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Thomas A. Schmidt

In 1976, F. Hirzebruch [9] showed that the smooth model Y_2 of the Hilbert modular surface of level 2 for the field $Q(\sqrt{5})$ is isomorphic to the Klein surface of the icosahedron obtained from a blowing-up of $\mathbb{P}_2(\mathbb{C})$. Later, T. Schmidt [16] showed that under this isomorphism, the Cohen-Wolfart embedding of the non-arithmetic Hecke group of signature $(2, 5, \infty)$ has a lift in Y_2 that corresponds to six specific exceptional divisors of the Klein surface.

In this dissertation, we consider the Klein A_5-invariants of $\mathbb{P}_2(\mathbb{C})$ as seen in the Klein surface. We show, using Hirzebruch's isomorphism, that their images in the Hilbert modular surface X of level one are curves uniformized by non-compact and non-arithmetic triangle groups contained in the Hilbert modular group. We also give a correspondence between the A_5-orbits of $\mathbb{P}_2(\mathbb{C})$ and the elliptic singularities of the surface X.

Hilbert Modular Surfaces and Uniformizing Groups of Klein Invariants

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Andrea Moreira Bell

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HILBERT MODULAR SURFACES AND UNIFORMIZING GROUPS OF KLEIN INVARIANTS

1 INTRODUCTION

A modular curve is the quotient of the Poincaré half-plane $\mathbb{H}$ by a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, acting by fractional linear transformations. Similarly, a Hilbert modular surface is the quotient of $\mathbb{H}^2$ by a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O})$, with $\mathcal{O}$ the ring of integers of a real quadratic field K. These surfaces are particular examples of Shimura varieties. In 1990, P. Cohen and J. Wolfart showed [3] that the algebraic curve uniformized by any triangle Fuchsian group can be embedded into some Shimura variety.

On the other hand, numerous interesting results about the geometry of Hilbert modular surfaces have been obtained by F. Hirzebruch. We will be considering in this dissertation the particular case of the number field $K = \mathbb{Q}(\sqrt{5})$ and its corresponding compactified Hilbert modular surface of level 2, X_2. In 1976, Hirzebruch proved [10] that the smooth surface Y_2 obtained by resolving the singularities of X_2 is isomorphic to a surface obtained by blowing-up 10 points of the cubic surface of 27 lines and is also isomorphic to the Klein surface of the icosahedron.

Furthermore, it was shown by Hirzebruch that 15 of the 27 lines of the cubic surface and the 10 exceptional divisors of a blowing-up of that surface, correspond in Y_2 to the lines resolving the cuspidal resolutions of X_2 and to the image of the diagonal $\{z_1 = z_2\}$ of $\mathbb{H}^2$ respectively. In 1997, T. Schmidt identified an isomorphism between the remaining 12 lines of the cubic surface and the lift to Y_2 of the curve arising from the Cohen-Wolfart embedding of the non-arithmetic Hecke group of signature $(2, 5, \infty)$ in $\mathrm{PSL}_2(\mathcal{O}_K)$. Therefore one has now a modular interpretation of all 27 lines and a geometric interpretation of the modular embedding of the groups of signature $(2, 5, \infty)$. Moreover, in [16], C. McMullen of Harvard University has recently shown the existence of algebraic curves on Hilbert modular surfaces which are not uniformized by any subgroup of a triangle group.

We consider the Klein invariants A, B and C and their corresponding images in the Klein surface of the icosahedron given by the curves $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. We show that under Hirzebruch's isomorphism, the images of these curves in the Hibert modular surface X of level 1 are curves uniformized by non-arithmetic, non-compact triangle groups of signatures $(3, 5, \infty)$ and $(5, \infty, \infty)$. In passing, we give two interpretations of the A_5-orbits of length less than 60 of $\mathbb{P}_2(\mathbb{C})$, in terms of the elliptic singularities of the surface X. The first interpretation uses the A_5-covering of X by X_2. The second one uses the A_5-covering of X by the Hilbert modular surface of level $\sqrt{5}$.

We also study the Cohen-Wolfart embedding of a group of signature $(5, \infty, \infty)$ and give an attempt to interpret its pre-image in Y_2 in terms of a curve of the Klein surface. This last part is inconclusive at this point and the author intends to return to it in the near future.

Finally, in order to illustrate the complexity that lies behind understanding the modular embedding of curves in Hilbert modular surfaces, we give the example of the modular curve F_5 and we correct some small errors found in the literature.

2 BACKGROUND

2.1 Fuchsian Groups

We give here a brief overview of Fuchsian groups. For more details refer, for example, to [12, 21].

2.1.1 The Group $\mathrm{PSL}_2(\mathbb{R})$

Consider the Poincaré half-plane $\mathbb{H} = \{z \in \mathbb{C};\ \Im(z) > 0\}$ with the hyperbolic metric and the group of matrices

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

Denote by I_2 the identity matrix in $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\}$ acts on $\mathbb{H}$ by fractional linear transformations. For $\gamma \in \mathrm{SL}_2(\mathbb{R})$ as above, denote again by γ its image in $\mathrm{PSL}_2(\mathbb{R})$. The action is defined by:

$$\gamma : z \in \mathbb{H} \mapsto \gamma(z) := \frac{az + b}{cz + d}.$$

Under this action, $\mathrm{PSL}_2(\mathbb{R})$ can be seen as the group $\mathrm{Isom}^+(\mathbb{H})$ of orientation-preserving isometries of $\mathbb{H}$. An element γ of $\mathrm{PSL}_2(\mathbb{R})$ is said to be parabolic if its trace, $\mathrm{tr}(\gamma)$, equals 2. It is called elliptic if $\mathrm{tr}(\gamma) < 2$ and hyperbolic if $\mathrm{tr}(\gamma) > 2$. Elliptic elements have a unique fixed point in $\mathbb{H}$. Parabolic elements do not have fixed points in $\mathbb{H}$ but have one in its Euclidean boundary, $\mathbb{R} \cup \{\infty\}$; and hyperbolic elements have two fixed points in $\mathbb{R} \cup \{\infty\}$.

The group $\mathrm{PSL}_2(\mathbb{R})$ is also a topological space, for the element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ can be identified with the point $(a, b, c, d) \in \mathbb{R}^4$.

