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Application of a Mellin transform to a series which represents a generalization of the Lerch zeta function yields a transformation series. One obtains the asymptotic behavior of the series, together with some associated expansions and limit relations, and moreover, a specialization of parameters yields several classical results.
ON A GENERALIZATION OF THE LERCH ZETA FUNCTION

by

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ON A GENERALIZATION OF THE LERCH ZETA FUNCTION

CHAPTER I

INTRODUCTION

The transcendental function defined by the power series

\[ \phi(z, s, a) = \sum_{n=0}^{\infty} (a + n)^{-s} z^n, \]

valid for \( |z| < 1, a \neq 0, -1, -2, \ldots \) and \( s \) arbitrary has been the subject of a large number of investigations. For \( z = 1 \) the restriction \( \text{Re} \, s > 1 \) is necessary and

\[ \phi(1, s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} = \zeta(s, a), \text{Re} \, s > 1 \]

is the Hurwitz zeta function. The difference equation

\[ \phi(z, s, a) = z^m \phi(z, s, m + a) + \sum_{n=0}^{m-1} (a + n)^{-s} z^n, \]

\( m = 1, 2, 3, \ldots \),

which is a consequence of \( (1) \), makes it sufficient to restrict the parameter \( a \) such that \( 0 < \text{Re} \, a \leq 1 \). The main properties of \( (1) \) (for a short condensation see [7,
vol. 1, p. 27]) were investigated by Lipschitz [16], Lerch [13], Hardy [10] and Barnes [3]. Important results of the above mentioned contributions are:

The series (1) (set \( z = e^{-\tau} \)) admits the transformation into a Fourier series [13]

\[
(4) \quad \phi(e^{-\tau}, s, a) = \sum_{n=0}^{\infty} (a + n)^{-s} e^{-n\tau} = \Gamma(1 - s)e^{\tau a} \sum_{n=-\infty}^{\infty} e^{i2\pi na}(\tau + i2\pi n)^{s-1},
\]

\( \text{Re } s < 1, 0 < a < 1 \) or \( \text{Re } s < 0, 0 < a \leq 1 \)

which leads to the Lerch transformation formula

\[
(5) \quad \phi(e^{-\tau}, s, a) - e^{a\tau} \Gamma(1 - s)\tau^{s-1} =
\]

\[
= -i(2\pi)^{s-1} \Gamma(1 - s)e^{a\tau} \cdot \left[ e^{i(2\pi a + \frac{\pi}{2} s)} \phi\left(e^{i2\pi a}, 1 - s, 1 + \frac{\tau}{2\pi i}\right) -
\right.
\]

\[
- e^{-i(2\pi a + \frac{\pi}{2} s)} \phi\left(e^{-i2\pi a}, 1 - s, 1 - \frac{\tau}{2\pi i}\right) \]

or, if we set \( \tau = \log \left(\frac{1}{z}\right) \),
\[(6) \quad \phi(z, s, a) = \frac{i(2\pi)^{s-1} \Gamma(1 - s) z^{-a}}{\phi(e^{-i2\pi a}, 1 - s, \frac{\log z}{2\pi i}) - e^{i\pi(s + 2a)} \phi(e^{i2\pi a}, 1 - s, 1 - \frac{\log z}{2\pi i})}.\]

**Hardy's relation**

\[(7) \quad \lim_{z \to 1}[\phi(z, s, a) - z^{-a} \Gamma(1 - s) (\log \frac{1}{z})^{s-1}] = \zeta(s, a)\]

gives the character of the singularity of \(\phi\) at \(z = 1\), which is the only singular point in the finite part of the \(z\)-plane.

**Barnes's result.** The function

\[\phi(z, s, a) - z^{1-a} \phi(z, s, 1)\]

has no singularities in the finite part of the \(z\)-plane, except the singularity due to \(z^{1-a}\) at the origin, and admits the expansion

\[(8) \quad \phi(z, s, a) - z^{1-a} \phi(z, s, 1) =
\]

\[= z^{-a} \sum_{k=0}^{\infty} \frac{\log z}{k!} [\zeta(s - k, a) - \zeta(s - k, 1)],\]

convergent for \(|\log z| < 2\pi\).
In all these results the analysis employed is based on the conversion of the infinite series (1) into a loop integral, a subsequent deformation of the loop, and finally the evaluation of the integral by the residue theorem. In more recent contributions a different approach has been chosen. Apostol [1] proved (6) by a method based on the transformation theory of the theta functions and in a further paper [2] it was shown that Hurwitz's series for \( \zeta(s, a) \),

\[
(9) \quad \zeta(s, a) = 2(2\pi)^{s-1}\Gamma(1 - s) \sum_{n=1}^{\infty} n^{s-1}\sin(2\pi n a + \frac{\pi}{2} s),
\]

\( 0 < a < 1, \Re s < 1 \) or \( 0 < a \leq 1, \Re s < 0 \)

can be obtained from (6). This procedure, however, made a rearrangement of (1) necessary, since (9) cannot be obtained from (6) by putting \( z = 1 \). It was shown later in a paper by Oberhettinger [20] that the transition \( z \to 1 \) in (6), or \( \tau \to 0 \) in (5), can be made utilizing Hardy's limit relation (7). In addition, a new proof of (5) and (7) was given applying the summation formulas of Poisson and Plana respectively. The special case \( a = 1 \) in (1) leads to a function
\[
F(z, s) = z\phi(z, s, 1) = \sum_{n=1}^{\infty} n^{-s}z^n, \quad |z| < 1,
\]

which is also known as Jonquières' function [17, p. 33], whose properties have been extensively investigated [11, 12, 19, 21, 23, 26, 27]. It has been widely overlooked, however, that as a special case of \( \phi \) many results concerning \( F(z, s) \) are simple consequences of Lerch's transformation formula (6). Furthermore, a large number of functions occurring in various branches of physics, for example the Fermi-Dirac functions [4, 18, 22], the Bose-Einstein functions [5], the Hahn functions [9], and a number of others [14] are likewise special cases of (1). In view of the fact that the various results concerning \( \phi \) have been obtained by rather different methods, an attempt is made in this thesis to present a more direct and unified approach to this important transcendental function. In addition a generalization of (1) is obtained. This is achieved by the use of a particular summation formula which is based on the Mellin transform and its asymptotic properties. These general considerations are the subject of Chapter II. In Chapter III we apply these results to a simple special case. This leads to the transformation of a function defined by a generalization of (1) containing four parameters. The result is that
can be transformed into the convergent or asymptotic (depending on $\delta$) series

$$
(11) \quad \frac{1}{\delta} \Gamma\left(\frac{1}{\delta} - \frac{\nu}{\delta}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(\nu - \delta k, a) \tau^k.
$$

