FORMING DIMENSIONLESS PRODUCTS BY USING AN ALGORITHM DEVELOPED FROM MATRIX THEORY

by

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FORMING DIMENSIONLESS PRODUCTS BY USING AN ALGORITHM DEVELOPED FROM MATRIX THEORY

INTRODUCTION

There is some scattered literature on dimensional analysis embracing the use of matrices. In most of these references, the ideas of matrix algebra and of transformations have not been adopted. But the matrix concepts are easily mastered. Consequently, the author thinks it is opportune to exploit these ideas.

Some definitions pertaining to matrix algebra are given. Elementary transformations are stated and exemplified. The elementary approach to dimensional analysis found in most texts is presented. An alternative treatment is then examined. This treatment consists of applying matrix algebra to the same dimensional analysis. A transformation technique is demonstrated and leads to an algorithm adopted for the solution of the associated homogeneous underdetermined matrix equations. A matrix solution similar to that obtained by the elementary approach is sought. This comparison exposes the merits of the matrix method.

Usually in dimensional analysis, convention dominates. Thus, in a good part of the literature available, "conventional" concepts of mass, length, and time are used. However, there is good reason to introduce what might be considered "unconventional" concepts. It is by the use of such "unconventional" concepts that different dimensional products are calculated. It is suggested that the strangeness of some of the products is not sufficient cause for us to disregard them.

As a mode of representation of data, a rearrangement of some dimensionless products may be profitable. Certainly this rearrangement involves combinations of already available dimensionless products. Hence the result of the reshuffling is not new. But the different dimensionless products mentioned above are obtained by using "unconventional" concepts. They are not a consequence of reshuffling or of any combination. Therefore, they are not to be thrown out. They should serve to open up new areas of experimentation. It is felt that numerical solutions obtained from such experiments should be the criterion used in judging their retention or rejection.

MATRIX ALGEBRA

Definitions

General

A rectangular array of numbers or of functions is called a matrix and will be designated by a capital letter. Thus

$$B \Longleftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

is read, "B represents the array of numbers $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$."

B is a matrix. Brackets are usually used to enclose the array in question.

The numbers or functions are called the elements of the matrix and are denoted by a_{ij}, where subscripts i and j identify the particular row and column respectively, and show where that element is situated in the array.

A matrix M has m rows and n columns. That is,

$$i = 1, 2, 3 \ldots\ldots m$$

$$j = 1, 2, 3 \ldots\ldots n.$$

M is said to be of order (m, n) or to be an $m \times n$ matrix.

Any matrix with $a_{ij} = 0$ for all i and j is called the zero matrix. It is designated by 0.

The identity matrix is a square array, m = n, for which

$$a_{ij} = \begin{cases} 1, \text{ for all } i = j \\ 0, \text{ for all } i \neq j. \end{cases}$$

For example, the 3 x 3 identity matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The symbol I will be used to designate the identity matrix.

Product of Matrices

The product of matrix P and Q is denoted as PQ. It is defined as that matrix whose element in the ith and jth column is found by multiplying corresponding elements of the ith row of P by the elements in the jth column of Q and adding the results.

Symbolically, let the elements of P, Q and PQ be described by p_{ij}, q_{jl}, r_{il} in that order where, in p_{ij}, i represents rows and j represents columns.

From the definition of the product PQ, we have

$$r_{il} = \sum_{j=1}^{k} p_{ij}\, q_{jl} \qquad \begin{cases} i = 1, 2, 3 \ldots m \\ l = 1, 2, 3 \ldots n \end{cases}$$

Notice that multiplication has meaning only when the number of columns of P is the same as the number of rows of

Q. When this is so, the matrices involved are said to be <u>conformable</u> for multiplication.

Multiplication, in general, is not commutative. That is, the order of multiplication is important. In fact, reversing the order of multiplication will not have meaning unless the matrices P and Q are square.

With P conformable to Q for multiplication, P is said to be <u>postmultiplied</u> by Q, with the product written as PQ. This operation is also described as Q <u>premultiplied</u> by P.

Other General Definitions

Corresponding to the system of equations

$$\begin{aligned} v + w - x + y + z &= 1 \\ - w + x + y - z &= 0 \\ 2w \quad\quad + y + z &= 2 \end{aligned}$$

is the <u>coefficient matrix</u>,

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 \end{pmatrix}$$

which will be designated by A. Its elements are the coefficients of the variables v, w, x, y and z appearing in the system of linear equations.

The matrix

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 2 \end{pmatrix}$$

which contains the coefficients of the variables and the constant terms on the right-hand side of the equations is called the augmented matrix (7, Ch. 1).

The variables can be used to form the 5 x 1 matrix

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}$$

which will be designated by X. Such a matrix is usually referred to as a <u>column matrix</u>.

The rows and columns of a matrix can be interchanged so as to produce another matrix of order n x m. This new matrix is called the <u>transpose of the original matrix</u> and is designated by a superscript T. For example, corresponding to

$$X \Leftrightarrow \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}$$

is the transpose matrix $(v \quad w \quad x \quad y \quad z)$, a <u>row matrix</u>, denoted by X^T.

<u>Illustration 1</u>. <u>Multiplication</u>

The number of columns of the coefficient matrix referred to in the section above is 5. The number of rows of X is 5. Thus matrix A is conformable to X for multiplication, and the product AX can be formed.

Applying the definition of multiplication,

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} [v + w - x + y + z] \\ [- w + x + y - z] \\ [2w \quad + y + z] \end{pmatrix} \Leftrightarrow AX$$

and AX is of order 3 x 1. The reader can see that the product XA cannot be formed since X is a 5 x 1 matrix, and A is a 3 x 5 matrix.

The choice of X above has been specially made to demonstrate the multiplication. However, notice the relation between AX and the left-hand side of the system of equations. If D is the array

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

it is noted that the system of equations is obtained by equating corresponding elements of matrices AX and D. So in compact matrix form that system is written as AX = D.

Inverse Matrix

If A, B are matrices such that

$$AB = I,$$

the matrix A is said to be the inverse of B (and vice versa). The inverse matrix is designated by a superscript -1, such as B^{-1}. Note that an inverse does not necessarily exist for any matrix. In order that a matrix have an inverse, it is necessary that it be a square matrix and that its determinant be non-zero. Sometimes B, as it is, may not have an inverse.

Successive Multiplication

Question: Is $(PQ)R = P(QR)$?

This, in general mathematical language, is asking whether the multiplication of matrices is associative. The answer is yes (7, p. 8).

Following the rules of multiplication, a product $(PQ)R$ can be formed from matrices PQ and R if PQ and R are conformable. The result may be written without parenthesis as PQR. Again $(PQR)A$ can be formed if PQR and A are conformable. Successive multiplication then is possible and can be extended indefinitely.

Illustration 2

$$\text{Let } R = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & -3 & -1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Then,

$$RA \Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & -3 & -1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

Next,

$$Q(RA) \Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

Also,

$$P[Q(RA)] \Leftrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \Leftrightarrow D.$$

Since, due to associativity, the brackets on the left-hand side may be removed, we write PQRA = D.

<u>Partitioning Matrices</u>

It is sometimes convenient to subdivide matrices into rectangular blocks of elements. This is called <u>partitioning the matrix</u>. Examine the matrix D.

$$D \Leftrightarrow \left(\begin{array}{ccc|cc} -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

D may be subdivided in several ways. The partitioning shown has submatrices of order 2 x 3 in the upper left block, 2 x 2 in the upper right block, 1 x 3 in the lower left block, and 1 x 2 in the lower right block; namely,

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (0 \quad 1 \quad 1), \text{and } (0 \quad 0)$$

respectively. Designate these submatrices by W, I, Y, and Z in that order. Then the matrix D is designated in partitioned form as

$$D \Leftrightarrow \left(\begin{array}{c|c} W & I \\ \hline Y & Z \end{array}\right).$$

Use of Partitioning into Submatrices in Multiplication

Let the partitioning of matrices A and X be indicated by

$$A \Leftrightarrow \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$$

$$X \Leftrightarrow \left(\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right).$$

Assuming that the partitioning indicated is such that A_{11}, A_{21} are conformable to X_1, and A_{12}, A_{22} are conformable to X_2 for multiplication, it can be shown (7) that the product AX is

$$AX \Leftrightarrow \left(\begin{array}{c} A_{11}\,X_1 + A_{12}\,X_2 \\ \hline A_{21}\,X_1 + A_{22}\,X_2 \end{array}\right) .$$

Elementary Transformations

Certain Products of Matrices

Suppose

$$M \Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Consider the following multiplications involving M (7, p. 95).

Premultiplications on M:

(a) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4p & 5p & 6p & p \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(c) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1+4g & 2+5g & 3+6g & 4+g \end{pmatrix}$$

Postmultiplications on M:

(d) $$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 4 & 5 & 1 & 6 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

(e) $$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2q & 3 & 0 \\ 4 & 5q & 6 & 1 \\ 1 & 2q & 3 & 4 \end{pmatrix}$$

(f) $$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ h & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+2h & 2 & 3 & 0 \\ 4+5h & 5 & 6 & 1 \\ 1+2h & 2 & 3 & 4 \end{pmatrix}$$

Effects of These Multiplications

The products obtained in cases (a), (b) and (c) are of the form

$$GM = H.$$

Those obtained in cases (d), (e) and (f) are of the form

$$MJ = K.$$

Examine the equations (a) through (c). The effect on M of the premultiplication by G is observed to be a change involving the rows. In (a) the second and third rows of M are interchanged as seen in H. The effect in (b) is that the second row of M is multiplied by a constant p. In (c) g times the second row is added to the third row of M to obtain H.

Now, examine equations (d), (e) and (f). The effect of the postmultiplication of M by J is to change the columns of M. In (d) there is an interchange of the third and fourth columns of M to produce K. In (e), the second column of M is multiplied by a constant q. In (f), h times the second column is added to the first column of M.

Types of Transformations

The Matrix as an Operator

In all cases shown in the preceeding section, the multiplications can be considered to be operations on M producing changes in rows or columns. The operations which are accomplished are three in number (7, p. 90). They are:

1) Cases (a) and (d)

   The interchange of any two parallel lines of a matrix

2) Cases (b) and (e)

   The multiplication of all the elements of any line by the same constant

3) Cases (c) and (f)

   The addition of an arbitrary multiple of any parallel line to another parallel line.

