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Title THE EXTENDED EXPONENTIAL OF A MEASURE SPACE

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This paper defines the extended exponential of a measure space $\mathrm{Exp}\,(X, S, \mu)$, and then proves the measure theoretic analogue of the ordinary exponential law,

$$\prod_{k \in I} \mathrm{Exp}\, A_k = \mathrm{Exp}\left[\sum_{k \in I} A_k\right],$$

where $I$ is some countable index set. The results are an extension of those of D. S. Carter on the exponential of a measure space $\exp\,(X, S, \mu)$ -- manuscript to be published.

The construction begins with a totally $\sigma$-finite measure space $(X, S, \mu)$. For each ordinal number $n \leq \omega$, let $X^n$ be the set of ordered sequences of length $n$ in $X$; in particular $X^0 = \{\emptyset\}$. Let $\langle X^n \rangle$ be the set of unordered sequences of length $n$ in $X$; that is, the set of equivalence classes in $X^n$, two ordered sequences being equivalent if one is a rearrangement of the other. Carter defines the exponential of $X$ to be the disjoint union of finite

unordered sequences, $\exp X = \sum_{n < \omega} \langle X^n \rangle$. This notation is motivated by the following exponential law: Let $Y$ and $Z$ be a pair of disjoint sets. Then there is a natural one-to-one map $\overline{\phi}$ on the Cartesian product $\exp Y \cdot \exp Z$ onto $\exp(Y + Z)$. Carter shows how the measure space structure $S, \mu$ on $X$ serves to induce a corresponding structure $\exp S, \exp \mu$ on $\exp X$. The resulting measure space, $\exp(X, S, \mu) = (\exp X, \exp S, \exp \mu)$ satisfies the conditions,

a) $(\exp \mu)(\exp X) = \exp(\mu(X))$

b) the exponential law holds in the sense that if $(X, S, \mu) = (Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)$, then the function $\overline{\phi}$ above is a measure isomorphism from the direct product

$$\exp(Y, S_Y, \mu_Y) \cdot \exp(Z, S_Z, \mu_Z)$$

to

$$\exp[(Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)] .$$

In this paper the extended exponential of $X$ is defined by adjoining the unordered infinite sequences $\langle X^\omega \rangle$ to $\exp X$; that is, $\operatorname{Exp} X = \sum_{n \leq \omega} \langle X^n \rangle$. The measure space structure is also extended to obtain the extended exponential $\operatorname{Exp}(X, S, \mu)$. The measure extension is trivial; $(\operatorname{Exp}\mu)(\langle X^{\omega}\rangle) = 0$.

Now let $\{(X_n, S_n, \mu_n): n \in I\}$ be a family of pairwise disjoint measure spaces in $(X, S, \mu)$ such that

$$(X, S, \mu) = \sum_{n \in I} (X_n, S_n, \mu_n)$$

where $I$ is either a finite or countable index set. The exponential law is then extended to show that there is a natural measurable isomorphism $\phi$ from $\prod_{n \in I} \operatorname{Exp}(X_n, S_n)$ to $\operatorname{Exp}[\sum_{n \in I} (X_n, S_n)]$. In the event $I$ is finite, it is shown that $\phi$ is actually a measure isomorphism from $\prod_{n \in I} \operatorname{Exp}(X_n, S_n, \mu_n)$ to $\operatorname{Exp}[\sum_{n \in I} (X_n, S_n, \mu_n)]$. In the case $I$ is infinite it is shown that $\phi$ need not be a measure isomorphism, for the infinite product measure $(\prod_{n \in I} \operatorname{Exp}\mu_n)$ may not exist.

# THE EXTENDED EXPONENTIAL OF A MEASURE SPACE

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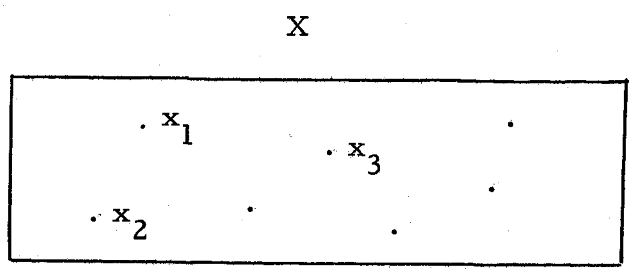
# THE EXTENDED EXPONENTIAL OF A MEASURE SPACE

## CHAPTER 1. INTRODUCTION

In his studies of the mathematical foundations of statistical mechanics, D.S. Carter [1] has constructed what he calls "the exponential of a measure space." To visualize this construction in terms of a simple example, consider a random experiment in which the outcome is a variable but finite number of points $x_1, x_2, \cdots, x_n$ in a plane rectangle $X$. This outcome might be represented by the <u>ordered</u> sequence

$$x = (x_1, x_2, \cdots, x_n) ;$$




that is by a point in the $n^{th}$ direct power $X^n$ of $X$. The event that there are exactly $n$ points in $X$ would then be represented by the set of all such sequences of length $n$; i.e., by the set $X^n$. Now suppose that the order of the points in unimportant, so that the outcome is the same if $(x_1, \cdots, x_n)$ is replaced by any rearrangement $(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$, where $i_1, i_2, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$. Then it is natural to represent the outcome not by the single sequence $x$, but by the unordered sequence $\langle x\rangle$, consisting of all rearrangements of $x$.

Definition 1.1. Let $x = (x_1, x_2, \cdots, x_n) \in X^n$. Then

$$\langle x\rangle = \{(x_{i_1}, x_{i_2}, \cdots, x_{i_n}) : i_1, i_2, \cdots, i_n \text{ is a permutation of } 1, 2, \cdots, n\}.$$

The event that there are exactly $n$ points in $X$ is now represented by the set of all such unordered n-tuples, called the symmetric $n^{th}$ power of $X$, and denoted by $\langle X^n\rangle$.

Definition 1.2. $\langle X^n\rangle = \{\langle x\rangle : x \in X^n\}$.

To include the event that there are no points in $X$, one adjoins a special set $\langle X^0\rangle$, consisting of a single element $e_X$ (for "empty sequence" or "unordered sequence of length zero" in $X$).

Definition 1.3. $\langle X^0\rangle = \{e_X\}$.

The event that there is some finite number $n$,

$$n = 0 \text{ or } 1 \text{ or } 2 \cdots.$$

of points in $X$, which is the universal event for the experiment in question, is represented by the union of the symmetric powers $<X^n>$. Carter calls this disjoint union the "exponential of $X$," written[1]

$$\exp X = \sum_{n<\omega} <X^n> .$$

The reason for this terminology lies in the following analog of the familiar exponential law

$$\exp(a+b) = (\exp a)(\exp b).$$

Let $X$ be partitioned into the disjoint union of two subsets $Y$, $Z$. Let the exponential construction be carried out for the sets $Y$ and $Z$ separately, as it was carried out above for $X$. Then there is a natural equivalence between $\exp X = \exp(Y+Z)$ and the direct product set $\exp Y \cdot \exp Z$. This equivalence is given by the concatenation function $\bar{\phi}$ which associates with each ordered pair $(y, z) \epsilon \exp Y \cdot \exp Z$ the concatenation $\bar{\phi}(y, z)$, defined by the

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[1] Here and elsewhere throughout the discussion, disjoint unions will be denoted as sums. Thus we write $A+B$ or $\sum_{n=0}^{\infty} A_n$ iff the intersections $A \cap B$, or $A_i \cap A_j$ $(i \neq j,\ j \epsilon \omega)$ are void.

following equations:

Definition 1.4.

$$\overline{\phi}(\langle y_1, y_2, \cdots, y_m\rangle, \langle z_1, z_2, \cdots, z_n\rangle)$$

$$= \langle y_1, y_2, \cdots, y_m, z_1, \cdots, z_n\rangle$$

$$\overline{\phi}(e_y, \langle z_1, z_2, \cdots, z_n\rangle) = \langle z_1, z_2, \cdots, z_n\rangle$$

$$\overline{\phi}(\langle y_1, \cdots, y_m\rangle, e_z) = \langle y_1, y_2, \cdots, y_m\rangle$$

$$\overline{\phi}(e_y, e_z) = e_x$$

The reader will easily verify that $\overline{\phi}$ is a set-theoretic equivalence between $\exp Y \cdot \exp Z$ and $\exp(Y + Z)$.

Clearly, the exponential construction can be carried out for arbitrary sets, and the "exponential law" is generally valid.

Now suppose the set $X$ carries the structure of a totally $\sigma$-finite <u>measure space</u> -- such as plane Lebesgue measure in the example of the rectangle. It turns out that this measure on the set $X$ induces a natural measure on $\exp X$. Furthermore, the exponential law above extends directly to such measure spaces, in the sense that the concatenation function $\overline{\phi}$ is a measure isomorphism from the <u>measure space</u> $\exp(Y + Z)$ into the direct product measure space $(\exp Y) \cdot (\exp Z)$.

The practical significance of these results stems from the fact that the set $\exp X$ provides the sample space (set of outcomes) of random experiments as described above. The $\sigma$-algebra of measurable sets associated with the measure space $\exp X$ then serves an an algebra of events. Thus, $\exp X$ becomes a probability space for an experiment by assigning an appropriate probability measure to this $\sigma$-algebra. To appreciate the significance of the "exponential law", let $X$ again be decomposed into a disjoint union of subsets, $X = Y + Z$. Any random experiment on $X$ induces separate experiments on $Y$ and $Z$, in which only those points occurring in $Y$ alone, or in $Z$ alone, are observed. The exponential law asserts that the sample space for the experiment on $X$ factors into the direct product of the sample spaces for the induced experiments on $Y$ and $Z$. This shows that the standard techniques of probability theory for product spaces are applicable.

The object of this dissertation is to extend Carter's results to include denumerably infinite sequences, corresponding to outcomes in which denumerably many points of $X$ occur. Chapter 2 gives a brief summary of those results from measure theory pertinent to this paper and Chapter 3 a summary of Carter's results. In Chapter 4 the set $\langle X^{\omega} \rangle$ of unordered infinite seqences is defined, and adjoined to $\exp X$, giving the "extended exponential of $X$", namely:

$$\text{Exp } X = \sum_{n \le \omega} \langle X^n \rangle .$$

A natural measurable space structure, Exp (X, S) is also defined for Exp X.

In Chapters 5 and 6 we show that the exponential law extends to the set Exp X and the measurable space Exp (X, S). Indeed, the exponential law now holds not only for decompositions of X into a pair or a finite number of subsets, but for arbitrary denumerable disjoint decompositions, by analogy with the law

$$\text{Exp } \left( \sum_{n=1}^{\infty} a_i \right) = \prod_{n=1}^{\infty} (\text{Exp } a_i) .$$

As far as the exponential measure itself is concerned, the extension is trivial, for $\langle X^{\omega} \rangle$ is simply assigned measure zero. (This implies that for an experiment in which denumerable distributions of points can occur with positive probability, the probability measure cannot be absolutely continuous with respect to the natural measure on Exp X).

In Chapter 7 we show under what conditions the exponential law can be extended to the measure space Exp (X, S, $\mu$).

# CHAPTER 2. MATHEMATICAL PRELIMINARIES

This chapter gives a brief review of those definitions and theorems from measure theory which are pertinent to this paper, and establishes notational convention.

## I. Measure Spaces

A ring $R$ in a set $X$ is a non-empty class of subsets of $X$ which is closed under differences and under finite unions. The ring is an algebra if $X \in R$. A $\sigma$-ring $S$ in a set $X$ is a non-empty class of subsets of $X$ which is closed under countable unions and under differences. If $\mathcal{A}$ is any non-empty class of subsets of $X$, then the $\sigma$-ring $S(\mathcal{A})$ is the smallest $\sigma$-ring in $X$ which contains $\mathcal{A}$. It is also called the $\sigma$-ring generated by $\mathcal{A}$.

A measurable space $(X,S)$ is a set $X$ together with a $\sigma$-ring $S$ of subsets of $X$. A real-valued function $\mu$ on a family $C$ of sets is finitely additive if for each pair of disjoint sets $A$ and $B$ belonging to $C$,

(a) $$\mu(A \cup B) = \mu(A) + \mu(B).$$

The function $\mu$ is countably additive if for each pairwise disjoint sequence of sets $\{A_n\}$ in $C$ whose union is in $C$, we have

(b) $$\mu\left(\bigcup_{n\in\omega} A_n\right) = \sum_{n\in\omega} \mu(A_n) .$$

<u>Convention 2.1.</u> Here and elsewhere throughout the discussion, disjoint unions will be denoted as sums. Thus, we write $A + B$ or $\sum_{n\in\omega} A_n$ iff the intersections $A \cap B$ or $A_i \cap A_j$ $(i \neq j;\ i, j \in \omega)$ are void. Then, formulas (a) and (b) become

$$\mu(A + B) = \mu(A) + \mu(B)$$

and

$$\mu\left(\sum_{n\in\omega} A_n\right) = \sum_{n\in\omega} \mu(A_n) .$$

A measure $\mu$ is an extended real valued, non-negative countably additive set function $\mu$ defined on a ring $R$ such that $\mu(\phi) = 0$. A measure $\mu$ on a ring $R$ is <u>finite</u> if $\mu(E) < \infty$ for every $E \in R$. If $X \in R$ and $\mu(X) < \infty$, then $\mu$ is called <u>totally finite</u>. If $E \in R$ and there exists a sequence $\{E_n\}$ of sets in $R$ such that $\mu(E_n) < \infty$ for each $n = 1, 2, \cdots$ and $E \subset \bigcup_{n\in\omega} E_n$, then the measure of $E$ is said to be <u>$\sigma$-finite.</u> If $X \in R$ and $\mu(X)$ is $\sigma$-finite, $\mu$ is said to be <u>totally $\sigma$-finite.</u>

We have been talking about measures $\mu$ on a $\sigma$-ring $S$ of subsets of some set $X$. This structure has a special name. A <u>measure space</u> <u>$(X, S, \mu)$</u> is a measurable space $(X, S)$ together with a measure $\mu$ on $S$.

<u>Convention 2. 2.</u> Throughout the remainder of the paper a measure space $(X, S, \mu)$ will always be totally $\sigma$-finite, and a measurable space $(X, S)$ will always be an algebra.

We can use a measurable space $(X, S)$ to induce a $\sigma$-ring $S_Y$ on any subset $Y$ of $X$ by letting

$$S_Y = \{B \cap Y : B \in S\} .$$

Since $X \in S$, it follows that $Y \in S_Y$. <u>In our applications $Y$ will always belong to $S$.</u>

Similarly, if $(X, S, \mu)$ is a measure space, we can use $\mu$ to induce a measure $\mu_Y$ on $S_Y$ simply be letting $\mu_Y$ be the function $\mu$ restricted by $S_Y$. The measure space $(Y, S_Y, \mu_Y)$ is the <u>measure space induced on $Y$</u> by $(X, S, \mu)$.

## <u>II. Sums of Disjoint Measure Spaces</u>

Let $\{(X_k, S_k, \mu_k) : k \in I\}$ be an indexed family of disjoint measure spaces; that is, the $X_k$'s are pairwise disjoint. (For our applications, the index set $I$ will be either a finite ordinal

$N = \{1, 2, 3, \cdots, n\}$ or the first limit ordinal $\omega = \{1, 2, \cdots\}$.

We can use this indexed family to form a new measure space

$$(X, S, \mu) = \sum_{k \in I} (X_k, S_k, \mu_k)$$

called the <u>sum</u> of the family. The construction is as follows. Let

$X = \sum_{k \in I} X_k$ and $S = \sum_{k \in I}^{*} S_k$ where this sum of $\sigma$-algebras is defined by

$$\sum_{k \in I}^{*} S_k = \{ \sum_{k \in I} A_k : A_k \in S_k \quad \text{for all} \quad k \in I \} .$$

Let $\mu = \sum_{k \in I} \mu_k$ where this sum of measures is defined by

$$(\sum_{k \in I} \mu_k)(\sum_{k \in I} A_k) = \sum_{k \in I} \mu_k(A_k) .$$

It is easy to verify that $\sum_{k \in I}^{*} S_k$ is actually a $\sigma$-ring and that $\sum_{k \in I} \mu_k$ is a totally $\sigma$-finite measure on this $\sigma$-ring. Thus, $(X, S, \mu)$ is actually a measure space.

The following results, which bring out the relationship between induced spaces and sums, are easily verified.

Theorem 2.3. Consider the sum of disjoint measure spaces

$$(X, S, \mu) = \sum_{k \in I} (X_k, S_k, \mu_k)$$

as defined above. Then, for each $k \in I$, $(X_k, S_k, \mu_k)$ is the measure space induced by $(X, S, \mu)$ on the subset $X_k$.