Since for $\delta = 1$,

$$
(12) \quad \psi(\tau, \nu, a, 1) = \sum_{n=0}^{\infty} (a + n)^{-\nu} e^{-\tau(n+a)}
= e^{-\tau a} \phi(e^{-\tau}, \nu, a),
$$

the series (11) includes a generalization of Barnes's series (8) and Hardy's relation (7). In fact all the relations (4) - (9) can be derived from (11) upon specializing parameters. This is carried out in Chapter IV.
CHAPTER II

A SUMMATION FORMULA INVOLVING THE MELLIN TRANSFORM

Consider the two-sided Laplace transform of a function $F(t)$ and its inversion formula

\begin{equation}
    g(s) = \int_{-\infty}^{\infty} F(t) e^{-st} dt
\end{equation}

\begin{equation}
    F(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} g(s) e^{st} ds.
\end{equation}

With $x = e^{-t}$ and $F(-\log x) = f(x)$ one has the equivalent Mellin transform of a function $f(x)$ and its inversion formula

\begin{equation}
    g(s) = \int_{0}^{\infty} f(x) x^{s-1} dx
\end{equation}

\begin{equation}
    f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} g(s) x^{-s} ds.
\end{equation}

The theory of this type of an integral transform is well developed [6]. We note here a few simple properties:
A sufficient condition for the existence of (15), (16) is that the integral (13) be absolutely convergent. The transform function \( g(s) \) represents an analytic function of \( s \) in a strip \( \alpha_1 < \text{Re} \, s < \alpha_2 \), where \( \alpha_1 \) and \( \alpha_2 \) are the abscissas of ordinary convergence of the integral (15). The strip of analyticity may extend to infinity in one or both directions of \( \text{Re} \, s \). In the latter case, \( g(s) \) represents an entire function. If \( \alpha_1 = \alpha_2 \), then \( g(s) \) is defined on the line \( \text{Re} \, s = \alpha_1 = \alpha_2 \) only. We denote the abscissas of absolute convergence of the integral (15) by \( \beta_1, \beta_2 \) (\( \beta_1 < \beta_2 \)). The constant \( \sigma \) in the inversion integral (16) is chosen such that \( \beta_1 < \sigma < \beta_2 \); that is, the path of integration in (16), consisting of a straight line parallel to the imaginary \( s \)-axis at a distance \( \sigma \), lies in the strip of absolute convergence.

We consider now a function \( f(x) \) of a real variable \( x > 0 \) such that (15) and (16) hold. With this function we form the series

\[
S = \sum_{n=0}^{\infty} f(n + a).
\]  

(17) 

Then by (16)
\[ S = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} g(s)(n + a)^{-s} ds, \]
\[
\text{Max}(1, \beta_1) < \sigma < \beta_2,
\]

where \( g(s) = \int_{0}^{\infty} f(x)x^{s-1} dx \). Again, \( \beta_1 \) and \( \beta_2 \) are the abscissas of absolute convergence. Interchanging the order of summation and integration, we have the series \( \sum_{n=0}^{\infty} (n + a)^{-s} \), which is Hurwitz's zeta function [7, vol. 1, p. 24]

\[ (18) \quad \zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}. \]

This series is absolutely convergent for \( \Re s > 1 \). Moreover, \( \zeta(s, a) - \frac{1}{s - 1} \) is an entire function of \( s \).

We have then

\[ (19) \quad S = \sum_{n=0}^{\infty} f(n + a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s, a)g(s) ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h(s) ds, \]

where
Formula (19) represents the transformation of the left side series into a complex integral, the path of integration being of the type in (16). It may be noted that (19) is a generalization of a formula given by Ferrar [8] (see also [25, p. 63]). Of special interest is the case when the Mellin transform $g(s)$ of the function $f(x)$ involved in the summation is such that it is analytic in a half-plane $\Re s < c$ to the left of the path of integration except for a finite or infinite number of singularities of one-valued character. The same behavior holds then for the function $h(s)$. This suggests the possibility of completing the path of integration in (19) to a closed path by adding a suitable simple contour $C$ such that all or part of the singularities of $h(s)$ in $\Re s < c$ lie inside the closed contour. The application of the residue theorem combined with a suitable behavior of $h(s)$ may then effect the transformation of the left side series in (19) into a so-called residue series for the integral at the right side. This residue series may turn out to be finite, in which case a summation of the series $\sum_{n=0}^{\infty} f(n + a)$ has been effected. In the case of a convergent infinite series, this procedure leads to
transformation of two infinite series into each other. Of special importance is the case in which $h(s)$ contains a parameter $z$, say, such that $h(s)$ as given by (20) is of the form

$$h(s) = z^{-s} \varphi(s),$$

where $\varphi(s)$ is independent of $z$. Then (under certain conditions) an asymptotic series of the sum $S$ in (19) for small positive $z$ can be given and this series, if convergent, represents the exact result. This is expressed in the following theorem [6, vol. 2, p. 115]:

**Theorem.** Let

\begin{equation}
\phi(z) = \frac{1}{2\pi i} \int_{c-i \infty}^{c+i \infty} z^{-s} \varphi(s) ds,
\end{equation}

such that

(a) $\varphi(s)$ is analytic in a left half-plane

$\text{Re } s \leq c$ except for singular points of one-valued character (poles or essential singularities)

$\lambda_0, \lambda_1, \lambda_2, \ldots; c > \text{Re } \lambda_0 > \text{Re } \lambda_1 > \ldots + - \infty$.

The principal part of the Laurent expansion of $\varphi(s)$ at $\lambda_v$ has the form
In each strip of finite width \( c_0 \leq x \leq c \),
\( \varphi(x + iy) \to 0 \) as \( |y| \to \infty \) uniformly in \( x \).