<u>Matrices and Elementary Transformations</u>

The operations listed above are called elementary transformations of a matrix. The set of matrices designated by G and J above are special. They are like identity matrices. In fact, any of them can be obtained from an identity matrix by changing just one element or two. In case (c), for instance, the matrix $G_{(c)}$ multiplied with M is the identity matrix, except for the element in the third row, second column. When g times the second row is added to the third row of M, matrix H is obtained. In this case, observe that g times the second row of the 3 x 3 identity makes the row read $(0 \quad g \quad 0)$. Adding this to the third row, the third row becomes $(0 \quad g \quad 1)$. This is the third row of $G_{(c)}$ of the last section. So $G_{(c)}$ has the same form as H.

What these special matrices or elementary transformations achieve is stated in the theorem which follows.

Theorem

To effect an elementary transformation on any matrix, first perform the intended elementary transformation on an identity matrix of appropriate order, then premultiply the given matrix by the result if the operation is on rows, or postmultiply if it is on columns (7, p. 96).

DIMENSIONAL ANALYSIS. THE ELEMENTARY APPROACH

What is Dimensional Analysis?

Since E. Buckingham published his paper (3) in 1914, a procedure has been introduced into physical science known as dimensional analysis. It is one that allows a set of variables having an undefined relationship to be organized. The arrangement does not destroy the generality of the relationship (18), but makes possible an easier determination of it. [See (2) and (16)]. When the relationship involves three variables, the associated graphs or nomograms may not be complicated. As the number of variables increases, aids such as double alighment charts, Eiffel charts or logographs (17) become useful. So that apart from the immense experimental difficulty in finding the influence of any one variable in the presence of many others, there exists also the problem of pictorial or graphical representation.

Dimensional analysis is suggested to eliminate some of these difficulties. A grouping of the physical variables results from the analysis. The groups are dimensionless with respect to the concepts or dimensions (mass, length, time, etc.) used. Usually, because the dimensionless groups are of a fewer number than the original number of variables, the graphical representation is easier. The dimensional analysis does not give the distinct form of the functional relationship. This form has to be determined experimentally.

Buckingham's Theorem

The theory of dimensional analysis is based on the hypothesis that the solution of the problem is expressible by means of a dimensionally homogeneous equation in terms of specified variables. A theory of the mechanism of the phenomenon being considered is formed, and upon this, the decision is made as to which variables enter the problem.

Definition

An equation is said to be dimensionally homogeneous if the form of the equation does not depend upon the fundamental units of measurement.

Buckingham's Theorem

"If an equation is dimensionally homogeneous, it can be reduced to a relationship among a complete set of dimensionless products." (11, p. 18)

Comments on the theorem

From the statement of the theorem, any dimensionally homogeneous equation,

$$V = f(Q_1, Q_2 \dots Q_n) \quad \text{----------} \quad (1)$$

can be reduced to the form

$$\pi = F(\pi_1, \pi_2 \dots \pi_p) \quad \text{----------} \quad (2)$$

in which π, π_1, $\pi_2 \dots \pi_p$ are dimensionless.

The Greek letter π stands for product.

Each of the products, π, is assumed to be a product of powers of the variables. That is,

$$= Q_1^{k_1} Q_2^{k_2} \dots Q_n^{k_n} V^{k_{n+1}} \quad \text{-----} \quad (3)$$

Generally $p < n$. That is, the number of π's is less than the number of Q's.

By considering dimensional homogeneity with respect to the physical dimensions (mass, length, time, etc.), the products are determined, and the final form,

$$\pi = F(\pi_1, \pi_2 \dots \pi_p)$$

is obtained.

The steps in the proof of the theorem, implying that this reduction is possible, depend on various theorems and lemmas found in the literature [(11, p. 58), (18)].

A Typical Problem

From experience with turbulent flow (11, p. 99), it is observed that the flow in a pipe may depend upon variables such as

u = average velocity at distance y from the pipe wall,

e = roughness height of the wall,

l = length or diameter of the pipe,

ρ = fluid density,

ν = kinematic viscosity,

τ = shear stress at the wall.

It is necessary to relate the above variables by considering the dimensions of mass, length and time.

Procedure

The procedure of obtaining π, π_1, π_2 .... π_p in equation (2) is illustrated in the following.

Consider the product

$$\pi = u^{k_1} y^{k_2} \nu^{k_3} \rho^{k_4} \tau^{k_5} e^{k_6} l^{k_7} \quad \text{---- (4)}$$

where k_i denotes an exponent to be determined such that the π is dimensionless.

The dimensionless ratios $\frac{y}{l}$, $\frac{y}{e}$ can be formed by inspection so that the variables l and e need not be used in the product [See (3)]. The product relation that needs to be considered, in this case, is the following.

$$\pi = u^{k_1} y^{k_2} \nu^{k_3} \rho^{k_4} \tau^{k_5} \quad \text{------- (5)}$$

Method, Using Elementary Approach

Rewriting equation (5) in terms of the dimensions of mass (M), length (L), and time (T) (refer to Appendix I, if necessary) gives

$$(MLT)^0 = (LT^{-1})^{k_1} (L)^{k_2} (L^2T^{-1})^{k_3} (ML^{-3})^{k_4} (ML^{-1}T^{-2})^{k_5} \quad \text{---------- (6)}$$

Equate the indices for each dimension.

For M: $k_4 + k_5 = 0$

For L: $k_1 + k_2 + 2k_3 - 3k_4 + k_5 = 0$ ------- (7)

For T: $-k_1 - k_3 - 2k_5 = 0$

In solving for the indices, k_i, we obtain relationships between them, e.g.,

$$k_2 = -k_3$$
$$k_5 = -(k_1 + k_3)\tfrac{1}{2} \quad \text{-------- (8)}$$
$$\text{and } k_4 = (k_1 + k_3)\tfrac{1}{2}$$

Substitute for k_2, k_5, and k_4 in equation (5)

$$\pi = u^{k_1}\, y^{-k_3}\, \nu^{k_3}\, \rho^{\frac{1}{2}(k_1+k_3)}\, \tau^{-\frac{1}{2}(k_1 + k_3)}$$

Group terms with k_1, k_3 separately; we obtain

$$\pi = \left(u\sqrt{\frac{\rho}{\tau}}\right)^{k_1} \left(\frac{\nu}{y}\sqrt{\frac{\rho}{\tau}}\right)^{k_3} \quad \text{---------- (9)}$$

Because the system of equations, (3) above, is underdetermined, values have to be chosen for k_1 and k_3.

Choosing $k_1 = 1$, and $k_3 = 0$ yields

$$\pi_1 = \left(u\sqrt{\frac{\rho}{\tau}}\right)$$

Choosing $k_1 = 0$, and $k_3 = 0$ gives

$$\pi_2 = \left(\frac{\nu}{y}\sqrt{\frac{\rho}{\tau}}\right)$$

Rewriting equation (9) in the form

$$\pi = F(\pi_1, \pi_2 \cdots \pi_p)$$

we obtain

$$\left(u\sqrt{\frac{\rho}{\tau}}\right) = g\left(\frac{y}{\nu}\sqrt{\frac{\tau}{\rho}}\right) \text{ ---------- } (10)$$

By including the length ratios, the equation takes the form

$$u = \sqrt{\frac{\tau}{\rho}}\, f\left(\frac{y}{\nu}\sqrt{\frac{\tau}{\rho}}, \frac{y}{l}, \frac{y}{e}\right) \text{---------- } (11)$$

<u>Note</u>

It is well to observe that there are different types of variables which might enter into a problem. For purposes of model theory, there will be three types (16).

a) <u>Geometrical Variables</u>: Length, diameter, thickness, chord, span, etc. These variables have a length dimension.

b) <u>Kinematic Variables</u>: Velocity, mass flow, acceleration, angular velocity, revolutions/minute.

c) <u>Dynamic Variables</u>: (1) <u>Fluid properties</u> - density, specific weight, viscosity, elasticity, surface tension. These variables give rise to various forces. (2) <u>Characteristics of performance</u> - pressure, hydraulic head, torque, resistance, stress, shear, lift, drag, etc.

A MATRIX SOLUTION

Dimensional Table

Matrix Equations

Before proceeding further, re-examine equations (7) which are obtained by the elementary approach.

$$
\begin{aligned}
k_4 + k_5 &= 0 \\
k_1 + k_2 + 2k_3 - 3k_4 - k_5 &= 0 \\
-k_1 \qquad - k_3 \qquad\quad -2k_5 &= 0
\end{aligned}
$$

Adopting matrix notation, this set is rewritten as

$$
\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & -3 & -1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{-----} \ (12)
$$

The Table

From any given functional form, e.g., equation (5), a table, called the dimensional table, can be formed. In general, the skeleton of this table consists of the variables as the column headings, and the physical concepts (or dimensions -- M, L, T, etc.) as the row headings.

Any dimension of a particular variable in the equation appears in the column headed by that variable, and will be located on the row (labeled M, L, T, etc.) corresponding to that dimension. For convenience, the undetermined power, k_i, of the variable is placed in the last row.

Table 1, which follows, is self-explanatory.

<u>Check</u>: $\tau^{k_5} = (M^1\ L^{-1}\ T^{-2})^{k_5}$, from equation (6).

In the τ column, Table 1 has 1, -1, -2 which are the dimensions (M, L, T) of τ.

Table 1. <u>Dimensional Table</u>

Variable	u	y	ν	ρ	τ
M	0	0	0	1	1
L	1	1	2	-3	-1
T	-1	0	-1	0	-2
Unknown Powers	k_1	k_2	k_3	k_4	k_5

Note that the array of numbers in the table is the first matrix on the left-hand side of equation (12). In other words, if equation (12) is rewritten in the notation form,

$$AX = 0 \qquad \text{----------} \quad (13)$$

the matrix A is the array appearing in the dimensional table (11, p. 33).

The Matrix Transformation Technique

The Pivot Element

From equations (12) or Table 1, the matrix shown below is obtained.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & -3 & -1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

This matrix is operated upon in the section Successive Multiplication (see page 8) to give

$$\left(\begin{array}{ccc|cc} -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

in accordance with the rules in that section. The partitioning indicated first appeared in the section entitled Partitioning Matrices. In the upper right-hand corner is found an array which is the identity matrix of order 2 x 2. This form is very desirable in computations, and the way to achieve it is discussed next.

Pick any element. Call this the "pivot" element. This element is to be transformed into unity by any or all of the elementary transformations. After this is achieved, all other elements appearing in the same column as the "pivot" element are changed to zero by similar operations.

This procedure is the basis of the matrix transformation technique.

The idea is to change "pivot columns" (i.e., the columns containing the chosen pivot elements) into columns having

$a_{ij} = 1$, for a certain i and j, and

$a_{ij} = 0$, for all other elements in the jth column.