Definition 1. *A Fuchsian group is a discrete subgroup of* $PSL_2(\mathbb{R})$*, with respect to the induced topology.*

The following definitions can be given in a more general context but here we will consider only the metric space $\mathbb{H}$ and groups G of homeomorphisms of $\mathbb{H}$.

Definition 2. *We say that a group* G *acts properly discontinuously on* $\mathbb{H}$ *if the* G*-orbit* Gz *of any point* z *of* $\mathbb{H}$ *is locally finite. That is, if for any compact subset* K *of* $\mathbb{H}$, $K \cap (gz) \neq \emptyset$ *for only finitely many* $g \in G$*. Note that we consider a left action of* G *in* $\mathbb{H}$*.*

Definition 3. *Let* G *be a group of homeomorphisms acting properly discontinuously on* $\mathbb{H}$*. A closed region* $\mathcal{R}$ *is defined to be a fundamental domain for* G *if*

i) $\bigcup_{g \in G} g(\mathcal{R}) = \mathbb{H}$

ii) $\mathring{\mathcal{R}} \bigcap g(\mathring{\mathcal{R}}) = \emptyset \quad \forall g \in G - \{Id\}$,

where $\mathring{\mathcal{R}}$ *represents the interior of* $\mathcal{R}$*.*

The family $\{g(\mathcal{R});\ g \in G\}$ *is called a tessellation of* $\mathbb{H}$*.*

Theorem 2.1.1. *Let* G *be a subgroup of* $PSL_2(\mathbb{R})$*. Then* G *is a Fuchsian group if and only if* G *acts properly discontinuously on* $\mathbb{H}$*.*

Proof. See for example [12].

Any Fuchsian group has a connected and convex fundamental domain.

2.1.2 Triangle Fuchsian Groups

Definition 4. *A triangle Fuchsian group of signature* (m_1, m_2, m_3)*, with* $m_i \in \mathbb{N} \cup \{\infty\}$*, is a Fuchsian group generated by three elements* $\gamma_1, \gamma_2, \gamma_3$ *which may be elliptic or parabolic and satisfy the fundamental relations:*

$$\begin{cases} \gamma_1\gamma_2\gamma_3 = I_2 \\ \gamma_i^{m_i} = I_2, \quad (i = 1, 2, 3), \end{cases} \tag{2.1}$$

as well as the inequality

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1 .$$

The parabolic elements correspond to values $m_i = \infty$. *We set* $\frac{1}{m_i} = 0$ *for* $m_i = \infty$.

Geodesics in the hyperbolic plane $\mathbb{H}$ are semicircles and straight lines orthogonal to the real axis $\mathbb{R}$ (see, say [12]). Consider a hyperbolic triangle $\mathcal{F}$ with angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$, where the vertices are the points of $\mathbb{H}$ or $\mathbb{R} \cup \{\infty\}$ fixed by the γ_i, and the edges are the geodesics joining those points. The fundamental domain for a triangle Fuchsian group of signature (m_1, m_2, m_3) is $\mathcal{R} = \mathcal{F} \cup \widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}$ is the reflection of $\mathcal{F}$ about one of its edges. The hyperbolic area of $\mathcal{R}$ is finite and equals

$$\text{area}(\mathcal{R}) := \int_{\mathcal{R}} \frac{\mathrm{d}x\mathrm{d}y}{y^2} = 2\pi(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}), \text{ for } z = x + iy \in \mathbb{H} . \qquad (2.2)$$

The hyperbolic area depends only on the signature of the group. Two triangle groups of the same signature are conjugate in $\mathrm{PSL}_2(\mathbb{R})$.

Consider a triangle Fuchsian group Λ of signature (m_1, m_2, m_3) and its fundamental domain $\mathcal{R}$. The action of Λ on $\mathbb{H}$ induces a natural projection

$$\pi : \mathbb{H} \to \Lambda \backslash \mathbb{H} ,$$

where the points of $\Lambda\backslash\mathbb{H}$ are the Λ-orbits of $\mathbb{H}$. The quotient space $\Lambda\backslash\mathbb{H}$ has finite hyperbolic volume and is an oriented genus zero surface with marked points (Λ-orbits of elliptic fixed points) and cusps (Λ-orbits of parabolic points). The volume is defined as the area of $\mathcal{R}$ given by (2.2). If the group Λ has no parabolic elements, the space $\Lambda\backslash\mathbb{H}$ is compact and Λ is said to be cocompact.

2.1.3 Arithmetic Fuchsian groups

Defining arithmetic Fuchsian groups would require going over some material that is not relevant in the present work. For an accurate definition of arithmeticity, the reader may refer to [12, 20, 21]. Arithmetic groups are characterized by the following theorem proven, for example, in [15].

Theorem 2.1.2. *Let Λ be a Fuchsian group with parabolic elements. Then Λ is arithmetic if and only if it is commensurable with $PSL_2(\mathbb{Z})$.*

Another characterization of arithmetic groups is given in [20] in terms of the field generated by the trace and the square of the trace of all the elements of Λ.

Nevertheless, one can interpret the notion of arithmeticity by saying that if a Fuchsian group Λ is arithmetic then the quotient space $\Lambda\backslash\mathbb{H}$ parameterizes isomorphism classes of abelian varieties.

For example, the group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I_2\}$ given by matrices with coefficients in $\mathbb{Z}$ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting properly discontinuously on $\mathbb{H}$. The $\mathrm{PSL}_2(\mathbb{Z})$-orbit points of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ are in one-to-one correspondence with isomorphism classes of elliptic curves. The group $\mathrm{PSL}_2(\mathbb{Z})$, called the modular group, is in fact a triangle Fuchsian group of signature $(2, 3, \infty)$, which is arithmetic. The quotient $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ is an example of a modular curve. The curves considered in the main result of this text have the interesting property of being uniformized by non-arithmetic triangle groups.

The arithmeticity of a triangle Fuchsian group depends only on its signature, therefore we often say that the signature (m_1, m_2, m_3) is arithmetic (or non-arithmetic). In [21], Takeuchi shows — and gives an explicit list — that there are only finitely many arithmetic triangle signatures.

2.2 Hilbert Modular Surfaces

Hilbert modular surfaces are a generalization of modular curves in dimension 2. Here we are interested on the action of a group on two copies of the Poincaré half-plane, $\mathbb{H} \times \mathbb{H}$. The following is essentially based on [9].

