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An Adiabatic Approach to Analysis of Time Inhomogeneous Markov Chains: a Queueing Policy Application

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In this thesis, convergence of time inhomogeneous Markov chains is studied using an adiabatic approach. The adiabatic framework considers slowly changing systems and the adiabatic time quantifies the time required for the change such that the final state of the system is close to some equilibrium state. This approach is used in Markov chains to measure the time to converge to a stationary distribution. Continuous time reversible Markov chains on a finite state space with generators changing at fixed time intervals are studied. This characterization is applied to a Markovian queueing model with unknown arrival rate. The time inhomogeneous Markov chain is induced by a queueing policy dependent on uncertainties in arrival rate estimation. It is shown that the above convergence happens with high probability after a sufficiently large time. The above evolution is studied via simulations as well and compared to the bounds suggested by the analysis. These results give the sufficient amount of time one must wait for the queue to reach a stationary, stable distribution under our queueing policy.
An Adiabatic Approach to Analysis of Time Inhomogeneous Markov Chains: a Queueing Policy Application

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Leena Zacharias, Author
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A Markov chain is a finite or countable state space random process where the probability of any particular future behavior of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behavior [1]. If these probabilities, called transition probabilities, are independent of time, it is called a stationary or time homogeneous Markov chain or else, a non-stationary or time inhomogeneous Markov chain. The theory of Markov chains is extensively used in queueing theory [2] and consequently in communication networks [3], among other fields like biology, physics, economics etc.

It is a well-known result that an irreducible, ergodic Markov chain converges to a unique stationary distribution. The time taken to converge to this distribution is of interest in applications such as Monte Carlo Markov chain [4] which is a widely used algorithm to sample from a desired probability distribution and works by constructing a Markov chain which has the desired distribution as the stationary distribution. After a large number of steps the Markov chain must be “$\epsilon$-close” according to some distance notion to its stationary distribution. This is the idea of Markov chain mixing and the time is quantified by mixing time [5]. In the case of time inhomogeneous Markov chains, the evolution of Markov chains must be slow enough such that the distribution is “$\epsilon$-close” to the stationary distribution of the
final transition matrix. This is the so-called adiabatic approach and this time is quantified by adiabatic time \([6]\).

The adiabatic approach follows along the lines of the adiabatic evolution in quantum mechanics \([7]\). The quantum adiabatic theorem studies the evolution of a system from an initial Hamiltonian to a final Hamiltonian (operator corresponding to total energy of the system) and says that if the evolution is slow enough, the system will be “\(\epsilon\)-close” in \(\ell^2\) norm to the ground state of the final Hamiltonian. We consider \(\ell^1\) norms in our study of Markov chains.

We study the convergence, in adiabatic sense, of time inhomogeneous Markov chains, where the time inhomogeneity can be due to an underlying nonstationary process or uncertainties in measurement of an underlying stationary process. We apply the above convergence results to a Markovian queueing model where the time inhomogeneity arises due to the latter of the two conditions mentioned above.

1.1 Queueing Models in Networking

Queues are extensively used in the analysis of communication networks. Packets are forwarded from source to destination by intermediate servers called as routers. These routers look at the address in the packets and decide on an output link which sends the packet to its next hop. If the selected output link is busy, the packets are kept in a queue and served when the link is free. There can also be other concerns like congestion in the network and quality of service for different kinds of traffic. Another use of queueing in networks is in wireless nodes where packets are sent out on a wireless medium and if the medium is not free, they form a queue at the node. Here the concerns are power spent in transmission, collision
in the medium etc.

A queue is typically defined by the arrival rate, service or departure rate, number of servers and buffer size. In a finite buffer size queue, we are interested in maintaining a distribution which is more biased towards smaller queue lengths, since otherwise we will have high blocking probabilities. Such a stable queue is achieved by keeping the departure rate strictly above the arrival rate. In a network situation for example, there might be other constraints also on the departure rate. If the departure rate is too large, it might cause congestion somewhere else in the network. Or in the case of multiple queues of different kinds of traffic, each of which has to satisfy some quality of service conditions, sending packets from just one queue will affect the other queues. In wireless networks, we might require to maintain such a departure rate and nothing more, due to power restrictions. Also, in a network of multiple such nodes we might have problems of collision in the wireless medium. Hence it becomes necessary to monitor the departure rate and keep it at such a level to keep our queue stable and at the same time achieve the desired network objectives.

We consider a queue in which packets arrive at a fixed but unknown rate and we use an estimate of this arrival rate to decide a queueing policy designed to ensure stable queues. In particular, we let the departure rate be always higher than the estimated arrival rate anticipating the estimate to be correct eventually. The evolution of the time inhomogeneous Markov chain is dictated by this adaptive departure policy. We study the time required for the queue to reach this stable distribution using the above outlined adiabatic approach and under suitable estimation and departure policies for the unknown arrival rate.
1.2 Related Work

The adiabatic theorem in quantum mechanics was first stated by Born and Fock [8]. A version of the adiabatic theorem [7] considers two Hamiltonians and the evolution of the system from the initial to final Hamiltonian. The theorem states that for sufficiently large time, the final state of the system and the ground state of the final Hamiltonian will be $\epsilon$-close in $\ell^2$ norm. The lower bound on time was found to be inversely proportional to the cube of the least spectral gap of the Hamiltonian over all time.

The adiabatic theorem for Markov chains was studied by Kovchegov [6] where the adiabatic evolution was studied for discrete time and continuous time Markov chains. The linear evolution of a time inhomogeneous Markov chain from an initial to a final probability transition matrix was studied and the adiabatic time was found to be proportional to the square of the mixing time or inversely proportional to the square of the spectral gap of the final transition matrix. This result has also been generalized to a more general adiabatic dynamics in [9]. [10] studies the sufficient condition on adiabatic time for transition of general birth-death chains which is a general case of our specific queueing application.

1.3 Overview

- Adiabatic evolution of continuous time reversible Markov chains with bounded generators where the changes happen at fixed time intervals.
- Upper bound on distance between the distribution of the Markov chain and
the stationary distribution at large enough time.

- Application of the above upper bound to an M/M/1/K queueing model with unknown but constant arrival rate.

- Estimate of the arrival rate and corresponding dependent departure policy to ensure an eventually stable queue.

- Evolution of continuous time generator matrices determined by the changing, random departure rates based on the estimates of the arrival rate.

- The main result gives the sufficient time needed for the queue to converge to the eventual stable distribution.

Chapter 2 gives some of mathematical preliminaries including definitions and general results which we use in the analysis. In Chapter 3, we look at the evolution of continuous time Markov chains whose generator matrices change at fixed time intervals and find an upper bound on the distance to a stationary distribution. In Chapter 4, the above result is applied to a queue and we consider a specific form of birth and death processes dictated by our estimation of the unknown arrival rate to the queue. The adiabatic time for this system is evaluated. We also compare the distance predicted by our upper bounds to the actual distance via simulations. Finally, Chapter 5 concludes the present work and looks at some possible future directions.
Chapter 2 – Mathematical Preliminaries

In this chapter we present the main definitions and some results which will be useful in our analysis later.

2.1 Notation

- A vector $x$ is a column vector whose $i$th entry is denoted by $x(i)$ and transpose denoted by $x^T$.
- A matrix $P$ is a square matrix whose $(i,j)$th entry is denoted by $P_{ij}$.

2.2 Reversible Markov chains

We consider reversible Markov chains which are defined as follows.

**Definition 1 (Reversibility)** Let $P$ be a transition matrix and $\pi$ a strictly positive probability distribution on a finite state space $\Omega$. The pair $(P, \pi)$ is reversible if the detailed balance equations

$$\pi(i)P_{ij} = \pi(j)P_{ji}$$

hold for all $i, j \in \Omega$. 

\[\square\]
If $P$ is irreducible, then $\pi$ is the unique stationary distribution of $P$.

Define the following scalar products and the corresponding norms in terms of a stationary distribution.

**Definition 2** Let $\pi$ be a strictly positive probability distribution on a finite state space $\Omega$ with $|\Omega| = r$ and let $\ell^2(\pi)$ be the real vector space $\mathbb{R}^r$ with the following scalar product and norm,

$$
\langle x, y \rangle_{\pi} = \sum_{i \in \Omega} x(i)y(i)\pi(i),
$$

$$
\|x\|_{\pi} = \left(\sum_{i \in \Omega} x(i)^2\pi(i)\right)^{\frac{1}{2}},
$$

and let $\ell^2(\frac{1}{\pi})$ be the real vector space $\mathbb{R}^r$ with the following scalar product and norm,

$$
\langle x, y \rangle_{\frac{1}{\pi}} = \sum_{i \in \Omega} \frac{x(i)y(i)}{\pi(i)},
$$

$$
\|x\|_{\frac{1}{\pi}} = \left(\sum_{i \in \Omega} \frac{x(i)^2}{\pi(i)}\right)^{\frac{1}{2}}.
$$

The following result is from Chapter 6 of [4, p. 202].

**Proposition 1 ([4])** The pair $(P, \pi)$ is reversible if and only if

$$
P^* = D^{\frac{1}{2}}PD^{-\frac{1}{2}},
$$

(2.1)
is a symmetric matrix where

\[ D = \text{diag}\{\pi(1), \ldots, \pi(r)\}. \] (2.2)

Moreover, a reversible matrix \( P \) has real eigenvalues with right eigenvectors orthonormal in \( \ell^2(\pi) \) and left eigenvectors orthonormal in \( \ell^2(\frac{1}{\pi}) \).

The proof is in Appendix A. The above proposition is used in the proof of the following result from Chapter 6, Theorem 3.3 of [4, p. 209].