Such columns are similar to columns of an identity matrix. Then a permutation of columns or rows or both will produce an identity matrix of appropriate order as a submatrix. In the above case, an identity matrix of order 2 x 2 is obtained in the upper right-hand corner.

Illustration 3.

Let the array shown in Table 1 be designated as a matrix A.

$$A \Longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & -3 & -1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix}$$

Using a_{14} and a_{25} as the pivot elements, the idea of obtaining an identity matrix, I, is pursued. To effect the elementary operations, premultiply matrix A by such matrices as can accomplish that operation. These operations are depicted in section Successive Multiplication whence

$$(PQR)A = D \qquad \text{----------} \ (14)$$

Partition as in section <u>Partitioning Matrices</u>, the operation is completed.

We have, finally,

$$D = \left(\begin{array}{c|c} W & I \\ \hline Y & Z \end{array}\right)$$

where

$$I \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W \Leftrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

$$Y \Leftrightarrow (0 \quad 1 \quad 1)$$

$$Z \Leftrightarrow (0 \quad 0).$$

<u>Solving the Matrix Equation DX = 0.</u>

<u>A Comparison</u>

From equations (13) and (14),

$$DX = 0 \qquad \text{----------} \ (15)$$

Write

$$D \Leftrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \left(\begin{array}{c} D_1 \\ \hline D_2 \\ \hline D_3 \end{array}\right)$$

where D_1, D_2, D_3 designate the three row matrices.

Hence, computing D_3X, D_1X and D_2X, the following equations result.

$$\begin{aligned} k_2 + k_3 \qquad\qquad &= 0 \\ -\tfrac{1}{2}k_1 - \tfrac{1}{2}k_2 - k_3 + k_4 \qquad &= 0 \qquad \text{-----} \ (16) \\ \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2 + k_3 \qquad + k_5 &= 0 \end{aligned}$$

From these, equations (8) are again obtained, namely,

$$\begin{aligned} k_2 &= -k_3 \\ k_4 &= \tfrac{1}{2}(k_3+k_1) \qquad \text{----------} \ (8) \\ k_5 &= -\tfrac{1}{2}(k_3+k_1) \end{aligned}$$

By way of comparison, refer to equations (7), (8) and (13). Equations (7) and (13) are the same. And above, they lead naturally to the same result, equations (8). But it is agreed that the result is obtained directly from equation (15), whereas it is obscure in the elementary approach which yielded equations (7).

The above convinces the reader that the transformation technique can change a matrix into a more suitable form for computation. Hence it is seen that there is a justification for all these operations.

$\underline{X_1 = EZ}$

The solution to equation (13) is equation (8)

$$\begin{aligned} k_2 &= -k_3 \\ k_5 &= -(k_1+k_3)\tfrac{1}{2} \qquad \text{----------} \ (8) \\ k_4 &= +(k_1+k_3)\tfrac{1}{2} \end{aligned}$$

This is the equation,

$$X_1 = EZ \qquad \text{----------} \; (17)$$

where

$$\begin{pmatrix} k2 \\ k5 \\ k4 \end{pmatrix} \Longleftrightarrow X_1$$

$$\begin{pmatrix} k_1 \\ k_3 \end{pmatrix} \Longleftrightarrow Z$$

and

$$E \Longleftrightarrow \begin{pmatrix} 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Since X_1 and Z consist of elements, k_i, it is possible to write

$$X = \begin{pmatrix} Z \\ \hline X_1 \end{pmatrix}$$

where the column matrix X has elements, k_i. In equation (8), if values are chosen for k_1 and k_3, then k_2, k_4, and k_5 can be determined.

<u>Example</u>

a. Choose for Z, $k_1 = 1$ and $k_3 = 0$.

Then $X_1 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

b. Choose for Z, $k_1 = 0$ and $k_3 = 1$.

$$X_1 = \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

The product corresponding to the first choice is

$$\pi_{(1)} = u\rho^{\frac{1}{2}}\tau^{-\frac{1}{2}} = u\sqrt{\frac{\rho}{\tau}}$$

and corresponding to the second choice is

$$\pi_{(2)} = y^{-1} \nu \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} = \frac{\nu}{y}\sqrt{\frac{\rho}{\tau}}$$

A Goal

The reduction from

$$u = f(y, \nu, \rho, \tau, e, l)$$

to

$$\pi = F(\pi_1, \pi_2, \ldots \pi_p)$$

is completed by writing in the π-values. Thus, equation (11) is obtained again.

$$u\sqrt{\frac{\rho}{\tau}} = f\left(\frac{\nu}{y}\sqrt{\frac{\rho}{\tau}}, \frac{y}{l}, \frac{y}{e}\right) \quad \text{-----------} \quad (11)$$

During the reduction process, the equation

$$AX = 0$$

is obtained. It is the primary purpose of the analysis to make the relationships easier to handle. Therefore, it is desirable to alter the matrix form such that X or its submatrix X_1 can be written down by inspection. This results in a matrix equation,

$$DX = 0,$$

where D is in a suitable form.

Note:

X can be partitioned thus

$$X \Leftrightarrow \left(\frac{Z}{X_1}\right).$$

X_1 is then determined from

$$X_1 = EZ.$$

The features of the above equation will be discussed in the next section.

FEATURES OF AN IMPORTANT MATRIX EQUATION

Deriving $X_1 = -B^{-1}CZ$ from $AX = 0$

Consider the non-homogeneous matrix equation $HY = Q$ in its equivalent form $HY-Q = 0$. By using partitions, write it compactly as

$$(H \mid -Q)\left(\frac{Y}{1}\right) = 0$$

With $A \Leftrightarrow (H \mid -Q)$

and $X \Leftrightarrow \left(\frac{Y}{1}\right)$,

the associated homogeneous equation is

$$AX = 0.$$

So in general it is sufficient to consider $AX = 0$ alone.

Consider A and X (in the equation $AX = 0$) partitioned as follows

$$A = (B \mid C)$$

$$X = \left(\frac{X_1}{Z}\right),$$

where B is of order m x m, and B and X_1 are such that they are conformable for multiplication. C and Z are also conformable. Therefore, the orders of C, X_1 and Z are $m \times (n-m)$, $m \times 1$, and $(n-m) \times 1$, respectively.

Now $AX = 0$

$$(B \mid C)\left(\frac{X_1}{Z}\right) = 0$$

$$BX_1 + CZ = 0$$

$$\therefore BX_1 = -CZ$$

Then if B^{-1} exists,

$$X_1 = -(B^{-1}C)Z \qquad \text{----------} \; (18)$$

This is an important matrix equation. Note that X_1 and Z are parts of the same X for which a solution is sought.

It is seen from equations (17) and (18) that

$$E = (-B^{-1}C).$$

In section 4, A MATRIX SOLUTION, X_1 is solved by arbitrarily specifying the elements in Z. Hence, by the same token, the resulting X in the above derivation is made up as follows.

$$X = \left(\frac{Z \text{ assumed}}{X_1 \text{ calculated}}\right)$$

Consider the matrix B. Its number of rows is fixed, so is its order which is $m \times m$. Consequently, X_1 also is of fixed order, $m \times 1$.

How to Obtain $B^{-1}C$

Attention is now focused on equation (18), namely,

$$X_1 = (-B^{-1}C)Z.$$

The question is asked, "Is there a way of evaluating $B^{-1}C$ separately?" The answer to this is "yes."

Procedure

After having selected a matrix B, reduce it to I by a series of elementary transformations. This involves the successive use of pivot elements. Then as a by-product, the original C matrix, also a part of the A matrix, reduces to $B^{-1}C$, which is what is being sought. The procedure is illustrated below with an example.

Example

The relation

$$V = f(P, Q, R, S, T, U) \quad ------ (19)$$

is given. To reduce it to a relation involving products, a product of the following form is assumed.

$$\pi = P^{k_1}Q^{k_2}R^{k_3}S^{k_4}T^{k_5}U^{k_6}V^{k_7} \quad ----- (20)$$

It is assumed that the dimensional table shown below is obtained from equation (20), in accordance with the section entitled Matrix Transformation Technique.

Table 2. Dimensional Table

Matrices	C				B		
Variables	P	Q	R	S	V	U	T
M	2	-1	3	0	1	-2	0
L	1	0	-1	0	2	1	2
T	0	1	0	3	2	-1	1
Matrices	Z				X_1		
Indices	k_1	k_2	k_3	k_4	k_7	k_6	k_5

Note: It does not matter actually whether A is partitioned as

$$A = (B|C) \quad \text{or} \quad (C|B).$$

It is only required that X_1 and Z be conformable to matrix B and matrix C respectively. This idea is used in the table above.

Transformations

To perform the requisite transformations, suppose M, N, F, G and H are chosen as follows, for row operations on matrix A. (A is the array in Table 2, above).

$$M \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$N \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F \Leftrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$G \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Particular pivot elements are

$$A_{17}, A_{35}, \text{ and } A_{26}.$$

(The suffixes have their usual meanings.)

Let $(MNFGH)A = D$.

Then performing the transformations, we obtain,

$$D \Longleftrightarrow \left(\begin{array}{cccc|ccc} 11 & -9 & 9 & -15 & 1 & 0 & 0 \\ -5 & 4 & -5 & 6 & 0 & 1 & 0 \\ -8 & 7 & 7 & 12 & 0 & 0 & 1 \end{array}\right)$$

Partitioning D as indicated above, we have

$$D = (E^1 | I)$$

where I is of order 3 x 3.

By performing row transformations, I has been obtained under the columns which once were B. Hence the products of the matrices must amount to B^{-1}. Since this is the case (checked), then under the columns constituting C before, we have also premultiplied by B^{-1}. This is now $B^{-1}C$. The submatrix E^1 is then $B^{-1}C$ which we set out to find.

What Z's to Assume

In the section entitled, Solving the Matrix Equation DX = 0, above, a choice of matrix Z made up of elements k_i

which were either unity or zero led to $\pi_{(1)}$, $\pi_{(2)}$, or simply π_1, π_2. To generalize this idea, always assume Z as a "pivot column." Recall that such columns are similar to columns of the identity matrix.

That is, for a 4 x 1 Z matrix,

$$Z^{(1)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad Z^{(2)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad Z^{(3)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{and}$$

$$Z^{(4)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Matrix Z is different in each case. Let a superscript indicate a specific choice. Generally Z will be given as

$$Z^{(M)} = \begin{pmatrix}0\\\hline 1\\\hline 0\end{pmatrix}$$

where 0 is the zero matrix.

$$(M) = 1, 2 \ldots. (n-m)$$

That is, there are (n-m) choices of Z that can be made.