Theorem 2.4. Let $(X, S, \mu)$ be a measure space, and $X = \sum_{k \in I} X_k$ be a partition of $X$ into disjoint measurable subsets. For each $k$, let $(X_k, S_k, \mu_k)$ be the measure space induced on $X_k$ by $(X, S, \mu)$. Then

$$(X, S, \mu) = \sum_{k \in I} (X_k, S_k, \mu_k) .$$

Remark: Let $\{(X_k, S_k) : k \in I\}$ be a sequence of disjoint measurable spaces. Let $X = \sum_{k \in I} X_k$ and $S = \sum_{k \in I}^{*} S_k$ be defined as above. Then,

$$(X, S) = \sum_{k \in I} (X_k, S_k)$$

is the measurable space $\left( \sum_{k \in I} X_k, \sum_{k \in I}^{*} S_k \right)$, and the analogs to Theorems 2.3 and 2.4 for measurable spaces are true.

## III. Products of Measure Spaces

Again let $I$ be either a finite ordinal $N = \{1, 2, \cdots, n\}$ or the first limit ordinal $\omega = \{1, 2, \cdots\}$. Let $\{(X_k, S_k) : k \in I\}$ be a family of measurable spaces indexed by the set $I$. We can use the family $\{(X_k, S_k) : k \in I\}$ to form a new measurable space

$$\prod_{k \in I} (X_k, S_k) = \left( \prod_{k \in I} X_k, \prod_{k \in I} S_k \right)$$

called the <u>product</u> of the family. To construct this product let

$\prod_{k \in I} S_k$ be the $\sigma$-ring $S(C)$ generated by the family of <u>measurable rectangles</u> $C$ in $\prod_{k \in I} X_k$ where

$$C = \left\{ \prod_{k \in I} A_k : A_k \in S_k \right\} .$$

If $I = N$, $C$ will be denoted $C^n$. Clearly $\left( \prod_{k \in I} X_k, \prod_{k \in I} S_k \right)$ is a measurable space. In the event $I = \omega$, it can be shown [4] that $S(C) = S(C^{\omega})$ where $C^{\omega}$ is the class of <u>measurable rectangular cylinders</u> defined by

$$C = \{ \prod_{k \in I} A_k : A_k \in S_k \quad \text{and} \quad A_k = X_k \quad \text{for all but finitely many} \quad k \}.$$

This result is important in the sequel.

A special notational convention is needed for the product of two $\sigma$-rings. It would be natural to write $S_1 \cdot S_2$ for $\prod_{k=1}^{2} S_k$. However, we wish to reserve this notation for

$$S_1 \cdot S_2 = \{A_1 \cdot A_2 : A \in S_1, \quad B \in S_2 \} .$$

Therefore we will denote the product $\sigma$-ring by

$$S_1 * S_2 = S(S_1 \cdot S_2) .$$

If $(X_1, S_1)$ and $(X_2, S_2)$ are measurable spaces, the product space $\prod_{k=1}^{2} (X_k, S_k)$ will be denoted by $(X_1, S_1) \cdot (X_2, S_2)$.

Now let $\{ (X_k, S_k, \mu_k) : k \in I \}$ be a family of measure spaces. With proper restrictions on the measures $\mu_k$, we can use the family to form a new measure space

$$\prod_{k \in I} (X_k, S_k, \mu_k) = ( \prod_{k \in I} X_k, \prod_{k \in I} S_k, \prod_{k \in I} \mu_k) .$$

Here $\prod_{k \in I} S_k$ is defined as above. The construction of $\prod_{k \in I} \mu_k$ differs according to whether $I = N$ or $I = \omega$.

We start with the case $I = N$. In this case the only restriction is that each $\mu_k$ shall be $\sigma$-finite, and this condition is guaranteed by Convention 2. 2. Consider the set function $\prod_{k \in N} \mu_k$, defined on the measurable rectangles of $X^n$ by the equation

$$\left(\prod_{k \in N} \mu_k\right)\left(\prod_{k \in N} A_k\right) = \prod_{k \in N} \mu_k(A_k),$$

where $\prod_{k \in N} \mu_k(A_k) = 0$ if any factor $\mu_k(A_k) = 0$. This function admits a unique extension to a $\sigma$-finite measure on $\prod_{k \in N} S_k$ (see [4]). This is the <u>product measure</u>, also denoted by $\prod_{k \in N} \mu_k$.

The infinite product measure $\prod_{k \in \omega} \mu_k$ is defined similarly except that undefined infinite products of real numbers must be avoided. For this reason we assume that each $(X_k, S_k, \mu_k)$ is a probability space -- that is, $\mu_k(X_k) = 1$ for each $k \in \omega$. As in the case of finite product measures, the set function $\prod_{k \in \omega} \mu_k$ defined on the measurable rectangles of $X^\omega$ by

$$\Big(\prod_{k\in\omega}\mu_k\Big)\Big(\prod_{k\in\omega}A_k\Big) = \prod_{k\in\omega}\mu_k(A_k)$$

admits a unique extension to a probability measure on $\prod_{k\in\omega} S_k$, called the infinite product measure $\prod_{k\in\omega}\mu_k$.

## IV. Isomorphisms

Multiple use will be made of the word isomorphic. Two sets $X$ and $Y$ will be called isomorphic if there exists a one-to-one map $\phi$ from $X$ onto $Y$. The function $\phi$ will be called a set isomorphism. In the event $X$ is isomorphic to $Y$ via the function $\phi$ we shall write

$$X \underset{\phi}{\simeq} Y .$$

Let $(X, S)$ and $(Y, T)$ be measurable spaces. A function $\phi$ from $X$ into $Y$ is measurable iff $\phi^{-1}(B) \in S$ for each $B \in T$. The function $\phi$ is a measurability isomorphism if both $\phi$ and $\phi^{-1}$ are measurable and $\phi$ is a set isomorphism. Two measure spaces $(X, S)$ and $(Y, T)$ will be called measurably isomorphic if there exists a measurability isomorphism between them. When $\phi$ is a measurable isomorphism from $(X, S)$ to $(Y, T)$ we shall write

$$(X, S) \underset{\phi}{\simeq} (Y, T) .$$

Let $(X, S, \mu)$ and $(Y, T, \nu)$ be two measure spaces which are measurably isomorphic, and let $\phi$ be a measurability isomorphism between the two spaces. The function $\phi$ is said to be <u>measure preserving</u> provided

$$\mu(A) = \nu[\phi(A)]$$

for each $A \epsilon S$. A measurability isomorphism which is measure preserving is called a <u>measure isomorphism</u>. Two measure spaces $(X, S, \mu)$ and $(Y, T, \nu)$ will be called isomorphic if there exists a measure isomorphism between them. Whenever $\phi$ is a measure isomorphism from $(X, S, \mu)$ to $(Y, T, \nu)$ we shall write

$$(X, S, \mu) \underset{\phi}{\simeq} (Y, T, \nu) .$$

Now suppose we have three measurable spaces $(X, S)$, $(Y, T)$, $(Z, U)$, and suppose

$$(Y, T) \underset{\phi}{\simeq} (Z, U) .$$

Then there is a natural measuribility isomorphism $\sigma$ from the product space $(X, S) \cdot (Y, T)$ into the product space $(X, S) \cdot (Z, U)$ as follows.

<u>Theorem 2. 5.</u> Let $(X, S, \mu)$, $(Y, T, \nu)$ and $(Z, U, \lambda)$ be measure spaces such that

$$(Y, T, \nu) \underset{\phi}{\simeq} (Z, U, \lambda) .$$

Then the function $\sigma$ defined by

$$\sigma(x, y) = (x, \phi(y))$$

is a measure isomorphism from the product measure space $(X, S, \mu) \cdot (Y, T, \nu)$ onto the product measure space $(X, S, \mu) \cdot (Z, U, \lambda)$. That is

$$(X, S, \mu) \cdot (Y, T, \nu) \underset{\sigma}{\simeq} (X, S, \mu) \cdot (Z, U, \lambda).$$

Proof: (outline of proof)

Clearly $\sigma$ is a set isomorphism such that

$$\sigma^{-1}(x, y) = (x, \phi^{-1}(z)) .$$

To show that $\sigma$ is a measurability isomorphism, let $S*T$ denote the product $\sigma$-ring and let $A \epsilon S$ and $B \epsilon T$. Then

$$\sigma(A \cdot B) = A \cdot \phi(B) \epsilon S*U.$$

But then Theorem 2. 6 (see below) implies that $\sigma(S*T) \subset S*U$.

Conversely, if $A \epsilon S$ and $C \epsilon U$, then

$$\sigma^{-1}(A \cdot C) = A \cdot \phi^{-1}(C) \epsilon S{*}T.$$

Again applying Theorem 2. 6, we have $\sigma^{-1}(S{*}T) = S{*}U$; hence $\sigma$ is a measurability isomorphism.

To show $\sigma$ is a measure isomorphism, notice that there is a one-to-one correspondence between the measurable rectangles of $S{*}T$ and those of $S{*}U$ through the equation

$$\sigma\,(A \cdot B) = A \cdot \phi(B).$$

Moreover, since $\nu(A) = \lambda(\phi(B))$, we have

$$(\mu \cdot \nu)\,(A \cdot B) = (\mu \cdot \lambda)\,[\sigma(A \cdot B)] .$$

Since the measures $\mu \cdot \nu$ and $\mu \cdot \lambda$ are characterized by their values on the measurable rectangles, it follows that

$$(\mu \cdot \nu)(E) = (\mu \cdot \lambda)\,[\sigma(E)]$$

for every $E \epsilon S{*}T$.

## V. Probability Spaces

A totally $\sigma$-finite measure space $(X, S, \mu)$ such that $\mu(X) = 1$ is called a probability space. The measure $\mu$ is called a probability and the sets belonging to $S$ are called events.

## VI. Some Special Theorems

This section contains some theorems which are of use in the sequel but which are not usually found in textbooks. For this reason their proofs are sketched.

Theorem 2.6. Given a set $X$ and a measurable space $(Y, S(\mathcal{A}))$ where $\mathcal{A}$ is any non-empty family of subsets of $Y$, let $f$ be a function from $X$ into $Y$. Let

$$f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$$

$$f^{-1}(S(\mathcal{A})) = \{f^{-1}(A) : A \in S(\mathcal{A})\}.$$

Then

$$f^{-1}(S(\mathcal{A})) = S(f^{-1}(\mathcal{A})).$$

Proof: Since $\mathcal{A} \subset S(\mathcal{A})$, $f^{-1}(\mathcal{A}) \subset f^{-1}(S(\mathcal{A}))$. But, since inverse functions preserve Boolean operations and countable sums, $f^{-1}(S(\mathcal{A}))$ is a $\sigma$-ring. Thus

$$S(f^{-1}(\mathcal{A})) \subset f^{-1}(S(\mathcal{A})).$$

To prove the converse, let

$$C = \{E \subset Y : E \in S(\mathcal{A}) \text{ and } f^{-1}(E) \in S(f^{-1}(\mathcal{A}))\}.$$

It is easily shown that $C$ is a $\sigma$-ring.

But $\mathcal{A} \subset C$; therefore $S(\mathcal{A}) \subset C$.

Thus $f^{-1}(S(\mathcal{A})) \subset S(f^{-1}(\mathcal{A}))$.

<u>Theorem 2.7.</u> Let $(X, S(\mathcal{A}))$ be a measurable space, where $\mathcal{A}$ is a non-empty family of subsets of $X$. Let $Y$ be a set, and let $\mathcal{A} \cdot Y = \{A \cdot Y : A \in \mathcal{A}\}$ and $S(\mathcal{A}) \cdot Y = \{A \cdot Y : A \in S(\mathcal{A})\}$. Then

$$S(\mathcal{A} \cdot Y) = S(\mathcal{A}) \cdot Y .$$

Proof: Let $\pi$ be the projection of $X \cdot Y$ onto $X$.

That is $\pi(x, y) = x$ for each $(x, y) \in X \cdot Y$.

Then, for each $A \in \mathcal{A}$, $\pi^{-1}(A) = A \cdot Y$.

Thus, $\pi^{-1}(\mathcal{A}) = \mathcal{A} \cdot Y$, so that

$$S(\pi^{-1}(\mathcal{A})) = S(\mathcal{A} \cdot Y) .$$

But by Theorem 2.6,

$$S(\pi^{-1}(\mathcal{A})) = \pi^{-1}(S(\mathcal{A})) = S(\mathcal{A}) \cdot Y .$$

Thus

$$S(\mathcal{A} \cdot Y) = S(\mathcal{A}) \cdot Y.$$

<u>Theorem 2.8.</u> Let $\{(X_n, S(\mathcal{A}_n)) : n \in N\}$ be a sequence of disjoint measurable spaces indexed by a set $N$. For each $n \in N$, let

$\mathcal{a}_n$ be a non-empty family of subsets of $X_n$. Let

$$\sum_{n\in N}^{*} S(\mathcal{a}_n) = \{ \sum_{n\in N} A_n : A_n \in S(\mathcal{a}_n)\}$$

Then

$$S(\sum_{n\in N} \mathcal{a}_n) = \sum_{n\in N}^{*} S(\mathcal{a}_n) .$$

Proof: Since $\mathcal{a}_n \subset S(\mathcal{a}_n)$ for each $n \in N$, $\sum_{n\in N} \mathcal{a}_n \subset \sum_{n\in N}^{*} S(\mathcal{a}_n)$.
We know (see Section III) that $\sum_{n\in N}^{*} S(\mathcal{a}_n)$ is a $\sigma$-ring.
Thus, $S(\sum_{n\in N} \mathcal{a}_n) \subset \sum_{n\in N}^{*} S(\mathcal{a}_n)$.

Conversely, since $\mathcal{a}_k \subset \sum_{n\in N} \mathcal{a}_n$ for each $k \in N$,

$$S(\mathcal{a}_k) \subset S(\sum_{n\in N} \mathcal{a}_n).$$

Therefore, $\sum_{k\in N}^{*} S(\mathcal{a}_k) \subset S(\sum_{n\in N} \mathcal{a}_n)$.

There is a similar result for products, namely:

<u>Theorem 2. 9.</u> Let $(X, S(\mathcal{A}))$ and $(Y, S(\mathcal{A}))$ be two measurable spaces where $\mathcal{A}$ and $\mathcal{B}$ are non-empty families of subsets of $X$ and $Y$ respectively. Suppose $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Then

$$S(\mathcal{A}) * S(\mathcal{B}) = S(\mathcal{A} \cdot \mathcal{B})$$

where $S(\mathcal{A}) * S(\mathcal{B})$ is the product $\sigma$-ring, and

$$\mathcal{A} \cdot \mathcal{B} = \{A \cdot B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Proof: Since $\mathcal{A} \cdot \mathcal{B} \subset S(\mathcal{A}) * S(\mathcal{B})$, it follows that $S(\mathcal{A} \cdot \mathcal{B}) \subset S(\mathcal{A}) * S(\mathcal{B})$. To prove that $S(\mathcal{A}) * S(\mathcal{B}) \subset S(\mathcal{A} \cdot \mathcal{B})$, it suffices, in view of the definition of the product $\sigma$-ring, to show that $S(\mathcal{A}) \cdot S(\mathcal{B}) \subset S(\mathcal{A} \cdot \mathcal{B})$. Let $E = A \cdot B$, where $A \in S(\mathcal{A})$ and $B \in S(\mathcal{B})$. Then

$$E = (A \cdot Y) \cap (X \cdot B)$$

But $A \cdot Y \in S(\mathcal{A}) \cdot Y = S(\mathcal{A} \cdot Y)$ by Theorem 2. 7, and $X \cdot B \in X \cdot S(\mathcal{B}) = S(X \cdot \mathcal{B})$ by the same theorem. Since $S(\mathcal{A} \cdot Y) \subset S(\mathcal{A} \cdot \mathcal{B})$ and $S(X \cdot \mathcal{B}) \subset S(\mathcal{A} \cdot \mathcal{B})$, we have $A \cdot Y$ and $X \cdot B \in S(\mathcal{A} \cdot \mathcal{B})$. Thus

$$E = (A \cdot Y) \cap (X \cdot B) \in S(\mathcal{A} \cdot \mathcal{B})$$

so that

$$S(\mathcal{A}) \cdot S(\mathcal{B}) \subset S(\mathcal{A} \cdot \mathcal{B}).$$

# CHAPTER 3. THE EXPONENTIAL OF A MEASURE SPACE

This chapter gives a brief summary of the exponential of a measure space as developed by D. S. Carter. The construction requires a measure space $(X, S, \mu)$ where, according to Convention 2. 2, $S$ is a $\sigma$-algebra in $X$ and $\mu$ is a totally $\sigma$-finite measure. Recall from Chapter 1 that the symmetric $n^{th}$ power $<X^n>$ of a set $X$ is

$$\{<x> : x \in X^n\}$$

where $<x>$ is the unordered sequence of length $n$ corresponding to $x$ (see Definition 1. 1); and that the exponential of a set $X$, written $\exp X$, is the disjoint union of symmetric $n^{th}$ powers of $X$. That is

$$\exp X = \sum_{n \in \omega} < X^n> \ .$$

Since to each point in $X^n$ there corresponds a unique point in $<X^n>$, there exists a naturally induced function $f^n$ from $X^n$ onto $<X^n>$, given by

<u>Definition 3. 1.</u> For each $x \in X^n$, the function $f^n$ maps $x$ onto $<x>$. That is,

$$f^n(x) = \langle x \rangle .$$

To keep notation compact, multiple use will be made of the symbol $\langle\,\rangle$ throughout the sequel according to the following convention.

