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Every inventory system requires answers to two basic questions: when to order and how much to order.

In this paper we have described and compared four alternate methods for obtaining the answers for these two questions. Four different types of demand distribution during the lead time are considered: normal, exponential, triangular and uniform. Two types of stockout penalty are assumed; one when the stockout penalty is proportional to the number of occasions out of stock and the other when the stockout penalty is proportional to the average annual number of units out of stock. The necessary conditions for optimality are developed for these eight cases. These provide one method of finding the optimal order quantity, the optimal reorder point and the minimum cost. The optimal values are also compared using a direct search technique, a graphical method and a numerical approximation approach.

An Evaluation of Alternative Approaches for the Computation of Order Quantities and Reorder Points

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AN EVALUATION OF ALTERNATIVE APPROACHES FOR THE COMPUTATION OF ORDER QUANTITIES AND REORDER POINTS

I. INTRODUCTION AND SUMMARY

1.1 Transactions Recording <Q, r> Inventory Models

An important requirement for the effective control of any inventory system is that we know its inventory levels on-hand and on-order stock at any given time. There are two ways that these levels are commonly obtained. They are known as the "Transaction Recording" and the "Periodic Review" methods. The former determines the state of the system after each transaction, i.e., demand, receipt of items, order placed, etc., while the latter determines the state of the system only at discrete points in time. The model which we are going to study is commonly denoted by the <Q, r> model, which means to place an order for a quantity Q of stock when the inventory level reaches the reorder point r. This model is applicable to single item inventory control for which transaction recording is used to update the system.

The <Q, r> inventory model requires estimates of different types of costs which are as follows:

Items variable cost: This is the unit price of the stock item which may or may not depend on the quantity purchased.

Reordering cost: This may include transportation cost, crating cost, uncrating cost, inspection cost, administrative cost etc. In the case of manufacturing it is the set up cost of the machines.

Inventory carrying cost: This might include insurance, cost of floor-space, capital cost, overhead cost, risk of obsolescence etc. For manufacturing it may include a warehouse cost.

Stockout costs: These costs are difficult to measure but they must be taken into account when an inventory system is analyzed. Two types of stockout penalties are considered for our model. One is used when the stockout penalty is assumed to be proportional to the number of occasions out of stock and the other when the stockout penalty is assumed proportional to the average annual number of units out of stock.

For the costs included in our model, the assumptions of D. Herron (1967) are employed which are as follows.

(a) Out-of-stock items may be back ordered and delivered when available.

(b) The variable costs of re-ordering, carrying inventory and of stockouts are known quantities.

(c) Periods of stockout are of short duration, so that the average inventory on hand can be approximated as the sum of the safety stock plus one-half the order quantity.

(d) A stockout penalty should be imposed for either actual or impending stockouts. The inventory should be controlled so as to balance the stockout avoidance costs against the expected cost if an actual stockout occurs.

The assumptions made regarding demand are as follows:

(a) The mean rate of demand over a period is constant.

(b) The units are demanded one at a time, not in a batch.

(c) The lead time may be constant or variable, but the probability distribution of demand during the lead time is known.

(d) Successive lead times are independent of each other.

There are three methods commonly used to determine the operating policies for any inventory system; Simulation, the Heuristic-Intuitive technique, and an Analytical approach.

In the simulation method we begin with a model of the system and a complete set of (possibly hypothetical) rules for operating the system. Then the operation of the system is simulated as if the operating rules were in effect. This technique is useful in comparing a set of operating rules with the procedure currently used to operate the real-world system. This method may be used as a complement to (or alternative to) the analytical approach or the heuristic-intuitive analysis.

The heuristic-intuitive technique can provide the inputs to analytical or simulation approach. This technique is based upon the formulator's experience with the intuitive feelings concerning the system under consideration.

Our approach is the analytical method which consists of constructing a mathematical model of the system to be studied. This ususally consists of the following two procedures. The first procedure is to determine a specific (set of) operating rule(s) which will optimize the model's operation. The second procedure (and the one studied here) is to assume a specific (set of) operating rule(s) that involve one or more parameters and to optimize with respect to these parameters. With the help of the high speed digital computers, fairly complex optimizations can be solved.

1.2 Statement of the Problem

The purpose of this thesis is to evaluate the various alternative methods for computing the optimal values of the parameters of the $\langle Q, r \rangle$ model for a given type of the demand distribution during the lead time and the stockout penalty. One method to find the optimal values for Q and r is an analytical determination which is discussed in Chapter II. For a given type of demand distribution and stockout penalty if it is possible to find an expression for the optimal value of Q which is independent of r, the optimal values for Q and r can be calculated easily. But when it is either not possible or prohibitively expensive, it is desirable to consider other solution.

methods. The search techniques are considered in Chapter III to find the values for Q and r which minimize the total costs. To save the computer time required and the difficulties created by the search techniques, the graphical and numerical approximation methods are used for computing the optimal values of the parameters of the <Q, r> model. These methods are discussed in Chapters IV and V respectively. Chapter VI summarizes the conclusions and recommendations.

1.3 Review of Literature and Area Covered by This Thesis

Several authors have treated various combinations of demand distribution and stockout penalty using an analytical, graphical or approximation approach. Their contributions are summarized, together with ours, in Table 1. In this thesis we have also developed a FORTRAN IV program for the direct search technique and solved a few examples for normal and exponential distribution cases using the direct search approach. An attempt is made to apply "gradient" or "steepest ascent" methods and the difficulties are pointed out in using these "semi-direct" methods for our models. Expressions are developed to show the convexity of the total cost functions for eight cases, i.e., four different types of demand distribution and two types of stockout penalty. For a graphical method of solution expressions are developed to find dimensionless ratios for exponential, triangular and uniform demand distributions taking both types of stockout penalty into consideration. This approach is demonstrated by constructing the graphs for the exponential case. For normal and exponential demand distributions a few examples are solved and a comparison is made in Chapter V for the optimal values of the order quantity Q, reorder point r and the total cost K by using the various methods. Conclusions and recommendations based on results are presented in Chapter VI.

Table 1. Present state of specific lead time demand distribution and stockout penalty.

Type of Distribution	Type of Stockout Penalty	Analytical Method	Direct Search Technique	Graphical Approach	Approximation Method
Normal	Type 1	Herron (1967)	Puranmalka (1971)	Herron (1967)	Herron (1967)
	Type 2	Herron (1967)	"	"	"
Exponential	Type 1	Puranmalka (1971)	"	Puranmalka (1971)	Hunt (1965)
	Type 2	Buchan and Koenigsberg (1963)	"	"	"
Triangular	Type 1	Puranmalka (1971)	"	"	"
	Type 2	"	"	"	______
Uniform	Type 1	"	"	"	______
	Type 2	"	"	"	Hunt (1965)

II. ANALYTIC DETERMINATION OF Q AND r

In this chapter four types of lead-time-demand distributions will be considered: normal, exponential, triangular and uniform. Each distribution is considered for both types of stockout penalty. The function for total average annual cost is developed for each type of distribution and each type of stockout penalty. Then for each cost function necessary optimality conditions are used to find the optimal values of Q and r, which will minimize the total cost.

2.1 Explicit Development of Two Cost Expressions and the Necessary Conditions

When the stockout penalty is proportional to the average number of stockout occasions, the total average annual cost K is given by the following expression:

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r - \mu) + \frac{V\lambda}{Q}\int_r^\infty h(x)dx$$

Similarly for the stockout penalty proportional to the average annual number of units out of stock, the total cost K is given by the following expression:

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r - \mu) + \frac{W\lambda}{Q}\int_r^\infty (x-r)h(x)dx$$

where

K = Total annual average cost (dollars)

A = Re-ordering cost (dollars)

λ = Mean rate of demand during the year (units/year)

Q = Quantity of items for which order is placed at a time (units)

I = Rate of inventory carrying charge, per dollar, per year

C = Price of unit item (dollars/unit)

t = Safety stock expressed as number of standard deviations

x = Actual demand during lead time

r = Reorder point (units)

μ = Mean demand during lead time (units)

σ = Standard deviation for forecasted demand compared to actual demand during replenishment lead time, often called the forecast error.

V = Stockout penalty per replenishment occasion at which stock-out occurs (dollars/replenishment occasion)

W = Stockout penalty per unit out of stock (dollars/unit out of stock)

S = Safety stock, defined as the expected stock on hand at the time of arrival of a procurement. In terms of reorder point, $S = r - \mu = t\sigma$.

$h(x)$ = Demand distribution during lead time

Q^* = Optimal quantity to be ordered for minimum cost

r^* = Optimal reorder point

t^* = Optimal "t"

$\hat{Q}^*$ = Approximated optimal quantity

$\hat{r}^*$ = Approximated optimal reorder point

$\hat{t}^*$ = Approximated optimal "t".

The necessary conditions to find the optimal values for Q and t (or r) are that the first derivatives of cost function with respect to Q and t (or r) should be equal to zero, i.e.,

$$\frac{\partial K}{\partial Q} = \frac{\partial K}{\partial t} = 0 \quad \text{or} \quad \frac{\partial K}{\partial Q} = \frac{\partial K}{\partial r} = 0.$$

2.2 Analytic Solution of the Necessary Conditions for Optimality

Here we are developing the cost functions by using the necessary conditions to find the optimal values for Q and t (or r) for four different types of demand distribution and two different types of stock-out penalty. D. Herron (1967) has given cost expressions and the necessary conditions for optimality for the normal distribution case.

2.2.1 Lead Time Demand Distribution--Normal

When the stockout penalty is proportional to the number of occasions out of stock, the average annual cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda}{Q}\int_r^\infty \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2} dx\,.$$

Writing this in terms of t and u, where $t = \frac{r-\mu}{\sigma}$ and $u = \frac{x-\mu}{\sigma}$, we get

$$K = \frac{A\lambda}{Q} + IC(0.5Q+t\sigma) + \frac{V\lambda}{Q}\int_t^\infty \frac{1}{\sqrt{2\pi}}\, e^{-0.5u^2} du,$$

or

$$K = \frac{A\lambda}{Q} + IC(0.5Q+t\sigma) + \frac{V\lambda}{Q}\,\Phi\,, \tag{1}$$

where

$$\Phi = \int_t^\infty \frac{1}{\sqrt{2\pi}}\, e^{-0.5u^2} du.$$

The necessary conditions for optimality are given by

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + 0.5IC - \frac{V\lambda}{Q^2}\,\Phi\,, \tag{2}$$

and

$$\frac{\partial K}{\partial t} = 0 = IC\sigma - \frac{V\phi\lambda}{Q}\,, \tag{3}$$

where

$$\phi = \frac{-d\Phi}{dt} = \frac{1}{\sqrt{2\pi}}\, e^{-0.5t^2}\,.$$

Expressions (2) and (3) may be solved for Q^* and t^* by an iterative procedure. And then from Equation (1) we can get the minimum cost K^*.

When the stockout penalty is proportional to the average annual number of units out of stock, the average total cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{W\lambda}{Q}\int_r^\infty (x-r)\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Writing this in terms of t and u, we get

$$K = \frac{A\lambda}{Q} + IC(0.5Q+t\sigma) + \frac{W\lambda}{Q}\int_t^\infty (u-t)\frac{\sigma}{\sqrt{2\pi}}\, e^{-0.5u^2} du,$$

or

$$K = \frac{A\lambda}{Q} + IC(0.5Q+t\sigma) + \frac{W\lambda}{Q}\sigma(\phi-t\Phi). \qquad (4)$$

The necessary conditions for optimality are as follows:

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + 0.5IC - \frac{W\lambda\sigma}{Q^2}(\phi-t\Phi) \qquad (5)$$

$$\frac{\partial K}{\partial t} = 0 = IC\sigma - \frac{W\lambda\sigma\,\Phi}{Q} \qquad (6)$$

In the same way as described earlier, the iterative solution of Equations (5) and (6) will give the optimal Q^* and t^*. Putting these values in Equation (4) we get the minimum cost K^*.

2.2.2 Lead Time Demand Distribution -- Exponential

For the case where the stockout penalty is proportional to the number of occasions out of stock, the total average cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda}{Q}\int_r^\infty \frac{1}{\mu} e^{-\frac{x}{\mu}} dx,$$

or

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda}{Q} e^{-\frac{r}{\mu}}.$$

For the exponential case $\mu = \sigma$, so we get

$$t = \frac{r-\mu}{\mu} = \frac{r}{\mu} - 1$$

or

$$\frac{r}{\mu} = 1 + t.$$

In terms of t the cost function is

$$K = \frac{A\lambda}{Q} + IC(0.5Q+t\sigma) + \frac{V\lambda}{Q} e^{-(1+t)} . \tag{7}$$

The necessary conditions for optimality are given by the following expressions.

