 Students' Treat Each Other as Mathematical Authorities

One major concern for the instructor with the transition to remote learning was how to run an IBL class in an online format, which she had never done before. However, I show in this section that even with the remote transition, the students continued to grow in the dimensions of rehumanizing mathematics around *participation/positioning, ownership, and creation*, by treating each other as mathematical authorities who were capable of creating new ideas during class. In particular, I explicate a day in Week 13 of the term where the instructor and I agreed that the students had fully developed a mathematical community among themselves. While it is beyond the scope of this paper to explicate all of the student conversations that occurred in this class period, I hope to give enough context to provide the reader with a sense of how invested the students were in their learning, without the professor's leadership.

At the beginning of the online transition, the instructor decided to make more use of the virtual learning platform by picking students to present problems a day early and have them post their proofs to the online platform so that others could read and leave comments before class began. The statement discussed in this example is "Prove that if a function f is continuous on $[a,b]$ and there exists an x in (a,b) such that $f(x) > 0$, then there exists an open interval T, containing x, such that $f(t) > 0$ for all t in T". First, the presenter student uploaded a version of his proof to the learning platform the night before class. Six students left comments and questions on the proof before class began, which were then discussed during class time, and the presenter responded to everyone's comments after class as well (see Figure 6-2 for an example). I note that the casualness of the conversations and inclusion of humor shows both the *body/emotion* of mathematics coming into play and provides some evidence of the level of comfort students had reached with each other in class.



Re: Discussion on #65
by [redacted] Wednesday, April 15, 2020, 11:33 AM

This [redacted] guy, made impressive. I particularly like the artistic background, the gradient affect really adds to the proof.

Do we know that there exists an f(a_i) (line 4) because x is in the open interval so it will never be at the endpoints?

Re: Discussion on #65
by [redacted] Thursday, April 16, 2020, 12:03 PM

I strive to provide the best quality lighting in all of my artistic expression.

And yes, that is why I claimed that. because I know that their will be some arbitrary a_i such that f(a_i) exists between x and 0.

Figure 6-2: A student comment and response to Proof 65

During class time, Dr. Miya facilitated the discussion by reading comments aloud, but primarily left the explanation of the comment and its resolution to the students. In particular, one student highlighted an error in the presenter's proof in that his proof technique would not work if the *x* picked was the maximum of the function.

Classroom Excerpt 4

Easton: Yeah so, so line 6 states that there, there exists an n such that f(n) > f(x) and we can pick um, our uh, sequence such that, that while I mean, n isn't even in the sequence, so we can just take a point such that, so this will not work if um if like x equals zero and the function is -x^2.
Jo: Can you say that interval one more time, Easton?
Easton: So, so the, so a counter example
Jo: Ok
Easton: to statement 6 (line 6)
Jo: uh huh
Easton: Is the function f(x) = -x^2 +1,
Jo: Ok

Easton: Where x equals 0, at the point x equals 0.
Jo: Ok
Easton: There doesn't exist a point n around x that is greater than x.
Jo: Ok… So Easton you said if x is zero, um. Yeah. Ok. Cause you can't find an f(n) such that it's greater… Yeah that makes sense… Easton, do you think I could put something in there like, it could be like if f(x), if x is a maximum on the interval, pick an f, pick an n, like pick an f(n) that's like less than f(x) but less than zero? Could I do something like that? To still be able to create an interval?

At this point, the instructor personally messaged me saying, "They don't need me anymore". I interpreted this moment as her saying that the students were now capable of catching each other's errors and discussing them on their own; they did not need her help anymore, she was an observer of the class like myself in many ways. The presenter did not have time to fix his proof in class but put up a revised version for his classmates by the next week that was met with full approval. In sum, this class period showed that the students continued to *position* each other as sources of mathematical authority and had gotten to a place in their class *participation* that the instructor felt like she was a secondary resource in the classroom. Furthermore, they were taking full *ownership* over their mathematical content before and during class, and able to *create* new mathematics together in the moment, in other words, they were working mathematicians.

6.4.2.3 Students' Share Their Experiences on the Last Day of Class

On the last day of class, Dr. Miya asked students to bring some sort of creative summary of the term to share with their classmates. I share pieces from three such creative summaries that highlight aspects of the *body/emotion* dimension, and I interpret the students' eager participation in this creative exercise as evidence that they had internalized the *broadening mathematics* dimension. Figure 6-3 shows part of a thank you letter that one student wrote, in which they thanked every student in the class individually for their unique contributions. I note that the student found different ways to compliment how every student in class helped them whether through providing a *window* into new reasoning, *positioning* each other as sources of mathematical authority who can give feedback, or providing a positive and *creative* attitude during class. Finally, I note that reading a personalized thank you letter out loud to one's entire class is a deep act of vulnerability and made several students in class cry and/or smile.

> As my summary of this course, I tossed between several ideas but then I settled on something that I thought would accurately represent the time that I spent in this classroom and with these people, so here it goes:
> Thank you too.....
> [redacted] for always starting out your proofs with "well so this is how I thought to do it," you destressed an environment that was easy for all of us to get worked up in.
>
> [redacted] thank you for giving feedback and any and all times, and always making time for me if I individually asked you for help or criticism, it meant and still continues to mean a lot to me.
>
> [redacted] thank you for having the best attitude of this entire class, even if its when people have found something wrong with the first line of your proof! You think on the fly on the board like a teacher.

Figure 6-3: Part of a student's thank you letter

Figure 6-4 shows part of a poem that a student wrote about their experiences in class. This student also had several stanzas that called each student out by name with their unique class contributions. However, I share just the final four lines of the poem as evidence of the deep love and care that this class community had for each other. This poem emphasizes both aspects of the *body/emotion* dimension that I described in Section 6.4.1.5: laughter and vulnerability.

> Proving convergence and limits and continuity,
>
> And laughing till we cry explains real to a tee.
>
> I said I didn't know where or how to start,
>
> But here you go; this is how my real analysis family touched my heart.

Figure 6-4: Part of a student's poem

Finally, Figure 6-5 is a popular meme from the T.V. show *The Office*, in which two pictures are framed as different by the corporation, when the well-informed reader knows that they are in fact the same thing (in this case, both pictures are mathematical definitions of continuity used in an undergraduate Real Analysis course). Memes were highly encouraged by the instructor in the creative summaries, and several of the memes pertained to inside jokes established among the students that were representative of their unique experiences together. This *Office* meme in particular strikes me as an interesting way to engage in *broadening mathematics* through meme culture; only a Real Analysis level student would understand both statements in the picture as definitions of continuity and "get" the meme. Thus, memes *broadened* the ways these students did and shared mathematical knowledge, and in turn show each other the level of mathematical sophistication they had reached.

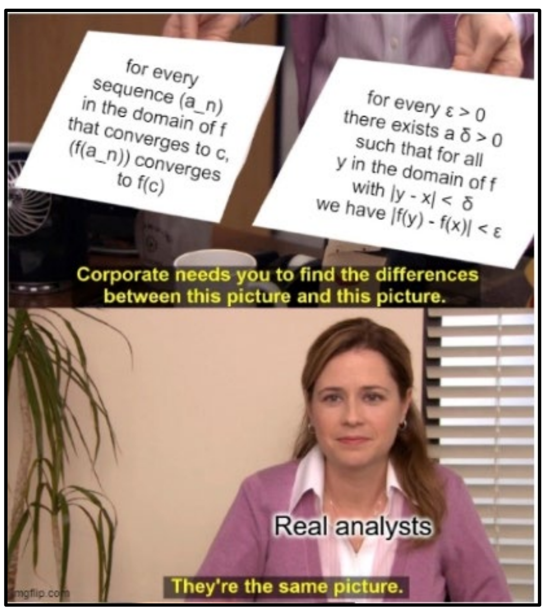


Figure 6-5: A meme that was created and shared by a student

6.4.3 Student Reflections in Final Interviews

In this final section of Results, I share a number of student quotes from their final individual interviews that address what students felt they were taking away overall from their class experience. Before each quote, I had asked students, “What will you take away from this course long term and remember in five years?”. Notably, none of the students talked about remembering specific Real Analysis content in five years, but instead they focused on how this class had larger lessons that they saw as tangible and important to the rest of their lives. I share each quote and provide some commentary on how I see dimensions of rehumanization reflected in their thoughts on the class.

Interview Excerpt 9

Ash: I feel like this class really focused on like who we were in Real Analysis, with all the extra little things we would do. And so, I would definitely take away that like, ***the math that I do is my math****. And I can learn from so many other people, but at the end of the day it's me that's gonna produce something that's worthwhile for me… I think this class was great in like, I think I already said this, in like* ***molding who we are as mathematicians****.*

In this excerpt, I interpret Ash as taking away a deeper sense of *ownership* over the mathematics she created, and that this *ownership* was responsible for an overall growth in her sense of mathematical identity. Furthermore, I note that she answered the question in part using the pronoun “we” in “who we were in Real Analysis” and “molding who we are as mathematicians, indicating that she had established a sense of community with her peers in this classroom. Finally, Ash referenced “all the little things” at the beginning of her quote, by which I interpret she meant the added activities (like *Friend of a Set* and the *This I Believe* essay) Dr. Miya added. Thus, we have evidence that the intentional work of Dr. Miya to include activities beyond the necessity of an IBL structure were beneficial in providing Ash with a rehumanizing experience.

Interview Excerpt 10

Sloan: I mean, I think my biggest takeaway is that ***we need other people and we need other people to be successful and to be successful with us****. If one of us is struggling, then all of us are, and we have responsibility as classmates and as peers to help each other when we're stuck and when people are asking us for help, which I think that's a super cool thing to be cultivated within a classroom, especially a mathematics classroom… And* ***my***

> ***perception of when you would ask who I think is doing the best in the class, it changed throughout the semester*** *because I was realizing that the people who were taking time to really listen to other people's questions, and address them or be like, "I had that same question. Can we talk about that?" Those were the people who were doing well in real analysis, which is not what I initially would have said…*

Here, I interpret Sloan as reflecting on the dimension of *participation/positioning* in class. Throughout the term she found herself changing her perceptions of what success looks like and what it takes to be successful in a mathematics class; namely that it takes *participating* in mathematics with your peers and *positioning* them as authorities that can both answer and ask questions that further mathematical understanding. Recall that the second pillar of IBME states that students have opportunities to collaborate with each other, and in this quote we see Sloan fully embracing this pillar and taking it as a large lesson into her future mathematical endeavors.

Interview Excerpt 11

Taylor: I think it was a very quirky class, it was very different. ***It's definitely one of the courses that I'll remember, when I think about math classes that I took.*** *I think just saying about what we did, like being able to solve really simple proofs, makes it... When I talked to someone about this class and I try to maybe brag about why it's so interesting, I usually go on and be like, "We're trying to prove the basics of math, the things that we, you've taken math since you were a little kid and you understand, but we're trying to understand why that's true." Like why is x*0=0, or why -x actually equals -1*x.* ***We know that's true, but like we have to show it.***

I interpret Taylor's words as addressing both the first pillar of IBME, deep engagement in rich mathematics, and the rehumanizing dimension of *living practice*. In his quote, he emphasized what their class learned about proof and deductive reasoning in mathematics. He found personal empowerment through his new understanding of mathematics as something that could be rigorously proved and debated over with classmates; it was no longer a series of basic facts that one can just assume are true. I also note that he said he would definitely remember this course, which I take to mean that it had a significant impact on him.

Interview Excerpt 12

Hayden: I feel like ***Dr. Miya is a human in the way she runs this class****, is very much kind of put in me that it's okay to fail. It's okay to not know what you're doing. And it's okay to like ask for help, which is things that have not been super easy for me before. So, I think that that's a lesson that I took away from it, is she has a very open and home feeling environment, to where it's like,* ***I don't have to have everything together here. It's completely okay to be like, I don't know what I'm doing****. And I think that's an important lesson for all of us. And I think that's something that I'll hang on to for a while past this class.*

Finally, in Hayden's excerpt, I interpret Dr. Miya as engaging in the fourth pillar of IBME regarding the fostering of an equitable environment, and attending to the *body/emotion* dimension of rehumanization through vulnerability. Hayden described Dr. Miya as *human,* which brings up an interesting question of whether other mathematics teachers run classes in a way that feels if not inhuman, then removed from humanity. Thus, this quote provides evidence of the work required in engaging a class in rehumanization; it is not something we can take for granted as happening in our classrooms.

6.5 Discussion

6.5.1 Summary of Results

In this paper, I have presented an empirically grounded case study that contributes to the conversation of how the dimensions of rehumanizing mathematics can occur in an IBL classroom, and more broadly to IBME classrooms in general. To summarize this work, I recall my two research questions:

1) *In what ways did the instructor use the IBL class structure, and add elements beyond the IBL class structure, to promote dimensions of rehumanizing mathematics with students?*
2) *How was the class' engagement in rehumanizing mathematics reflected in the Spring 2020 remote transition and the end of the term?*

In the first seven sections of 6.4.1, we saw several ways in which Dr. Miya encouraged the rehumanizing dimensions of *participation/positioning*, *windows/mirrors*, *living practice*, *creation*, *broadening mathematics*, *body/emotion*, and *ownership* among her students. Some of these dimensions, like *participation/positioning* and *creation*, were embedded in the IBL structure of the class regarding student proof presentations. In these cases, we saw examples of how Dr. Miya leveraged the IBL format, such as when she helped students to work through the Law of Cancellation themselves in Section 6.5.4.1.1. Other dimensions, like *windows/mirrors* and *broadening mathematics*, were less informed by the IBL structure of the class. Here I shared examples of how Dr. Miya included extra class elements that addressed these areas such as the *This I Believe* essay, which offered an opportunity to engage in a level of personal vulnerability on par with students' intense academic vulnerability with each other over the term. I also explored why the dimension of *cultures/history* did not appear in my data analysis and ways in which that might be problematic or explored in future IBL curriculums. In Section 6.4.2 I gave three examples of how the class exemplified dimensions of rehumanization during their transition to remote learning, as well as excerpts from students' final individual interviews that explored ways in which the longevity of the class in their memory was deeply connected to rehumanizing experiences.

Overall, I interpret that the range of activities and intentional facilitation choices that Dr. Miya put into her IBL class went above and beyond what is required of an IBL class structure (based on the four-pillar characterization of IBME) as well what most students are used to in their mathematics classes, IBL or not. Ultimately, Dr. Miya chose to confront the emotional burden of COVID-19 on the classroom by leaning on the dimensions of rehumanizing mathematics that I saw her build with students over the course of the term. There was more to this class than just Real Analysis content, over time they became a Real Analysis community. Dr. Miya's focus on her students as mathematical authorities, as dynamic mathematicians, as creative humanistic beings, coupled with a classroom full of laughter and vulnerability, brought about one of Gutiérrez's main goals with rehumanizing mathematics, namely that "rehumanizing mathematics seeks to not only decouple mathematics from wealth, domination, and compliance (O'Neil 2016); it also recouples it with connection, joy, and belonging" (Gutiérrez, 2018, p. 4).

There are several takeaways from this snapshot of one inquiry-based learning classroom and their experience with COVID-19. First, this case study emphasizes the importance of setting up norms and activities that promote rehumanizing mathematics early in the term, as they provided a sense of normalcy and support during a crisis such as the emergency shift to remote instruction. The findings suggest that the students' overall experience was improved thanks to the work that the instructor had done throughout the term to create a classroom community. Second, this article offers some specific examples of ways that teachers can engage in the four pillars of IBME through rehumanization. Third, Dr. Miya provides us with several concrete examples that extended beyond what an IBL class structure provides to promote rehumanization. We saw specific ways in which an instructor explicitly sought to find ways for students to feel empowered, to be given authority, and to feel comfortable and confident in themselves as people and as mathematicians.

6.5.2 Limitations and Future Directions

As well as things went considering the COVID-19 emergency, and while the overall experience might have been positive for students, it was by no means a perfect example of equity in action. I certainly acknowledge this class has a much more nuanced story as to how these episodes were developed and experienced over term. Furthermore, Dr. Miya's status as a tenured professor and the students' being junior and senior mathematics majors in an already collaborative department, and a relatively small class size, made this class well-primed for the rehumanization activities discussed here. And, although participation was incredibly rich for a percentage of the class pre- and post- transition, there were a handful of students who chose not to engage in the IBL structure fully. Thus, more work needs to be done looking at IBL classrooms, and IBME classrooms more broadly, to see how these dimensions can be developed in other environmental situations.

Nonetheless, while the specifics of this particular case make it unique in some ways, I have provided an existence proof of what may be possible for an IBL classroom both in terms of reinforcing and adding layers of rehumanization to a mathematics class and in navigating unexpected trials and circumstances (in this case, the transition to remote instruction during COVID-19). In future studies, it would be beneficial to follow students' journeys and growth after participating in a class with intentionally designed rehumanization activities, or follow an instructor's reflections through multiple iterations of teaching a class that engages in rehumanization activities. As Gutiérrez says, "rehumanizing is a verb; it reflects an ongoing process and requires constant vigilance to maintain and to evolve with contexts" (Gutiérrez, 2018 p. 3). Thus, I see this study as the beginning of a long journey to document and grow our understanding of rehumanizing mathematics in relation to IBME classrooms.

References

Adiredja, A., & Zandieh, M. (2020). Everyday examples in linear algebra: Individual and collective creativity. *Journal of Humanistic Mathematics*, *10*(2), 40–75. *https://doi.org/10.5642/jhummath.202002.05*

Anderton, L., & Wright, D. (2012). We could all be having so much more fun! A case for the history of mathematics in education. *Journal of Humanistic Mathematics*, *2*(1), 88–103. *https://doi.org/10.5642/jhummath.201201.08*

Brown, S. (2018). E-IBL: an exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 1–15.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. *In A. E. Kelly & R. A. Lesh (Eds.), Handbook of Research Design in Mathematics and Science Education*, 307–333.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190.

Cook, S., Murphy, S., & Fukawa-Connelly, T. (2016). Divergent definitions of inquiry-based learning in undergraduate mathematics. *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*, 660–665. *https://doi.org/Available from: http://sigmaa.maa.org/rume/RUME19v3.pdf*

Davies, B., & Harre, R. (2007). Positioning: The discursive production of selves. *Journal for the Theory of Social Behaviour*, *20*(1), 43–63.

Dawkins, P. C. (2014). How students interpret and enact inquiry-oriented defining practices in undergraduate real analysis. *The Journal of Mathematical Behavior*, *33*, 88–105. *https://doi.org/10.1016/j.jmathb.2013.10.002*

Dawkins, P. C., Oehrtman, M., & Mahavier, W. T. (2019). Professor goals and student experiences in traditional IBL real analysis: A case study. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 315–336. *https://doi.org/10.1007/s40753-019-00095-4*

Ernst, D. C., Hodge, A., & Yoshinobu, S. (2017). What is inquiry-based learning? *Notices of the American Mathematical Society*, *64*(06), 570–574. *https://doi.org/10.1090/noti1536*

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. *https://doi.org/10.1073/pnas.1319030111*

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Kluwer Academic Publishers, 101 Philip Drive, Norwell, MA 02061 ($69.

Gordon, M. (2019). Mathematics students as artists: Broadening the mathematics curriculum. *Journal of Humanistic Mathematics*, *9*(2), 192–210. *https://doi.org/10.5642/jhummath.201902.13*

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*. *https://doi.org/10.1207/s15327833mtl0102_4*

Gutiérrez, R. (2009). Framing equity" helping students “play the game” and “change the game”. *Teach Excellence Equity Math*, *1*(1), 5–7.

Gutiérrez, R. (2018). The need to rehumanize mathematics. In *Annual Perspectives in Mathematics Education: Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). The National Council of Teachers of Mathematics, Inc.

Haberler, Z., Laursen, S. L., & Hayward, C. N. (2018). What's in a name? Framing struggles of a mathematics education reform community. International Journal of Research in Undergraduate Mathematics Education, 4(3), 415–441. https://doi.org/10.1007/s40753-018-0079-4

Hassi, M.-L., & Laursen, S. L. (2015). Transformative learning: Personal empowerment in learning mathematics. *Journal of Transformative Education*, *13*(4), 316–340. *https://doi.org/10.1177/1541344615587111*

Jennifer M. Langer-Osuna. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education*, *48*(3), 237. *https://doi.org/10.5951/jresematheduc.48.3.0237*

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504–516. *https://doi.org/10.5951/jresematheduc-2020-0043*

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *PRIMUS*, *28*(1), 13–30. *https://doi.org/10.1080/10511970.2017.1338807*

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725. *https://doi.org/10.1016/j.jmathb.2013.04.006*

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406. *https://doi.org/10.5951/jresematheduc.45.4.0406*

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–146. *https://doi.org/10.1007/s40753-019-00085-6*

Lipka, J., Adams, B., Wong, M., Koester, D., & Francois, K. (2019). Symmetry and measuring: Ways to teach the foundations of mathematics inspired by yupiaq elders. *Journal of Humanistic Mathematics*, *9*(1`), 107–157. *https://doi.org/10.5642/jhummath.201901.07*

Matthews, M., & Hodge, A. (2016). Using IBL in a history of mathematics course: A skeptic's success. *Journal of Humanistic Mathematics*, *6*(2), 23–37.