Calculating X_1

Each choice of Z, $Z^{(M)}$, leads to a specific column matrix, $X_1{}^{(M)}$, $X^{(M)}$ corresponding to a $Z^{(M)}$ is obtained from equation (18).

$$X_1{}^{(M)} = (-B^{-1}C)Z^{(M)} \qquad \text{----------} \; (18)$$

Example

From page 35 we obtained matrix $B^{-1}C$, namely,

$$B^{-1}C = \begin{pmatrix} 11 & -9 & 9 & -15 \\ -5 & 4 & -5 & 6 \\ -8 & 7 & -7 & 12 \end{pmatrix}.$$

The following X_1 values are calculated using equation (18).

$$X_1^{(1)} = \begin{pmatrix} -11 \\ 5 \\ 8 \end{pmatrix}, \quad X_1^{(2)} = \begin{pmatrix} 9 \\ -4 \\ -7 \end{pmatrix}, \quad X_1^{(3)} = \begin{pmatrix} -9 \\ 5 \\ 7 \end{pmatrix}$$

and $X_1^{(4)} = \begin{pmatrix} 15 \\ -6 \\ -12 \end{pmatrix}.$

The Features - Summary

Consider equation (18)

$$X_1 = -(B^{-1}C)Z.$$

The part of the right-hand side involving B^{-1} has to be obtained by using elementary transformations. Elements in matrix Z have to be assumed. By completing the multiplications indicated in the equation, X_1 is obtained. Hence, it is seen that to solve the underdetermined matrix equation $AX = 0$ essentially means solving

$$X_1 = -(B^{-1}C)Z.$$

This is an important observation.

DEVELOPMENT OF AN ALGORITHM

Algorithm

From pages 22, 31, and 32, the reduction of

$$V = f(p, q, r, s, t, u) \text{ to}$$

$$\pi = F(\pi_1, \pi_2 \ldots\ldots \pi_p)$$

actually is equivalent to solving for X in the relation $AX = 0$. Consider the partitioning of this equation (see page 31).

$$A = (B \mid C)$$

$$X = \left(\frac{X_1}{Z} \right) .$$

With B conformable to X_1, and C conformable to Z for multiplication, we can rewrite the equation $AX = 0$ as

$$X_1 = -(B^{-1}C)Z \qquad \text{----------} \quad (18)$$

It has been shown that to solve for X, Z has to be assumed and X_1 calculated from it. This then forms the basis of an algorithm.

An algorithm is developed based on the features of equation (18). The method is stated in three steps as follows.

1. By successive operations on rows, transform submatrix B into the identity matrix I. When this is completed, the submatrix which is originally C changes into $B^{-1}C$.

(Submatrices B and C of matrix A are obtained from the dimensional table developed for the particular dimensional analysis problem.)

2. Assume Z such that each $Z^{(M)}$ is a column of the identity matrix of order (n-m) x (n-m). Then obtain $X_1^{(M)}$ from the respective columns of the matrix $-B^{-1}C$. (If the array shows only $B^{-1}C$, obtain X_1 by changing the sign of each element.)

   (M) = 1, 2 .... (n-m). See page 36.

3. Form $X = \left(\frac{Z \text{ assumed}}{X_1 \text{ calculated}}\right).$

   Each $X^{(M)}$ gives a π_M. Hence the reduction is completed.

   (Note that since X is a column, it will be preferable to save space by writing its transpose, X^T. The M columns become M rows).

   Finally, introduce π_M into the relationship.

$$\pi = F(\pi_1, \pi_2 \dots \pi_p)$$

__Form of the Transformation Table__

A transformation table will be set up that shows the steps given in the previous paragraph. The dimensional table will be included in the transformation table. It will be referred to as the array $(C|B)$. I, an identity matrix of order m x m is merely attached to the right-hand side of the $(C|B)$ matrix to give $(C|B|I)$. Also appended on the left-hand side of $(C|B)$ is matrix T. Matrix T is that transformation matrix operating on $(C|B|I)$ at that point in the calculation.

So $(C|B)$ and I are written down first. Then T is filled in to achieve the row operation that the calculator has in mind. As the table is set up, the matrices T, $(C|B)$ and I are arranged in order.

Example

An example follows, using Table 2 as given on page 33.

Notice that the single vertical line indicates a partition. The double vertical lines distinguish the beginning and end of one matrix from another.

STEP 1

As shown below, write the transformation table for matrix A, obtained from Table 2.

<u>Transformation Table or Algorithm</u>

T			C				B			I		
OPERATION			P	Q	R	S	V	U	T	1	2	3
1	0	0	2	-1	3	0	1	-2	0	1	0	0
-2	1	0	1	0	-1	0	2	1	2	0	1	0
-2	0	1	0	1	0	3	2	-1	1	0	0	1
1	0	0	2	-1	3	0	1	-2	0	1	0	0
0	1	-2	-3	2	-7	0	0	5	2	-2	1	0
0	0	1	-4	3	-6	3	0	3	1	-2	0	1
1	2	0	2	-1	3	0	1	-2	0	1	0	0
0	-1	0	5	-4	5	-6	0	-1	0	2	1	-2
0	-3	1	-4	3	-6	3	0	3	1	-2	0	1
			-8	7	-7	12	1	0	0	-3	-2	4
			-5	4	-5	6	0	1	0	-2	-1	2
			11	-9	9	-15	0	0	1	4	3	-5
			$B^{-1}C$				I			B^{-1}		

<u>STEP 2</u>

Now choose the following values for Z.

$$Z^{(1)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad Z^{(2)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad Z^{(3)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},$$

$$Z^{(4)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

STEP 3

Using $X = \left(\frac{Z}{X_1}\right)$ where $X_1 = -(B^{-1}C)Z$, set out the table for X as follows.

The X-Table

$$\begin{array}{c|cccc|c}
 & \multicolumn{4}{c}{X} & \\
 & 1 & 2 & 3 & 4 & \\
\hline
Z & \left\{\begin{array}{c}1\\0\\0\\0\end{array}\right. & \begin{array}{c}0\\1\\0\\0\end{array} & \begin{array}{c}0\\0\\1\\0\end{array} & \begin{array}{c}0\\0\\0\\1\end{array} & \left\{\begin{array}{c}k_1\\k_2\\k_3\\k_4\end{array}\right. \\
X_1 & \left\{\begin{array}{c}8\\5\\-11\end{array}\right. & \begin{array}{c}-7\\-4\\9\end{array} & \begin{array}{c}7\\5\\-9\end{array} & \begin{array}{c}-12\\-6\\15\end{array} & \left\{\begin{array}{c}k_5\\k_6\\k_7\end{array}\right.
\end{array}$$

Matrix of Solutions

In matrix form the solution X is given by

$$X = \left(\frac{Z_1}{X_1}\right)$$

where $X_1 = -B^{-1}CZ$ with rows equal in number to the dimensional matrix A, and Z is assumed to be a pivot column. (Preferably, the result is written as X^T.) Call this the matrix of solutions.

Matrix of Solutions, (i.e., $X^T = (Z^T | X_1{}^T)$)

P	Q	R	S	T	U	V
1	0	0	0	-11	5	8
0	1	0	0	9	-4	-7
0	0	1	0	-9	5	7
0	0	0	1	15	-6	-12

Each row gives a π_M. (See section Calculating X_1)

With i in the order 1, 2, 3, 4 from top row to the bottom,

$$\pi_1 = PT^{-11}U^5V^8$$
$$\pi_2 = QT^9U^{-4}V^{-7}$$
$$\pi_3 = RT^{-9}U^5V^7$$
$$\pi_4 = ST^{15}U^{-6}V^{-12}$$

and it is observed that P, Q, R, S occur only once in their corresponding dimensionless products π_1, π_2, π_3, π_4 (11, p. 36).

FINAL STEP

Introduce π_M into the relationship

$$\pi = F(\pi_1, \pi_2 \ldots\ldots \pi_p)$$

$$PT^{-11}U^5 \ V^8 = F(QT^9U^{-4}V^{-7},\ RT^{-9}U^5V^7,\ ST^{15}U^{-6}V^{-12}).$$

Observation

Since $B^{-1}I = B^{-1}$, the columns of I transform into B^{-1} in the end. They also indicate the products of the various T matrices. That is, in the end, with $T_1 = E$, $T_2 = F, \ldots\ldots, T_k = L$

$$B^{-1} = (EF \ldots\ldots L).$$

To check for errors, determine whether this final matrix, B^{-1}, and matrix B form a product $B^{-1}B$ equal to I.

Check:

$$\begin{pmatrix} -3 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 3 & -5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Modification

After this section, the matrices I and T will not be included. The dimensional table will be used as given, (C B), and the three steps followed. Horizontal lines drawn across the page will indicate the end of each transformation.

The Typical Problem

Try this algorithm on the problem that was done in chapter 3 and again in chapter 4.

	u	v	y	e	τ	
M	0	0	0	1	1	(1)
L	1	2	1	-3	-1	(2)
T	-1	-1	0	0	-2	(3)
	k_1	k_2	k_3	k_4	k_5	

Indicate the rows by (1), (2), (3) in order. Omit the intermediate steps. The array containing I is as follows.

0	1	1	0	0	(2)
$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	(1)
$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	(3)
$B^{-1}C$			I		

Notice the row permutation above. Matrix X^T is given by the following matrix of solutions.

	u	v	y	e	τ
π_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
2	0	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$
	Z^T		$X_1{}^T$		

$$\pi_1 = u\sqrt{\frac{e}{\tau}}, \quad \pi_2 = \frac{v}{y}\sqrt{\frac{e}{\tau}}$$

and $\quad u\sqrt{\frac{\rho}{\tau}} = f\left(\frac{y}{\nu}\sqrt{\frac{\tau}{\rho}},\ \frac{y}{e},\ \frac{y}{L}\right)$

Permutation

A permutation of columns and/or rows results in a different array. Hence, the resulting 's are also different.