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + IC(0.5) - \frac{V\lambda}{Q^2} e^{-(1+t)} \tag{8}$$

$$\frac{\partial K}{\partial t} = 0 = IC\sigma - \frac{V\lambda}{Q} e^{-(1+t)} \tag{9}$$

From Equations (8) and (9) we get

$$Q* = \mu + \sqrt{\mu^2 + \frac{2A\lambda}{IC}} , \tag{10}$$

and from Equation (9) we get

$$t* = [\ln \frac{V\lambda}{IC\sigma Q*} - 1.0] . \tag{11}$$

As Equation (10) is independent of t, $Q*$ can be directly calculated from Equation (10) and then $t*$ can be calculated from Equation (11). Putting these values of $Q*$ and $t*$ in Equation (7) we get the minimum cost $K*$. Therefore in this case there is no need of an iterative procedure to determine the optimal values.

When the stockout penalty is proportional to the average annual number of units out of stock, the average total cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{W\lambda}{Q} \int_r^\infty (x-r) \frac{1}{\mu} e^{-\frac{x}{\mu}} dx.$$

Proceeding in the same way as above, we get

$$K = \frac{A\lambda}{Q} + IC(0.5Q+t\sigma) + \frac{W\lambda}{Q} \sigma e^{-(1+t)}. \tag{12}$$

And the necessary conditions for optimality are:

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + IC(0.5) - \frac{W\lambda}{Q^2}\sigma e^{-(1+t)}, \tag{13}$$

and

$$\frac{\partial K}{\partial t} = 0 = IC\sigma - \frac{W\sigma\lambda}{Q} e^{-(1+t)}. \tag{14}$$

From Equations (13) and (14) we get

$$Q^* = \mu + \sqrt{\mu^2 + \frac{2A\lambda}{IC}} , \tag{15}$$

and from Equation (14) we get

$$t^* = [\ln \frac{W\lambda}{ICQ^*} - 1.0]. \tag{16}$$

In this case also the Equation (15) for Q^* is independent of t. Therefore without using an iterative procedure the optimal values may be calculated as described earlier.

2.2.3 Lead Time Demand Distribution -- Triangular

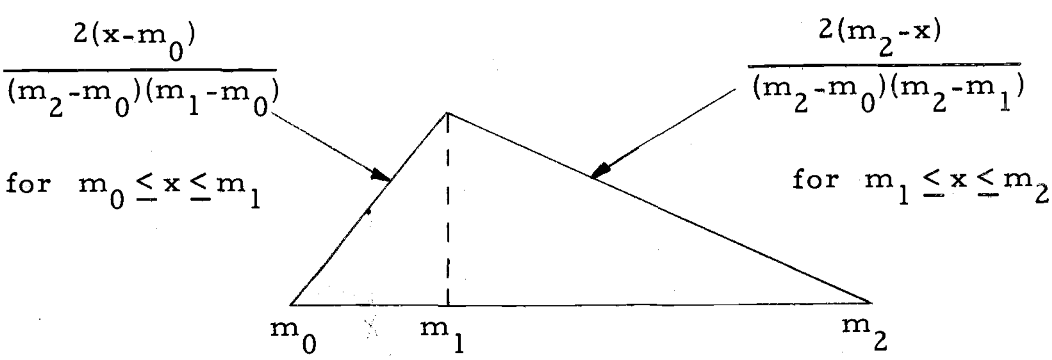

Figure 1. Triangular demand distribution.

mean $\mu = \int_0^{\infty} xh(x)dx$

$$= \int_{m_0}^{m_1} x \frac{2}{m_2-m_0} \frac{x-m_0}{m_1-m_0} dx + \int_{m_1}^{m_2} x \frac{2}{m_2-m_0} \frac{m_2-x}{m_2-m_1} dx$$

$$= \frac{1}{3(m_2-m_0)} \left[\frac{2m_1^3-3m_0m_1^2+m_0^3}{m_1-m_0} + \frac{m_2^3-3m_2m_1^2+2m_1^3}{m_2-m_1} \right]$$

For the case where the stockout penalty is proportional to the number of occasions out of stock, the average annual cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda}{Q} \int_r^{m_2} \frac{2}{m_2-m_0} \frac{m_2-x}{m_2-m_1} dx,$$

where $m_1 \leq r \leq m_2$, or

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda}{Q} \frac{(m_2-r)^2}{(m_2-m_0)(m_2-m_1)}. \tag{17}$$

The necessary conditions for optimality are given by the following equations:

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + 0.5\,IC - \frac{V\lambda}{Q^2} \frac{(m_2-r)^2}{(m_2-m_0)(m_2-m_1)} \tag{18}$$

$$\frac{\partial K}{\partial r} = 0 = IC - \frac{2V\lambda}{Q} \frac{(m_2-r)}{(m_2-m_0)(m_2-m_1)} \tag{19}$$

From Equations (18) and (19) we get

$$Q^* = \left[\frac{2A\lambda}{IC[1 - \frac{IC(m_2-m_0)(m_2-m_1)}{2V\lambda}]} \right]^{0.5}, \tag{20}$$

and

$$r^* = \left[m_2 - \frac{ICQ^*(m_2-m_0)(m_2-m_1)}{2V\lambda} \right]. \tag{21}$$

Since expression (20) for Q^* is independent of r, there is no need of an iterative procedure for finding optimal values.

When the stockout penalty is proportional to the average annual number of units out of stock, the total cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{W\lambda}{Q} \int_r^{m_2} (x-r) \frac{2}{(m_2-m_0)} \frac{m_2-x}{(m_2-m_1)} dx,$$

where $m_1 \leq r \leq m_2$, or

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{W\lambda(m_2-r)^3}{3Q(m_2-m_0)(m_2-m_1)}. \tag{22}$$

The necessary conditions for optimality are given by

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + IC(0.5) - \frac{W\lambda(m_2-r)^3}{3Q^2(m_2-m_0)(m_2-m_1)}, \quad (23)$$

and

$$\frac{\partial K}{\partial r} = 0 = IC - \frac{W\lambda(m_2-r)^2}{Q(m_2-m_0)(m_2-m_1)} . \quad (24)$$

In this case expressions (23) and (24) must be solved iteratively to get the optimal values for Q and r.

2.2.4 Lead Time Demand Distribution -- Uniform

Buchan and Koenigsberg (1963) have given expressions for some specialized case of uniform distribution. Here the expressions are developed for the general uniform distribution case.

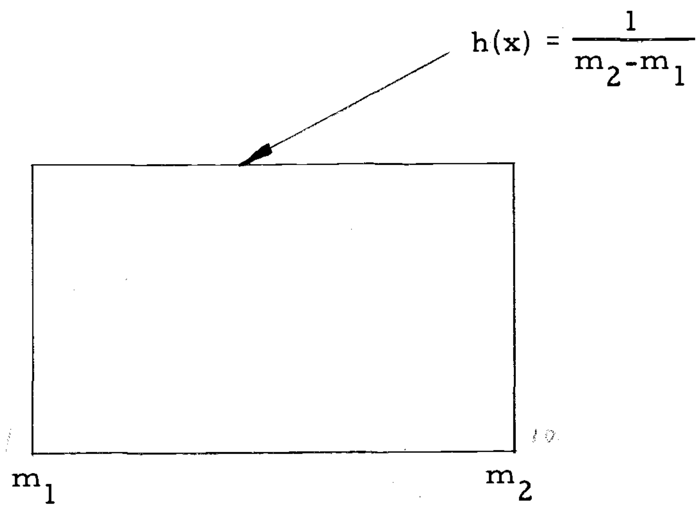

Figure 2. Uniform demand distribution.

$$\text{mean} \quad \mu = \int_0^{\infty} xh(x)dx = \int_{m_1}^{m_2} x \frac{1}{(m_2-m_1)} dx = \frac{(m_1+m_2)}{2}$$

For the case where the stockout penalty is proportional to the number of occasions out of stock, the cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda}{Q}\int_{r}^{m_2} \frac{1}{(m_2-m_1)} dx,$$

where $m_1 \leq r \leq m_2$, or

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{V\lambda(m_2-r)}{Q(m_2-m_1)} . \tag{25}$$

The necessary conditions for optimality are:

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + 0.5IC - \frac{V\lambda(m_2-r)}{Q^2(m_2-m_1)} , \tag{26}$$

and

$$\frac{\partial K}{\partial r} = 0 = IC - \frac{V\lambda}{Q(m_2-m_1)} . \tag{27}$$

From Equations (26) and (27) we get

$$Q* = \frac{V\lambda}{IC(m_2-m_1)} \tag{28}$$

and

$$r^* = \left[\frac{A(m_2 - m_1)}{V\lambda} - \frac{V}{2IC(m_2 - m_1)} + m_2 \right] . \tag{29}$$

Expression (28) for Q^* is independent of r and expression (29) for r^* is independent of Q. Therefore there is no need of using an iterative procedure for finding the optimal values.

When the stockout penalty is proportional to the average annual number of units out of stock, the total cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r - \mu) + \frac{W\lambda}{Q} \int_r^{m_2} (x - r) \frac{1}{m_2 - m_1} \, dx ,$$

or

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r - \mu) + \frac{W\lambda}{2Q(m_2 - m_1)} (m_2 - r)^2 . \tag{30}$$

The necessary conditions for optimality are:

$$\frac{\partial K}{\partial Q} = 0 = -\frac{A\lambda}{Q^2} + IC(0.5) - \frac{W\lambda(m_2 - r)^2}{2Q^2(m_2 - m_1)} , \tag{31}$$

and

$$\frac{\partial K}{\partial r} = 0 = IC - \frac{W\lambda(m_2 - r)}{Q(m_2 - m_1)} . \tag{32}$$

From Equations (31) and (32) we get

$$Q^* = \left[\frac{A\lambda}{\frac{IC}{2} - \frac{1}{2W\lambda} I^2 C^2 (m_2 - m_1)} \right]^{0.5} , \tag{33}$$

and

$$r^* = \left[m_2 - \frac{Q^* IC(m_2 - m_1)}{W\lambda} \right] . \tag{34}$$

Since expression (33) for Q^* is independent of r, there is no need of any iterative procedure for getting optimal values of Q and r.

III. DIRECT SEARCH METHOD

3.1 The Direct Search Technique

Hooke and Jeeves (1961) have defined the "direct search" as a sequential examination of trial solutions involving comparison of each trial solution with the best obtained up to that time, together with the strategy for determining what the next trial solution will be.

Hooke and Jeeves (1961) state that this method can be applied to a problem where there is a set of points which represents all the possible solutions and there is a certain criterion which can be used to decide that one solution, P_2, is better than another solution, P_1, (written $P_2 \subset P_1$), for any two points in the solution space. There is presumably some point, P^*, which is the optimal solution having the property $P^* \subset P$ for all $P \neq P^*$.

For example, suppose we have a function $f(X_1, X_2, \ldots, X_n)$ which we wish to minimize. The value of this function depends upon the vector $(X_1, X_2, \ldots, X_n)$. A solution point, P_i, is said to be better than another solution point, P_j, (written $P_i \subset P_j$), if and only if

$$f(X_{1i}, X_{2i}, \ldots, X_{ni}) < f(X_{1j}, X_{2j}, \ldots, X_{nj}).$$

A point, B_0, is selected arbitrarily to be the first "base point". The second point, P_1, is chosen and compared with B_0. If $P_1 \subset B_0$, P_1 becomes the second base point, B_1. If not, B_1 is the same as B_0. This process continues, each new point being compared with the current base point B_r. A trial at the point, P_{r+1}, may be defined as a potential "move" or "step" from the base point B_r. This move is a success if $P_{r+1} \subset B_r$ and it will be a failure otherwise.

There is a logical procedure for determining which point around B_r should be chosen for the next trial solution. There is also provision for determining when the process should be terminated.

As an alternative to the direct search technique we considered certain "semi-direct" methods, in particular those which are referred to as "gradient" or "steepest ascent" methods. In brief such a method can be described as follows (Saaty and Bram, 1964). For any function $f(X_1, X_2, \ldots, X_n)$ the first derivatives with respect to $X_1, X_2, \ldots, X_n$ are evaluated at some base point, $(X_{10}, X_{20}, \ldots, X_{n0})$, which is chosen arbitrarily. The vector of these derivatives, denoted as $(X'_{10}, X'_{20}, \ldots, X'_{n0})$, is called the gradient of the function $f(X_1, X_2, \ldots, X_n)$. For a minimization problem we move in a direction opposite to that of the gradient vector. To determine the step size $J \geq 0$ (how far to move in the specified direction), the values $X_{10} - JX'_{10}, X_{20}-JX'_{20}, \ldots, X_{n0}-JX'_{n0}$ are substituted for $X_1, X_2, \ldots, X_n$ respectively in the cost function which becomes a function of only one variable, J. Setting the first derivative with respect to J, equal to zero, the step size, J is evaluated. Then the point, $(X_{10}-JX'_{10}, X_{20}-JX'_{20}, \ldots, X_{n0}-JX'_{n0})$, becomes the new base point. The process is repeated until we reach such a base point where the first derivatives $X'_1, X'_2, \ldots, X'_n = 0$, which means the gradient vector is equal to zero.

The difficulty in using a Gradient Method for our models, especially in the normal and exponential distribution cases, is that in order to find step size, J, we must be able to calculate the gradient. Because the cost function involves integrals or other complex expressions, calculation of the gradient is cumbersome.