Consider the quadratic number field $K = \mathbb{Q}(\sqrt{5})$. Its ring of integers $\mathcal{O}$ is the rank two $\mathbb{Z}$-module generated by the unit $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$. The Hilbert modular group for the field K is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ defined by

$$\mathrm{SL}_2(\mathcal{O}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathcal{O} \right\} .$$

The group $\Gamma := \mathrm{PSL}_2(\mathcal{O}) = \mathrm{SL}_2(\mathcal{O})/\{\pm I_2\}$ acts properly discontinuously on $\mathbb{H}^2$. For any

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma ,$$

the action is defined by fractional linear transformations

$$(z_1, z_2) \in \mathbb{H}^2 \mapsto (\gamma^{(1)} z_1, \gamma^{(2)} z_2) ,$$

where $\gamma^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ and for any $x \in K$, $x \mapsto x^{(i)}$ is the i-th Galois embedding of K in $\mathbb{R}$. In particular, for $i = 1$ we have the identity and $\gamma^{(1)} = \gamma$.

The Hilbert modular surface of K is the quotient $\Gamma \backslash \mathbb{H}^2$. It is a non-compact surface with finite volume, where the volume is given by the element

$$\omega = \left(\frac{1}{2\pi} \right)^2 \frac{\mathrm{d}x_1 \mathrm{d}y_1}{y_1^2} \frac{\mathrm{d}x_2 \mathrm{d}y_2}{y_2^2}, \quad \text{for } z_j = x_j + i y_j . \tag{2.3}$$

The cusps of Γ are defined as the orbits of Γ in $\mathbb{P}_1(K) = K \cup \{\infty\}$. The number of cusps equals the class number of the field K. For $K = \mathbb{Q}(\sqrt{5})$, that number equals 1. We represent this single cusp by the point ∞ of $\mathbb{P}_1(K)$. The surface $\Gamma\backslash\mathbb{H}^2$ is compactified by adding the cusp ∞. Let

$$X := \overline{\Gamma\backslash\mathbb{H}^2} = \Gamma\backslash\mathbb{H}^2 \cup \{\infty\}$$

be its compactification. It is an algebraic surface with singularities at ∞ and at the elliptic fixed points of Γ. These consist of two points of order 2, two points of order 3 and two points of order 5. Resolving the cusp singularity yields a surface that we denote by Y and we let Z be the smooth surface obtained by resolving the remaining elliptic singularities.

2.3 The Ideal $2\mathcal{O}$

The principal congruence subgroup of level 2 of Γ is defined as

$$\Gamma_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv \pm 1 \bmod 2\mathcal{O}; b \equiv c \equiv 0 \bmod 2\mathcal{O} \right\} .$$

The group Γ_2 is a normal subgroup of Γ. The quotient Γ/Γ_2 is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_4)$ because $\mathcal{O}/2\mathcal{O} \simeq \mathbb{F}_4$. On the other hand, $\mathrm{PSL}_2(\mathbb{F}_4)$ is isomorphic to the alternating group A_5. This isomorphism is induced by the action of $\mathrm{PSL}_2(\mathbb{F}_4)$ on the five points of $\mathbb{P}_1(\mathbb{F}_4)$ (see for example [2]). Therefore we have

$$\Gamma/\Gamma_2 \simeq A_5 .$$

2.3.1 The Cusps of Γ_2

The cusps of the surface Γ_2 are the various orbits of $\mathbb{P}_1(K)$ under the action of Γ_2. There are 5 such orbits. Indeed, consider two points $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$ of $\mathbb{P}_1(K)$ with $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ and $\gcd(\alpha, \beta) = \gcd(\gamma, \delta) = 1$. We will also denote these points by $\frac{\alpha}{\beta} = [\alpha : \beta]$, with the condition $[\alpha : \beta] = [k\alpha : k\beta]$ for any $k \in K^*$. With this notation we have $\infty = [1 : 0]$. The points $[\alpha : \beta]$ and $[\gamma : \delta]$ are conjugate under the action of Γ_2 if and only if there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ_2 such that

$$[a\alpha + b\beta : c\alpha + d\beta] = [\gamma : \delta]$$

in $\mathbb{P}_1(K)$, that is

$$\begin{cases} a\alpha + b\beta = k\gamma \\ c\alpha + d\beta = k\delta\,. \end{cases}$$

for some $k \in K^*$. In fact, we can suppose $k \in \mathcal{O}$ with $k \neq 0$. We know that

$$\begin{cases} a \equiv d \equiv 1 \bmod 2\mathcal{O} \\ b \equiv c \equiv 0 \bmod 2\mathcal{O} \end{cases}$$

and $\alpha, \beta \in \mathcal{O}$, so

$$\begin{cases} a\alpha \equiv \alpha \bmod 2\mathcal{O} \\ b\beta \equiv 0 \bmod 2\mathcal{O} \\ c\alpha \equiv 0 \bmod 2\mathcal{O} \\ d\beta \equiv \beta \bmod 2\mathcal{O}\,; \end{cases}$$

thus we have

$$\alpha \equiv k\gamma \bmod 2\mathcal{O}\,, \quad \beta \equiv k\delta \bmod 2\mathcal{O}$$

and $\mathcal{O}/2\mathcal{O} \simeq \mathbb{F}_4$. Therefore, in $\mathbb{P}_1(\mathbb{F}_4)$ we have

$$[\alpha : \beta] \sim [k\gamma : k\delta] = [\gamma : \delta]\,.$$

It follows that the points $[\alpha : \beta]$ and $[\gamma : \delta]$ belong to the same orbit by the action of Γ_2 in $\mathbb{P}_1(K)$ if and only if they define the same point on $\mathbb{P}_1(\mathbb{F}_4)$. Since $\mathbb{P}_1(\mathbb{F}_4)$ consists of 5 points, there are 5 orbits for the action of Γ_2.

2.3.2 The Surface X_2

The group Γ_2 acts freely on $\mathbb{H}^2$ with 5 cusps. Indeed, the elliptic elements of Γ are given in [7] and one can check that none of them are in Γ_2. The surface $\Gamma_2\backslash\mathbb{H}^2$ is a non-compact surface with finite volume that we compactify by adding five points. The surface

$$X_2 := \overline{\Gamma_2\backslash\mathbb{H}^2}$$

is an algebraic surface with singularities only at the 5 cusps. We denote by Y_2 the surface obtained by resolving these singularities.