Convention 3.2.

a. Let $x \in X^n$. Then $\langle x \rangle$ denotes the unordered sequence of length $n$ corresponding to $x$.

b. Let $A \subset X^n$ $(n \in \omega)$. Then

$$\langle A \rangle = \{\langle x \rangle : x \in A\} .$$

c. Let $C$ be a family of subsets of $\sum_{n \in \omega} X_n$. Then

$$\langle C \rangle = \{\langle A \rangle : A \in C\} .$$

That is, $\langle\,\rangle$ applied to a point in $X^n$ means "symmterize" the point, $\langle\,\rangle$ applied to a set means symmetrize every point in the set, and $\langle\,\rangle$ applied to a family of sets means apply $\langle\,\rangle$ to every set in the family.

Let $C_X^n$ be the family of all measurable rectangles in $X$. That is,

Definition 3.3. $$C_X^n = \{ \prod_{k=1}^{n} A_k : A_k \in S\} , \quad n \neq 0 .$$

Then $S(C_X^n) = S^n$ is the ordinary product $\sigma$-ring of subsets of $X^n$ generated by the family $C_X^n$ (2; III). Since $S$ is an algebra, $S^n$

is also an algebra. We now let the family $< C_X^n >$ (see Convention 3.2) generate a $\sigma$-algebra $S < C_X^n >$ for $< X^n >$.

Another $\sigma$-algebra is $S^n$, defined by

Definition 3.4. For each $n \neq 0$

$$\hat{S}^n = \{A \subset < X^n > : (f^n)^{-1}(A) \in S^n\}.$$

$\hat{S}^n$ is clearly a $\sigma$-algebra since inverse functions preserve unions and differences. A third possibility is $< S^n >$ (see Convention 3.2) provided $< S^n >$ is a $\sigma$-ring. Carter not only proves $< S^n >$ is a $\sigma$-ring but also

Theorem 3.5. For each $n \in \omega$, $n \neq 0$

$$S < C_X^n > = \hat{S}^n = < S^n > .$$

He uses this result to prove the isomorphism theorem.

An analogous theorem is, as yet, unavailable in the infinite product case. However, the difficulty can be by-passed by avoiding the infinite analogue of Definition 2.4 (see Chapter 4).

In the special case $n = 0$, the only subsets of $< X^0 > = \{e_X\}$ are $\emptyset$ and $\{e_X\}$. Thus, if we define $< S^0 >$ by

Definition 3.6. $$< S^0 > = \{\emptyset, \{e_X\}\},$$

then $<S^0>$ is trivially a $\sigma$-algebra. We shall, on occasion, also denote $<S^0>$ by $<C^0>$ or $S<C^0>$.

The machinery is now available to obtain a $\sigma$-ring for $\exp X$. Referring to 2; II, the sum

$$\sum_{n \in \omega}^{*} <S^n> \;=\; \{\sum_{n \in \omega} A_n : A_n \in <S^n>\}$$

is a $\sigma$-algebra for $\exp X$. Since by Definition 3.5 $<S^n> = S<C_X^n>$ for each $n$, $\sum_{n \in \omega}^{*} <S^n> = \sum_{n \in \omega}^{*} S<C_X^n>$. We give this $\sigma$-algebra the special name $\exp S$.

Definition 3.7. $\exp S = \sum_{n \in \omega}^{*} S<C_X^n>$.

The pair $(\exp X, \exp S)$ is a measurable space denoted by $\exp (X, S)$.

Definition 3.8. $\exp (X, S) = (\exp X, \exp S)$

Convention 3.9. When it is necessary to emphasize the fact that $S$ is a $\sigma$-algebra of subsets of $X$, a subscript $X$ will be appended to $S$. Thus $(X, S_X)$ is the set $X$ together with the $\sigma$-ring $S_X$ in $X$. Using this convention we write $\exp (X, S_X)$.

Now let $X$ be decomposed into a disjoint union, and let $\phi$

be the concatenation function of Definition 1.4. Then one easily obtains

Theorem 3.10. Let $X = Y + Z$ be any decomposition of $X$ into a pair of disjoint, measurable subsets. The function $\bar{\phi}$ is a set isomorphism of $\exp Y \cdot \exp Z$ into $\exp (Y + Z)$. That is

$$\exp Y \cdot \exp Z \underset{\bar{\phi}}{\simeq} \exp (Y + Z).$$

But $\bar{\phi}$ is more than just a set isomorphism. It is actually a measurability isomorphism (see 2; IV). To state this precisely, consider the measurable spaces $(Y, S_Y)$ and $(Z, S_Z)$ induced in $Y$ and $Z$ by $(X, S)$. Let $\exp (Y, S_Y) \cdot \exp (Z, S_Z)$ denote the product of the measurable spaces $\exp (Y, S_Y)$ and $\exp (Z, S_Z)$ according to 2; III. Then Carter has established

Theorem 3.11. Let $(X, S) = (Y, S_Y) + (Z, S_Z)$. Then $\bar{\phi}$ is a measurability isomorphism of

$$\exp (Y, S_Y) \cdot \exp (Z, S_Z) \quad \text{into} \quad \exp [(Y, S_Y) + (Z, S_Z)] .$$

Thus,

$$\exp (Y, S_Y) \cdot \exp (Z, S_Z) \underset{\bar{\phi}}{\simeq} \exp [(Y, S_Y) + (Z, S_Z)] .$$

Our ultimate goal is to do probability theory. For this reason we shall refer to members of $S_X$, $S_Y$, $S_Z$, etc. as <u>events.</u> The significance of the isomorphism theorem is that it enables us to "factor" an event occurring in $X$ into an event occurring in $Y$ alone and an event occurring in $Z$ alone. That is, a random experiment on $X$ induces separate experiments in $Y$ and $Z$.

Meanwhile, there is available a very natural measure $<\mu>$ for $\exp(X,S)$. Let $(X^n, S^n, \mu^n)$ denote the product of the measure space $(X,S,\mu)$ n-times. That is, $(X^n, S^n, \mu^n) = \prod_{t=1}^{n} (X_t, S_t, \mu_t)$ where $X_t = X$, $S_t = S$, and $\mu_t = \mu$ for each $t = 1, 2, \cdots, n$. Then there is a measure $\hat{\mu}^n$ on $(<X^n>, <S^n>)$ defined by

<u>Definition 3.12.</u> Let $A \in <S^n>$. Define the set function $\hat{\mu}^n$ on $<S^n>$ by

$$\hat{\mu}^n(A) = \frac{\mu^n\{(f^n)^{-1}(A)\}}{n!} \quad \text{when} \quad n \neq 0 .$$

In the special case $n = 0$, let $\hat{\mu}^0[\{e_X\}] = 1$ and $\hat{\mu}^0(\emptyset) = 0$.

It follows easily [4] that $\hat{\mu}^n$ is a measure for each $n \in \omega$. We now define $<\mu>$.

<u>Definition 3.13.</u> Let $A = \sum_{n \in \omega} A_n$ be any set in $\exp S$ (see Definition 3.7). Define the set function $<\mu>$ on $\exp S$ by

$$<\mu>(A) = \sum_{n=0}^{\infty} \hat{\mu}^n(A_n)$$

Then $<\mu>$ is also a measure [4] and $(\exp X;\ \exp S_X;\ <\mu>)$, denoted by $\exp(X, S, \mu)$, is a totally $\sigma$-finite measure space. $<\mu>$ is certainly not the only measure available to us, but it has interesting properties. One of these is that it satisfies the equation

$$<\mu>[\exp X] = \exp[\mu(X)] = e^{\mu(X)} .$$

To see this, let $A \in S$. Then, for any $n = 1, 2, \cdots$

$$f^n(\underbrace{A \cdot A \cdot \cdots \cdot A}_{n\text{-times}}) = <A^n> ,$$

and $<A^n> \in <C_X^n> \subset <S^n>$. Furthermore, since $A^n$ is symmetric,

$$(f^n)^{-1}<A^n> = A^n ,$$

and therefore,

$$\hat{\mu}^n[(f^n)^{-1}<A^n>] = \frac{[\mu(A)]^n}{n!}$$

by definition of a product measure. Thus

$$\text{(i)} \qquad <\mu>\left(\sum_{n=1}^{\infty} <A^n>\right) = \sum_{n=1}^{\infty} \frac{[\mu(A)]^n}{n!} = e^{\mu(A)} - 1 .$$

In the special case $n = 0$, $<X^0> = \{e_X\}$; so that

$$<\mu>[\exp X] = \exp[\mu(X)] = e^{\mu(X)} .$$

This result is more, however, than a piece of mathematical sophistry. For example, assume $X$ is a subset of $R^3$ (3-dimensional Euclidean space), and that $X$ is the family of Lebesgue measurable subsets of $X$ with Lebesgue measure $\mu$. Then, provided $0 \leq \mu(X) < \infty$, the set function $<p>$ on $\exp S$ defined by

$$<p>(A) = e^{-\mu(X)} <\mu>(A)$$

is a probability (2; V). Referring to the physical model of Chapter 1, $<p>$ can be interpreted as a "Poisson distribution". To see this, let $A \in S$. Then using equation (i),

$$<p>[\exp A - \{e_A\}] = e^{-\mu(X)} \sum_{n=1}^{\infty} \frac{[\mu(A)]^n}{n!}$$

$$= e^{-\mu(X)}[e^{\mu(A)} - 1]$$

is the probability of the event "A contains at least one particle."

That is

$A$ contains exactly 1 particle

or

$A$ contains exactly 2 particles

.
.
.

or

$A$ contains exactly $n$ particles

.
.
.
.

where $n \in \omega$. The distribution is uniform (homogeneous) in that the larger $\mu(A)$, the more likely the event $\exp(A) - \{e_A\}$.

Theorem 3.11 stated that $\overline{\phi}$ was a measurability isomorphism of $\exp(Y, S_Y) \cdot \exp(Z, S_Z)$ into $\exp[(Y, S_Y) + (Z, S_Z)]$. Actually $\overline{\phi}$ is a measure isomorphism (see 2; IV) in the following sense. Let $X = Y + Z$ and let

$$(X, S, \mu) = (Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)$$

be a decomposition of $(X, S, \mu)$ according to Theorem 2.4. This decomposition of $(X, S, \mu)$ induces a decomposition of $\exp(X, S, \mu)$ into $\exp(Y, S_Y, \mu_Y)$ and $\exp(Z, S_Z, \mu_Z)$. Let

$\exp(Y, S_Y, \mu_Y) \cdot \exp(Z, S_Z, \mu_Z)$ denote the direct product of the measure spaces $\exp(Y, S_Y, \mu_Y)$ and $\exp(Z, S_Z, \mu_Z)$. Then we have

Theorem 3.14. Let $X = Y + Z$ be a decomposition of $X$ into a pair of disjoint measurable sets and let $(X, S, \mu) = (Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)$ be the corresponding decomposition of $(X, S, \mu)$. Then $\bar{\phi}$ is a measure isomorphism of $\exp(Y, S_Y, \mu_Y) \cdot \exp(Z, S_Z, \mu_Z)$ into $\exp[(Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)]$. That is

$$\exp(Y, S_Y, \mu_Y) \cdot \exp(Z, S_Z, \mu_Z) \underset{\bar{\phi}}{\simeq} \exp[(Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)]$$

In the next chapter $\exp X$ is extended to $\operatorname{Exp} X$ so as to include infinite symmetric sequences.

# CHAPTER 4. THE EXTENDED EXPONENTIAL OF $(X, S)$

In this chapter we adjoin the set $<X^{\omega}>$ of unordered infinite sequences to $\exp X$ to obtain the extended exponential $\operatorname{Exp} X$. We also induce a $\sigma$-algebra, $S<C^{\omega}>$, in $<X^{\omega}>$ determined by an arbitrary $\sigma$-algebra $S$ in $X$. In this way we endow $<X^{\omega}>$, and hence also $\operatorname{Exp} X$, with the structure of a measurable space. Finally, the exponential of a measure is extended in a trivial way from $\exp X$ to $\operatorname{Exp} X$.

To define $<X^{\omega}>$, let the Cartesian product

$$X^{\omega} = \pi \{X_t : X_t = X, \ \ t \in \omega\}$$

be the *infinite direct product of* $X$. Alternatively,

$$X = \{(x_1, x_2, \cdots) : x_i \in X, \ i \in \omega\}$$

is the set of all *ordered infinite sequences in* $X^{\omega}$. Now let $G_{\omega}$ be the *group of permutations on* $\omega$; that is, let $G_{\omega}$ be the set of all one-to-one functions on $\omega$ onto $\omega$. Each permutation $g \in G_{\omega}$ induces a corresponding function, also denoted by $g$, on $X^{\omega}$ through the equation

$$g(x_1, x_2, \cdots) = (x_{g(1)}, x_{g(2)}, \cdots).$$

For each ordered sequence $x \in X^{\omega}$, the corresponding unordered infinite sequence $<x>$ is the class of all "permutations of $x$". That is,

Definition 4. 1.

$$<x> = \{g(x) : g \in G_{\omega}\} .$$

The symmetric $\omega^{th}$ power of $X$, denoted $<X^{\omega}>$, is the set of all unordered infinite sequences in $X$.

Definition 4. 2.

$$<X^{\omega}> = \{<x> : x \in X^{\omega}\} .$$

Finally, the extended exponential of $X$, Exp $X$, is the infinite disjoint union of symmetric powers.

Definition 4. 3.

$$\operatorname{Exp} X = \sum_{n \leq \omega} <X^{n}>$$

The extension of Convention 3. 2 to Exp $X$ reads identically. That is

Convention 4.4.

a. Let $x \in X^n$ for $n \leq \omega$. Then $<x>$ denotes the unordered sequence of length $n$ corresponding to $x$.

b. Let $A \subset \sum_{n \leq \omega} X^n$. Then

$$<A> = \{<x> : x \in A\} .$$

c. Let $C$ be a family of subsets of $\sum_{n \leq \omega} X^n$. Then

$$<C> = \{<A> : A \in C\} .$$

Now let $C_X^\omega$ be the family of all measurable cylinders in X. That is,

Definition 4.5.

$$C_X^\omega = \{\prod_{k \in \omega} A_k : A_k \in S \quad \text{and} \quad A_k = X \text{ for all but finitely many } k\} .$$

Then $<C_X^\omega>$ is the family of symmetrized cylinder sets. The $\sigma$-algebra, $S<C_X^\omega>$ (see 2; I), associated with a given $\sigma$-algebra S on X is then a $\sigma$-algebra for $<X^\omega>$. Combining Definitions 4.5 and 3.7 we are able to define a $\sigma$-algebra for Exp X, given by

Definition 4. 6.

$$\mathrm{Exp}\, S = \sum_{n \leq \omega}^{*} S < C_X^n >$$

Clearly $\exp S \subset \mathrm{Exp}\, S$. The set Exp X together with the $\sigma$-algebra Exp S forms a measurable space (Exp X, Exp S), given by

Definition 4. 7.

$$\mathrm{Exp}\,(X, S) = (\mathrm{Exp}\, X,\ \mathrm{Exp}\, S).$$

The extension of the exponential measure $<\mu>$ to Exp X has physical ramifications. Suppose the model you have in mind is a rectangle (measurable) X in $R^3$ of finite volume (measure) throughout which a variable, but countable, number of points are in motion. If "points" correspond to some physical entity, such as gas molecules, then the event X described by

"there is a countably infinity of points in X"

never occurs. That is, the probability $p < X^{\omega} >$ equals zero. If we are again thinking of a Poisson (homogeneous) distribution, we must choose $<\mu>$ on Exp X so that

$$<\mu> < X^{\omega} > = 0 .$$

In this case everything works out nicely; that is,

$$<\mu> [\operatorname{Exp} X] = e^{\mu(X)} ;$$

while

$$<p> <X^{\omega}> = 0 .$$

If however, $\mu(X) = \infty$ (that is, our rectangle has "infinite volume"), the event $<X^{\omega}>$ may certainly occur. We still choose $<\mu> <X^{\omega}> = 0$ even though the probability of the event $<X^{\omega}>$ may be greater than zero. This means that the probability measure on $\operatorname{Exp}(X,S)$ will not be absolutely continuous with respect to $<\mu>$ .