3.2 Local and Global Minima; Convexity

Like any other search technique, "direct search" finds a local minimum. If the function is convex in all its variables then this local minimum is a global minimum. One definition of convexity is the following: Any function f(X) is said to be convex if

$$f(\alpha X' + (1-\alpha)X'') \leq \alpha f(X') + (1-\alpha)f(X''), \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

For any convex function, the second derivatives of the function with respect to all of its variables are always non-negative. We use this fact to demonstrate convexity of our cost functions. The convexity of the cost functions is demonstrated here for only two cases which are discussed in Section 2.2.1. The convexity of the remaining six cases discussed in Section 2.2 is shown in Table 8 of Appendix I.

For the case where the lead time demand distribution is normal and the stockout penalty is proportional to the number of occasions out of stock, the second derivatives of the cost function given by Equation (1), Chapter II, with respect to Q and t are given by

$$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2V\lambda}{Q^3}\Phi , \tag{35}$$

and

$$\frac{\partial^2 K}{\partial t^2} = \frac{V\lambda t}{Q}\phi . \tag{36}$$

For positive values of Q and t Equations (35) and (36) are positive. But for negative values of t Equation (36) becomes negative.

For this reason, we took the cost equation given by Wagner (1970) into consideration to see whether its second derivative with respect to t is positive or not for all values of t (positive or negative). The only difference between the cost function given by Equation 1 and Wagner's cost function is an extra holding cost term. This extra term (Wagner, 1970) is given by

$$\frac{IC\mu}{2Q} \int_r^\infty (x-r) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2} dx,$$

which in terms of t becomes

$$\frac{IC\mu}{2Q} \sigma (\phi - t\Phi).$$

Adding this extra term in Equation (1) for total cost and then differentiating it twice with respect to Q and t we get,

$$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2V\lambda}{Q^3} \Phi + \frac{IC\mu}{Q^3} \sigma (\phi - t\Phi), \tag{37}$$

and

$$\frac{\partial^2 K}{\partial t^2} = \frac{V\lambda t}{Q} \phi + \frac{IC\mu}{2Q} \sigma \phi. \tag{38}$$

When t is negative, Equation (38) is not necessarily positive. It is interesting to compare the accuracy of Herron's and Wagner's holding cost terms.

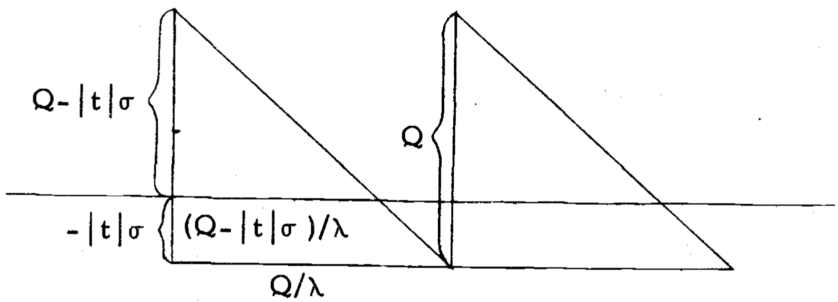

Figure 3. Pattern showing the condition for negative safety stock.

For a given Q, the holding cost is a function of t. Herron's holding cost for different values of t is given as: $IC(0.5Q + t\sigma)$, for $t > 0$; $IC(0.5Q)$, for $t=0$; and $IC(0.5Q - |t|\sigma)$ for $t < 0$. But when $t = (-0.5Q/\sigma)$, this term becomes zero. The approximate true value of the holding cost for different values of t is given by $IC(0.5Q + t\sigma)$, for $t > 0$. At $t = 0$, this term becomes $IC(0.5Q)$. From Figure 3 we see that for the negative values of t, within the range $(-Q/\sigma) \leq t \leq 0$, the approximate true holding cost is given by $IC(0.5)(Q- |t|\sigma)^2 (1/Q)$. Therefore when $t = (-0.5Q/\sigma)$, this true value becomes $IC(0.5)^3 Q$. The approximate true value of the holding cost becomes zero when $t = (-Q/\sigma)$. Therefore it can be said that the Herron's holding cost underestimates the total cost for negative values of t. Apparently Wagner is compensating for this by adding the extra term.

Wagner's holding cost for different values of t is given by $IC(0.5Q + t\sigma) + (IC\mu/2Q)\sigma(\phi - t\Phi)$, for $t > 0$. For any value of t, the term $(\phi - t\Phi)$ is positive. Comparing with the approximate true value of the holding cost for $t > 0$, we find that the Wagner's holding cost is greater than the approximate true value. When $t = 0$, the Wagner's holding cost becomes $IC(0.5Q) + (IC\mu/2Q)\sigma\phi$. But at $t = 0$, the value of ϕ is equal to 0.39886, therefore the Wagner's holding cost at this point is greater than the approximate true value of the holding cost. For the negative values of t, the Wagner's holding cost is given by $IC(0.5Q-|t|\sigma) + (IC\mu/2Q)\sigma(\phi + |t|\Phi)$. When $t = (-0.5Q/\sigma)$, the sum of these terms becomes $(IC\mu/2Q)\sigma(\phi + (0.5Q/\sigma)\Phi)$, which is a positive quantity. Whether this quantity is greater or less than the approximate true value for $t = (-0.5Q/\sigma)$, will depend upon the values of I, C, μ, Q and σ. The same holds true for $t < (-0.5Q/\sigma)$. Therefore it can be said that the Wagner's holding cost overestimates the cost for $t \geq 0$. However, as t becomes large (more than 4.0), Wagner's extra term approaches zero so that the overestimation becomes negligible.

For the case where the stockout penalty is proportional to average annual number of units out of stock, we find that (from Chapter II)

$$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2W\lambda}{Q^3}\sigma(\phi - t\Phi),$$

and

$$\frac{\partial^2 K}{\partial t^2} = \frac{W\lambda}{Q}\sigma\phi.$$

These expressions are positive for any value of t either negative or positive, (written $-\infty \leq t \leq \infty$), and for positive value of Q. Therefore the cost function given by Equation (4) is convex in Q and t.

3.3 Pattern Search

Hooke and Jeeves (1961) have defined the "pattern search" as a direct search routine for minimizing a function of n variables $f(X_1, X_2, \ldots, X_n)$. The pattern search routine makes two types of moves. The first type of move is an exploratory move desired to acquire knowledge concerning the behavior of the function $f(X_1, X_2, \ldots, X_n)$. The second type of move is a pattern move designed to utilize the information acquired in exploratory moves and accomplish the actual minimization of the function by moving in the direction of the established pattern. The intuitive basis for the second type of move is the assumption that a successful set of moves (in some direction) in the past is likely to be successful again.

Wilde (1964) has explained how the pattern is established. In brief it is presented here. The search begins at a base point B_0 chosen arbitrarily. A step size δ_i for each independent variable X_i $(i = 1, 2, \ldots, n)$ is chosen. The value of the function is evaluated at the point B_0 and the point $B_0 + \delta_1$. The new point $B_0 + \delta_1$ becomes the temporary point if the point $B_0 + \delta_1$ is better than the base point B_0. If the point $B_0 + \delta_1$ is not better than B_0 then the value of the function is evaluated at the point $B_0 - \delta_1$. If this point $B_0 - \delta_1$ is better than B_0 then the point $B_0 - \delta_1$ becomes the temporary point, otherwise B_0 remains the temporary point.

Thus, for the case of minimization of a function:

$$\text{Temporary point } P_{11} \equiv \begin{cases} B_0 + \delta_1 & \text{if } f(B_0+\delta_1) < f(B_0) \\ B_0 - \delta_1 & \text{if } f(B_0-\delta_1) < f(B_0) < f(B_0+\delta_1) \\ B_0 & \text{if } f(B_0) < \text{MINIMUM}[f(B_0+\delta_1), f(B_0-\delta_1)]. \end{cases}$$

Now the second variable X_2 is considered in the same way, but about the temporary point P_{11}. The values of the function are evaluated at $P_{11} + \delta_2$ and $P_{11} - \delta_2$. These values of the function are compared with the value of the function at the point P_{11}. Then the second temporary point P_{12} is established in the same way. The first subscript of P_{12} shows the development of the first pattern and the second subscript shows that the second of n variables are considered. Similarly after considering all the n variables, the temporary point P_{1n} can be found, which becomes the second base point B_1, i.e., $P_{1n} \equiv B_1$.

The original base point B_0 and the newly determined base point B_1 together establish the first pattern. Hoping that similar results may be obtained if the test starts from second base point B_1, the local excursions are skipped and the arrow joining B_0 and B_1 is extended, (arbitrarily) to twice its length. This establishes the new temporary point P_{20} for the second pattern based at B_1.

When the number of variables is two, the pattern moves are as shown below in Figure 4.

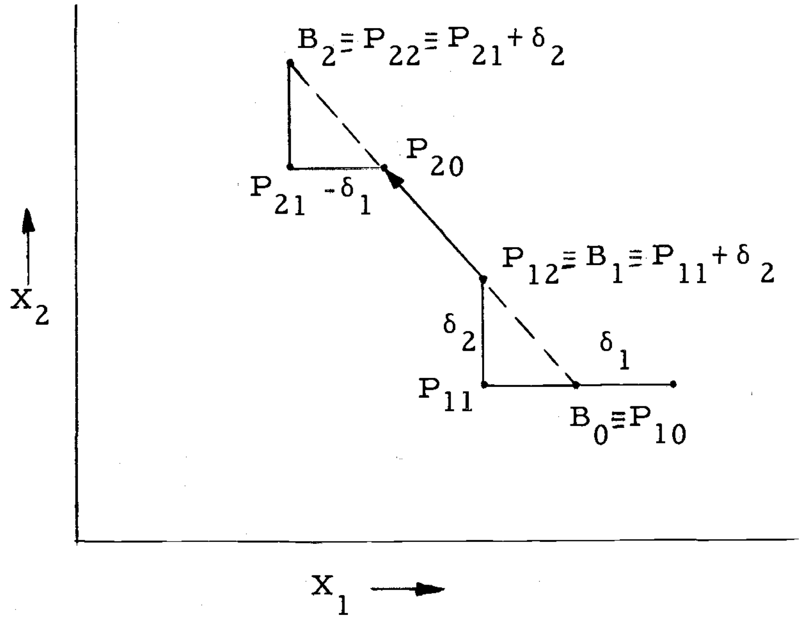

Figure 4. Exploratory and pattern moves in case of two variables.

3.4 A Direct Search Computer Program Technique

The Appendices contain a computer routine for the direct search technique, together with related flow diagrams. The symbols used for the flow chart are defined in Appendix II and the flow chart is shown in Figure 6 of Appendix III. This flow chart shows that a cost function of any number of variables can be minimized and the optimal values for the variables can be found. For our model there are only two variables, the order quantity Q and the safety stock expressed as number of standard deviations t (or the reorder point r). The computer used to solve our model is a CDC 3300; OS3 time sharing system with FORTRAN IV compiler.

The initial step sizes for both the variables, Q and t (or r), are fed in as input data. As the search continues, movement takes place from one solution to another, always moving towards the point which gives minimum cost. The step size increase or decrease depends upon the results of the trials made. Figure 7 in Appendix IV shows how the local excursions are skipped without exploratory movements. Actually, in case of a success, we do not change the step size for the variable by increasing it but instead move the pattern in such a way in the direction of the success that it has the same effect as that of increasing the step size.

The failure of exploratory moves in both directions from the current base point indicates the need for a reduction in the step size. Figure 8 of Appendix IV, which details a part of Figure 6, shows how this reduction in the step size is carried out. Here the step size is reduced to half of the previous step size until the step size becomes smaller than the minimum step size allowed for that avariable.

The minimum step size allowed for the variable Q is unity (1.0) because fractional values of the order quantity should not be allowed. Similarly for the reorder point r, the minimum step size allowed is unity. For the variable t, the minimum step size allowd is 0.01. The reasons for this are as follows.

In the case of normal distribution, for any value of t between 0.0 and ∞ (written $0 \leq t \leq \infty$), if Δt (step size) is 0.01, the value of Φ varies by at most 0.0040, which is itself a very small value, i.e., $|\Phi(t) - \Phi(t \mp .01)| \leq 0.0040$.

For any value of t between 0.0 and ∞ (written $0 \leq t \leq \infty$), if Δt is .01, the value of ϕ varies by at most 0.0025, i.e., $|\phi(t) - \phi(t \mp .01)| \leq .0025$.

For the range $-4.0 \leq t \leq 4.0$, if the step size for t is taken to be 0.01, there are 801 points for which the computer can calculate the value of Φ and store in its memory. If the step size is further reduced then more values of Φ must be calculated and stored which will increase the time taken by the computer.

We have also provided for and specified lower and upper bounds on the decision variables. The order quantity Q should not be less than unity $(Q \geq 1.0)$, because the fractional quantity should not be allowed, and if Q is equal to zero, the cost will become infinite. No upper bound for Q is specified.