The isomorphism $\Gamma/\Gamma_2 \simeq \mathrm{PSL}_2(\mathbb{F}_4)$ and the action of Γ on $\mathbb{H}^2$ induce an action of A_5 on X_2 and an A_5-cover

$$\begin{array}{c} X_2 \\ \Big\downarrow{\scriptstyle \Gamma/\Gamma_2\simeq A_5} \\ X \end{array}$$

The group A_5 acts also on Y_2 and in particular on the resolution of the cusp singularities.

2.4 The Cusp Resolution

In order to fix notation, we will recall without giving details, some of the steps of the resolution of cuspidal singularities as in [9].

Let K be a real quadratic number field with ring of integers $\mathcal{O}$ and M be a $\mathbb{Z}$-module of K of rank 2 (usually we will take $M = \mathcal{O}$). An element $x \in K$ is said to be totally positive if $x^{(1)} > 0$ and $x^{(2)} > 0$, where $x^{(i)}$, $i = 1, 2$ are the images of x under the two Galois embeddings of K into $\mathbb{R}$. Let U_M^+ be the group of totally positive units ε of K such that $\varepsilon M = M$. The elements of U_M^+ are algebraic integers. Let V be a finite index subgroup of U_M^+. Consider also the subgroup of $GL^+(K)$ given by

$$G(M,V) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} ;\ a \in V,\ b \in M \right\}.$$

The group $G(M,V)$ acts on $\mathbb{H}^2$ and the quotient $G(M,V)\backslash\mathbb{H}^2$ is a complex manifold that we compactify by adding a point ∞ and denote

$$\overline{G(M,V)\backslash\mathbb{H}^2} = (\,G(M,V)\backslash\,\mathbb{H}^2\,) \cup \{\infty\}\,.$$

We construct the local ring of $\overline{G(M,V)\backslash\mathbb{H}^2}$ by taking continuous functions on $G(M,V)\backslash\mathbb{H}^2$ such that their restriction to a neighborhood of $\{\infty\}$ are holomorphic. We wish to resolve the singularity of $\overline{G(M,V)\backslash\mathbb{H}^2}$ at the point $\{\infty\}$.

The module M acts on $\mathbb{C}^2$ by translations $(z_1, z_2) \mapsto (z_1 + \lambda^{(1)}, z_2 + \lambda^{(2)})$, where $\lambda \in M$. Consider the subgroup M_+, of totally positive elements of M. For $k \geq 0$, $k \in \mathbb{Z}$, there are bases (A_{k-1}, A_k) of M given by successive boundary points of the convex hull of M_+ in $\mathbb{R}_+$, in such a way that $A^{(1)}_{k-1} > A^{(1)}_k$ and $A^{(2)}_{k-1} < A^{(2)}_k$. For each k there are integers $r \geq 0$ and $b_k \geq 2$ and an $\varepsilon \in V$ such that

$$\begin{cases} A_{k-1} + A_{k+1} = b_k A_k, \\ A_{k+r} = \varepsilon A_k\,. \end{cases} \tag{2.4}$$

One may define a group isomorphism for $k \geq 0$

$$\begin{array}{rcl} \phi:\ M\backslash\mathbb{C}^2 & \longrightarrow & \mathbb{C}^* \times \mathbb{C}^* \\ (z_1, z_2) \bmod M & \longmapsto & (u_k, v_k) \end{array} \tag{2.5}$$

as follows.

Consider the map

$$\begin{array}{rcl} M\backslash\mathbb{C}^2 & \longrightarrow & \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \\ (z_1, z_2) \bmod M & \longmapsto & (y_1, y_2) \bmod \mathbb{Z}^2\,, \end{array} \tag{2.6}$$

such that $z_i = A^{(i)}_{k-1} y_1 + A^{(i)}_k y_2 \bmod M^{(i)}$. This is well defined because M is a $\mathbb{Z}$-module of rank 2 and M acts on $\mathbb{C}^2$ as follows. For all $x = aA_{k-1} + bA_k \in M$; $a, b \in \mathbb{Z}$,

$$(z_1, z_2) \longmapsto (z_1 + x^{(1)}, z_2 + x^{(2)}),$$

with

$$\begin{aligned} z_1 + x^{(1)} &= z_1 + aA^{(1)}_{k-1} + bA^{(1)}_k \\ &= (y_1 + a)A^{(1)}_{k-1} + (y_2 + b)A^{(1)}_k , \\ z_2 + x^{(2)} &= z_2 + aA^{(2)}_{k-1} + bA^{(2)}_k \\ &= (y_1 + a)A^{(2)}_{k-1} + (y_2 + b)A^{(2)}_k . \end{aligned}$$

Finally, we define the map

$$\begin{aligned} \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z} &\longrightarrow \mathbb{C}^* \times \mathbb{C}^* \\ (y_1, y_2) \bmod \mathbb{Z}^2 &\longmapsto (u_k, v_k) = (e^{2i\pi y_1}, e^{2i\pi y_2}) . \end{aligned} \tag{2.7}$$

Therefore ϕ is defined by

$$\begin{cases} 2i\pi z_1 = A^{(1)}_{k-1} \log u_k + A^{(1)}_k \log v_k \bmod (2i\pi M^{(1)}) \\ 2i\pi z_2 = A^{(2)}_{k-1} \log u_k + A^{(2)}_k \log v_k \bmod (2i\pi M^{(2)}) . \end{cases} \tag{2.8}$$

Take two copies of $\mathbb{C}^* \times \mathbb{C}^*$ and the map:

$$\begin{aligned} \psi : \mathbb{C}^* \times \mathbb{C}^* &\longrightarrow \mathbb{C}^* \times \mathbb{C}^* \\ (u_k, v_k) &\longmapsto (u_{k+1}, v_{k+1}) \end{aligned}$$

defined by

$$\begin{cases} u_{k+1} = u_k^{b_k} v_k \\ v_{k+1} = u_k^{-1} . \end{cases} \tag{2.9}$$

The maps ϕ and ψ are compatible under the change of basis of (A_{k-1}, A_k) into (A_k, A_{k+1}), using (2.4). We can extend ψ to $\{(u_k, v_k) \in \mathbb{C}^2;\ u_k \neq 0\}$ and if we glue different copies of $\mathbb{C}^2$ for $k \geq 0$ by this extension $\tilde{\psi}$ (provided it is well defined), we obtain a complex manifold Y which contains a family of rational curves S_k given by $v_k = 0$ on the k-th coordinate system and by $u_{k+1} = 0$ on the $(k+1)$-st coordinate system. The curves S_k and S_{k+1} intersect transversally in one point; the curves S_k and S_{k+l} with $|l| > 1$ intersect nowhere.

