Expectational Stability of Sunspot Equilibria in Non-Convex Economies

Bruce McGough[a], **Qinglai Meng**[a] **and Jianpo Xue**[b]

[a] *Department of Economics, Oregon State Univeristy, Corvallis, OR 97331, USA*

[b] *School of Finance, Renmin University of China, Beijing 100871, China*

January 18, 2012

Abstract

We examine the stability under learning (E-stability) of sunspot equilibria in non-convex real business cycle models. The production technology is Cobb-Douglas with externalities generated by factor inputs. We establish that, with a general utility function, the well-known Benhabib-Farmer condition (Benhabib and Farmer, 1994) – that the labor-demand curve is upward-sloping and steeper than the Frisch labor-supply curve – is necessary for joint indeterminacy and E-stability. Then, with a separable utility function and allowing for negative externalities from capital inputs, we discover large regions in parameter space corresponding to stable indeterminacy, that is, learnable sunspot equilibria. These existence results overturn the conventional wisdom that sunspot equilibria in RBC-type models are inherently unstable, and provide concise closure to the *stability puzzle* of Evans and McGough (2005b).

JEL classification: C62; D11; D83; E32.

Keywords: Real business cycle models; Indeterminacy; Expectational stability; Externalities; Preferences.

1 Introduction

Work by Shell (1977), Case and Shell (1983), Azariadas (1981), and others, demonstrated the potential for macroeconomic models, including those couched in general equilibrium theory, to be indeterminate, and thus exhibit equilibrium outcomes that depend on extrinsic shocks, i.e. shocks that affect the economy *only* because the economy's agents expect them to: these shocks are called sunspots and the associated equilibrium outcomes are called sunspot equilibria. Naturally interpreted as self-fulfilling prophecies, these sunspot equilibria gave rise to a new explanation of the business cycle: fluctuations in economic activity may be the result of variation in agents' beliefs.

Separately, the literature on learning in macroeconomics developed in part as a justification for the strong informational assumptions required to support the rational expectations hypothesis. In a learning environment, agents are assumed to form expectations using the same types of econometric forecasting models as economists. If the resulting endogenous variables converge to a rational expectations equilibrium (REE), that equilibrium is said to be stable under learning, and the case for relying on rational expectations as a modeling assumption, and for focusing attention on the REE, is correspondingly strengthened.

Stability under learning is not generic. Even for a given functional structure, some parameterizations of a model may yield learnable equilibria while for others the equilibria may be unstable. In this way, learnability may be viewed as a selection criterion: the relevance of a particular model's implications may be strengthened if the associated equilibria are stable under learning. Via this notion, Woodford (1990) lent additional credence to the sunspot explanation of the business cycle by showing in a simple overlapping generations model that the associated sunspot equilibria are stable.

With the pioneering work of Benhabib and Farmer (1994) and Farmer and Guo (1994), sunspot driven business cycles were joined to applied dynamic stochastic general equilibrium (DSGE) modeling. These authors developed calibrated non-convex real business cycle (RBC) models that well-matched the data using only sunspot processes as exogenous stochastic drivers. Their work spawned a large and still growing literature dedicated to exploring and relaxing conditions under which RBC-type and other DSGE models exhibit sunspot equilibria.

As mentioned above, stability under learning is model dependent, and while Woodford (1990) found that sunspot equilibria *may* be stable under learning, he did not show that they have to be stable; and, in fact, Evans and Honkapohja (2001) found that the equilibria analyzed by Farmer and Guo (1994), at least for the particular calibration used, are not stable under learning. This instability result was explored in detail by Evans and McGough (2005a), and they studied multiple parameterizations of a variety of RBC-type models and found no stable sunspot equilibria: they dubbed the contrast of their instability results with Woodford's stability results "the stability puzzle."[1]

Unraveling the stability puzzle has proven to be complicated. Evans and McGough (2005b) and (2010) emphasize the representation dependence of learning stability. A given sunspot equilibrium may be characterized by a number of natural recursions, or "representations," depending on what type of sunspot process is taken as observable by agents; and, the chosen recursion dictates the functional form of the forecasting model used by learning agents when forming expectations. Further, it can be shown that whether a given equilibrium is stable under learning may depend on the agents' forecasting model, and thus on the representation used by the researcher when conducting stability analysis. Evans and McGough (2010) showed that Woodford (1990) and Evans and Honkapohja (2001) used fundamentally different representation types, which could, in principle, explain their differing stability results; however, using the same representation type as Woodford (1990), Evans and McGough (2005b) still failed to find stable sunspot equilibria.[2] Duffy and Xiao (2007) also studied the stability puzzle. In particular, they emphasized a particular restriction on the reduced form parameters necessary for stable sunspots – that the coefficient modifying expectations of future aggregate consumption be negative – and then showed that the models examined by Evans and McGough (2005b) never met this condition.

While the work described above represents important progress towards understanding the stability puzzle, the fundamental question remains: are there non-convex RBC models which exhibit stable sunspot equilibria? The central contribution of this paper is to demonstrate that the answer to this question is an unqualified "yes." We take, as our point of departure, a discrete-time version of the model studied by Meng and Yip (2008), which may be interpreted as the Benhabib-Farmer-Guo model extended to incorporate a general utility function and possibly negative capital externalities. We begin studying the restrictions on the model's deep structure necessary for sunspot equilibria to be stable under learning. We find the generic condition that if stable sunspot equilibria exist then the labor demand curve crosses the Frisch labor supply curve from below.[3]

[1]The stability puzzle involves the additional observation that the a-theoretic reduced-form system of expectational difference equations (see eqs. (14) – (15)) associated to non-convex RBC models of the form studied by Evans and McGough (2005b) does allow for stable sunspot equilibria: it is only when the reduced form parameters are derived from an underlying DSGE model that instability appears to necessarily obtain. See Section 2.3.4 for details.

[2]Using a alternative timing convention, Evans and McGough (2005b) did find small regions of the model's parameter space corresponding to stable sunspots: see Section 3.3.1 for details.

[3]In their model with positive capital externalities, Benhabib and Farmer (1994) established this condition as necessary for the *existence* of sunspot equilibria. Meng and Yip (2008) found that provided the capital externality is negative, sunspot equilibria may exist even if labor demand can

Furthermore, by requiring that utility is separable we obtain additional conditions for learnable sunspots: the marginal effect of capital on the externality must be negative and the steady-state relative risk aversion in consumption must be less than unity. Using these necessary restrictions to provide guidance, we determine that in large regions of the model's parameter space, stable sunspot equilibria exist.

The organization of this paper is as follows: Section 2 presents the model and background on learning; Section 3 reports the analytic and numeric results, including subsections detailing the conditions necessary for learnable sunspots, numerical evidence of their existence, and a discussion of caveats; Section 4 concludes. The Appendix contains all technical arguments.

2 Model, indeterminacy, E-stability

In this section we develop the analytic notions, framework and tools necessary to obtain our results on sunspot stability. We begin by generalizing the Benhabib-Farmer-Guo model (BFG-model) to allow for the possibilities of non-separable utility and negative capital externalities. Then, within the context of the model's reduced form system of expectational difference equations, we review the notions of indeterminacy and expectational stability.

2.1 The model

Our model is a discrete-time version of the generalized BFG-model studied by Meng and Yip (2008). The model's broad structure is consistent with a standard real competitive economy. There is a homogenous good that may be used for capital or consumption, and there are two types of economic agents – households and firms – who interact in competitive markets. We describe the behavior of these agents in turn, and then characterize the equilibria of the economy.