**Proposition 2 ([4])** Let \( P \) be a reversible irreducible transition matrix on the finite state space \( \Omega \), with the stationary distribution \( \pi \) and second largest eigenvalue modulus \( |\lambda_2(P)| \). Then for any probability distribution \( \nu \) on \( \Omega \) and for all \( n \geq 1 \)

\[ \| \nu^T P^n - \pi^T \|_{\frac{1}{\pi}} \leq |\lambda_2(P)|^n \| \nu - \pi \|_{\frac{1}{\pi}}. \]

See Appendix B for proof.

### 2.3 Continuous time Markov chains

For a continuous time Markov chain \( \{X_t\}_{t \geq 0} \) on a finite state space, with a bounded generator matrix, we can find the transition probabilities by the technique of uniformization [11].

**Proposition 3 (Uniformization)** For a continuous time Markov chain \( \{X_t\}_{t \geq 0} \) on a finite state space \( \Omega \) with a bounded generator matrix \( Q = [Q_{ij}]_{i,j \in \Omega} \), and \( q = \)
\( \max_{i \in \Omega} \sum_{j \neq i} Q_{ij} \), the upper bound on the departure rates of all states, transition probabilities are given by

\[
P(t) = \sum_{n=0}^{\infty} e^{-qt} \frac{(qt)^n}{n!} P^n = e^{Qt},
\]

(2.3)

where the matrix \( P = I + \frac{1}{q} Q \).

See Appendix C for proof. The following result is used to bound the distance of the continuous chain in terms of a discrete one and is based on [4, p. 364].

**Proposition 4** For a continuous time Markov chain on a finite state space \( \Omega \) with generator matrix \( Q = q(P - I) \) with reversible, irreducible \( P \) and stationary distribution \( \pi \), for any probability distribution \( \nu \) on \( \Omega \),

\[
\| \nu^T e^{Qt} - \pi^T \|_1 \leq \| \nu - \pi \|_1 e^{-q(1 - |\lambda_2(P)|))t},
\]

where \( |\lambda_2(P)| \) is the second largest eigenvalue modulus of \( P \).

The proof is in Appendix D.

### 2.4 Other Results

We use total variation distance to measure the distance between two probability distributions.

**Definition 3 (Total variation distance)** For any two probability distributions...
\( \nu \) and \( \pi \) on a finite state space \( \Omega \), we define

\[
\|\nu - \pi\|_{TV} = \frac{1}{2} \sum_{i \in \Omega} |\nu(i) - \pi(i)|.
\]

Note that the above function measures the distance on a scale of 0 to 1. The following result is used to relate total variation distance to the \( \frac{1}{\pi} \) norm defined in Definition (2) and is from Chapter 6, Theorem 3.2 of [4, p. 209].

**Proposition 5 ([4])** For any probability distribution \( \nu \) and a strictly positive probability distribution \( \pi \),

\[
\|\nu - \pi\|_{TV} \leq \frac{1}{2}\|\nu - \pi\|_{\pi}.
\]

The proof is in Appendix E.

The following result gives the exact values for eigenvalues of matrices with a special structure and is from [12].

**Proposition 6** The eigenvalues of an \( r \times r \) matrix,

\[
P = \begin{pmatrix}
1 - \beta & \beta & 0 & 0 & \cdots \\
1 - \beta & 0 & \beta & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\cdots & 0 & 1 - \beta & 0 & \beta \\
\cdots & 0 & 0 & 1 - \beta & \beta \\
\end{pmatrix}
\]
where $0 < \beta < 1$ are given by $1, 2\sqrt{\beta(1-\beta)} \cos \left( \frac{\pi j}{r} \right), j = 1, \ldots, r-1$ and therefore the second largest eigenvalue modulus is $|\lambda_2(P)| = 2\sqrt{\beta(1-\beta)} \cos \left( \frac{\pi}{r} \right)$.

Refer Appendix F for proof. This matrix arises in the queueing example which we consider in Chapter 4.

2.5 Mixing and Adiabatic Time

Mixing time of a Markov chain measures the time needed for the Markov chain to converge to its stationary distribution.

**Definition 4** For a continuous time Markov chain $P(t)$, with stationary distribution $\pi$, given an $\epsilon > 0$, the mixing time $t_{mix}(\epsilon)$ is defined as

$$t_{mix}(\epsilon) = \inf \left\{ t : \| \nu^T P(t) - \pi^T \|_{TV} \leq \epsilon, \text{ for all probability distributions } \nu \right\}.$$

To define adiabatic time, consider a linear evolution [6] of generator matrices described by

$$Q \left( \frac{t}{T} \right) = \left( 1 - \frac{t}{T} \right) Q_{\text{initial}} + \frac{t}{T} Q_{\text{final}},$$

for $T > 0$ and bounded generators $Q_{\text{initial}}$ and $Q_{\text{final}}$. If $\pi_f$ is the unique stationary distribution for $Q_{\text{final}}$, adiabatic time measures the time needed for the chain to converge to $\pi_f$.

**Definition 5** Given the above transitions generating a continuous time Markov
chain $P(0, T)$ and $\epsilon > 0$, adiabatic time is defined as

$$T_\epsilon = \inf \left\{ T : \| \nu^T P(0, T) - \pi^T \|_{TV} \leq \epsilon, \text{ for all probability distributions } \nu \right\}.$$

We will look at a different evolution of generator matrices in Chapter 3.

2.6 Queueing Theory Basics

A queue is specified by

- Input process (interarrival distribution)
- Service mechanism (service time distribution)
- Number of channels or servers
- Buffer size or holding capacity for waiting items

We consider an $M/M/1/K$ queue in Chapter 4 defined by

- Poisson input process
- Exponential service time
- Single server
- Buffer size $K$
Chapter 3 – Analysis via Adiabatic Framework

In this chapter we define an evolution of continuous time Markov generator matrices and look at the convergence in Definition (5) in terms of this new evolution. Consider the following evolution: time is divided into slots of size $\Delta t$ and the generator matrix changes at these intervals. A bounded generator matrix $Q_i$ determines the transition probabilities in the time interval $(i\Delta t, (i+1)\Delta t]$. The method of uniformization gives the corresponding transition probability matrix $P(i\Delta t, (i+1)\Delta t)$ as in (2.3). Let the upper bound on departure rates over all states be $q_i$ for each $Q_i$. Therefore,

$$P(i\Delta t, (i+1)\Delta t) = e^{Q_i\Delta t} = e^{q_i(P_i-I)\Delta t},$$

where $P(t_1, t_2)$ denotes the matrix of transition probabilities from time $t_1$ to $t_2$. Let the matrix $P_i$ be irreducible and reversible with stationary distribution $\pi_i$ and second largest eigenvalue modulus $|\lambda_2(P_i)|$. Note that this evolution can be the result of any kind of time inhomogeneity in the system resulting in a changing $Q_i$ which is updated at fixed intervals of time. As noted before, the time inhomogeneity can be due to the nature of the underlying process or due to uncertainties in measurements of parameters and the following theorem captures either kind.
3.1 A General Result for Time Inhomogeneous Markov Chains

Let $\nu_n$ be the distribution of the chain at time $n\Delta t$. We are interested in the distance between $\nu_n$ and the stationary distribution $\pi_n$ corresponding to matrix $P_n$ at time $n\Delta t$. The following theorem gives an upper bound on the distance at time $n\Delta t$ in terms of the distance at time $n_0\Delta t$ for $n_0 < n$.

**Theorem 1** For the time inhomogeneous Markov chain generated by the matrices $\{Q_i\}_{i=0}^n = \{q_i(P_i - I)\}_{i=0}^n$ from time 0 to $n\Delta t$,

$$
\|\nu_n - \pi_n\|_{TV} \leq \frac{1}{2}\|\nu_{n_0} - \pi_{n_0}\|_{\pi_{n_0}} \prod_{i=n_0}^{n-1} e^{-q_i(1-|\lambda_2(P_i)|)\Delta t} \sqrt{\max_{k \in \Omega} \frac{\pi_i(k)}{\pi_{i+1}(k)}}
$$

$$
+ \frac{1}{2} \sum_{i=n_0}^{n-1} \|\pi_i - \pi_{i+1}\|_{\pi_{i+1}} \prod_{j=i+1}^{n-1} e^{-q_j(1-|\lambda_2(P_j)|)\Delta t} \sqrt{\max_{k \in \Omega} \frac{\pi_j(k)}{\pi_{j+1}(k)}},
$$

where $\nu_n$ is the distribution at time $n\Delta t$, $\nu_{n_0}$ is the distribution at time $n_0\Delta t$ for $n_0 < n$ and $\{P_i\}_i$ are irreducible and reversible with stationary distribution $\pi_i$ and second largest eigenvalue modulus $|\lambda_2(P_i)|$.