Maxwell, K., & Chahine, I. (2013). Cultural immersion and mathematics teacher education: Explorations in morocco and south africa. *Journal of Humanistic Mathematics*, *3*(2), 62–75. *https://doi.org/10.5642/jhummath.201302.04*

Nunez, R. E. |Edwards. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*, 45–65.

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*, *26*(3), 189–194. *https://doi.org/10.1016/j.jmathb.2007.10.001*

Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006). Capitalizing on advances in mathematics and k-12 mathematics education in undergraduate mathematics:

An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85–93. *https://doi.org/10.1007/BF03036787*

Rasmussen, C., Marrongelle, K., Kwon, O. N., & Hodge, A. (2017). Opinion: Four goals for instructors using inquiry-based learning. *Notices of the American Mathematical Society*, *64*(11), 1. *https://doi.org/10.1090/noti1597*

Shannon, K. (2018). It's not the moore method, but… a student-driven, textbook-supported, approach to teaching upper-division mathematics. *PRIMUS*, *28*(2), 118–127. *https://doi.org/10.1080/10511970.2017.1348416*

Stone-Johnstone, A., Reinholz, D., Brooke Mullins, S., Smith, J., & Andrews-Larson, C. (2019). Inquiry without equity: A case study of two undergraduate math classes. *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education*.

Strand, S. R. (2016). The intermediate value theorem as a starting point for inquiry-oriented advanced calculus. In *ProQuest LLC*. ProQuest LLC.

Tang, G., El Turkey, H., Cilli-Turner, E., Savic, M., Karakok, G., & Plaxco, D. (2017). Inquiry as an entry point to equity in the classroom. *International Journal of Mathematical Education in Science and Technology*, *48*. *https://doi.org/10.1080/0020739X.2017.1352045*

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., … Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences*, *117*(12), 6476–6483. *https://doi.org/10.1073/pnas.1916903117*

von Renesse, C., & Ecke, V. (2015). Inquiry-based learning and the art of mathematical discourse. *PRIMUS*, *25*(3), 221–237. *https://doi.org/10.1080/10511970.2014.921799*

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577–599. *https://doi.org/10.1080/10511970.2012.667516*

7 (Paper 3) Students' Shifting Values and Norms on Proof in an IBL Real Analysis Course

(Submitted to the International Journal of Research in Undergraduate Mathematics Education in May of 2021)

Abstract: In this paper, I provide three examples of how students' perceptions of proof shifted over the term in an inquiry-based learning undergraduate real analysis course. I rely on Dawkins and Weber's (2017) theoretical framework for conceptualizing proof in terms of values, and norms that uphold those values. My work uses student data to exemplify, extend, and elaborate upon the framework and share how an inquiry-based learning classroom provided valuable opportunities for students to engage in proof activity that helps them internalize these proof values and norms. I also share instructor actions in the classroom that were instrumental in this process, as students recalled these moments specifically in final individual interviews where they were asked to reflect on what they had learned about themselves as mathematicians and their proof activity over the term.
Keywords: *proof, proof activity, proof frameworks, values, norms, inquiry-based learning,*

7.1 Introduction

Reading, writing, interpreting, and creating proofs are all notoriously difficult activities for mathematics students to learn (Stylianou et al., 2015) and instructors to teach (G. Stylianides et al., 2017). Dawkins and Weber (2017) theorized that students do not come into class with the same proof values and norms as mathematicians and that these differences in belief can cause difficulty for students when learning how to participate in proof activity in ways that align with the broader mathematical community. To this end, Dawkins and Weber (2017) generated a theoretical framework that puts forth four proof values that mathematicians hold, and several norms that align with those values. Their goal here was to provide researchers a way to reframe students' struggles with proof not as inherent to them or their mathematical abilities, but as a call to the mathematical community to unearth, identify, and explicitly promote proof values and norms in the classroom, in order to make proof a comprehensible activity for students.

Inquiry-based learning (IBL) classrooms provide students opportunities to engage with mathematics material, and many advanced (300-400 level) undergraduate IBL classrooms offer students opportunities to develop their own proof practice, as opposed to passively watching an instructor present proofs. In this paper, I provide data from an IBL undergraduate Real Analysis classroom that sheds light on how students' perceptions of their proof activity shifted over the term, using Dawkins and Weber's (2017) theoretical framework as a guide. My goal is to exemplify, extend, and elaborate upon the framework in the context of a real analysis classroom, thus demonstrating its effectiveness as a framework. Additionally, I share how the IBL structure provided valuable opportunities for students to engage in proof activity that helped them develop these proof values and norms. Finally, I show how the instructor's actions in the classroom were instrumental in this process, as students' recalled these moments specifically in final individual interviews when asked to reflect on what they had learned about themselves and their proof activity over the term. To this end, my research questions are the following:

1) *How did shifts in students' perceptions of proof at the end of the term exemplify, elaborate, or extend Dawkins & Weber's (2017) theoretical framework on proof norms and values?*
2) *What facets of the IBL class structure and the instructor's actions promoted these shifts in perceptions of proof among students?*

Together, these examples of classroom episodes, together with students' reflections at the end of the term, bring a practical lens to Dawkins and Weber's (2017) framework, and they extend contribute to the framework by highlighting more reasons why students may find it difficult to identify with the values and norms of mathematicians with regards to proof.

7.2 Literature Review

In Section 7.2.1, I position inquiry-based learning (IBL) within the larger framework of inquiry-based mathematics education (IBME) and provide a review of IBME literature to better frame the contributions of this study. Then, in Section 7.2.2 I define proof, proving, and proving practice, and define my approach to studying proof in the classroom at the socially-embedded level. Finally, in Section 7.2.3 I provide context for the Real Analysis content addressed in this paper through examples of other studies that have also explored these areas.

7.2.1 Relevant Literature and Characterization of Inquiry-Based Learning

Overall, several large studies have shown that active learning environments, and IBL classrooms in particular, provide improved learning outcomes for mathematics students (Freeman et al., 2014; Laursen et al., 2014; Theobald et al., 2020). There are also studies that have taken a more fine-grain case study approach to capture student experiences in IBL classrooms. For example, Dawkins (2009) looked at the development of sociomathematical norms in a non-traditional (meaning it had elements of inquiry) undergraduate Real Analysis classroom. In his analysis, he found three clusters of sociomathematical norms around valuing visualization, mathematical communication, and developing mathematics (i.e., creating definitions). Other case studies include Hassi and Laursen's (2015) work on students' transformative experiences in IBL classrooms and Dawkins et al. (2019) case study of one IBL teacher's goals in relation to students' experiences. Notably, recent studies have pushed back on the notions of improved learning outcomes by suggesting that IBL classrooms do not necessarily guarantee more equitable outcomes for all students equally (Brown, 2018; Johnson et al., 2020; Stone-Johnstone et al., 2019).

I use the four-pillar characterization of *inquiry-based mathematics education* (IBME) to define the IBL classroom structure I observed[16]. This term was created to unite several strands of inquiry-based teaching and research, including IBL. Notably, the *Academy of Inquiry Based Learning*[17] uses this pillars to define IBL, signifying a large amount of overlap between the two

16 (1) students engage deeply with coherent and meaningful mathematical tasks, (2) students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, and (4) instructors foster equity in their design and facilitation choices (Laursen & Rasmussen, 2019, p. 138)
17 http://www.inquirybasedlearning.org/

terms. IBL itself developed along multiple parallel tracks among mathematicians and practitioners who teach with active learning methods in their classrooms. In this paper, I focus on how students engaged in deep rich mathematics through their proof activity (Pillar 1), and how the instructor inquired into student thinking (Pillar 3) through guided class activities that helped to support shifts students' perceptions of their proof activity. Thus, this study contributes to the IBME literature by providing evidence of ways in which IBL classes afford students opportunities to engage in proof activity, a window into student experiences of proof activity in an IBL class, and practical activities that instructors could add to their IBME classroom to help promote proof norm development for students.

7.2.2 Relevant Literature and Characterization of Proof

7.2.2.1 Defining Proof

Proof is a notoriously difficult practice for students to engage with (Stylianou et al., 2015; Weber, 2001), and researchers and mathematicians to define (Hanna, 1990; Harel & Sowder, 2007a; Raman, 2003). I employ Czocher and Weber's (2020) descriptive definition of proof as a *cluster category*, which they define as "a collection of properties that an object can satisfy to 'count toward' category membership, but no single property is necessary or sufficient for category membership" (p. 59-60). Czocher and Weber's categories include a proof as a convincing justification, a perspicuous justification, an a priori justification, a transparent justification, or a justification that has been sanctioned by the mathematical community. In this work, I focus on the categories "a priori justification" and "justification that has been sanctioned by the mathematical community", from Czocher and Weber's definition, and relate the broader mathematical community to the classroom level as described in Stylianides et al. (2017). Since my work focuses on the social context of proof activity, I additionally subscribe to Dawkins &

Weber's definitions of *proving* as "the activity that a mathematician engages in to produce proofs" and *proving practice* as "the constellation of activities that a mathematician engages in with respect to proof, including producing proofs, presenting proofs to her colleagues, evaluating the proofs that her colleagues present to her, and appreciating, understanding, and learning from the proofs that she and her colleagues produce" (p. 124). Combined, these characterizations of proof, proving, and proving practice allowed me to analyze students' shifts in perception of proof based on activities that occurred at the classroom level.

7.2.2.2 Viewing Proof as a Socially-Embedded Activity

Due to the nature of the IBL classroom and my data collection, I adopt the socially-embedded view of proof from Stylianides et al. (2017). The goal of this perspective is to understand proof as it occurs within a social context, and to question how instruction engages students "in authentic mathematical activity of proving as it is practiced in the mathematical community" (p. 247).

Studies from this perspective have found that "students' perceptions of proof are largely shaped by regularities that students observe in their classroom" (p. 248), that the format of proofs can constrain how students reason in mathematics classes, that "mathematicians usually do not read proofs to gain certainty in theorems but to advance their mathematical agenda" (p. 248), and that students and secondary teachers often see proofs as primarily a conviction tool and not for explanation or communication. Stylianides et al. emphasize that research around "the role of proof within a classroom or mathematical community and how this role might be socially negotiated" (p. 248) is still relatively new. One study that took this perspective is Fukawa-Connelly (2012b), who categorized several social and sociomathematical norms from an Abstract Algebra class around student participation in class proof presentations. These norms included presenter responsibilities (explaining and defending your work, responding to questions), audience responsibilities (reading carefully, convincing yourself, asking questions), and norms such as only using peer-validated knowledge, and working with others (p. 413). However, these norms pertained specifically to proof presentation, whereas my work considers norms that address broader values that mathematicians have when it comes to proof. Thus, my study adds to the literature by providing a new example and way of proof being studied in the context of student classroom activity, using a socially-embedded perspective.

7.2.2.3 Proof Frameworks

One way that students learn to create proofs that conform to the norms of the broader mathematical community is through the use of *proof frameworks* (Selden et al., 2018; Selden & Selden, 2013). They describe two aspects of a final written proof as the formal-rhetorical part and the problem-centered part. They describe the first aspect as "the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results" (p. 308). The problem-centered part is "the part that *does* depend on genuine mathematical problem solving, intuition, and a deeper understanding of the concepts involved" (p. 308). In particular, Selden and Selden delineate between two types of problem-solving that occur when a student engages in proof activity: (1) solving the mathematical problem that takes one from the hypotheses to the given conclusion, and (2) converting one's informal solution into a formal proof (p. 310). In this paper, I contribute directly to the literature on proof frameworks by providing an example in the Results (Section 7.5.2) of how students' utilized a proof framework for proofs using the definition of convergence that had direct implications for their classroom norms on the relationship between formal proof and problem-solving.

7.2.3 Relevant Literature and Characterization of Real Analysis

Real Analysis, often introduced as advanced calculus at the undergraduate level, has been a popular context for studying students' proving activity (e.g., Alcock & Weber, 2005; Lew et al., 2016; Weber & Alcock, 2009) due to it being one of the first proof-based mathematics classes that students encounter. The class is also required of most undergraduate mathematics majors and as a prerequisite for mathematics graduate school programs, which heightens its importance as a field in which we need deeper understanding of how students learn, and thus how we can better teach, the material of the course.

This paper uses examples of students' interpretation and engagement with the following mathematical contexts: formal mathematical definitions, the epsilon-N definition of convergence, and statements with multiple quantifiers. While the examples that address these areas do so for the purpose of explaining how proof norms developed in the classroom, they also provide some insight into how students conceptualize these topics in Real Analysis. In the next three sections, I provide background literature to contextualize these three mathematical areas and how they have been studied.

7.2.3.1 Formal Mathematical Definitions

In Section 7.5.1, I explore how students' perceptions of the relationship between formal definitions and their proof activity changed over the term. To contextualize that section, I provide some background literature and information related to students' thinking about formal mathematical definitions. In explaining why students' have difficulty transitioning to a class like

Real Analysis, which uses formal mathematical definitions, Alcock and Simpson (2002) write the following:

> … what eludes the students is the distinction between a dictionary definition as a *description* of pre-existing objects and a mathematical definition as the chosen basis for deduction, one which serves to *determine* the nature of the objects. (p. 33).

In other words, mathematical definitions are difficult for students because the definitions have been constructed by mathematicians to determine the nature of certain categories of mathematical objects. . Furthermore, students are not necessarily privy to the process of construction of these definitions, which can make the use of such definitions feel ambiguous and disconnected from their reality. Several other researchers have also looked at defining as a mathematical activity (Zandieh & Rasmussen, 2010), and defining as a disciplinary practice (Rasmussen et al., 2015). While I do not focus on students' use or interpretation of definitions as in the previously cited studies, understanding the nature of formal definitions and their use in proof was central to the class I observed.

7.2.3.2 Sequence Convergence and the Range-First Perspective

In Section 7.5.2, I discuss a classroom episode in which a student presented a proof that involved the ε-N definition of sequence convergence (see Figure 7-1), and I analyze the class' subsequent conversations around how to construct proofs that use the definition properly. To frame this section, I now discuss literature on sequence convergence (particularly the epsilon-N definition).

> A sequence $(a_n)_{n=1}^{\infty}$ **converges** to L if for every $\varepsilon > 0$ there exists a positive integer N such that $|a_n - L| < \varepsilon$ for all $N \leq n$. The value L is called a **limit** of the sequence.

Figure 7-1: Epsilon-N definition of convergence as it appeared in class packet

Students' understandings of limiting processes have been studied in depth (Adiredja, 2014; Adiredja & James, 2013; Bezuidenhout, 2001; Cottrill et al., 1996; Dawkins & Roh, 2016; Roh, 2008; Roh & Lee, 2017; Tall & Vinner, 1981) due to the complexity of concurrently relating the domain (in the case of sequence convergence, the sequence index 'n') and the range (again in the case of sequence convergence, the error bound between the a_n element and the limit). Swinyard and Larsen (2012) found "that finding limit candidates and verifying limit candidates involve distinct mental processes" and they discuss "the need for students to shift from the x-first process used to identify limit candidates to a y-first process necessary for verifying that a given candidate is indeed a limit" (p. 466).

This *range-first* perspective considers the interval of possible output values around the limit candidate (by choosing an epsilon), and then determines an index "N" value beyond which the outputs of the sequence fall within the designated interval, which can be counterintuitive for students (Swinyard, 2011). The range-first perspective has influenced several studies on limits and convergence in particular, such as Oehrtman et al. (2014), which looked at how two students reinvented the definition of sequence convergence using approximation schemes and concrete examples. Reed (2017) also used the range-first perspective to explain how one student, Kyle, made sense of the convergence of functions, nothing that the range-first perspective "transformed his understanding of point-wise convergence of functions" (p. 245).

My aim in referencing these studies is to support the fact that understanding the definition of sequence convergence is difficult for students, and that in particular they can find it challenging to coordinate the relationship between the range (epsilon error-bound) and the domain (sequence index '*n*'). The focus of my study is on the way in which a particular classroom episode (and the elements of course design, instructional moves, and student interactions entailed in that episode) supported shifts in students' perceptions of proof. An understanding of the conceptual difficulties students can have with the definition of convergence may prove helpful to the reader when understanding the instructor's actions in class and the distinctions drawn between epsilon and N.

7.2.3.3 Statements using Multiple Quantifiers

In Section 7.5.3, I share a classroom excerpt that emphasizes the placement of the quantifiers "for every epsilon > 0" and "there exists an *a* in *A*" in a theorem. Statements such as AE (for all… there exists) and EA (there exists… for all) are called *multiple quantifier statements*. Several studies have shown that students have difficulty interpreting statements with multiple quantifiers (e.g., Dubinsky & Yiparaki, 2000; Durand-Guerrier & Arsac, 2005). Additionally, researchers have designed tasks specifically to help students interpret statements with multiple quantifiers (e.g., Dawkins & Roh, 2016; Parr et al., 2018; Roh & Lee, 2011, Vroom, 2020a). Vroom (2020b) considered multiple quantifier statements in detail, specifically looking at the grammatical complexities students pay attention to when writing and interpreting multiple quantifier statements, and developed an instructional theory for supporting students "in learning about dynamic processes that are encoded with statements with multiple quantifiers" (p. 7), and how instructors might support students in refining their mathematical statement writing. In particular, Vroom considered how pedagogical content tools (Rasmussen & Marrongelle, 2006) can be used to leverage students' progressive mathematizing when writing mathematical statements. Though my study does not necessarily add new insights to the multiple quantifier literature, it does emphasize ways in which students' work with multiple quantifier statements can impact their perceptions of proof as a whole and I provide an example of the instructor utilizing pedagogical content tools in a similar way to Vroom (2020b)'s conceptual framework.

7.3 Theoretical Perspectives

Given my socially-embedded perspective on proof, I employ Dawkins and Weber's (2017) framework on proof values and norms to understand how students' classroom proof practice developed over the term. Dawkins and Weber developed this framework theoretically, meaning without use of student data, as a way in which to help make sense of why numerous research studies have shown that it is difficult to "apprentice students into the mathematical practices associated with proof" (Dawkins & Weber, 2017, p. 123). Many students find proof problematic or confusing, and in their paper, Dawkins and Weber argue that these struggles stem from students not having adopted the values and norms of mathematicians. In this section, I begin by defining what Dawkins and Weber mean by both values and norms. Then I delineate the values and norms as discussed in Dawkins and Weber (2017) in Table 7-1. More detail on each of the norms pertinent to this paper will be discussed in the Results section.

By proof *values*, Dawkins and Weber mean the axiology of the mathematical community, or the "shared values, goals, and principles that the discipline is trying to achieve in the theories that it produces" (Dawkins & Weber, 2017, p. 125). In general, values are assumed by the community without justification, and may not always be upheld or attained (for example, if the value is an unattainable ideal or in conflict with other values).

Similarly, by proof *norms*, Dawkins and Weber mean the methodology of the mathematical community, or the "acceptable means for developing and justifying theories" (Dawkins & Weber, 2017, p. 125). Norms are meant to be expectations of practice that work to uphold values, in other words they facilitate activity and provide a roadmap which "allow practitioners to practice their craft without constantly evaluating the nature of their craft" (p. 126). Dawkins and Weber note that norms are most clearly recognizable when they have been breached and through observation of how people identify, react to, and repair such breaches. They also note that there is an emergent distinction (Cobb & Yackel, 1996) between the individual and social levels of individual behavior and adherence to community norms. In other words, "one can distinguish an individual's *personal* beliefs and expectations from what she *perceives* to be the values and norms of the community in which they participate" (p. 127).

In Table 7-1, I provide the values and norms around proof as listed in the section headings of Dawkins & Weber (2017); the numbering of norms and values is specific to this paper for ease of discussion. I note that there are no norms associated with the fourth value. I provide a brief description of the norms as needed in the Results section and refer the reader to Dawkins & Weber (2017) for an in-depth description of each value and norm.