2.1.1 Households

There is a unit mass of identical infinitely-lived households. At the beginning of each period, the representative household faces standard consumption/savings and

be downward sloping. Here we find that the Behabib-Farmer condition re-emerges even with negative capital externalities if we insist upon learnability: see Section 3.1 for details.

labor/leisure decisions. The representative household's problem is given by

$$\max_{C_t,S_t,L_t} \quad E_0 \sum_{t=0}^{\infty} \rho^t u(C_t, L_t) \tag{1}$$

$$\text{s.t.} \quad C_t + S_t = w_t L_t + (1+r_t)S_{t-1} + \pi_t \tag{2}$$

Here E_0 is the conditional expectations operator, C_t is consumption, S_t is savings held from time t to time $t+1$, L_t is the quantity of labor supplied, w_t is the wage, r_t is the net return to savings, and π_t is the dividend flow, taken as given by the household. All prices are written in real terms. All markets are competitive, so that prices are taken as given. Finally, initial savings S_{-1} is given and all households face the usual no Ponzi game condition that the present value of consumption must not exceed the present value of income plus initial wealth.

The representative household's problem, as given by (1) is standard, and versions of it have been studied in detail in both the real business cycle literature (King and Rebello,1999) and the indeterminacy literature (Farmer,1999). However, within the literature on indeterminacy, it is usually assumed that the utility function is separable in consumption and leisure ($u_{CL} = 0$), and, in fact, particular functional forms are often imposed.[4] In our study, we remain agnostic about the nature of preferences, assuming only that the utility function satisfies the following standard concavity and normality properties:

A. 1: $u_C > 0, u_{CC} \leq 0$;

A. 2: $u_L < 0, u_{LL} \leq 0$;

A. 3: $u_c u_{CL} - u_L u_{CC} < 0$, $u_L u_{CL} - u_C u_{LL} < 0$, $u_{CC} u_{LL} - u_{CL}^2 < 0$.

Maintaining this level of generality allows us to realize a deep connection between labor supply, labor demand, and joint E-stability and indeterminacy.

The first-order conditions for the representative household are

$$u_C(C_t, L_t) = \rho E_t (1 + r_{t+1}) u_C(C_{t+1}, L_{t+1}), \tag{3}$$

$$u_L(C_t, L_t) + w_t u_C(C_t, L_t) = 0, \tag{4}$$

$$\lim_{n\to\infty} E_t \rho^n u_C(C_{t+n}, L_{t+n}) S_{t+n} = 0. \tag{5}$$

Eq. (3) is the usual Euler equation capturing the intertemporal trade-off between consumption in time t and time $t+1$, and E_t is the rational expectations operator conditional on time t information. Eq. (4) measures the period t trade-off between consumption and leisure, and finally, eq. (5) is the transversality condition. Given prices w_t and r_t, and dividends π_t, the representative household's behavior is completely characterized by (2) – (5).

[4]For example, in Benhabib and Farmer (1994), $u(C, L) = \log C - (1+\chi)^{-1} L^{1+\chi}$, $\chi \geq 0$.

2.1.2 Firms

There is a unit mass of identical firms. Each firm has access to a technology which depends on the quantities of capital and labor employed. The representative firm's problem is given by

$$\max_{K_t, N_t} \quad Y_t - R_t K_t - w_t N_t \tag{6}$$

$$\text{s.t.} \quad Y_t = A_t K_t^a N_t^b \tag{7}$$

Here K is capital, N is labor demanded, and $R = r + \delta$ is the rental rate of capital (δ captures depreciation). Eq. (7) establishes the assumed functional form for technology, where A_t is total factor productivity. Note that the production function exhibits constant returns to scale in the choice variables K and N provided $a + b = 1$, which, in turn, implies that all firms earn zero profits.

The first order conditions for the firm are particularly simple because markets are competitive and the firm's problem is static. Given prices R_t and w_t, and total factor productivity A_t, the representative firm chooses time t labor and capital to satisfy

$$R_t = a A_t K_t^{a-1} N_t^b \tag{8}$$

$$w_t = b A_t K_t^a N_t^{b-1}. \tag{9}$$

The interpretation of these FOC is entirely standard: eq. (8) states that the firm rents capital until the marginal product of capital equals the rental rate; and eq. (9) says that the firm hires labor until the marginal product of labor equals the wage.

2.1.3 Factor-generated externalities

If total factor productivity is exogenous then the economy has a unique equilibrium which may be identified with the solution to the social planner's problem. Following Benhabib and Farmer (1994) and Meng and Yip (2008), we endogenize total factor productivity by assuming factor-generated externalities. Specifically, we assume that

$$A_t = \bar{K}_t^{\alpha - a} \bar{N}_t^{\beta - b}, \tag{10}$$

where barred variables are aggregate levels and $\alpha, \beta > 0$. Consistent with Meng and Yip (2008), and in contrast to much of the literature on factor generated externalities, we do not assume *apriori* that $\alpha > a$. When $\alpha < a$ then there is a negative capital externality: total factor productivity is decreasing in aggregate capital. As will be clear below, such a relaxation is crucial for a main result of this paper for the case of separable utility function (see Section 3.2).

The endogenous nature of A_t does not affect the private returns to scale. Social returns to scale are given by $\alpha + \beta$, and, as is standard in the indeterminacy literature, a necessary condition for the presence of multiple equilibrium is increasing social returns: $\alpha + \beta > 1$. Notice it is possible to have increasing social returns even in case there is a negative capital externality; further, in this case, both the private and the social production function are increasing in capital.

2.1.4 The linearized model

The model is closed by requiring clearing in the factor markets in equilibrium: $S_t = K_{t+1}$, $L_t = N_t$. In addition, $\bar{K}_t = K_t$, $\bar{N}_t = N_t$. Imposing these identities, together with the pricing eqs. (8) and (9) into the household's budget constraint and the FOC (2)-(4) yields

$$u_C(C_t, L_t) = \rho E_t(1 + aK_{t+1}^{\alpha-1} L_{t+1}^{\beta} - \delta) u_C(C_{t+1}, L_{t+1}), \tag{11}$$

$$u_L(C_t, L_t) + bK_t^{\alpha} L_t^{\beta-1} u_C(C_t, L_t) = 0, \tag{12}$$

$$K_{t+1} = K_t^{\alpha} L_t^{\beta} + (1 - \delta)K_t - C_t. \tag{13}$$

We assume that there is a unique deterministic steady state of the system (11)–(13), denoted by (C^*, K^*, L^*), and log-linearize (11)–(13) about the steady state.[5] Denote by lowercase variables the proportional deviation of the variables from the steady state: for example,

$$k_t = \frac{K_t - K^*}{K^*} \approx \log\left(K_t / K^*\right).$$

The log-linearized model can be written as

$$\begin{aligned} E_t k_{t+1} &= d_k k_t + d_c c_t, & (14)\\ c_t + e_k k_t &= b_k E_t k_{t+1} + b_c E_t c_{t+1}, & (15) \end{aligned}$$

where the reduced form parameters are defined in terms of the model's steady-state and deep parameters: see Appendix. Here we have substituted out labor using the log-linearized version of the contemporaneous relation (12). As is standard in the RBC literature and the related literature on indeterminacy and E-stability, we study the number and nature of bounded solutions to the linearized reduced form model (14) – (15).[6]

[5] As in Hintermaier (2003), we assume the existence of a unique steady state in this paper. Otherwise, we need some boundary conditions for the utility function to guarantee steady-state uniqueness.

[6] For unrestricted reduced form parameters, the indeterminacy and E-stability properties of the model (14) – (15) have been studied by Evans and McGough (2005b) and Duffy and Xiao (2007).

2.2 Indeterminacy

A rational expectations equilibrium (REE) is any bounded (uniformly bounded almost everywhere) solution to (14) – (15). Standard real business cycle models have a unique equilibrium; however, if the model incorporates some form of non-convexity – such as increasing social returns – then there may be many equilibria; furthermore, even if there is no fundamental stochastic component to the economy, a given equilibrium may be stochastic in nature. As is standard, we say that the model is determinate if there is a unique equilibrium and indeterminate if there are multiple equilibria.