For the normal demand distribution, the value of cost is not calculated when t is outside the range $-4.0 \leq t \leq 4.0$, because the right tail probability when t is greater than 4.0 and the left tail probability when t is less than -4.0 are too low.

Near the region of the optimal solutions, the step sizes for the variables will continue to decrease until they all become less than the minimum step size allowed. At this point the step size is put equal to zero, so that when the optimal solution is reached, all the step sizes will be zero. The checking is done as shown in Figure 9 of Appendix V. When all the step sizes are zero, the search is ended and we get the values for minimum cost K^*, optimal quantity Q^* and optimal t^* (or r^*).

3.5 Computational Results

For Example 1 (which is stated below), the optimal values for Q, t and K are found by using direct search technique. Because the initial step sizes allowed for the variables Q and t and the initial values for the variables Q and t affect the number of times cost function is evaluated, the optimal values for Q, t and K and the computational time required taking different initial values for the variables and different initial step sizes, are discussed below. For comparison Example 1is also solved by the approximation method and the results are stated in Chapter V.

To study the effect of different initial step sizes on the number of cost function evaluations and computational time (compilation and run) required to arrive at the optimum solution, tests were carried out taking initial Q and initial t to be 40.0 and 0.30 respectively, in Example 1. The results are shown in Table 2, below.

Table 2. Effect of varying initial step sizes on the number of cost function evaluations and computational time.

Initial ΔQ	Initial Δt	Number of Cost Function Evaluations	Q*	t*	K*	Computational Time (Compilation and Run)
1	.04	59	45	.45	331.5	5.1 secs
4	.04	86	45	.45	331.5	5.4 secs
4	.01	146	45	.44	331.7	5.5 secs
8	.01	171	45	.44	331.7	5.9 secs
16	.02	105	45	.44	331.7	5.57 secs
32	.02	121	45	.44	331.7	5.836 secs
32	.04	127	45	.45	331.5	5.6 secs

Example 1. Normal distribution case where the stockout penalty is proportional to the average annual number of units out of stock.

$N = 2$

$A = 6.0$ dollars/order

$\lambda = 960$ units/year

$IC = 7.0$ dollars/unit/year

$\sigma = 6.0$ units

$W = 1.0$ dollar/unit/year

From Table 2 we see that when the initial values of Q and t are nearly optimal then small initial ΔQ and Δt will require less cost function evaluations.

The number of times the cost function is evaluated depends upon the initial values for Q and t, i.e., the first base point B_0. Example 1 is solved keeping initial step sizes for variables Q and t constant (initial $\Delta Q = 4.0$ and initial $\Delta t = .04$) and varying initial values for Q and t. Table 3 shows the results obtained.

From Table 3 we see that if the initial values of Q and t are nearly optimal, the number of times the cost function is evaluated is less.

The following example illustrates the direct search technique applied to exponential distribution:

Table 3. Effect of varying initial values for variables on the number of cost function evaluations and computational time.

Initial Q	Initial t	Number of Cost Function Evaluations	Q*	t*	K*	Computational Time (Compilation and Run)
21	-1.65	524	45	.45	331.5	6.9 secs
42	0.35	59	45	.44	331.7	5.1 secs
49	0.45	25	45	.43	331.6	5.1 secs
53	0.11	116	45	.44	331.7	5.238 secs
85	2.33	514	45	.45	331.5	6.1 secs
112	-0.50	301	45	.45	331.5	6.5 secs
112	0.50	199	45	.45	331.5	5.206 secs
150	-1.20	492	45	.45	331.5	6.9 secs
163	1.12	310	45	.45	331.5	5.512 secs
219	1.60	421	45	.45	331.5	5.227 secs
251	-2.04	1132	45	.45	331.5	8.02 secs

Example 2. $\lambda = 4850$, $A = \$11.50$, $IC = \$25.0$, $\mu = 25$, $V = \$57.50$/occasion out of stock. The following results were found:

$Q^* = 96.0$, $r^* = 38.0$ and $K^* = \$2741.3$.

For comparison, this problem is also solved using graphical method and approximation method (see Chapters IV and V).

IV. GRAPHICAL METHOD

4.1 Introduction

In order to exactly compute the minimum cost K^*, optimum Q^* and optimum r^* (or t^*) we have to use either a direct search method or an iterative method for normal demand distribution during lead time for both types of stockout penalty and for triangular demand distribution during lead time for the case where the stockout penalty is proportional to the average annual number of units out of stock. The reason of this is that in these cases the expressions for Q^* and r^* (or t^*) are not independent of each other. To avoid the necessity of using these methods Herron (1966) has developed a graphical method for a normal demand distribution during lead time for both types of stockout penalty.

For the normal distribution case where the stockout penalty is proportional to the number of occasions out of stock, graphs are drawn plotting Q_w/σ on the x-axis and Q^*/Q_w, t^* and K^*/ICQ_w on the y-axis with V/A as parameter. For any given problem where we have to find out K^*, Q^* and t^*, first we calculate Q_w/σ and V/A. Then, from the graph, corresponding to these values we read t^*, Q^*/Q_w and K^*/ICQ_w. Multiplying Q^*/Q_w and K^*/ICQ_w by Q_w and ICQ_w respectively the values for Q^* and K^* are found.

Similarly for a normal demand distribution during lead time, where the stockout penalty is proportional to the average annual number of units out of stock, graphs are drawn plotting Q_w/σ on the x-axis and Q^*/Q_w, t^* and K^*/ICQ_w on the y-axis with $W\sigma/A$ as parameter. In the same way, as discussed above, the optimal values for Q^*, t^* and K^* may be found.

Proceeding in the same way the expressions are herein developed for the exponential, triangular and uniform demand distributions during lead time for both types of stockout penalty. With the help of these expressions graphs can be drawn for all these three types of distributions and two types of penalty costs. In this chapter graphs are shown only for the exponential distribution case taking both types of stockout penalty costs into consideration. In the exponential distribution case when expressions were developed for both types of stockout penalty, an interesting similarity was found between the expressions. In one type of penalty, wherever V/A appears, $W\sigma/A$ appears in the other type of penalty. Except for this difference the expressions found are identical. This leads to an additional economy in the graphical method, in which on a single graph both the penalty cases can be shown.

For four different types of distribution and two types of stockout penalty, the parameters and the ratios taken on the x and y axes are summarized in Table 9 of Appendix VI.

4.2 Development of the Dimensionless Ratios and Construction of the Graphs

In this section we develop the expressions containing the dimensionless ratios for the exponential, triangular and uniform distributions of demand during lead time taking both types of stockout penalty into consideration. It is also explained how the graphs can be drawn with the help of these expressions. To illustrate, graphs are drawn for the exponential distribution case.

4.2.1 Lead Time Demand Distribution -- Exponential

For the case where the stockout penalty is proportional to the number of occasions out of stock, from Equation (9), we get

$$Q^* = \frac{V\lambda e^{-(1+t^*)}}{IC\sigma} .$$

Putting this value of Q^* in Equation (8) and introducing Q_w, we get

$$(.5)(Q_w/\sigma)(V/A) = \left(\frac{1}{e^{-(1+t^*)}}\right)\left[1+(V/A)e^{-(1+t^*)}\right]^{0.5} . \qquad (39)$$

Introducing Q_w in Equation (9), we get

$$(Q^*/Q_w) = 0.5(Q_w/\sigma)(V/A)e^{-(1+t^*)}. \qquad (40)$$

From Equations (7) and (8), we get

$$K^* = IC(Q^* + t^*\sigma),$$

or

$$K^*/ICQ_w = (Q^*/Q_w) + t^*/(Q_w/\sigma). \qquad (41)$$

One value for the parameter V/A is assumed, say 50.0. Now different values for t^* are assumed, i.e., $t^* = .2, .4, \ldots, 5.0$. For $V/A = 50.0$ and for each value of t^*, Q_w/σ is calculated from Equation (39). The corresponding values for Q^*/Q_w are calculated from Equation (40) and the values for K^*/ICQ_w are calculated from Equation (41). Then the graphs are drawn with Q_w/σ on the x-axis and t^*, Q^*/Q_w and K^*/ICQ_w respectively on the y-axis. The parameter for these three curves is $V/A = 50.0$.

Similarly different values of V/A are assumed. They are 10.0, 5.0, 3.0, 2.0 and 1.0. For each value of V/A the process is repeated and the graphs are drawn. Figure 5 shows these curves. For the case where the stockout penalty is proportional to the average annual number of units out of stock, proceeding in the same way, from Equations (12), (13) and (14), we get

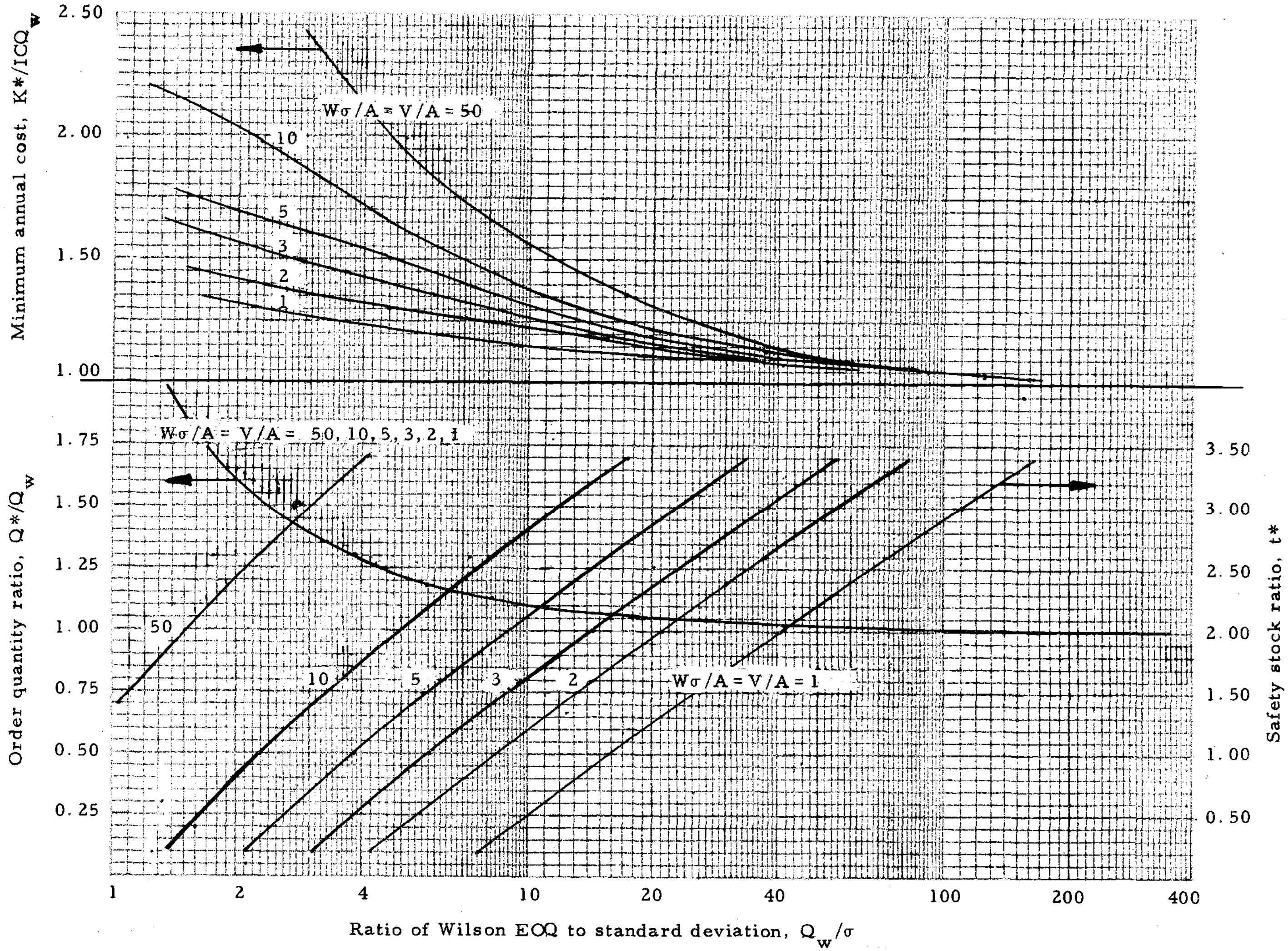

Figure 5. Minimum-cost conditions for stockout penalty proportional to the number of occasions out of stock or the average annual number of units out of stock.

$$(0.5)(Q_w/\sigma)(W\sigma/A) = \frac{1}{e^{-(1+t^*)}}\left[1+(W\sigma/A)e^{-(1+t^*)}\right]^{0.5}, \tag{42}$$

$$(Q^*/Q_w) = (0.5)(Q_w/\sigma)(W\sigma/A)[e^{-(1+t^*)}], \tag{43}$$

and

$$K^*/ICQ_w = Q^*/Q_w + t^*/(Q_w/\sigma) \tag{44}$$

In the same way, as explained before, graphs can be drawn between Q_w/σ on the x-axis and t^*, Q^*/Q_w and K^*/ICQ_w on the y-axis with $W\sigma/A$ as the parameter. The expressions (39), (40), (41) and (42), (43), (44) are identical except for the difference that in place of V/A, $W\sigma/A$ appears. Therefore for both types of penalty costs, the relation between Q_w/σ and t^*, Q^*/Q_w and K^*/ICQ_w is shown by a single graph. Another interesting characteristic found in the exponential distribution case is that the curves for the different values of the parameter, drawn with Q_w/σ on the x-axis and Q^*/Q_w on the y-axis fall on top of each other, i.e., for any value of the parameter this curve is the same.