Value	Norm	Description
Value 1		Mathematical knowledge is justified by a priori arguments.
	Norm 1.1	Justification in proof must be based on stipulated definitions.
	Norm 1.2	Justification in a proof should be deductive and not admit rebuttals.
Value 2		Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
	Norm 2.1	Mathematical proof is written without reference to author or reader's agency.
	Norm 2.2	A proof is an autonomous object, not a description of a problem-solving process.
Value 3		Mathematicians desire to increase their understanding of mathematics.
	Norm 3.1	Routine calculations and obvious justifications are omitted from a proof.
	Norm 3.2	Irrelevant statements are not presented in a proof.
	Norm 3.3	Published proofs are typeset to reveal their mathematical structure.
	Norm 3.4	Symbol choice follows conventions.
Value 4		Mathematicians desire a set of consistent proof standards.

Table 7-1: Proof Values and Norms, adapted from Dawkins & Weber (2017)

I use this theoretical framework by Dawkins and Weber as my guiding theoretical perspective in this paper. In the methods section, I elaborate specifically how I used the framework to analyze my data, and my results focus on exemplifying and elaborating this framework.

7.4 Methods

7.4.1 Data Site and Class Description

The data for this paper comes from an inquiry-based learning, undergraduate real analysis (advanced calculus) classroom. The class consisted of 19 students[18], at a small liberal arts college in the western United States. The professor had been teaching with IBL methods for over twelve years and is an active member of the *Academy of Inquiry Based Learning*. The title of the course was Real Analysis, and according to the class syllabus, the purpose of the class was to "prov[e] all of those theorems you accepted as true back in calculus." Most of class time was spent with students at the board presenting proofs from a printed set of mathematical content that was given to the students on the first day. These presentations included time for feedback from peers and occasionally the instructor would step to the board to help students unpack a definition or question that emerged during discussion. Students spent time outside of class working alone or with peers on their proofs for class presentations.

The printed materials given to the students were fairly sparse, beginning with the definition of the real numbers, the field axioms, a definition of subtraction and division, and a list of theorems to prove in numerical order. Because the theorems were listed in order, the expectation was that each proof could only be completed using the definitions, and proofs given to that point. The IBL structure of the class meant that students were expected not to use any resources other than the provided definitions and axioms, the instructor's office hours, and each other when writing their proofs. Final class grades were based on class participation, weekly homework

---

18 The students were junior/seniors who were completing either a mathematics major or minor, or secondary mathematics education degree.

assignments of writing up proofs that were presented in class, two exams, and a portfolio that combined commentary on proofs done throughout the term with a final reflective essay.

7.4.2 Data Collection

7.4.2.1 Classroom Observations

I observed the classroom every day of the term, taking detailed field notes of proofs as they were presented and any inferences I had about classroom interactions as they occurred. I also filmed the classroom with a single camera at the back of the room, focused on the front whiteboard where students would present proofs. I sat at the back of the classroom, near the professor, with the camera, and I did not interact with the students or instructor during class time. The rest of the classroom was set up into small table groups of four to five students. Occasionally, as I ran the camera, I would zoom out to capture whole class discussions more easily, but any small group discussions or side conversations between students and/or the instructor were not captured. In general, I used my fieldnotes as a guide to highlight certain video episodes that might benefit from further observation and transcribed these smaller portions for analysis.

7.4.2.2 Individual Student Interviews

In addition to the classroom observations, I completed 3-4 individual interviews with five students throughout the term. These students were picked from selection interviews that were advertised to the whole class based on their interest, availability, and diverse post-college plans. In the student interviews, I used a standardized interview guide (Patton, 2015 p. 344), so that each student was asked the same questions about their course experience. However, I left myself freedom to combine the guide with a more informal conversation approach (Patton, 2015 p. 342) that allowed me to probe deeper if I felt a student had more to say about a particular topic. I made this decision because I believed it likely that treating the interview as a jointly produced conversation would help interviewees give more robust and honest answers about their classroom experiences. My interview questions were designed to draw out each individual's experience of the classroom and how they interpreted others' actions in class. Example interview questions included:

- What are the most challenging parts of the class, if any?
- What are the most rewarding parts of the class, if any?
- How do you feel about presenting proofs in class?
- How do you feel watching others present proofs in class?
- How would you describe the students, professor, and classroom environment?

I also recounted class episodes that I had observed to gain a secondary source of data and opinion on how those events were interpreted by other students in the class. In this paper, I focus primarily on students' final interviews. In this interview I asked overarching questions for the entire term such as the following:

- How has your perception of proof changed over the term, if at all?
- How has your perception of yourself as a mathematician changed over the term, if at all?
- What do you think the professor's goals were for you all in this class?
- What will you remember about this class in five years, if anything?

All of the student interviews were audio recorded and transcribed for further analysis.

7.4.3 Data Analysis

Broadly, I used thematic analysis (Braun & Clarke, 2006) as my method of data analysis. Thematic analysis is "a method for identifying, analyzing, and reporting patterns (themes) within data" (p. 6) that works flexibly among various theoretical frameworks; however, a number of explicit choices need to be made as the researcher to clarify the purpose and direction of the analysis. For instance, my analysis is *theoretical* in that it was driven by my analytic interest in understanding students' shifts in mindset about proof and was informed by the proof values and norms listed in Dawkins & Weber (2017), as opposed to an *inductive,* approach. I also describe my work as *latent,* as opposed to *semantic,* since the themes I identified went beyond the surface level of the data and began to "examine the *underlying* ideas, assumptions, and conceptualizations – and ideologies – that are theorized as shaping or informing the semantic content of the data" (p. 13). I made this choice because I read and analyzed my data from the perspective of trying to draw conclusions about students underlying proof norms from their interview responses. Finally, I approached the data with a *constructionist* epistemology, as opposed to an *essentialist/realist* view, which means I considered that meaning and experience as socially produced, rather than inhering within the individuals. I choose this view because it coincided with how I interpreted the students' interview answers within the socio-cultural context of their classroom and as indicative of norms occurring at the classroom level.

With these distinctions in mind, I went through the six phases of thematic analysis: (1) familiarizing myself with the data, (2) generating initial codes, (3) searching for themes, (4) reviewing themes, (5) defining and naming themes, and (6) producing the result.

I began familiarizing myself with the data by rereading all of the final individual student interviews, making note of any classroom episodes or earlier interviews in the term that students referenced in relation to proof, and I compiled this data as well for further analysis. Once I had a broad understanding of the data that I would be analyzing for this paper, I applied Dawkins and Weber's proof norms as a coding system to use with students' final interviews. I specifically chose the norms for my coding scheme because they related to behaviors of mathematicians that could be identified in the student interviews, as opposed to the proof values which are more philosophical in nature. Based on my rereading of the interviews, I decided to focus on two interview questions, "How has your perception of yourself as a mathematician changed over the term, if at all?" and "How has your perception of proof changed over the term, if at all?". From student responses, I coded for items such as students discussing definitions in proofs (Norm 1.1), discussion of problem solving in relation to proof (Norm 2.2), and discussion around the types of justifications used in proofs (Norm 3.1 and 3.2). I also maintained a list of interesting student interview excerpts that did not fit into Dawkins and Weber's existing theoretical framework and any of my interpretations or reflections on these excerpts. From this initial list of codes, I began developing and reviewing themes of how these norms related to students' overall values of proof, such as students' shifting use of definitions in proofs.

Finally, I read across the broader set of student interviews and classroom observations that I had set aside to help triangulate and verify anything my students said in their final interviews with regards to proof. The classroom data in particular helped me to further define and name themes in relation to the impact of the IBL class structure and instructor's actions on students' shifts in perception of proof. In total, I developed three central themes discussed in Sections 7.5.1, 7.5.2, and 7.5.3. Altogether, I went through several iterative rounds of re-reading interviews and observational data in comparison to what I had written, and iteratively refined these themes to produce the final result.

7.5 Results

Since I coded my data using Dawkins and Weber's (2017) theoretical framework on proof values and norms, and since I view my findings in this paper as exemplifying, extending, and elaborating upon their framework, I have organized the results according to their three values. In Section 7.5.1, I consider Value 1 (*Mathematical truth is a priori*) and provide two examples that exemplify ways in which students attend to Norm 1.1 (*Justification in proof must be based on stipulated definitions*) in their proof writing. In Section 7.5.2, I consider Value 2 (*Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author*), and I elaborate on Norm 2.2 (*A proof is an autonomous object, not a description of a* *problem-solving process*) by providing an example of how students came to recognize the distinction that Norm 2.2 represents. Finally, in Section 7.5.3, I elaborate on how the norms associated with Value 3 (*Proofs should increase mathematicians' understanding*) may carry implicit difficulties for students as they debate what counts as "irrelevant", "routine", or "obvious" in their proofs.

7.5.1 Relating Proofs and Definitions

The first value in Dawkins and Weber's (2017) framework is that *mathematical knowledge is justified by a priori arguments*. This value delineates one primary way in which mathematics differs from other areas in STEM – mathematicians value *a priori* knowledge, which Dawkins and Weber define as knowledge created independently of experience or empirical evidence. I focus on the first norm associated with this value, namely that *justification in proof must be based upon stipulated definitions*.

In advanced calculus, the definitions often involve abstract concepts that cannot easily be visualized – indeed, this is one reason why proofs in advanced calculus can be difficult for students to digest. Instead, definitions are constructed to be representative of a particular mathematical construct or set of objects that relates to a larger mathematical system. Recall that the materials students received to work through contained axioms, definitions, and theorems to prove in a linear order, so a particular theorem could only be justified using material from earlier in the packet. Thus, students were required to engage with definitions consistently and had many opportunities to participate in Norm 2.1, *justification in proof must be based upon stipulated definitions*, which was central to the classroom's proving activity. Due to the students' advanced mathematical level, and the requirements of the class, it seems plausible that they had already established this norm in earlier classes and applied it regularly throughout the term. However, in my data analysis, I found two ways that students reflected on definitions during their final interviews that illuminated nuanced aspects of this norm.

7.5.1.1 Definitions require interpretation and consensus

On the first day of class, Dr. Miya put a slide up at the front of the room stating "*We Understand Calculus Well*"[19] and asked students to write open-ended questions about the statement (not yes/no questions) on Post-It notes. After five minutes, the students transitioned to working with their groupmates and the class as a whole to categorize and discuss their questions. They came up with questions such as "Who counts as 'we'?", "What does 'well' mean?", and "How did we gain our understanding?" In her first interview, Dr. Miya described her purpose in doing this activity on the first day of class as wanting to start the class off in a way that aligned with the IBL structure. In other words, she expected students to be constantly questioning and probing the depths of their mathematical understanding and wanted the first activity of the term to be symbolic of that expectation. In his final interview, Taylor recalled this activity while reflecting on Dr. Miya's goals for the class.

Interview Excerpt 1

*Taylor: Kind of like how you understand what, like how you define something really changes how you're able to categorize things**... If we really want to prove something, we all have to have the same consensus on a definition, we all have to understand the definitions the same way, and the axioms the same way.** That came back to the beginning of the class, you know what the phrase that we did was like… We understand math well, or something like that, or calculus well. And it was just like **how you define those words, changes how you interpret the meaning of the message.** So that's the same with our proofs, like we need to understand the definitions really well. And **we all have to come to the same consensus of what that definition means in order to prove something or to understand how a proof flows logically.***

---

19 The instructor cited the Question Formulation Technique for this idea: https://rightquestion.org/what-is-the-qft/

In this quote, Taylor reflected on how definitions can have multiple interpretations and that to him, an important part of this class to him was that he learned the importance of making sure everyone understood and interpreted definitions in the same way. This level of scrutiny and appreciation for the purpose of definitions in one's proving practice, and how others use definitions as well, reflects notable depth of insight for an undergraduate student. It is also remarkable that Taylor drew a connection between their classroom proof activity and the "*We Understand Calculus Well*" activity, providing evidence that the ways in which norms are set and presented at the beginning of the term can have a lasting impact on students. This episode provides insight into Norm 2.1, by showing a way in which the relationship between proofs and definitions became more salient for a student. Namely, this student considered his ability to write proofs as deeply reliant on his ability to interpret definitions in a way that was in consensus with his peers' interpretations.

7.5.1.2 Using definitions as a starting place when writing a proof

In their final interviews, a major theme across the data was that after taking this class, students were more comfortable in taking the first steps to write a proof. As a prime example, I asked Hayden in her final interview how her perception of proof and writing proofs had changed over the term. Her response is seen in Interview Excerpt 2.

Interview Excerpt 2

Hayden: I feel like this is the first semester in which proof-writing has actually like got into my brain in a point that made sense. Like with discrete, I was like, okay, well, if I know how to start this, and then I would just kind of write until I figured out where I was trying to go and then would be like, yeah, I think that's where I want to go. ***Where now I sit down and I'm like, okay, what am I given and, what definitions go along with what I'm given, what axioms, how much information can I flesh out of what I'm given?*** *And then I need to look back and see if the things we have previously proved that we know to be true, to see which one of those I can arrange in an order in which I can get to where I need to be.* ***So for me, it finally made sense to sit down and be like, okay, here's the definitions I have. Where does this take me and kind of follow a road*** *instead of just*

> *being like, Oh, let me just kind of write like everything I think I know until I get to where I'm going to be, if that makes sense.*

For Hayden, this class was instrumental in growing her understanding of how to write a proof. In particular, she was able to focus in on the definitions and axioms as a starting place for a proof and to allow herself to follow those definitions, and any hypotheses of the theorem, to their necessary conclusions. Not only did Hayden recognize that proofs should depend on stipulated definitions, as Norm 2.1 suggests, but she also saw that definitions were a helpful place to begin one's process when writing a proof. I note that Hayden's use of definitions is also reminiscent of the *formal-rhetorical* part of a proof (Selden et al., 2018; Selden & Selden, 2013), which provides further evidence that the experiences afforded in an IBL classroom lend students opportunities to take on the proof norms and values of mathematicians.

Overall, the rigorous proof-centered nature of the Real Analysis content was prime for students to develop experience with Value 1. One benefit of the IBL structure is that it provided students with multiple opportunities to engage in authentic proof activity and brought their perceptions of the relationship between definitions and formal proofs to be closer to the proof norms of the broader mathematical community. We also saw how the instructor's activity, "We Understand Calculus Well", at the beginning of the term supported this norm development; recall Taylor reflected on this activity and its connection the mathematical definitions in his final interview. This activity provided evidence that a proof norm can be addressed through non-mathematical activities in ways that students' recognize and value.

7.5.2 Relating Proofs and Problem-Solving Processes

The second value in Dawkins and Weber's (2017) framework is that *mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author*. This value follows from the first; an a priori argument should not need to reference a particular author, time, or experience. Overall, the students I observed attended to this value but did not draw attention to it, and I hypothesize that they had internalized this norm to a point where they did not engage deeply with the value during class time (again, they were junior/senior level mathematics majors who had completed an introductory proof course and were used to writing proofs in line with Value 2). However, Norm 2.2, *a proof is an autonomous object, not a description of a problem-solving process*, did arise in final interviews as students considered how their understanding of proof had changed and what they would take away from the course. In the following section, I share an example of how students learned to separate their construct of proof from that of problem-solving, with a process that the instructor called *pre-cognitive scratchwork*, which I recognize as a proof framework (Selden & Selden, 2013).

7.5.2.1 Pre-Cognitive Scratchwork

In the fifth week of class, students began using the definition of convergence to prove that various sequences converge. Recall that the definition uses multiple quantifiers "*for every*" and "*there exists*", along with the sequence index *N*, to determine a point in the sequence after which the terms are sufficiently close to the limit, *L* (see Figure 7-1).

In the following excerpt, a student named Sloan had just finished writing her proof to show - that $\left(\frac{10n-1}{2n+1}\right)_{n=1}^{\infty}$ converges, and she had picked the limit *L=5* (see Figure 7-2). She told her classmates that she was stuck and slightly confused in her proof about how to find a "big *N*". The instructor stepped in and wrote down student suggestions at the board, and the class ultimately rewrote the equation $\frac{6}{2n+1} < \varepsilon$ to find that $n > \frac{3}{\varepsilon} - \frac{1}{2}$.

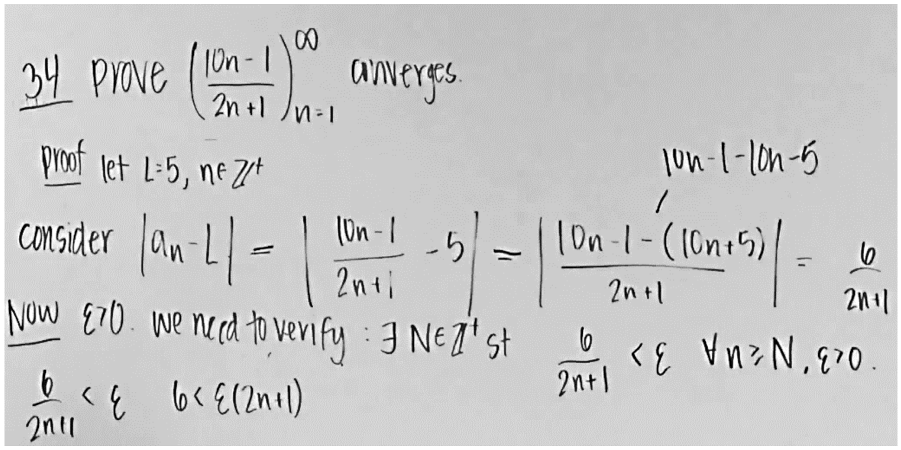


Figure 7-2: Sloan's board work on Statement 34.

This was one of the rare occasions in which the instructor led the class. Drawing on her prior experience of teaching this class multiple times, she seemed to anticipate this moment when students would begin working with the definition of convergence as one in which she would need to offer more guidance. In Class Excerpt 2, she offered this guidance by explaining a process she called "*pre-cognitive scratchwork*". In particular, she identified Sloan's board work as the pre-cognitive scratchwork necessary to find a "big *N*" used in the definition of convergence, and that after this work, the proof would be rewritten to follow the logic of the definition of convergence. In Figure 7-3, we see Dr. Miya's board work of her proof for this problem.

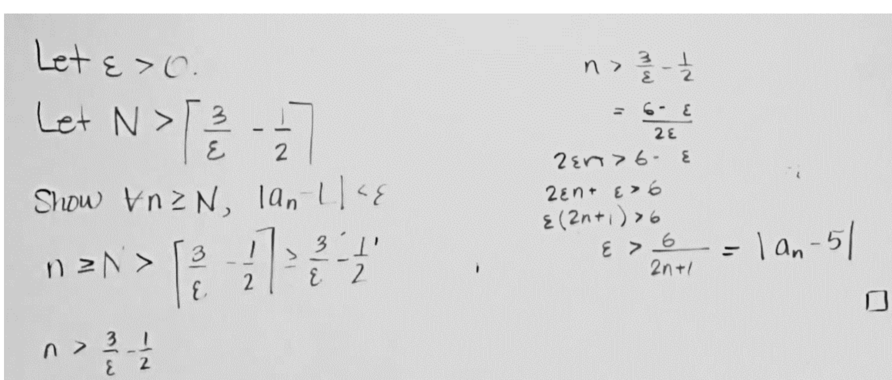


Figure 7-3: Dr. Miya's board work on Statement 34

Dr. Miya's proof started with the end of Sloan's board work by first letting $\varepsilon > 0$, and then picking "big *N*" to be greater than the ceiling of $\frac{3}{\varepsilon} - \frac{1}{2}$. She then moved backwards through all of the algebra that the class and Sloan had completed to end back at Sloan's first line, which is that $\varepsilon > |a_n - 5|$, thus proving that 5 is the limit of the sequence. She talked about this seemingly backwards approach and the purpose of Sloan's work in Classroom Excerpt 2.

Classroom Excerpt 2

Miya : ...This [her proof] doesn't assume anything is less than epsilon, but has this weird like "where did you get that?", and the answer is we got that by working out the problem backwards. Assuming what we wanted to prove, finding what N should be, and then sticking it in and proving that it works. And this step is super important because this is a proof in the right order. Start with epsilon, find big N, and show that beyond that point, we are always close, and close means within epsilon, of our limit. How's that? Easton?
Easton: I've always had a problem with this way of writing a proof
Miya: I know!
Easton: Because we do things, we find the answer the only way we can, and then we pretend like we didn't do it that way.
Miya: Yes and you mentioned a couple weeks ago that induction proofs feel like this to you.
Easton: Yes.
Miya: And I remember you saying that and thinking "oh he's in for it, it's happening".
(class laughter)
Miya: So I agree with you, that it is a little disturbing cause this like comes from nowhere. And for that reason, when you are writing this up, it's helpful to your reader to include your scratchwork. Put it before! Label it as "my pre-cognitive scratchwork"... You're doing the only thing that you CAN do. Right? If Sloan hadn't guessed on the limit of 5, we could not continue the proof. At some point we are making that leap of faith like in Indiana Jones. You are actually guessing something and seeing if it works.