To assess the determinacy properties of the model, we proceed as follows: write $\xi_{t+1} = c_{t+1} - E_t c_{t+1}$. Because agents are rational, we have that $E_t \xi_{t+1} = 0$, that is, ξ_t is a martingale difference sequence (mds). Noting that $E_t c_{t+1} = c_{t+1} - \xi_{t+1}$ and $E_t k_{t+1} = k_{t+1}$, we may rewrite the system (14) – (15) as

$$\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = J \begin{pmatrix} k_t \\ c_t \end{pmatrix} + \begin{pmatrix} 0 \\ \xi_t \end{pmatrix}, \tag{16}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ b_k & b_c \end{pmatrix}^{-1} \begin{pmatrix} d_k & d_c \\ e_k & 1 \end{pmatrix}.$$

We conclude that if (k_t, c_t) is an REE then there is an mds ξ_t so that (k_t, c_t) satisfies (16).

Whether the model is indeterminate may be assessed by analyzing the matrix J. If J has one eigenvalue inside the unit circle and one eigenvalue outside – as would be the case if total factor productivity were exogenous – then given an initial value for k_0, the linearized system has a unique non-explosive solution and the model is determinate. If, on the other hand, both eigenvalues of J are inside the unit circle, then there are multiple equilibria and the model is indeterminate. In particular, given any uniformly bounded martingale difference sequence ξ_t, the VAR determined by (16) identifies a bounded solution to the log-linearized model. In this case, the process ξ_t captures extrinsic fluctuations in agents' expectations and is referred to as a sunspot, and the corresponding solution to the model is called a sunspot equilibrium.

2.3 E-stability

The rational expectations hypothesis requires that agents form expectations conditional on the true distributions of the model's endogenous variables, and it requires that agents coordinate their expectations in case these endogenous variables depend on extraneous sunspot processes; however, the hypothesis is silent on how agents arrive at a sufficient understanding of the endogenous variables' distributions, and how agents ultimately manage to coordinate their expectations. The macroeconomics learning literature fills this gap by providing a mechanism through which agents may be able gain understanding and achieve coordination; in this way, learning can provide strong support for the relevance of sunspot equilibria.

To model learning, we back off the assumption that agents know the equilibrium's endogenous distribution when forming expectations. We emphasize this by rewriting the reduced form model with a modified expectations operator:

$$
\begin{aligned}
E_t^* k_{t+1} &= d_k k_t + d_c E_t^* c_t, \qquad (17)\\
c_t + e_k k_t &= b_k E_t^* k_{t+1} + b_c E_t^* c_{t+1}. \qquad (18)
\end{aligned}
$$

Note that agents are required to forecast contemporaneous consumption in order to avoid the simultaneous determination of c_t and $E_t^* c_{t+1}$: see Section 3.3.1 for further discussion. At time t, agents are assumed to use available data to estimate a forecasting model, and then use this estimated model to form expectations. These expectations are imposed into the reduced form system (17) – (18), new data are realized and the process repeats. If, via this process, the economy converges in a natural sense to an REE, we say that the associated equilibrium is stable under learning.

2.3.1 Representations

For a particular equilibrium to be stable under learning, agents must use a forecasting model of sufficient richness to capture the endogenous variables' conditional distributions; furthermore, a sunspot equilibrium may be well-captured by forecasting models of several different functional forms, depending on what type of sunspot process is take as observable by agents; and, whether the equilibrium is stable under learning may depend on which type of forecasting model agents use. To clarify these issues and make precise our stability results, we rely on the notion of "equilibrium representation," as introduced by Evans and McGough (2005a).

In case of indeterminacy, a rational expectations equilibrium may be characterized using one of several different recursions; a *representation* of the REE is a particular recursion, and it is the representation that determines the functional form of the forecasting model used by agents. To be precise and explicit, let (k_t, c_t) be an equilibrium to the model (17) – (18). We now develop the two representations of this equilibrium germane to our analysis.

As above, define $\xi_t = c_t - E_{t-1} c_t$. Then ξ_t is a uniformly bounded martingale difference sequence and the computation above shows that the equilibrium (k_t, c_t) satisfies

$$
\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = J \begin{pmatrix} k_t \\ c_t \end{pmatrix} + \begin{pmatrix} 0 \\ \xi_t \end{pmatrix}. \qquad (19)
$$

We call the recursion (19) the *general form representation* of the sunspot equilibrium (k_t, c_t). Note that it is the recursion, not the equilibrium, that is called a representation.

To develop the common factor representation of the sunspot equilibrium (k_t, c_t), define ξ_t as above. Write $J = S(\lambda_1 \oplus \lambda_2)S^{-1}$ where the λ_i are the eigenvalues of J and S is a matrix whose columns are the associated eigenvectors. Assume that $\lambda_i \in \mathbb{R}$.[7] Change coordinates: $z_t = S^{-1}(k_t, c_t)'$, $\tilde{\xi}_t = S^{-1}(0, \xi_t)'$. It follows that

$$z_{it} = \lambda_i z_{it-1} + \tilde{\xi}_{it}.$$

Write $\eta_{it} = (1 - \lambda_i L)^{-1}\tilde{\xi}_{it}$, and let $S^{ij} = (S)^{-1}_{ij}$. Then, along the equilibrium path,

$$c_t = -\frac{S^{i1}}{S^{i2}} k_t + \frac{1}{S^{i1}} \eta_{it}.$$

Since $k_t = d_k k_{t-1} + d_c c_{t-1}$, we conclude that the sunspot equilibrium (k_t, c_t) satisfies

$$\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = \begin{pmatrix} d_k & d_c \\ -\frac{S^{i1}}{S^{i2}} d_k & -\frac{S^{i1}}{S^{i2}} d_c \end{pmatrix} \begin{pmatrix} k_t \\ c_t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{S^{i2}} \end{pmatrix} \eta_{it}. \tag{20}$$

We call (20) the *common factor representation* of the sunspot equilibrium (k_t, c_t) associated to λ_i, and we call η_{it} the associated *common factor sunspot*.

Like ξ_t, η_{it} is an exogenous process, but unlike ξ_t, it is serially correlated; also, there are two common factor representations, one for each eigenvalue. Finally, notice that *every* sunspot equilibrium has a general form representation and, if the eigenvalues of J are real, then every sunspot equilibrium also has two common factor representations. For extensive discussion of common factor representations, see Evans and McGough (2005a, 2005b).[8]

2.3.2 PLM's, ALMS, and T-maps

To conduct our analysis of stability under learning, we provide agents with forecasting models that derive their functional form from the representations developed above. Following Evans and McGough (2005b), we assume that agents know the values of the reduced form parameters d_k and d_c, as well as the serial correlation of the common factor sunspot variables.[9]

[7] The theory of common factor representations in case of complex roots has been developed, but is not yet available.

[8] We note that there is an important connection between common factor representations and minimal state variable solutions (MSV). In particular, a common factor representation may be viewed as an MSV solution with a common factor sunspot appended to it.

[9] We could instead assume that the agents are required to learn about these parameters as well; however, our results would be unaffected because there is no expectational feedback in associated processes.

We begin by endowing agents with forecasting models consistent with the general form representation of the sunspot equilibrium. We assume that when forming expectations, agents view lagged endogenous variables as well as the current realization of the sunspot ξ_t, and use the following forecasting model:

$$c_t = A + Bk_{t-1} + Dc_{t-1} + F\xi_t. \tag{21}$$

To forecast capital, agents use (21) together with the known capital accumulation equation $k_t = d_k k_{t-1} + d_c c_{t-1}$.