**Proof:** We start with the $\frac{1}{\pi}$ norm so that we can use the result in Proposition 4.

$$
\|\nu_n - \pi_n\|_{\frac{1}{\pi_{n}}} = \|\nu_{n-1}^T e^{Q_{n-1}\Delta t} - \pi_{n-1}^T\|_{\frac{1}{\pi_{n}}}
$$

$$
\leq \|\nu_{n-1}^T e^{Q_{n-1}\Delta t} - \pi_{n-1}^T\|_{\frac{1}{\pi_{n}}} + \|\pi_{n-1} - \pi_n\|_{\frac{1}{\pi_n}}
$$

$$
\leq \|\nu_{n-1}^T e^{Q_{n-1}\Delta t} - \pi_{n-1}^T\|_{\frac{1}{\pi_{n}}} \sqrt{\max_{k \in \Omega} \frac{\pi_{n-1}(k)}{\pi_n(k)}}
$$

$$
+ \|\pi_{n-1} - \pi_n\|_{\frac{1}{\pi_n}},
$$

$$
\leq \|\nu_{n-1} - \pi_{n-1}\|_{\frac{1}{\pi_{n-1}}} e^{-q_{n-1}(1-|\lambda_2(P_{n-1})|)\Delta t} \sqrt{\max_{k \in \Omega} \frac{\pi_{n-1}(k)}{\pi_n(k)}}
$$

$$
+ \|\pi_{n-1} - \pi_n\|_{\frac{1}{\pi_n}}.
$$
where (3.1) follows from triangle inequality, (3.2) follows from the definition of the norm and (3.3) follows from Proposition 4. We can expand the above iteratively using triangle inequality for every time step which gives us

\[
\| \nu_n - \pi_n \| \leq \| \nu_{n_0} - \pi_{n_0} \| \prod_{i=n_0}^{n-1} e^{-q_i(1-|\lambda_2(P_i)|)|\Delta t} \sqrt{\max_{k \in \Omega} \frac{\pi_i(k)}{\pi_{i+1}(k)}} 
+ \sum_{i=n_0}^{n-1} \| \pi_i - \pi_{i+1} \| \prod_{j=i+1}^{n-1} e^{-q_j(1-|\lambda_2(P_j)|)|\Delta t} \sqrt{\max_{k \in \Omega} \frac{\pi_j(k)}{\pi_{j+1}(k)}} .
\]

The result follows from Proposition 5.

Note that Theorem 1 involves terms from time $n_0 \Delta t$ onwards. The motive behind this is, at large enough $n_0$ we can put an upper bound on all the terms in Theorem 1.

This theorem gives the distance upper bounds for evolutions of the kind described where the generator matrices change at regular time intervals. One way in which this can happen is because of changes in the underlying process itself, for example, changes in the arrival rate of a queueing process. Another way is when lack of knowledge of parameters of the system forces us to use estimates in place of actual parameters, for example, when the arrival rate to a queue is constant but unknown. The above theorem applies to either kind whereas in the next chapter, we look at a specific queueing example where the latter is the cause.
Chapter 4 – Application to Queueing Model

In this chapter we apply the above defined adiabatic evolution model to a queueing process. In particular, we consider time inhomogeneity due to uncertainty in a parameter. Consider an M/M/1/K queue with unknown packet arrival rate $\lambda$ per unit time. We estimate $\lambda$ at time $i\Delta t$ denoted by $\hat{\lambda}_i$ and decide packet departure rate, $\mu_i = f(\hat{\lambda}_i)$ based on this estimate.

**Definition 6** Queueing policy is defined as the sequence $\{\hat{\lambda}_i, \mu_i = f(\hat{\lambda}_i)\}_{i \geq 1}$ where $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mu_i$ is applied for time from $(i\Delta t, (i+1)\Delta t]$.

This decides the following generator matrix from $(i\Delta t, (i+1)\Delta t]$:

$$Q_i = \begin{pmatrix}
-\lambda & \lambda & 0 & 0 & \ldots \\
\mu_i & -(\mu_i + \lambda) & \lambda & 0 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\ldots & 0 & \mu_i & -(\mu_i + \lambda) & \lambda \\
\ldots & 0 & 0 & \mu_i & -\mu_i
\end{pmatrix}$$  \hspace{1cm} (4.1)

The corresponding transition probability matrix $P(i\Delta t, (i+1)\Delta t)$ is obtained as in (2.3). The upper bound on departure rates over all states is $\lambda + \mu_i$. Therefore,

$$P(i\Delta t, (i+1)\Delta t) = e^{Q_i \Delta t} = e^{(\lambda + \mu_i)(P_i - I) \Delta t},$$
where the matrix, \( P_i \) is

\[
P_i = \begin{pmatrix}
1 - \beta_i & \beta_i & 0 & 0 & \ldots \\
1 - \beta_i & 0 & \beta_i & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\ldots & 0 & 1 - \beta_i & 0 & \beta_i \\
\ldots & 0 & 0 & 1 - \beta_i & \beta_i \\
\end{pmatrix},
\]

(4.2)

where \( \beta_i = \frac{\lambda}{\mu_i + \lambda} \). The \( P_i \)'s are reversible and irreducible with stationary distribution given by

\[
\pi_i = \frac{1}{\sum_{\tau=0}^{K} \rho_i^\tau} \left[ 1, \rho_i, \rho_i^2, \ldots, \rho_i^{K-1}, \rho_i^K \right]^T,
\]

where \( \rho_i = \frac{\beta_i}{1 - \beta_i} = \frac{\lambda}{\mu_i} \). Notice that for \( \rho_i < 1 \) the stationary distribution is more biased towards the smaller queue lengths and for \( \rho_i > 1 \) the queue has higher probability to be fuller. The second largest eigenvalue modulus is given by

\[
|\lambda_2(P_i)| = 2\sqrt{\frac{\rho_i}{1 + \rho_i}} \cos\left(\frac{\pi}{K+1}\right)
\]

which follows from Proposition 6. Also, the matrix \( Q_0 \) could be decided from a random departure rate and corresponds to the transition probability matrix \( P(0, \Delta t) \).

4.1 Performance of an Adaptive Queueing Policy

Now we look at a specific queueing policy determined by the time average of number of packets arrived.
Definition 7

$$\hat{\lambda}_i = \frac{1}{i\Delta t} \sum_{k=1}^{i} X_k,$$

$$\mu_i = f(\hat{\lambda}_i) = (1 + \delta)\hat{\lambda}_i,$$

where $X_k \sim \text{Poisson}(\lambda\Delta t)$ is the number of packets in the $k$th slot of duration $\Delta t$ and $\delta > 0$ is a constant.

This particular queueing policy ensures that the departure rate is always higher than the estimated arrival rate and since the estimated arrival rate itself must approach the actual one, should ensure a stable queue. Remember our observation about stationary distributions being biased towards smaller queue lengths for $\rho_i < 1$.

With this queueing policy, we have the adiabatic evolution generated by the matrices $Q_i$ in (4.1). The corresponding $P_i$'s are given by (4.2). The ratio $\rho_i = \frac{\lambda}{\mu_i} = \frac{\lambda}{(1+\delta)\hat{\lambda}_i}$. With full knowledge of the arrival rate, the above ratio becomes $\rho = \frac{1}{1+\delta}$ and let the corresponding matrix $P$ has stationary distribution $\pi$. We need to redefine the adiabatic time for the above evolution as the evolution itself is random now and we can only talk about distances in probabilistic terms.

Definition 8 Given the above transitions generating a continuous time Markov chain $P(0, n\Delta t)$ and $\epsilon > 0$, adiabatic time is defined as

$$T_{\epsilon, \gamma} = \Delta t \cdot \inf \{ n : \Pr[\|\nu^T P(0, n\Delta t) - \pi^T\|_{TV} < \epsilon] > 1 - \gamma \},$$

for all probability distributions $\nu$. 

where \( \pi \) is the stationary distribution corresponding to the queueing policy \( \{ \lambda, (1 + \delta) \lambda \} \).

Note that here \( \nu \) stands for the initial distribution and \( \nu^T P(0, n\Delta t) = \nu_n \), the distribution at time \( n \).

**Theorem 2 (Main result)** Given \( 0 < \epsilon < 1, 0 < \gamma < 1 \) and \( \lambda \), the unknown arrival rate, the queueing policy in Definition (7) with \( \delta > 0 \) for an M/M/1/K queue has

\[
T_{\epsilon, \gamma} \leq \frac{2 \log \frac{2}{\gamma_1}}{\lambda(\epsilon_0^2 - \epsilon_0^3)} + \frac{\log \left( 2 \left[ (1 + \epsilon_0)(1 + \delta) \right]^{K+1} \right) - \log(\epsilon \delta)}{\frac{1}{2\Delta t} \log \left( \frac{\left[ (1-\epsilon_0)(1+\delta) \right]^{K+1} - 1}{\left[ (1-\epsilon_0+\epsilon_1)(1+\delta) \right]^{K+1} - 1} \right) + \lambda \left( \sqrt{1+\delta}(1-\epsilon_0) - 1 \right)^2},
\]

where \( \epsilon_0 \) satisfies

\[
e^{-\lambda \Delta t(\sqrt{1+\delta}(1-\epsilon_0)-1)^2} \sqrt{\frac{\left[ (1 - \epsilon_0 + \epsilon_1)(1 + \delta) \right]^{K+1} - 1}{\left[ (1 - \epsilon_0)(1 + \delta) \right]^{K+1} - 1}} \leq 1,
\]

\[
1 - e^{-\lambda \Delta t(\sqrt{1+\delta}(1-\epsilon_0)-1)^2} \sqrt{\frac{\left[ (1 - \epsilon_0 + \epsilon_1)(1 + \delta) \right]^{K+1} - 1}{\left[ (1 - \epsilon_0)(1 + \delta) \right]^{K+1} - 1}} \leq \frac{\epsilon}{2},
\]

\[
\epsilon_0 \leq \frac{\delta \epsilon}{(1 + \delta)(1 + \epsilon)},
\]

and \( \epsilon_1 = \frac{\lambda \Delta t(\epsilon_0^2 - \epsilon_0^3)}{2 \log \frac{1}{\gamma_1}} (\frac{1}{\sqrt{\lambda \Delta t \gamma_2}} + \epsilon_0), 0 < \gamma_1 < \gamma, \gamma_2 = \gamma - \gamma_1. \)

The above theorem gives a sufficient condition on the time we must wait before the distribution of the queue length converges to the desired stationary distribution \( \pi \). Note that \( \pi \) is decided by \( \delta \) and can be designed to give a stable stationary distribution. Hence the theorem gives the sufficient amount of time to converge
to a stable distribution with a very high probability. The choice of $\epsilon_0$ to be the largest which satisfies all three conditions will give the lowest time in the theorem. At large enough time the estimated arrival rate must approach the actual arrival rate and the difference can be bounded by $\epsilon_0$ with a high probability. Furthermore two consecutive estimates, can differ only by a maximum of $\epsilon_1$.