Example

The same example as before:

	u	y	ν	ρ	τ		
M	0	0	0	1	1	(1)	
L	1	1	2	-3	-1	(2)	$\Longleftrightarrow$ A
T	-1	0	-1	0	-2	(3)	
	k_1	k_2	k_3	k_4	k_5		

0	0	0	1	1	
1	1	2	0	2	
-1	0	-1	0	2	
$-\frac{1}{2}$	$-\frac{1}{2}$	-1	1	0	
$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	$\Longleftrightarrow$ D
0	1	1	0	0	
$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0	
$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	
0	1	1	0	0	

0	1	1	0	0	(3)
$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0	(1)
$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	(2)

	X	
Z	1	0
	0	1
	0	1
X_1	$\frac{1}{2}$	$-\frac{1}{2}$
	$-\frac{1}{2}$	$\frac{1}{2}$

X^T is the following array

	u	y	v	ρ	τ
π_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
π_2	0	1	1	$-\frac{1}{2}$	$\frac{1}{2}$
	k_1	k_2	k_3	k_4	k_5

$$\pi_1 = u\sqrt{\frac{\rho}{\tau}}$$

$$\pi_2 = yv\sqrt{\tau/\rho}$$

Here π_2 is different from the previous result.

The Algorithm versus Other Methods

There are numerous ways of solving a set of equations (generally called underdetermined) that consist of m equations in n unknowns, and with $m < n$. Among the methods are the use of flow-graphs, and the gradient methods (6).

There are also matrix procedures. In the matrix procedures elementary transformations play a major role. It is noted that once rows and columns of numbers or functions have been arranged in a specified way, simple operations can be performed on them without literally knowing what the numbers or functions stand for. Thus, one can proceed to obtain an identity matrix (as a submatrix of a partitioned matrix) with only the concepts of row operations in mind. Hence the algorithm that is developed from matrix theory is easy to use. This cannot be said of some of the other methods.

List of Symbols for the Next Section

Mechanical

a = acceleration

A = area

F = force

m = mass

K = constant in $F = K \frac{m_1 m_2}{r^2}$

k = conductivity

ρ = mass density

V = volume

τ = shear stress

μ = dynamic coefficient of viscosity

ν = kinematic coefficient of viscosity

Q flow = flow rate, ft.^{3}/ sec., say

$\frac{du}{dy}$ = velocity gradient

I = moment of inertia

E = modulus of elasticity

θ = temperature

$\Delta\theta$ = temperature difference

Q = quantity of heat, thermal energy

h = heat transfer coefficient

U = energy

P = pressure, head, or power

β = coefficient of thermal expansion

C = specific heat

Electrical

c = capacitance

L = inductance

J = current density

V = voltage, electrical potential

Q = charge

m = mass, e.g., mass of electron

l = distance, as between electrodes, say

ϵ = permittivity

μ = permeability

ϕ = flux

P = power

I = current

U = energy, in watts, say.

Symbols of Dimensions (or Concepts)

M_q = mass, as quantity of matter

M_i = mass, as inertia

L_x = length, axial

L_y = length, radial

P = power: thermal, electrical or radiant

ϕ = flux: thermal, electrical or radiant

Q = quantity of heat

Q = electrical charge

I = electric current

V = voltage

T = time

FURTHER CONCEPTS

"Unconventional" Concepts

In the preceding sections, the concepts of mass, length and time are used as fundamental concepts upon which the analysis is based. This procedure amounts to sticking to an old and rather restrictive idea. This is the idea of describing physical things in terms of mechanics only. It is also in order to regard temperature (designated by θ) and electric charge (Q) as fundamental. In fact, the concepts are arbitrary. However, it is noted that these concepts of mass, length and time form the basis of other concepts. Others are derived from them through the definitions and statements of physical laws.

As indicated on page 16, when describing some phenomenon, previous knowledge is used to determine which variables are worth considering. The variables can be chosen. It seems reasonable that the concepts or dimensions of the variables must also be chosen. Such an idea of choice is promoted in this section.

The Length Concept

Consider length in two mutually perpendicular directions: l_x and l_y. Or consider an axial and a tangential direction: l_r and l_t. Though the lengths are in different directions, they are measured in the same units (feet, say).

The Mass Concept

In some relations, mass appears as a quantity of matter. For example, mass appears as such in the heat equation,

$$Q = mC\Delta\theta$$

where Q = quantity of heat

m = mass

C = specific heat

and $\Delta\theta$ = change in temperature.

In others, mass is inertia. For instance, inertia mass is under consideration in the equation of motion,

$$F = ma$$

where F = force

m = mass -- inertia

a = acceleration.

Distinguishing between these two kinds of mass, let M_q and M_i refer to mass as a quantity of matter, or as inertia, respectively. These considerations follow the lines of Moon and Spencer (13) and Huntley (8).

a) It is advisable to consider mass as a quantity of matter when dealing with mass flow (8, p. 125).

b) When dealing with pressure gradient, pressure, dynamic coefficient of viscosity and heat transfer coefficient, consider mass as inertia.

c) When specific heat is involved, use the ratio M_i / M_q. (8)

Some Suggested Combinations of Concepts

Hoping to see various kinds of legitimate and reasonable combinations that will enhance the meaning of dimensional analysis, the following combinations of concepts are presented. This presentation is meant to introduce flexibility.

Table 3. Table Showing Various Concepts

Concepts					
Mechanics	M	L_x	L_y	T	
Heat	M	L	T	θ	Q
Heat	M_q	M_i	L_x	L_y	T
Electricity	M	L	T	Q	
Not using mass					
Heat	L_x	L_y	T	θ	P
Electricity	L	T	I	V	
Electricity	L	T	I	P	
Electricity	L_x	L_y	T	I	P
Light	L_x	L_y	T	ϕ	P

See Appendix

Examples

An Electrical Example

Problem: Determine the nature of the functional relation

$$J = f(V, Q, m, l, \epsilon).$$

Case I: Use the basic concepts of mass (M), length (L), time (T), and electric charge (Q) in forming the dimensional table.

Variables	J	ϵ	V	Q	m	l	
L	-2	-3	2	0	0	1	(1)
M	0	-1	1	0	1	0	(2)
T	-1	2	-2	0	0	0	(3)
Q	1	2	1	1	0	0	(4)
Indices	k_1	k_2	k_3	k_4	k_5	k_6	

A pivot element to be selected is a_{33}. After completing the transformations, permute the rows as shown by the numbers in brackets below.

$-\frac{1}{2}$	-1	1	0	0	0	(3)
3/2	1	0	1	0	0	(4)
$-\frac{1}{2}$	0	0	0	1	0	(2)
-3	-1	0	0	0	1	(1)

The same methods as in the previous section were followed to obtain $B^{-1}C$. This is made up of the first two columns in the array above.

Write X^T, indicating the rows as π_i.

	J	$\in$	V	Q	m	l
π_1	1	0	$-\frac{1}{2}$	$-3/2$	$\frac{1}{2}$	3
π_2	0	1	1	-1	0	1
	k_1	k_2	k_3	k_4	k_5	k_6

$$\pi = c\left(J\,\frac{m^{\frac{1}{2}}l^3}{V^{\frac{1}{2}}Q^{3/2}}\right)^{k_1}\left(\frac{Vl\in}{Q}\right)^{k_2}$$

or in the usual form,

$$J = c\sqrt{\frac{V}{m}}\left(\frac{Q^{3/2}}{l^3}\right)\cdot f\left(\frac{Vl\in}{Q}\right).$$

<u>Case II:</u> Instead of L, M, T and Q, use L_r, L_t, T, I and P. I stands for current and P for electrical power (13, p. 507).

Variables	J	$\in$	V	Q	m	l	
L_r	-1	-1	0	0	0	0	(1)
L_t	-1	0	0	0	-2	1	(2)
T	0	1	0	1	3	0	(3)
I	1	2	-1	1	0	0	(4)
P	0	-1	1	0	1	0	(5)
Indices	k_1	k_2	k_3	k_4	k_5	k_6	

Follow through the transformations, using the rows as numbered above. Finally, rearrange rows to obtain an identity above k_2 k_3 .... k_6; thus,

1	1	0	0	0	0	(1)
3/2	0	1	0	0	0	(4)
½	0	0	1	0	0	(3)
-½	0	0	0	1	0	(5)
-2	0	0	0	0	1	(2)

Write X^T

	J	ϵ	V	Q	m	l
π_1	1	-1	-3/2	-½	½	2
	k_1	k_2	k_3	k_4	k_5	k_6

$$\pi_1 = C_1 J \frac{m^{\frac{1}{2}} l^2}{\epsilon V^{3/2} Q^{\frac{1}{2}}}$$

or

$$J = C \sqrt{\frac{Q}{m}} \frac{\epsilon V^{3/2}}{l^2}$$

<u>Comparison</u>

Substituting $K \frac{V l \epsilon}{Q}$

for $f\left(\frac{V l \epsilon}{Q}\right)$

in the final result in Case I, we have

$$J = C_1 \sqrt{\frac{V}{m}} \cdot \frac{Q^{3/2}}{l^2} \quad K \frac{V l \epsilon}{Q}$$

$$= C \sqrt{\frac{Q}{m}} \frac{V^{3/2} \epsilon}{l^2}$$

This substitution then gives

$$J = C\sqrt{\frac{Q}{m}} \cdot \frac{V^{3/2}\epsilon}{l^2}$$

as in Case II.

Comment

It is worth noting that the evaluation of $f\left(\frac{Vl\epsilon}{Q}\right)$ would require an experiment.

But here, by a judicious choice of concepts, the function $f\left(\frac{Vl\epsilon}{Q}\right)$ comes out of the analysis directly as $K\frac{Vl\epsilon}{Q}$. This is a remarkable feature of the idea of unrestricted concepts. The equation obtained in Case II is often referred to as Child's Equation.

Heat Transfer Example

Consider next the heat transfer coefficient evaluated for a body immersed in a fluid.

$$h = f(L, \Delta\theta, \beta, \rho, \mu, g, c, K)$$

where C is the specific heat.

Case I: We use as basic concepts M, L, T and Θ.

Dimensional Table

Variables	h	g	C	L	ρ	μ	K	
M	1	0	0	0	1	1	1	(1)
L	0	1	2	1	-3	-1	1	(2)
T	-3	-2	-2	0	0	-1	-3	(3)
	-1	0	-1	0	0	0	-1	(4)
Indices	k_1	k_2	k_3	k_4	k_5	k_6	k_7	

Assume from the relation

$$\Delta L = \beta L \Delta\theta$$

that $\beta\Delta\theta$ must be dimensionless. After the matrix transformations, permute the rows and rewrite them below.