Eq. (21) identifies the way in which agents perceive consumption will evolve, and because of this, it is called the perceived law of motion (PLM) associated to the general form representation of a sunspot equilibrium. We provide the PLM with the following interpretation: agents hold beliefs summarized by the coefficients A, B, D and F (they obtain these beliefs through an estimation procedure as discussed below). At time t, agents view the realization of the sunspot variable ξ_t, and with this realization and lagged data, the agents use their beliefs captured by the PLM to form forecasts of consumption and capital.

Assuming that agents do not view contemporaneous endogenous variables when forming forecasts,[10] we find

$$\begin{aligned} E_t^* c_{t+1} &= (1+D)A + B(d_k + D)k_{t-1} + (Bd_c + D^2)c_{t-1} + DF\xi_t, \\ E_t^* k_{t+1} &= d_c A + (d_k^2 + d_c B)k_{t-1}) + d_c(d_k + D)c_{t-1} + d_c F\xi_t. \end{aligned}$$

Incorporating these forecasts into (17) yields

$$\begin{aligned} c_t = {} & (b_c(1+D) + b_k d_c)A + \big(b_k(d_k^2 + d_c B) + b_c B(d_k + D) - e_k d_k\big)\, k_{t-1} \\ & + \big(b_k d_c(d_k + D) + b_c(Bd_c + D^2) - e_c d_c\big)\, c_{t-1} + (b_k d_c + b_c D)\, F\xi_t. \end{aligned} \tag{22}$$

Eq.(22) is called the actual law of motion (ALM) associated to the general form representation of a sunspot equilibrium: it identifies the endogenous behavior implied by the beliefs encoded into the PLM. Notice that the ALM also captures the self-fulfilling nature of sunspots: they affect the economy if and only if agents believe in their relevance. For example, the realization of ξ_t affects consumption precisely when $F \neq 0$.

Because of the careful specification of the perceived law of motion, the PLM and the ALM have the same functional form. This allows us to define a map $T : \mathbb{R}^4 \to \mathbb{R}^4$ taking the perceived coefficients to actual or implied coefficients. The map is given by

$$\begin{aligned} A &\to (b_c(1+D) + b_k d_c)A, \\ B &\to b_k(d_k^2 + d_c B) + b_c B(d_k + D) - e_k d_k, \\ D &\to b_k d_c(d_k + D) + b_c(Bd_c + D^2) - e_c d_c, \\ F &\to (b_k d_c + b_c D)\, F. \end{aligned}$$

[10]We discuss an alternative timing assumption in Section 3.3.1.

A fixed point of the T-map corresponds to the coincidence of the perceived and actual parameters: at a fixed point, agents are using the correct forecasting model. Thus, a fixed point of the T-map identifies a representation of a rational expectations equilibrium.

2.3.3 The E-stability principle

To introduce the notion of E-stability, we begin at a general, abstract level. Suppose that agents in a given model have perceptions summarized by the parameter vector Φ, and that $T(\Phi)$ provides the implied parameters of the ALM. Let Φ^* be fixed point of the T-map. Then the fixed point, and its associated equilibrium representation, is said to be *Expectationally Stable* (E-stable) if it is a Lyapunov stable fixed point of the differential system

$$\dot{\Phi} = T(\Phi) - \Phi. \tag{23}$$

Let DT be the Jacobian of T evaluated at the fixed point. If real parts of its eigenvalues are less than one then the representation is E-stable.

E-stability is of interest because of its deep connection to the stability of the equilibrium representation under real time learning. According to the E-stability principle, if a representation is E-stable then it is locally learnable using recursive least squares or related learning algorithms. Establishing the validity of the E-stability principle for a given model requires the use of sophisticated machinery from the theory of stochastic recursive algorithms; however, the intuition for the connection between E-stability and real time learning is straightforward. Recall the learning story summarized above: agents use data to estimate their forecasting model, or PLM, and they used the estimated model to form expectations, thus generating new data. Finally, the new data are used to re-estimate the model. Recursive estimation methods imply that the parameter estimates are updated in the direction implied by the associated forecast error. Now look at the differential equation: it says to move the perceptions Φ in the direction of $T(\Phi) - \Phi$, which is essentially a forecast error. If the fixed point Φ^* is Lyapunov stable then by moving in the direction of the forecast error, Φ eventually converges to Φ^*; thus, it makes sense that the associated learning algorithm, which moves estimates in the direction dictated by the forecast error, will also lead to convergence to the fixed point. For complete details on E-stability, see Evans and Honkapohja (2001).

Our discussion of E-stability has focused on the analysis of an isolated fixed point of the T-map; however, when sunspots are involved, things are more complicated. Let ξ_t be any uniformly bounded mds and let

$$\Gamma = \{\Phi \in \mathbb{R}^4 : T(\Phi) = \Phi\}.$$

If ξ_t is a uniformly bounded mds then so too is $\gamma\xi_t$ for any $\gamma \in \mathbb{R}$; it follows that

Γ is a one dimensional affine space. Because instead being finite, the set of fixed points of the T-map is an unbounded continuum, our formulation of E-stability must be altered. We say that the general form representations associated to the set Γ are E-stable provided that if V is an open set containing Γ then there is an open set $U(V)$ containing Γ so that any solution of the differential equation (23) with initial conditions in U remains in V and converges to a point in Γ. Again, let DT be the derivative of the T-map evaluated at any point in Γ, and note that DT is independent of the point chosen. Also, notice that the eigenvalue of DT associated to the perceived coefficient of the sunspot is necessarily unity. It can be shown if the remaining eigenvalues have real parts less than unity then Γ is E-stable: See Evans and McGough (2005a) for details. In case Γ is E-stable, we say that sunspot equilibria represented in general form are stable under learning.

The stability analysis so far has been developed assuming that agents use a forecasting model consistent with a general form representation of the sunspot equilibrium; however, analysis based on common factor representations proceeds in an analogous fashion. Given the sunspot process ξ_t, we may form the associated common factor sunspots $\eta_{it} = (1 - \lambda_i L)^{-1}\tilde{\xi}_{it}$: see Section 2.3.1 for notation. We now assume that these are observable and agents regress on them when forming expectations. Specifically, we provide agents with a forecasting model of the form

$$c_t = A + Bk_{t-1} + Dc_{t-1} + G\eta_{it}. \tag{24}$$

Without loss of generality, we assume that agents know the values of the λ_i, and thus form forecasts as follows:

$$\begin{aligned} E_t^* c_{t+1} &= (1+D)A + B(d_k + D)k_{t-1} + (Bd_c + D^2)c_{t-1} \\ &\quad + (D + \lambda_1)G\eta_{1t}, \\ E_t^* k_{t+1} &= d_c A + (d_k^2 + d_c B)k_{t-1} + d_c(d_k + D)c_{t-1} + d_c G\eta_{it}. \end{aligned}$$

These forecasts may be imposed into the reduced form system and the T-map may be obtained as above. Given the T-map, stability analysis is conducted in precisely the same manner. We note that provided the roots of J are real, there are two common factor representations of the associated sunspot equilibria, and both must be checked for stability. In case stability obtains, we say that sunspot equilibria represented in common factor form are stable under learning.

2.3.4 Stable sunspot equilibria

E-stability is representation dependent: a given sunspot equilibrium may be stable under learning if agents use one type of representation, and unstable under learning otherwise. For example, Evans and McGough (2005a) studied a univariate model with lag and found that for all parameterizations the general form representations of sunspot equilibria were not E-stable, whereas, for a subset of the model's parameter space, the common factor representations of sunspot equilibria were stable under learning. In fact, common factor representations seem to inherit the stability properties of the associated minimal state variable solution: for more details, see Evans and McGough (2005a), and for an interesting connection between MSV solutions and E-stability, see McCallum (2007).