4.2 Proof of Theorem 2

To prove Theorem 2, we need to bound the terms in Theorem 1. Since $\hat{\lambda}_i \Delta t$ is an empirical average of $i$ iid Poisson($\lambda \Delta t$) distributed random variables, it follows from the law of large numbers that it must approach the actual value of $\lambda \Delta t$ at large enough $i$. Furthermore, two consecutive estimates must be very near to each other. We need this fact to bound terms which have both $i$ and $i+1$ in them like $\sqrt{\max_{k \in \Omega} \pi_i(k) \over \pi_{i+1}(k)}$ and $\|\pi_i - \pi_{i+1}\|_1 \pi_{i+1}$. The above facts are made precise in the following lemma.

**Lemma 3** For $0 < \epsilon_0 < 1$ and $0 < \gamma_1 < 1$, there exists $n_0 = \frac{2 \log \frac{\lambda}{\lambda - \epsilon_0}}{\lambda \Delta t (\epsilon_0 - \epsilon_1)}$ such that

- $\forall i \geq n_0$, with probability at least $1 - \gamma_1$

\[
|\hat{\lambda}_i - \lambda| \leq \lambda \epsilon_0, \quad (4.3)
\]

\[
e^{-\lambda_i (1 - |\lambda_2(P_i)|) \Delta t} < e^{-\lambda \Delta t (\sqrt{(1+\delta)(1-\epsilon_0)} - 1)^2}, \quad (4.4)
\]

\[
\|\pi_i - \pi\|_{TV} < \frac{1}{2 \delta} \frac{\epsilon_0 (1+\delta)}{\epsilon_0 (1+\delta)}, \quad (4.5)
\]

- For $\epsilon_1 = \frac{1}{n_0} \epsilon_0 \frac{(1+\delta)}{(\sqrt{\lambda \Delta t}) + \epsilon_0}$ and $0 < \gamma_2 < 1$ and $\forall i \geq n_0$, with probability at
least \( 1 - \gamma_1 - \gamma_2 \)

\[
|\hat{\lambda}_{i+1} - \hat{\lambda}_i| < \lambda \epsilon_1, \tag{4.6}
\]

\[
\sqrt{\max_{k \in \{0,1,...,K\}} \frac{\pi_i(k)}{\pi_{i+1}(k)}} < \sqrt{\left[ \frac{(1 - \epsilon_0 + \epsilon_1)(1 + \delta)}{(1 - \epsilon_0)(1 + \delta)} \right]^{K+1} - 1}, \tag{4.7}
\]

\[
\|\pi_i - \pi_{i+1}\|_{\pi_{i+1}} \frac{1}{\pi_{i+1}} < \sqrt{\frac{(1 - \epsilon_0)(1 + \delta) \epsilon_1}{(1 - \epsilon_0)^2(1 + \delta) - (1 - \epsilon_0 + \epsilon_1)}}. \tag{4.8}
\]

- With probability at least \( 1 - \gamma_1 \),

\[
\|v_{n_0} - \pi_{n_0}\| \frac{1}{\pi_{n_0}} < \left[ \frac{(1 + \epsilon_0)(1 + \delta)}{\delta} \right]^\frac{K+1}{2}. \tag{4.9}
\]

Refer Appendix G for proof.

Rewriting Theorem 1 for the queueing model we have described,

\[
\|v_n - \pi_n\|_{TV} \leq \frac{1}{2} \left\|v_{n_0} - \pi_{n_0}\right\| \prod_{i=n_0}^{n-1} e^{-(\lambda+\mu_i)(1 - |\lambda_2(P_i)|)\Delta t} \sqrt{\max_{k \in \{0,1,...,K\}} \frac{\pi_i(k)}{\pi_{i+1}(k)}}
\]

\[
+ \frac{1}{2} \sum_{i=n_0}^{n-1} \left[ \|\pi_i - \pi_{i+1}\|_{\pi_{i+1}} \prod_{j=i+1}^{n-1} e^{-(\lambda+\mu_j)(1 - |\lambda_2(P_j)|)\Delta t} \right] \sqrt{\max_{k \in \{0,1,...,K\}} \frac{\pi_j(k)}{\pi_{j+1}(k)}}. \]
According to Lemma 3, the above inequality can be bounded, with probability at least \(1 - \gamma_1 - \gamma_2 = 1 - \gamma\),

\[
\|\nu_n - \pi_n\|_{TV} < \frac{1}{2} \left[ \frac{(1 + \epsilon_0)(1 + \delta)}{\delta} \right]^{\frac{K+1}{2}} \left[ e^{-\lambda \Delta t(\sqrt{(1+\delta)(1-\epsilon_0)} \cdot 1)} \right]^{n-n_0} \sqrt{\frac{[(1 - \epsilon_0 + \epsilon_1)(1 + \delta)]^{K+1} - 1 \gamma n-n_0}{[(1 - \epsilon_0)(1 + \delta)]^{K+1} - 1}} \\
+ \frac{1}{2} \frac{\sqrt{(1 - \epsilon_0)(1 + \delta)} \epsilon_1}{(1 - \epsilon_0)^2(1 + \delta) - (1 - \epsilon_0 + \epsilon_1)} \sum_{i=n_0}^{n-1} \left[ e^{-\lambda \Delta t(\sqrt{(1+\delta)(1-\epsilon_0)} \cdot 1)} \right]^{n-i-1} \sqrt{\frac{[(1 - \epsilon_0 + \epsilon_1)(1 + \delta)]^{K+1} - 1 \gamma n-n_0}{[(1 - \epsilon_0)(1 + \delta)]^{K+1} - 1}} \\
< \frac{1}{2} \left[ \frac{(1 + \epsilon_0)(1 + \delta)}{\delta} \right]^{\frac{K+1}{2}} \left[ e^{-\lambda \Delta t(\sqrt{(1+\delta)(1-\epsilon_0)} \cdot 1)} \right]^{n-n_0} \sqrt{\frac{[(1 - \epsilon_0 + \epsilon_1)(1 + \delta)]^{K+1} - 1 \gamma n-n_0}{[(1 - \epsilon_0)(1 + \delta)]^{K+1} - 1}} \\
+ \frac{1}{2} \frac{\sqrt{(1 - \epsilon_0)(1 + \delta)} \epsilon_1}{(1 - \epsilon_0)^2(1 + \delta) - (1 - \epsilon_0 + \epsilon_1)}.
\]

\[
\sum_{i=0}^{\infty} \left[ e^{-\lambda \Delta t(\sqrt{(1+\delta)(1-\epsilon_0)} \cdot 1)} \right]^{i} \sqrt{\frac{[(1 - \epsilon_0 + \epsilon_1)(1 + \delta)]^{K+1} - 1 \gamma n-n_0}{[(1 - \epsilon_0)(1 + \delta)]^{K+1} - 1}} \\
= \frac{1}{2} \left[ \frac{(1 + \epsilon_0)(1 + \delta)}{\delta} \right]^{\frac{K+1}{2}} \left[ e^{-\lambda \Delta t(\sqrt{(1+\delta)(1-\epsilon_0)} \cdot 1)} \right]^{n-n_0} \sqrt{\frac{[(1 - \epsilon_0 + \epsilon_1)(1 + \delta)]^{K+1} - 1 \gamma n-n_0}{[(1 - \epsilon_0)(1 + \delta)]^{K+1} - 1}} \\
+ \frac{1}{2} \frac{\sqrt{(1 - \epsilon_0)(1 + \delta)} \epsilon_1}{(1 - \epsilon_0)^2(1 + \delta) - (1 - \epsilon_0 + \epsilon_1)} \le \frac{\epsilon}{2}. \tag{4.10}
\]
where (4.10) holds if \( \epsilon_0 \) satisfies
\[
e^{-\lambda \Delta t (\sqrt{(1+\delta)(1-\epsilon_0)})^2} \sqrt{\frac{[(1-\epsilon_0+\epsilon_1)(1+\delta)]^{K+1}}{[(1-\epsilon_0)(1+\delta)]^{K+1}} - 1} < 1.
\]
Now the distance between \( \nu_n \) and \( \pi \) which is the stationary distribution corresponding to the matrix \( P \) with full knowledge of \( \lambda \) is given by
\[
\|\nu_n - \pi\|_{TV} = \|\nu_n - \pi_n + \pi_n - \pi\|_{TV} \leq \|\nu_n - \pi_n\|_{TV} + \|\pi_n - \pi\|_{TV} \leq \epsilon,
\]
using triangle inequality.

Our aim is to find the \( n \) such that \( \|\nu_n - \pi\|_{TV} < \epsilon \). (4.10) holds if both terms are \( \leq \frac{\epsilon}{4} \). For the first term we get,
\[
\frac{[(1+\epsilon_0)(1+\delta)]^{K+1}}{\delta} \cdot \sqrt{\frac{[(1-\epsilon_0+\epsilon_1)(1+\delta)]^{K+1} - 1}{[(1-\epsilon_0)(1+\delta)]^{K+1} - 1}^{n-n_0}} \leq \frac{\epsilon}{2}
\]
or
\[
n - n_0 \geq \log \left( 2 \frac{[(1+\epsilon_0)(1+\delta)]^{K+1}}{\delta} \right) - \log(\epsilon \delta) + \frac{1}{2} \log \left( \frac{[(1-\epsilon_0)(1+\delta)]^{K+1} - 1}{[(1-\epsilon_0+\epsilon_1)(1+\delta)]^{K+1} - 1} \right) + \lambda \Delta t (\sqrt{(1+\delta)(1-\epsilon_0)} - 1)^2
\]
which gives a condition on \( n \). Adding with \( n_0 \) and multiplying by \( \Delta t \) gives the adiabatic time as defined in Definition (8). For the second term of (4.10),
\[
\frac{\sqrt{(1-\epsilon_0+\epsilon_1)(1+\delta)}}{(1-\epsilon_0)^2(1+\delta) - (1-\epsilon_0+\epsilon_1)} \leq \frac{\epsilon}{2},
\]
which gives a second condition on $\epsilon_0$.