4.2.2 Lead Time Demand Distribution -- Triangular

For the case where the stockout penalty is proportional to the number of occasions out of stock, if we proceed in the same way, then from Equations (17), (18) and (19), we get

$$(Q_w)\left(\frac{V}{A}\right)\frac{1}{(m_2-m_0)(m_2-m_1)} = \frac{1}{(m_2-r^*)}\left[1+\left(\frac{V}{A}\right)\frac{(m_2-r^*)^2}{(m_2-m_0)(m_2-m_1)}\right]^{0.5}, \tag{45}$$

$$\left(\frac{Q^*}{Q_w}\right) = (Q_w)\left(\frac{V}{A}\right)\frac{(m_2-r^*)}{(m_2-m_0)(m_2-m_1)}, \tag{46}$$

and

$$\left[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}\right] = \left(\frac{Q^*}{Q_w}\right) - \frac{(m_2-r^*)}{Q_w}. \tag{47}$$

In this case the parameter is $\frac{V}{A(m_2-m_0)(m_2-m_1)}$. For one value of this parameter different values of (m_2-r^*) are assumed and corresponding values of Q_w, Q^*/Q_w and $\left[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}\right]$ are found from expressions (45), (46) and (47). Then the graphs are drawn taking Q_w on the x-axis and Q^*/Q_w, (m_2-r^*) and $\left[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}\right]$ on the y-axis respectively. The same process is repeated to draw graphs for other values of the parameter.

For the case where the stockout penalty is proportional to the average annual number of units out of stock, from Equations (22), (23) and (24), we get

$$(Q_w)\frac{W}{A(m_2-m_0)(m_2-m_1)} = \frac{2}{(m_2-r^*)^2}\left[1+\frac{(m_2-r^*)^3}{3}\frac{W}{A(m_2-m_0)(m_2-m_1)}\right]^{0.5}, \tag{48}$$

$$2\left(\frac{Q^*}{Q_w}\right) = (Q_w) \frac{W}{A(m_2-m_0)(m_2-m_1)} (m_2-r^*)^2, \tag{49}$$

and

$$\left[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}\right] = \left(\frac{Q^*}{Q_w}\right) - \frac{(m_2-r^*)}{Q_w}. \tag{50}$$

In the same way, as expalined earlier, graphs can be drawn with Q_w on the x-axis and Q^*/Q_w, (m_2-r^*) and $[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}]$ on the y-axis respectively with $\frac{W}{A(m_2-m_0)(m_2-m_1)}$ as the parameter.

4.2.3 Lead Time Demand Distribution -- Uniform

For the case where the stockout penalty is proportional to the number of occasions out of stock, from Equations (25), (26) and (27), we get

$$(Q_w) \frac{V}{A(m_2-m_1)} = 2\left[1+(m_2-r^*) \frac{V}{A(m_2-m_1)}\right]^{0.5}, \tag{51}$$

$$\left(\frac{Q^*}{Q_w}\right) = (0.5)(Q_w) \frac{V}{A(m_2-m_1)}, \tag{52}$$

and

$$\left[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}\right] = \left(\frac{Q^*}{Q_w}\right) - \frac{(m_2-r^*)}{Q_w}. \tag{53}$$

In this case the graphs can be drawn taking Q_w on the x-axis and Q^*/Q_w, (m_2-r^*) and $[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}]$ on the y-axis respectively with $\frac{V}{A(m_2-m_1)}$ as the parameter.

For the case where the stockout penalty is proportional to the number of units out of stock, from Equations (30), (31) and (32), we get

$$(Q_w) \frac{W}{2A(m_2-m_1)} = \left[\frac{1}{(m_2-r^*)^2} + \frac{W}{2A(m_2-m_1)}\right]^{0.5}, \quad (54)$$

$$\left(\frac{Q^*}{Q_w}\right) = (Q_w) \frac{W}{2A(m_2-m_1)} \quad (m_2-r^*), \quad (55)$$

and

$$\left[\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}\right] = \left(\frac{Q^*}{Q_w}\right) - \frac{(m_2-r^*)}{Q_w} \quad . \quad (56)$$

The ratios taken on the x-axis and the y-axis are the same as for the case where stockout penalty is proportional to the number of occasions out of stock but in this case the parameter is $\frac{W}{2A(m_2-m_1)}$.

4.3 Graphical Method Used to Solve Example 2 in Chapter III

Example 2 is stated in Chapter III. For this problem, we have $V/A = 57.5/11.5 = 5.0$,

$$Q_w = \sqrt{\frac{2A\lambda}{IC}} = \sqrt{\frac{2 \times 11.5 \times 4850}{25.0}} = 66.80 \quad \text{units},$$

standard deviation $\sigma = \mu = 25.0$, and

$$\frac{Q_w}{\sigma} = \frac{66.80}{25.0} = 2.672.$$

From the graph (Figure 5), we get $t^* = .54$, $Q^*/Q_w = 1.44$ and $K^*/ICQ_w = 1.635$. When we calculate for the optimal values for Q, r and K, we get

$$Q^* = (Q^*/Q_w)Q_w = 1.44 \times 66.80 = 96.0 \text{ units},$$

$$r^* = t^*\sigma + \mu = .54 \times 25.0 + 25.0 \simeq 38.0 \text{ units}, \quad \text{and}$$

$$K^* = (K^*/ICQ_w)ICQ_w = 1.635 \times 25.0 \times 66.80 = \$2734.45.$$

Table 4 summarizes the results found by using the direct search technique and the graphical method.

Table 4. Comparison of results by using the direct search technique and the graphical method.

Method	Optimum Quantity Q*	Re-order Point r*	Minimum Cost K*
Direct search	96	38	$2741.3
Graphical	96	38	$2734.45

V. NUMERICAL APPROXIMATIONS FOR DETERMINING OPTIMAL PARAMETER VALUES

5.1 Introduction

Many industries do not have computers to use. So they need a method by which the optimal values for order quantity Q, reorder point r (or t) and total annual average cost K may be calculated without the need for complex iterative solutions of the analytic or direct search types. It is for these reasons that Herron (1967) has developed numerical approximations for the normal distribution case taking both types of stockout penalty into consideration, and Hunt (1965) has given numerical approximations for the exponential distribution case for both types of stockout penalty, for the triangular distribution case where the stockout penalty is proportional to the number of occasions out of stock, and for the uniform distribution case where the stockout penalty is proportional to the average annual number of units out of stock. In this chapter, these approximations are reviewed and a few examples for the normal and exponential distribution cases are solved by (i) the direct search technique and (ii) Herron's and Hunt's numerical approximation methods. The data and results are summarized in Tables 5 and 6. It is found that although the approximated optimal values for Q and t (written $\hat{Q}^*$ and $\hat{t}^*$) may vary considerably from the exact optimal values Q^* and t^*

respectively, the approximated minimum cost K^* does not vary much from the exact minimum cost K^*.

5.2 The Approximations

In this section we review the approximations given by Herron (1967) for the normal distribution case for both types of stockout penalty, and by Hunt (1965) for the exponential, triangular and uniform distribution cases. The types of stockout pentalty taken into consideration for these cases are stated in Section 5.1.

5.2.1 Lead Time Demand Distribution -- Normal

For the case where the stockout penalty is proportional to the number of occasions out of stock, Herron (1967) has given the following approximation for Φ.

$$\Phi \cong 0.60\, e^{-t/0.65} - .0049 \tag{57}$$

For the range $0 \leq t \leq 3$, the approximated value of Φ from Equation (57) deviates from the true value of Φ by at most 30%. Employing this value of Φ in Equations (2) and (3), and introducing Q_w, we get

$$\frac{\hat{Q}^*}{Q_w} = 0.65(Q_w/\sigma) + \{1-0.0049(V/A) + [-0.65/(Q_w/\sigma)]^2\}^{0.5}. \tag{58}$$

Equation (58) is independent of t so the approximate optimal order quantity $\hat{Q}*$ may be calculated. From Equation (3), we get

$$IC\sigma = \frac{V\lambda}{Q*} \frac{1}{\sqrt{2\pi}} e^{-0.5t*^2} .$$

In this expression, setting $\hat{Q}*$ in place of $Q*$, we get

$$\hat{t}* = [\, 2 \ln \frac{V\lambda}{\sqrt{2\pi}\, \hat{Q}*IC\sigma} \,]^{0.5}, \qquad (59)$$

from which the value for the approximate optimum t may be calculated. Substituting these values of $Q*$ and $t*$ into Equation (1), the approximate minimum cost $K*$ may be calculated.

For the case where the stockout penalty is proportional to the average annual number of units out of stock, Herron (1967) has given an approximation for $(\phi\text{-}t\Phi)$ as:

$$\phi - t\Phi \cong 0.50\, e^{-t/.50} - .0009. \qquad (60)$$

For the range $0 \leq t \leq 3$, the approximated value of $\phi - t\Phi$ given by Equation (60) deviates from its true value by at most 30%. Employing this approximated value of $\phi - t\Phi$ in Equation (4) and setting the partial derivatives of Equation (4) equal to zero, we get

$$\frac{\hat{Q}*}{Q_w} = 0.50/(Q_w/\sigma) + \{1 - .0009(W\sigma/A) + [-0.50/(Q_w/\sigma)]^2\}^{0.5}. \qquad (61)$$

Since this expression is independent of t, the value for approximated optimum order quantity $\hat{Q}*$ may be calculated. From Equation (6), we get

$$IC = \frac{W\lambda}{Q*}\Phi.$$

Substituting $\hat{Q}*$ for $Q*$ and employing the approximated value for $\phi - t\Phi$, we get

$$\hat{t}* = \{0.879 + 2.5[\ln(W\lambda/ICQ*) - 0.693]\}^{0.5} - 0.938. \qquad (62)$$

Then by putting these values of $\hat{Q}*$ and $\hat{t}*$ in Equation (4) the approximated minimum cost $K*$ may be calculated.

A few examples for the normal distribution case are solved. The type of stockout penalty and the data for these examples are listed in Table 5. The results for optimum Q, optimum t and minimum cost K as calculated by the direct search method are denoted as $Q*$, $t*$ and $K*$ respectively, and by Herron's approximation method are denoted as $\hat{Q}*$, $\hat{t}*$ and $\hat{K}*$ respectively. The percentage variation in the approximated values $\hat{Q}*$, $\hat{t}*$ and $\hat{K}*$ from the exact values $Q*$, $t*$ and $K*$ respectively, are also calculated and shown in the same table. Table 5 also shows the difference between the computational times for the two methods mentioned above.

5.2.2 Lead Time Demand Distribution--Exponential

Hunt (1965) has indicated that the expression for total cost can be broken into two parts to disconnect the two decision variables Q and r, and then we can solve for each optimum value sequentially.

For the case where the stockout penalty is proportional to the number of occasions out of stock, Equation (7) for total cost, in terms of r, may be written as:

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r - \mu) + \frac{V\lambda}{Q} e^{-\frac{r}{\mu}} . \tag{63}$$

Breaking this equation into two parts, we get

$$K_1(Q) = \frac{A\lambda}{Q} + IC(0.5Q), \tag{64}$$

and

$$K_2(r) = IC(r-\mu) + \frac{V\lambda}{Q} e^{-\frac{r}{\mu}} . \tag{65}$$

Differentiating Equation (64) with respect to Q, we get

$$(Q_w)^2 = \frac{2A\lambda}{IC} . \tag{66}$$

The optimal value for r (written as r_w) is obtained by substituting Q_w for Q in Equation (65) and setting the partial derivative of Equation (65) with respect to r, equal to zero. In this way, we get

$$r_w = \mu \ln \frac{V\lambda}{IC\mu Q_w} . \qquad (67)$$

However, because Equation (65) is differentiated with respect to r taking Q constant as Q_w, which is logically wrong, errors are introduced in the results. To reduce the effect of these errors it is necessary to know the difference between the costs calculated, $K(Q_w, r_w) - K(Q^*, r^*)$, where $K(Q_w, r_w)$ denotes the value of the total cost when Q_w and r_w are substituted for Q and r respectively in Equation (63) and $K(Q^*, r^*)$ denotes the value of the total cost when Q^* and r^* are substituted for Q and r respectively in the same equation.