Restrictions on the reduced form parameters necessary to guarantee E-stability of general form and common factor representations of sunspot equilibria are complicated, and provided in Evans and McGough (2005b) and Duffy and Xiao (2007); however, because of the important role that they will play in our analysis below, we emphasize on some of the results in the following two lemmas:[11]

Lemma 1 *If the reduced form system (14) – (15) exhibits E-stability of general form and/or common factor representation sunspot equilibria then*

$$b_c < 0. \tag{25}$$

Lemma 2 *If the reduced form system (14) – (15) exhibits E-stability of general form representation sunspot equilibria then*

$$b_c < 0,\ b_c < d_k - d_c e_k < 0,\ b_c + d_k - d_c e_k < 1 - b_k d_c + d_k b_c < 0. \tag{26}$$

2.4 The stability puzzle

Evans and McGough (2005b) established a stability puzzle: there are reduced form parameter constellations for the model (14) – (15) consistent with stable general form representations and stable common factor representations; however, reduced form parameters derived from non-convex RBC models do not lie within these constellations: the sunspot equilibria of the real indeterminacy literature appear to be unstable under learning.[12] The finding is all the more puzzling since calibrated New-Keynesian models are known to be consistent with stable common factor representations of sunspot equilibrium: see Evans and McGough (2005c).[13] Duffy and Xiao (2007) shed light on the underlying mechanism driving the puzzle by showing that while a necessary condition for the stability of either general form or common factor representations is that

[11]These are necessary conditions for E-stability. Sufficient conditions are difficult to obtain because of the presence of zero eigenvalues, which necessarily obtain when the model is indeterminate.

[12]Evans and McGough (2005b) found small regions of stability for certain non-convex RBC models provided common factor representations are used and the contemporaneous timing assumption is adopted.

[13]There are no examples of stable general form representations of sunspot equilibria derived from New Keynesian models.

the coefficient b_c modifying agents' expectations of future consumption be negative (see Lemma 1), several existing non-convex RBC models, including Schmitt-Grohé and Uribe (1999), Benhabib and Farmer (1994), and Wen (1998) all impart $b_c > 0$. However, a fundamental open question remains: are there non-convex RBC models which exhibit stable sunspot equilibria?

3 Results

We begin this section by developing analytic results characterizing conditions necessary for the presence of stable sunspot equilibria. Then, using these conditions for guidance, we provide numerical examples of stable sunspot equilibria in calibrated models. We close with a numerical investigation of the link between E-stability and real time statistical learning of sunspot equilibria in our model.

3.1 Necessary conditions for E-stable sunspots: the case of general utility function

In their classic paper, Benhabib and Farmer (1994) established a necessary condition for indeterminacy (the Benhabib-Farmer condition): labor demand must be upward-sloping and steeper than the Frisch labor supply. Farmer (1999) provides the following intuition: an increase in capital stock shifts the labor demand curve upward; if the demand curve is downward sloping, labor hours increase thus increasing the capital stock further and leading to an explosive (and thus non-optimal) path; however, if the demand curve is upward-sloping and crosses the supply curve from below then the increase in capital stock is dampened by the associated decrease in labor supplied.

The Benhabib-Farmer condition is imposed in many papers on model indeterminacy; however, Meng and Yip (2008) found, in a model with negative capital externalities and with a general utility function, that sunspot equilibria and downward sloping labor demand are compatible. To gain intuition for their result, we review a special case here. Assuming that $U(C,L) = \frac{1}{1-\sigma}\left(C^{1-\sigma}-1\right) - \frac{1}{1+\chi}L^{1+\chi}$ ($\sigma \geq 0$, $\chi \geq 0$), and log-linearizing the intra-temporal first order condition (4), yields a log-linearized Frisch labor supply given by

$$\sigma c_t + \chi l_t = w_t. \tag{27}$$

Labor demand is obtained by log-linearizing the wage equation (9):

$$\alpha k_t + (\beta - 1) l_t = w_t. \tag{28}$$

Meng and Yip (2008) found that provided social production exhibits increasing returns ($\alpha + \beta > 1$) and provided the intertemporal elasticity of consumption (σ^{-1}) and labor supply elasticity (χ^{-1}) are sufficiently large, indeterminacy obtains even when demand is downward-sloping (there exists $\bar{\beta} \in (0,1)$ so that $\beta \in (\bar{\beta}, 1)$ implies indeterminacy). Revisiting the thought experiment envisioned by Farmer, suppose that k_t increases. Just as above, with downward-sloping labor demand, the direct effect is an increase in wage rates and labor supplied, which creates the positive feedback loop leading to an explosive path for capital. However, the resulting increase in savings (and thus k_{t+1}) lowers the expected marginal productivity of capital (MPK) tomorrow through two channels: the usual channel capturing diminishing marginal returns; and the additional channel introduced by the negative capital externality. The large expected decrease in the MPK significantly reduces the expected real interest rate, and because $\sigma < 1$, this causes a significant increase in current consumption. Finally, the increase in c_t shifts the labor supply (27) left, mitigating the increase in labor hours. Provided χ is small, the effect of this shift on real wage is not significant and the positive feedback loop is broken.

While a downward sloping labor demand curve allows for indeterminacy in a model with negative capital externalities, also requiring the associated sunspot equilibria to be stable under learning brings back the Benhabib-Farmer condition in full force: in fact, their condition proves to be necessary even for the model with a general concave utility function. We state this result in Proposition 1 below. Evaluated at the steady state, we denote

$$\begin{aligned} \delta_{cc} &= \frac{u_{cc}}{u_c} c^*, \delta_{cl} = \frac{u_{cl}}{u_c} l^*, \\ \delta_{lc} &= \frac{u_{lc}}{u_l} c^*, \delta_{ll} = \frac{u_{ll}}{u_l} l^*, \end{aligned}$$

and (from utility concavity assumption)

$$\chi^* = \delta_{ll} - \delta_{cl} \frac{\delta_{lc}}{\delta_{cc}} \geq 0.$$

Proposition 1 follows from Lemma 1.

Proposition 1 *With a general concave utility function $u = u(C_t, L_t)$, if the model exhibits sunspot equilibria which are stable under learning when represented either in general or common factor form then the Benhabib-Farmer condition that labor demand cross the Frisch labor supply from below, and given by*

$$\beta - 1 > \chi^*, \tag{29}$$

must hold.

Proof: See Appendix.

3.2 E-stable sunspot equilibria: the case of separable utility

By focusing our attention on a separable utility specification, additional necessary conditions for joint indeterminacy and E-stability can be obtained. Assume the utility function is given by

$$u\left(C_t\right) - v\left(L_t\right),$$

where $u' > 0$, $u'' \leq 0$, $v' > 0$, $v'' \geq 0$. Let (evaluated at the steady state)

$$\sigma^* = -\frac{u''\left(c^*\right)}{u'\left(c^*\right)} c^* \geq 0,\ \chi^* = \frac{v''\left(l^*\right)}{v'\left(l^*\right)} l^* \geq 0.$$

From Lemma 2, we can obtain the following result on E-stability.

Proposition 2 *If the model with separable utility exhibits sunspot equilibria which are stable under learning when represented in general form then*

$$\beta - 1 > \chi^*, \tag{30}$$

$$\alpha < a, \tag{31}$$

$$\sigma^* < 1. \tag{32}$$

Proof: See Appendix.

As Lemma 2, Proposition 2 holds for E-stability of sunspot equilibria represented in general form. The result for common factor representations cannot be obtained analytically; however, we have conducted an extensive numerical analysis and found that Proposition 2 holds for equilibria represented in common factor form as well.