<u>Definition 4. 8.</u> Let $<\mu>$ be the exponential measure of Definition 3. 13. To extend $<\mu>$ to $\operatorname{Exp} X$, let

$$\hat{\mu}^{\omega}(A) = 0$$

for each $A \in S<C_X^{\omega}>$ .

Obviously $\hat{\mu}^{\omega}$ is a measure, albeit trivial, on $S<C_X^{\omega}>$ . In particular $\hat{\mu}^{\omega}<X^{\omega}> = 0$. Extending Definition 3. 13 we have

<u>Definition 4. 9.</u> Let $A = \sum_{n \leq \omega} A_n$ be any set in $\operatorname{Exp} S$ (see Definition 4. 6). We define the set function $<\mu>$ on $\operatorname{Exp} S$ by

$$<\mu> (A) = \sum_{n \leq \omega} \hat{\mu}^n (A_n) .$$

It is worth noting that $<\mu> (\mathrm{Exp}\ X) = e^{\mu(X)}$, and that $<\mu> <X^{\omega}> = 0$. The measure space $(\mathrm{Exp}\ X, \mathrm{Exp}\ S, <\mu>)$, which is denoted by $\mathrm{Exp}\ (X,S,\mu)$, is totally $\sigma$-finite. This follows trivially from the total $\sigma$-finiteness of $\exp\ (X,S,\mu)$. Thus, we have

Definition 4.10.

$$\mathrm{Exp}\,(X,S,\mu) = (\mathrm{Exp}\ X,\ \mathrm{Exp}\ S,\ <\mu>)$$

Lemma 4.11. Let $(X,S,\mu)$ be a totally $\sigma$-finite measure space. Then $\mathrm{Exp}\ (X,S,\mu)$ is totally $\sigma$-finite.

# CHAPTER 5. THE EXPONENTIAL LAWS FOR SETS

In this chapter we will extend the exponential law

$$\exp Y \cdot \exp Z \underset{\phi}{\simeq} \exp (Y+Z)$$

to $\operatorname{Exp} Y \cdot \operatorname{Exp} Z$. That is, we will construct a set isomorphism $\phi$ such that

$$\operatorname{Exp} Y \cdot \operatorname{Exp} Z \underset{\phi}{\simeq} \operatorname{Exp} (Y+Z)$$

and such that $\phi$ is actually an extension of $\bar{\phi}$. This occurs as a special case of a more general construction as follows: Let $\{X_k : k \in I\}$ be a disjoint family of sets indexed by a finite or countable set $I$, as in Chapter 2; II. We will construct a set isomorphism $\phi$ such that

$$\prod_{k \in I} \operatorname{Exp} X_k \underset{\phi}{\simeq} \operatorname{Exp} (\sum_{k \in I} X_k)$$

In the case $I = \{1, 2\}$, the restriction of $\phi$ to $\exp X_1 \cdot \exp X_2$ is $\bar{\phi}$.

It is easy to describe $\phi$. Let

$$(\alpha^k : k \in I) = (\alpha^1, \alpha^2, \cdots )$$

be an ordered sequence in $\prod_{k \in I} \operatorname{Exp} X_k$. Let $a^k \in \alpha^k$. Then $a^k$

is an ordered sequence in $X_k$. To construct $\phi$, we must combine the terms of these sequences $a^k$ into one long sequence $\bar{a}$, and then take $\phi(a^k : k \in I)$ to be the unordered sequence $<\bar{a}>$. This can be done in many different ways; e.g., by any "diagonalization process" applied to the family of sequences $a^k$.

To make these notions precise, we shall use the interpretation of an ordered sequence $(x_1, x_2, \cdots, x_n)$ of length $n$ in a set $X$ as a function $x$ from the set $n = \{0, 1, \cdots, n-1\}$ <u>into</u> the set $X$ such that $x(k) = x_{k+1}$. Similarly, an ordered sequence of length $\omega$ in $X$ is a map $x$ from $\omega$ into $X$ such that $x(k) = x_{k+1}$. The case $n = 0$, corresponding to the event that there are no points in $X$, may be treated as a special case, as follows: We define a sequence of length zero to be a function on $0 = \emptyset$ into $X$. There is only one such sequence, namely the empty function $\emptyset$. We therefore write $X^0 = \{\emptyset\}$. Since there is only one equivalence relation on a set with one element, there is only one choice for the unordered sequence of length zero, namely $<\emptyset> = \{\emptyset\}$. This provides a concrete interpretation for the empty sequence $e_X$ introduced in Chapter 3, namely,

$$e_X = <\emptyset> .$$

Notice that with this interpretation, the empty sequence $e_X$ is independent of the set $X$. However, we shall continue to attach subscripts $X, Y, Z$ to $e$ whenever we wish to think of $e$ as associated with the particular sets $X, Y, Z$.

It is also convenient for purposes of defining $\phi$ to work with the notion of a generalized sequence in the sense of

Definition 5.1. A generalized sequence in a set $X$ is a function whose range is in $X$ and whose domain is a finite or countably infinite set. The length of a generalized sequence is the cardinality of its domain. That is, if $\bar{a}$ is a generalized sequence with domain $A$, then its length $L$ is

$$L = \omega \quad \text{if} \quad A \quad \text{is infinite}$$

$$L = n = \{0, 1, \cdots, n-1\} \quad \text{if} \quad A \quad \text{has exactly} \quad n \quad \text{points}$$

$$L = 0 = \emptyset \quad \text{if} \quad A \quad \text{is empty} .$$

Now the terms of a generalized sequence $\bar{a}$ can be arranged into an ordered sequence as follows: Let $A$ be the domain of $\bar{a}$, $L$ be its length, and $f$ be any set isomorphism of $L$ into $A$. Then $\bar{a} \circ f$ is an ordered sequence of length $L$, whose terms are just those of $\bar{a}$. Although there are many such ordered sequences, depending on the choice of $f$, the unordered sequence $\langle \bar{a} \circ f \rangle$ is clearly independent of $f$. For if $g$ is a second isomorphism of $L$ into $A$, then $\pi = g^{-1} \circ f$ is a permutation on $L$ such that $\bar{a} \circ f = (\bar{a} \circ g) \circ \pi$, so that $\langle \bar{a} \circ f \rangle = \langle \bar{a} \circ g \rangle$. We may therefore regard this unordered sequence as determined directly by $\bar{a}$. This justifies

Definition 5.2. Let $\bar{a}$ be a generalized sequence in $X$ with domain $A$ and length $L$, and let $f$ be any set isomorphism of $L$ into $A$. Then $<a \circ f>$ will be called the unordered sequence determined by $\bar{a}$, and will be denoted by $<\bar{a}>$:

$$<\bar{a}> = <\bar{a} \circ f> .$$

Note that if $\bar{a}$ is an ordinary ordered sequence, then the unordered sequence $<\bar{a}>$, as given by Definition 5.2, coincides with those of Definitions 1.1 and 4.1.

For later application we need

Lemma 5.3. Let $a, b$ be two generalized sequences with domains $A, B$ respectively. Then $<a> = <b>$ if and only if there is a set isomorphism $\pi: A \to B$ such that $a = b \circ \pi$.

Proof: Suppose there is such a set isomorphism $\pi$. Then the lengths (cardinalities) of $A, B$, which we shall denote by $L, M$, respectively, are equal. Thus, if $f$ is any isomorphism of $L$ into $A$, then $\pi \circ f$ is an isomorphism of $M$ into $B$, and $a \circ f = b \circ (\pi \circ f)$. Therefore,

$$\langle a \rangle = \langle a \circ f \rangle = \langle b \circ (\pi \circ f) \rangle = \langle b \rangle .$$

Conversely, suppose $\langle a \rangle = \langle b \rangle$. Let $f, g$ be isomorphisms of $L, M$ into $A, B$ respectively. Then $\langle a \circ f \rangle = \langle b \circ g \rangle$, and the sequences $a \circ f$, $b \circ g$ are equivalent. That is, they have the same domain, $L = M$, and there is a permutation $p$ on $L$ such that $a \circ f = (b \circ g) \circ p$. Thus the function $\pi = g \circ p \circ f^{-1}$ is an isomorphism of $A$ into $B$ such that $a = b \circ \pi$.

We are now ready to define $\phi$ precisely.

Definition 5.4. Let $\alpha = (\alpha^k : k \in I) = (\alpha^1, \alpha^2, \cdots)$ be an ordered sequence in $\prod_{k \in I} \operatorname{Exp} X_k$; so that each $\alpha^k$ is an unordered sequence in $X_k$. For each $k \in I$, let $a^k \in \alpha^k$ be an ordered sequence representing $\alpha^k$, and let $A_k$ be the domain (length) of $a^k$. Let $a$ be the generalized sequence with domain

$$A = \{(k, j) : k \in I, \ \ j \in A_k\}$$

such that $a(k, j)$ is the $j^{th}$ term of $a^k$; i.e.,

$$a(k, j) = a^k(j) .$$

Then $\phi$ is that function on $\prod_{k \in I} \operatorname{Exp} X_k$ such that

$$\phi(\alpha) = \langle a \rangle .$$

To be sure that $\phi$ is properly defined we must show that if the $a^k$ are replaced by equivalent sequences $b^k \epsilon \alpha^k$, and $b$ is the corresponding generalized sequence such that $b(k,j) = b^k(j)$, then $<b> = <a>$.

Lemma 5.5. $\phi$ is well-defined.

Proof: Since the sequences $b^k$ and $a^k$ are equivalent, they have the same domain $A_k$, and there is a permutation $\pi_k$ on $A_k$ such that $a^k = b^k \circ \pi_k$. Thus the generalized sequences $a$ and $b$ have the same domain $A$. Moreover the function $\pi$ defined on $A$ by

$$\pi(k,j) = (k, \pi_k(j))$$

is a set isomorphism such that $a = b \circ \pi$. Hence, by Lemma 5.3, $<a> = <b>$.

Lemma 5.6. $\phi$ is one-to-one.

Proof: Let $\alpha = (\alpha^k : k \epsilon I)$ and $\beta = (\beta^k : k \epsilon I)$ be two sequences such that $\phi(\alpha) = \phi(\beta)$. For each $k$ let $a^k \epsilon \alpha^k$ and $b^k \epsilon \beta^k$ be representative sequences with domains $A_k$, $B_k$, and let $a, b$ be the generalized sequences determined by the $a^k$, $b^k$ with domains $A, B$ respectively. Then $<a> = <b>$. By Lemma 5.3 there is a set isomorphism $\pi : A \rightarrow B$ such that $a = b \circ \pi$. We will show that for each $k$ there is an isomorphism $\pi_k : A_k \to B_k$ such that

$$\pi(k,j) = (k, \pi_k(j)) \quad \text{for all} \quad j \in A_k .$$

Hence $a(k,j) = b(\pi(k,j)) = b(\pi_k(j))$ or alternatively, $a^k(j) = b^k(\pi_k(j))$, so that $a^k = b^k \circ \pi_k$. Since $\pi_k$ is a set isomorphism, the cardinalities $A_k$ and $B_k$ are equal, and $\pi_k$ is a permutation. Therefore $\langle a^k \rangle = \langle b^k \rangle$, so $\alpha_k = \beta_k$ and $\phi$ is one-to-one.

To show that $\pi$ has the required form, choose any $j \in A_k$, and write $\pi(k,j) = (\bar{k}, \bar{j})$. Then $a(k,j) = b(\bar{k}, \bar{j})$, and since $a(\bar{k}, \bar{j}) \in X_k$ we have $b(\bar{k}, \bar{j}) = b^{\bar{k}}(\bar{j}) \in X_k$. Therefore $\bar{k} = k$ and we may write $\pi(k,j) = (k, \pi_k(j))$, where $\pi_k$ is a function on $A_k$ into $B_k$. If $\sigma$ denotes the inverse of $\pi$, and $\sigma_k$ is the corresponding function on $B_k$ such that $\sigma(k,j) = (k, \sigma_k(j))$, then it is easy to check that $\sigma_k$ is both a left and right inverse to $\pi_k$. Hence $\pi_k$ is both one-to-one and onto, and the lemma follows.

<u>Lemma 5.7.</u> $\phi$ is onto.

Proof: Let $\bar{\alpha}$ be any element of $\operatorname{Exp}\left(\sum_{k \in I} X_k\right)$, and $\bar{a} \in \bar{\alpha}$ be any representative sequence. Let $L$ be the length of $\bar{a}$, and for each $k \in I$ let

$$L_k = \bar{a}^{-1}[X_k] = \{ j \in L : \bar{a}(j) \in X_k \} .$$

Since the $X_k$'s are disjoint, the $L_k$'s form a disjoint decomposition of L. Let $A_k$ be the cardinality of $L_k$ (the number of terms of $\overline{a}$ in $X_k$), and let $g_k$ be any set isomorphism on $A_k$ onto $L_k$. Define the <u>ordered sequence</u> $a^k$ in $X_k$ by $a^k = \overline{a} \circ g_k$, and the unordered sequence $\alpha^k \in \text{Exp } X_k$ by $\alpha^k = \langle a^k \rangle$. Then the sequence $\alpha = (\alpha^k ; k \in I)$ belongs to $\prod_{k \in I} \text{Exp } X_k$. We will show that $\overline{\alpha} = \phi(\alpha)$ ; i.e., that $\langle \overline{a} \rangle = \langle a \rangle$, where a is the generalized sequence determined by the $a^k$. Consider the function g on $A = \{(k, j): k \in I, \; j \in A_k\}$ into L, defined by

$$g(k, j) = g_k(j) \quad .$$

Then g is clearly a set isomorphism, and $a(k, j) = a^k(j) = \overline{a}(g_k(j)) = \overline{a}(g(k, j))$. Hence $a = \overline{a} \circ g$ or $\overline{a} = a \circ g^{-1}$, and $\langle \overline{a} \rangle = \langle a \rangle$ by Definition 5.2.

We have now proved

<u>Theorem 5.8.</u> For each finite or countably infinite index set I, $\phi$ is a <u>set isomorphism</u> from $\prod_{k \in I} \text{Exp } X_k$ onto $\text{Exp } (\sum_{k \in I} X_k)$.

That is

$$\prod_{k \in I} \text{Exp } X_k \underset{\phi}{\simeq} \text{Exp } \left( \sum_{k \in I} X_k \right)$$

Actually, for each $I$ there is a $\phi$; so it would be more correct to write $\phi_I$. However, we shall employ the symbol $\phi_n$ in case $I = \{1, 2, \cdots, n\}$ only, for purposes of emphasis. In the case $I = \omega$, we shall always write $\tilde{\phi}$ in accordance with

Convention 5.9. If $\underline{I = \omega}$ we shall always write

$$\prod_{k \in \omega} \operatorname{Exp} X_k \underset{\tilde{\phi}}{\simeq} \operatorname{Exp} \left( \sum_{k \in I} X_k \right).$$

The reader will easily verify that in the case $n = 2$ we have

Theorem 5.10. If $n = 2$, the restriction of $\phi$ to $\exp X_1 \cdot \exp X_2$ is $\bar{\phi}$.

This will enable us to use the results of Chapter 3 in proving

$$\prod_{k=1}^{2} \operatorname{Exp} (X_k, S_k) \underset{\phi}{\simeq} \operatorname{Exp} \left[ \sum_{k=1}^{2} (X_k, S_k) \right].$$

Aside from the fact that $\phi$ is a set isomorphism, the two most important results of this chapter are the next two lemmas.

Lemma 5.11. Let $X = \sum_{k=1}^{n} X_k$ be a disjoint decomposition. For each $k \in N$, let $a^k \in \operatorname{Exp} X_k$. Let $m$ be an integer, $1 \leq m < n$.