From (4.11) and (4.5),

\[ \| \pi_n - \pi \|_{TV} < \frac{1}{2} \frac{\epsilon_0(1 + \delta)}{\delta - \epsilon_0(1 + \delta)} \leq \frac{\epsilon}{2}, \]  \hspace{1cm} (4.12)

which gives

\[ \epsilon_0 \leq \frac{\delta \epsilon}{(1 + \delta)(1 + \epsilon)}, \]

and a third condition on $\epsilon_0$. This completes the proof of Theorem 2.

4.3 Simulations

4.3.1 Distance variations over time

Now we look at the distance of the queue from a stable distribution with increasing time. This distance must be small for large enough time from the discussions above. Here we look at distance as a function of time for a single sample path to verify whether the distance at the adiabatic time predicted is indeed lower than $\epsilon$.

**Simulation 1** $\lambda = 10, \delta = 0.1, K = 100, \Delta t = 0.5, \epsilon = 0.1, \gamma = 0.05, \gamma_1 = 0.04$.

Theorem 2 predicts $n_0 = 174391$ and $n = 175438$ for $\epsilon_0 = 0.003$ (the highest which satisfies all the conditions in Theorem 2) or adiabatic time for $\epsilon = 0.1, \gamma = 0.05$, $T_{0.1,0.05} = 87719$. Figure 4.1 shows the results of a simulation which measures the distance at each time slot vs. the distance upper bound given by triangle
inequality in Theorem 1. The triangle inequality upper bound very closely follows the actual distance and both are much smaller than the distance upper bound predicted by (4.10) and (4.12), which is 0.056. Note that this is only a sample path. This indicates that even though Theorem 2 gives $T_{0.1,0.05} = 87719$ as a sufficient condition, the distance of $\epsilon = 0.1$ is achieved much before. For example, Figure 4.2 shows the simulation results for $T = 5000$ or $n = 10000$. (The upper bound starts from $n_0 = 500$.) This was also found to be true for all 100 sample paths tested. The Figure 4.3 shows the histogram of distances in these 100 sample paths. Note that all of them have distance $< 0.1$.

Notice the initial increase in the upper bound due to Theorem 1 in both Figures 4.1 and 4.2. This is because of the second term of the bound which adds terms with time. After a sufficient number of time slots, however, the decrease in the first term overshadows the increase in the second and the overall bound starts coming down.

![Figure 4.1: Comparison of distance upper bound in Theorem 1 with actual distance](image_url)
Figure 4.2: Comparison of distance upper bound with actual distance at lesser time than predicted by Theorem 2

Figure 4.3: Histogram of distances of 100 sample paths at $T = 5000$

4.3.2 Effect of Changing Parameters

Now we look at the effects of increasing $\delta$ which decides how fast the departures are compared to the arrivals.

**Simulation 2** $\lambda = 10, K = 100, \Delta t = 0.5, \epsilon = 0.1, \gamma = 0.05, \gamma_1 = 0.04$ with
different $\delta = 0.1 : 0.1 : 0.9$.

Note that increase in $\delta$ corresponds to emptying out the queue faster or in other words, the stable distribution should be approached faster. Theorem 2 predictions are as shown in Figure 4.4 which shows that the adiabatic time decreases with increases in $\delta$. Or, at the same time, the distance decreases with increasing $\delta$. The distances at $T = 2500$ averaged over 100 sample paths are shown in Figure 4.5 which confirms this.

![Figure 4.4: Adiabatic time for different $\delta$](image)

Another parameter of interest is the sampling time $\Delta t$ which decides how frequently the estimates are done.

**Simulation 3** $\lambda = 10, K = 100, \delta = 0.1, \epsilon = 0.1, \gamma = 0.05, \gamma_1 = 0.04$ with different $\Delta t = 1, 5, 10, 20, 25, 50$.

The increase in $\Delta t$ corresponds to applying the information gained from estimates less frequently and must result in an increased distance at the same time. The
Figure 4.5: Comparison of distance for different $\delta$ distances at time $T = 100$ averaged over 100 sample paths are shown in Figure 4.6 which shows that this is true. This shows the benefits of updates which are more frequent and hence the system following a smoother path towards the stable distribution rather than attempting to reach there at one go.

Figure 4.6: Comparison of distance for different $\Delta t$. 
Chapter 5 – Conclusion

Adiabatic time of time inhomogeneous Markov chains measures the time needed for the chain to converge to a stationary distribution. The motivation for this approach is the adiabatic evolution in quantum mechanics which measures the time needed for a system to change such that it remains close to its equilibrium state in $\ell^2$ norm. We considered convergence of Markov chains in terms of total variation distance which is an $\ell^1$ norm. Time inhomogeneous, reversible Markov chains were considered with its evolution governed by bounded generator matrices which change at fixed time intervals. We followed the adiabatic approach to study this chain and bounded the distance to a stationary distribution. This is a general theorem which is applicable to evolutions of the above kind which can be due to changes in underlying process itself or due to uncertainties in parameters of the system.

We used this model of time inhomogeneous Markov chains to characterize a queue in which the time inhomogeneity is due to uncertainty in parameters. We considered a Markovian queue with unknown arrival rate along with an adaptive queueing policy and derived sufficient conditions on time after which the distribution of the queue can be considered to be stable with high probability.
5.1 Future Work

One possible future direction would be to set up the problem as a general Markov decision process and investigate the performance of online learning algorithms which try to maximize a reward. It might also be of interest to look at cases where the underlying process is changing as for example when the arrival rate to the queue varies over time. The queueing example can also be set in terms of a discrete time chain where the input and output processes are Bernoulli. Also, when deciding the queueing policy we fixed a $\delta$; it might be of interest to decide the optimal $\delta$ in systems where a tradeoff exists between emptying the queue and other network objectives.
APPENDICES
Appendix A – Proof of Proposition 1

This proof follows from [4]. The matrix defined in (2.1) is symmetric if and only if

\[
P_{ij}^* = P_{ji}^* \\
\Leftrightarrow \frac{\sqrt{\pi(i)}}{\sqrt{\pi(j)}} P_{ij} = \frac{\sqrt{\pi(j)}}{\sqrt{\pi(i)}} P_{ji},
\]

(A.1)

\[
\Leftrightarrow \pi(i) P_{ij} = \pi(j) P_{ji},
\]

(A.2)

or if and only if \( P \) is reversible as (A.2) is the definition of reversibility. (A.1) follows from the definition of \( P^* \) in (2.1) and \( D \) in (2.2).

Since \( P^* \) is symmetric, it has real eigenvalues with the same set of left and right eigenvectors which are orthonormal in Euclidean space. Let \( w_j \) be the eigenvector of \( P^* \) associated with eigenvalue \( \lambda_j \). Define vectors \( v_j \) and \( u_j \) by

\[
w_j = v_j^T \sqrt{\pi}, \\
w_j = u_j^T \sqrt{\frac{1}{\pi}},
\]

where square root and reciprocal of a vector is the vector of square roots and reciprocals respectively. From the definition of \( D \) in (2.2),

\[
P v_j = D^{-\frac{1}{2}} P^* D^{\frac{1}{2}} v_j \\
= D^{-\frac{1}{2}} P^* w_j
\]
\[
= \lambda_j D^{-\frac{1}{2}} w_j = \lambda_j v_j,
\]

Therefore, \( \lambda_j \) is also the eigenvalue of \( P \) with right eigenvector \( v_j \). Similarly, \( u_j \) is the left eigenvector corresponding to \( \lambda_j \).

Also, since \( w_j \) are orthonormal in Euclidean space,

\[
\langle v_i, v_j \rangle_{\pi} = (v_i^T \sqrt{\pi})^T (v_j^T \sqrt{\pi}) = w_i^T w_j = 0.
\]

Therefore, \( v_j \) are orthonormal in \( \ell^2(\pi) \) space and similarly \( u_j \) are orthonormal in \( \ell^2(\frac{1}{\pi}) \) space.