Between two singular points \( \lambda_v \) and \( \lambda_{v+1} \) there is a \( \beta_v \) (real) with
\( \text{Re } \lambda_{v+1} < \beta_v < \text{Re } \lambda_v \), \( (v = 0, 1, 2, \ldots) \)
such that the integral
\[
\int_{-\infty}^{\infty} z^{-iY} \varphi(\beta_v + iy) dy
\]
converges uniformly for \( 0 < z \leq Z_v \).

[This is the case, for example, if
\( \varphi(s) = \varphi(x + iy) = 0(|y|^\alpha) \) for fixed \( x \) with \( \text{Re } \alpha < 0 \).]

Then
\[
\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} z^{-s} \varphi(s) ds = \Phi(z)
\]
converges for \( 0 < z \leq Z_0 \) and
\[(23) \quad \phi(z) = \sum_{\nu=0}^{n} \left[ \frac{b_1^{(\nu)}}{\nu!} (-\log z) + \ldots + \right.
\[+ \frac{b_{r_\nu}^{(\nu)}}{(r_\nu - 1)!} (\log z)^{r_\nu-1} + \ldots \right]\]
\[+ \frac{1}{2\pi i} \int_{\beta_n-i\infty}^{\beta_n+i\infty} z^{-s} \varphi(s) ds\]
\[\text{as } z \to 0 \quad \text{(through positive values)},\]

where the last term is \(o(z^{-n})\). We have then the following asymptotic expansion for \(\phi(z):\)

\[(24) \quad \phi(z) \approx \sum_{\nu=0}^{\infty} \left[ \frac{b_1^{(\nu)}}{\nu!} (-\log z) + \ldots + \right.
\[+ \frac{b_{r_\nu}^{(\nu)}}{(r_\nu - 1)!} (\log z)^{r_\nu-1} + \ldots \right]\]
\[\text{as } z \to 0 \quad \text{(through positive values)}.\]
The first term \((v = 0)\) in the series (24) is due to a shift to the left of the line of integration from \(c - i\omega\) to \(c + i\omega\) parallel to itself such as to pass between \(\lambda_0\) and \(\lambda_1\) under consideration of the residue of \(z^{-s}\varphi(s)\) at \(s = \lambda_0\). The following terms result from a repeated operation of the same kind. A similar theorem holds for the asymptotics of an integral of the same kind as (21) for the case \(z \to \infty\) provided the singularities of \(\varphi(s)\) are in a right half-plane \(\text{Re } s > c\) [6, vol. 2, p. 115].
CHAPTER III

A GENERALIZATION OF LERCH'S ZETA FUNCTION

The analysis outlined in Chapter II is now applied to a special case. This procedure will lead to expansion formulas concerning a function which represents a generalization of the Lerch zeta function. Specialization of certain parameters will yield previously known results. We apply now (19) to the function

\[ f(x) = x^{-\nu} e^{-\tau x^\delta} \]

Here \( \delta > 0 \) and we choose, for the time being \( \tau \) to be real and positive. The analytic continuation to complex \( \tau \) will be obvious later [See formula (34)]. Then from (15) and (25), with the substitution \( x^\delta = t \),

\[ g(s) = \frac{1}{\delta} \int_0^\infty \frac{1}{t^{s-\nu}-1} e^{-\tau t} \, dt \]

This is the gamma function integral and

\[ g(s) = \frac{1}{\delta} \Gamma \left( \frac{s-\nu}{\delta} \right) \]

\( \text{Re } s > \text{Re } \nu \), \( \tau > 0 \), \( \delta > 0 \).
The abscissas of absolute convergence are $\beta_1 = \text{Re } \nu$, $\beta_2 = \infty$. By (19), (25) and (26) we have

$$
(27) \quad \sum_{n=0}^{\infty} (n + a)^{-\nu} e^{-\tau (n+a)} = \\
\frac{1}{2\pi i \delta} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s}{\tau - s} \frac{\zeta(s, a)}{\Gamma(s - \nu)} ds,
$$

$c > \text{Max}(1, \text{Re } \nu)$.

Replacing $s$ by $\delta s$, one obtains

$$
(28) \quad \sum_{n=0}^{\infty} (n + a)^{-\nu} e^{-\tau (n+a)} = \\
\frac{1}{2\pi i \delta} \int_{\gamma-i\infty}^{\gamma+i\infty} \tau^{-s} \zeta(\delta s, a) \Gamma(s - \nu) ds,
$$

$c > \frac{1}{\delta} \text{Max}(1, \text{Re } \nu)$.

This integral is of the form (21). We proceed now to evaluate the integrals (27) and (28). First we make use of (24) in order to establish the asymptotic behavior of (28) for small $\tau$. Having done this, we use (27) for a more detailed investigation of the possible convergence of the asymptotic expansion obtained by the previously mentioned theorem concerning an integral of the form (21). Comparing (28) and (21), we have $z = \tau$ and
(29) \[ \varphi(s) = \tau^{\delta \nu} \xi(\delta s, a) \Gamma(s - \frac{\nu}{\delta}) . \]

It is easily verified that all conditions for \( \varphi(s) \) hold [see (3), (4), (9), (10) Appendix I]. The only singularities of \( \varphi(s) \) in the finite \( s \)-plane are poles of the first order. Hence only the coefficients \( b^{(v)}_1 \) are different from zero; that is, in the expansion (24) only the first term inside the summation sign gives a contribution. Since \( \zeta(z, a) - \frac{1}{z - 1} \) is an entire function of \( z \) and since \( \Gamma(z) - \frac{(-1)^k}{k!} \frac{1}{z + k} \) is analytic near \( z = -k \), \( (k = 0, 1, 2, \ldots) \), it follows that for the set of singularities (all of them simple poles)

\[
\begin{align*}
\text{(a)} & \quad \lambda_0 = \frac{1}{\delta}, \quad b^{(0)}_1 = \tau^{\delta \nu} \Gamma\left(1 - \frac{\nu}{\delta}\right) \quad [\text{due to } \zeta(s, a)], \\
\text{(b)} & \quad \lambda_k = \frac{\nu}{\delta} - k, \quad b^{(k)}_1 = \tau^{\delta \nu} \zeta(\nu - \delta k, a) \frac{(-1)^k}{k!} \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{align*}
\]