To reduce the cost difference, an adjustment may be made for the value of $\hat{Q}^*$ and $\hat{r}^*$, where

$$\hat{Q}^* = f_1(Q_w, r_w), \quad \text{and} \quad \hat{r}^* = f_2(Q_w, r_w).$$

Hunt (1965) has given these approximations as:

$$\hat{Q}^* \cong Q_w e^d, \qquad (68)$$

and

$$\hat{r}^* \cong r_w - d^2 Q_w, \qquad (69)$$

where

$$d = \frac{\mu}{Q_w}, \quad \text{and} \quad \hat{r}^*, r_w > 0.$$

Hunt also mentions that the approximated minimum cost is given by

$$K(\hat{Q}*, \hat{r}*) = K(Q_w, r_w) - K_1(Q_w) \frac{d^2}{2} . \tag{70}$$

According to Hunt, to get optimal values for the order quantity, re-order point and the total cost, the values of Q_w and r_w should be calculated from Equations (66) and (67) respectively. Then from Equations (68), (69) and (70), the desired approximate optimal values may be calculated.

It is worth mentioning here that as soon as we know the approximate optimal values $\hat{Q}*$ and $\hat{r}*$, we can put these values in Equation (63) to get the approximate minimum cost $K(\hat{Q}*, \hat{r}*)$. The approximate minimum cost is calculated by both the methods. The data and results are summarized in Table 6. This table also shows the optimal values found by using the direct search technique.

It was mentioned in Chapter II that Equation (10) is independent of t. Therefore without using any search technique the optimal order quantity $Q*$ may be calculated from Equation (10). From Equation (11), we know that

$$t* = \ln \frac{V\lambda}{IC\mu Q*} - 1.0.$$

Putting this value of $t*$ in the expression, $r* = t*\sigma + \mu$, we get

$$r^* = \mu[\ln \frac{V\lambda}{IC\mu Q^*}], \tag{71}$$

because for the exponential distribution case $\sigma = \mu$. Putting the value for Q^* in Equation (71), the value for r^* may be calculated. Then with these values of Q^* and r^*, the value for K^* may be calculated from Equation (63). The optimum values calculated this way are also shown in Table 6. These values are exactly the same as found by the direct search technique. Table 6 also shows the percentage variation in the approximate optimal values from the respective exact optimal values, and the computational times of these different methods.

For the case where the stockout penalty is proportional to the average annual number of units out of stock, the Equation (12) for total cost, in terms of r, may be written as:

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r - \mu) + \frac{W\lambda}{Q} \sigma e^{-\frac{r}{\mu}}. \tag{72}$$

The only difference between Equations (63) and (72) is that in the place of V, the term $W\sigma$ appears. Therefore if we proceed in the same way, we find

$$K_1(Q) = \frac{A\lambda}{Q} + IC(0.5Q), \tag{73}$$

and

Table 5. Comparison between exact solutions by using direct search technique and Herron's approximate solutions.

Demand Distribution	Stockout penalty is proportional to the number of	Average annual demand λ	Inventory carrying charge IC	Standard deviation σ	Stockout penalty V or W	Optimal Q by using direct search Q*	Appro. opt. Q by using Herron's appro. method $\hat{Q}$*	Percentage varriation in Q	Optimal t by using direct search t*	Appro. opt. t by using Herron's appro. method $\hat{t}$*	Percentage varriation in t	Minimum cost K by using direct search K*	Appro. min. cost K by using Herron's appro. method $\hat{K}$*	Percentage varriation in K	Computational time (compilation and run) for direct search	Computational time (compilation and run) for Herron's appro. solution
Normal	occasions out of stock	3400	14.0	30	30.0	95	76	20%	0.20	0.6966	248%	1415.3	1419.51	0.30%	9.625 secs	3.68 secs
Normal	units out of stock	3430	14.0	60	1.0	116	92	20.7%	0.08	0.3250	306%	1695.0	1716.15	1.25%	8.200 secs	3.18 secs
Normal	units out of stock	2000	12.0	30	1.0	71	62	12.7%	0.19	0.3343	76%	925.7	931.48	0.62%	8.700 secs	2.63 secs
Normal	units out of stock	1091	11.0	15	1.0	48	43	10.4%	0.04	0.1737	334%	536.3	538.43	0.40%	6.200 secs (average)	2.7 secs
Normal (Example 1)	units out of stock	960	7.0	6	1.0	45	44	2.22%	0.44	0.4722	7.3%	331.7	331.77	0.02%	6.200 secs (average)	2.6 secs

Where the ordering cost A = 6.0.

Table 6. Comparison of exact solutions with Hunt's approximate solutions.

Demand distribution	Stockout penalty is proportional to the number of	A =	λ =	IC =	b =	V =	Q* Direct search	Q* From Equation (10)	$\hat{Q}$*	Percentage deviation in Q	r* Direct search	r* From Equation (71)	$\hat{r}$*	Percentage deviation in r	K* Direct search	K* From Equation (63)	Hunt's minimum cost $\hat{K}$*	Percentage deviation in Hunt's cost	Cost taking $\hat{Q}$* and $\hat{r}$*	Percentage deviation in cost K($\hat{Q}$*, $\hat{r}$*)	Computational time (compilation and run) Direct search	Computational time (compilation and run) Using Eq. (10, 71, 63)	Computational time (compilation and run) approximate method
Exponential (Example 2)	occasions out of stock	11.5	4850	25.0	.04	57.5	96	96	97	1.04%	38	38	38	0%	2741.30	2741.30	2739.93	.05%	2741.30	0%	4.4 secs	2.7 secs	2.918 secs
Exponential	occasions out of stock	9.5	4150	22.0	.05	95.0	83	83	84	1.2%	48	48	47	2.08%	2434.91	2434.91	2434.15	.03%	2434.96	.002%	4.7 secs	2.6 secs	2.954 secs

$$K_2(r) = IC(r-\mu) + \frac{W\sigma\lambda}{Q} e^{-\frac{r}{\mu}} . \tag{74}$$

Differentiating Equation (73) with respect to Q and setting it equal to zero, we get

$$(Q_w)^2 = \frac{2A\lambda}{IC} . \tag{75}$$

Substituting Q_w for Q in Equation (74) and setting the partial derivative of Equation (74) with respect to r, equal to zero, we get

$$r_w = \mu \ln \frac{W\lambda}{ICQ_w} . \tag{76}$$

For this case Hunt has given the approximation for the optimal values of Q and r as:

$$\hat{Q}^* \cong Q_w e^d, \tag{77}$$

and

$$\hat{r}^* \cong r_w - d^2 Q_w, \tag{78}$$

where

$$\hat{r}^*, r_w > 0, \quad \text{and} \quad d = \frac{\mu}{Q_w} .$$

Equations (77) and (78) are the same as Equations (68) and (69) respectively. Hunt also mentions that the approximated minimum cost may be calculated from the following equation which is exactly the same as Equation (70).

$$K(\hat{Q}*, \hat{r}*) = K(Q_w, r_w) - K_1(Q_w)\frac{d^2}{2}$$

In this case the approximate optimal values may be calculated in the same way as discussed earlier. It is worth mentioning here that there is no need of using these approximations when the exact optimal values are easily found from Equations (12), (15) and (16).

5.2.3 Lead Time Demand Distribution--Triangular

Hunt has also developed the approximations for the optimal values of Q, r and K, for the case where the stockout penalty is proportional to the number of stockout occasions. But his approximations for the optimal order quantity and reorder point are exactly the same as the values of $Q*$ and $r*$ found from Equations (20) and (21). If we know $Q*$ and $r*$, then the minimum cost $K*$ may be calculated from Equation (17). Therefore Hunt's approximation for the minimum cost is not of much importance to us.

5.2.4 Lead Time Demand Distribution--Uniform

Hunt has developed approximations for optimal values of Q, r and K, for the case where the stockout penalty is proportional to the average annual number of units out of stock. Due to the similar reasons as given in Section 5.2.3, his approximations are not of much importance to us.

VI. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions and Recommended Methods

The purpose of this thesis was to evaluate different methods that may be used to find out the optimal values of Q, r (or t) and K for a given type of demand distribution and stockout penalty. Although all the methods are potentially applicable for each case, one particular method usually has advantages over another. These advantages may be accuracy, simplicity, economy, etc. From Table 5 we see that the optimal values for Q, t (or r) and K were found by using the direct search technique and Herron's approximation method for the normal distribution case. Although the optimal values for Q and t found by using each method differed significantly, the optimal values of K did not differ much. The Herron's approximation method is easy to program, takes less computational time but gives approximate results, whereas the direct search technique is accurate but requires a good knowledge of computer programming. From Table 6 we see that for the exponential distribution case, the optimal values found by using direct search, Equations 10, 71 and 63, or Hunt's approximation method, are almost identical.

Of the four types of demand probability of distributions discussed in this thesis, the search technique is recommended for the normal distribution for both types of stockout penalties, and for the triangular distribution for the case where the stockout penalty is proportional to the average annual number of units out of stock. In these three cases the graphical method may also be used instead of the direct search technique. In the case of the exponential distribution the optimal values for Q, t and K may be found directly from the solution of the necessary conditions for optimality (see Equations 10, 11 and 7, Chapter II, for the case where the stockout penalty is proportional to the number of occasions out of stock, and Equations 15, 16 and 12 for the case where the stockout penalty is proportional to the average annual number of units out of stock, discussed in Section 2. 2. 2). Similarly for the triangular distribution where the stockout penalty is proportional to the number of occasions out of stock, the optimal values for Q, r and K may be found directly from Equations 20, 21 and 17 (Section 2. 2. 3). For the uniform distribution the optimal values for Q, r and K may be found from Equations 28, 29 and 25 for the case where the stockout penalty is proportional to the number of occasions out of stock and from Equations 33, 34 and 30 for the case where the stockout penalty is proportional to the average annual number of units out of stock (Section 2. 2. 4). All these results are summarized in Table 7, below.

Table 7. Suggestions for the methods to be used in some specific cases.

Type of Distribution	The Stockout Penalty is Proportional to the Number of	Recommended Methods
Normal	Occasions out of stock	Direct search or graphical
	Units out of stock	Direct search or graphical
Exponential	Occasions out of stock	Equations 10, 11 and 7 (Section 2.2.2)
	Units out of stock	Equations 15, 16 and 12 (Section 2.2.2)
Triangular	Occasions out of stock	Equations 20, 21 and 17 (Section 2.2.3)
	Units out of stock	Direct search or graphical
Uniform	Occasions out of stock	Equations 28, 29 and 25 (Section 2.2.4)
	Units out of stock	Equations 33, 34 and 30 (Section 2.2.4)

6.2 Recommendations for Future Research

The direct search technique is used in this thesis for the $\langle Q, r \rangle$ model where no constraints were imposed on Q and r. There appears to be no reason why one could not use this technique for the optimization of a function where the variables of the function are constrained. There would probably be little increase in complexity of the direct search procedure, whereas a corresponding analytical approach (using constrained indirect optimization methods) would require more sophistication.

It appears that if one were interested in a discrete demand model (using, for example a Poisson demand distribution), the direct (pattern) search method might be appropriate, since it only looks at discrete points, separated by an increment δ, anyway.

Perhaps an increase in accuracy could be obtained if the graphical results were converted to Nomogram scale.

In this thesis we have looked upon only two types of stockout penalty; one when the stockout penalty is proportional to the number of occasions out of stock and the other when it is proportional to the average annual number of units out of stock. Analogous research may be done for some other types of stockout penalty, e.g., when the stockout penalty is proportional to the fraction of time out of stock, when it is proportional to the average time and units out of stock, etc.

When the stockout penalty is proportional to the fraction of time out of stock, the total expected cost per year is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q+r-\mu) + \frac{Y\lambda}{Q}\int_{r}^{\infty} \frac{(x-r)}{x} h(x)dx,$$

where Y is the stockout penalty based on the fraction of time out of stock and the other terms are as defined earlier.

When the stockout penalty is proportional to the average time and units out of stock, the total expected cost is given by

$$K = \frac{A\lambda}{Q} + IC(0.5Q + r-\mu) + \frac{Z\lambda}{Q} \int_{r}^{\infty} \frac{(x-r)}{2} \frac{(x-r)}{x} h(x)dx,$$

where Z is the stockout penalty based on average time and units of stock and the other terms are as defined earlier.

Such formulations could be the basis for further research developments which extend the techniques developed in this thesis.

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APPENDICES

APPENDIX I

Table 8. Demonstration of convexity for Sections 2.2.2 through 2.2.4.