Then

$$\phi_n(\alpha^1, \cdots, \alpha^n) = \phi_2(\phi_m(\alpha^1, \cdots, \alpha^m), \phi_{n-m}(\alpha^{m+1}, \cdots, \alpha^n))$$

Proof: For each $k \in I$, let $a^k \in \alpha^k$, and let $A_k$ be the domain of $a^k$. Let $a, b, c$ be the generalized sequences determined respectively by $(a^k : 1 \leq k \leq n)$, $(a^k : 1 \leq k \leq m)$ and $(a^k : m+1 \leq k \leq n)$. The domains of these generalized sequences are

$$A = \{(k, j) : 1 \leq k \leq n, \ j \in A_k\}$$

$$B = \{(k, j) : 1 \leq k \leq m, \ j \in A_k\}$$

$$C = \{(k, j) : 1 \leq k \leq n-m, \ j \in A_{k+m}\}$$

Notice that $C$ is isomorphic to the relative complement of $B$ in $A$. That is

$$\overline{C} = B\text{-}A = \{(k, j) : m < k \leq n, j \in A_k\}$$

is isomorphic to $C$ through the function $h : C \to \overline{C}$ given by

$$h\,(k, j) = (k+m, j) .$$

Notice also that $b$ is the restriction of $a$ to $B$, and that $a \circ h$ is equal to $C$.

To determine $\langle b \rangle = \phi_m(\alpha^1, \cdots, \alpha^m)$ and

$<c> = \phi_{n-m}(\alpha^{m+1}, \cdots, \alpha^{n})$, let $L, M$ be the lengths of $B, C$, and let $f, g$ be the isomorphisms of $L, M$ into $B, C$, respectively. Then $<b>$, $<c>$ are represented by the sequences $b \circ f \epsilon <b>$ and $c \circ g \epsilon <c>$. Therefore $\phi_2(<b>, <c>)$ is determined by the generalized sequence $d$, with domain

$$D = \{(i, j) : i=1, \ j \epsilon L \quad \text{or} \quad i = 2, \ j \epsilon M\}$$

such that

$$\alpha(i, j) = \begin{cases} b \circ f(j) & \text{if} \quad i = 1 \\ c \circ g(j) & \text{if} \quad i = 2 \end{cases}$$

We are to show that the generalized sequences $a$ and $d$ determine the same unordered sequence $<a> = <d>$. For this purpose consider the function $\pi : D \to A$ defined by

$$\pi(i, j) = \begin{cases} f(j) & \text{if} \quad i = 1 \\ h(g(j)) & \text{if} \quad i = 2 \end{cases}$$

Since $f, g, h$ are set isomorphisms, so also is $\pi$. Moreover $d = a \circ \pi$, for

$$a(\pi(i,j)) = \begin{cases} a(f(j)) = b(f(j)) & \text{if} \quad i = 1 \\ a(h(g(j))) = c(g(j)) & \text{if} \quad i = 2 \end{cases}$$

Hence by Lemma 5. 3 $\langle a \rangle = \langle d \rangle$ .

A completely analogous proof establishes

Lemma 5. 12. Let $X = \sum_{k \in \omega} X_k$ be a disjoint decomposition of X. For each $k \in \omega$, let $a^k \in \operatorname{Exp} X_k$. Then for each $k \in \omega$

$$\tilde{\phi}(a^1, a^2, \cdots) = \phi_2(\phi_k(a^1, a^2, \cdots, a^k), \ \tilde{\phi}(a^{k+1}, a^{k+2}, \cdots))$$

The remainder of this chapter is devoted to some lemmas about $\phi$ which are needed for later proofs. They can be easily verified by the reader. Lemmas 5. 14 and 5. 15 examine the effect of $\phi_2$ and $\tilde{\phi}$ on certain distinguished subsets of $\operatorname{Exp} Y \cdot \operatorname{Exp} Z$. Similarly, Lemmas 5. 16 and 5. 17 examine the effect of $\phi_n$ and $\tilde{\phi}$ on certain distinguished subsets of their domains. We will need

Definition 5. 13. Let A and B be sets. Then $(A \cdot B)^\omega$ consists of all sequences of the form

$$(x_1, y_1, x_2, y_2, \cdots)$$

where $x_i \in A$ and $y_i \in B$ for all $i \in \omega$.

Lemma 5.14. Let $B, A_1, \cdots, A_p \subset Y$ and let $C, A_{p+1}, A_{p+2}, \cdots, A_n \subset Z$. Then, using Convention 4.4, we have

(a) $\phi_2[<A_1 \cdot \cdots \cdot A_p \cdot B^{\omega}> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^{\omega}] = <\prod_{k=1}^{n} A_k \cdot (B \cdot C)^{\omega}>$

(b) $\phi_2[<A_1 \cdot \cdots \cdot A_p \cdot B^k> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^{\omega}>] = <\prod_{i=1}^{n} A_i \cdot B^k \cdot C^{\omega}>$

(c) $\phi_2[<A_1 \cdot \cdots \cdot A_p \cdot B^{\omega}> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^k>] = <\prod_{i=1}^{n} A_i \cdot C^k \cdot B^{\omega}>$

(d) $\phi_2[<A_1 \cdot \cdots \cdot A_p> \cdot <A_{p+1} \cdot \cdots \cdot A_n>] = <A_1 \cdot A_2 \cdot \cdots \cdot A_n>$

Exp Y and Exp Z each contain the unordered empty sequences; that is, $e_Y \in \text{Exp } Y$ and $e_Z \in \text{Exp } Z$. It is easy to show that

$$\phi_2(e_Y, <z>) = <z>$$

$$\phi_2(<y>, e_z) = <y>$$

$$\phi_2(e_Y, e_Z) = e_X = e_{Y+Z}$$

Continuing the idea of Lemma 5.14, we therefore have

Lemma 5.15. Let $A \subset \mathrm{Exp}\ Y$, then

$$\phi_2[A \cdot \{e_Z\}] = A$$

$$\phi_2[\{e_Y\} \cdot \{e_Z\}] = \{e_X\} .$$

[Note that if $A = \{e_Y\}$, these formulas imply that $\{e_X\} = \{e_Y\}$, in keeping with the fact that the empty sequence is independent of the set. If we did not wish to indicate the sets with subscripts, we would write $\phi_2(e, e) = e$.]

Now suppose $X = \sum_{k=1}^{n} X_k$ where $n > 2$. Generalizing Lemma 5.15, we have

Lemma 5.16. For each $k \in N$, let $e_k$ be the empty sequence in $X_k$, let $a_k \in \mathrm{Exp}\ X_k$, and let $A_k \subset \mathrm{Exp}\ X_k$. Then

a. $\phi_n(e_1, e_2, \cdots, e_n) = e_X$

b. $\phi_n(a_1, a_2, \cdots, a_m, e_{m+1}, \cdots, e_n) = \phi_m(a_1, a_2, \cdots a_m)$

for each $m$ where $1 \leq m < n$.

c. $\phi_n[A_1 \cdot A_2 \cdot \cdots \cdot A_m \cdot \prod_{k=m+1}^{n} \{e_k\}] = \phi_m[A_1 \cdot A_2 \cdot \cdots \cdot A_m]$ .

Extending these results to the infinite case, we obtain

Lemma 5. 17. Let $X = \sum_{k \in \omega} X_k$. For each $k \in \omega$, let $e_k$ be the empty sequence in $X_k$, let $a_k \in \operatorname{Exp} X_k$, and let $A_k \subset \operatorname{Exp} X_k$. Then

a. $\tilde{\phi}(e_1, e_2, \cdots) = e_X$

b. For each $n \in \omega$

$$\tilde{\phi}(a_1, a_2, \cdots, a_n, e_{n+1}, e_{n+2}, \cdots) = \phi_n(a_1, \cdots, a_n)$$

c. For each $n \in \omega$

$$\tilde{\phi}\left[\prod_{k=1}^{n} A_k \cdot \prod_{k>n} \{e_k\}\right] = \phi_n[A_1 \cdot A_2 \cdot \cdots \cdot A_n]$$

# CHAPTER 6. THE EXPONENTIAL LAWS FOR MEASURABLE SPACES

In the last chapter we showed that

$$\prod_{k=1}^{n} \operatorname{Exp} X_k \underset{\phi}{\simeq} \operatorname{Exp}\left(\sum_{k=1}^{n} X_k\right)$$

for each $n \in \omega$, and that

$$\prod_{k \in \omega} \operatorname{Exp} X_k \underset{\tilde{\phi}}{\simeq} \operatorname{Exp}\left(\sum_{k \in \omega} X_k\right).$$

In this chapter we show that $\phi$ and $\tilde{\phi}$ are actually <u>measurability isomorphisms</u> (see 2; IV). Let $I$ be any finite or countably infinite index set, and let $\prod_{k \in I} \operatorname{Exp}(X_k, S_k)$ denote the <u>ordinary direct product</u> of the measurable spaces $\operatorname{Exp}(X_k, S_k)$, $k \in I$ (see 2; III) where

$$(X, S) = \sum_{k \in I} (X_k, S_k).$$

We shall prove that if $I = \{1, 2, \cdots, n\}$, then

$$\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k) \underset{\phi}{\simeq} \operatorname{Exp} \sum_{k=1}^{n} (X_k, S_k)$$

for each $n \in \omega$; and if $I = \omega$, then

$$\prod_{k \in \omega} \operatorname{Exp}(X_k, S_k) \underset{\phi}{\simeq} \operatorname{Exp}\left(\sum_{k \in \omega} (X_k, S_k)\right).$$

We begin with the case $I = \{1, 2\}$, proving that if $(X, S) = (Y, S_Y) + (Z, S_Z)$, then

$$\operatorname{Exp}(Y, S_Y) \cdot \operatorname{Exp}(Z, S_Z) \underset{\phi}{\simeq} \operatorname{Exp}[(Y, S_Y) + (Z, S_Z)].$$

Let $C^X$ denote the family of all measurable rectangles in Exp X. That is, using Convention 2. 1,

Definition 6. 1. $$C^X = \sum_{n \leq \omega} < C_X^n > .$$

We first establish

Lemma 6. 2. $$\operatorname{Exp} S_X = S(C^X)$$

Proof: By Definition 4. 6

$$\operatorname{Exp} S_X = \sum_{n \leq \omega}^{*} S < C_X^n >$$

Applying Theorem 2. 8, we have

$$\sum_{n \leq \omega}^{*} S < C_X^n > \;=\; S\left( \sum_{n<\omega} < C_X^n > \right) .$$

Therefore

$$\operatorname{Exp} S_X = S(C^X) .$$

Lemma 6. 3. $\qquad S(C^Y \cdot C^Z) \;=\; \operatorname{Exp} S_Y * \operatorname{Exp} S_Z$

Proof: From Lemma 6. 2

$$S(C^Y) = \operatorname{Exp} S_Y \quad \text{and} \quad S(C^Z) = \operatorname{Exp} S_Z .$$

Thus

$$\operatorname{Exp} S_Y * \operatorname{Exp} S_Z \;=\; S(C^Y) * S(C^Z) .$$

But by Theorem 2. 9

$$S(C^Y) * S(C^Z) = S(C^Y \cdot C^Z) .$$

For the statement of the next lemma, recall Definition 5. 13 for $(A \cdot B)^\omega$ .

Lemma 6. 4. Let $A \subset Y$, $B \subset Z$, and $C \subset X^n$ for some $n \in \omega$. Then

$$<C \cdot (A+B)^{\omega}> \;=\; <C \cdot (A \cdot B)^{\omega}> + \sum_{k=0}^{\infty} [<C \cdot A^{k} \cdot B^{\omega}> + <C \cdot B^{k} \cdot A^{\omega}>]$$

where, in the case $k = 0$, $<C \cdot A^{0} \cdot B^{\omega}>$ is interpreted to mean $<C \cdot B^{\omega}>$, and $<C \cdot B^{0} \cdot A^{\omega}>$ means $<C \cdot A^{\omega}>$.

Proof: $<C \cdot (A+B)^{\omega}>$ consists of all sequences of the form

$$a = <x_1, x_2, \cdots, x_n, w_1, w_2, \cdots>$$

where $(x_1, x_2, \cdots, x_n) \epsilon C$ and $w_i \epsilon A$ or $B$ for each $i \epsilon \omega$. But if $w_i \epsilon A$ or $B$ for each $i = 1, 2, \cdots$, then

<u>Case one</u>: $w_i \epsilon A$ for only finitely many $i$ and
$w_i \epsilon B$ for only infinitely many $i$

or

<u>Case two</u>: $w_i \epsilon A$ for infinitely many $i$ and
$w_i \epsilon B$ for only finitely many $i$

or

<u>Case three</u>: $w_i \epsilon A$ for infinitely many $i$ and
$w_i \epsilon B$ for infinitely many $i$.

Furthermore, these three possibilities are mutually exclusive because $A$ and $B$ are disjoint.

Case one implies $a \epsilon \sum_{k=0}^{\infty} <C \cdot A^k \cdot B^\omega>$ for some $k \epsilon \omega$.

Case two implies $a \epsilon \sum_{k=0}^{\infty} <C \cdot B^k \cdot A^\omega>$ for some $k \epsilon \omega$.

Since, in case three, the sequence $(w_1, w_2, \cdots)$ can be rearranged so that its terms alternate between A and B, case three implies

$$a \epsilon <C \cdot (A \cdot B)^\omega> \quad .$$

This coupled with the mutual exclusiveness of the possibilities implies

$$a \epsilon <C \cdot (A \cdot B)^\omega> + \sum_{k=0}^{\infty} [<C \cdot A^k \cdot B^\omega> + <C \cdot B^k \cdot A^\omega>]$$

Conversely

$$<C \cdot (A \cdot B)^\omega> + \sum_{k=0}^{\infty} [<C \cdot A^k \cdot B^\omega> + <C \cdot B^k \cdot A^\omega>]$$

consists of all sequences

$$b = <x_1, x_2, \cdots, x_n, \quad w_1, w_2, \cdots>$$

where either

1. $b \in \langle C \cdot (A \cdot B)^{\omega} \rangle$, in which case $w_{2i} \in B$ for each $i = 1, 2, \cdots$ and $w_{2i-1} \in A$ for each $i = 1, 2, \cdots$,

or

2. $b \in \langle C \cdot A^k \cdot B^{\omega} \rangle$ for some $k \in \omega$ (in which case $w_1, w_2, \cdots, w_k \in A$ and $w_i \in B$ for all $i = k+1, k+2, \cdots$),

or

3. $b \in \langle C \cdot B^k \cdot A^{\omega} \rangle$ for some $k \in \omega$ (in which case $w_1, w_2, \cdots, w_k \in B$ and $w_i \in A$ for all $i = k+1, k+2, \cdots$).

In either of cases 1, 2, or 3

$$b \in \langle C \cdot (A + B)^{\omega} \rangle \quad .$$

Turning now to the proof that the set isomorphism $\phi$ from $\mathrm{Exp}\, Y \cdot \mathrm{Exp}\, Z$ onto $\mathrm{Exp}\, X$ is also a measurability isomorphism, we will show that $\phi[C^Y \cdot C^Z] \subset \mathrm{Exp}\, S_X$ and that $\phi^{-1}[C^X] \subset \mathrm{Exp}\, S_X * \mathrm{Exp}\, S_Z$. Then we will invoke Theorem 2.6 to obtain

$$S(\phi^{-1}[C^X]) = \phi^{-1}(S[C^X]) ,$$

and

$$S(\phi[C^Y \cdot C^Z]) = \phi(S[C^Y \cdot C^Z]) .$$

It will then follow that $\phi[\operatorname{Exp} S_Y \cdot \operatorname{Exp} S_Z] = \operatorname{Exp} S_X$.

Lemma 6.5. $\phi^{-1}[C^X] \subset S[C^Y \cdot C^Z)$ .

Proof: $\phi^{-1}[C^X] = \{B \subset \operatorname{Exp} Y \cdot \operatorname{Exp} Z : B = \phi^{-1}(A)$ for some $A \in C^X\}$.