Note that for joint indeterminacy and E-stability Proposition 2 requires the capital externality to be negative ($\alpha < a$) and the labor externality to be positive ($\beta > 1$), and at the same time increasing returns at the social level ($\alpha + \beta > 1$). Although there has been much research in the literature on the estimation of returns to scale and external effects of production functions, we cannot judge the empirical plausibility of negative capital externalities based on available evidence: almost all of the relevant analyses impose that the elasticities with respect to the capital externality and the labor externality have the same sign (and indeed even with the same proportion in terms of factor shares). With this restriction, some studies such as Burnside (1996) found that some industries display negative external effects (from both capital stock and labor input) whereas some other industries display positive external effects. In a recent investigation, Harrison (2003) also found positive external effects for the investment good sector and negative effects in the consumption good sector. To further assess on the empirical plausibility of negative capital externalities and other conditions here one needs to relax the restriction typically imposed on the elasticities of the production with respect to capital and labor external effects (i.e., not restricting that they have the same sign).[14]

3.2.1 E-stable sunspot equilibria

Proposition 2 establishes necessary conditions for E-stable sunspot equilibria; however sufficient conditions are intractable, and thus we turn to numerical analysis to demonstrate existence. We begin by fixing all parameters but α and σ^*: set $a = 0.3$, $b = 0.7$, $\delta = 0.025$, $\rho = 0.99$, $\chi^* = 0.08$, $\beta = 1.1$. Notice that the Benhabib-Farmer condition($\beta - 1 > \chi^*$) is satisfied. We now vary α and σ^* within the region dictated by Proposition 2 and determine which parameter pairs (α, σ^*) correspond to joint E-stability and indeterminacy. Because the E-stability conditions differ depending on which type of forecasting model agents use, we report the general form and common factor stability results separately

We begin with a search for E-stable sunspots using general form representations. Figure 1 shows a large region of parameters (α, σ^*) for which the model with negative capital externalities and separable utility has sunspot equilibria that are stable under learning provided that agents use a forecasting model consistent with a general form representation.

We view Figure 1 as surprising, and it captures the main result of the paper. It provides the first positive response to the stability puzzle: non-convex RBC models – even two dimensional versions – can exhibit E-stable sunspot equilibria.[15] Perhaps equally surprising, the sunspot equilibria are stable when agents use a general form representation to form forecasts: this is an outcome never witnessed in the New Keynesian framework. Finally, while the intertemporal elasticity of consumption is admittedly large, we note that it is in line with some values used in the literature: for example, Evans, Honkapohja and Romer (1998) specify $\sigma < 0.3$.[16] For alternative calibrations of the model we find E-stable sunspots for larger values of σ^*.[17]

Figure 1 also provides the corresponding stability analysis assuming agents use a forecasting model consistent with a common factor representation of the sunspot

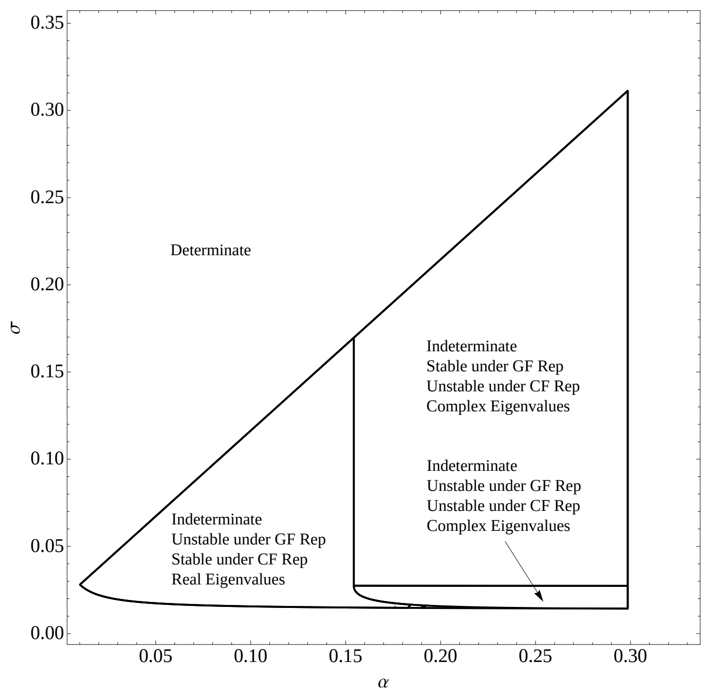
[14] Kehoe (1991, pp. 2133-2134) studied a one-sector growth model with exogenous labor supply where indeterminacy happens when production is subject to a negative externality from capital, but he did not study the E-stability issue.

[15] As discussed in the next section, under a different timing assumption and in a different model, Evans and McGough (2005b) found small regions of parameters corresponding to sunspot equilibria provided that agents used common factor representations.

[16] Also, in our model, $(\sigma^*)^{-1}$ is the local, steady-state value of the intertemporal elasticity of consumption – we do not require that $(\sigma^*)^{-1}$ capture consumers' global behavior.

[17] For example, with $\delta = \chi = 0, a = b = .5$, and $\beta = 1.005$, we find stable sunspots with $\alpha = \sigma^* = .49$.

Figure 1: Indeterminacy and E-stability



equilibrium. We again find a large region of parameter space corresponding to E-stable sunspot equilibria. Note that the stability regions associated to alternative representations are disjoint: for our specifications of the model, stability of general form and common factor representations are mutually exclusive.

3.3 Additional issues

Our results so far are presented in stark terms: E-stable sunspot equilibria exist in calibrated versions of our model and the Benhabib-Farmer condition is generically necessary for their existence. While this presentation style serves to highlight the principal contributions of our paper, completeness requires a discussion of several issues germane to E-stability of sunspot equilibria and their empirical relevance.

3.3.1 Expectations and information: the timing assumption

E-stability analysis is conducted by positing a forecasting model for the agents (a PLM) and then establishing how expectations are formed given the forecasting model. For example, if agents use a PLM consistent with a general form representation, their forecasting model takes the form (21), rewritten here for convenience:

$$c_t = A + Bk_{t-1} + Dc_{t-1} + F\xi_t.$$

Boundedly rational forecasts are determined using (21) as follows:

$$E_t^* c_{t+1} = A + BE_t^* k_t + DE_t^* c_t.$$

Since k_t is predetermined, it is assumed that $E_t^* k_t = k_t$; however, we must take a stand on how agents form expectations of contemporaneous consumption, that is, we must specify $E_t^* c_t$. It is common in the learning literature to assume agents that do not view contemporaneous (non-predetermined) aggregates when forming expectations: intuitively, agents form expectations of the future and perform actions conditional on the expectations; contemporaneous endogenous variables obtain as a result of these actions. Modeled this way,

$$E_t^* c_t = A + Bk_{t-1} + Dc_{t-1} + F\xi_t,$$

which is the assumption we have made in our above analysis. However, it is also reasonable to assume that agents' expectations and contemporaneous endogenous variables are simultaneously determined: intuitively, agents submit schedules of actions contingent on expectations of the future, where these expectations condition on realizations of contemporaneous variables. Modeled this way, $E_t^* c_t = c_t$. Importantly, the resulting T-map, while identifying the same fixed points (and thus REE), imparts different learning dynamics. Whether a particular equilibrium is stable under learning may depend on the timing assumption imposed by the modeler. Evans and McGough (2005b) studied a variety of models using both timing conventions, and did find small regions of parameter space corresponding to stable sunspot equilibria provided agents used forecasting models corresponding to common factor representations, and provided the alternative, contemporaneous timing assumption was employed. Because the contemporaneous timing assumption tends to be stabilizing (i.e. stability under the usual timing assumption imparts stability under the alternative timing assumption, but not necessarily vice-versa), and because we have already located large regions of parameter space corresponding to sunspot stability, we refrain from investigating the alternative timing assumption in our model.

3.3.2 Sunspot equilibrium dynamics

Duffy and Xiao (2007, p. 887) say (in effect) that the dynamics of the equilibrium (16), reproduced here for convenience,

$$\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = J \begin{pmatrix} k_t \\ c_t \end{pmatrix} + \begin{pmatrix} \xi_t \\ 0 \end{pmatrix},$$

are empirically plausible only if the eigenvalues of J have positive real part. Intuitively, they wish to avoid "period-by-period oscillatory convergence to the steady state." We note that for many specifications of our model, the stable sunspot equilibria satisfy this notion of empirical plausibility. For example, using the base parameters from Figure 1, and setting $\sigma = .3$ and $\alpha = .29$ yields stable sunspots and a J matrix with eigenvalues given by $0.9694 \pm .0.0306i$.