Demand Distribution	Stockout Penalty is Proportional to the Number of	Section	Second Derivatives of Cost Function	Whether Positive or Negative	Reasons of Being Positive or Negative
Exponential	occasions out of stock	2.2.2	$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2V\lambda}{Q^3} e^{-(1+t)}$	positive	Q is positive t may be positive or negative
	occasions out of stock	2.2.2	$\frac{\partial^2 K}{\partial t^2} = \frac{V\lambda}{Q} e^{-(1+t)}$	positive	Q is positive t may be positive or negative
	units out of stock	2.2.2	$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2W\sigma\lambda}{Q^3} e^{-(1+t)}$	positive	Q is positive t may be positive or negative
	units out of stock	2.2.2	$\frac{\partial^2 K}{\partial t^2} = \frac{W\sigma\lambda}{Q} e^{-(1+t)}$	positive	Q is positive t may be positive or negative
Triangular	occasions out of stock	2.2.3	$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2V\lambda}{Q^3} \frac{(m_2 - r)^2}{(m_2 - m_0)(m_2 - m_1)}$	positive	$m_2 \geq r$ $m_2 > m_0$ $m_2 > m_1$

(continued)

Table 8. (Continued)

Demand Distribution	Stockout Penalty is Proportional to the Number of	Section	Second Derivatives of Cost Function	Whether Positive or Negative	Reasons of Being Positive or Negative
Triangular	occasions out of stock	2.2.3	$\frac{\partial^2 K}{\partial r^2} = \frac{2V\lambda}{Q(m_2-m_0)(m_2-m_1)}$	positive	$m_2 > m_0$ $m_2 > m_1$
	units out of stock	2.2.3	$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2W\lambda}{3Q^3} \frac{(m_2-r)^3}{(m_2-m_0)(m_2-m_1)}$	positive	$m_2 \geq r$ $m_2 > m_0$ $m_2 > m_1$
	units out of stock	2.2.3	$\frac{\partial^2 K}{\partial r^2} = \frac{2W\lambda(m_2-r)}{Q(m_2-m_0)(m_2-m_1)}$	positive	$m_2 \geq r$ $m_2 > m_0$ $m_2 > m_1$
Uniform	occasions out of stock	2.2.4	$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{2V\lambda}{Q^3} \frac{(m_2-r)}{(m_2-m_1)}$	positive	$m_2 \geq r$ $m_2 > m_1$
	occasions out of stock	2.2.4	$\frac{\partial^2 K}{\partial r^2} = 0$		
	units out of stock	2.2.4	$\frac{\partial^2 K}{\partial Q^2} = \frac{2A\lambda}{Q^3} + \frac{W\lambda(m_2-r)^2}{Q^3(m_2-m_1)}$	positive	$m_2 \geq r$ $m_2 > m_1$
	units out of stock	2.2.4	$\frac{\partial^2 K}{\partial r^2} = \frac{W\lambda}{Q(m_2-m_1)}$	positive	$m_2 > m_1$

APPENDIX II

Glossary of Symbols Used in Flow Chart for Direct Search

N	= Number of variables.
XB(I)	= Arbitrarily chosen initial value for variable I, where I = 1, 2, . . . , N.
DELT(I)	= Initial step size for variable I.
ERR(I)	= Minimum value of step size for variable I.
Cost(FA)	= Value of the cost function at the base points B_0, B_1, B_2... .
Cost(FB)	= Value of the cost function at the temporary points P_{11}, P_{12}, P_{21}, P_{22}, etc.
Cost(FC)	= Value of the cost function at temporary points to check for concavity.
Cost(FD)	= Value of the cost function at optimal point.
DELTA(I)	= Variable step size for variable I, may be reduced if the move is a failure.
X(I)	= Value for variable I.
CHECK	= Minimum [DELTA(I); I = 1, 2, . . , N.].
SIGN	= 1.0 or -1.0 denotes whether the step size is taken in the positive direction or opposite to it.
FIGN(I)	= SIGN for variable I.

APPENDIX III

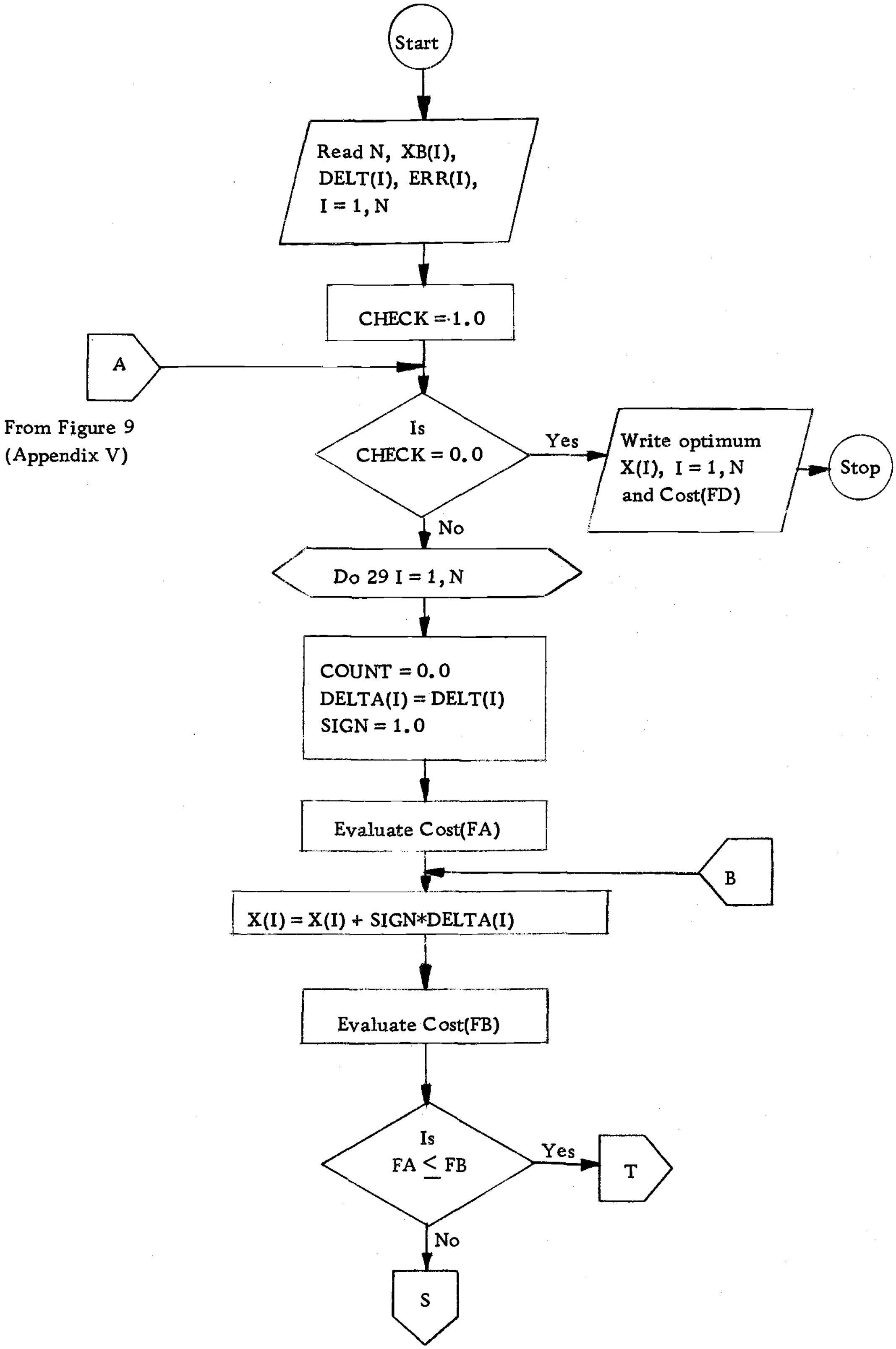

Figure 6. Flow chart for the direct search technique.

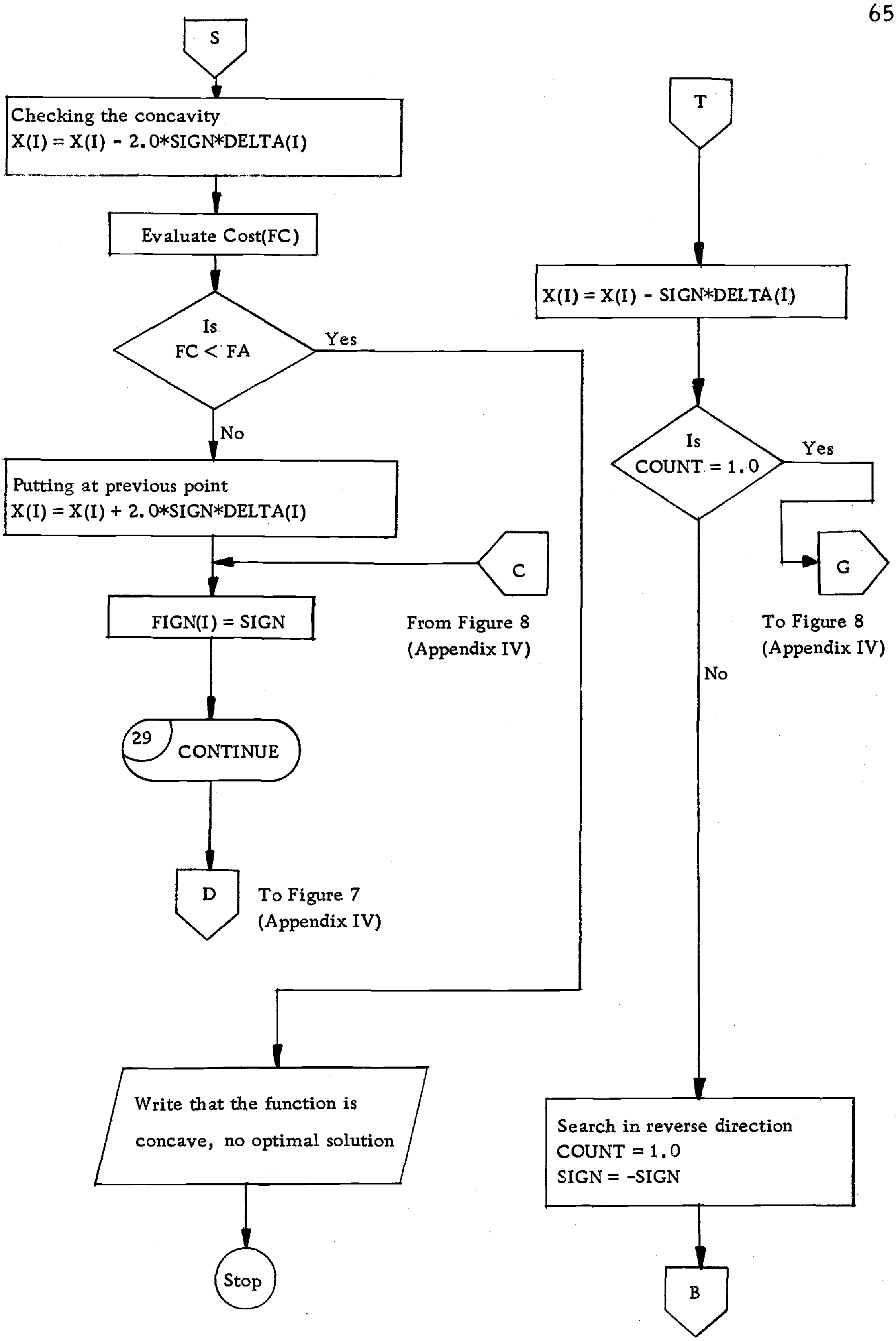

Figure 6. (Continued)

APPENDIX IV

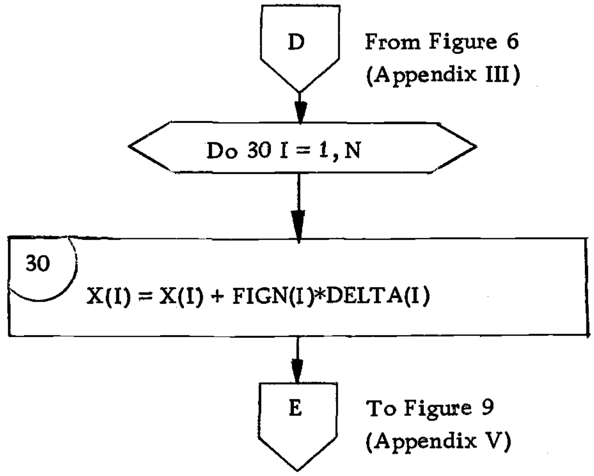

Figure 7. The incremental procedure.