$A \in C^X$ implies that $A \in \langle C_X^p \rangle$ for some $p \in \omega$, or that $A \in \langle C_X^\omega \rangle$. But, in view of Theorem 3.11,

$$\phi^{-1}[\exp S] = \exp S_Y * \exp S_Z \subset \operatorname{Exp} S_Y * \operatorname{Exp} S_Z .$$

Since $\langle C_X^p \rangle \subset \exp S_X$ for each $p \in \omega$, clearly

$$\phi^{-1}[\langle C_X^p \rangle] \subset \operatorname{Exp} S_Y * \operatorname{Exp} S_Z .$$

Thus it suffices to prove

$$\phi^{-1}[\langle C_X^\omega \rangle] \subset \operatorname{Exp} S_Y * \operatorname{Exp} S_Z .$$

Let $A \in \langle C_X^\omega \rangle$ (see Convention 4.4 and Definition 4.5). Then, for some $n \in \omega$

$$A = <\prod_{k=1}^{n} A_k \cdot \prod_{t \in \omega} A_t> \quad \text{where} \quad A_t = \bar{A} \quad \text{for all} \quad t \in \omega$$

$$A_k, \ \bar{A} \in S, \quad k = 1, \cdots, n,$$

$$= <\prod_{k=1}^{n} A_k \cdot \bar{A}^{\omega}> \quad .$$

But $S$ is the sum of the $\sigma$-algebras $S_Y$ and $S_Z$ (see 2; II). Therefore we can write

$$\bar{A} = B + C \qquad \text{and} \qquad A_k = B_k + C_k$$

where $B_k \in S_Y$ and $C_k \in S_Z$ for each $k = 1, 2, \cdots, n$, and $B \in S_Y$, $C \in S_Z$. Then

$$\prod_{k=1}^{n} A_k = \prod_{k=1}^{n} (B_k + C_k)$$

can be expressed as a <u>finite</u> <u>union</u> of sets of the form

$$D_1 \cdot D_2 \cdot \cdots \cdot D_n = \prod_{k=1}^{n} D_k$$

where $D_k = B_k$ or $D_k = C_k$. But then $A$ is a finite union of sets of the form

$$<\prod_{k=1}^{n} D_k \cdot \bar{A}^{\omega}>$$

and $\phi^{-1}(A)$ is a finite union of inverse images of such sets. Thus, we can assume that

$$A = <\prod_{k=1}^{n} A_k \cdot \bar{A}^{\omega}> = <\prod_{k=1}^{n} A_k \cdot (B + C)^{\omega}>$$

where $A_k \epsilon S_Y$ or $A_k \epsilon S_Z$ for each $k$ and $\bar{A} = B + C$ where $B \epsilon S_Y$ and $C \epsilon S_Z$. But using Lemma 6.4

$$<\prod_{k=1}^{n} A_k \cdot (B+C)^{\omega}> = <\prod_{k=1}^{n} A_k \cdot (B+C)^{\omega}> + \sum_{k=0}^{\infty} [<\prod_{k=1}^{n} A_k \cdot B^k \cdot C^{\omega}>$$

$$+ <\prod_{k=1}^{n} A_k \cdot C^k \cdot B^{\omega}>]$$

Now reorder the $A_k$'s (if necessary) so that

$$A_1, A_2, \cdots, A_p \epsilon S_Y \quad \text{and} \quad A_{p+1}, \cdots, A_n \epsilon S_Z .$$

Then from Lemma 5.14

$$<\prod_{k=1}^{n} A_k \cdot (B \cdot C)^{\omega}> = \phi[<A_1 \cdot A_2 \cdot \cdots \cdot A_p \cdot B^{\omega}> \cdot <A_{p+1}, \cdots, A_n \cdot C^{\omega}>] ;$$

and for each $k \epsilon \omega$

$$<\prod_{k=1}^{n} A_k \cdot B^k \cdot C^{\omega}> = \phi[<A_1 \cdot \cdots \cdot A_p \cdot B^k> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^{\omega}>] \quad \text{and}$$

$$<\prod_{k=1}^{n} A_k \cdot C^k \cdot B^\omega> = \phi[<A_1 \cdot \cdots \cdot A_p \cdot B^\omega> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^k>] .$$

Thus

$$\phi^{-1}(A) = <A_1 \cdot \cdots \cdot A_p \cdot B^\omega> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^\omega>$$

$$+ \sum_{k=0}^{\infty} <A_1 \cdot \cdots \cdot A_p \cdot B^k> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^\omega>$$

$$+ \sum_{k=0}^{\infty} <A_1 \cdot \cdots \cdot A_p \cdot B^\omega> \cdot <A_{p+1} \cdot \cdots \cdot A_n \cdot C^k>$$

where each set on the right belongs to $C^Y \cdot C^Z$. Since $\phi^{-1}(A)$ is a countable union of members of $C^Y \cdot C^Z$,

$$\phi^{-1}(A) \in S(C^Y \cdot C^Z).$$

But, by Lemma 6.3

$$S(C^Y \cdot C^Z) = \text{Exp } S_Y * \text{Exp } S_Z .$$

Thus

$$\phi^{-1}[C^X] \subset \text{Exp } S_Y * \text{Exp } S_Z .$$

Now we prove that $\phi$ maps each set $A \cdot B$ belonging to $C^Y \cdot C^Z$

onto a set in Exp S. That is,

<u>Lemma 6.6.</u> $$\phi(C^Y \cdot C^Z) \subset S(C^X) .$$

Proof: Let $A \cdot B \epsilon (C^Y \cdot C^Z)$. Since

$$\phi(\exp S_Y * \exp S_Z) = \exp S_X \subset \operatorname{Exp} S_X ,$$

it suffices to consider the following cases only.

<u>Case one:</u> $A \epsilon < C_Y^p >$ for some $p \epsilon \omega$ and $B \epsilon < C_Z^\omega >$,

or

<u>Case two:</u> $A \epsilon < C_Y^\omega >$ and $B \epsilon < C_Z^p >$ for some $p \epsilon \omega$,

or

<u>Case three:</u> $A \epsilon < C_Y^\omega >$ and $B \epsilon < C_Z^\omega >$ .

The proofs of cases one and two are analogous; so it suffices to consider cases one and three.

Case one: $A \epsilon < C_Y^p >$ for some $p \epsilon \omega$ implies $A = < A_1 \cdot A_2 \cdot \cdots \cdot A_p >$ where $A_k \epsilon S_Y$ for each $k = 1, 2, \cdots, p$. $B \epsilon < C_Z^\omega >$ implies $B = < \prod_{k=1}^{n} B_k \cdot \overline{B}^\omega >$ for some $n \epsilon \omega$ where $B_1, B_2, \cdots B_n$, $\overline{B} \epsilon S_Z$. But then, from Lemma 5.13

$$\phi[<A_1 \cdot \cdots \cdot A_p> \cdot <\prod_{k=1}^{n} B_k \cdot \overline{B}^{\,\omega}>]$$

$$= <A_1 \cdot A_2 \cdot \cdots \cdot A_p \cdot B_1 \cdot \cdots \cdot B_n \cdot \overline{B}^{\,\omega}> .$$

The expression on the right belongs to $C^X$ since $S_X$ is the sum of the $\sigma$-algebras $S_X$ and $S_Z$ (i.e., $S_Y \subset S_X$ and $S_Z \subset S_X$ ).

Thus

$$\phi(A \cdot B) \in C^X .$$

Case three: $A \in <C_Y^{\omega}>$ and $B \in <C_Z^{\omega}>$ imply that for some $m, n \in \omega$

$$A = <\prod_{k=1}^{n} A_k \cdot \overline{A}^{\omega}>$$

$$B = <\prod_{k=1}^{m} B_k \cdot \overline{B}^{\,\omega}>$$

where $A_k, \overline{A} \in S_Y$ for $k = 1, 2, \cdots, n$ and

$B_k, \overline{B} \in S_Z$ for $k = 1, 2, \cdots, m$.

Then, again using Lemma 5.13,

$$\phi(A \cdot B) = <\prod_{k=1}^{n} A_k \cdot \prod_{k=1}^{m} B_k \cdot (\overline{A} \cdot \overline{B})^{\omega}> .$$

Let $C = \prod_{k=1}^{n} A_k \cdot \prod_{k=1}^{m} B_k$. Then $\phi(A \cdot B) = \langle C \cdot (\overline{A} \cdot \overline{B})^{\omega} \rangle$.

Using Lemma 6.4

$$\phi(A \cdot B) = \langle C \cdot (\overline{A} + \overline{B})^{\omega} \rangle - \sum_{k=0}^{\infty} [\langle C \cdot A^k \cdot \overline{B}^{\omega} \rangle + \langle C \cdot B^k \cdot \overline{A}^{\omega} \rangle].$$

But $\langle C \cdot (\overline{A} + \overline{B})^{\omega} \rangle$, $\langle C \cdot \overline{A}^k \cdot \overline{B}^{\omega} \rangle$, and $\langle C \cdot \overline{B}^k \cdot \overline{A}^{\omega} \rangle$ are all members of $C^X$ since $S_Y \subset S_X$ and $S_Z \subset S_X$. Then $\phi(A \cdot B)$ is the difference between a member of $C^X$ and a countable union of members of $C^X$. Thus,

$$\phi(A \cdot B) \in S(C^X)$$

so that

$$\phi(C^Y \cdot C^Z) \subset S(C^X).$$

The last two lemmas suffice to prove that $\phi$ is an isomorphism. That is,

<u>Theorem 6.7.</u> Let $(X,S) = (Y,S_Y) + (Z,S_Z)$. Then

$$\text{Exp}\,(Y,S_Y) \cdot \text{Exp}\,(Z,S_Z) \underset{\phi}{\simeq} \text{Exp}\,(Y + Z,\ S_Y + S_Z)$$

Proof: By the previous two lemmas

$$\phi[C^Y \cdot C^Z] \subset S(C^X),$$

and

$$(\phi)^{-1}[C^X] \subset S(C^Y \cdot C^Z) .$$

But both $\phi$ and $\phi^{-1}$ are functions. Therefore, using Theorem 2.6,

a. $$\phi[S(C^Y \cdot C^Z)] = S\{\phi[C^Y \cdot C^Z]\} \subset S(C^X)$$

and

b. $$\phi^{-1}[S(C^X)] = S\{\phi^{-1}[C^X]\} \subset S(C^Y \cdot C^Z).$$

Combining (a) and (b)

$$\phi[S(C^Y \cdot C^Z)] = S(C^X) .$$

That is, $\phi$ is a measurability isomorphism.

Now let $X = X_1 + X_2 + \cdots + X_n$ be a decomposition of $X$ and let

$$(X, S) = (X_1, S_1) + (X_2, S_2) + \cdots + (X_n, S_n)$$

be the corresponding decomposition of $(X, S)$ according to Theorem 2.4. Let $\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k)$ denote the ordinary direct product of the measurable spaces $\text{Exp}\,(X_k, S_k)$, $k = 1, 2, \cdots, n$. To prove

$$\prod_{k=1}^{n} \text{Exp}\,(X_k, S_k) \underset{\phi_n}{\simeq} \text{Exp}\,[\sum_{k=1}^{n} (X_k, S_k)]$$

for each $n \in \omega$, we use mathematical induction and a decomposition of $\phi_n$. For each $A \in \prod_{k=1}^{n} \text{Exp}\, S_k$, we must show that $\phi(A) \in \text{Exp}\,(\sum_{k=1}^{n} S_k)$; and, for each $B \in \text{Exp}\,(\sum_{k=1}^{n} S_k)$, we must show that $\phi^{-1}(B) \in \prod_{k=1}^{n} \text{Exp}\, S_k$.

<u>Theorem 6.8.</u> Let $(X, S) = \sum_{k=1}^{n} (X_k, S_k)$. Then

$$\prod_{k=1}^{n} \text{Exp}\,(X_k, S_k) \underset{\phi_n}{\simeq} \text{Exp}\,[\sum_{k=1}^{n} (X_k, S_k)]$$

Proof: We know the theorem is true for $n = 1$ and $n = 2$. For the inductive proof we assume, for some integer $n > 2$, that the theorem holds for all $\ell < n$. From Lemma 5.11, if $(x^1, x^2, \cdots, x^n) \in \prod_{k=1}^{n} \text{Exp}\, X_k$, then

$$\phi_n(x^1, x^2, \cdots, x^n) = \phi_2(x^1, \phi_{n-1}(x^2, x^3, \cdots, x^n)).$$

Let

$$\sigma_1(x^1, x^2, \cdots, x^n) = (x^1, \phi_{n-1}(x^2, x^3, \cdots, x^n)) .$$

By the induction hypothesis

$$\prod_{k=2}^{n} \operatorname{Exp}(X_k, S_k) \underset{\phi_{n-1}}{\simeq} \operatorname{Exp}\Big[\sum_{k=2}^{n} (X_k, S_k)\Big] .$$

Since forming the Cartesian product of sets is an associative operation, we shall treat $(x_1, x_2, \cdots, x_n)$ and $(x_1, (x_2, \cdots, x_n))$ as if they were identical. We conclude from Theorem 2.5 that

$$\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k) \underset{\sigma_1}{\simeq} \operatorname{Exp}(X_1, S_1) \cdot \operatorname{Exp}\Big[\sum_{k=2}^{n} (X_k, S_k)\Big]$$

Now let $\sigma_2 = \phi_2$ applied to $\operatorname{Exp} X_1 \cdot \operatorname{Exp}\big(\sum_{k=2}^{n} X_k\big)$. By Theorem 6.7 (or the induction hypothesis),

$$\operatorname{Exp}(X_1, S_1) \cdot \operatorname{Exp}\Big[\sum_{k=2}^{n} (X_k, S_k)\Big] \underset{\sigma_2}{\simeq} \operatorname{Exp}\Big[\sum_{k=1}^{n} (X_k, S_k)\Big]$$

But $\phi_n = \sigma_2 \circ \sigma_1$, and $\sigma_2 \circ \sigma_1$ is clearly a measurable isomorphism from $\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k)$ into $\operatorname{Exp}[\sum_{k=1}^{n} (X_k, S_k)]$, since the composition of two measurable isomorphisms is a measurable isomorphism.

Now let $X = X_1 + X_2 + \cdots$ be a countable decomposition of $X$ and

$$(X, S) = (X_1, S_1) + (X_2, S_2) + \cdots$$

be the corresponding decomposition of the measurable space $(X, S)$. We wish to show that the function $\tilde{\phi}$ (see Convention 5.9) is an isomorphism of the infinite product space

$$\prod_{n\in\omega} \operatorname{Exp}(X_n, S_n) = (\prod_{n\in\omega} \operatorname{Exp} X_n, \prod_{n\in\omega} \operatorname{Exp} S_n).$$

That is, we will prove that

$$\prod_{n\in\omega} \operatorname{Exp}(X_n, S_n) \underset{\tilde{\phi}}{\simeq} \operatorname{Exp}[\sum_{n\in\omega} (X_n, S_n)] .$$

Recall from Chapter 2; III that the infinite product $\sigma$-algebra $\prod_{k\in\omega} \operatorname{Exp} S_k$ is defined as the $\sigma$-algebra $S(C)$ where

$$C = \{\prod_{k\in\omega} A_k : A_k \in \operatorname{Exp} S_k \text{ and } A_k = \operatorname{Exp} X_k \text{ for all but finitely many } k\}.$$

By analogy with the proof of Theorem 6.7, we begin by showing that $\tilde{\phi}$ maps sets in $C$ into sets in $\text{Exp } S = S(C^X)$.

<u>Lemma 6.9.</u> $\tilde{\phi}[C] \subset \text{Exp } S$.

Proof: Let $A \in C$. Then for some $n \in \omega$

$$A = \prod_{k=1}^{n} A_k \cdot \prod_{k>n} \text{Exp } X_k$$

where $A_k \in \text{Exp } S_k$ for each $k = 1, 2, \cdots, n$. But by Lemma 5.12,

$$\tilde{\phi}(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots) = \phi[\phi(x_1, x_2, \cdots, x_n), \tilde{\phi}(x_{n+1}, x_{n+2}, \cdots)]$$

for each $(x_1, x_2, \cdots) \in \prod_{n \in \omega} \text{Exp } S_n$.

Thus

$$\tilde{\phi}[A] = \phi[\phi[\prod_{k=1}^{n} A_k] \cdot \tilde{\phi}[\prod_{k>n} \text{Exp } X_k]] .$$

But by Theorem 5.8

$$\tilde{\phi}[\prod_{k>n} \text{Exp } X_k] = \text{Exp } (\sum_{k>n} X_k) \in \text{Exp } (\sum_{k>n} S_k) .$$

By Theorem 6.8

$$\phi[\prod_{k=1}^{n} A_k] \in \operatorname{Exp}(S_1 + S_2 + \cdots + S_n).$$

Then using Theorem 6.7

$$\phi[\,\phi[\prod_{k=1}^{n} A_k] \cdot \tilde{\phi}[\prod_{k>n} \operatorname{Exp} X_k]\,] \in \operatorname{Exp} S.$$

That is

$$\tilde{\phi}[C] \subseteq \operatorname{Exp} S.$$

We again turn to the proof that $\tilde{\phi}^{-1}[\operatorname{Exp} S] \subset \prod_{k\in\omega} \operatorname{Exp} S_k$.