Therefore, by (24) and (28)

\[
(31) \sum_{n=0}^{\infty} (n + a)^{-\nu} e^{-\tau(n+a)\delta} \approx \frac{1}{\delta} \Gamma\left(1 - \frac{\nu}{\delta}\right) \tau^{\frac{\nu-1}{\delta}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(\nu - \delta k, a) \tau^k .
\]
Now the series on the left side of (31) represents a function of \( \tau \) analytic in \( \Re \tau > 0 \). The possible convergence of the power series in \( \tau \) on the right side depends on the behavior of its coefficients for large indices \( k \). We find from (5) and (1), Appendix I, that for large \( k \) and fixed \( \nu, \delta, a \)

\[
(32) \quad \frac{1}{k!} \zeta(\nu - \delta k, a) = O[(2\pi)^{-\delta}k^{-k(1-\delta)\log k}].
\]

We obtain then the following result:

The function

\[
(33) \quad \Psi(\tau, \nu, a, \delta) = \sum_{n=0}^{\infty} (n + a)^{-\nu}e^{-\tau(n+a)\delta}
\]

\[\delta > 0, \ a \neq 0, -1, 2, \ldots, \]

is an analytic function of \( \tau \) in \( \Re \tau > 0 \). By (32) one has

\[
(34) \quad \Psi(\tau, \nu, a, \delta) - \frac{1}{\delta} \left( \frac{1}{\delta} \right)^{\nu-1} =
\]

\[= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(\nu - \delta k, a) \tau^k,
\]

where the series is convergent for all \( \tau \) if \( \delta < 1 \) and convergent for \( |\tau| < 2\pi \), if \( \delta = 1 \). If \( \delta > 1 \), the series is divergent and in this case the series on the right in (34) represents an asymptotic expansion of the
function of $\tau$ on the left as $\tau \to 0$. Putting $\tau = \log(\frac{1}{z})$, we can write (34) as follows.

\begin{equation}
\sum_{n=0}^{\infty} (n + a)^{-v} z^{(n+a)\delta - \frac{1}{\delta}} \Gamma\left(\frac{1-v}{\delta}\right) (\log \frac{1}{z})^{\frac{v-1}{\delta}} = \sum_{k=0}^{\infty} \frac{(\log z)^k}{k!} \zeta(v - \delta k, a).
\end{equation}

The series on the right is valid for all $z \neq 0$ if $0 < \delta < 1$. For $\delta > 1$, the relation (35) holds asymptotically as $z \to 1$. For $\delta = 1$, the series on the right in (35) converges for $|\log z| < 2\pi$. Thus, we obtain the limit relations

\begin{equation}
\lim_{\tau \to 0} \left[ \sum_{n=0}^{\infty} (n + a)^{-v} e^{-\tau(n+a)\delta} - \frac{1}{\delta} \Gamma\left(\frac{1-v}{\delta}\right) (\log \frac{1}{z})^{\frac{v-1}{\delta}} \right] = \zeta(v, a),
\end{equation}

\begin{equation}
\lim_{z \to 1} \left[ \sum_{n=0}^{\infty} (n + a)^{-v} z^{(n+a)\delta} - \frac{1}{\delta} \Gamma\left(\frac{1-v}{\delta}\right) (\log \frac{1}{z})^{\frac{v-1}{\delta}} \right] = \zeta(v, a).
\end{equation}

For $\delta = 1$, we have

$$\Psi(\tau, v, a, 1) = \sum_{0}^{\infty} (n + a)^{-v} e^{-\tau(n+a)} = e^{-\tau a} \phi(e^{-\tau}, v, a),$$
\[ \psi(\log \frac{1}{z}, \nu, a, 1) = \sum_{0}^{\infty} (n + a)^{-\nu} z^{n+a} = z^a \phi(z, \nu, a). \]

Consequently,

\[
\phi(e^{-\tau}, \nu, a) - e^{a\tau} \Gamma(1 - \nu) \tau^{\nu-1} =
\]

\[
= e^{\tau a} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(\nu - k, a) \tau^k,
\]

\[ |\tau| < 2\pi, \]

\[
(38)
\]

\[
\phi(z, \nu, a) - z^{-a} \Gamma(1 - \nu) (\log \frac{1}{z})^{\nu-1} =
\]

\[
= z^{-a} \sum_{k=0}^{\infty} \frac{(\log z)^k}{k!} \zeta(\nu - k, a),
\]

\[ |\log z| < 2\pi \]

(39)

(The domain of convergence for this series is discussed in Appendix III.)

and

\[
\lim_{\tau \to 0} \left[ \sum_{0}^{\infty} (n + a)^{-\nu} e^{-n\tau} - \Gamma(1 - \nu) e^{\tau a} \tau^{\nu-1} \right] = \zeta(\nu, a),
\]

\[ (40) \]

\[
\lim_{z \to 1} \left[ \sum_{0}^{\infty} (n + a)^{-\nu} z^n - z^{-a} \Gamma(1 - \nu) (\log \frac{1}{z})^{\nu-1} \right] = \zeta(\nu, a).
\]

\[ (41) \]

The last two equations represent Hardy's limit relation (7), while (38) and (39) lead to Barnes's expansion (8).
It is worthwhile to note that the integral occurring in (27) can also be evaluated directly without recourse to the general theorem, as reflected by equations (21) - (24). The integrand in (27)

\[ h(s) = \tau \frac{-s}{\delta} \Gamma \left( \frac{s - \nu}{\delta} \right) \zeta(s, a) \]

has poles of the first order at \( s = 1 \) and \( s = \nu - \delta k \), \( (k = 0, 1, 2, \ldots) \). Their respective residues are equal to

\[ \tau \frac{-1}{\delta} \Gamma \left( \frac{1 - \nu}{\delta} \right) \text{ at } s = 1 \text{ and } \]

\[ \frac{(-1)^k}{k!} \delta \tau \frac{k - \nu}{\delta} \zeta(\nu - k\delta, a) \text{ at } s = \nu - k\delta . \]