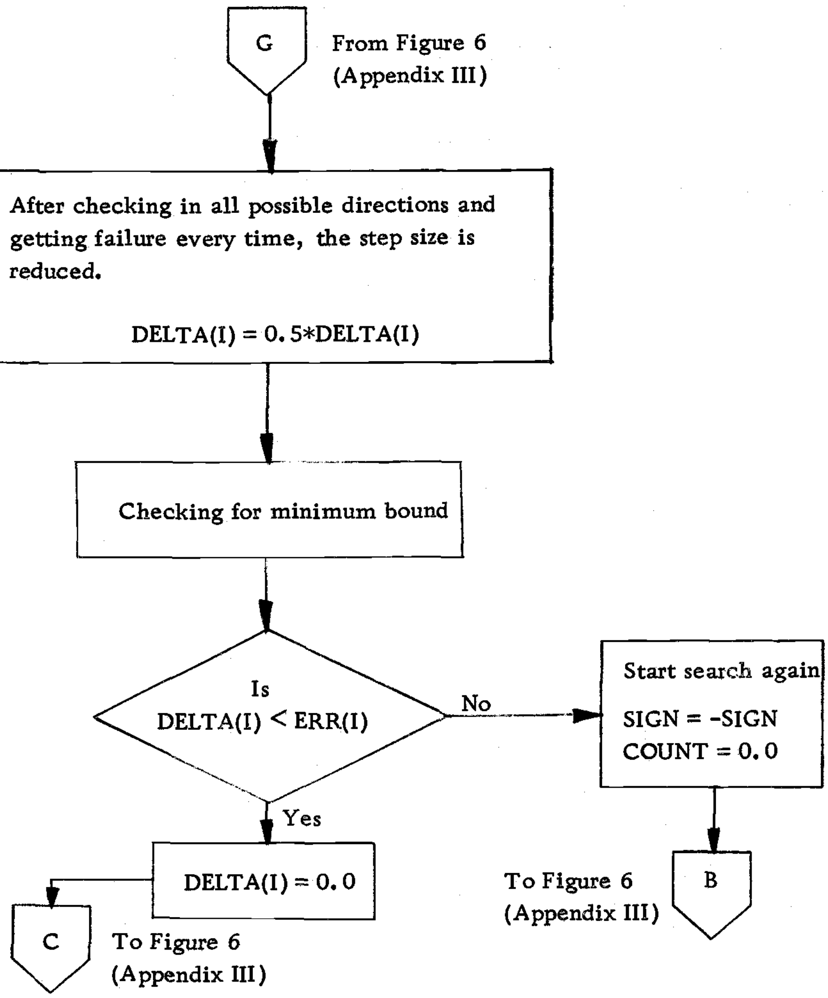

Figure 8. Decrease in the step size.

APPENDIX V

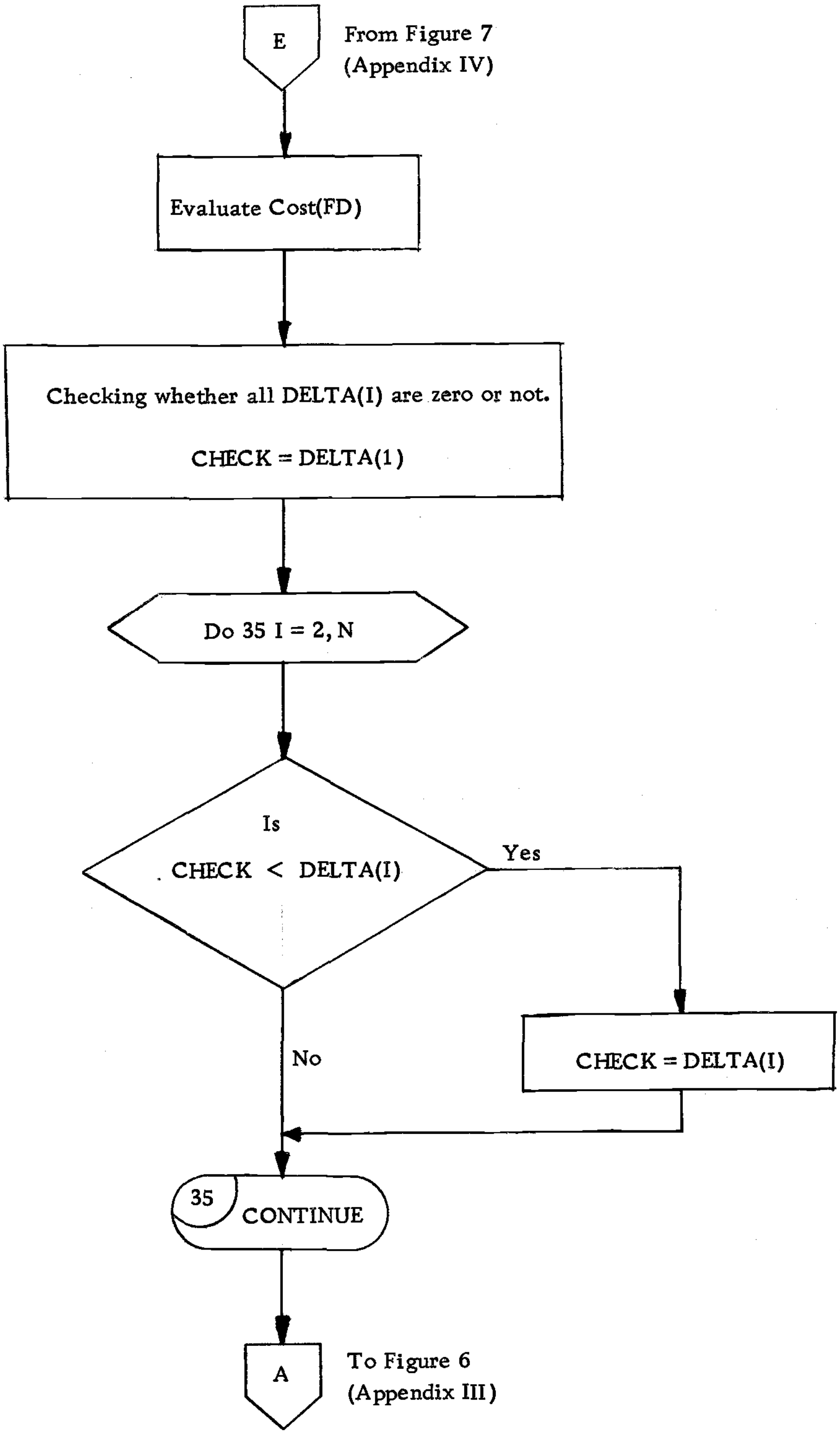

Figure 9. End of the search.

```
      PROGRAM DIRECT SEARCH
      COMMON A,SAMDA,CIN,SIGMA,PENAL,X,KAM
      DIMENSION X(2),DELT(2),ERR(2),DELTA(2),FIGN(2)
C    **DEFINITION OF THE TERMS USED**
C     A=REORDERING COST
C     SAM=SAMDA=AVERAGE DEMAND DURING A YEAR
C     CIN=HOLDING COST PER UNIT PER YEAR
C     SIG=SIGMA=STANDARD DEVIATION OF LEAD TIME DEMAND
C     PEN=PENAL=STOCKOUT PENALTY PER UNIT OUT OF STOCK
C     X(1)=VALUE FOR THE ORDER QUANTITY Q, OUR FIRST VARIABLE
C     X(2)=VALUE FOR THE SECOND VARIABLE T
C     N=NUMBER OF VARIABLES
C     XA=INITIAL VALUE FOR X(1)
C     XB=INITIAL VALUE FOR X(2)
C     DELT(1)=INITIAL STEP SIZE FOR VARIABLE Q
C     DELT(2)=INITIAL STEP SIZE FOR VARIABLE T
C     ERR(1)=MINIMUM STEP SIZE FOR FIRST VARIABLE
C     ERR(2)=MINIMUM STEP SIZE FOR SECOND VARIABLE
C    **DATA INPUT**
      N=TTYIN(2HN=)
      A=TTYIN(2HA=)
      SAMDA=TTYIN(4HSAM=)
      CIN=TTYIN(4HCIN=)
      SIGMA=TTYIN(4HSIG=)
      PENAL=TTYIN(4HPEN=)
      X(2)=TTYIN(3HXB=)
      DO 1 I=1,N
    1 DELT(I)=TTYIN(4HDEL=)
      DO 2 I=1,N
    2 ERR(I)=TTYIN(4HERR=)
      YAM=SQRT(2.0*A*SAMDA/CIN)
      WRITE(61,46)YAM
      X(1)=TTYIN(3HXA=)
      NTIME=0
      KAM=1
      CHECK=1.0
    3 CONTINUE
C    **SEARCH FOR END**
      IF(CHECK.EQ.0.0)GO TO 36
      DO 29 I=1,N
      COUNT=0.0
      DELTA(I)=DELT(I)
      SIGN=1.0
      CALL COST(FA)
      NTIME=NTIME+1
    4 CONTINUE
      X(I)=X(I)+SIGN*DELTA(I)
      CALL COST(FB)
      NTIME=NTIME+1
      KAM=2
      IF(FA-FB)5,5,15
C    **CHANGING THE DIRECTION OF SEARCH**
    5 X(I)=X(I)-SIGN*DELTA(I)
      IF(COUNT.EQ.1.0)GO TO 7
      SIGN=-1.0*SIGN
      COUNT=1.0
      GO TO 4
C    **DECREASING THE STEP SIZE**
    7 DELTA(I)=0.5*DELTA(I)
      IF(DELTA(I).LT.ERR(I))GO TO 10
      COUNT=0.0
```

```
      SIGN=-1.0*SIGN
      GO TO 4
   10 DELTA(I)=0.0
      GO TO 28
   15 X(I)=X(I)-2.0*SIGN*DELTA(I)
C     **CHECKING FOR CONVEXITY IF THE FIRST MOVE WAS A SUCCESS,
C       OTHERWISE CHECKS IN THE OPPOSITE DIRECTION**
      CALL COST(FC)
      NTIME=NTIME+1
      IF(FC-FA)37,20,20
   20 X(I)=X(I)+2.0*SIGN*DELTA(I)
   28 CONTINUE
      FIGN(I)=SIGN
   29 CONTINUE
      DO 30 I=1,N
   30 X(I)=X(I)+FIGN(I)*DELTA(I)*0.0
C     **FINDING THE MINIMUM OF DELTA(1) AND DELTA(2)**
      CHECK=DELTA(1)
      DO 35 I=2,N
      IF(CHECK.LT.DELTA(I))GO TO 33
      GO TO 34
   33 CHECK=DELTA(I)
   34 CONTINUE
   35 CONTINUE
      GO TO 3
   36 WRITE(61,43)(X(J),J=1,N)
      CALL COST(FD)
      NTIME=NTIME+1
      WRITE(61,47)NTIME
      WRITE(61,44)FD
      GO TO 38
   37 WRITE(61,45)
   38 CONTINUE
   43 FORMAT(' THE OPTIMUM QUANTITY IS',F7.1,/
     1' AND OPTIMUM SAFETY STOCK STANDARD DEVIATION IS',F6.4)
   44 FORMAT(' THE MINIMUM COST IS',F9.1)
   45 FORMAT(' AS CONCAVE FUNCTION SO I FAIL TO FIND MIN.')
   46 FORMAT(' WILSONS QUANTITY IS',F9.4)
   47 FORMAT(' NO. OF TIMES SUBCOST IS CALLED =',I4)
      END
      SUBROUTINE COST(TAR)
      COMMON A,SAMDA,CIN,SIGMA,PENAL,X,KAM
      DIMENSION X(2),FINTEG(801)
      IF(KAM.EQ.2)GO TO 2
C     **CALCULATION OF PROB. FOR NORMAL DISTRIBUTION**
      FINTEG(801)=.0001
      H=.01
      DO 1 IX=1,800
      Z=801-IX
      M=Z
      GT=(Z-400.5)*.01
    1 FINTEG(M)=H*EXP(-.50*GT*GT)/SQRT(6.2857142857)+FINTEG(M+1)
    2 Y=(X(2)+4.01)*100.0
      N=Y
C     **CALCULATION FOR TOTAL COST**
      PHEE=FINTEG(N)
      DEPHEE=(1.0/SQRT(6.2857142857))*EXP(-.5*X(2)*X(2))
      TAC=(A*SAMDA/X(1))+CIN*(X(1)/2.0+X(2)*SIGMA)
      TAR=TAC+(PENAL*SAMDA*SIGMA*(DEPHEE-X(2)*PHEE)/X(1))
      RETURN
      END
```

APPENDIX VI

Table 9. Ratios taken on the x-axis and the y-axis in the graphical method.

Type of the Distribution	Stockout Penalty is Proportional to	Ratio on x-axis	Ratio 1 on y-axis	Ratio 2 on y-axis	Ratio 3 on y-axis	Parameter
Normal	The number of occasions out of stock	Q_w/σ	Q^*/Q_w	t^*	k^*/ICQ_w	V/A
Normal	The number of units out of stock	Q_w/σ	Q^*/Q_w	t^*	K^*/ICQ_w	$W\sigma/A$
Exponential	The number of occasions out of stock	Q_w/σ	Q^*/Q_w	t^*	K^*/ICQ_w	V/A
Exponential	The number of units out of stock	Q_w/σ	Q^*/Q_w	t^*	K^*/ICQ_w	$W\sigma/A$
Triangular	The number of occasions out of stock	Q_w	Q^*/Q_w	m_2-r^*	$\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}$	$V/(A(m_2-m_0)(m_2-m_1))$
Triangular	The number of units out of stock	Q_w	Q^*/Q_w	m_2-r^*	$\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}$	$W/(A(m_2-m_0)(m_2-m_1))$
Uniform	The number of occasions out of stock	Q_w	Q^*/Q_w	m_2-r^*	$\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}$	$V/(A(m_2-m_1))$
Uniform	The number of units out of stock	Q_w	Q^*/Q_w	m_2-r^*	$\frac{K^*}{ICQ_w} - \frac{m_2-\mu}{Q_w}$	$W/(2A(m_2-m_1))$