Since, by Lemma 6.2, $\operatorname{Exp} S = S(C^X)$, we need only prove, by analogy with Theorem 6.7, that $\tilde{\phi}^{-1}[C^X] \subset \prod_{k\in\omega} \operatorname{Exp} S_k$. We will need the fact (see 2; III) that

$$\prod_{k\in\omega} \operatorname{Exp} S_k = S(C) = S(H)$$

where $H = \{\prod_{k\in\omega} A_k : A_k \in \operatorname{Exp} S_k\}$. We will also need a definition and some lemmas. Lemma 5.12 states

(1) $\quad \tilde{\phi}(a_1, a_2, \cdots) = \phi_2(\phi_N(a_1, \cdots, a_N),\ \tilde{\phi}(a_{N+1}, \cdots))$

for each $N \epsilon \omega$ and for each $(a_1, a_2, \cdots) \epsilon \prod_{k \epsilon \omega} \operatorname{Exp} X_k$. Actually, the right-hand side of equation (1) is a composite function

$$\phi_2 \circ \sigma_N$$

where $\phi_2$ is $\phi$ applied to

$$\operatorname{Exp}(X_1 + \cdots + X_N) \cdot \operatorname{Exp}(X_{N+1} + \cdots)$$

and $\sigma_N$ is defined by,

<u>Definition 6.10.</u> Let $(a_1, a_2, \cdots) \epsilon \prod_{k \epsilon \omega} \operatorname{Exp} X_k$. For each $n \epsilon \omega$ define

$$\sigma_N(a_1, a_2, \cdots) = (\phi_N(a_1, \cdots, a_N), \quad \tilde{\phi}(a_{N+1}, \cdots))$$

With this definition, we have

<u>Lemma 6.11.</u> $\sigma_N$ maps $\prod_{n \epsilon \omega} \operatorname{Exp} X_n$ one-to-one onto

$$\operatorname{Exp}(X_1 + \cdots + X_N) \cdot \operatorname{Exp}(X_{N+1} + \cdots).$$

Proof: Follows immediately from the fact that $\phi_N$ maps $\prod_{k=1}^{N} \operatorname{Exp} X_k$ one-to-one onto $\operatorname{Exp}(X_1 + \cdots + X_N)$, and that $\tilde{\phi}$

maps $\prod_{k>n} \operatorname{Exp} X_k$ one-to-one onto $\operatorname{Exp}(X_{N+1} + \cdots)$.

The following lemma is also needed.

<u>Lemma 6.12.</u> Let $A$ be a subset of $\operatorname{Exp}(X_1 + \cdots + X_N)$ and let $B$ be a subset of $\operatorname{Exp}(\sum_{k>n} X_k)$. Then

$$\sigma_N^{-1}[A \cdot B] = \phi_N^{-1}[A] \cdot \tilde{\phi}^{-1}[B]$$

Proof: By the definition of $\sigma_N$,

$$\sigma_N[\phi_N^{-1}[A] \cdot \tilde{\phi}^{-1}[B]] = (\phi_N \circ \phi_N^{-1}[A]) \cdot (\tilde{\phi} \circ \tilde{\phi}^{-1}[B]) = A \cdot B$$

We are now ready to prove

<u>Lemma 6.13.</u> $\tilde{\phi}^{-1}[C^X] \subset \prod_{k \in \omega} \operatorname{Exp} S_k$

Proof: Recall Definition 6.1 for $C^X$. If $A \in C^X$, then

a. $A \in \langle C_X^n \rangle$ for some $n \in \omega$, or

b. $A \in \langle C_X^\omega \rangle$ .

We begin with

Case (a): Suppose $A \in \langle C_X^n \rangle$.

Then $A = \langle A_1 \cdot A_2 \cdot \cdots \cdot A_n \rangle$ where $A_k \in S$ for each $k = 1, 2, \cdots, n$.
But $S = \sum^*_{k \in \omega} S_k$. Thus, each $A_k$ can be uniquely expressed as

$$A_k = \sum_{i \in \omega} B_{ki}$$

where each $B_{ki} \in S_i$. Then

$$\prod_{k=1}^{n} A_k = \prod_{k=1}^{n} [\sum_{i \in \omega} B_{ki}]$$

can be expressed as the union of all sets $C_1 \cdot C_2 \cdot \cdots \cdot C_n$ where $C_k = B_{ki}$ for some $i \in \omega$. But the number of such sets is countable. Thus $\tilde{\phi}^{-1}(A)$ can be expressed as a countable union of sets of the form

$$\tilde{\phi}^{-1} \langle C_1 \cdot \cdots \cdot C_n \rangle$$

where each $C_k \in S_{i_k}$ for some $i_k \in \omega$. Then without loss of generality, we can assume

$$A = \langle C_1 \cdot C_2 \cdot \cdots \cdot C_n \rangle$$

where each $C_k \in S_{i_k}$ for some $i_k \in \omega$. Let

$$N = \max\{i_1, i_2, \cdots, i_n\}.$$

Then,

$$C_k \in \sum_{k=1}^{N}{}^{*} S_k \quad \text{for each} \quad k = 1, \cdots, n.$$

Thus,

(1) $$A = \langle C_1 \cdot C_2 \cdot \cdots \cdot C_n \rangle \in \operatorname{Exp}\left(\sum_{k=1}^{N}{}^{*} S_k\right).$$

Since $\tilde{\phi} = \phi_2 \circ \sigma_N$, we have $\tilde{\phi}^{-1} \circ \phi_2^{-1}$. Here $\phi_2^{-1}$ maps $\operatorname{Exp}(X_1 + \cdots + X_N) \cdot \operatorname{Exp}(X_{N+1} + \cdots)$. Thus, using Lemma 5. 16,

$$\phi_2^{-1}[A] = \langle C_1 \cdot \cdots \cdot C_n \rangle \cdot \{e^N\}] .$$

Therefore

$$\tilde{\phi}^{-1}[A] = \sigma_N^{-1} \circ \phi_2^{-1}[A] = \sigma_N^{-1}[\langle C_1 \cdot \cdots \cdot C_n \rangle \cdot \{e^N\}] .$$

But by Lemmas 6. 12 and 5. 17,

$$\sigma_N^{-1}[\langle C_1 \cdot \cdots \cdot C_n \rangle \cdot \{e^N\}] = \phi_N^{-1}\langle C_1 \cdot \cdots \cdot C_n \rangle \cdot \prod_{k=N+1}^{\infty} \{e_k\} .$$

Since, using equation (1), $\langle C_1 \cdot \cdots \cdot C_n \rangle \in \sum_{k=1}^{N}{}^{*} S_k$ it follows from

Theorem 6.8 (the isomorphism theorem for finite products) that

$$\phi_N^{-1} < C_1 \cdot \cdots \cdot C_N > \epsilon \prod_{k=1}^{N} \operatorname{Exp} S_k .$$

But, since $\{e_k\} \epsilon \operatorname{Exp} S_k$ for each $k > N$, it follows from the definition of an infinite product $\sigma$-ring (see 2; III) that

$$\prod_{k>n} \{e_k\} \ \epsilon \prod_{k>n} \operatorname{Exp} S_k .$$

But then

$$\phi_N^{-1} < C_1 \cdot C_2 \cdot \cdots \cdot C_n > \cdot \ \tilde{\phi}^{-1}\{e^N\} \ \epsilon \prod_{k \epsilon \omega} \operatorname{Exp} S_k .$$

Thus

$$\tilde{\phi}^{-1}[A] \ \epsilon \prod_{k \epsilon \omega} \operatorname{Exp} S_k .$$

Therefore

$$\tilde{\phi}^{-1} < C_X^n > \subset \prod_{k \epsilon \omega} \operatorname{Exp} S_k \quad \text{for each} \quad n \epsilon \omega .$$

Case (b) : $\quad A \epsilon < C_X^{\omega} >$

Then, for some $n \epsilon \omega$

$$A = \langle \prod_{k=1}^{n} A_k \cdot \bar{A}^{\omega} \rangle$$

where $\bar{A} \in S$ and $A_1, A_2, \cdots, A_n \in S$.

Using the same argument employed in case (a), we can assume each $A_k \in \sum_{k=1}^{N} {}^{*} S_k$ for some $N \in \omega$. Then, since $S = \sum_{k=1}^{N} {}^{*} S_k$, we have $\bar{A} = B + D$ where $B \in \sum_{k=1}^{N} {}^{*} S_k$ and $D \in \sum_{k>N}^{*} S_k$.

Thus

$$A = \langle A_1 \cdot A_2 \cdot \cdots \cdot A_n \cdot (B + D)^{\omega} \rangle .$$

Using Lemma 6. 4

$$\langle A_1 \cdot A_2 \cdot \cdots \cdot A_n \cdot (B+D)^{\omega} \rangle = A_1 \cdot \cdots \cdot A_n \cdot (B \cdot D)^{\omega} \rangle$$

$$+ \sum_{k \in \omega} [\langle A_1 \cdot \cdots \cdot A_n \cdot B^k \cdot D^{\omega} \rangle$$

$$+ \langle A_1 \cdot \cdots \cdot A_n \cdot D^k \cdot B^{\omega} \rangle ] .$$

Using Lemmas 5. 14 and 6.4 on the right-hand side of this equation, we have

$$\phi_2^{-1}[\langle A_1 \cdot \cdots \cdot A_n \cdot (B+D)^{\omega} \rangle] = \langle A_1 \cdot \cdots \cdot A_n \cdot B^{\omega} \rangle \cdot \langle D^{\omega} \rangle$$

$$+ \sum_{k \in \omega} [\langle A_1 \cdot \cdots \cdot A_n \cdot B^k \rangle \cdot \langle D^{\omega} \rangle$$

$$+ \langle A_1 \cdot \cdots \cdot A_n \cdot B^{\omega} \rangle \cdot \langle D^k \rangle ] \; .$$

But

$$(0) \quad \langle A_1 \cdot \cdots \cdot A_n \cdot B^{\omega} \rangle \quad \text{and} \quad \langle A_1 \cdot \cdots \cdot A_n \cdot B^k \rangle \in \mathrm{Exp}\,( \sum_{k=1}^{N}{}^{*} S_k ) \, ,$$

whereas

$$\langle D^k \rangle \quad \text{and} \quad \langle D^{\omega} \rangle \in \mathrm{Exp}\,( \sum_{k > N}^{*} S_k ) \, .$$

Since $\widetilde{\phi}^{-1} = \sigma_N^{-1} \circ \phi_2^{-1}$,

$$\widetilde{\phi}^{-1}(A) = \sigma_N^{-1}[\langle A_1 \cdot \cdots \cdot A_n \cdot B^{\omega} \rangle \cdot \langle D^{\omega} \rangle]$$

$$+ \sum_{k=1}^{\infty} \sigma_N^{-1}[\langle A_1 \cdot \cdots \cdot A_n \cdot B^k \rangle \cdot \langle D^{\omega} \rangle ]$$

$$+ \sum_{k=1}^{\infty} \sigma_N^{-1}[\langle A_1 \cdot \cdots \cdot A_n \cdot B^{\omega} \rangle \cdot \langle D^k \rangle ] \; .$$

Using Lemma 6. 12,

$$(1) \qquad \tilde{\phi}^{-1}(A) = \phi_N^{-1}\langle A_1 \cdot \cdots \cdot A_n \cdot B^\omega \rangle \cdot \tilde{\phi}^{-1}\langle D^\omega \rangle$$

$$+ \sum_{k=1}^{\infty} [\, \phi_N^{-1}\langle A_1 \cdot \cdots \cdot A_n \cdot B^k \rangle \cdot \tilde{\phi}^{-1}\langle D^\omega \rangle$$

$$+ \phi_N^{-1}\langle A_1 \cdot \cdots \cdot A_n \cdot B^\omega \rangle \cdot \tilde{\phi}^{-1}\langle D^k \rangle \,] \, .$$

Theorem 6.8 tells us that

$$(2) \qquad \prod_{k=1}^{N} \operatorname{Exp}(X_k, S_k) \underset{\phi_n}{\simeq} \operatorname{Exp}\left[\sum_{k=1}^{N} (X_k, S_k)\right] .$$

Thus (0) implies that $\phi_N^{-1}\langle A_1 \cdot \cdots \cdot A_n \cdot B^\omega \rangle$ and $\phi_N^{-1}\langle A_1 \cdot \cdots \cdot A_n \cdot B^k \rangle \in \prod_{k=1}^{N} \operatorname{Exp} S_k$. From case (a) we have

$$(3) \qquad \tilde{\phi}^{-1}\langle D^k \rangle \in \prod_{i>N} \operatorname{Exp} S_i \quad \text{for each} \quad k \in \omega .$$

It remains to prove that

$$\tilde{\phi}^{-1}\langle D^\omega \rangle \in \prod_{k>N} \operatorname{Exp} S_k .$$

But

$$(4) \qquad \langle D^\omega \rangle = \operatorname{Exp} D - \exp D.$$

Thus

$$\tilde{\phi}^{-1}\langle D^{\omega}\rangle = \tilde{\phi}^{-1}[\operatorname{Exp} D] - \tilde{\phi}^{-1}[\exp D].$$

Again using case (a),

$$(5) \qquad \tilde{\phi}^{-1}[\exp D] \in \prod_{k>N} \operatorname{Exp} S_k .$$

For each $k > N$, let $D_k = X_k \cap D$. Since $D_k \subset X_k$, we have $\prod_{k>N} \operatorname{Exp} D_k \subset \prod_{k>N} \operatorname{Exp} X_k$. Moreover, if $\alpha = (\alpha^k : k > N) \in \prod_{k>N} \operatorname{Exp} X_k$, it is clear that $\tilde{\phi}(\alpha)$ is an unordered sequence in $D$ if and only if each $\alpha^k$ is also in $D$, and hence in $D_k$. That is

$$\tilde{\phi}^{-1}[\operatorname{Exp} D] = \prod_{k>N} \operatorname{Exp} D_k .$$

But since $D_k \in S_k$, $\operatorname{Exp} D_k \in \operatorname{Exp} S_k$ for each $k > N$, so that

$$(6) \qquad \tilde{\phi}^{-1}[\operatorname{Exp} D] \in \prod_{k>N} \operatorname{Exp} S_k .$$

Equations (4), (5), and (6) imply that

$$(7) \qquad \tilde{\phi}^{-1}\langle D^{\omega}\rangle \in \prod_{k>N} \operatorname{Exp} S_k .$$

Equations (1), (2), (3) and (7) now imply that

$$\tilde{\phi}^{-1}[A] \in \prod_{k\in\omega} \operatorname{Exp} S_k .$$

Thus

$$\tilde{\phi}^{-1}[<C_X^{\omega}>] \subset \prod_{k\in\omega} \operatorname{Exp} S_k .$$

Cases (a) and (b) together imply

$$\tilde{\phi}^{-1}[C^X] \subset \prod_{k\in\omega} \operatorname{Exp} S_k .$$

This leads us to,

Theorem 6.14.

$$\prod_{k\in\omega} \operatorname{Exp}(X_k, S_k) \underset{\tilde{\phi}}{\simeq} \operatorname{Exp}\left[\sum_{k\in\omega} (X_k, S_k)\right]$$

Proof: The proof is completely analogous to that of Theorem 6.7. Simply invoke Theorem 2.6.

# CHAPTER 7. THE EXPONENTIAL LAWS FOR MEASURE SPACES

Let $(X, S, \mu)$ be a measure space. Let this space be decomposed (see 2; II) so that

$$(X, S, \mu) = \sum_{n \in I} (X_n, S_n, \mu_n)$$

where $I$ is a finite or countable index set. In the last chapter we showed that the set isomorphism $\phi$ of Chapter 5 is a measurability isomorphism of $\prod_{n \in I} \text{Exp}(X_n, S_n)$ into $\text{Exp}(X, S)$. In this chapter we will show that $\phi$ is also a measure isomorphism of

$\prod_{n \in I} \text{Exp}(X_n, S_n, \mu_n)$ into $\text{Exp}(X, S, \mu)$ provided $I$ is finite.

When $I$ is infinite, it is shown that $\tilde{\phi}$ need not be a measure isomorphism, for the infinite product measure

$(\prod_{n \in \omega} < \mu_n >)$ may not exist. We do show, however, that $\tilde{\phi}$ is a measure isomorphism in the trivial case $\mu(X) = 0$.