We consider now the integral

\[ \int h(s)ds \]

taken over the closed contour as indicated in Figure 2, \[ [c > \text{Max}(1, \text{Re} \nu)]. \]
The radius $R_N$ of the half-circle in the left half-plane is chosen such that it separates the poles $s = v - N\delta$ and $s = v - (N + 1)\delta$ of $h(s)$. We then let $R_N$ tend to infinity. The contributions to the integral along the horizontal lines vanish as $R_N \to \infty$. This follows
from the behavior of $h(s)$ due to (3), (4), (10), Appendix I. On the circumference of the half-circle $s = R e^{i\varphi}$. We have for large $R$ by (1), (7), (8), Appendix I,

\begin{equation}
|h(s)| = 0 \left\{ R^{\frac{\nu}{\delta}} \exp[R(\frac{1}{\delta} - 1)\cos \varphi \log R +
+ R \cos \varphi(\log 2\pi - \frac{1}{\delta} \log \tau) - R\varphi(\frac{1}{\delta} - 1)\sin \varphi -
- R \cos \varphi(\frac{1}{\delta} - 1 + \frac{1}{\delta} \log \delta)] \right\}, \quad \frac{\pi}{2} < \varphi < \pi.
\end{equation}

In this interval $\cos \varphi < 0$, $\sin \varphi > 0$ and the term $\exp[R(\frac{1}{\delta} - 1)\cos \varphi \log R]$ is the dominant one provided $\delta \neq 1$. That is to say, the contribution to the integral (43) along the half-circle vanishes as $R_{N} \to \infty$ regardless of $\tau$ if $\delta < 1$. If $\delta = 1$, the contribution along the half-circle vanishes if $\tau < 2\pi$ (this is the condition for the convergence of the residue series). The case $\delta > 1$ leads to an exponential increase of $h(s)$ on the half-circle as $R_{N}$ increases. Thus, the residue theorem leads exactly to our formerly obtained result (34), where in addition the asymptotic character of the residue series in the case $\delta > 1$ was stated.
Let us apply now the results obtained in Chapter III to two special cases. First we wish to derive Hurwitz's series (9). In (34) we set \( \tau = \pm i2\pi\beta \) \((0 \leq \beta < 1)\), \( a = 1 \), and \( \delta = 1 \). Then

\[
\sum_{n=0}^{\infty} e^{\mp 2\pi i(n+1)} (n + 1)^{-\nu} = \mp i\Gamma(1 - \nu)(2\pi\beta)^{-\nu-1} e^{\mp \pi i \nu / 2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (2\pi\beta)^n e^{\pm \pi i n / 2} \zeta(\nu - n).
\]

This relation is valid for \( \text{Re } \nu > 1, \ 0 \leq \beta < 1 \). Multiplying the above relation first by \( e^{\pm \pi i \nu / 2} \), then by \( e^{-i \pi \nu / 2} \), and adding the results, we obtain

\[
\sum_{n=0}^{\infty} (n + 1)^{-\nu} \cos 2[(n + 1)\beta - \pi / 2 \nu] = (2\pi\beta)^{-\nu-1} \sin(\pi\beta)\Gamma(1-\nu) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (2\pi\beta)^n \zeta(\nu-n) \cos[\pi / 2(\nu+n)],
\]

\( \text{Re } \nu > 1, \ 0 \leq \beta < 1 \).
If the series on the right is split into even and odd terms, it follows (with $\sin(\pi v)\Gamma(1 - v) = \frac{\pi}{\Gamma(v)}$ ) that

$$\sum_{n=1}^{\infty} n^{-v} \cos\left(2\pi n\beta - \frac{\pi}{2}v\right) = \frac{\pi (2\pi \beta)^{v-1}}{\Gamma(v)} +$$

$$+ \cos\left(\frac{\pi}{2}v\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2\pi \beta)^{2n} \zeta(v - 2n) +$$

$$+ \sin\left(\frac{\pi}{2}v\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} (2\pi \beta)^{2n+1} \zeta(v - 2n - 1),$$

Re $v > 1, \ 0 \leq \beta < 1$.

If $v$ is replaced by $1 - s$, we obtain by means of (5), Appendix II,

(45) $$\sum_{1}^{\infty} n^{s-1} \sin(2\pi n\beta + \frac{\pi}{2}s) = \frac{1}{2} (2\pi)^{1-s} \frac{\zeta(s, \beta)}{\Gamma(1 - s)} ,$$

which is Hurwitz's series (9).

We proceed now to derive Lerch's transformation formula (5). For this purpose we use the expansion formula (38). We divide (38) by $\Gamma(1 - v)$ and replace the Hurwitz zeta function in the series at the right by the just derived expression (45). We have
\[
\frac{(-1)^k}{k!} \frac{\zeta(\nu - k, a)}{\Gamma(1 - \nu)} = 2(2\pi)^{\nu-1}(2\pi)^{-k} \frac{\Gamma(1 - \nu + k)}{k! \Gamma(1 - \nu)} \cdot \sum_{n=1}^{\infty} n^{\nu-1} n^k \sin(2\pi na + \frac{\pi}{2} \nu - \frac{\pi}{2} k).
\]

We observe that (binomial coefficients)
\[
(-1)^k \frac{\Gamma(1 - \nu + k)}{k! \Gamma(1 - \nu)} = \binom{\nu - 1}{k}.
\]

Then (interchanging the order of summation),
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\zeta(\nu - k, a)}{\Gamma(1 - \nu)} \tau^k = 2(2\pi)^{\nu-1}.
\]

\[
\cdot \sum_{n=1}^{\infty} n^{\nu-1} \left[ \sum_{k=0}^{\infty} \binom{\nu - 1}{k} \left( \frac{\tau}{2\pi n} \right)^k \sin(2\pi na + \frac{\pi}{2} \nu - \frac{\pi}{2} k) \right].
\]