Appendix B – Proof of Proposition 2

The proof is from [4]. \( P \) is a reversible, irreducible transition matrix. Since \( P \) is reversible we have real eigenvalues and left eigenvectors orthonormal in \( \ell^2(\frac{1}{\pi}) \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be the eigenvalues of \( P \) ordered such that

\[
\lambda_1 = 1 > |\lambda_2| \geq \cdots \geq |\lambda_r|.
\]

If \( \{u_1, \ldots, u_r\} \) are the left eigenvectors of \( P \), we have for any vector \( x \in \mathbb{R}^r \),

\[
x^T = \sum_{j=1}^{r} \langle x, u_j \rangle_{\frac{1}{\pi}} u_j^T.
\]
since $u_j$ form an orthonormal basis of $\mathbb{R}^r$. For all $n$, we have $u_j^T P^n = \lambda^n_j u_j^T$, where $u_j$ is the left eigenvector corresponding to the eigenvalue $\lambda_j$. Therefore,

$$x^T P^n = \sum_{j=1}^r \lambda^n_j \langle x, u_j \rangle \frac{1}{\pi} u_j^T.$$ 

Note that $u_1 = \pi$ since $\lambda_1 = 1$ and $\pi^T P = \pi^T$. Therefore we have $\langle \nu - \pi, u_1 \rangle \frac{1}{\pi} = 0$. Therefore

$$\|\nu^T P^n - \pi^n\|_\frac{1}{\pi} = \|\nu^T P^n - \pi^n P^n\|_\frac{1}{\pi} = \|\nu^T P^n - \pi^n\|_\frac{1}{\pi}$$

$$= \sum_{j=2}^r \lambda^n_j \langle \nu - \pi, u_j \rangle \frac{2}{\pi}$$

$$\leq |\lambda_2| 2^n \sum_{j=2}^r \langle \nu - \pi, u_j \rangle \frac{2}{\pi}$$

$$= |\lambda_2| 2^n \|\nu - \pi\|_\frac{1}{\pi}.$$ 

Appendix C – Proof of Proposition 3

For a continuous time Markov chain with generator matrix $Q$, we have the time spent in each state $i$ exponentially distributed with rate $q_i = \sum_{j:j\neq i} Q_{ij}$. If $q = \max_{i \in \Omega} q_i$, we can consider $\frac{q}{q}$ of the transitions to happen to out of the state and the remaining $1 - \frac{q}{q}$ transitions to the same state. The transition to any other state
$j$ happens with probability $\frac{Q_{ij}q_i}{q}$. These define the following transition matrix $P$,

$$P_{ij} = \begin{cases} 
1 - \frac{q_i}{q} & \text{if } j = i, \\
\frac{Q_{ij}}{q} & \text{if } j \neq i.
\end{cases}$$

Since, $Q_{ii} = -q_i$ the matrix $P = I + \frac{1}{q}Q$.

The transition probabilities $P_{ij}(t)$ conditioned on the number of transitions $N(t)$ which is a Poisson process with rate $q$, are given by

$$P_{ij}(t) = \Pr\{X(t) = j|X(0) = i\} = \sum_{n=0}^{\infty} \Pr\{X(t) = j, N(t) = n|X(0) = i\} \Pr\{N(t) = n|X(0) = i\} = \sum_{n=0}^{\infty} (P^n)_{ij}e^{-qt}(qt)^n/n!,$$

$$P(t) = e^{q(P-I)t} = e^{Qt}.$$ 

Appendix D – Proof of Proposition 4

From (2.3),

$$\nu^T P(t) - \pi^T = \sum_{n=0}^{\infty} e^{-qt}(qt)^n/n! \left(\nu^T P^n - \pi^T\right),$$

and therefore

$$\left\|\nu^T e^{Qt} - \pi^T\right\|_1 = \left\|\nu^T P(t) - \pi^T\right\|_1.$$
\[ = \left\| \sum_{n=0}^{\infty} \frac{e^{-qt\left(\frac{qt}{n!}\right)^n}}{n!} \left( \nu^T P^m - \pi^T \right) \right\|_1 \]
\[ \leq \sum_{n=0}^{\infty} \frac{e^{-qt\left(\frac{qt}{n!}\right)^n}}{n!} \left\| \nu^T P^m - \pi^T \right\|_1 \quad (D.1) \]
\[ \leq \sum_{n=0}^{\infty} \frac{e^{-qt\left(\frac{qt}{n!}\right)^n}}{n!} |\lambda_2|^n \|\nu - \pi\|_1 \quad (D.2) \]
\[ = \|\nu - \pi\|_1 e^{-q(1-|\lambda_2|)t}, \]

where (D.1) follows from Jensen’s inequality since every norm is a convex function and (D.2) follows from Proposition 2.

Appendix E – Proof of Proposition 5

The proof is from [4]. From the definition of the total variation distance,

\[ (\|\nu - \pi\|_{TV})^2 = \frac{1}{4} \left( \sum_{i \in \Omega} |\nu(i) - \pi(i)| \right)^2 \]
\[ = \frac{1}{4} \left( \sum_{i \in \Omega} \left( \frac{\nu(i)}{\pi(i)} - 1 \right) \pi(i)^{1/2} \pi(i)^{1/2} \right)^2 \]
\[ \leq \frac{1}{4} \sum_{i \in \Omega} \left( \frac{\nu(i)}{\pi(i)} - 1 \right)^2 \pi(i) \quad (E.1) \]
\[ = \frac{1}{4} \sum_{i \in \Omega} \frac{(\nu(i) - \pi(i))^2}{\pi(i)} \]
\[ = \frac{1}{4} \|\nu - \pi\|_1^2, \quad (E.2) \]

where Cauchy-Bunyakovsky-Schwarz inequality is used in (E.1) and the definition of the \( \frac{1}{\pi} \) norm is used in (E.2). The result follows by taking square roots.
Appendix F – Proof of Proposition 6

The proof is from [12]. Let $\lambda_k, k = 1, \ldots, r$ be the eigenvalues of $P$. Define $s$ as the reciprocal of the eigenvalue. The eigenvector $x$ satisfies the following set of equations,

\[
x(1) = s((1 - \beta)x(1) + \beta x(2)),
\]
\[
x(j) = s((1 - \beta)x(j - 1) + \beta x(j + 1)), j = 2, \ldots, r - 1,
\]
\[
x(r) = s((1 - \beta)x(r - 1) + \beta x(r)).
\]

This system of equations admits the solution $x = [1, \ldots, 1]^T$ corresponding to $s = 1$. To find the remaining $r - 1$ roots, we use the method of particular solutions. (F.2) is satisfied by $x(j) = a^j$ if $a$ is a root of the quadratic equation $a = (1 - \beta)s + a^2 \beta s$. The two roots of this equation are

\[
a_1(s) = \frac{1 + \sqrt{1 - 4\beta(1 - \beta)s^2}}{2\beta s},
\]
\[
a_2(s) = \frac{1 - \sqrt{1 - 4\beta(1 - \beta)s^2}}{2\beta s}.
\]

Therefore, the general solution of (F.2) is

\[x(j) = A(s)a_1^j(s) + B(s)a_2^j(s),\]

where $A(s)$ and $B(s)$ are arbitrary. Equations (F.1) and (F.3) will be satisfied by the above solution if and only if $x(0) = x(1)$ and $x(r) = x(r + 1)$. This requires
that \( A(s) \) and \( B(s) \) satisfy the conditions

\[
A(s)(1 - a_1(s)) + B(s)(1 - a_2(s)) = 0,
\]
\[
A(s)a_1^r(s)(1 - a_1(s)) + B(s)a_2^r(s)(1 - a_2(s)) = 0.
\]

These two equations can be satisfied together only if

\[
a_1^r(s) = a_2^r(s). \tag{F.6}
\]

We have to determine the values of \( s \) for which the above equation is possible.

From (F.4) and (F.5), we have \( a_1(s)a_2(s) = \frac{1-\beta}{\beta^2} \), and therefore, \( a_1(s)(\frac{\beta}{1-\beta})^{\frac{i}{2}} \) and \( a_2(s)(\frac{\beta}{1-\beta})^{\frac{i}{2}} \) must be \( 2r \)th roots of unity which can be written as \( e^{i\pi k/r} \) where \( \beta = -1 \) and \( k = 0, 1, \ldots, 2r - 1 \). Thus all solutions of (F.6) are among the roots of

\[
a_1(s) = \left( \frac{1-\beta}{\beta} \right)^{\frac{i}{2}} e^{i\pi k/r},
\]
\[
a_2(s) = \left( \frac{1-\beta}{\beta} \right)^{\frac{i}{2}} e^{-i\pi k/r}.
\]

From (F.4) and (F.5), we have \( a_1(s) + a_2(s) = \frac{1}{\beta s} \). Therefore we can solve for \( s \) and therefore for its reciprocal \( \lambda_k \) as

\[
\lambda_k = 2\sqrt{\beta(1-\beta)} \cos\left(\frac{\pi k}{r}\right).
\]

The solution corresponding to \( k = r \) must be disregarded since this leads to the trivial solution \( x = [0, \ldots, 0]^T \). Corresponding to \( k = 0 \), we have the solution
$x = [1, \ldots, 1]^T$ which we have already considered. We have $r - 1$ distinct solutions corresponding to $k = 1, 2, \ldots, r - 1$ and for $k = r + 1, r + 2, \ldots, 2r - 1$ we get the same solutions. Therefore we have corresponding to $\lambda_k$, the eigenvector

$$x_k(j) = \left(\frac{1 - \beta}{\beta}\right)\frac{\pi kj}{r} - \left(\frac{1 - \beta}{\beta}\right)\frac{\pi k(j+1)}{r}, k = 1, \ldots, r - 1.$$

Appendix G – Proof of Lemma 3

Proof of (4.3)

For a random variable $X$ which is the empirical average of $i$ iid random variables $\{X_k\}_{k=1}^i$, Chernoff bound for tail probabilities gives,

$$\Pr\{X \geq a\} \leq \exp \left( -i \sup_{t>0} (at - \log(\mathbb{E}[e^{tX_i}])) \right),$$
$$\Pr\{X \leq a\} \leq \exp \left( -i \sup_{t>0} (-at - \log(\mathbb{E}[e^{-tX_i}])) \right).$$

Since $X_i$ is Poisson($\lambda \Delta t$) in our case $\log(\mathbb{E}[e^{tX_i}]) = \lambda \Delta t (e^t - 1)$. Substituting this in the above, we get

$$\Pr\{\hat{\lambda}_t \Delta t \geq \lambda \Delta t (1 + \epsilon_0)\} \leq \exp \left( -i \lambda \Delta t \sup_{t>0} ((1 + \epsilon_0)t - (e^t - 1)) \right).$$