The coordinate u_{k+1} given by (2.9) can be viewed as a meromorphic function on Y (see [9], Section 2.2). Suppose $r \geq 2$. The divisor of u_{k+1} is a finite linear combination of the S_k. On the curve S_{k-1}, it has the multiplicity b_k, on S_k the multiplicity 1 and on S_{k+1} the multiplicity 0. So this divisor is linearly equivalent to

$$b_k S_{k-1} + S_k + 0 S_{k+1} = (0) .$$

The intersection of this divisor with S_k is then zero and we obtain

$$b_k + S_k \cdot S_k = 0 .$$

Therefore, the intersection number of S_k is $-b_k$ (and is $(2 - b_k)$ if $r = 1$; see [9], Section 2.4).

We have an exact sequence of groups

$$1 \longrightarrow M \longrightarrow G(M, V) \longrightarrow V \longrightarrow 1 .$$

The group $G(M, V)$ acts on $\mathbb{H}^2$ and M also acts on $\mathbb{H}^2$ by translations, so V acts on $M \backslash \mathbb{H}^2$.

Consider the surface:

$$Y^+ = \phi(M\backslash\mathbb{H}^2) \cup (\cup_{k\geq 0} S_k)\,,$$

where ϕ is the embedding

$$\phi : M\backslash\mathbb{H}^2 \hookrightarrow Y$$

induced by (2.5).

The subgroup V acts on Y and its action is given by a generator $\varepsilon \in V$ as in (2.4) in such a way that ε^n sends a point of Y with coordinates (u_k, v_k) on the k-th coordinate system, to the point with the same coordinates on the $(k + nr)$-th coordinate system.

This action is compatible under ϕ to the action of V on $M\backslash\mathbb{H}^2$. It leaves invariant the surface Y^+ and the action of V on Y^+ is free and properly discontinuous (see [22], lemma 3.1).

We consider now the surface $Y(M, V) = Y^+/V$ which contains a cycle of rational curves S_k; $k \in \mathbb{Z}/r\mathbb{Z}$, with intersection numbers as given above.

The surface $Y(M, V)$ is the resolution at the point $\{\infty\}$ of a normal complex surface isomorphic to $\overline{G(M, V)\backslash\mathbb{H}^2}$, (see [22]).

The choice of the bases of M is realized using continued fractions in the following way. Suppose that the module M is generated by

$$A_{-1} = \omega_0, \quad A_0 = 1\,,$$

with $\omega_0 \in M$ satisfying $0 < \omega_0^{(2)} < 1 < \omega_0^{(1)} = \omega_0$. By setting for all $k \geq 0$,

$$\omega_k = \frac{A_{k-1}}{A_k}\,,$$

the equation (2.4) becomes equivalent to

$$\omega_k = b_k - \frac{1}{\omega_{k+1}}$$

and we have a continued fraction

$$\omega_0 = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots}}}$$

denoted by $((b_0, \ldots, b_r))$.

If we take other values for A_{-1} and A_0, the module M generated by such a basis (A_{-1}, A_0) is strictly equivalent to the module $A_0^{-1}M$ generated by $(\omega_0, 1)$, with $\omega_0 = A_{-1}/A_0$ satisfying $0 < \omega_0^{(2)} < 1 < \omega_0^{(1)} = \omega_0$. We know that there exists a one-to-one correspondence between strict equivalence classes of complete $\mathbb{Z}$-modules in K and isomorphism classes of cyclic singularities with a primitive admissible cycle $((b_0, \ldots, b_r))$; see [9], Section 2.5. So we can always consider the module M as generated by $(\omega_0, 1)$ as above.

2.4.1 The Cusp Resolution of X

Consider again the number field $K = \mathbb{Q}(\sqrt{5})$ and its associated Hilbert modular group $\mathrm{SL}_2(\mathcal{O})$. Denote by U the subgroup of units of $\mathcal{O}$. A fundamental unit is $\varepsilon_0 = \frac{1}{2}(1 + \sqrt{5})$ and the subgroup of totally positive units of $\mathcal{O}$ is

$$U^+ = U^2 = \{\varepsilon_0^{2n};\, n \in \mathbb{Z}\}.$$

As we said in Section 2.2, the compact surface $X = \Gamma \backslash \mathbb{H}^2 \cup \{\infty\}$ has a cusp at the point ∞. The isotropy subgroup Γ_∞ of this cusp is of type $G(M, V)$, with $M = \mathcal{O}$ and $V = U^+ = U^2$.

In the notation of Section 2.4, we can take $\omega_0 = \varepsilon_0^2 = \frac{1}{2}(3+\sqrt{5})$ which satisfies $0 < \omega_0^{(2)} < 1 < \omega_0^{(1)} = \omega_0$ and considering the continued fraction

$$\omega_0 = \frac{1}{2}(3+\sqrt{5}) = 3 - \frac{1}{\omega_0} = 3 - \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{\ddots}}}$$

we obtain, for all $k \geq 0$, $\omega_k = \omega_0$, and $b_k = b_0 = 3$.

On the other hand we have, using (2.4), Section 2.4,

$$A_{-1} = \omega_0$$
$$A_0 = 1$$
$$A_1 = 3 - \omega_0 = \tfrac{1}{2}(3-\sqrt{5})A_0,$$

with $\frac{1}{2}(3-\sqrt{5}) = \omega_0^{-1} \in V$. Thus we have $\varepsilon = \frac{1}{2}(3-\sqrt{5})$ and $r = 1$. Therefore the resolution of the cusp ∞ of X corresponds to a configuration of a unique curve S_0 of self-intersection number $S_0 \cdot S_0 = 2 - b_0 = -1$.

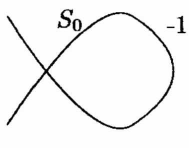

Figure 2.1: Configuration on Y. Resolution of the cusp.