I find this episode to be especially illuminating as to how students might be uncomfortable with certain proof frameworks that are based on constructing the necessary values for the definition (such as the index "N" in sequential convergence proofs and "δ" in epsilon-delta continuity proofs). These processes can feel unfamiliar and contrived to students, as Easton points out in his line "we find the answer the only way we can, and then pretend like we didn't do it that way". Dr. Miya acknowledged this concern among her students, and then as a solution encouraged them to share their problem-solving process with their classmates as part of their mathematical communication, as long as they separated it from their formal proof using the definition of convergence. Thus, pre-cognitive scratchwork allowed for the students was a concrete way to separate their two types of problem-solving into the mathematical problem solving and the formal proof writing. Overall, I see this example as one way of exemplifying why a-contextual proofs, which Dawkins and Weber cite as being valued by mathematicians, can be confusing for students, and one instructor's solution to aid students in this transition to working with a-contextual proofs.

The students held on to this concept of pre-cognitive scratchwork for the remainder of the term, and several students reflected on it in their final interviews. I provide two examples below of student perspectives. First, Ash commented on how separating her problem-solving from proof writing helped her in the beginning stages of proof writing, similar to Hayden's use of definitions in Section 7.5.1.2.

Interview Excerpt 4

Ash: *I think that I have gotten more comfortable in like where I begin. …* ***And the pre-cognitive scratchwork was a really big thing.*** *Um I don't like to write things down if they're not right. And so trying things is so annoying. Because I just want to write it down and get on with it. But so,* ***the pre-cognitive scratchwork seminar that we had in class was helpful, because now I start on like a blank piece of paper and I write down a bunch of stuff and a bunch of ideas before I actually start my proof.*** *And that really helps a lot too.*

Ash found that by recognizing and separating her problem-solving from proof writing using the concept of pre-cognitive scratchwork, she was able to engage with a theorem more quickly and confidently than before this class. While Parker also recognized this distinction between problem-solving and proof writing, she took a different lesson away from the concept than Ash.

Interview Excerpt 5

*Parker: A lot of people, when they think of learning math, they think of somebody up at the board lecturing and taking down notes and you practice with homework. But I think the biggest thing, especially because I want to be a math teacher is math is so much more than just numbers and notes**. It's also about discussing your thought process and embracing every part of your cognitive trail, not just the final product**, especially when Dr. Miya started talking about the pre-cognitive scratch work and how important it is. **And even though it feels like the proof, it's not, but it's still as important.***

Here, Parker saw the benefit of recognizing problem-solving as separate from proof in that being explicit about one's problem-solving was important in terms of mathematical explanation and communication.

In sum, I have elaborated on Value 2 by sharing a way in which the class brought about the use of Norm 2.2. During work on a proof using the epsilon-N definition of sequential convergence, the instructor leveraged a student's question on how to finish their proof to introduce process she called "pre-cognitive scratchwork". Overall, this data provides an example of how Dawkins and Weber's framework can make sense of students' proof difficulties (a-contextual proofs might not be representative of their problem-solving process) and a potential solution to help students solidify the norm (by separating the problem-solving process from the proof via pre-cognitive scratchwork, proof framework (Selden & Selden, 2013) for proofs using the epsilon-N definition of sequential convergence). In Ash and Parker's final interviews, we saw that this activity was impactful on their overall views of proof, particularly in how they began proof and how they saw formal proofs fitting into their proof activity as a whole. While the instructor's use of pre-cognitive scratchwork was not an intrinsic feature of the IBL classroom structure, she did embody the student-centered experience of IBL by waiting to introduce the concept to students until they had an intellectual need (Harel, 2008) for it, as opposed to simply introducing pre-cognitive scratchwork as part of the material for the section.

7.5.3 Relating Proofs and Understanding

The third value from Dawkins and Weber's (2017) framework states that "*mathematicians desire to increase their understanding of mathematics*". They comment that this value is sometimes at odds with the first value of wanting proofs to demonstrate a priori knowledge. A fully formal proof supplies all of the logical steps necessary and leaves less room to appeal to the mathematician's intuition, leaving the proof in danger of being unreadable and supplying less understanding of the mathematics it is trying to convey. Thus, the norms that Dawkins and Weber write pertain to the choices that mathematicians make in their proofs to balance the needs of logical rigor and more intuitive arguments. These norms include omitting routine calculations, justifications, and irrelevant statements, typesetting the proof in conventional ways, and using conventional symbol choices. However, within the socially-embedded view of proof, it can be difficult for novice students to distinguish between what counts as "routine" or "irrelevant", and what is a logical necessity in their proofs.

In this section I provide an example that extends Dawkins and Weber's third proof value, which is that it can be difficult for students to interpret what counts as personal preference or logical necessities when deciding how to write their proof for a reader's understanding or how to give feedback on a peer's proof. I begin by describing an activity from the first day of class. I then share an episode from class in which Dr. Miya referenced this activity, and I provide some insight from student interviews to show how impactful this moment was on them and their subsequent shift in proof values.

7.5.3.1 Dear John Activity

One of the students' first assignments in class was to take an un-punctuated paragraph letter called "*Dear John*" and punctuate it in two different ways that changed the meaning of the paragraph. Figure 7-4 shows two punctuated versions of the letter, the version on the left construes a meaning of ill-will towards John and the version on the right construes a meaning of love and appreciation for John.

Dear John: I want a man who knows what love is. All about you are generous, kind, thoughtful people who are not like you. Admit to being useless and inferior; you have ruined me. For other men I yearn. For you I have no feelings whatsoever. When we're apart, I can be forever happy. Will you let me be? Yours, Gloria.	Dear John: I want a man who knows what love is all about. You are generous, kind, thoughtful. People who are not like you admit to being useless and inferior. You have ruined me for other men. I yearn for you. I have no feelings whatsoever when we're apart. I can be forever happy, will you let me be yours? Gloria.

Figure 7-4: Two Versions of the Dear John Punctuation Activity

Dr. Miya had students read versions of the letter aloud during class and emphasized that just as punctuation matters in their writing, clear communication and attention to detail would be an important part of their proof writing process throughout the term. The important part of this exercise was that it emphasized for students that things that might seem irrelevant, such as a comma or period, can actually change the meaning of an entire paragraph. This sort of attention to detail was especially important for the students to consider in an IBL class where they were in charge of providing mathematical material to their classmates as well as confirming the validity of each other's work as we will see in the next section.

7.5.3.2 Distinguishing between formatting and logic

In the third week of class, students began working with infimums and supremums in preparation for the definition of a limit. The classroom excerpt presented in this section uses the definition of the supremum of a set *A* (see Figure 7-5).

> If u is an upper bound for the set A such that $u \leq a$ for all upper bounds a of A then u is called a **supremum** or **least upper bound** of A.

Figure 7-5: Definition of supremum as it appeared in class packet

In the following classroom excerpt, Easton had just finished presenting a proof of the following statement: "Suppose a set A of the real numbers is bounded above. Show that for all $\varepsilon > 0$, there exists an a in A such that $sup(A) - \varepsilon < a \leq sup(A)$". At the end of his proof, Easton concluded with the statement: "Thus, $sup(A) - \varepsilon < a < sup(A)$, for all $\varepsilon > 0$". Dr. Miya then went to the board and wrote the opening and closing statements of Easton's proof for the class to consider (see Figure 7-6), I note that she uses the symbol ∀ for "for all" and ∃ for "there exists" which were both familiar notation to the students. Again, this was one of the rare occasions in which Dr. Miya went to the board and led the class through an exercise, and I hypothesize that she predicted the students would need her guidance in recognizing the importance of where quantifiers are placed in mathematical statements based on years of teaching the course. This is an example of a generative alternative pedagogical content tool (Rasmussen & Marrongelle, 2006), which helped Dr. Miya transition from a facilitator in the back of the classroom to assuming more responsibility for the mathematical content by taking two different student ideas and asking everyone to compare them. Rasmussen and Marrongelle write that generative alternatives offer "an occasion for students to provide explanations and justification for why they favored one option over another" and that "the mathematical agenda and students' mathematical reasoning [are] furthered through student reflection on their own thinking and the explanations of others" (p. 415).

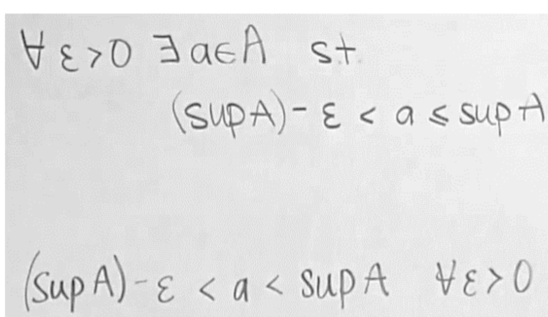


Figure 7-6: Dr. Miya's writing of Easton's opening and closing proof statements

One notable difference between these two statements is the position of "for all epsilon greater than zero" which is at the beginning of the first statement and the end of the second statement. Dr. Miya asked students to discuss in their small groups any differences between the two statements that they saw. When she asked for responses, several students commented on how the first line used *"≤ sup(A)*" and the second line used *"< sup(A)*". Dr. Miya asked students if they noticed any other differences between the two statements and got no responses. She then asked students to nominate someone in their small group to read both statements aloud to the entire group. After reading the statements aloud, a student commented that the second statement did not contain "there exists an *a* in *A*" and, when prompted by Dr. Miya, said that it should go at the beginning of the second statement. I emphasize this part of the class discussion to note that students did not attend to the position of "for all epsilon greater than zero" in either sentence, which drastically changes the mathematical meaning. To draw students' attention to the importance of this difference, Dr. Miya introduced a new example using the set $A = \{1 - \frac{1}{n+1} \mid n \in N\}$. Classroom Excerpt 5 starts with Dr. Miya walking students through different choices of epsilon and finding an element *a* such that *sup(A) – ε < a < sup(A)*.

Classroom Excerpt 5

Miya: So we know 1 = sup(A). Fantastic. Ok what if I say epsilon = ¼… what is an element of the set –
Emory: 4/5
Miya: 4/5. Is between [sup(A) – epsilon] and the supremum. Ok, are we good? Is there any other a? Um.
Taylor: 5/6
Miya: 5/6, 7/8, 99/100 and what have you. Is the supremum in the set? No. Okay so I take a step of length ¼. So if epsilon is ¼, then 4/5 is an a that will work. Because 4/5 is between ¾ and 1. Ok… What if epsilon is… pi/272? How might we find this guy? What are we looking for, what should this thing [a] look like?
Sloan: An element of the set.
Miya: And what do elements of the set look like? n/n+1. We could do this algebraically. 100/101 works? Ok I'll believe you. Will 4/5 work here? No. It's too far to the left. Our

epsilon isn't big enough to put 4/5 in between. ***Epsilon is how big our step is. This says (first statement), we pick an epsilon and then find an a that works. Jordan just got it, that's my favorite moment, when a student is like "oh!". This one says we pick our step size, and we find the a that goes with it. Jordan what does this (second statement). say?***

Jordan: ***It says you pick an a and that it works for all epsilon.***

Jose: ***That's not true.***

Miya: *Jose what are you saying is not true?*

Jose: *Because in our example, like 4/5 for epsilon, I don't know how to word it correctly but I'm thinking it doesn't work.*

Miya: *So, you're right! It can't be that way. You can't find an a that works for all epsilon, unless the supremum is in the set. Then a is the supremum and it works for all epsilon, it doesn't matter how far you walk to the left. But if supremum is not in the set, as you make epsilon smaller, you start excluding things from the set.* ***Think back to punctuation and the Dear John letter. The only difference between this and this is where "for all epsilon greater than zero" sits. Position in the sentence is not commutative. Right? We can't just put "for all epsilon greater than zero" at the end and have it mean the same thing as what it means at the beginning.***

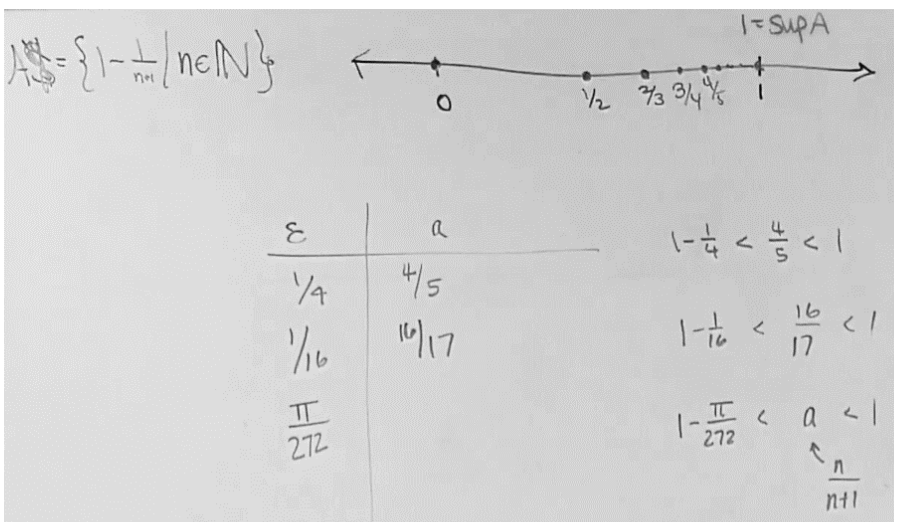


Figure 7-7: Dr. Miya's Example Using Multiple Quantifiers

Dr. Miya started with picking smaller and smaller epsilons and showed that as smaller values for epsilon are chosen, the options for picking a member of the set, a, that satisfies the inequality *sup(A) – ε < a < sup(A)* become more limited (see Figure 7-7 for her board work). Through this process, she showed how the choice of set member *a* depended on the epsilon picked, and so the statement must begin with choosing the epsilon. As she explained this, several students had breakthrough “a-ha” moments, such as Jordan and Jose as seen in the transcript. Dr. Miya reinforced this understanding by referring back to the Dear John activity (Section 7.5.3.1), stating that punctuation sentence structure mattered deeply in their class.

I included this example from the classroom data because in their final interviews, several students reflected on a shift in their perspectives on writing proofs and how much choice is left up to the writer, whether that be formatting, typesetting, or sentence structure. In her second interview, Ash recalled this classroom episode with Easton’s proof.

Interview Excerpt 6

Ash: *Um I remember a moment in class when like Easton presented a proof and we talked about it the entire class. And we talked about the quantifiers and like what position they needed to be in. And um we came to the conclusion that Easton's last line needed to be written differently for the quantifiers to be in the right spot.* ***And then I remember him saying, I remember him saying "oh that was just my preference, I know that it's supposed to be written that way but I just put it on the board like that".***

Int.: *Mmm.*

Ash: *And like that definitely happened. Um I feel like those are really weird moments for me. I don't really know how to respond. Cause part of me is like "did you really know? or like are you just saying that was your preference and you know what they're talking about? Or like, you know?* ***Sometimes I feel like it's easier to like use "yeah it was my preference" instead of saying like "oh yeah you're right, the logic of my proof is confused".***

Here, Ash brought up an interesting point regarding the social dynamics of presenting in front of peers and that the word “preference” may be used to cushion against the embarrassment of writing an illogical statement on the board. This was an issue that the class had to navigate, and learn to read work carefully and delineate between what they personally thought were more intuitive or easy to understand proofs, and whether those proofs accurately represented the formal mathematics they were trying to convey. In a similar vein, Sloan also commented in her second interview on how this class had shifted her understanding of the difference between preference and logic.

Interview Excerpt 7

Sloan: Yeah I think that, it's definitely more of a gray scale than black and white of preference and logic. Um, which makes it harder to decipher which one it is… ***And I think I'm learning that a lot of the things that I thought were preference were actually logic. Um, so now I'm a little, I'm like more open to people saying things****. And I think when you're uncertain as to whether something's preference or logic it's better to say it.*

In particular, Sloan commented that the concept had become more of a gray scale for her, and that "things that I thought were preference were actually logic". Interestingly, this shift in perspective had made her more open to hearing her classmates' proofs and questioning whether a proof needs to be written a certain way. This was a natural consequence of the IBL structure, every student was coming to class with slightly different versions of proofs for the same problem and they had to spend a decent amount of time deciding whether the differences between their personal proof and the one being presented signified a logic error on someone's part, a different approach to the problem, or a more formal/intuitive version of their own work.

I have extended Value 3 by considering the difficulties involved in determining what a proof needs in order to be considered understandable. In particular, I questioned the intrinsic piece of the norms of this value by noting that students may have difficulty perceiving what "obvious" or "routine" pieces can be left out of a proof and furthermore how much choice is left to the proof writer when structuring the proof argument. I started with a classroom excerpt example of the instructor utilizing a generative example pedagogical content tool (Rasmussen & Marrongelle, 2006), through juxtaposing two mathematical statements with multiple quantifiers in different orders. The goal of this generative example was to show students the importance of quantifier placement. In students' interviews after this class period, Ash and Sloan reflected on how eye-opening this moment was for them. Notably, it made them more aware of nuance in proof, the difference between proof writing preference and logical necessity, and made them feel more open to being corrected by their peers. The IBL structure of this class was incredibly beneficial in developing this third value for students because it provided them opportunities to debate and engage in authentic discussion around what in a proof could be omitted or written in different ways. Since the students had no guidance from external resources such as a lecture or textbook it was up to them as a class to determine these qualities. Similar to Section 7.5.1, I shared another activity that the instructor gave at the beginning of the term, “Dear John”, which again points to the potential for instructor’s to develop proof norms through non-mathematical activities in deep and impactful ways for students.

7.6 Limitations and Future Directions

One large limitation of this study was that my observations of students proof activity was limited to the classroom in which they were mainly presenting or giving feedback on proofs. Much of students’ proof activity was occurring outside of the classroom either alone, in study groups, or in Dr. Miya’s office hours. Thus, one natural future direction for this work would be to observe students in more settings outside of the IBL classroom in order to develop a more well-informed picture of their proof presentations in class and to look for other ways in which their norms and values for proof might shift over the course of the term.

In each section of the Results, we saw Dr. Miya provide activities that facilitated shifts students’ in proof perceptions at the end of the term. These were the “*We Do Calculus Well*” activity, the “*Dear John*” activity, the concept of “*pre-cognitive scratchwork*”, the generative example of comparing multiple quantifier statements, and the guided example of using the multiple quantifier statements with the set $A = \{1 - \frac{1}{n+1} \mid n \in N\}$. These activities were spontaneous for her and intuitively developed over years of teaching the course. Thus, one limitation of this study is the small case size and the question of whether the activities would work with different instructors in other Real Analysis classrooms and whether different students would also experience these activities in ways that shift their perceptions of proof. However, another future direction for this study is to use these activities, and potentially others, in more Real Analysis classrooms with Dawkins & Weber's framework as a guiding lens, with the explicit intention of having students reflect on this activities and whether they more generally promote shifts in students' perceptions of proof from the beginning to end of term.

References

Adiredja, A. (2014). *Leveraging students' intuitive knowledge about the formal definition of a limit* [UC Berkeley]. *https://escholarship.org/uc/item/0jk8m1fd*

Adiredja, A. P., & James, K. (2013). Students' knowledge resources about the temporal order of delta and epsilon. *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, 140–152.

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics*, *22*(2), 28–34.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior*, *24*(2), 125–134. *https://doi.org/10.1016/j.jmathb.2005.03.003*

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487–500.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. *https://doi.org/10.1191/1478088706qp063oa*

Brown, S. (2018). E-IBL: an exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 1–15.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *The Journal of Mathematical Behavior*, *15*(2), 167–192.

Czocher, J. A., & Weber, K. (2020). Proof as a cluster category. *Journal for Research in Mathematics Education*, *in press*.

Dawkins, P. C. (2009). *Non-traditional socio-mathematical norms in undergraduate real analysis*. ProQuest LLC.