-1	-3	0	1	0	0	0	(2)
0	-2	0	0	1	0	0	(1)
0	2	-1	0	0	1	0	(3)
1	0	1	0	0	0	1	(4)

Then X^T is given as below.

	h	g	c	L	ρ	μ	K
π_1	1	0	0	1	0	0	-1
π_2	0	1	0	3	2	-2	0
π_3	0	0	1	0	0	1	-1
	k_1	k_2	k_3	k_4	k_5	k_6	k_7

$$\pi_1 = \frac{hL}{K}$$

$$\pi_2 = \frac{gL^3\rho^2}{\mu^2}$$

$$\pi_3 = \frac{c\mu}{K}$$

and $\pi_4 = \beta\Delta\theta$, as assumed.

The Final Relation

$$\left(\frac{hL}{K}\right) = f\left(\frac{gL^3\rho^2}{\mu^2}, \frac{C\mu}{K}, \beta\Delta\theta\right).$$

Case II: The heat transfer problem is reconsidered with the basic concepts of M, L, T, θ, and Q.

Variables	h	g	β	C	L	$\Delta\theta$	ρ	μ	K	
M	0	0	0	-1	0	0	1	1	0	(1)
L	-2	1	0	0	1	0	-3	-1	-1	(2)
T	-1	-2	0	0	0	0	0	-1	-1	(3)
θ	-1	0	-1	-1	0	1	0	0	-1	(4)
Q	1	0	0	1	0	0	0	0	1	(5)
Indices	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	

After the transformation, re-arrange the rows.

-1	3	0	0	1	0	0	0	0	(2)
0	0	-1	0	0	1	0	0	0	(4)
0	-2	0	0	0	0	1	0	0	(1)
0	2	0	-1	0	0	0	1	0	(3)
1	0	0	1	0	0	0	0	1	(5)

Hence π_1 is given in tabular form as follows:

π_1	1	0	0	0	1	0	0	0	-1
π_2	0	1	0	0	3	0	2	-2	0
π_3	0	0	1	0	0	1	0	0	0
π_4	0	0	0	1	0	0	0	1	-1
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9

$$\pi_1 = \frac{hL}{K}$$

$$\pi_2 = gL^3\rho^2/\mu^2$$

$$\pi_3 = \beta\Delta\theta$$

$$\pi_4 = C\mu/K$$

and $\frac{hL}{K} = f\left(\frac{gL^3\rho^2}{\mu^2}, \beta\Delta\theta, \frac{C\mu}{K}\right).$

Note: The product $\pi_3 = \beta\Delta\theta$ was assumed in Case I. By using all the variables given in Case II, the same results as in Case I were obtained.

<u>Modification</u>

The relation

$$(B|C)\begin{pmatrix} Z \\ X_1 \end{pmatrix} = 0$$

or

$$X_1 = -B^{-1}CZ$$

is very important, as it underlies all the above work. Since B is m x m, Z is consequently m x (n-m).

m = number of rows in matrix A

n = number of columns in matrix A.

A legitimate increase in m would cause (n-m) to decrease. That is, we would have fewer elements in Z to name arbitrarily.

The result in Case II had been obtained by increasing the number of concepts. The dimensional table for the same heat transfer problem will now be modified by reconsidering length. Suppose a case of axial symmetry were considered. The appropriate designation of length would be as follows:

L_x for axial length, and

L_y for radial length.

These lengths essentially refer to the length of the pipe and the radius of the pipe, respectively.

Case III: Use the concepts of M, L_x, L_y, T, θ and Q.

The resulting table follows.

(a)

	h	g	β	C	L	$\Delta\theta$	ρ	μ	K	
M	0	0	0	-1	0	0	1	1	0	(1)
L_x	-2	0	0	0	0	0	-1	-1	-1	(2)
L_y	0	1	0	0	1	0	-2	0	0	(3)
T	-1	-2	0	0	0	0	0	-1	-1	(4)
θ	-1	0	-1	-1	0	1	0	0	-1	(5)
Q	1	0	0	1	0	0	0	0	1	(6)
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	

Put the rows in the order (3), (2), (5), (1), (4) and (6). The resulting products are given below:

$$\pi_1 = \frac{h\mu^2}{gL^2\rho^2K}, \qquad \pi_2 = \beta\Delta\theta, \qquad \pi_3 = \frac{CgL^3\rho^2}{K}.$$

(b) By choosing another B matrix, the final arrangement of rows follows in the order (6), (2), (5), (1), (4) and (3).

The Result

	h	β	g	C	L	$\Delta\theta$	ρ	μ	K
π_1	1	0	0	-3	-1	0	0	-3	-2
π_2	0	1	0	0	0	1	0	0	0
π_3	0	0	1	-3	2	0	2	1	1
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9

The Products

$$\pi_1 = \frac{h}{c^3 L \mu^3 K^2}$$

$$\pi_2 = \beta \Delta\theta$$

$$\pi_3 = \frac{g L^2 \rho^2}{c^3 \mu K}$$

(c) In this case, β and $\Delta\theta$ are omitted .

	h	K	g	C	L	ρ	μ
π_1	1	0	-2	-1	-5	-4	8
π_2	0	1	-1	-1	-3	-2	1

$$\pi_1 = \frac{h \mu^8}{\rho^4 g^2 c L^5}$$

$$\pi_2 = \frac{K \mu}{c g L^3 \rho^2}$$

Also $\pi_3 = \beta \Delta\theta$, (assumed).

Comment:

All theses products are as weird as Grashof's Number, namely

$$G = \frac{\beta \Delta\theta \, g L^3 \rho^2}{\mu^2} .$$

The resemblance (even if slight) of these products to already accepted numbers calls for further experimental investigation.

Forced Convection Problem

In the forced convection problem that follows, the dual role of mass is introduced.

$$h = f(\overline{\rho v}, \Delta\theta, k, c, L)$$

Recognize here that $c = C\frac{M_i}{M_q}$ (8, p. 125).

Case I. Use M_q, M_i, L, T and θ as concepts

	h	$\overline{\rho v}$	$\Delta\theta$	k	c	L	
M_q	0	1	0	0	-1	0	(1)
M_i	1	0	0	1	1	0	(2)
L	0	-2	0	1	2	1	(3)
T	- 3	-1	0	-3	-2	0	(4)
θ	0	0	1	-1	-1	0	(5)
	k_1	k_2	k_3	k_4	k_5	k_6	

Perform the necessary transformations. After permuting rows to read in order (5), (2), (1), (3) and (4), we obtain X^T below.

	h	$\overline{\rho v}$	$\Delta\theta$	k	c	L
π_1	1	0	-1	-1	0	1
π_2	0	1	0	-1	1	1
	k_1	k_2	k_3	k_4	k_5	k_6

Products

$$\pi_1 = \left(\frac{hL}{\Delta\theta K}\right)$$

$$\pi_2 = \left(\frac{\rho\bar{v} L c}{K}\right)$$

Case II: Consider μ as a pertinent variable.

$$h = f(\rho\bar{v}, \Delta\theta, K, c, L, \mu)$$

Concepts: Use M_q, M_1, L, T, and θ as before.

Resulting Products

We will obtain the products underneath.

$$\pi_1 = \left(\frac{hL}{\Delta\theta K}\right)$$

$$\pi_2 = \left(\frac{\rho\bar{v} c L}{K}\right)$$

and $$\pi_3 = \left(\frac{C\mu}{K}\right) .$$

The Typical Problem is Modified

We will return now to the initial problem. Table 1 (see page 23) is modified by using the dimensions L_x, L_y in place of L.

Case I: Here volume is considered to have the dimensions

$$L_y^{\,2} \cdot L_x$$

<u>Modified Table</u>

Variables	u	v	y	ρ	τ	
M	0	0	0	1	1	(1)
L_x	0	-1	0	-1	-1	(2)
L_y	1	3	1	-2	0	(3)
T	-1	-1	0	0	-2	(4)
Indices	k_1	k_2	k_3	k_4	k_5	

By the algorithm we obtain the following array:

0	1	0	0	0	(4)
0	0	1	0	0	(3)
$-\frac{1}{2}$	0	0	1	0	(1)
$\frac{1}{2}$	0	0	0	1	(2)

Hence X^T is given by

	u	v	y	ρ	τ
π_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$

$$\pi_1 = u\sqrt{\frac{\rho}{\tau}}$$

This means

$$u\sqrt{\frac{\rho}{\tau}} = \text{constant.}$$

<u>Case II:</u> Consider the dimensions of volume to be L_x^3.

<u>Resulting Dimensional Table</u>

Variables	u	ν	y	ρ	τ	
M	0	0	0	1	1	(1)
L_x	0	2	0	3	-1	(2)
L_y	1	0	1	0	0	(3)
T	-1	-1	0	0	-2	(4)
	k_1	k_2	k_3	k_4	k_5	

The product obtained from X^T (when the rows follow in order (4), (3), (1), (2)) is

$$\pi = u\frac{\nu}{y}\cdot\frac{\rho}{\tau}$$

That is

$$u\sqrt{\frac{\rho}{\tau}} = K\sqrt{\frac{\tau}{\rho}}\cdot\frac{\nu}{y}$$

<u>Physical Implications</u>:

Both results (Cases I and II) disagree with the result on page 21. However, one or the other of the above results have appeared earlier in the literature when assumptions such as a smooth surface, infinite wall, etc., were used. The question here is whether the above results are not the true ones and whether the earlier result is not a very general but less informative result.

Significance of the Results

Some interesting results have been made possible by a foresight that had not been utilized before. In the above examples many more dimensionless products were found by the inclusion of non-conventional concepts. We did not have these products before. Their significance can only be checked by experiment.

COMPLETING THE ANALYSIS

A Back-Reference

Let us refer back to the problem from which we obtained

$$\pi_1 = PT^{-11}U^5V^8$$
$$\pi_2 = QT^9U^{-4}V^{-7}$$
$$\pi_3 = RT^{-9}U^5V^7$$
$$\pi_4 = ST^{15}U^{-6}V^{-12}$$

on page 43.

Expressing π_3, π_2, π_4 as functions of π_1,

$$\pi_1 = K\pi_2^{\,a}\,\pi_3^{\,b}\,\pi_4^{\,c}$$
$$(PT^{-11}U^5V^8) = K(QT^9U^{-4}V^{-7})^a \cdot (RT^{-9}U^5V^7)^b \cdot (ST^{15}U^{-6}V^{-12})^c$$

The constants K, a, b, c are yet to be determined. We could choose to determine these constants by assuming $\pi_1 = f(\pi_2)$, $\pi_1 = f(\pi_3)$, $\pi_1 = f(\pi_4)$ each separately. That is, we hold the others constant and determine the function in question by a plotting of data.

Thanks to Levenspiel, Weinstein and Li (12), we simply calculate the dimensionless groups π_i for each run and tabulate them. The procedure for evaluating K, a, b, c then is one developed from multiple regression analysis.