2.4.2 The Cusp Resolution of X_2

The compact surface X_2 contains 5 cusps with respect to the congruence subgroup Γ_2 which we resolve by a surface Y_2. These cusps are conjugated by some element of $\mathrm{SL}_2(\mathbb{R})$, thus it is enough to study the isotropy subgroup of one of these cusps, say ∞ . Denote this subgroup by $\Gamma_{2\infty}$; we have

$$\Gamma_{2\infty} = \left\{ \begin{pmatrix} \varepsilon & \lambda \\ 0 & \varepsilon^{-1} \end{pmatrix} : \varepsilon \in U,\ \varepsilon \equiv 1 \bmod 2\mathcal{O},\ \lambda \in 2\mathcal{O} \right\},$$

But we can also write (see [10])

$$\Gamma_{2\infty} = \left\{ \pm \begin{pmatrix} \varepsilon_0^p & \lambda \\ 0 & \varepsilon_0^{-p} \end{pmatrix} : p \in 3\mathbb{Z}, \lambda \in 2\mathcal{O} \right\}.$$

The image of $\Gamma_{2\infty}$ in $\mathrm{PSL}_2(K)$ is in fact of type $G(2M, V^3) = G(2\mathcal{O}, (U^2)^3)$ which is conjugated by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ to $G(\mathcal{O}, (U^2)^3)$.

We obtain then for each cusp of X_2 a configuration of 3 curves which intersect transversally and whose self-intersection number is (-3). Therefore the surface Y_2 contains five such configurations.

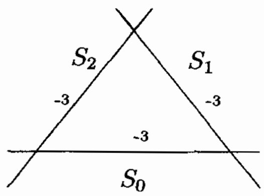

Figure 2.2: Resolution of one cusp on Y_2

2.4.3 The Action of A_5 at the Cusp Resolutions

In order to determine the intersection of some curves in Y and their pull-back in Y_2 with the resolution of the cusp singularities of X and X_2 respectively, we need to understand the action of $\Gamma_\infty/\Gamma_{2\infty}$ on the resolution of the cusp ∞. The following summarizes the discussion given in [10].

The isotropy group of the cusp ∞ for the action of A_5 on X_2 is isomorphic to the alternating group A_4. This subgroup contains a Klein group V. When acting on the resolution of ∞, a given involution $\tau \in V$ leaves invariant each line of the triangle, fixing one of them pointwise and fixing also the opposite vertex. We blow-up that vertex as in Figure 2.3.

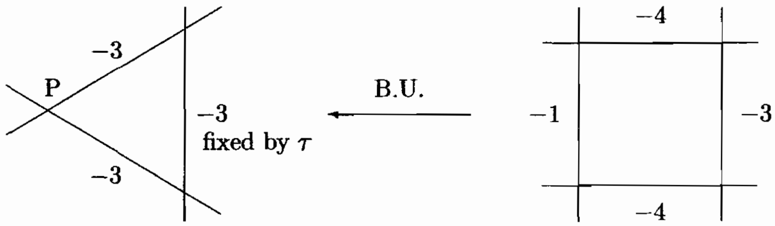

Figure 2.3: The A_5-action on the cusps (a)

The vertical lines are fixed by τ while the horizontal lines are invariant but not pointwise fixed. The other two involutions of V fix the horizontal lines and leave the vertical lines invariant. When we factor out by the action of τ and then by the action of $V/\{1,\tau\}$, we obtain the configuration shown in Figure 2.4.

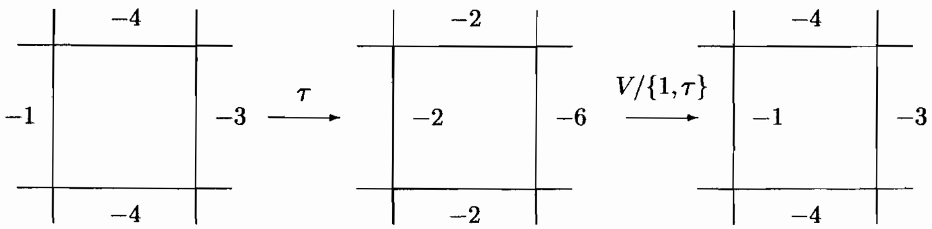

Figure 2.4: The A_5- action on the cusps (b)

Blowing down the new exceptional curve, we get the configuration 2.5.

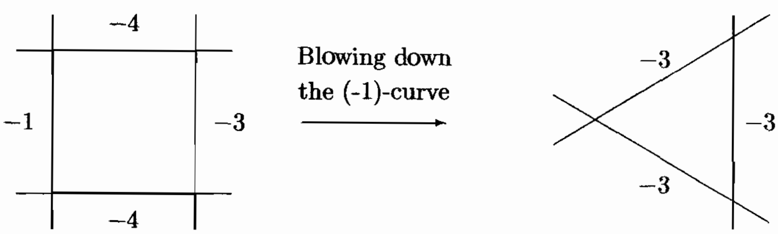

Figure 2.5: The A_5- action on the cusps (c)

Finally, the group $A_4/V \simeq \mathbb{Z}_3$ acts by permuting the lines of this triangular configuration, so the quotient by this action yields the same configuration as the resolution of ∞ on Y (see Figure 2.6).

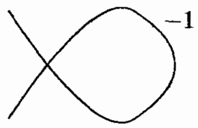

Figure 2.6: The A_5- action on the cusps (d)

This is true for each cusp. The five resolutions on Y_2 are identical and the elements of order 5 of A_5 act by permuting these five configurations, so the image in Y is the resolution of ∞ of X. Following the steps above, one can follow from Y_2 to Y or vice versa, the behavior of a given curve at the neighborhood of the cusps.

For instance, the image of the diagonal $z_1 = z_2$ of $\mathbb{H}^2$ in X is the modular curve $F_1(X)$, which is irreducible. Its image $F_1(Y)$ in Y intersects the cusp resolution of Figure 2.1 transversally at a single point, distinct from the self-intersection point P. Tracing this behavior up to Y_2 and using the fact that the stabilizer in A_5 of $F_1(Y)$ is isomorphic to the symmetric group S_3, Hirzebruch determines in [10] the intersection of the 10 components of the pull-back of $F_1(Y)$ to Y_2 with the resolutions of the five cusps. That is, each component intersects transversally 3 of the five resolutions at a single point and avoids the other two, in such a way that each of the 15 lines of the resolutions intersects exactly two different components of $F_1(Y_2)$.