Collecting the even and odd terms of this sum,
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\zeta(\nu - k, a)}{\Gamma(1 - \nu)} \tau^k =
\]
\[
= 2(2\pi)^{\nu-1} \left\{ \sum_{n=1}^{\infty} n^{\nu-1} \sin(2\pi na + \frac{\pi}{2} \nu) \left[ \sum_{k=0}^{\infty} \binom{\nu-1}{2k} (-1)^k \left( \frac{\tau}{2\pi n} \right)^{2k} \right] - \sum_{n=1}^{\infty} n^{\nu-1} \cos(2\pi na + \frac{\pi}{2} \nu) \left[ \sum_{k=0}^{\infty} \binom{\nu-1}{2k+1} (-1)^k \left( \frac{\tau}{2\pi n} \right)^{2k+1} \right] \right\}.
\]
We use now the elementary formulas

\[
\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{2k} z^{2k} = \frac{1}{2} \left[ (1 + iz)^{\alpha} + (1 - iz)^{\alpha} \right]
\]

\[
\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{2k+1} z^{2k+1} = -\frac{1}{2} i[(1 + iz)^{\alpha} - (1 - iz)^{\alpha}]
\]

from which it follows that

\[
-\frac{1}{2} i(1 - iz)^{\alpha} e^{i\delta} + \frac{1}{2} i(1 + iz)^{\alpha} e^{-i\delta} =
\]

\[
= \sin \delta \sum_{k=0}^{\infty} \binom{\alpha}{2k} (-1)^k z^{2k} - \cos \delta \sum_{k=0}^{\infty} \binom{\alpha}{2k+1} (-1)^k z^{2k+1}
\]

so that

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\xi(\nu - k, a)}{\Gamma(1 - \nu)} \tau^k =
\]

\[
= -i(2\pi)^{\nu-1} \left\{ \sum_{n=1}^{\infty} n^{\nu-1} \left[ \frac{e^{i(2\pi n + \pi/2)\nu}}{(1 - i\frac{\tau}{2\pi n})^{1-\nu}} - \frac{e^{-i(2\pi n + \pi/2)\nu}}{(1 + i\frac{\tau}{2\pi n})^{1-\nu}} \right] \right\}
\]

Therefore,
\[ \phi(e^{-\tau}, \nu, a) - \Gamma(1-\nu)e^{\tau a \Gamma(1-\nu)} = -i(2\pi)^{\nu-1}e^{\tau a \Gamma(1-\nu)}. \]

\[ \sum_{n=1}^{\infty} \left[ \frac{i(2\pi n a + \frac{\pi}{2} \nu)}{(n + \frac{\tau}{2\pi})^{1-\nu}} - \frac{-i(2\pi n a + \frac{\pi}{2} \nu)}{(n - \frac{\tau}{2\pi})^{1-\nu}} \right] \]

which is (5) written in explicit form (see also [20, equation 11]). The conditions given in [20],

\[ \text{Re } \nu < 0, \; 0 < a \leq 1, \; 0 \leq \text{Im } \tau < 2\pi, \]

are sufficient to justify interchanging the order of summation carried out previously.


12. Lawden, D.F. The function \( \sum_{n=1}^{\infty} n^r z^n \) and associated polynomials. Proceedings of the Cambridge
13. Lerch, M. Note sur la fonction \( K(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{i2\pi kx}}{(w+k)^s} \). Acta Mathematica 11:19-24. 1887.


APPENDICES
APPENDIX I

Asymptotic Behavior of $\Gamma(s)$ and $\zeta(s)$

The asymptotic behavior of certain analytic functions is needed for the evaluation of the integral in (27) or (28) as outlined in Chapter III. This concerns mainly the functions $g(s)$, $\Gamma(s)$ and $\zeta(s, a)$. First we notice that if $g(s)$ is the Mellin transform of $f(x)$, then $g(s) \to 0$ as $\text{Im } s \to \pm \infty$ for fixed $\text{Re } s$ (in the strip of absolute convergence). This follows by the Riemann-Lebesgue lemma for Fourier integrals. With regard to $\Gamma(s)$ and $\zeta(s, a)$ we consider two possibilities of the transition as $|s| \to \infty$.

A. Let $s = x + iy = \text{Re } e^{i\varphi}$, $\varphi = \text{arg } s$, and let $R \to \infty$ in a given sector $\alpha < \text{arg } s < \beta$.

B. Let $s = x + iy$ and let $y \to \pm \infty$ for fixed $x$.

We have (Stirling's formula [7, vol. 1, p. 47])

\begin{align}
(1) \quad \Gamma(s) &= \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} e^{-s} e^{s \log s} \left[1 + O\left(\frac{1}{s}\right)\right]; \\
(2) \quad \frac{\Gamma(s + \alpha)}{\Gamma(s + \beta)} &= s^{\alpha - \beta} \left[1 + O\left(\frac{1}{s}\right)\right], \quad (\alpha, \beta \text{ fixed}).
\end{align}

Both formulas are valid as $|s| \to \infty$ in $-\pi < \text{arg } s < \pi$. 
Also

\[
(3) \quad |\Gamma(x \pm iy)| = O\left(|y|^{x-\frac{1}{2}} e^{-\frac{\pi}{2}|y|}\right)
\]
as \(y \to \infty\) for fixed \(x\).

As for the asymptotic behavior of \(\zeta(s, a)\) for large \(s\) (\(0 < a \leq 1\)), it is clear (for instance from the integral expression [17, p. 23] or from Hurwitz's series (9)) that the dependency on the parameter \(a\) is of negligible importance. That is to say, the asymptotic behavior of \(\zeta(s, a)\) for large \(s\) can be replaced by that of

\[
\zeta(s, 1) = \zeta(s).
\]

From the definition of \(\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\)

(Re \(s > 1\)) as a Dirichlet series, it is obvious that

\[
(4) \quad |\zeta(s)| = O(1), \quad \text{Re } s \geq 1.
\]

Using (4) and the well-known functional equation

\[
(5) \quad \zeta(s) = 2(2\pi)^{s-1} \sin(\frac{\pi}{2}s) \Gamma(1-s)\zeta(1-s)
\]

\[
= \frac{\pi (2\pi)^{s-1} \zeta(1-s)}{\cos(\frac{\pi}{2}s) \Gamma(s)}
\]

with \(s = x + iy\), the result is

\[
(6) \quad |\Gamma(s)\zeta(s)| = O\left(e^{-\frac{\pi}{2}|y|} e^{x \log(2\pi)}\right), \quad \text{Re } s \leq 0.
\]
[The equal sign in (4) and consequently in (6) follows by continuity if one considers (10)].