3.3.3 Real-time learning

The E-stability Principle provides the connection between E-stability and the convergence of real time adaptive algorithms; however, the E-stability Principle is just that – a principle – and whether it holds may depend on modeling particulars. Formally establishing whether E-stability governs real time convergence requires applying the theory of stochastic recursive algorithms; and application of this theory requires that the set of fixed points to the T-map be compact: see Evans and Honkapohja (2001). Because our T-map – indeed any T-map in any study of indeterminate linear models – violates this assumption, the E-stability Principle does not formally apply.[18]

To justify reliance on E-stability when analyzing sunspot equilibria, researchers in the learning literature often turn to simulation, and because stable sunspot equilibria in the context of non-convex RBC models are a newly discovered phenomenon, we perform the simulation exercise here. Let $\Phi = (A, B, D, F)'$ be the perceived coefficients of an agent using a forecasting model consistent with a general form representation, let Φ_t be the agent's estimate of Φ given data dated time t and before, and let $X_t = (1, k_{t-1}, c_{t-1}, \xi_t)'$ be the vector of regressors, where ξ_t is the observable sunspot variable. Real time learning via recursive least squares is then summarized via the following dynamic system written in causal ordering:

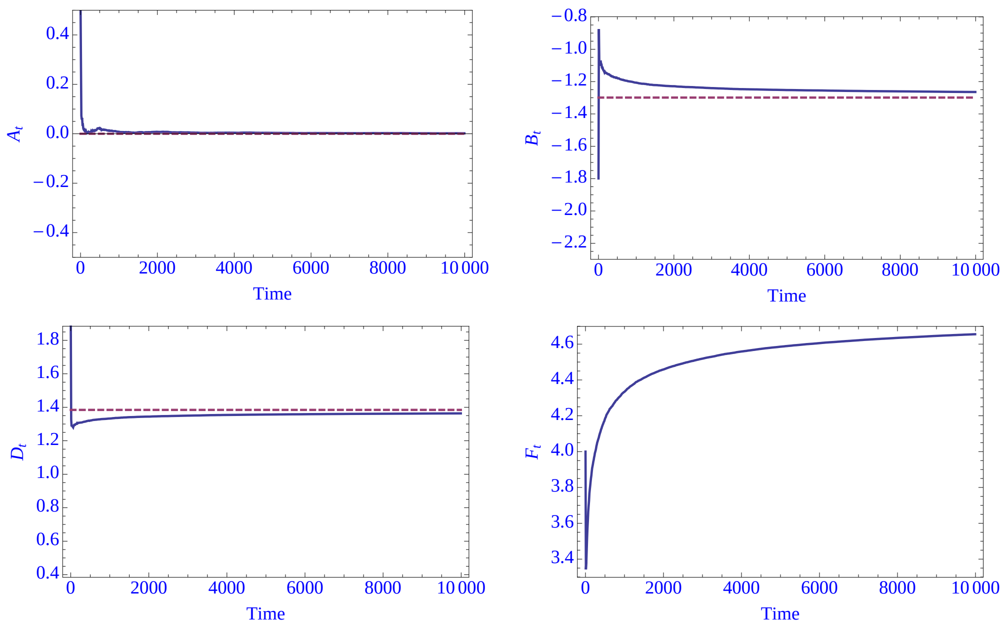
$$\begin{aligned}
R_t &= R_{t-1} + \frac{\gamma}{t}\left(X_t X_t' - R_{t-1}\right), \\
c_t &= T(\Phi_{t-1})' X_t, \\
k_t &= \delta_k k_{t-1} + \delta_c c_{t-1}, \\
\Phi_t &= \Phi_{t-1} + \frac{\gamma}{t} R_t^{-1} X_t \left(c_t - \Phi_{t-1}' X_t\right). \qquad (33)
\end{aligned}$$

Here R_t captures the sample covariance of the regressors and γ/t is the scaled gain measuring how much weight is placed on new information. Notice that (33) advises the agent to adjust his forecasting model estimates Φ_{t-1} in the direction indicated by the forecast error $c_t - \Phi_{t-1}' X_t$. Using the same parameter specification as Figure 1, and setting $\sigma = 0.2$ and $\alpha = 0.1$, Figure 2 demonstrates that our E-stable sunspot equilibria are learnable, as indicated by the E-stability Principle.[19]

[18] This issue is well-known in the learning literature: see Evans and Honkapohja (2001) for details.

[19] The learning algorithm is initialized near the REE. Also, a similar dynamic system obtains in

Figure 2: Realtime Learning



4 Conclusion

The case for self-fulfilling prophecy to drive the business cycle would be bolstered by establishing that the associated equilibria are stable under learning; however, the original enthusiasm engendered by the stability results of Woodford (1990) have been dampened by the subsequent findings of instability in more mainstream work such as RBC models; in fact, many have wondered whether stable sunspot equilibrium can even exist in these types of models. We explore this issue of existence in one-sector RBC model with factor-generated externalities. Our first result extends the conclusions of Benhabib and Farmer (1994): a necessary condition for joint indeterminacy and E-stability is that the labor-demand curve is upward-sloping and steeper than the Frisch labor-supply curve. We find also that when the utility function is separable in consumption and leisure, then additional necessary conditions obtain: capital externalities must be negative and the steady-state value of intertemporal elasticity of consumption must be larger than one.

Using these necessary conditions for guidance, we numerically explore our model's parameter space and establish the existence of stable sunspot equilibria. This result provides an explicit answer to the stability puzzle: non-convex RBC-type models can exhibit stable sunspot equilibria; moreover, we find that even if agents use general

case agents use a forecasting model consistent with a common factor representation; and similar numerical indication of converge is obtained: we suppress these findings here.

form representations, stability may obtain.

We should note that the main results obtained in this paper do have their limitations. First, stability of sunspot equilibria can only happen when the Benhabib-Farmer condition holds, that is, the labor-demand curve is upward-sloping and steeper than the Frisch labor-supply curve. This condition has sometimes been criticized for its empirical plausibility. In addition, the steady-state value of relative risk aversion required is admittedly low (for the case with separable utility). However, the principal result of this paper is to show that simple structural (one-sector RBC) models may produce stable sunspot equilibria. Our finding suggests that sunspot equilibria may also exist in alternative or more complex structural models. For example, future research may consider models with a more general form of the production function or models with multiple sectors.

5 Appendix

5.1 Proof of Proposition 1

Log-linearizing the first-order conditions (11)-(13) yields

$$E_t k_{t+1} = d_k k_t + d_c c_t, \tag{34}$$

$$c_t + e_k k_t = b_k E_t k_{t+1} + b_c E_t c_{t+1}, \tag{35}$$

where

$$b_c = 1 + \frac{\left(1 - \frac{\delta_{lc}}{\delta_{cc}}\right)\rho\theta\beta}{1 - \beta + \delta_{ll} - \delta_{cl}\frac{\delta_{lc}}{\delta_{cc}}},$$

$$b_k = \frac{\rho\theta}{\delta_{cc}}[\frac{(\alpha-1)(1+\delta_{ll}-\delta_{cl})+\beta}{1+\delta_{ll}-\delta_{cl}\frac{\delta_{lc}}{\delta_{cc}}-\beta}] + e_k,$$

$$e_k = \frac{\alpha\delta_{cl}}{\delta_{cc}(1+\delta_{ll}-\delta_{cl}\frac{\delta_{lc}}{\delta_{cc}}-\beta)},$$

$$d_k = 1 - \delta + \frac{\alpha\theta}{a}\left(1 + \frac{\beta}{1+\delta_{ll}-\delta_{cl}-\beta}\right),$$