2.5 The Clebsch Surface and the Klein Icosahedral Surface

The icosahedron of $\mathbb{R}^3$ is a regular polyhedron formed by 20 triangles, with 12 vertices and 30 edges. Consider an icosahedron I whose vertices lie on the unit sphere S^2.

Under the antipodal identification

$$S^2/\{\pm 1\} \simeq \mathbb{P}_2(\mathbb{R}) \subset \mathbb{P}_2(\mathbb{C}),$$

the 12 vertices form 6 projective points. The 20 mid-points of the faces form 10 points in $\mathbb{P}_2(\mathbb{C})$. Finally, two opposite edges of I can be joined by a circle in S^2, that is by a projective line. We have therefore 16 points and 15 lines in $\mathbb{P}_2(\mathbb{C})$. Consider now the surface $\mathcal{K}$ obtained by blowing-up the 16 points in $\mathbb{P}_2(\mathbb{C})$.

$$\mathcal{K} := \mathbb{P}_2(\mathbb{C}) \text{ with 16 blown-up points.}$$

We call this surface the Klein icosahedral surface. The proper transforms on $\mathcal{K}$ of the 15 lines form five disjoint sets of triangles, giving a configuration identical to the resolution of the cusp singularities in Y_2 (see Figure 2.2). The exceptional curves corresponding to the 10 mid-points intersect the five triangles of $\mathcal{K}$ in the same way as the 10 components of $F_1(Y_2)$ intersect the cuspidal resolutions in Y_2.

Consider now the Clebsch diagonal surface. This is the cubic surface defined by:

$$\mathcal{S} = \left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}_4(\mathbb{C});\ \sum_{i=0}^{4} x_i = 0,\ \sum_{i=0}^{4} x_i^3 = 0 \right\}.$$

The hyperplane $x_i = 0$ cuts $\mathcal{S}$ in 3 lines intersecting pairwise in a point. These sections thus define five sets of 3 lines. Moreover, given three distinct sets, there are three lines, one from each set that coincide at a single point. There are 10 such points of intersection (see for example [10]). The surface

$$\widetilde{\mathcal{S}} = \mathcal{S} \text{ with } 10 \text{ blown-up points}$$

is obtained by blowing-up these 10 points of intersection. Following Hirzebruch, we also call this surface the Clebsch diagonal surface. Again, the configuration obtained in $\widetilde{\mathcal{S}}$ with the 15 lines and the 10 exceptional divisors is identical to the triangular configuration of

Y_2. Considering a particular invariant of the three surfaces $\mathcal{K}, \widetilde{\mathcal{S}}$ and Y_2, namely the Euler number, Hirzebruch shows in [10] the following result:

Theorem 2.5.1 (Hirzebruch). *The Hilbert modular surface Y_2, the Klein icosahedral surface $\mathcal{K}$ and the Clebsch diagonal surface $\widetilde{\mathcal{S}}$ are isomorphic.*

This is actually a corollary of a stronger result proven in [10]. The surface $\mathcal{S}$ is also known as the cubic of 27 lines. We have identified 15 of them. The remaining 12 lines correspond under $\widetilde{\mathcal{S}} \simeq \mathcal{K}$ to the exceptional divisors of the fundamental points and the proper transform of the six conics passing through 5 fundamental points.

2.6 The Klein A_5 Invariants of $\mathbb{P}_2(\mathbb{C})$

The set of invariants for the action of A_5 on $\mathbb{P}_2(\mathbb{C})$ is generated by a conic A, a sextic B, a curve C of degree 10 and a curve D of degree 15, with a relation among A, B, C and D^2:

$$D^2 = -1728B^5 + C^3 + 720AB^3C - 80A^2BC^2 + 64A^3(5B^2 - AC)^2 .$$

We call these curves the Klein invariants. In the following we mostly use Klein's [13] notation. Consider 6 points e_i on $\mathbb{P}_2(\mathbb{C})$ so that no three of them are collinear. We can think of these points as the ones coming from the vertices of the icosahedron in Section 2.5. We call these points the fundamental points.

For each fundamental point e_i, there is a conic G_i avoiding e_i and containing the other 5 points. Consider the polar line to e_i with respect to G_i. This line intersects G_i in 2 points, see Figure 2.7 that we call the polars to e_i with respect to G_i. We have in total 12 such points.

The conic A passes through the 12 polars and avoids the 6 fundamental points. In fact, A and G_i ($\forall i = 1, \ldots, 6$), share tangents at the polars to e_i. Therefore, the 12 polars with

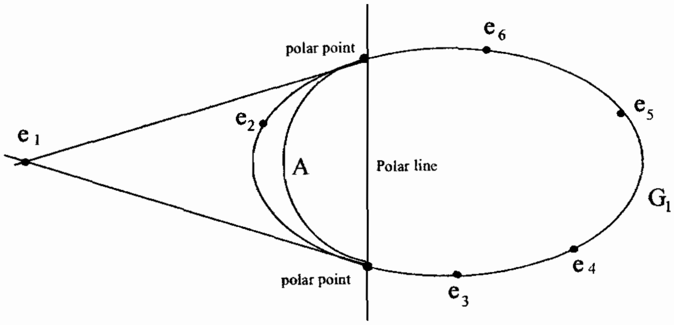

Figure 2.7: Polars to fundamental points

respect to G_i are also polars to the fundamental points with respect to A.

There are 15 lines in the projective plane joining 2 fundamental points each. The union of these lines gives a curve D of degree 15. Any three of these lines passing through 3 distinct pairs of fundamental points form a triangle. So we have 5 triangles. We denote by $L_{ij},\ j = 1, \ldots, 6$, the line joining e_i and e_j, and by $\delta_s,\ s = 0, \ldots, 4$, the five triangles. Thus if we label by $s \in \{0, 1, 2, 3, 4\}$ the five disjoint partitions $\{(i,j)(k,l)(m,n)\}$ of 2 elements of $\{1, 2, 3, 4, 5, 6\}$, we have:

$$\delta_s = L_{ij} L_{kl} L_{mn},\ s = 0, \ldots, 4; \quad \text{and} \quad D = \delta_0 \delta_1 \delta_2 \delta_3 \delta_4 .$$

Three lines L_{ij} from 3 different triangles intersect at a point called a Brianchon point. There are 10 such points and we denote them by $w_i,\ i = 1, \ldots, 10$. For each Brianchon point consider its 2 polars with respect to A, these are the intersection points of A and the polar line to w_i with respect to A. There are 20 such points.