Applying (1) to (5), we obtain

\begin{equation}
|\zeta(s)| = O\left(\frac{1}{R^2} e^{v(R,\varphi)}\right)
\end{equation}

where,

\begin{equation}
v(R, \varphi) = -R \log R \cos \varphi + R \cos \varphi + \\
+ R \cos \varphi \log(2\pi) - (\frac{\pi}{2} - \varphi)R \sin \varphi
\end{equation}

as \( R \to \infty \) in \( \frac{\pi}{2} < \varphi < \pi \) and \( -\pi < \varphi < -\frac{\pi}{2} \)

(the validity in the latter interval follows by Schwartz's reflection principle).

Similarly, (3) applied to (6) gives

\begin{equation}
|\zeta(s)| = |\zeta(x \pm iy)| = O\left(|y|^{\frac{1}{2} - \chi}\right)
\end{equation}

as \( y \to \infty \) and \( x < 0 \).

Finally, in the critical strip \( 0 < x < 1 \) [24, p. 82], we have

\begin{equation}
|\zeta(x \pm iy)| = O\left(|y|^{\frac{1}{2} - \frac{1}{2} \chi}\right)
\end{equation}

as \( y \to \infty \) and \( 0 < x < 1 \).
It is not difficult to show (see the integral expression [7, vol. 1, p. 26] or use Hurwitz's series for $\zeta(s, a)$) that the same asymptotic relations hold also for $\zeta(s, a)$. That is to say, in the asymptotic sense the terms involving $a$ in $\zeta(s, a)$ are of a lower order compared with the dominant part which is independent of $a$. 
APPENDIX II

A Taylor Series Expansion for $\zeta(s, a)$

It follows from

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}, \text{ Re } s > 1$$

that

$$\left[ \frac{d^n}{da^n} \zeta(s, a) \right]_{a=1} = (-1)^n \frac{\Gamma(n + s)}{\Gamma(s)} \zeta(n + s).$$

Hence, by Taylor's theorem

(1) $\Gamma(s)\zeta(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a - 1)^n \Gamma(n + s) \zeta(n + s),$ \ 

$|a - 1| < 1, \ s \neq 1$;

or, with Riemann's functional equation

$$\Gamma(n + s)\zeta(n + s) = \frac{1}{2} (2\pi)^{s+n} \frac{\zeta(1 - s - n)}{\cos[\frac{\pi}{2}(s + n)]}$$

(2) $\Gamma(s)\zeta(s, a) = \pi(2\pi)^{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{[2\pi(a - 1)]^n}{\cos[\frac{\pi}{2}(s + n)]} \zeta(1 - s - n),$ \ 

$|a - 1| < 1, \ s \neq 1.$
If (2) is split into even and odd terms in the summation, the result is

\[ \Gamma(s)\zeta(s, a) = \]

\[ = \pi(2\pi)^{s-1} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{[2\pi(a-1)]^{2n}}{\cos\left(\frac{\pi}{2}s\right)} \zeta(1-s-2n) + \right. \]

\[ + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{[2\pi(a-1)]^{2n+1}}{\sin\left(\frac{\pi}{2}s\right)} \zeta(-s-2n) \right\} , \]

\[ |a - 1| < 1, \ s \neq 1. \]

Upon multiplying by \( \sin\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s\right) \), we obtain

\[ \zeta(s, a) = \]

\[ = 2(2\pi)^{s-1} \Gamma(1-s) \left\{ \sin\left(\frac{\pi}{2}s\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} [2\pi(a-1)]^{2n} \zeta(1-s-2n) + \right. \]

\[ + \cos\left(\frac{\pi}{2}s\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} [2\pi(a-1)]^{2n+1} \zeta(-s-2n) \right\} , \]

\[ |a - 1| < 1, \ s \neq 1 ; \]

or, with \( a = 1 + \alpha \) and \( \zeta(s, 1 + \alpha) = \zeta(s, \alpha) - \alpha^{-s} \).
(5) \( \zeta(s, \alpha) - \alpha^{-s} = \zeta(s, 1 + \alpha) = \)

\[= 2(2\pi)^{s-1} \Gamma(1-s) \left\{ \sin\left(\frac{\pi}{2} s\right) \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!} (2\pi \alpha)^{2n} \zeta(1-s-2n) + \right. \]
\[+ \cos\left(\frac{\pi}{2} s\right) \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2\pi \alpha)^{2n+1} \zeta(-s-2n) \right\}, \]

\(|\alpha| < 1, \ s \neq 1 .\)
APPENDIX III

Some Mapping Properties

The transition from (34) to (35) represents a mapping of the \( \tau \)-plane into the \( z \)-plane by means of the mapping function

\[ z = e^{-\tau}. \]

Let \( \tau = \sigma + it, \ z = x + iy, \) then

\[ x = e^{-\sigma} \cos t, \quad y = -e^{-\sigma} \sin t. \]

The strip \( 0 \leq t < 2\pi \) in the \( \tau \)-plane is mapped into the entire \( z \)-plane and the half-strip \( 0 < \sigma < \infty, \ 0 \leq t < 2\pi \) is mapped into the interior of the circle \( |z| = 1 \). Of particular importance is the mapping of the \( \tau \)-domain \( (|\tau| < 2\pi) \) of the power series in (38) into that \( (|\log z| < 2\pi) \) of the corresponding series (39). Setting

\[ \tau = \rho e^{i\varphi}, \quad z = e^{-\tau} = e^{-\rho e^{i\varphi}} = re^{i\theta} \]

one obtains

\[ |z| = r = e^{-\rho} \cos \theta, \quad \theta = -\rho \sin \varphi. \]

It is sufficient to investigate the mapping of the upper half-circle \( |\tau| < 2\pi \) into the \( z \)-plane, since the
same considerations apply to the lower half-circle. For this purpose we observe the image point $z$ as $\tau$ traverses the arc ABC of the upper half-circle (Figure 1).