The Equations

A matrix equation of the type

$$Q\,Y = P$$

$$\text{or} \quad (Q|P)\begin{pmatrix} Y \\ \hline -1 \end{pmatrix} = 0$$

is obtained from the multiple regression analysis.

If det $Q \neq 0$,

then $Q^{-1}QY = Q^{-1}P$. Therefore, $Y = Q^{-1}P$.

So starting with $(Q\,|\,P)$, the same methods as before will be applied to obtain

$$(I\,|\,Q^{-1}P)$$

and that column matrix, $Q^{-1}P$, would equal Y.

The Nature of Y and (Q | P)

First,

$$Y = \begin{pmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{pmatrix}$$

where $b_0 = \log K$,

K being the constant in the relation

$$\pi_1 = K\,\pi_2^{\,a}\,\pi_3^{\,b}\,\pi_4^{\,c} \qquad \text{----------} \quad (9)$$

That is, $y = b_0 + b_1x_1 + b_2x_2 + b_3x_3$.

$b_1 = a$, $b_2 = b$, $b_3 = b_k = c$.

For N runs, $(Q \mid P)$ is given by

$$(Q \mid P) = \left(\begin{array}{ccccc|c} N & \Sigma x_1 & \Sigma x_2 & \cdots & \Sigma x_k & \Sigma y \\ \Sigma x_1 & \Sigma x_1^2 & \Sigma x_1 x_2 & \cdots & \Sigma x_1 x_k & \Sigma x_1 y \\ \Sigma x_2 & \Sigma x_2 x_1 & \Sigma x_2^2 & \cdots & \Sigma x_2 x_k & \Sigma x_2 y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Sigma x_k & \Sigma x_k x_1 & \Sigma x_k x_2 & \vdots & \Sigma x_k^2 & \Sigma x_k y \end{array}\right)$$

This is a symmetric matrix obtained (in 12) by statistical methods. N = number of runs.

$$x_1 = \log \pi_1$$
$$x_2 = \log \pi_2$$

$$x_k = \log \pi_k$$
$$y = \log \pi$$

and x_1, x_2, x_1^2, x_2^2, etc., are products computed for each run, and $y = b_0 + b_1 x_1 + b_2 x_2 + \ldots\ldots b_k x_k$.

<u>Comments:</u>

The matrix Q is a square matrix. Hence since a solution exists other than the trivial $Y = 0$, all the constants can be found by using the experimental data to compute $(Q \mid P)$.

The data helps to determine Z in the matrix equation

$$X_1 = -(B^{-1}C)Z.$$

Similar matrix procedures applied to the data lead to definite values of the elements of matrix Z.

The fact that Y could be calculated from values of the square matrix Q and P, shows that if we were to obtain an m x n matrix A from the start that had $m = (n-1)$, experimentation would not be necessary to determine the exponent. It is realized though, that experimentation would be necessary to confirm or reject the product.

CONCLUSIONS

The dimensional analysis leads to a system of under-determined equations. In other areas similar equations are encountered. What is done is to develop certain criteria to supplement the equations? Instances of this appear in feedback theory. In dimensional analysis, this is not possible, for the analysis is not meant to give a clear-cut solution. It is true that one encounters a scaling criterion (5), but it must be pointed out here that the scaling criterion does not belong to the category mentioned above.

This difference makes it imperative on the analyst to use other means. Since a system of under-determined equations is arrived at, some parameters, Z must be arbitrary. This statement is amply made in the matrix equation

$$X_1 = -(B^{-1}C)Z$$

where X, the solution sought, is

$$X \Leftrightarrow \left(\frac{Z}{X_1}\right)$$

This matrix equation is arrived at through simple operations on the matrices. For instance, it is to be observed that in the process of obtaining X_1, the parallel lines (i.e., the rows or columns) of the matrix are, sometimes, to be rearranged. This is because some variables that are easily measurable are thought fit to appear in only one product, π_1. The simple nature of all the operations or transformations adopted in this thesis makes the resulting algorithm stand out clearly as one that can be adopted wherever the system of under-determined equations rears its head.

A great attribute of the algorithm developed is that it is adoptable for machine use. Owing to the various number of ways in which the products can be formed, it is concluded that machine computations will greatly help in this work. The programming will follow the lines of routines which are available in most computing centers. [See Appendix]. It merely involves a matrix inversion with careful rounding off of answers. This is one of the many reasons why this method is discussed here instead of other equally useful ones, namely: the relaxation method (15), or any of the gradient methods (6).

By the application of an unrestricted number of concepts, it has been shown that the number of rows of the matrix A can be made almost equal to the number of columns. That is, m could be brought closer to n. It is concluded that there must be some better ways to use all concepts. Regarding the old mechanical concepts of mass, length, and time, it is apparent that we are not being specific in our definitions. It has been demonstrated that we can associate some sort of "direction" to length. The question remains unanswered whether other concepts like energy and power are not equally basic. Lastly it is hoped that by the use of the methods mentioned in this thesis, there will be some further thought on the subject of dimensionless analysis. It is noted that the theory of models is based in part on dimensional analysis. Would this analysis reveal some disturbing phenomenon in the theory of models? We do not know yet.

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APPENDIX

APPENDIX 1.

Dimensions of Some Physical Quantities

RELATION, ETC.	DIMENSIONS
Mechanics and Materials	
$F = ma$	$[F] = [MLT^{-2}]$; $[M] = [FL^{-1}T^{2}]$
$F = \frac{km_1m_2}{r^2} = ma$	$[F] = [L^4T^{-4}]$; $[M] = [L^3T^{-2}]$
$\rho = \frac{m}{v}$ (density)	$[\rho] = [ML^{-3}]$
$\tau = \frac{F}{A}$ (shear)	$[\tau] = [MLT^{-2}/L^2] = [ML^{-1}T^{-2}]$
$P = \frac{F}{A}$	$[P] = [F/L^2]$
$E = \frac{\text{stress}}{\text{strain}}$	$[E] = [F/L^2]$
Mass Moment of Inertia	$[I_{mass}] = [ML^2]$
Moment of Inertia	$[I_{area}] = [L^4]$
Fluid Mechanics	
Mass Flow	$[Q_{flow}] = [L^3T^{-1}]$
$\mu = \tau/\frac{du}{dy}$	$[\mu] = [FL^{-2}TLL^{-1}] = [ML^{-1}T^{-1}]$
$\nu = \mu/\rho$	$[\nu] = [ML^{-1}T^{-1}M^{-1}L^{3}] = [L^2T^{-1}]$
Momentum	$[mV] = [FT] = [MLT^{-1}]$
Energy	$[U] = [FL] = [ML^2T^{-2}] = [TIV]$

RELATION, ETC.	DIMENSIONS
Fluid Mechanics (cont'd)	
Power	$[P] = [UT^{-1}] = [ML^2T^{-3}] = [IV]$
Heat Flow	
Quantity of Heat	$[Q] = [FL] = [ML^2T^{-2}]$
Heat Equation	$[Q] = [mc\,\Delta\theta]$
Specific Heat	$[C] = [QM^{-1}\theta^{-1}] = [L^2T^{-2}\theta^{-1}]$
Thermal Capacity	$[mC] = [Q\theta^{-1}]$
Thermal Capacity/Unit Volume	$[mCL^{-3}] = [Q\theta^{-1}L^{-3}]$
Coefficient of Thermal Expansion, $\Delta L = \beta L \Delta\theta$	$[\beta] = [\theta^{-1}]$
$Q/T = KA\,\Delta\theta/L$	$[k] = [(Q)(LT\theta)^{-1}] = [MLT^{-3}\theta^{-1}]$
$Q/T = hA\,\Delta\theta$	$[h] = [(Q)(TL^2\theta)^{-1}] = [MT^{-3}\theta^{-1}]$
Electricity	Note: Q = charge; V = voltage
$I = dQ/dT$	$[I] = [QT^{-1}]$
$L = V/\frac{dI}{dt}$	$[L] = [VQ^{-1}T^{-2}]$
$J = I/A$	$[J] = [IA^{-1}] = [QT^{-1}L^{-2}]$
$E = F/Q$ (field intensity)	$[E] = [MLT^{-2}Q^{-1}]$
$V = -E$ (potential)	$[V] = [E] = [MLT^{-2}Q^{-1}]$
$C = dQ/dV$	$[c] = [(Q)(MLT^{-2}Q^{-1})^{-1}] = [M^{-1}L^{-1}T^2Q^2]$
$R = V/I$	$[R] = [(V)(QT^{-1})^{-1}] = [MLT^{-1}Q^{-2}]$

RELATION, ETC.	DIMENSIONS
Electricity (cont'd)	
Magnetic Field Intensity	$[H] = [IL^{-1}]$
$F = \frac{1}{\epsilon} \frac{q_1 q_2}{4\pi r^2}$	$[\epsilon] = [M^{-1}L^{-3}T^2Q^2] = [L^{-1}TIV^{-1}]$
$a = \frac{1}{\sqrt{\epsilon\mu}}$	$[\mu] = [a^{-2}\epsilon^{-1}] = [(LT^{-1})^{-2}\epsilon^{-1}] =$ $[MLQ^{-2}]$
$B = \mu H$ (induction)	$[B] = [(mLQ^{-2})(L^{-1}T^{-1}Q)] =$ $[mT^{-1}Q^{-1}]$
$\phi =$ BA, flux	$[\phi] = [(MT^{-1}Q^{-1})L^2] = [ML^2T^{-1}Q^{-1}]$

APPENDIX 2.

TABLE OF CONCEPTS

Dimensions	L_x	L_y	T	Θ	P
Energy	0	0	1	0	1
Entropy	0	0	1	-1	1
K	1	-2	0	-1	1
C_p	2	0	-2	-1	0
ELECTRICAL	L_x	L_y	T	I	P
Field Strength	0	-1	0	-1	1
V	0	0	0	-1	1
D	-1	-1	1	1	0
ϵ	1	0	1	2	1
Capacitance	0	0	1	2	-1
Inductance	0	0	1	-2	1
Flux	0	0	1	-1	1
	L_x	L_y	T	M	Q
Field Strength	1	0	-2	1	-1
Flux	2	0	-1	1	-1
LIGHT	L_x	L_y	T	ϕ	P
Energy, E	0	0	1	0	1
ϕ	0	0	0	1	0
P	0	0	0	0	1

	L_x	L_y	T	m
Energy, P	0	2	-3	1
ϕ	0	2	-2	1
MECHANICAL	L_x	L_y	T	
g	0	1	-2	
ν	2	0	-1	

APPENDIX 3

Machine Computation

$$AX = 0$$

$$(B\ C)\begin{pmatrix} X_1 \\ \hline Z \end{pmatrix} = 0$$

$$BX_1 = -CZ$$

If A is m x n with $m < n$, make B an m x m.