We begin with the case $I = \{1, 2\}$. Let $X = Y + Z$ and let

$$(X, S, \mu) = (Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)$$

be the corresponding decomposition of $(X, S, \mu)$. By Convention

2.2, each of the component measure spaces is $\sigma$-finite. Carry out the exponential construction on each of the three measure spaces obtaining $\mathrm{Exp}\,(X,S,\mu)$, $\mathrm{Exp}\,(Y,S_Y,\mu_Y)$ and $\mathrm{Exp}\,(Z,S_Z,\mu_Z)$. Each of these is totally $\sigma$-finite (see Lemma 4.11). Let

$$\mathrm{Exp}\,(Y,S_Y,\mu_Y) \cdot \mathrm{Exp}\,(Z,S_Z,\mu_Z)$$

denote the product of the measure space $\mathrm{Exp}\,(Y,S_Y,\mu_Y)$ with the measure space $\mathrm{Exp}\,(Z,S_Z,\mu_Z)$. We know (see Theorem 6.7) that

$$\mathrm{Exp}\,(Y,S_Y) \cdot \mathrm{Exp}\,(Z,S_Z) \underset{\phi}{\simeq} \mathrm{Exp}\,[(Y,S_Y) + (Z,S_Z)]$$

and (see Theorem 3.14) that

$$\text{(1)} \quad \exp\,(Y,S_Y,\mu_Y) \cdot \exp\,(Z,S_Z,\mu_Z) \underset{\phi}{\simeq} \exp\,[(Y,S_Y,\mu_Y) + (Z,S_Z,\mu_Z)]\,.$$

Equation (1) tells us that if $A \in \exp S_Y * \exp S_Z$, then

$$<\mu_Y> <\mu_Z> (A) = <\mu> (\overline{\phi}[A]).$$

But $\overline{\phi}$ is $\phi$ restricted to $\exp X$, so that

$$<\mu_Y> <\mu_Z> (A) = <\mu> (\phi[A]).$$

Thus, in order to prove

$$\mathrm{Exp}\,(Y,S_Y,\mu_Y) \cdot \mathrm{Exp}\,(Z,S_Z,\mu_Z) \underset{\phi}{\simeq} \mathrm{Exp}\,[(Y,S_Y,\mu_Y) + (Z,S_Z,\mu_Z)]\,,$$

we must show $\langle\mu_Y\rangle\langle\mu_Z\rangle(B) = \langle\mu\rangle(\phi[B])$ for every $B \in \mathrm{Exp}\, S_Y * \mathrm{Exp}\, S_Z - \exp S_Y * \exp S_Z$.

The following lemmas are needed in the proof.

<u>Lemma 7.1.</u> $\mathrm{Exp}\, S_Y * \mathrm{Exp}\, S_Z = \sum^*_{m,n \leq \omega} S\langle C^m_X\rangle * S\langle C^n_Y\rangle$

Proof: By Definition 4.6, $\mathrm{Exp}\, S_Y = \sum^*_{m \leq \omega} S\langle C^m_Y\rangle$ and $\mathrm{Exp}\, S_Z = \sum^*_{n \leq \omega} S\langle C^n_Z\rangle$. Since $S\langle C^m_Y\rangle \subset \mathrm{Exp}\, S_Y$ for each $m \leq \omega$, and since $S\langle C^n_Z\rangle \subset \mathrm{Exp}\, S_Z$ for each $n \leq \omega$,

$$S\langle C^m_Y\rangle * S\langle C^n_Z\rangle \subset \mathrm{Exp}\, S_Y * \mathrm{Exp}\, S_Z.$$

Thus,

$$\sum^*_{m,n \leq \omega} S\langle C^m_Y\rangle * S\langle C^n_Z\rangle \subset \mathrm{Exp}\, S_Y * \mathrm{Exp}\, S_Z .$$

To prove the converse we need only show that for each $A \in \mathrm{Exp}\, S_Y$ and $B \in \mathrm{Exp}\, S_Z$,

$$A \cdot B \in \sum^*_{m,n \leq \omega} S\langle C^m_Y\rangle * S\langle C^n_Z\rangle .$$

Thus, let $A \in \operatorname{Exp} S_Y$ and let $B \in \operatorname{Exp} S_Z$.

Then $A = \sum_{m \leq \omega} A_m$ where $A_m \in S < C_Y^m >$ and

$$B = \sum_{n \leq \omega} B_n \quad \text{where} \quad B_n \in S < C_Z^n > .$$

Then

$$A \cdot B = \left(\sum_{m \leq \omega} A_m\right) \cdot \left(\sum_{n \leq \omega} B_n\right) = \sum_{m, n \leq \omega} A_m \cdot B_n .$$

But

$$\sum_{m, n \leq \omega} A_m \cdot B_n \in \sum_{m, n \leq \omega}^{*} S < C_Y^m > * S < C_Z^n > .$$

Hence

$$\operatorname{Exp} S_Y * \operatorname{Exp} S_Z \subset \sum_{m, n \leq \omega}^{*} S < C_Y^m > * S < C_Z^n > .$$

We can show similarly that

<u>Lemma 7.2.</u> $\exp S_Y * \exp S_Z = \sum_{m, n < \omega}^{*} S < C_Y^m > * S < C_Z^n > .$

Theorem 7.3. Let $(X, S, \mu) = (Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)$.

Then

$$\operatorname{Exp}(Y, S_Y, \mu_Y) \cdot \operatorname{Exp}(Z, S_Z, \mu_Z) \underset{\phi}{\simeq} \operatorname{Exp}[(Y, S_Y, \mu_Y) + (Z, S_Z, \mu_Z)] .$$

Proof: Let $A \in \operatorname{Exp} S_Y * \operatorname{Exp} S_Z$. To show that $<\mu_Y> \cdot <\mu_Z> (A) = <\mu> ([A])$ it suffices, in view of Theorem 3.14, to let

$$A \in \operatorname{Exp} S_Y * \operatorname{Exp} S_Z - \exp S_Y * \exp S_Z .$$

Using Lemmas 7.1 and 7.2,

$$\operatorname{Exp} S_Y * \operatorname{Exp} S_Z - \exp S_Y * \exp S_Z = \sum_{m \leq \omega}^{*} S < C_Y^{\omega} > * S < C_Z^{m} >$$

$$+ \sum_{n < \omega}^{*} S < C_Y^{n} > * S < C_Z^{\omega} > .$$

Therefore $A$ may be decomposed into a disjoint union of the form

$$A = \sum_{m \leq \omega} A_m + \sum_{n < \omega} B_m$$

where $A_m \in S < C_Y^{\omega} > * S < C_Z^{m} >$ and $B_n \in S < C_Y^{n} > * S < C_Z^{\omega} >$ .

This being the case, we can, without loss of generality, assume that

$$A \in S\langle C_Y^{\omega}\rangle * S\langle C_Z^{n}\rangle \quad \text{for some} \quad n \in \omega,$$

or that

$$A \in S\langle C_Y^{m}\rangle * S\langle C_Z^{\omega}\rangle \quad \text{for some} \quad m \leq \omega .$$

We prove only the first case, the proof of the second case being completely analogous.

Since $A \in S\langle C_Y^{\omega}\rangle * S\langle C_Z^{n}\rangle$, we have $A \subset \langle Y^{\omega}\rangle \cdot \langle Z^{n}\rangle$. But

$$\langle\mu_Y\rangle \cdot \langle\mu_Z\rangle (\langle Y^{\omega}\rangle \cdot \langle Z^{n}\rangle) = \hat{\mu}_Y^{\omega}(\langle Y^{\omega}\rangle) \cdot \hat{\mu}_Z^{n}(\langle Z^{n}\rangle) = 0$$

since $\hat{\mu}_Y^{\omega}(\langle Y^{\omega}\rangle) = 0$.

Thus $\langle\mu_Y\rangle \cdot \langle\mu_Z\rangle (A) = 0$.

Now consider $\langle\mu\rangle[\phi(A)]$. Since $A \subset \langle Y^{\omega}\rangle \cdot \langle Z^{n}\rangle$, it follows that $\phi(y, z) \in \langle X^{\omega}\rangle$ for each $(y, z) \in A$. Thus

$$\phi[A] \subset \langle X^{\omega}\rangle$$

so that

$$\langle\mu\rangle (\phi[A]) = 0 .$$

Therefore

$$\langle\mu_Y\rangle \langle\mu_Z\rangle (A) = \langle\mu\rangle (\phi[A])$$

for each $A \in \operatorname{Exp} S_Y * \operatorname{Exp} S_Z$.

The previous theorem generalizes easily to the case $I = \{1, 2, \cdots, n\}$. Let $X = \sum_{k=1}^{n} X_k$ and (see 2; II) let

$$(X, S, \mu) = \sum_{k=1}^{n} (X_k, S_k, \mu_k)$$

be the corresponding decomposition of $(X, S, \mu)$. Carry out the exponential construction on each of these measure spaces. By Lemma 4.11 each of the measure spaces $\operatorname{Exp}(X_k, S_k, \mu_k)$ is totally $\sigma$-finite. Let $\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k, \mu_k)$ denote the product of the $n$ measure spaces $\operatorname{Exp}(X_k, S_k, \mu_k)$, $k = 1, 2, \cdots, n$. By Theorem 6.8

$$\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k) \underset{\phi_n}{\simeq} \operatorname{Exp}\left[\sum_{k=1}^{n} (X_k, S_k)\right]$$

We now prove

<u>Theorem 7.4.</u> Let $(X, S, \mu) = \sum_{k=1}^{n} (X_k, S_k, \mu_k)$. Then

$$\prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k, \mu_k) \underset{\phi}{\simeq} \operatorname{Exp}\Big[\sum_{k=1}^{n} (X_k, S_k, \mu_k)\Big]$$

Proof: The theorem is trivial in case $n = 1$, since $\phi_1$ is the identity function on $\operatorname{Exp} X$. It is true in case $n = 2$ by Theorem 7.3. Proceeding by induction, suppose the theorem is true for $k = n-1$, where $n > 2$. Then

$$(1) \qquad \prod_{k=2}^{n} \operatorname{Exp}(X_k, S_k, \mu_k) \underset{\phi_{n-1}}{\simeq} \operatorname{Exp}\Big[\sum_{k=2}^{n} (X_k, S_k, \mu_k)\Big]$$

Let $\sigma(x_1, \cdots, x_n) = (x_1, \phi_{n-1}(x_2, \cdots, x_n))$ for each $(x_1, x_2, \cdots, x_n) \in \prod_{k=1}^{n} \operatorname{Exp} X_k$. Then using Theorem 2.5 and equation (1), we have

$$(2) \qquad \prod_{k=1}^{n} \operatorname{Exp}(X_k, S_k, \mu_k) \underset{\sigma}{\simeq} \operatorname{Exp}(X_1, S_1, \mu_1) \cdot \operatorname{Exp}\Big[\sum_{k=2}^{n} (X_k, S_k, \mu_k)\Big]$$

But

$$(3) \qquad \operatorname{Exp}(X_1, S_1, \mu_1) \cdot \operatorname{Exp}\Big[\sum_{k=2}^{n} (X_k, S_k, \mu_k) \underset{\phi_2}{\simeq} \operatorname{Exp}\Big[\sum_{k=1}^{n} (X_k, S_k, \mu_k)\Big]$$

and $\phi_n = \phi_2 \circ \sigma$ .

Thus equations (2) and (3), plus the transitivity of measure isomorphisms, imply that

$$\prod_{k=1}^{n} \text{Exp }(X_k, S_k, \mu_k) \underset{\phi_n}{\simeq} \text{Exp }[\sum_{k=1}^{n} (X_k, S_k, \mu_k)] \ .$$

We now turn to the question of whether $\tilde{\phi}$ is a measure isomorphism in the case of a countably infinite decomposition,

$$(X, S, \mu) = \sum_{k \in \omega} (X_k, S_k, \mu_k) \ .$$

We first show that $\tilde{\phi}$ is an isomorphism in the trivial case $\mu(X) = 0$. In this case we have

$$<\mu> (\text{Exp } X) = e^{\mu(X)} = 1$$

so that $<\mu>$ is a probability measure. Furthermore this measure is concentrated at the single point $e$, since by Definitions 3. 12 and 4. 9

$$<\mu> (<X^0>) = <\mu> (\{e\}) = 1$$

That is,

$$\langle\mu\rangle\,(A) = \begin{cases} 1 & \text{if } e \in A \\ 0 & \text{if } e \notin A \end{cases}$$

Similarly, for each $k \in \omega$, the measure $\langle\mu_k\rangle$ is a probability measure on $\operatorname{Exp} X_k$, concentrated at the single point $e_k = e_{X_k}$; for $X_k \subset X$ so that $\mu(X_k) = 0$. It follows that the infinite product measure space $\prod_{k \in \omega} \operatorname{Exp}(X_k, S_k, \mu_k)$ is defined, and is a probability space (see 2; III). Moreover, the infinite product measure $\prod_{k \in \omega} \langle\mu_k\rangle$ is concentrated at the single point $(e_1, e_2, \cdots)$ since

$$\left(\prod_{k \in \omega} \langle\mu_k\rangle\right)(\{e_1\} \cdot \{e_2\} \cdot \cdots) = \prod_{k \in \omega} \langle\mu_k\rangle(\{e_k\}) = 1$$

The fact that $\tilde{\phi}$ is a measure isomorphism will now follow from

<u>Lemma 7.5.</u> Let $(X, S, \mu)$ and $(Y, T, \nu)$ be measure spaces such that $\mu$ is concentrated at a single point $x \in X$ and $\nu$ is concentrated at a single point $y \in Y$. Suppose there is a measurability isomorphism $\phi$ from $(X, S)$ to $(Y, T)$, such that $\phi(x) = y$. If $\mu(\{x\}) = \nu(\{y\})$, then $\phi$ is a measure isomorphism.

Proof: Let $A \in S$. If $x \in A$, then $y \in \phi[A]$ and we have

$$\mu(A) = \mu(\{x\}) = \nu(\{y\}) = \nu(\phi[A]).$$

If $x \notin A$ then $y \notin \phi(A)$ and

$$\mu(A) = \nu(\phi[A]) = 0$$

Thus $\phi$ is a measure isomorphism.

Theorem 7.6. Let $(X, S, \mu) = \sum_{k \in \omega} (X_k, S_k, \mu_k)$. If $\mu(X) = 0$, then

$$\prod_{k \in \omega} \operatorname{Exp}(X_k, S_k, \mu_k) \underset{\phi}{\cong} \operatorname{Exp}\left[\sum_{k \in \omega} (X_k, S_k, \mu_k)\right]$$

Proof: By Theorem 6.14, $\tilde{\phi}$ is a measurability isomorphism of $\prod_{k \in \omega} \operatorname{Exp}(X_k, S_k)$ into $\operatorname{Exp}\left[\sum_{k \in \omega} (X_k, S_k)\right]$. By Lemma 5.17

$$\tilde{\phi}(e_1, e_2, \cdots) = e \quad .$$

The theorem now follows directly from Lemma 7.5 and the remarks preceding it.

The last theorem is not particularly interesting. Is a more general theorem possible? In general, if $\langle\mu\rangle(\operatorname{Exp} X_n) \neq 1$ for infinitely many $n$, the measure space $\prod_{n \in \omega} \operatorname{Exp}(X_n, S_n, \mu_n)$ may not exist. To see this, let $(X, S, \mu)$ be the real line with Lebesgue measure $\mu$. Let $X = \sum_{n \in \omega} X_n$, and choose the sequence $\{X_n\}$ so that $\mu(X_{2n}) = 2$ and $\mu(X_{2n-1}) = \frac{1}{2}$. Then $(\prod_{n \in \omega} <\mu_n>)(\prod_{n \in \omega} <X_n>)$ is not defined. Thus, $\prod_{n \in \omega} \operatorname{Exp}(X_n, S_n, \mu_n)$ does not exist and there is no isomorphism to discuss.

There may be many cases where $0 < <\mu> (\operatorname{Exp} X_n) \neq 1$ for infinitely many $n \in \omega$, and yet $\prod_{n \in \omega} \operatorname{Exp}(X_n, S_n, \mu_n)$ is defined. If this is the case, must it follow that $\prod_{n \in \omega} \operatorname{Exp}(X_n, S_n, \mu_n)$ is isomorphic under $\tilde{\phi}$ to $\operatorname{Exp}[\sum_{n \in \omega} (X_n, S_n, \mu_n)]$ ? This remains an interesting open question.

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