The maximum above is attained at $t = \log(1 + \epsilon_0) > 0$. Similarly for

$$\Pr\{\hat{\lambda}_t \Delta t \leq \lambda \Delta t (1 - \epsilon_0)\} \leq \exp \left( -i \lambda \Delta t \sup_{t>0} ((1 - \epsilon_0)t - (e^{-t} - 1)) \right).$$
The maximum here is attained at $t = \log(\frac{1}{1-\epsilon_0}) > 0$. Using the above bounds, we can bound the probability of the complement of the event in (4.3).

\[
\Pr\{|\hat{\lambda}_i - \lambda| \geq \lambda \epsilon_0\} = \Pr\{\hat{\lambda}_i \Delta t \geq \lambda \Delta t(1 + \epsilon_0)\} + \Pr\{\hat{\lambda}_i \Delta t \leq \lambda \Delta t(1 - \epsilon_0)\} \\
\leq \exp\left(-i \lambda \Delta t((1 + \epsilon_0) \log(1 + \epsilon_0) - \epsilon_0)\right) + \exp\left(-i \lambda \Delta t((1 - \epsilon_0) \log(1 - \epsilon_0) + \epsilon_0)\right) \\
< 2 \exp\left(-i \lambda \Delta t((1 + \epsilon_0) \log(1 + \epsilon_0) - \epsilon_0)\right) \quad \text{(G.1)} \\
< 2 \exp\left(-i \lambda \Delta t((1 + \epsilon_0)(\epsilon_0 - \frac{\epsilon_0^2}{2}) - \epsilon_0)\right) \quad \text{(G.2)} \\
= 2 \exp\left(-i \lambda \Delta t\left(\frac{\epsilon_0^2 - \epsilon_0^3}{2}\right)\right) \leq \gamma_1. \quad \text{(G.3)}
\]

(G.1) holds since $(1 - \epsilon_0) \log(1 - \epsilon_0) + \epsilon_0 > (1 + \epsilon_0) \log(1 + \epsilon_0) - \epsilon_0$ for $0 < \epsilon_0 < 1$ and (G.2) is true since $\log(1 + \epsilon_0) > \epsilon_0 - \frac{\epsilon_0^2}{2}$ according to Taylor’s theorem for $0 < \epsilon_0 < 1$. From (G.3),

\[
\exp\left(-i \lambda \Delta t\left(\frac{\epsilon_0^2 - \epsilon_0^3}{2}\right)\right) \leq \frac{\gamma_1}{2} \\
i \geq \frac{2 \log \frac{2}{\gamma_1}}{\lambda \Delta t \left(\frac{\epsilon_0^2 - \epsilon_0^3}{2}\right)}.
\]

This implies (4.3) must hold for

\[
n_0 = \frac{2 \log \frac{2}{\gamma_1}}{\lambda \Delta t \left(\frac{\epsilon_0^2 - \epsilon_0^3}{2}\right)}.
\]
Proof of (4.6)

We have the following equation from the estimation of $\lambda$,

$$
\hat{\lambda}_{i+1} \Delta t = \hat{\lambda}_i \Delta t \frac{i}{i + 1} + \frac{1}{i + 1} X_{i+1}
$$

$$
\hat{\lambda}_i \Delta t - \hat{\lambda}_i \Delta t = \frac{1}{i + 1} (X_{i+1} - \hat{\lambda}_i \Delta t)
$$

$$
< \frac{1}{n_0} (X_{i+1} - \lambda \Delta t (1 - \epsilon_0)), \quad (G.4)
$$

with a probability of at least $1 - \gamma_1$, $\forall i + 1 > n_0$ since we have $\hat{\lambda}_i \geq \lambda (1 - \epsilon_0)$ by (4.3). Since $X_{i+1}$ is a Poisson($\lambda \Delta t$) random variable, we can find a constant $c$ such that $\Pr\{|X_{i+1} - \lambda \Delta t| \geq c \lambda \Delta t\} \leq \gamma_2$. From Chebyshev’s inequality, for Poisson($\lambda \Delta t$) random variable

$$
\Pr\{|X_{i+1} - \lambda \Delta t| \geq c \lambda \Delta t\} \leq \frac{1}{c^2 \lambda \Delta t} \leq \gamma_2.
$$

A choice of $c = \frac{1}{\sqrt{\lambda \Delta t \gamma_2}}$ will satisfy the above equation. Therefore, (G.4) can be written as

$$
\hat{\lambda}_{i+1} - \hat{\lambda}_i < \frac{1}{n_0} (\lambda (1 + c) - \lambda (1 - \epsilon_0))
$$

$$
= \frac{\lambda}{n_0} (c + \epsilon_0),
$$

with a probability of at least $1 - \gamma_1 - \gamma_2 = 1 - \gamma$. Similarly we have

$$
\hat{\lambda}_{i-1} - \hat{\lambda}_i < \frac{\lambda}{n_0} (c + \epsilon_0),
$$
with a probability of at least $1 - \gamma$. Therefore (4.6) holds for $\epsilon_1 = \frac{1}{n_0} (c + \epsilon_0)$.

Proof of (4.4)

\[
|\lambda_2(P_i)| = 2 \cos \left( \frac{\pi}{K+1} \right) \frac{\rho_i^{1/2}}{1 + \rho_i} \\
= 2 \cos \left( \frac{\pi}{K+1} \right) \sqrt{1 + \frac{\hat{\lambda}_i}{\lambda}} / \sqrt{\frac{\hat{\lambda}_i}{\lambda}}.
\]

Therefore,

\[
\begin{align*}
e^{-(\lambda + \mu_i) (1 - |\lambda_2(P_i)|) \Delta t} &= e^{\lambda \Delta t [2 \cos \left( \frac{\pi}{K+1} \right) \sqrt{(1 + \delta) \frac{\hat{\lambda}_i}{\lambda} - 1 - (1 + \delta) \frac{\hat{\lambda}_i}{\lambda}}]}
< e^{\lambda \Delta t [2 \sqrt{(1 + \delta) \frac{\hat{\lambda}_i}{\lambda} - 1 - (1 + \delta) \frac{\hat{\lambda}_i}{\lambda}}]}
= e^{\lambda \Delta t (\sqrt{(1 + \delta) \frac{\hat{\lambda}_i}{\lambda} - 1})^2}.
\end{align*}
\]

(G.5) is an inequality since $\cos \left( \frac{\pi}{K+1} \right) < 1$ for $K > 1$. From (4.3) it follows that $\forall i \geq n_0$, with a probability at least $1 - \gamma_1$, $1 - \epsilon_0 \leq \frac{\hat{\lambda}_i}{\lambda} \leq 1 + \epsilon_0$. The RHS of (G.6) is maximized at $\frac{\hat{\lambda}_i}{\lambda} = 1 - \epsilon_0$. Therefore (G.6) can be bounded with a probability of at least $1 - \gamma_1$ by

\[
e^{-(\lambda + \mu_i) (1 - |\lambda_2(P_i)|) \Delta t} < e^{-\lambda \Delta t (\sqrt{(1 + \delta) (1 - \epsilon_0) - 1})^2},
\]
The RHS of (G.7) is maximized at $\lambda^*_i = 1 - \epsilon_0$ and therefore the above can be written as

$$
\|\pi_i - \pi\|_{TV} \leq 1 \frac{1}{2} \sum_{k=0}^{K} \left| \frac{\rho^k_i}{\sum_{r=0}^{K} \rho^r_i} - \frac{\rho^k}{\sum_{r=0}^{K} \rho^r} \right|
$$

where (G.8) is true since the quantity inside the modulus is always positive. This can be seen from

$$
\left| \frac{(1 - \epsilon_0)^{K-k}((1 - \epsilon_0)(1 + \delta) - 1)}{[(1 - \epsilon_0)(1 + \delta)]^{K+1} - 1} - \frac{\delta}{(1 + \delta)^{K+1} - 1} \right| 
$$

where (G.8) is true since the quantity inside the modulus is always positive.
Proof of (4.7)

We have

\[
\max_{k \in \{0, 1, \ldots, K\}} \frac{\pi_i(k)}{\pi_{i+1}(k)} = \max_{k \in \{0, 1, \ldots, K\}} \frac{\rho_i^k}{\rho_{i+1}^k} \sum_{r=0}^{K} \rho_{i+1}^r \\
= \max_{k \in \{0, 1, \ldots, K\}} \left( \frac{\lambda_{i+1}}{\lambda_i} \right)^k \frac{\sum_{r=0}^{K} (\frac{\lambda}{1+\delta})^{r+1}}{\sum_{r=0}^{K} (\frac{\lambda}{1+\delta})^r},
\]

(G.9)