Then C is an m x n - m

X_1 is m x 1

Z is (n-m) x 1.

The machine procedure is good for square matrices.

Example: Heat Transfer Problem

	h	g	c	L	ρ	μ	K	
m	1	0	0	0	1	1	1	(1)
L	0	1	2	1	-3	-1	1	(2)
T	-3	-2	-2	0	0	-1	-3	(3)
	-1	0	-1	0	0	0	-1	(4)

C			B				
0	1	2	1	-3	-1	1	(2)
1	0	0	0	1	1	1	(1)
-3	-2	-2	0	0	-1	-3	(3)
-1	0	-1	0	0	0	-1	(4)

It will be necessary to use C as a square matrix.

To make C square, add a column of zeros.

$$C = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

But we have a product $-CZ = D$, say. Therefore Z becomes $(\overline{n-m} + 1) \times 1$

$$\text{i.e.,} \quad Z = \begin{pmatrix} 0 \\ \hline Z \end{pmatrix}.$$

<u>Example</u>:

$$Z^1{}_1 = \begin{pmatrix} 0 \\ \hline 1 \\ 0 \\ 0 \end{pmatrix} \quad Z^1{}_2 = \begin{pmatrix} 0 \\ \hline 0 \\ 1 \\ 0 \end{pmatrix} \quad Z^1{}_3 = \begin{pmatrix} 0 \\ \hline 0 \\ 0 \\ 1 \end{pmatrix}$$

<u>Flow Chart for the Program So Far</u>

We need to input Matrix B

" Matrix C

" Vector X_1

Compute $X_1 = -B^{-1}CZ = B^{-1}D$

Output X_1

<u>ALWAC III E</u>:

Present Subroutines are

Column or Vector Input = 30

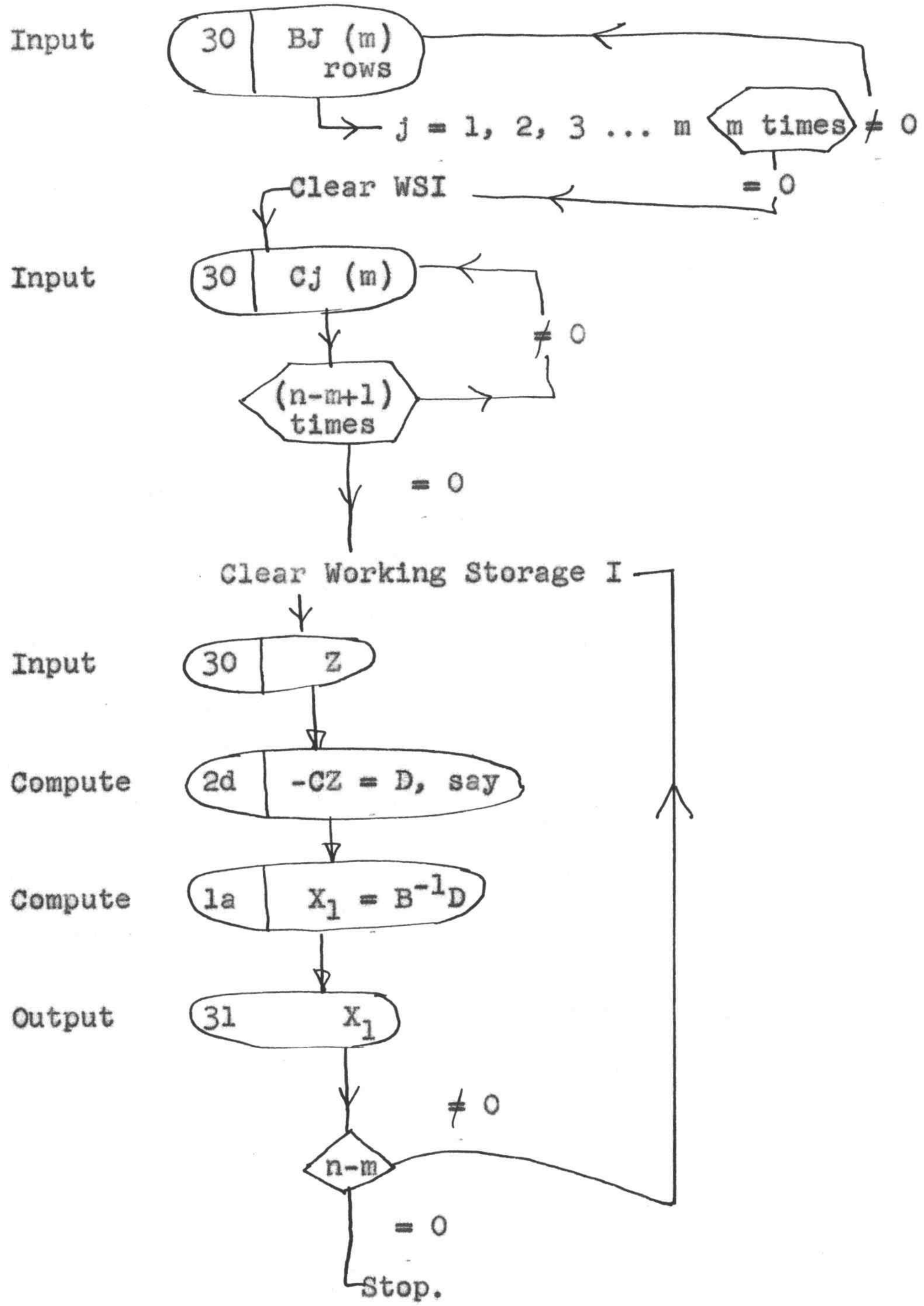
Vector Product = 2d, such as $-CZ = D$

" " = 1a, such as $B^{-1}D = X_1$

Vector Output = 3.1 X_1 out.

Flow Chart No. 1

Clear Working Storage I.



APPENDIX 4.

Some Examples

Example 1: (2, p. 92)

$$P = f(D, V, N, g, \rho, \nu)$$

Dimensional Table

	P	N	g	ν	D	V	ρ
L	1	0	1	2	1	1	-3
T	-2	-1	-2	-1	0	-1	0
M	1	0	0	0	0	0	1
	k_1	k_2	k_3	k_4	k_5	k_6	k_7

Matrix of Solutions

	P	N	g	ν	D	V	ρ
π_1	1	0	0	0	-2	-2	-1
π_2	0	1	0	0	1	-1	0
π_3	0	0	1	0	1	-2	0
π_4	0	0	0	1	-1	-1	0

$$\pi_1 = \left(\frac{P}{D^2V^2\rho}\right) \qquad \pi_2 = \left(\frac{ND}{V}\right)$$

$$\pi_3 = \left(\frac{gD}{\nu^2}\right) \qquad \pi_4 = \left(\frac{\nu}{DV}\right)$$

Example 2: (11, p. 43)

$$f = (P, M, L, D, C, \mu, N)$$

where c = clearance

f = friction coefficient

m = moment

Dimensional Table

	N	M	L	C	D	μ	P
L	0	2	1	1	1	-1	1
M	0	1	0	0	0	1	1
T	-1	-2	0	0	0	-1	-2
	k_1	k_2	k_3	k_4	k_5	k_6	k_7

Matrix of Solutions

	N	M	L	C	D	μ	P
π_1	1	0	0	0	0	1	-1
π_2	0	1	0	0	3	0	-1
π_3	0	0	1	0	-1	0	0
π_4	0	0	0	1	-1	0	0

$$\pi_1 = \left(\frac{N\mu}{P}\right) \qquad \pi_2 = \left(\frac{M}{PD^3}\right)$$

$$\pi_3 = \left(\frac{L}{D}\right) \qquad \pi_4 = \left(\frac{C}{D}\right)$$

Example 3:

$$u = f(\nu, y, \rho, \tau)$$

	u	ν	y	ρ	τ	
M_i	0	0	0	1	0	(1)
M_q	0	0	0	0	1	(2)
L	1	2	1	-3	-1	(3)
T	-1	-1	0	0	-2	(4)
	k_1	k_2	k_3	k_4	k_5	
	1	1	0	0	0	(4)
	-1	0	1	0	0	(3)
	0	0	0	1	0	(1)
	0	0	0	0	1	(2)

Example 4:

$$h = f(\rho, \nu, \mu, \Delta\theta, L, K, c)$$

	h	ρ	ν	μ	$\Delta\theta$	K	c	L
θ	0	0	0	0	1	-1	-1	0
L	0	-3	1	-1	0	1	0	1
M	1	1	0	1	0	1	0	0
T	-3	0	-1	-1	0	-3	-3	0
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
π_1	1	0	0	0	-1	-1	0	1
π_2	0	1	0	0	0	-1	1	4
π_3	0	0	1	0	-1/3	0	-1/3	-1
π_4	0	0	0	1	-1/3	-1	2/3	2

$$\pi_1 = \left(\frac{hL}{\Delta\theta K}\right) \qquad \pi_2 = \left(\frac{\rho c L^4}{K}\right)$$

$$\pi_3 = \left(\frac{\nu}{\Delta\theta^{1/3} L c^{1/3}}\right) \qquad \pi_4 = \left(\frac{\mu L^2 c^{2/3}}{\Delta\theta^{1/3} K}\right)$$

Example 5: Boussinesq's Problem

$$h = f(\rho\bar{v}, \Delta\theta, K, C, L)$$

	h	$\rho\bar{v}$	$\Delta\theta$	K	C	L	
M	1	1	0	1	0	0	(1)
L	0	-2	0	1	2	1	(2)
T	-3	-1	0	-3	-2	0	(3)
θ	0	0	1	-1	-1	0	(4)
	k_1	k_2	k_3	k_4	k_5	k_6	
	1	0	1	0	0	0	(4)
	1	1	0	1	0	0	(1)
	0	-1	0	0	1	0	(3)
	-1	-1	0	0	0	1	(2)

Write X^T as below

	h	$\rho\bar{v}$	$\Delta\theta$	K	C	L
π_1	1	0	-1	-1	0	1
π_2	0	1	0	-1	1	1

$$\pi_1 = \left(\frac{hL}{\Delta\theta K}\right) \qquad \pi_2 = \left(\frac{\rho\bar{v} CL}{K}\right)$$