Dawkins, P. C., Oehrtman, M., & Mahavier, W. T. (2019). Professor goals and student experiences in traditional IBL real analysis: A case study. *International Journal of Research in Undergraduate Mathematics Education*, *5*(3), 315–336. *https://doi.org/10.1007/s40753-019-00095-4*

Dawkins, P. C., & Roh, K. H. (2016). Promoting metalinguistic and metamathematical reasoning in proof-oriented mathematics courses: A method and a framework. *International Journal of Research in Undergraduate Mathematics Education*, *2*(2), 197–222. *https://doi.org/10.1007/s40753-016-0027-0*

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, *95*(2), 123–142. *https://doi.org/10.1007/s10649-016-9740-5*

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics Education IV*, *8*, 239–289.

Durand-Guerrier, V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics*, *60*(2), 149–172.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. *https://doi.org/10.1073/pnas.1319030111*

Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *The Journal of Mathematical Behavior*, *31*(3), 401–416. *https://doi.org/10.1016/j.jmathb.2012.04.002*

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6–13.

Harel, G., & Sowder, L. (2007). Toward comprehensive pserspectives on the learning and teaching of proof. In *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 805–842). *https://math.ucsd.edu/~harel/TowardComprehensivePerspective.pdf*

Hassi, M.-L., & Laursen, S. L. (2015). Transformative learning: Personal empowerment in learning mathematics. *Journal of Transformative Education*, *13*(4), 316–340. *https://doi.org/10.1177/1541344615587111*

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504–516. *https://doi.org/10.5951/jresematheduc-2020-0043*

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725. *https://doi.org/10.1016/j.jmathb.2013.04.006*

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406. *https://doi.org/10.5951/jresematheduc.45.4.0406*

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–146. *https://doi.org/10.1007/s40753-019-00085-6*

Lew, K., Timothy Patrick Fukawa-Connelly, Juan Pablo Mejía-Ramos, & Keith Weber. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162. *https://doi.org/10.5951/jresematheduc.47.2.0162*

Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior*, *33*, 131–148.

Parr, E., Roh, K. H., & Sellers, M. (2018). Teaching the representations of concepts in calculus: The case of the intermediate value theorem. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*. https://doi.org/10.1080/10511970.2018.1540023

Patton, M. (2015). *Qualitative research & evaluation methods: Integrating theory and practice*. SAGE.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, *52*(3), 319–325. *https://doi.org/10.1023/A:1024360204239*

Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *The Journal of Mathematical Behavior*, *20*(1), 55–87. *https://doi.org/10.1016/S0732-3123(01)00062-1*

Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006). Capitalizing on advances in mathematics and k-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85–93. *https://doi.org/10.1007/BF03036787*

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*, *37*(5), 388–420.

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259–281. *https://doi.org/10.1007/s10649-014-9583-x*

Reed, Z. (2017). The effects of the epsilon-n relationship on convergence of functions. *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, 236–248.

Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217–233.

Roh, K. H., & Lee, Y. H. (2011). The mayan activity: A way of teaching multiple quantifications in logical contexts. *PRIMUS*, *21*(8), 685–698. *https://doi.org/10.1080/10511970.2010.485602*

Roh, K. H., & Lee, Y. H. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 34–68. *https://doi.org/10.1007/s40753-016-0039-9*

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematical Enthusiast*, *10*(1), Article 14.

Selden, A., Selden, J., & Benkhalti, A. (2018). Proof frameworks: A way to get started. *PRIMUS*, *28*(1), 31–45. *https://doi.org/10.1080/10511970.2017.1355858*

Stone-Johnstone, A., Reinholz, D., Brooke Mullins, S., Smith, J., & Andrews-Larson, C. (2019). Inquiry without equity: A case study of two undergraduate math classes. *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education*.

Strand, S. R. (2016). The intermediate value theorem as a starting point for inquiry-oriented advanced calculus. In *ProQuest LLC*. ProQuest LLC.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In *Compendium for Research in Mathematics Education* (pp. 237–266). The National Council of Teachers of Mathematics, Inc.

Stylianou, D. A., Blanton, M. L., & Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. *International Journal of Research in Undergraduate Mathematics Education*, *1*(1), 91–134. *https://doi.org/10.1007/s40753-015-0003-0*

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of amy and mike. *The Journal of Mathematical Behavior*, *30*(2), 93–114.

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., … Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences*, *117*(12), 6476–6483. *https://doi.org/10.1073/pnas.1916903117*

Vroom, K. (2020a). A case study of learning how to use and order quantified variables by way of defining. *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*.

Vroom, K. (2020b). *Guided Reinvention as a Context for Investigating Students' Thinking about Mathematical Language and for Supporting Students in Gaining Fluency*. (Doctoral dissertation, Portland State University).

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577–599. *https://doi.org/10.1080/10511970.2012.667516*

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119. *https://doi.org/10.1023/A:1015535614355*

Weber, K., & Alcock, L. (2009). Proof in advanced mathematics classes: Semantic and syntactic reasoning in the representation system of proof. In *Teaching and Learning Proof Across the Grades: A K-16 Perspective*. Routledge.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education IV*, *IV*(8), 103–127.

8 Conclusion

8.1 Summary of Papers and Response to Overall Research Questions

In the previous three chapters, I have shared three papers that emphasized different aspects of students' experiences in an inquiry-based learning undergraduate advanced calculus classroom. Notably, this class was being taught by a highly experience instructor who was confident enough in her IBL teaching to include a number of humanistic or creative activities, sometimes spontaneously, into her classroom. Furthermore, she was able to help maintain the IBL structure of the course through the remote transition due to COVID-19 and facilitate a classroom space where students could continue to develop their classroom community. From this incredibly unique and powerful case study, my overall research questions were the following:

1) *In what ways did the IBL structure of the classroom influence and support interplay between the combined social and mathematical experiences of students in this classroom?*
2) *In what ways did the instructor influence and support interplay between the combined social and mathematical experiences of students in this classroom?*
3) *How, if at all, did the interplay between these combined social and mathematical experiences work to create an overall sense of classroom community?*

I now highlight a few ways in which my three papers addressed these overall research questions and summarize what I am taking away from my study as a whole.

My first question concerned the impact of the IBL structure of the classroom on the social and mathematical experiences of the students I observed. Overall, I found that the set-up of having students work from a minimal set of materials, working with their peers and the instructor, and presenting proofs during class time afforded many unique opportunities. For instance, Sloan's episode in Chapter 5 (Paper 1) with the Law of Cancellation occurred primarily because she was giving feedback to a peer's proof presentation. This resulted in several opportunities for the class to develop and negotiate their social and sociomathematical norms for the term (Cobb & Yackel, 1996). I also visited the Law of Cancellation in Chapter 6 (Paper 2), where I showed how this same episode could be viewed as an example of the participation/positioning dimension of rehumanizing mathematics (Gutiérrez, 2018). Several other dimensions were also accessible to the students via the IBL structure, such as creation (the class provided them opportunities to create proofs), and ownership (the class structure meant developing their own proofs outside of class). Finally, in Chapter 7 (Paper 3) I showed how the IBL structure of the class provided the fuel and necessity for students to engage authentically in proof activity, thus providing them several avenues by which to recognize and assimilate into the proof values and norms of the broader mathematical community. I see that the IBL structure of the class promoted an incredible amount of social activity between the students and professor, through the number of classroom discussions and interactions during proof presentations that I was able to capture in my data collection. Moreover, these social activities were all in the context of doing mathematics, and I argue that the IBL structure with the minimal set of materials promoted this social activity, as students were required to lean on each other to understand and create the mathematics as a class. In sum, my three papers have shown that the IBL structure of the classroom created a space for students to engage in mathematics as a social endeavor through their proof presentations, and in turn these social engagements helped students to develop further in their mathematical selves (through what I saw as their increased understanding of sociomathematical norms, proof values and norms, and dimensions of rehumanizing mathematics).

My second question regarded the nature of the instructor, and how her actions and choices in the classroom impacted the social and mathematical experiences of the students I observed. First, I acknowledge that she made the choice to run an IBL classroom and had a wealth of experience in IBL that made her an ideal instructor to observe. For instance, in Chapter 5 (Paper 1) with the Law of Cancellation episode, we saw several ways in which the instructor promoted the IBL student-centered nature of the classroom, and let the students come to their own conclusions on how to do the proofs, instead of stepping in and offering her own opinions. In Chapter 6 (Paper 2), I provided a number of examples of ways that she went above and beyond the IBL structure of the classroom to infuse her course with aspects of the humanities (such as the This I Believe Essay) or other ways of coming to know the mathematical material (such as the Friend of a Set activity). In Chapter 7 (Paper 3), I also gave examples of how the instructor promoted certain proof values and norms in the classroom through added activities that went beyond the IBL packet materials (such as the Dear John letter activity, and the concept of pre-cognitive scratchwork). Together, these episodes show how the instructor supported the existing IBL structure (which I have already shown was impactful on students' social and mathematical experiences) and extended the IBL structure into something more that offered a deeper level of social interaction and mathematical engagement among students.

My third research question pertained to community development in the classroom as a result of these social and mathematical experiences. While none of my papers used a theoretical framework that explicitly studied community (such as Wenger's communities of practice), I argue that by the end of the term, this classroom had developed into more than just a collection of students and an instructor; they had become a community. From my analysis, I argue that the biggest strength of teaching with IBL, or IBME methods in general, is that the social and mathematical experiences of the classroom occur concurrently. This reflexive relationship strengthens the bonds of the people within the classroom and enhances the boundaries of the classroom into something distinct from the rest of their lives. Every time a student presented a proof, participated in a class discussion, or completed one of the instructor's added activities, they were simultaneously doing mathematics and creating shared experiences with their peers. One of my main examples of this is the relationship between academic and personal vulnerability, which I discussed in Chapter 6 (Paper 2). Presenting proofs was an important aspect of the IBL structure that required an immense amount of academic vulnerability from students, as they shared their mathematical thinking with each other. Over time, these shared experiences built a level of personal vulnerability among students that were supported by the instructor as she shared her own vulnerability with students and attended to the body/emotion dimension of rehumanizing mathematics in her daily actions and activities such as the This I Believe essay. These mathematical and social experiences around vulnerability continued to build throughout the term, culminating in a powerful last day of class, as explored in Chapter 6 (Paper 2). Furthermore, the narrative of Chapter 5 (Paper 1), the other dimensions of rehumanizing mathematics in Chapter 6 (Paper 2), and the shifts in students' values and norms in Chapter 7 (Paper 2) all speak to the level of impact these students had on each other and the types of memories this classroom was capable of creating. In my personal experience, there are few undergraduate mathematics classrooms that attain such a memorable status among the hearts and minds of its' students, and I fully attribute the growth of classroom community that I witnessed in my study to the students' combined social and mathematical experiences that were afforded by the IBL structure and the intentional actions of the instructor.

8.2 Limitations

There were several limitations of my study that may have impacted the data collection and analysis that I was able to complete. In this section I consider limitations of the data collection, the uniqueness of the classroom I observed, and the general limitations of qualitative case studies.

The nature of the IBL structure meant that students were doing most of their proof activity outside of the classroom by themselves, with peers in study groups, or with the professor during office hours. I did not observe any of these locations and it is likely that experiences in these situations were a constant undercurrent to the classroom activity I observed. For example, I might have seen a student's final proof presentation that was widely accepted by the class, but the design of my study did not allow me to know how much time they spent with the professor working out the proof, or how they got ideas from a peer and developed a deeper sense of shared mathematical authority with their classmates via their shared understanding of the proof. I did not interview the professor as frequently as I had originally hoped due to the tumultuous nature of the term. More interviews with the professor would have been a helpful source of data to triangulate my findings and get more insight into her teaching moves and other ways she was working "behind the scenes" to facilitate community and positive experiences for the students. Third, while COVID-19 provided an interesting twist to my data collection and I was able to witness a great deal of rehumanizing during the remote transition, it also meant that my end of term data was quite atypical compared to the usual run of this class. The students did not get to present their *This I Believe* essays at an open-mic night as past years had done, and the students did not progress as far mathematically as they might have otherwise. The remote transition also shifted the ways in which students participated during class time, and some students participated more or less after the transition.

Many of this class' unique aspects stemmed from the instructor I chose to observe. She developed the IBL mathematics materials, the humanistic activities, had extensive expertise in teaching with IBL methods, and felt comfortable sharing her own humanity with students. One large limitation of my dissertation then, is that my focus on students' social and mathematical experiences meant that I did not necessarily have the data to share the journey and experience of a highly skilled instructor. As IBME becomes more popular within the education community, we need stories such as her journey to IBL teaching and more insights into her day-to-day teaching decisions to help other instructors become more comfortable with IBME. I would not necessarily expect an instructor teaching with IBL methods for the first time to have had the mental space or desire to a) include spontaneous humanistic activities throughout the term on top of the already unpredictable nature of an IBL classroom or b) continue teaching with the IBL structure through the transition to remote learning. However, I am also not implying that the experiences of this classroom will automatically happen just because of an experience instructor As a whole, my dissertation has demonstrated what is possible in the best possible circumstances of an extremely experienced, motivated, and effective instructor, which does not undermine the results of the study, but is a limitation that might incorrectly imply that such work is easy or automatic for instructors.

Finally, I acknowledge that this was a case study of one classroom and the results of my papers cannot necessarily be generalized to other IBL or IBME classrooms. In particular, the students in this class were highly familiar with each other and the professor before the beginning of the term. Additionally, many of them had experience with IBL teaching methods with this professor and others in the department (although not to the extremely scaffolded scale of the Real Analysis curriculum). Thus, I saw a fair amount of buy-in at the beginning of the term that other classrooms may not have had, which makes the likelihood of this study generalizing to other classrooms an unknown. Additionally, while COVID-19 was an interesting twist to my data collection, and provided an amazing amount of depth to Paper 2 on rehumanizing mathematics, it made the class an even more unique case study and the results are not necessarily generalizable or indicative of other iterations of this instructor's teaching.

8.3 Future Directions

Because I had so much data (an entire term's worth of classroom data), I necessarily limited myself in what all I focused on and analyzed for this dissertation. However, there is opportunity for further investigation, both based on the papers I did write, and based on the remaining data I have not yet written about. Each of my three papers offered multiple avenues for further research, many of them related to the limitations of this study. In this section, I summarize the future questions raised in each paper and relate them to my overall research questions. Then, I highlight some additional research investigations that seem promising based on my overall data collection.

8.3.1 Future Directions Based on Each Paper

In Paper 1, I considered how a future study could make better use of the entire interpretive framework and compare more of students' individual experiences with the first few weeks of class to see how classroom norms (Cobb & Yackel, 1996) are developed, negotiated, and internalized. I also considered how the introductory curriculum could be more explicitly utilized to instigate norm development in the classroom; recall that the norm developing benefits of the Law of Cancellation episode seemed to happen primarily due to spontaneous student ideas.

In Paper 2, I considered how a curriculum could be more intentionally designed with activities that are targeted towards specific aspects of rehumanizing mathematics (Gutiérrez, 2018). I also considered ways in which the structure of inquiry-based learning can limit the ways in which the dimension of *cultures/histories* can appear in the classroom, and I believe there are several opportunities for future researchers to address this current deficiency.

In Paper 3, I expressed a desire to observe students in more settings outside of the IBL classroom in order to develop a more well-informed picture of their final proof presentations in class and to look for other ways in which their proof norms and values might shift over the course of the term. I also considered a future study of a classroom that explicitly utilizes the instructor's activities, and potentially others, for norm development, and having students reflect on this activities in interviews to study whether these activities more generally promote shifts in students' perceptions of proof from the beginning to end of term.

In sum, these future directions support and extend the overall research goals of this dissertation by providing more avenues by which to study students' social and mathematical experiences in inquiry-based learning classrooms through intentional curriculum development that promotes norms, values, humanistic and social experiences, and mathematical experiences. I also would like to consider more varied types of data collection locations that could provide deeper insights into these experiences.

8.3.2 Remaining Questions for This Dataset

In addition to the questions that arose from the three papers I have presented in this dissertation; I have a number of future directions for remaining aspects of my data and ideas that arose more broadly during my data analysis.

8.3.2.1 Easton and Emory. There were two students in this classroom, Easton and Emory, whose mathematical contributions were somewhat contentious at the beginning of the term. In particular, the level of detail and logic that they desired in their proofs, and their lengthy explanations of those details, were beyond the accepted classroom sociomathematical norms of the time. I noticed an interesting arc throughout the classroom observation dataset, and the individual student interviews, of the classroom being in turns stymied, annoyed, frustrated, and eventually accepting of these students' behavior during class as everyone worked together to develop their knowledge of Real Analysis. An interesting future paper would follow the arc of these two students and their social experiences in the classroom as the entire class navigated the specific types of feedback they were willing to accept during proof presentations.

8.3.2.2 Mathematical Language Conventions. Due to the IBL packet materials being so sparse, and the large amount of freedom that students had in creating their proofs, there were a number of atypical and interesting conversations around language usage during the proof presentations. In particular, at the beginning of the term the students had a detailed conversation on whether there was a difference between starting a proof with the word "let", "suppose", or "assume" based on how the problem was stated. They also had an interesting and spontaneous class day spent of the definition of a definition, and the usage of the word "if" versus "if and only if" in the statement of a definition. An interesting future paper would look at these two episodes and more broadly across the dataset for other instances where the class picked up on mathematical distinctions that are not widely discussed in the current literature, thus providing educators with deeper knowledge of what novice proof writers might be paying attention to in their proof activity.

9 References

Adiredja, A. (2014). *Leveraging students' intuitive knowledge about the formal definition of a limit* [UC Berkeley]. https://escholarship.org/uc/item/0jk8m1fd

Adiredja, A. P. (2019). The Pancake Story and the Epsilon–Delta Definition. *PRIMUS*, 1–16. https://doi.org/10.1080/10511970.2019.1669231

Adiredja, A. P., & James, K. (2013). Students' knowledge resources about the temporal order of delta and epsilon. *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, 140–152.

Adiredja, A., & Zandieh, M. (2020). Everyday examples in linear algebra: Individual and collective creativity. *Journal of Humanistic Mathematics*, *10*(2), 40–75. https://doi.org/10.5642/jhummath.202002.05

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics*, *22*(2), 28–34.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior*, *24*(2), 125–134. https://doi.org/10.1016/j.jmathb.2005.03.003

Anderton, L., & Wright, D. (2012). We could all be having so much more fun! A case for the history of mathematics in education. *Journal of Humanistic Mathematics*, *2*(1), 88–103. https://doi.org/10.5642/jhummath.201201.08

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In *Yearbook of the National Society for the Study of Education, Constructivisim in Education* (pp. 193–224). University of Chicago Press.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In *A research companion to principles and standards for school mathematics* (pp. 27–44). National Council of Teachers of Mathematics.

Barnett, J. H., Lodder, J., & Pengelley, D. (2016). Teaching and Learning Mathematics From Primary Historical Sources. *PRIMUS*, *26*(1), 1–18. https://doi.org/10.1080/10511970.2015.1054010

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487–500.

Blair, R., Kirkman, E., & Maxwell, J. (2018). *Statistical abstract of undergraduate programs in the mathematical sciences in the united states: Fall 2015 cbms survey*. American Mathematical Society. https://doi.org/10.1090/cbmssurvey/2015

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Brown, S. (2018). E-IBL: an exploration of theoretical relationships between equity-oriented instruction and inquiry-based learning. *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 1–15.

Capaldi, M. (2015). Including inquiry-based learning in a flipped class. *PRIMUS*, *25*(8), 736–744. https://doi.org/10.1080/10511970.2015.1031303

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. *In A. E. Kelly & R. A. Lesh (Eds.), Handbook of Research Design in Mathematics and Science Education*, 307–333.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, *23*(1), 2–33. https://doi.org/10.2307/749161

Cook, J. P. (2014). The emergence of algebraic structure: Students come to understand units and zero-divisors. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 349–359. https://doi.org/10.1080/0020739X.2013.837520

Cook, S., Murphy, S., & Fukawa-Connelly, T. (2016). Divergent definitions of inquiry-based learning in undergraduate mathematics. *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*, 660–665. https://doi.org/Available from: http://sigmaa.maa.org/rume/RUME19v3.pdf

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *The Journal of Mathematical Behavior*, *15*(2), 167–192.

Creswell, J., & Poth, C. (2018). *Qualitative inquiry and research design: Choosing among five approaches* (Fourth Edition). SAGE.

Czocher, J. A., & Weber, K. (2020). Proof as a cluster category. *Journal for Research in Mathematics Education*, *in press*.

Darragh, L. (2014). Asking questions and performing mathematics identity. *J. Anderson, M. Cavanagh & A. Prescott (Eds.) Curriculum in Focus: Research Guided Practice (Proceedings of the 37th Annual Conference of the Mathematics Education Research Group of Australasia)*, 175–182.