From (4.3) it follows that \( \forall i \geq i_0 \), with a probability at least \( 1 - \gamma_1 \), \( 1 - \epsilon_0 \leq \frac{\hat{\lambda}_i}{\hat{\lambda}_0} \), \( \frac{\lambda_{i+1}}{\lambda_i} \leq 1 + \epsilon_0 \). In addition from (4.6), with a probability at least \( 1 - \gamma \), \( \frac{\hat{\lambda}_i}{\hat{\lambda}_0} - \epsilon_1 < \frac{\lambda_{i+1}}{\lambda_i} < \frac{\hat{\lambda}_i}{\hat{\lambda}_0} + \epsilon_1 \). The RHS of (G.9) is maximized at \( \frac{\lambda}{\lambda_i} = 1 - \epsilon_0 \), \( \frac{\lambda_{i+1}}{\lambda_i} = 1 - \epsilon_0 + \epsilon_1 \). Therefore (G.9) can be bounded with a probability of at least \( 1 - \gamma \) by

\[
\max_{k \in \{0, 1, \ldots, K\}} \frac{\pi_i(k)}{\pi_{i+1}(k)} \leq \max_{k \in \{0, 1, \ldots, K\}} \left( \frac{1 - \epsilon_0 + \epsilon_1}{1 - \epsilon_0} \right) \left( \frac{\sum_{r=0}^{K} (\frac{\rho}{1-\epsilon_0+\epsilon_1})^r}{\sum_{r=0}^{K} (\frac{\rho}{1-\epsilon_0})^r} \right)
\]

\[
= \left( \frac{1 - \epsilon_0 + \epsilon_1}{1 - \epsilon_0} \right) K \frac{\sum_{r=0}^{K} (\frac{\rho}{1-\epsilon_0+\epsilon_1})^r}{\sum_{r=0}^{K} (\frac{\rho}{1-\epsilon_0})^r}
\]

\[
= \frac{(1 - \epsilon_0 + \epsilon_1)^{K+1} - \rho^{K+1}}{(1 - \epsilon_0)^{K+1} - \rho^{K+1}} \frac{1 - \epsilon_0 - \rho}{1 - \epsilon_0 + \epsilon_1 - \rho}
\]

\[
< \frac{(1 - \epsilon_0 + \epsilon_1)^{K+1} - \rho^{K+1}}{(1 - \epsilon_0)^{K+1} - \rho^{K+1}}
\]

\[
= \frac{[(1 - \epsilon_0 + \epsilon_1)(1+\delta)]^{K+1} - 1}{[(1 - \epsilon_0)(1+\delta)]^{K+1} - 1},
\]

(G.10)

where the inequality in (G.10) holds since \( 1 - \epsilon_0 < 1 - \epsilon_0 + \epsilon_1 \) for \( \epsilon_1 > 0 \).
Proof of (4.8)

\[
\|\pi_i - \pi_{i+1}\|_{\pi_{i+1}}^{-1} = \left( \sum_{k=0}^{K} \frac{\left(\pi_i(k) - \pi_{i+1}(k)\right)^2}{\pi_{i+1}(k)} \right)^{1/2} = \left( \sum_{k=0}^{K} \frac{\left(\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k \frac{\lambda_{i+1}}{(1+\delta)\lambda_{i+1}}^k - \sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k \frac{\lambda_{i+1}}{(1+\delta)\lambda_{i+1}}^k\right)^2}{\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k} \right)^{1/2} = \left( \sum_{k=0}^{K} \frac{\left(\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k \frac{\lambda_{i+1}}{(1+\delta)\lambda_{i+1}}^k - \sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k \frac{\lambda_{i+1}}{(1+\delta)\lambda_{i+1}}^k\right)^2}{\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k} \right)^{1/2}. \tag{G.11}
\]

As in the above, the RHS of (G.11) is maximized at \(\hat{\lambda}_i = 1 - \epsilon_0, \hat{\lambda}_{i+1} = 1 - \epsilon_0 + \epsilon_1\). Therefore (G.11) can be bounded with a probability of at least \(1 - \gamma\) by

\[
\|\pi_i - \pi_{i+1}\|_{\pi_{i+1}}^{-1} \leq \left( \sum_{k=0}^{K} \frac{\rho_{i+1}^k}{\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k} \frac{\rho_{i+1}}{\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}} \right)^{1/2}
\]

\[
= \left( \sum_{k=0}^{K} \frac{\rho_{i+1}^k}{\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}^k} \frac{\rho_{i+1}}{\sum_{r=0}^{\lambda_{i+1}} \rho_{i+1}} \right)^{1/2}
\]

\[
= \left( \frac{bc}{a^2 - 1} \right)^{1/2}, \tag{G.12}
\]

where we define

\[
a = \sum_{k=0}^{K} \left(\frac{\rho}{1 - \epsilon_0}\right)^k
\]

\[
= \frac{(1 - \epsilon_0)^{K+1} - \rho^{K+1}}{(1 - \epsilon_0)^K (1 - \epsilon_0 - \rho)}.
\]
\[ b = \sum_{k=0}^{K} \left( \frac{\rho}{1 - \epsilon_0 + \epsilon_1} \right)^k = \frac{(1 - \epsilon_0 + \epsilon_1)^{K+1} - \rho^{K+1}}{(1 - \epsilon_0 + \epsilon_1)^K(1 - \epsilon_0 + \epsilon_1 - \rho)}, \]

\[ c = \sum_{k=0}^{K} \left( \frac{\rho(1 - \epsilon_0 + \epsilon_1)}{(1 - \epsilon_0)^2} \right)^k = \frac{(1 - \epsilon_0)^{2K+2} - \rho^{K+1}(1 - \epsilon_0 + \epsilon_1)^{K+1}}{(1 - \epsilon_0)^{2K}((1 - \epsilon_0)^2 - \rho(1 - \epsilon_0 + \epsilon_1))}. \]

Therefore, we can write using \( x = 1 - \epsilon_0, \ y = 1 - \epsilon_0 + \epsilon_1 \) for ease of notation

\[
\frac{bc}{a^2} = \frac{[y^{K+1} - \rho^{K+1}][x^{2K+2} - \rho^{K+1}y^{K+1}]}{(x^{K+1} - \rho^{K+1})^2 y^K} \frac{(x - \rho)^2}{(y - \rho)(x^2 - py)}
\]

\[
= \frac{x^{2K+2}y^{K+1} + \rho^{2K+2}y^{K+1} - \rho^{K+1}[x^{2K+2} + y^{2K+2}]}{x^{2K+2}y^{K+1} + \rho^{2K+2}y^{K+1} - 2\rho^{K+1}x^{K+1}y^{K+1}} \frac{(x - \rho)^2 y}{(y - \rho)(x^2 - py)}
\]

\[
< \frac{(x - \rho)^2 y}{(y - \rho)(x^2 - py)} \quad (G.13)
\]

\[
= \frac{x^2 y + \rho^2 y - 2\rho xy}{x^2 y + \rho^2 y - \rho(x^2 + y^2)}
\]

where (G.13) is true since \( x^{2K+2} + y^{2K+2} > 2x^{K+1}y^{K+1} \). Therefore (G.12) can be written as

\[
\|\pi_i - \pi_{i+1}\|_{\epsilon_{i+1}} < \left( \frac{x^2 y + \rho^2 y - 2\rho xy}{x^2 y + \rho^2 y - \rho(x^2 + y^2)} - 1 \right)^{1/2}
\]

\[
= \frac{\sqrt{\rho \epsilon_1}}{\sqrt{(y - \rho)(x^2 - py)}}
\]

\[
< \frac{\sqrt{x \epsilon_1}}{\sqrt{(x - \rho)(x^2 - py)}} \quad (G.14)
\]

\[
= \frac{\sqrt{x \rho \epsilon_1}}{\sqrt{(x^2 - \rho x)(x^2 - py)}}
\]
\[< \frac{\sqrt{x^2 - \rho y}}{|x^2 - \rho y|} \]
\[= \frac{\sqrt{(1 - \epsilon_0)(1 + \delta)\epsilon_1}}{|(1 - \epsilon_0)^2(1 + \delta) - (1 - \epsilon_0 + \epsilon_1)|}, \quad \text{(G.15)} \]

where (G.14) and (G.15) are true since \( y > x \).

Proof of (4.9)

\[
\|\nu_{n_0} - \pi_{n_0}\|_{\pi_{n_0}} \leq \left[ \sum_{k=0}^{K} \frac{\nu_{n_0}^2(k)}{\pi_{n_0}(k)} - 1 \right]^{1/2} \leq \left[ \sum_{r=0}^{K} \rho_{n_0}^r \sum_{k=0}^{K} \frac{\nu_{n_0}^2(k)}{\rho_{n_0}^k} - 1 \right]^{1/2} \leq \left[ \sum_{r=0}^{K} \rho_{n_0}^r \sum_{k=0}^{K} \frac{1}{\rho_{n_0}^k} \right]^{1/2} \leq \left[ \sum_{r=0}^{K} \frac{(1 + \delta)\lambda_{n_0}}{(1 + \delta)\lambda_{n_0}} \right]^{1/2} \sum_{k=0}^{K} \left( \frac{1 + \epsilon_0}{\lambda} \right)^k \left( \frac{1 + \delta}{\lambda} \right)^{\lambda_{n_0}} \right]^{1/2}, \quad \text{(G.16)}
\]

where (G.16) is true since \( \nu_{n_0}(k) \) is a probability. The RHS of (G.17) is maximized at \( \frac{\lambda_{n_0}}{\lambda} = 1 + \epsilon_0 \). Therefore (G.17) can be bounded with a probability of at least \( 1 - \gamma_1 \) by

\[
\|\nu_{n_0} - \pi_{n_0}\|_{\pi_{n_0}} \leq \left[ \sum_{r=0}^{K} \frac{(1 + \epsilon_0)^{K+1} - \rho^{K+1}}{(1 + \epsilon_0)^{K+1} - \rho^{K+1}} \right]^{1/2} \]
\[= \left[ \frac{(1 + \epsilon_0)^{K+1} - \rho^{K+1}}{(1 + \epsilon_0)^{K+1} - \rho^{K+1}} \right]^{1/2} \]
\[= \frac{(1 + \epsilon_0)^{K+1} - \rho^{K+1}}{(1 + \epsilon_0)^{K+1} - \rho^{K+1}} \]
\[
< \frac{(1 + \epsilon_0)^{K+1}}{(1 + \epsilon_0)^{\frac{K}{2}} \rho^\frac{K}{2}(1 + \epsilon_0 - \rho)} \\
= \frac{(1 + \epsilon_0)^{\frac{K}{2}+1}(1 + \delta)^{\frac{K}{2}+1}}{(1 + \epsilon_0)(1 + \delta) - 1} \\
< \frac{[(1 + \epsilon_0)(1 + \delta)]^{\frac{K}{2}+1}}{\delta}.
\]

This completes the proof of Lemma 3.
Bibliography