Then by (3),

\begin{align*}
(4) \quad |z| &= r = e^{-2\pi \cos \varphi} \\
(5) \quad \theta &= \arg z = -2\pi \sin \varphi.
\end{align*}
It is obvious from (5) that $\theta$ decreases monotonically from $\theta = 0$ to $\theta = -2\pi$ as $\tau$ describes the arc $AB$, and increases monotonically from $\theta = -2\pi$ back to $\theta = 0$ as $\tau$ describes the arc $BC$; while by (4), $r$ increases monotonically from $e^{-2\pi}$ to $e^{2\pi}$ along the entire arc $ABC$. Thus, for each $-2\pi < \theta \leq 0$ there exist two different values of $r$. Denote these values by $r$ and $r'$ such that $r > r'$ (the smaller is encountered if $0 \leq \varphi < \frac{\pi}{2}$; the larger, if $\frac{\pi}{2} < \varphi \leq \pi$, and $r = r'$ if $\varphi = \frac{\pi}{2}$). Therefore from (4) and (5),

\begin{align*}
(6) \quad r &= e^{\sqrt{4\pi^2 - \theta^2}}, \quad (\frac{\pi}{2} \leq \varphi \leq \pi) ; \\
(7) \quad r' &= e^{-\sqrt{4\pi^2 - \theta^2}} = \frac{1}{r}, \quad (0 \leq \varphi \leq \frac{\pi}{2}).
\end{align*}

These are the polar equations of two curves $C$ and $C'$ in the $z$-plane. We observe that

$r = r'$ for $\theta = -2\pi$ ;

\[
\frac{dr}{d\theta} = \frac{dr'}{d\theta} = 0 \quad \text{for} \quad \theta = 0.
\]

Furthermore,

\[
\frac{dr}{d\theta} \to -\infty \quad \text{and} \quad \frac{dr'}{d\theta} \to +\infty \quad \text{as} \quad \theta \to -2\pi.
\]
The last observation means that the slope of both $C$ and $C'$ is zero at $\theta = -2\pi$.

A few numerical examples will serve to illustrate the situation. In the following table, we give the correspondence of a point $\tau$ on the arc ABC with its image $z$-point as expressed by (5), (6), (7) ($r$ belongs to $C$ and $r'$ to $C'$).

<table>
<thead>
<tr>
<th>$\sin \phi$</th>
<th>$\phi$ (in degrees)</th>
<th>$\theta = -2\pi \sin \phi$</th>
<th>$r$</th>
<th>$r' = \frac{1}{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$e^{2\pi}$</td>
<td>$e^{-2\pi}$</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>7.2</td>
<td>$-\frac{1}{4\pi}$</td>
<td>$\frac{\pi}{4\sqrt{63}}$</td>
<td>$\frac{-\pi}{4\sqrt{63}}$</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>14.5</td>
<td>$-\frac{1}{2\pi}$</td>
<td>$\frac{\pi}{2\sqrt{15}}$</td>
<td>$\frac{-\pi}{2\sqrt{15}}$</td>
</tr>
<tr>
<td>$\frac{3}{8}$</td>
<td>22.0</td>
<td>$-\frac{3}{4\pi}$</td>
<td>$\frac{\pi}{4\sqrt{55}}$</td>
<td>$\frac{-\pi}{4\sqrt{55}}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
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<td>$-\pi$</td>
<td>$e^{\pi\sqrt{3}}$</td>
<td>$e^{-\pi\sqrt{3}}$</td>
</tr>
<tr>
<td>$\frac{5}{8}$</td>
<td>38.7</td>
<td>$-\frac{5}{4\pi}$</td>
<td>$\frac{\pi}{4\sqrt{39}}$</td>
<td>$\frac{-\pi}{4\sqrt{39}}$</td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td>48.6</td>
<td>$-\frac{3}{2\pi}$</td>
<td>$\frac{\pi}{2\sqrt{7}}$</td>
<td>$\frac{-\pi}{2\sqrt{7}}$</td>
</tr>
<tr>
<td>$\frac{7}{8}$</td>
<td>61.1</td>
<td>$-\frac{7}{4\pi}$</td>
<td>$\frac{\pi}{4\sqrt{15}}$</td>
<td>$\frac{-\pi}{4\sqrt{15}}$</td>
</tr>
<tr>
<td>1</td>
<td>90</td>
<td>$-2\pi$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
We see that

\[ r_{\text{max}} = e^{2\pi} \sim 530, \quad r_{\text{min}} = 1; \]

\[ r'_{\text{max}} = 1, \quad r'_{\text{min}} = e^{-2\pi} \sim 0.0019. \]

Thus, while \( \tau \) traverses the arc AB, the image point \( z \) traverses a spiral-like curve \( C' \) in the negative direction, starting at \( \theta = 0 \) with \( r'_{\text{min}} \) and terminating at \( \theta = -2\pi \) with \( r'_{\text{max}} = 1 \). As \( \tau \) proceeds along the arc BC the \( z \)-point travels along an "outer" spiral \( C \) in the positive direction, starting at \( \theta = -2\pi \) with \( r_{\text{min}} = r'_{\text{max}} = 1 \) and terminating at \( \theta = 0 \) with \( r_{\text{max}} = e^{2\pi} \). The "inner" spiral \( C' \) keeps (with the exception of the immediate neighborhood of \( \theta = -2\pi \)) very close to the origin (for \( \theta = -315^\circ \) the value of \( r' \) is still only \( \sim 0.05 \)), while the values of \( r \) for the "outer" spiral \( C \), starting with \( r = 1 \), increase rapidly.

This fact is illustrated in Figure 2 (the numbers at certain points of \( C \) denote the distance from the origin). It is, of course, not possible to sketch the "inner" spiral \( C' \) because of its relatively very small linear dimensions and we therefore omit it in Figures 2 and 3.

(Note that \( \frac{r'_{\text{max}}}{r'_{\text{min}}} = \frac{r_{\text{max}}}{r_{\text{min}}} = e^{2\pi} \sim 530 \).)
-2\pi \leq \arg z \leq 0

\text{z-plane}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Mapping of the upper half-circle of the \( \tau \)-plane into the \( z \)-plane.}
\end{figure}