Darragh, L. (2015). Recognising 'good at mathematics': Using a performative lens for identity. *Mathematics Education Research Journal*, *27*(1), 83–102. https://doi.org/10.1007/s13394-014-0120-0

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19–33. https://doi.org/10.1007/s10649-016-9696-5

Davies, B., & Harre, R. (1990). Positioning: The Discursive Production of Selves. *Journal for the Theory of Social Behaviour*. https://onlinelibrary-wiley-com.ezproxy.proxy.library.oregonstate.edu/doi/abs/10.1111/j.1468-5914.1990.tb00174.x

Dawkins, P. C. (2009). *Non-traditional socio-mathematical norms in undergraduate real analysis*. ProQuest LLC.

Dawkins, P. C. (2013). *Individual Adherence to Inquiry-Oriented Norms of Defining in Advanced Mathematics*. https://eric.ed.gov/?q=%22real+analysis%22+%22mathematics%22&pg=2&id=ED584495

Dawkins, P. C. (2014a). How students interpret and enact inquiry-oriented defining practices in undergraduate real analysis. *The Journal of Mathematical Behavior*, *33*, 88–105. https://doi.org/10.1016/j.jmathb.2013.10.002

Dawkins, P. C. (2014b). When proofs reflect more on assumptions than conclusions. *For the Learning of Mathematics*, *34*(2), 17–23.

Dawkins, P. C. (2018). Student interpretations of axioms in planar geometry. *Investigations in Mathematics Learning*, *10*(4), 227–239. https://doi.org/10.1080/19477503.2017.1414981

Dawkins, P. C., Oehrtman, M., & Mahavier, W. T. (2019). Professor goals and student experiences in traditional IBL real analysis: A case study. *International Journal of*

Research in Undergraduate Mathematics Education, *5*(3), 315–336. https://doi.org/10.1007/s40753-019-00095-4

Dawkins, P. C., & Roh, K. H. (2016). Promoting metalinguistic and metamathematical reasoning in proof-oriented mathematics courses: A method and a framework. *International Journal of Research in Undergraduate Mathematics Education*, *2*(2), 197–222. https://doi.org/10.1007/s40753-016-0027-0

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics Education IV*, *8*, 239–289.

Durand-Guerrier, V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics*, *60*(2), 149–172.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In *W. Zimmermann & S. Cunningham (Eds.), Visualization in teaching and learning mathematics.* (pp. 25–37). Mathematical Association of America.

Erickson, S. (2020). *Investigating provers' understandings of combinatorial proof*. Oregon State University.

Ernest, P. (1994). Varieties of constructivism: Their metaphors, epistemologies and pedagogical implications. *Hiroshima Journal of Mathematics Education*, *2*, 1–14.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Kluwer Academic Publishers, 101 Philip Drive, Norwell, MA 02061 ($69.

Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *The Journal of Mathematical Behavior*, *31*(3), 401–416. https://doi.org/10.1016/j.jmathb.2012.04.002

Fukawa-Connelly, T., Mejía-Ramos, J. P., Wasserman, N. H., & Weber, K. (2020). An evaluation of ULTRA; an experimental real analysis course built on a transformative theoretical model. *International Journal of Research in Undergraduate Mathematics Education*, *6*(2), 159–185. https://doi.org/10.1007/s40753-019-00102-8

Glasersfeld, E. von. (1989). Cognition, construction of knowledge, and teaching. *Synthese*, *80*(1), 121–140. https://doi.org/10.1007/BF00869951

Gordon, M. (2019). Mathematics students as artists: Broadening the mathematics curriculum. *Journal of Humanistic Mathematics*, *9*(2), 192–210. https://doi.org/10.5642/jhummath.201902.13

Gravemeijer, K. (1994). Educational Development and Developmental Research in Mathematics Education. *Journal for Research in Mathematics Education*, *25*(5), 443–471. https://doi.org/10.2307/749485

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*. https://doi.org/10.1207/s15327833mtl0102_4

Gutiérrez, R. (2009). Framing equity" helping students "play the game" and "change the game". *Teach Excellence Equity Math*, *1*(1), 5–7.

Gutiérrez, R. (2018). The need to rehumanize mathematics. In *Annual Perspectives in Mathematics Education: Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). The National Council of Teachers of Mathematics, Inc.

Haberler, Z., Laursen, S. L., & Hayward, C. N. (2018). What's in a name? Framing struggles of a mathematics education reform community. *International Journal of Research in Undergraduate Mathematics Education*, *4*(3), 415–441. https://doi.org/10.1007/s40753-018-0079-4

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6–13.

Hanna, G. (2000). Proof, Explanation and Exploration: An Overview. *Educational Studies in Mathematics*, *44*(1), 5–23. https://doi.org/10.1023/A:1012737223465

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part 1: Focus on proving. *ZDM Mathematics Education*, *40*, 487–500.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from an exploratory study. In *Research in College Mathematics Education III* (pp. 234–283). AMS.

Harel, G., & Sowder, L. (2007a). Toward comprehensive perspectives on the learning and teaching of proof. In *F.K. Lester (Ed.), Second handbook of research on mathematics teaching and learning* (pp. 805–842). Information Age Publishing. https://www.researchgate.net/publication/228388979_Toward_comprehensive_perspectives_on_the_learning_and_teaching_of_proof

Harel, G., & Sowder, L. (2007b). Toward comprehensive pserspectives on the learning and teaching of proof. In *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 805–842). https://math.ucsd.edu/~harel/TowardComprehensivePerspective.pdf

Hassi, M.-L. (2009). Empowering undergraduate students through mathematical thinking and learning. *A Declaration of Numeracy: Empowering Adults through Mathematics Education, Proceedings of the 15th International Conference of Adults Learning Mathematics (ALM)*, 17.

Hassi, M.-L., & Laursen, S. L. (2015). Transformative learning: Personal empowerment in learning mathematics. *Journal of Transformative Education*, *13*(4), 316–340. https://doi.org/10.1177/1541344615587111

Hayward, C. N., Kogan, M., & Laursen, S. L. (2016). Facilitating Instructor Adoption of Inquiry-Based Learning in College Mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 59–82. https://doi.org/10.1007/s40753-015-0021-y

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*(4), 389–399. https://doi.org/10.1007/BF01273372

Hersh, R. (2010). The best undergraduate math program you've never heard of. *Math Horizons*, *17*(3), 18–21.

Jennifer M. Langer-Osuna. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education*, *48*(3), 237. https://doi.org/10.5951/jresematheduc.48.3.0237

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, *51*(4), 504–516. https://doi.org/10.5951/jresematheduc-2020-0043

Kinsey, L. C., & Moore, T. E. (2015). Narrative structure in inquiry-based learning. *PRIMUS*, *25*(3), 212–220. https://doi.org/10.1080/10511970.2014.921650

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *PRIMUS*, *28*(1), 13–30. https://doi.org/10.1080/10511970.2017.1338807

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725. https://doi.org/10.1016/j.jmathb.2013.04.006

Laursen, S. L., Hassi, M.-L., & Hough, S. (2016). Implementation and outcomes of inquiry-based learning in mathematics content courses for pre-service teachers. *International Journal of Mathematical Education in Science and Technology*, *47*(2), 256–275. https://doi.org/10.1080/0020739X.2015.1068390

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, *45*(4), 406. https://doi.org/10.5951/jresematheduc.45.4.0406

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *5*(1), 129–146. https://doi.org/10.1007/s40753-019-00085-6

Lew, K., Timothy Patrick Fukawa-Connelly, Juan Pablo Mejía-Ramos, & Keith Weber. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162. https://doi.org/10.5951/jresematheduc.47.2.0162

Lipka, J., Adams, B., Wong, M., Koester, D., & Francois, K. (2019). Symmetry and measuring: Ways to teach the foundations of mathematics inspired by yupiaq elders. *Journal of Humanistic Mathematics*, *9*(1`), 107–157. https://doi.org/10.5642/jhummath.201901.07

Love, B., Hodge, A., Grandgenett, N., & Swift, A. W. (2014). Student learning and perceptions in a flipped linear algebra course. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 317–324. https://doi.org/10.1080/0020739X.2013.822582

Mahavier, Wm. T. (1997). A gentle discovery method (the modified texas method). *College Teaching*, *45*(4), 132–135.

Matthews, M., & Hodge, A. (2016). Using IBL in a history of mathematics course: A skeptic's success. *Journal of Humanistic Mathematics*, *6*(2), 23–37.

Maxwell, K., & Chahine, I. (2013). Cultural immersion and mathematics teacher education: Explorations in morocco and south africa. *Journal of Humanistic Mathematics*, *3*(2), 62–75. https://doi.org/10.5642/jhummath.201302.04

McLeod, D. B. (1994). Research on affect and mathematics learning in the JRME: 1970 to the present. *Journal for Research in Mathematics Education*, *25*(6), 637–647. https://doi.org/10.2307/749576

Mesa, V., Shultz, M., & Jackson, A. (2020). Moving away from lecture in undergraduate mathematics: Managing tensions within a coordinated inquiry-based linear algebra course. *International Journal of Research in Undergraduate Mathematics Education*, *6*(2), 245–278. https://doi.org/10.1007/s40753-019-00109-1

Mullen, E. T. (2012). Teaching an engaged analysis class through active learning. *PRIMUS*, *22*(3), 186–200. https://doi.org/10.1080/10511970.2010.497957

Nunez, R. E. |Edwards. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*, 45–65.

Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior*, *33*, 131–148.

Parr, E., Roh, K. H., & Sellers, M. (2018). Teaching the representations of concepts in calculus: The case of the intermediate value theorem. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*. https://doi.org/10.1080/10511970.2018.1540023

Patton, M. (2015). *Qualitative research & evaluation methods: Integrating theory and practice*. SAGE.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23–41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM*, *40*(3), 385–400.

Pedemonte, B., & Reid, D. (2011). The role of abduction in proving processes. *Educational Studies in Mathematics*, *76*(3), 281–303.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, *52*(3), 319–325. https://doi.org/10.1023/A:1024360204239

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*, *26*(3), 189–194. https://doi.org/10.1016/j.jmathb.2007.10.001

Rasmussen, C., Kwon, O. N., Allen, K., Marrongelle, K., & Burtch, M. (2006). Capitalizing on advances in mathematics and k-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. *Asia Pacific Education Review*, *7*(1), 85–93. https://doi.org/10.1007/BF03036787

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*, *37*(5), 388–420.

Rasmussen, C., Marrongelle, K., Kwon, O. N., & Hodge, A. (2017). Opinion: Four goals for instructors using inquiry-based learning. *Notices of the American Mathematical Society*, *64*(11), 1. https://doi.org/10.1090/noti1597

Rasmussen, C., Zandieh, M., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73.

Reed, Z. (2018). *Undergraduate Students' Generalizing Activity in Real Analysis: Constructing a General Metric*. Oregon State University.

Reed, Z. (2017). The effects of the epsilon-n relationship on convergence of functions. *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, 236–248.

Reinholz, D. L. (2020). Five practices for supporting inquiry in analysis. *PRIMUS*, *30*(1), 19–35. https://doi.org/10.1080/10511970.2018.1500955

Renesse, C., & DiGrazia, J. (2018). Mathematics, Writing, and Rhetoric: Deep Thinking in First-Year Learning Communities. *Journal of Humanistic Mathematics*, 24–63. https://doi.org/10.5642/jhummath.201801.04

Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217–233.

Roh, K. H., & Lee, Y. H. (2011). The mayan activity: A way of teaching multiple quantifications in logical contexts. *PRIMUS*, *21*(8), 685–698. https://doi.org/10.1080/10511970.2010.485602

Roh, K. H., & Lee, Y. H. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 34–68. https://doi.org/10.1007/s40753-016-0039-9

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematical Enthusiast*, *10*(1), Article 14.

Selden, A., Selden, J., & Benkhalti, A. (2018). Proof frameworks: A way to get started. *PRIMUS*, *28*(1), 31–45. https://doi.org/10.1080/10511970.2017.1355858

Shannon, C. A. (2016). Flipping the analysis classroom. *PRIMUS*, *26*(8), 727–735. https://doi.org/10.1080/10511970.2016.1162889

Shannon, K. (2018). It's not the moore method, but… a student-driven, textbook-supported, approach to teaching upper-division mathematics. *PRIMUS*, *28*(2), 118–127. https://doi.org/10.1080/10511970.2017.1348416

Starbird, M. (2015). *Century of advancing mathematics: Inquiry-based learning through the life of the MAA*. American Mathematical Scoeity.

Stone-Johnstone, A., Reinholz, D., Brooke Mullins, S., Smith, J., & Andrews-Larson, C. (2019). Inquiry without equity: A case study of two undergraduate math classes. *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education*.

Strand, S. R. (2016). The intermediate value theorem as a starting point for inquiry-oriented advanced calculus. In *ProQuest LLC*. ProQuest LLC.

Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and Argumentation in Mathematics Education Research. In Á. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education* (pp. 315–351). SensePublishers. https://doi.org/10.1007/978-94-6300-561-6_9

Stylianides, G., Stylianides, A., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In *Compendium for Research in Mathematics Education* (pp. 237–266). The National Council of Teachers of Mathematics, Inc.

Stylianou, D. A., Blanton, M. L., & Rotou, O. (2015). Undergraduate students' understanding of proof: Relationships between proof conceptions, beliefs, and classroom experiences with learning proof. *International Journal of Research in Undergraduate Mathematics Education*, *1*(1), 91–134. https://doi.org/10.1007/s40753-015-0003-0

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of amy and mike. *The Journal of Mathematical Behavior*, *30*(2), 93–114.

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493.

Tall, D. (1991). Preface. In *Advanced mathematical thinking* (pp. xii–xv). Kluwer Academic Publishers.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Tang, G., El Turkey, H., Cilli-Turner, E., Savic, M., Karakok, G., & Plaxco, D. (2017). Inquiry as an entry point to equity in the classroom. *International Journal of Mathematical*

Education in Science and Technology, *48*. https://doi.org/10.1080/0020739X.2017.1352045

Taylor, P., & Cambell-Williams, M. (1993). *A critical constructivist view on reforming the traditional teaching/learning process*. Annual Conference of the Australian Association for Educational Research.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., … Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences*, *117*(12), 6476–6483. https://doi.org/10.1073/pnas.1916903117

von Renesse, C., & Ecke, V. (2015). Inquiry-based learning and the art of mathematical discourse. *PRIMUS*, *25*(3), 221–237. https://doi.org/10.1080/10511970.2014.921799

Vroom, K. (2020). *Guided reinvention as a context for investigating students' thinking about mathematical language and for supporting students in gaining fluency*. (Doctoral dissertation, Portland State University).

Wasserman, N. H. (2014). Introducing algebraic structures through solving equations: Vertical content knowledge for K-12 mathematics teachers. *PRIMUS*, *24*(3), 191–214. https://doi.org/10.1080/10511970.2013.857374

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577–599. https://doi.org/10.1080/10511970.2012.667516

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119. https://doi.org/10.1023/A:1015535614355

Weber, K. (2002). Beyond proving and explaining: Proofs that justify the use of definitions and axiomatic structures and proofs that illustrate technique. *For the Learning of Mathematics*, *22*(3), 14–17.

Weber, K. (2008). The role of affect in learning real analysis: A case study. *Research in Mathematics Education*, *10*(1), 71–85.

Weber, K. (2010). Mathematics Majors' Perceptions of Conviction, Validity, and Proof. *Mathematical Thinking and Learning*, *12*(4), 306–336. https://doi.org/10.1080/10986065.2010.495468

Weber, K., & Alcock, L. (2004). Semantic and Syntactic Proof Productions. *Educational Studies in Mathematics*, *56*, 209–234. https://doi.org/10.1023/B:EDUC.0000040410.57253.a1

Weber, K., & Alcock, L. (2009). Proof in advanced mathematics classes: Semantic and syntactic reasoning in the representation system of proof. In *Teaching and Learning Proof Across the Grades: A K-16 Perspective*. Routledge.

Whyburn, L. S. (1970). Student oriented-teaching—The moore method. *The American Mathematical Monthly*, *77*(4), 351–359.

Yackel, E., & Rasmussen, C. (2003). Beliefs and norms in the mathematics classroom. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (Vol. 31, pp. 313–330). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47958-3_18

Zazkis, D., Weber, K., & Mejia Ramos, J. P. (2014). Activities that mathematics majors use to bridge the gap between informal arguments and proofs. *Proceedings of the Conference for Psychology of Mathematics Education*, 417–424.
Zazkis, D., Weber, K., & Mejía-Ramos, J. P. (2016). Bridging the gap between graphical arguments and verbal-symbolic proofs in a real analysis context. *Educational Studies in Mathematics*, *93*(2), 155–173. https://doi.org/10.1007/s10649-016-9698-3
Zwanch, K., Mullins, B., & Fortune, N. (2019). Understanding students' achievement and perceptions of inquiry-oriented instruction. *RUME Proceedings 2019*.

10 Appendices

Appendix A. Term Schedule

Week	Location	Topics	Activities	Interviews
Week 1: Jan 13-Jan 19	In-Person	Real Numbers	We Understand Calculus Well, Dear John Letter	
Week 2: Jan 20 – Jan 26	In-Person	Real Numbers	Curiosity Cabinet	
Week 3: Jan 27- Feb 2	In-Person	Real Numbers		
Week 4: Feb 3- Feb 9	In-Person	Real Numbers	Is a Hotdog a Sandwich? Definition Activity	Ind. Interviews 1 (Ash, Hayden, Parker, Sloan, Taylor)
Week 5: Feb 10 – Feb 16	In-Person	Real Numbers		
Week 6: Feb 17 – Feb 23	In-Person	Sequences and Convergence		
Week 7: Feb 24 – March 1	In-Person	Sequences and Convergence	Midterm Exam	
Week 8: March 2 – March 8	In-Person	Sequences and Convergence		Ind. Interviews 2 (Hayden, Parker, Sloan)
Week 9: March 9 – March 15	In-Person	Limit Points	Veritasium Video	
Week 10: March 16 – March 22	Virtual	Limit Points		
Week 11: March 23 – March 29	Virtual	Limit Points	This I Believe Essays	Ind. Interviews 3 (Ash, Hayden, Parker, Sloan, Taylor)
Week 12: March 30 – April 5	Virtual	Functions and Continuity	Epsilon-Delta Pancake Story	
Week 13: April 6 – April 12	Virtual	Functions and Continuity		
Week 14: April 13 – April 19	Virtual	Functions and Continuity		
Week 15: April 20 – April 26	Virtual	Functions and Continuity		
Week 16: April 27 – May 3	Virtual	Functions and Continuity	Final Exam, Proof Portfolio, Creative Summaries	Ind. Interviews 4 (Ash, Hayden, Parker, Sloan, Taylor)

Appendix B: Course Syllabus (Identifying Information Removed)

A NOTE FROM YOUR INSTRUCTOR:
Welcome to Real Analysis, where you'll be proving all of those theorems you accepted as true back in calculus. That's right. You'll be doing the proving. This course will be run using a method called Inquiry-Based Learning (IBL). The IBL method, which is student-centered, student-led, and student-oriented, fosters creativity, independent thinking, and mathematical intuition. There are Four Pillars of Inquiry-Based Learning, and we'll be making sure our work aligns with all of them.

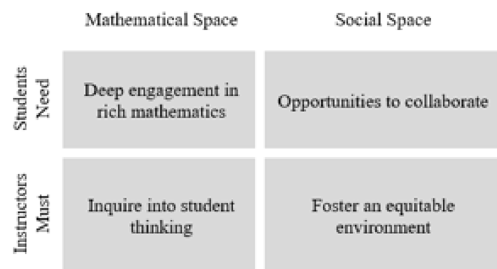


Figure 1: The Four Pillars of Inquiry-Based Learning

The following quotes from the Academy of Inquiry-Based Learning (AIBL) webpage give a good general description of the IBL method.

> Boiled down to its essence IBL is a teaching method that engages students in sense-making activities. Students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, and communicate... all those wonderful skills and habits of mind that Mathematicians engage in regularly. Rather than showing facts or a clear, smooth path to a solution, the instructor guides and mentors students via well-crafted problems through an adventure in mathematical discovery. Key components across effective IBL courses are (a) deep engagement in rich mathematical activities, and (b) opportunities to collaborate with peers (either through class presentations or group-oriented work).