$$d_c = \frac{\theta}{a}\left[\frac{(\delta_{cc}-\delta_{lc})\beta}{1+\delta_{ll}-\delta_{cl}-\beta} - 1\right] + \delta,$$

$$\delta_{cc} = \frac{u_{cc}}{u_c}c^*, \delta_{cl} = \frac{u_{cl}}{u_c}l^*, \delta_{lc} = \frac{u_{lc}}{u_l}c^*, \delta_{ll} = \frac{u_{ll}}{u_l}l^*,$$

$$\theta = 1/\rho - 1 + \delta.$$

From the necessary condition for joint E-stability and indeterminacy $b_c < 0$ in Lemma 1, we have

$$\left[1 - \left(1 - \frac{\delta_{lc}}{\delta_{cc}}\right)(1-\rho)\right]\beta < 1 + \delta_{ll} - \delta_{cl}\frac{\delta_{lc}}{\delta_{cc}} < \beta, \tag{36}$$

i.e.,

$$[1 - \eta^*(1 - \rho(1-\delta))]\beta < 1 + \chi^* < \beta, \tag{37}$$

where $\chi^* = \delta_{ll} - \delta_{cl}\frac{\delta_{lc}}{\delta_{cc}}$, $\eta^* = 1 - \frac{\delta_{lc}}{\delta_{cc}}$. From the utility concavity assumption, χ^* must be non-negative, i.e.,

$$\chi^* = \delta_{ll} - \delta_{cl}\frac{\delta_{lc}}{\delta_{cc}} \geq 0. \tag{38}$$

Thus

$$\beta - 1 > \chi^*, \tag{39}$$

which means that the slope of the labor-demand curve is positive and exceeds the slope of the Frisch labor-supply curve.

5.2 Proof of Proposition 2

If the utility function is separable, then we have

$$\begin{aligned} b_k &= \frac{\rho\theta}{\sigma^*}\left[1 - \alpha\left(1+\gamma\right)\right], \;\; b_c = 1 + \rho\theta\gamma, \; e_k = 0, \\ d_k &= 1 - \delta + \frac{\alpha}{a}\theta\left(1+\gamma\right), \;\; d_c = -\left[\frac{1}{a}\theta\left(\sigma^*\gamma + 1\right) - \delta\right], \end{aligned}$$

where

$$\gamma = \frac{\beta}{1 - \beta + \chi^*}.$$

From Lemma 2, the necessary conditions for joint indeterminacy and E-stability of general form representation sunspot equilibria are given by

$$b_c < 0, \; b_c < d_k - d_c e_k < 0, \; b_c + d_k - d_c e_k < 1 - b_k d_c + d_k b_c < 0. \tag{40}$$

The first inequality $b_c < 0$ gives constraints on the relation between β and χ^*:

$$\beta\rho(1-\delta) < 1 + \chi^* < \beta. \tag{41}$$

The second set of inequalities in (40) consists of two parts. The inequality $b_c < d_k$ gives an upper bound for $\frac{\alpha}{a} - 1$ and $d_k < 0$ gives its lower bound:

$$-\frac{1-\delta}{\theta\left(1+\gamma\right)} - 1 < \frac{\alpha}{a} - 1 < \frac{\rho - 1}{1 - \rho\left(1-\delta\right)}.$$

As $\rho < 1$, we have, $\frac{\alpha}{a} - 1 < 0$, i.e.

$$\alpha < a. \tag{42}$$

The third set of inequalities in (40) pin down the constraints on σ^*. By substituting the expressions for b_c, b_k, d_c, d_k into the inequalities, we have

$$\varphi\Gamma_\sigma < \sigma^* < \Gamma_\sigma,$$

where $\Gamma_\sigma = [1 - \frac{(1-\alpha)(1+\chi^*)}{\beta}]$ and $\varphi = [\frac{\frac{\theta}{a} - \delta}{\frac{2-\delta+\frac{\alpha}{a}\theta(1+\gamma)}{\rho\theta\gamma} + 1 - \delta + \frac{\theta}{a}}]$. It is obvious that $\Gamma_\sigma < 1$, and therefore

$$\sigma^* < 1. \tag{43}$$

References

[1] Azariadis, C., 1981. Self-fulfilling prophecies. Journal of Economic Theory 25, 380-396.

[2] Benhabib, J., Farmer R.E.A., 1994. Indeterminacy and increasing returns. Journal of Economics Theory 63, 19-41.

[3] Burnside, C., 1996. Production function regressions, returns to scale, and externalities. Journal of Monetary Economics 37, 177-201.

[4] Cass, D., Shell, K., 1983. Do sunspots matter?. Journal of Political Economy 91, 193-227.

[5] Duffy, J., Xiao, W., 2007. Instability of sunspot equilibria in RBC models under adaptive learning. Journal of Monetary Economics 54, 879-903.

[6] Evans G.W., Honkapohja, S., 2001. Learning and expectations in macroeconomics. Princeton University Press, Princeton.

[7] Evans G.W., Honkapohja, S., Romer P., 1998. Growth cycles. American Economic Review 88, 495-515.

[8] Evans, G.W., McGough, B., 2005a. Stable sunspot solutions in models with predetermined variables. Journal of Econmic Dynamics and Control 29, 601-625.

[9] Evans, G.W., McGough, B., 2005b. Indeterminacy and the stability puzzle in non-convex economies. Contributions to Macroeconomics 5, Iss. 1, Article 8.

[10] Evans, G. W., McGough, B., 2005c. Monetary policy, indeterminacy, and learning. Journal of Economic Dynamics and Control 29, 1809-1840.

[11] Evans, G.W., McGough, B., 2010. Implementing optimal monetary policy in New-Keynesian models with inertia. The B.E. Journal of Macroeconomics 10, Iss. 1, Article 5.

[12] Farmer, R.E.A., 1999. Macroeconomics of Self-Fulfilling Prophecies, 2nd. ed.. MIT Press, Cambridge, MA.

[13] Farmer R.E.A. and Guo J.T., 1994. Real business cycles and the animal spirits hypothesis. Journal of Economic Theory 63, 42-72.

[14] Harrison, S.G., 2003. Returns to scale and externalities in the consumption and investment sectors. Review of Economic Dynamics 6, 963-976.

[15] Hintermaier, T., 2003. On the minimum degree of returns to scale in sunspot models of the business cycle. Journal of Economic Theory 110, 400-409.

[16] Kehoe, T., 1991. Computation and multiplicity of equilibria, in: Hildenbrand, W., Sonnenschein, H. (Eds), Handbook of Mathematical Economics, Vol. IV. North-Holland, Amsterdam, pp. 2049-2143.

[17] King, R.G., Rebelo, S., 1999. Resuscitating real business cycles, in: Taylor J. B., Woodford M. (Eds.), Handbook of Macroeconomics, Vol. 1. Elsevier, pp. 927-1007.

[18] McCallum, B., 2007. E-stability via-a-vis determinacy results for a broad class of linear rational expectations models. Journal of Economic Dynamics and Control 31, 1376-1391.

[19] Meng, Q.L., Yip, C.K., 2008. On indeterminacy in one-sector models of the business cycle with factor-generated externalities. Journal of Macroeconomics 30, 97-110.

[20] Schmitt-Grohé, S., Uribe, M., 1997. Balanced budget rules, distortionary taxes, and aggregate instability. Journal of Political Economy 105, 976-1000.

[21] Shell, K., 1977. Monnaie at allocation intertemporelle. Working Paper, CNRS Seminaire de E. Malinvaud, Paris.

[22] Wen, Y., 1998. Capacity utilization under increasing returns to scale. Journal of Economic Theory 81, 7-36.

[23] Woodford, M., 1990. Learning to believe in sunspots. Econometrica 58, 277-307.