In short, this class won't "look" like other math classes you've taken. I won't lecture much, and you'll be serving as content experts as you decipher definitions, draw pictures to go along with axioms, prove theorems, and provide counterexamples to statements that are untrue. We'll explore the notion of \truth" as it pertains to mathematics. We will look closely at and reflect upon our individual proof processes to determine what works and what doesn't. We will discover the structure of mathematical arguments, and then you will create your own.
It's entirely reasonable that you would wonder why I'd run a class this way. I love talking about mathematics, so why take a back seat? The full answer to \WHY?" will unfold over the course of the semester. There will be times when you are certain this experiment in learning won't work, or that what you're doing won't possibly get you to whatever academic goal you've set for yourself. When that happens, please step back and trust me. Trust that I believe you can do this, and do it well. Trust that I believe that learning using IBL is the best way to learn to think like a mathematician. Trust that the takeaways from learning using IBL are much bigger than you can predict now. You'll see. I promise.

EQUITY, RESPECT, AND INCLUSIVITY

In our classroom, diversity and individual differences are respected, appreciated, and recognized as a source of strength. I support the use of mathematics as an analytic tool to challenge power, privilege, and oppression. It is our collective responsibility to create an emotionally safe space where ideas can be challenged while individuals are respected. I ask you to support one another as you develop as mathematicians and analytic thinkers. Students in this class are encouraged and expected to speak up and participate during class and to carefully and respectfully listen to each other. Every member of this class must show respect for every other member of this class. Any attitudes or actions that are destructive to the sense of community that we strive to create are not welcome and will not be tolerated. In summary: Be good to each other.

Pledge: I believe every assignment is a critical and effective learning experience. I promise to give you feedback in a timely manner. I promise to be respectful of you and your ideas. I promise to make you as uncomfortable as I can, asking you to do tasks that are just at the edge of what you can handle every day. I pledge to do everything I can to resist hated and bigotry in our classrooms and beyond. No matter where you come from, you are welcome in this classroom and you can belong here.

Rights of the Learner

As a student in this class, you have the right

1. to be confused,
2. to make a mistake and to revise your thinking,
3. to speak, listen, and be heard, and
4. to enjoy doing mathematics.

WHEN LIFE HAPPENS

When life happens to you, please email me. What do I mean? I am a planner. I am an over-planner, actually. I usually have contingency plans for my contingency plans. AND STILL, sometimes, unplanned events happen, both positive and negative, and there is a trickle-down of energy, grief, happiness, exhaustion, etc. into other aspects of my life. Emotional and physical support is a healing thing that humans can, and should, offer each other. I have been on the receiving end of this support many times in my life, but never as comprehensively as during my years at UP. This is a special place. In my time teaching, I have had students go through many major life events, ranging from the death of a parent to the birth of a child, breakups, homelessness and food insecurity, mental health crises, and long-term and short-term illnesses. You will have weeks when you're totally overwhelmed and may need an extension on an assignment. You may need someone to listen, or connect you to resources. You may feel like you are behind, and you may feel hopeless. You may be away from home for the first time, and you might be struggling with that. You may need to celebrate a major achievement, or even a minor one. Your success in this course, and in all of your courses, is important to me. Your success at this University is important to me. No matter what it is, I can help you get through it. We are a community, and a team, and I intend to hold up my end of supporting this community. When life happens, send me an email. You are not ever alone. I'm here.

COURSE/BULLETIN DESCRIPTION
A rigorous treatment of properties of the real numbers and functions of a single real variable. Topics include completeness, limits, continuity, differentiation, integration, and sequences. Additional topics may include series, an introduction to Euclidean or metric spaces. (Prerequisite: MTH 311)

TEXT and TECHNOLOGY
There is no text for this course. This course is concerned with theoretical, rather than computational mathematics. As such, you will have little use for a calculator. However, if you would like to have one for use in graphing functions, that's fine. It would be helpful to have a calculator on hand to graph examples or counterexamples. Additionally, we may use Desmos applets to help us visualize certain concepts and results. WolframAlpha and other online tools may well be useful. There will also be some assignments requiring you to submit your work using LATEX. LATEX is an open-source backend for document preparation and typesetting; it is the standard tool for publishing in mathematics. LATEX is loaded onto lab computers.

COURSE PERFORMANCE OBJECTIVES:
The main objective of this course is to discover proofs of theorems in real analysis. Along the way, you will develop skills needed to write a coherent, rigorous mathematical proof. You will also learn to present proofs to others, and to review others' proofs and mathematical processes.

Cognitive: The successful student will
1. demonstrate proof techniques such as direct proof, indirect proof, and proof by contrapositive
2. demonstrate increased understanding of logical structure by presenting proofs (both orally and written) and by critiquing their own proofs and those generated by classmates
3. demonstrate problem-solving abilities by reading and writing mathematical arguments
4. synthesize and articulate important definitions through discussion, develop mathematical models which _t those definitions, and apply the definitions in proofs of theorems and solutions to problems
5. present solutions and proofs to the class on a weekly basis
6. critique their own proofs and solutions through revisions, and critique classmates proofs during in-class presentations
7. apply major results of real analysis to new theorems in order to produce accurate proofs

Affective: The successful student will
1. value the challenges of doing mathematics independently and the pride of success in mathematics.
2. gain confidence in their ability to solve problems and write proofs.
3. gain confidence in their ability to present solutions and proofs in front of peers and faculty.
4. help others by creating and maintaining a supportive and inclusive learning environment, where differences are respected and valued deeply.[

COURSE METHODOLOGY
This course will be run using Inquiry-Based Learning. This method, which is student-centered, student-oriented, and student-led, fosters creativity, independent thinking, and mathematical intuition. You will receive a list of definitions and models to interpret and learn, and problems and theorems which you and your classmates must answer or prove. Each class, you will be asked to "claim" problems for which you have solutions, and then you and your colleagues will present problems on the board. These presentations comprise a major portion of the course. You will also be encouraged to find fault with the problems, solutions, and proofs, and to conjecture new theorems based on what you've learned, and to prove or disprove these conjectures. One model for creating independent mathematicians would be to have you each work individually on the problems, write up your solutions and proofs and hand them in, then present some of them to the class. However, for several reasons, including a wide range of student backgrounds and my desire to build a collaborative, equitable, and inclusive learning environment, I encourage you to work with other classmates on the problems. You may not ask faculty members (other
than me) for help, nor may you consult texts or the internet. You should keep the level of collaboration at discussing problems, and should each submit your own solutions. Even if you work with a partner, you should not turn in identical solutions, as you must write them up separately. Your voice is important, and for that reason I ask that you submit individual write-ups. As to presenting in class, you will have to do this regularly. You should expect to present at least once a week. If you are observing a presentation, it is your responsibility to follow the logic of the solution and verify that it is correct. If you cannot follow something, it is your responsibility to ask a question of the presenter. If you are presenting, you should be clear, accurate, and precise, as well as ready to _eld questions from me and your classmates. All students should make conjectures as we work through the class notes, particularly if a problem is incorrect, unclear, or if you can generalize a result. If you are truly stuck on a question, do not hesitate to ask your classmates for help. You should be working far enough ahead of the presentations so that there is time for this consultation.

LEARNING POLICIES:
This course is concerned with the creation and application of the theory that supports elementary calculus. Its conduct will be student-oriented in the sense that you will be asked to create solutions to problems, present your solutions for the scrutiny of the class, and to critique the work of others when presented. An important objective of the course is for you to develop your creative and critical mathematical skills; the dynamics of classroom interaction are thus critical. You are expected to be in class. Attendance will be taken at the start of every class period as students claim problems. There is no formal text for this course. Classes will be conducted from problem sets distributed by the instructor. Solutions to those problems should effectively allow the students to write their own text. You will receive lists of definitions, questions, and theorem statements, and will present your answers and proofs in class. These presentations are a major part of the course and will be graded according the rubric below. You may NOT consult books while creating your answers and proofs. Ideally, you will settle your questions and prove theorems independently or in groups prior to class. Each day in class, students will present solutions at the board. When you are presenting your proofs or solutions, strive to make your explanations clear and organized. If you are observing a presentation, it is your responsibility to follow the logic of the solution and verify for yourself that it is correct. If you cannot follow something, it is your responsibility to ask a question of the student presenting. Again, you may not use outside resources for this course, including texts, the internet, or other faculty members. You may talk to other students enrolled in MTH 401 this semester, and you may (of course) come talk to me. You may use your notes from class. You may not consult with students who took MTH 401 in a prior semester. Naturally, I expect that when other students are presenting solutions, you will be polite and respectful. I expect that you will be an active listener, and that you will not be distracted or paying attention to things other than what is happening in class.

METHODS OF ASSESSMENT
Course performance objectives are assessed by traditional means: graded homework assignments, journal assignments, a final portfolio, oral presentations, one midterm, and a cumulative take-home _nal exam. The development of analytical and logical reasoning skills are inherent in the nature of mathematics and are assessed in conjunction with the course performance objectives.

Attendance: You are expected to be in class. If you miss more than 3 classes, your _nal grade will be lowered by one full letter grade. The only exception to this rule is for a case beyond your control. If you miss a class, you are required to document and justify your absence. The reason for this policy is two-fold: you are expected to be in class to present solutions, and you are expected to be in class to help your classmates when they present solutions. We work in support of the idea that all of us is smarter than one of us. Your attendance and participation are critical to the success of the class.

Homework: You have a standing homework assignment to do the problems given to you in class. You should stay ahead of where we are in presentations. You should write up solutions to all of the problems. Selected problems will be selected for written submission, and will be graded according to the rubric below. Written homework assignments will be made weekly, and you are expected to submit clear, concise proofs. Often you will write several drafts of a problem before writing the best proof. This is OK. You need not type these solutions initially, but consider writing them in pen { by the time you write something for submission, it should be mistake-free, obviating the need for pencil. Again, you should write up neat, clear proofs for ALL of the problems. I will select some problems weekly for submission to me. You should work on problems roughly in the order in which they are presented, and should be ready to present solutions in class. Late work will not be accepted. If you fail to turn in 4 or more assignments (including journals), your final grade will be lowered one full letter grade. Homework may, and should be corrected and resubmitted. The higher of the two grades will be recorded. When you resubmit a solution or proof, you must staple all previous solutions to the new one. This will help me to chart your progress.

The Notebook: You are required to keep a 3-ring binder of your solutions and proofs. Your notebook must be neat and organized. Periodically I will examine your notebook to make sure you are keeping it current. Think of this notebook as your textbook that will serve as an important resource for future mathematics classes. You will be allowed to use this notebook on parts of the midterm and _nal exams. You need to use LaTeX to typeset the solutions and proofs in this notebook.

Class Presentations: There will be very little lecture in this course. Instead, class periods are dedicated to presentation of the problems. At the start of class, you will \claim" problems for which you think you have solutions. You will get 1 point for each problem that you claim. Group presentations are allowed; earned points will be divided among the group members at my discretion, and the presentation is still out of a maximum of 3 points. If you have an alternate solution or proof, you should present it in class and will earn credit if the solution is truly different than the others given.

Proof Portfolio We will have a summative portfolio assignment at the end of the term. You will receive more information on this after Spring Break.

Class Participation: Your participation is vital to the success of this course. Not only are you required to present problems weekly, you are required to be an active and engaged audience member. It is your responsibility to ask questions of the presenter if you don't understand something or disagree with the justification. Do not count on me to correct mistakes or clarify concepts during a presentation. In order to foster a comfortable mathematical community and to keep the whole experience fun for everyone, I expect you to treat your classmates with respect and kindness.

Note about outside resources: Since one of the primary goals of this class is to develop your ability to do mathematics independently, you may not use outside resources such as texts and the web. I do however, highly encourage teamwork among the students in this class. You may always ask me for help, but keep in mind, my goal is to encourage your independence and creativity.

Grading Rubric for Homework:
Written solutions for selected problems will be collected regularly.
5 Perfect (or nearly so): thorough but concise, rigorous, well-
structured using verifiably correct methods, and easy to read
4 Correct proof with slight technical/language issues
3 Correct proof with large technical/language issues
2 Incorrect proof with explanation of why proof is incorrect or
a discussion of what obstacles prevent completion of the proof
1 Incorrect proof
0 Incomprehensible proof or no progress
Grading Rubric for Presentations:
3 Perfect presentation
2 Mostly correct proof with technical/language issues
1 Some progress towards a correct proof but with much revision/work needed
0 Incorrect proof.

Journals: You will be asked to complete journals. You will be given prompts on the course Moodle page, and will respond electronically. These journals will be graded for completion only. Some journals will be posted to the course Moodle forum, so that you can read and respond to each others' thoughts. Other journals will have a more private turn-in option. I am always open to accepting a private submission if you are uncomfortable using the forum. Please keep this line of communication open. We will be looking closely at those things we choose to believe without proof and assumptions that we make and use in work. We'll consider these issues through a mathematical lens as we cover course content, as well as through a moral and humanistic lens, making connections between the two. Details on this project will be forthcoming, but we urge you to ask questions as they come up. Some of your work on this project will come in the form of journals. We will be hosting a public performance of one specific journal. Stay tuned for more information on this event.

Exams: There will be one midterm. We will discuss in-class and take-home options. The test date will be announced at least one week in advance but you can expect it will happen in mid-October. There will be a cumulative final exam.

Course grade based on the following percentages:
Written Homework Solutions and Journals and Notebook 30%
Class Presentations 20%
Midterm 15%
Final Exam 25%
Portfolio 10%

Appendix C: Class Materials

1. The Real Numbers

As the name of the course suggests, we will be working closely with the set of real numbers and so it might be a good idea to give some thought to the question of what exactly the real numbers are. Of course, we all have some experience working with real numbers - but as it turns out, giving a proper formal definition of the real numbers is not exactly trivial. Instead of a definition, we will instead simply describe the essential properties of the real numbers.

Notation: We denote the natural numbers with the symbol $\mathbb{N}$, the integers with $\mathbb{Z}$, and the rationals with $\mathbb{Q}$. We take for granted the basic properties of these number systems.

Definition: If A is any set (for our purposes, the elements of this set will typically be numbers), we write $x \in A$ to indicate that x is a member (or *element*) of A. If x is not a member of A, we write $x \notin A$. The set which contains no element is called the **empty set**. If a set has at least one element, it is called **nonempty**. If A and B are sets and if every element of A is an element of B, we say that A is a subset of B and write $A \subseteq B$.[1] If there is an element of B which is not in A, we say A is a **proper** subset and write $A \subset B$. If $A \subseteq B$ and $B \subseteq A$, we write $A = B$. Otherwise $A \neq B$.

Axiom: (Field Axioms) There exists an nonempty set $\mathbb{R}$ (which we will call the **real numbers**) with two binary operations on $\mathbb{R}$ called addition ($+$) and multiplication ($\cdot$) that have the following properties:

(i.) **Commutative Property**: $+$ and $\cdot$ are commutative operations, i.e. if a and b are any two real numbers then $a + b = b + a$ and $a \cdot b = b \cdot a$.

(ii.) **Associative Property**: $+$ and $\cdot$ are associative operations, i.e. if a, b, and c are real numbers then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c) = a \cdot (b \cdot c)$.

(iii.) **Additive and Multiplicative Identities**: There exists a real number, which we call 0 (zero) such that if x is any real number then $x + 0 = x$. There exists a real number distinct from zero, which we call 1 (one), such that if x is any real number then $x \cdot 1 = x$.

(iv.) **Additive and Multiplicative Inverses**: For each real number x, there is a real number which we call $-x$ (minus x) such that $x + (-x) = 0$. For each non-zero real number x there exists a real number which we call $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$. We will also denote the multiplicative inverse by x^{-1}.

(v.) **Distributive Property**: Multiplication distributes over addition, i.e. if a, b, and c are real numbers then $a \cdot (b + c) = a \cdot b + a \cdot c$. We adopt the usual order of operations rules so that the expression $a \cdot b + a \cdot b$ is equivalent to $(a \cdot b) + (a \cdot b)$.

---

[1]There is not universal agreement about the notation for subsets. Some authors will use the notation $A \subset B$ to mean that A is a subset of B even if it is not a proper subset.

Definition: Division and subtraction are defined in terms of multiplicative and additive inverses, so that $\frac{x}{y}$ means $x \cdot \frac{1}{y}$ and $x - y$ means $x + (-y)$. Furthermore, we adopt the usual convention of writing xy to mean $x \cdot y$.

1. Suppose a, b, and c are real numbers. Show that if $a + b = 0$ and $a + c = 0$ then $b = c$. In other words, the additive inverse of a is unique.

2. Prove that the multiplicative inverse is unique.

3. Prove that if x is a real number then $x \cdot 0 = 0$.

4. Prove that if x is any real number then $(-1)x = -x$.

5. Prove that if x and y are real numbers and $x \cdot y = 0$ then $x = 0$ or $y = 0$.

6. Which of the properties i.-v. above would still be satisfied if $\mathbb{R}$ were replaced with $\mathbb{Q}$? with $\mathbb{Z}$? with $\mathbb{N}$?

7. In general, a ***field*** is defined as any set together with two binary operations that satisfy properties i.-v. listed above. Let A be the set $\{E, O\}$ and the operations $+$ and $\cdot$ on A defined according to the following tables:

$+$	E	O
E	E	O
O	O	E

$\cdot$	E	O
E	E	E
O	E	O

Determine if A together with these operations is a field.

Axiom: (Order) There exists a subset of the real numbers called the **set of positive numbers**, denoted $\mathbb{R}^+$, that satisfies the following two conditions:

(a) The positive numbers are closed under addition and multiplication. In other words, for all $a, b \in \mathbb{R}^+$,

$$a + b \in \mathbb{R}^+ \quad \text{and} \quad a \cdot b \in \mathbb{R}^+.$$

(b) Given any real number a, one and only one of the following is true:
- (i) a is positive.
- (ii) a is zero.
- (iii) $-a$ is positive.

The compliment of the set $\mathbb{R}^+ \cup \{0\}$, i.e. the set of numbers a that aren't positive or zero, is called the set of **negative numbers** and is denoted $\mathbb{R}^-$.

8. Use the order axioms to verify the following facts.

(a) Let a be a real number. Show that if a is positive, then $-a$ is negative. Conversely, if a is negative, show that $-a$ is positive.

(b) The real number 1 is positive.

(c) There exists a negative number.

Definition: We define an order relation on $\mathbb{R}$ as follows: for all real numbers a and b we say that a is *less than* b ($a < b$) if $b - a \in \mathbb{R}^+$. In other words, $a < b$ means that the number $b - a$ is positive. For all real numbers, a and b, we say that $a \leq b$ if $b - a \in \mathbb{R}^+ \cup \{0\}$. In other words, $a \leq b$ means that the number $b - a$ is positive or zero. The symbols $>$ and $\geq$ are defined similarly.

9. Prove that for all real numbers a, b, and c:

(a) $a \leq a$.

(b) If $a \leq b$ and $b \leq a$ then $a = b$.

(c) If $a \leq b$ and $b \leq c$ then $a \leq c$.

10. Show that if $a < b$ then $-b < -a$.

11. Show that if $0 < a < b$ then $\frac{1}{a} > \frac{1}{b}$.

12. Show that for all real numbers a, b, c and d:

(a) If $a < b$ and $c \leq d$ then $a + c < b + d$.

(b) If $0 < a < b$ and $0 < c \leq d$ then $ac < bd$.

(c) If $a < b$ and $c < 0$ then $ac > bc$.

13. Show that given any $a \in \mathbb{R}$, there is a number that is strictly larger.

14. Prove that $a^2 \geq 0$ for any real number a.

15. Given real numbers $a, b \geq 0$ show that $a^2 \leq b^2$ if and only if $a \leq b$. Show that a similar result holds if the inequality is strict.

Definition: The **absolute value** of a real number x is denoted by $|x|$ and defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

16. Show that for $a \in \mathbb{R}$:

(a) $|a| \geq 0$

(b) $|a| = 0$ if and only if $a = 0$

(c) $|a| \geq a$

(d) $|-a| = |a|$

17. Show that for all $a \in \mathbb{R}$, $a^2 = |a|^2$.

18. Show that for all $a, b \in \mathbb{R}$,

(a) $|a \cdot b| = |a| \cdot |b|$

(b) $|a + b| \leq |a| + |b|$

19. Show that for $a, b, c \in \mathbb{R}$,

(a) $|a - b| \geq ||a| - |b||$

(b) $|a - b| \leq |a - c| + |b - c|$

20. Suppose that $x, a \in \mathbb{R}$ and $\epsilon > 0$ then

(a) $|x| < \epsilon$ if and only if $-\epsilon < x < \epsilon$

(b) $|x - a| < \epsilon$ if and only if $a - \epsilon < x < a + \epsilon$.

Definition: Given two distinct real numbers a and b such that $a < b$, we denote the set $\{x \in \mathbb{R} : a < x < b\}$ with the notation (a, b). We denote $\{x \in \mathbb{R} : a \leq x \leq b\}$ with the notation $[a, b]$. In words (a, b) is the set of all real numbers between a and b. The set (a, b) is called an **open interval**. The set $[a, b]$ is called a **closed interval**. The values a and b are called the **endpoints**.

We will generalize the notation for intervals in the usual ways. For instance, $[a, b)$ is understood to mean the set $\{x \in \mathbb{R} : a \leq x < b\}$. We may also replace a with $-\infty$ or b with ∞ in which case our interval is **unbounded**. If both a and b are finite, the interval is **bounded**.

At this point, we have established that the real numbers are an *ordered field*: they have a particular algebraic structure along with a notion of order. We have also seen how many familiar facts follow from just these few fundamental axioms. Of course there are many more such properties that we could state (and prove), but our goal was just to get a feel for the process of starting with a few definitions and axioms and then using them to prove more complicated statements. From this point on, therefore, we will refer to everything that depends on these axioms alone as "algebra" and will not worry about proving every last rule.

However, the concept of an ordered field is not yet enough to completely characterize the real numbers. The set of rational numbers, for example, is also an ordered field. So what are the important differences between $\mathbb{Q}$ and $\mathbb{R}$?

Definition: Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$.

(i.) a is an **upper bound** of A if $x \leq a$ for all $x \in A$. If a is an upper bound of A and $a \in A$, we say a is a **maximum** of A.

(ii.) a is a **lower bound** of A if $a \leq x$ for all $x \in A$. If a is a lower bound of A and $a \in A$, we say that a is a **minimum** of A.

(iii.) If a set A has an upper bound we say it is **bounded above**. If A has a lower bound we say it is **bounded below**. If A is both bounded both above and below, we say it is **bounded**.

Definition: If u is an upper bound for the set A such that $u \leq a$ for all upper bounds a of A then u is called a **supremum** or **least upper bound** of A.

Definition: If l is a lower bound for the set A and $l \geq a$ for all lower bounds a of A then l is called an **infimum** or **greatest lower bound** of A.

21. Prove that if x and y are both suprema of a set A then $x = y$.

Thus if a set A has a supremum, it is unique and we may speak of ***the*** supremum, which we denote $\sup A$. Likewise, we may speak of ***the*** infimum of a set A which we denote $\inf A$.

22. Find the minimum, maximum, infimum and supremum (if they exist) of the following subsets of $\mathbb{R}$:

(a) $(-3, 3]$

(b) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$

(c) $(0, \infty)$

(d) $\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\}$

Axiom: (Least Upper Bound Property) If a non-empty set $A \subseteq \mathbb{R}$ has an upper bound then it has a least upper bound, i.e. the supremum of A exists. Similarly, if A has a lower bound, then the infimum of A exists.

23. If $A \subseteq \mathbb{R}$ is non-empty and bounded below then A has a greatest lower bound.

24. Suppose that $A \subseteq \mathbb{R}$ is bounded above. Show that for any $\epsilon > 0$ there exists $a \in A$ such that $\sup A - \epsilon < a \leq \sup A$.

25. Prove or disprove: If $A, B \subseteq \mathbb{R}$ are nonempty sets such that for every $a \in A$ and $b \in B$ we have $a < b$ then $\sup A$ and $\inf B$ exist and $\sup A < \inf B$.

26. Suppose $A, B \subseteq \mathbb{R}$ are nonempty and bounded. Define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove that $\sup(A + B) = \sup(A) + \sup(B)$.

27. For any $x \in \mathbb{R}$, show there is a natural number n such that $n > x$.

28. For any $\epsilon > 0$, show that there exists a positive integer n such that $\frac{1}{n} < \epsilon$.

29. For any $a \in \mathbb{R}^+$, show that there is a number $x \in \mathbb{R}$ such that $x^2 = a$.

This completes our look at the real numbers. Every property of the real numbers is a consequence of the three axioms stated above. In fact these three axioms essentially characterize the set of real numbers. Of course, proving every last property of the real numbers would get a bit tedious and so from this point on, you are allowed to use all the basic or 'obvious' properties that you would have used before seeing these notes. Unfortunately, deciding what is 'obvious' is a subjective process and so if there is any doubt about whether something is obvious, you should go ahead and prove it, either from the axioms or other results.

2. Sequences and Convergence

Now that we know something about the real numbers, we begin our investigation of analysis. We start with the concept of a **sequence**. Informally, a sequence is an ordered list of numbers. Because order matters the sequence $(3, 1, 4, 1, 5, 9)$ is different from the sequence $(1, 1, 3, 4, 5, 9)$ even though they have the same elements. In general, sequences may be finite but we will focus our attention on infinite sequences, i.e. infinite ordered lists of real numbers. We can denote sequences with the notation $(s_n)_{n=1}^{\infty}$ where s_n refers to the nth element in the sequence. We will sometimes relax this notation and refer to a sequence by writing simply (s_n). This notation allows us to define many sequences by simply specifying s_n. For example, the sequence $(1, 4, 9, 16, 25, \ldots)$ can be specified by the declaration $s_n = n^2$. The variable n is called the **index** (and does not have to start at 1). This relationship between the elements of the ordered list and the index suggests the following formal definition.

Definition: A **sequence** of numbers is a function $s : \mathbb{N} \to \mathbb{R}$. Instead of the usual convention of writing the range of this function $s(1), s(2), \ldots$ and so on, we will instead list them as $s_1, s_2, \ldots$ as described above.

Definition: Let (a_n) be a sequence. We say that (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Likewise, the sequence (a_n) is **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **strictly** increasing or decreasing if the inequality is strict. A sequence is called **monotonic** (or **monotone**) if it is either increasing or decreasing.

Definition: A sequence (a_n) is **bounded** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded.

30. Graph each of the following sequences in two ways: first as points along the real line and secondly as functions from $\mathbb{N}$ to $\mathbb{R}$. For each sequence, carefully describe the behavior of the terms as the index n gets larger and larger.

(a) $\left(\frac{1}{n}\right)_{n=1}^{\infty}$

(b) $((-1)^n)_{n=1}^{\infty}$

(c) $(2^n)_{n=1}^{\infty}$

(d) $(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots)$.

Informally, we say that sequence *converges* to a number L if the terms a_n are getting closer and closer to L as n gets larger. To make this notion more precise, we need to carefully describe just how close a_n is to L and explain how that depends upon n.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ **converges** to L if for every $\varepsilon > 0$ there exists a positive integer N such that $|a_n - L| < \varepsilon$ for all $N \leq n$. The value L is called a **limit** of the sequence.

Notation: If a sequence $(a_n)_{n=1}^{\infty}$ converges to L, we use the notation $a_n \to L$ or equivalently $\lim\limits_{n\to\infty} a_n = L$.

31. Let $(a_n)_{n=1}^{\infty}$ be a sequence that converges to both L and L'. Show that $L = L'$ and therefore we are justified in talking about ***the*** limit of a sequence.

32. Using the definition of convergence show that the constant sequence $(a_n = c)$ converges to c.

33. Prove that $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges.

34. Prove that $\left(\frac{10n-1}{2n+1}\right)_{n=1}^{\infty}$ converges..

35. Prove that $((-1)^n)_{n=1}^{\infty}$ does not converge.

Definition: A sequence (a_n) is said to **diverge** or be **divergent** if it does not converge to L for any $L \in \mathbb{R}$.

There are two special types of divergence:

(i.) A sequence (a_n) is said to **diverge to** $+\infty$ if for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $a_n \geq M$ for all $n \geq N$.

(ii.) A sequence (a_n) is said to **diverge to** $-\infty$ if for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $a_n \leq M$ for all $n \geq N$.

36. Suppose M is a nonempty bounded set and let $\alpha = \sup M$. Prove there exists a sequence $(a_n)_{n=1}^{\infty}$ that converges to α such that $a_n \in M$ for all natural numbers n. (A similar statement is also true for $\inf M$.)

37. Prove that if the sequence (a_n) converges to L, then the set $M = \{a_n\}$ is bounded.

38. Prove that if a sequence (a_n) is increasing and bounded above then it converges. (Hint: Look at some examples, and consider the supremum.)

39. Suppose (a_n) converges to L and c is a constant. Prove (ca_n) converges to cL.

40. Suppose (a_n) converges to L and (b_n) converges to K. Prove that $(a_n + b_n)$ converges to $L + K$.

41. Suppose (a_n) converges to L and (b_n) converges to K. Prove that $(a_n b_n)$ converges to LK. (Hint: you may need to use one of the mathematician's favorite tricks: add and subtract the same thing from an expression.)

42. Suppose (a_n) converges to L, (b_n) converges to $K \neq 0$, and $b_n \neq 0$ for all natural numbers n. Prove that $(\frac{a_n}{b_n})$ converges to $\frac{L}{K}$.

43. Show that a sequence (a_n) converges to L if and only if every neighborhood of L contains all but finitely many of the terms of (a_n).

44. Prove that if $a_n \to L$ then $|a_n| \to |L|$. Is the converse true? Prove or disprove.

45. Prove that if $a_n \to L$ and $a_n \geq k$ for all $n \in \mathbb{N}$ then $L \geq k$.

46. Show that if $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ then $\lim_{n\to\infty} c_n = L$.

3. Limit Points

Definition: The term **point** refers to an element of $\mathbb{R}$. **A point set** is a collection of zero or more points.

Definition: If $M \subseteq \mathbb{R}$ is a point set and p is a point in $\mathbb{R}$, then p is a **limit point** of M if every open interval containing p also contains a point of M different from p.

47. Let $p \in (a, b)$. Prove p is a limit point of (a, b).

48. Prove or disprove: the point a is a limit point of the interval (a, b).

49. Suppose $p \notin [a, b]$. Prove p is not a limit point of $[a, b]$.

50. Suppose M is a point set that contains a limit point. Prove that M has at least 2 points. Can you say anything else about the size of M?

51. Prove that 2 is a limit point of $M = \{2 + \frac{1}{n} : n \in \mathbb{N}\}$

52. Prove $\mathbb{Z}$ has no limit points.

53. Let H, K be nonempty point sets. Prove or disprove the following statements:

(a) If p is a limit point of $H \cap K$, then p is a limit point of H and p is a limit point of K.
(b) If p is a limit point of $H \cup K$, then p is a limit point of H and p is a limit point of K.

54. Prove or disprove: If a sequence $(a_n)_{n=1}^{\infty}$ converges to p, then the point p is a limit point of the set $S = \{a_n : n \in \mathbb{N}\}$. Is the converse true?

55. Prove that if $M \subseteq \mathbb{R}$ and p is a limit point of M then there is a sequence $(a_n)_{n=1}^{\infty}$ in M such that $p = \lim\limits_{n\to\infty} a_n$.

4. Functions and Continuity

In the previous sections we introduced the concept of convergence and limits in the context of sequences. Now we will use these ideas to investigate *functions*. We begin with some basic definitions which should be familiar – we include them here for the sake of completeness.

Definition: Given sets A and B, a **function** f from A to B, denoted $f : A \to B$, is a subset of the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$ such that for each $a \in A$, there exists one and only one $b \in B$ with $(a, b) \in f$.

Definition: If $f : A \to B$ is a function, then A is called the **domain** and B is the **codomain** of f. The domain is the set of all first coordinates of points of f and the **range** is the set of all second coordinates (i.e. function values) of f.

Notation: We will use the usual functional notation and write $f(a) = b$ to mean that $(a, b) \in f$. We also say that f **maps** a to b. We refer to b as the **value** of the function at a. We will often use the functional notation to define a function without specifying the domain explicitly. In these cases, we will adopt the convention that the domain is assumed to be the largest set for which the functional formula makes sense.

The main distinction between limits of sequences and limits of functions is that in the case of a function, we must specify a particular point at which we are focusing our attention. Intuitively, we will say that a function f has a limit L at the point c if the values $f(x)$ get arbitrarily close to L as x gets very close to c. In particular, the limit does not depend on the value of f at c but only on the values of f at points x ***near*** c. In fact, it is not even necessary that f is defined at c. It is, however, necessary that f be defined at points x ***near*** c in a sense that we can make precise by using the concept of a limit point (indeed this is precisely why we introduced limit points in the first place).

Definition: Let c be a limit point of the domain of a function f. We say that L is a **limit of** f **at the point** c if for every sequence of points $(a_n)_{n=1}^{\infty}$ in the domain of f that converges to c, $(f(a_n))_{n=1}^{\infty}$ converges to L.

Notation: We express the statement that the function f converges to L at the point c with the usual notation $\lim_{x \to c} f(x) = L$.

56. Prove that the limit of a function at a point is unique, i.e. if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$ then $L = M$.

57. Let the function f be given by $f(x) = x$ and let $p \in \mathbb{R}$. Show that $\lim_{x \to p} f(x) = p$.

58. Given any point $p \in \mathbb{R}$, show that $\lim\limits_{x \to p} x^2 = p^2$.

Definition: (Alternative Definition of Limit) Let c be a limit point of the domain of a function f. We say that L is a **limit of f as x approaches** c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x in the domain of f satisfying $0 < |x - c| < \delta$.

59. Prove that the two definitions of the limit of a function are equivalent. (You will need to prove an if and only if statement.)

Definition: The function f is **continuous at** c if and only if

(i.) c is in the domain of f, and

(ii.) for every sequence $(a_n)_{n=1}^{\infty}$ with a_n in the domain of f such that $(a_n) \to c$, the sequence $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$.

In short, $\lim\limits_{x \to c} f(x) = f(c)$. If a function is continuous at every point in its domain, we say it is a continuous function.

60. Let $f(x) = 2x$ for all $x \in \mathbb{R}$. Prove that f is continuous.

61. Consider the function

$$f(x) = \begin{cases} 1 & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find all the points $c \in \mathbb{R}$ at which f is continuous (and justify your answer).

62. Consider the function

$$f(x) = \begin{cases} 0 & x \text{ is irrational,} \\ 1 & x \text{ is rational.} \end{cases}$$

Find all the points $c \in \mathbb{R}$ at which f is continuous (and justify your answer).

63. Consider the function

$$f(x) = \begin{cases} 0 & x \text{ is irrational,} \\ x & x \text{ is rational.} \end{cases}$$

Find all the points $c \in \mathbb{R}$ at which f is continuous (and justify your answer).

64. Prove or disprove: the function $f : \{0\} \to \mathbb{R}$ defined by $f(0) = 1$ is continuous.

65. Prove that if a function f is continuous on $[a, b]$ and there exists $x \in (a, b)$ such that $f(x) > 0$, then exists an open interval T, containing x, such that $f(t) > 0$ for all $t \in T$.

Appendix D: Student Selection Interview Protocol

Introduction:
Hello, thank you for taking the time to talk with me today. This interview is part of the study Mathematical Identity in Inquiry Based Learning Classrooms, in which I am doing research to learn more about student experiences' in undergraduate mathematics inquiry-based learning classrooms. Today I am particularly interested in learning more about you and your history with mathematics. Everything that you tell me is confidential and I will not attach your name to anything that you say or tell anyone else at the university what you have told me. If I ask you anything that you do not feel comfortable answering please feel free to tell me that you do not want to answer that question. To ensure that you understand what the study involves, would you please tell me…
What you think I am asking you to do?
What the biggest risk to you might be if you participate in this interview?
What remaining questions can I answer for you?

Interview Questions:
(These are the types of questions I expect to ask the selection interviewees)

1. Is there any demographic information you are comfortable sharing with me that could be included in the study? This can include race, ethnicity, gender, sexual orientation, religion, socioeconomic status, or anything else you see as a relevant part of your personal identity.

2. In your opinion, have any of the demographic qualities you talked about above influenced your participation in mathematics classes?

3. Can you tell me about a time when you were successful in mathematics?

4. Can you tell me about a time when you had a frustration in mathematics?

5. How would you characterize your experiences with mathematics so far?

6. What is important to you in a mathematics class?

7. How do you feel about doing mathematics at the board, in front of your classmates?

8. What are you plans with regards to mathematics in the future, after college?

9. Why are you taking this class?

10. If you're a math major, why are you a math major?

11. How would you describe the University of Portland math department?

12. How would you describe the students in your mathematics classes so far?

13. What does it mean for someone to be good at mathematics?

14. What are you most excited about for this class?

15. What are you most nervous about for this class?

Appendix E. Student Interviews Protocol

Introduction:
Hello, thank you for taking the time to talk with me today. This interview is part of the study Mathematical Identity in Inquiry Based Learning Classrooms, in which I am doing research to learn more about student experiences' in undergraduate mathematics inquiry-based learning classrooms. I am particularly interested in your classroom experience so far and thoughts on doing mathematics. Everything that you tell me is confidential and I will not attach your name to anything that you say or tell anyone else at the university what you have told me. If I ask you anything that you do not feel comfortable answering please feel free to tell me that you do not want to answer that question. To ensure that you understand what the study involves, would you please tell me…
What you think I am asking you to do?
What the biggest risk to you might be if you participate in this interview?
What remaining questions can I answer for you?

Interview Questions:
(These are the types of questions I expect to ask the primary participants)

1. Has this class been more or less like what you expected? How so?

2. What are the most challenging parts of the class?

3. What are the most rewarding parts of the class?

4. How do you feel about presenting proofs in class?

5. How do you feel watching others present proofs in class?

6. How would you describe the students in this class?

7. How would you describe the professor in this class?

8. How would you describe the classroom environment?

9. I noticed this (example) happen, what is your take on why this happened?

10. Can you tell me more about… what happened during (example)? Your thoughts on this proof presentation? Why this presentation worked well?

11. Can you give me an example of… students working well together? A classroom tension? An unexpected moment?

12. Who do you think is doing well in this class? What are they doing that makes you think this?

13. Are you doing these same traits that you mentioned earlier?

14. What makes a good mathematics student?

After This I Believe Project

1. What did you think of the This I Believe project?

2. What are some ways this experience made you think differently about yourself, if at all?

3. What are some ways this experience made you think differently your classmates, if at all?

4. How do you connect the purpose of this project to MTH 401?

End of Term Questions

1. Has this class been more or less like what you expected or different?

2. What do you think Dr. Miya's goals were for the class to be now that you've finished it? What do you think her big purpose was?

3. What would you say for you personally, what are your biggest takeaways from this class?

4. What would you say your favorite part of class was over the term?

5. What would you say your least favorite part of class was?

6. What sort of changes in yourself do you think you've seen throughout this term, if anything?

7. What have you learned about proofs this term, or how has your understanding of proofs changed over the term?

8. How has your understanding of calculus changed?

9. How has your understanding of mathematics as a field changed?

10. How has your understanding of what it means to be a mathematician changed?

11. Can you tell me about a time when you felt really proud of yourself in class?

12. Can you tell me about a time when you felt frustrated in class?

Appendix F. Professor Interview Protocol

Introduction:
Hello, thank you for taking the time to talk with me today. This interview is part of the study Mathematical Identity in Inquiry Based Learning Classrooms, in which I am doing research to learn more about student experiences' in undergraduate mathematics inquiry-based learning classrooms. I am particularly interested in your thoughts on the classroom dynamics over the past two weeks. I will ask you about general classroom activities and your perspectives on students in the classroom. Everything that you tell me is confidential and I will not attach your name to anything that you say or tell anyone else at the university what you have told me. Your data is primarily being used to triangulate and reflect on my own understandings of the classroom; it is not being used for analysis and presentation on its own. If I ask you anything that you do not feel comfortable answering please feel free to tell me that you do not want to answer that question. To ensure that you understand what the study involves, would you please tell me…
What you think I am asking you to do?
What the biggest risk to you might be if you participate in this interview?
What remaining questions can I answer for you?

Questions:
(These are the types of questions I expect to ask of the professor. I will ask about a variety of students, general and primary participants so that the professor is unaware of who primary participants are.)

1. How would you describe the classroom environment over the past two weeks?

2. What changes have you noticed in the classroom over the past two weeks?

3. Can you tell me more about… what happened during (example)? Your thoughts on this proof presentation? Why this presentation worked well?

4. Can you give me an example of… students working well together? A classroom tension? An unexpected moment?

5. I noticed this (example) happen, what is your take on why this happened?

6. How would you describe Student (name) in class?

7. What changes have you noticed in Student (name)?

8. I noticed you did this (example) in class, what was your reasoning? How did you expect students to respond? Was it successful?