The conic A therefore goes through the 20 polars to the Brianchon points and through the 12 polars to the fundamental points.

The sextic B goes through the 6 fundamental points having nodes at each of them, and these are the only singularities of B. This curve passes also by the 12 polar points where it intersects A transversally.

The curve C of degree 10 intersects B at the fundamental points. It has double cusps at these points and shares its tangents with B. The intersection multiplicity with B at each fundamental point equals 10. The curve C intersects A transversally in the 20 polars to the Brianchon points.

2.7 The Cohen-Wolfart Embedding

In [3], P. Cohen and J. Wolfart show that the quotient space $\Lambda\backslash\mathbb{H}$ of a Fuchsian triangle group (arithmetic or not) has a modular embedding in a Shimura variety V. The arithmetic case is trivial since we can take $V = \Lambda\backslash\mathbb{H}$ itself. The more interesting case is for non-arithmetic triangle groups. In the following we consider a specific type of triangle groups and explain their Cohen-Wolfart embedding in the Hilbert modular surface X.

Consider a Fuchsian triangle group Λ of signature (p, q, t) generated by elements $\gamma_P, \gamma_Q, \gamma_T$ satisfying (2.1) and so that Λ is contained in the Hilbert modular group $\Gamma = \mathrm{PSL}_2(\mathcal{O})$ for the field $\mathbb{Q}(\sqrt{5})$. The Cohen-Wolfart embedding of Λ is defined as follows: There is an injective, non-singular, complex analytic embedding

$$\begin{array}{rccc} F = (f_1, f_2): & \mathbb{H} & \to & \mathbb{H}^2 \\ & z & \mapsto & (f_1(z), f_2(z)) \end{array} \tag{2.10}$$

compatible with the inclusion $\iota : \Lambda \hookrightarrow \Gamma$ so that

$$F(\gamma z) = \iota(\gamma)F(z), \quad z \in \mathbb{H}, \ \gamma \in \Lambda. \tag{2.11}$$

Moreover, F can be extended continuously to the cusps of Λ, so that their images are again cusps of Γ. Finally, F induces on $\overline{\Lambda\backslash\mathbb{H}}$ a $\bar{\mathbb{Q}}$-rational morphism:

$$\Psi : \overline{\Lambda\backslash\mathbb{H}} \to X$$

Under an appropriate normalization, the compatibility condition (2.11) is

$$(f_1(\gamma z), f_2(\gamma z)) = (\gamma^{(1)} f_1(z), \gamma^{(2)} f_2(z)).$$

In fact, $\gamma^{(1)} = \gamma$ and f_1 is the identity map (see [3] for details). The map f_2 is defined in the following way:

Let $\mathcal{F}$ be the hyperbolic triangle of vertices P, Q and T and angles $(\pi/p, \pi/q, \pi/t)$. A fundamental domain for the action of Λ is $\mathcal{R} = \mathcal{F} \cup \widetilde{\mathcal{F}}$ as defined in Section (2.1.2). Consider the image $\Lambda^{(2)}$ of Λ under the non-trivial Galois automorphism of K. The corresponding hyperbolic triangle $\mathcal{F}^{(2)}$ is given by the vertices $P^{(2)}, Q^{(2)}, T^{(2)}$ which are the fixed points of $\gamma_P^{(2)}, \gamma_Q^{(2)}, \gamma_T^{(2)}$. The map f_2 is such that it maps the interior of $\mathcal{F}$ to the interior of $\mathcal{F}^{(2)}$. The region $\mathcal{R}$ gives a tessellation of $\mathbb{H}$. By the Schwartz reflection principle, the map f_2 can be continued across the sides of the triangle and extended to the images of P, Q and T under Λ, to define an analytic function on $\mathbb{H}$. This function also satisfies $f_2(\gamma z) = \gamma^{(2)} f_2(z)$.

As examples of their construction, Cohen and Wolfart give in [3] the modular embedding of groups of signature $(2, 5, \infty)$ and $(5, \infty, \infty)$ in the Hilbert modular surface X. The embedding of the Hecke group of signature $(2, 5, \infty)$ in X was studied by T. Schmidt who shows in [18] that the pull-back of the embedding to Y_2 is given by the exceptional divisors E_i obtained by blowing-up the fundamental points e_i in $\mathbb{P}_2(\mathbb{C})$.

2.8 Some Notation

The diagram in Figure 2.8 summarizes the correspondence between the surfaces mentioned above.

Let C be a curve in $\mathbb{P}_2(\mathbb{C})$. We denote by $\widetilde{C}$ the proper transform of C in $\mathcal{K}$. By $C(S)$ we mean the image of the curve $\widetilde{C}$ in the surface S. The latter being any of the Hilbert modular surfaces X, X_2, Y, Y_2 or Z.

$$\begin{array}{ccccccccc}
\mathbb{P}_4(\mathbb{C}) \supset \mathcal{S} & \xleftarrow{\text{B.U. 10 points}} & \widetilde{\mathcal{S}} & & & & & & \\
 & & \| & & & & & & \\
I \hookrightarrow \mathbb{P}_2(\mathbb{C}) & \xleftarrow{\text{B.U. 16 points}} & \mathcal{K} & = & Y_2 & \xrightarrow{\text{resol. cusps}} & X_2 = \overline{\Gamma_2\backslash\mathbb{H}^2} & \xleftarrow{\pi_2} & \mathbb{H}^2 \\
 & & & & \downarrow A_5 & & \downarrow A_5\simeq\Gamma/\Gamma_2 & & \\
 & & Z & \xrightarrow{\text{resol. elliptic}} & Y & \xrightarrow{\text{resol. cusp}} & X = \overline{\Gamma\backslash\mathbb{H}^2} & \xleftarrow{\pi} & \mathbb{H}^2
\end{array}$$

Figure 2.8: Correspondence between surfaces

For instance, the proper transform on $\mathcal{K}$ of the curve $B \subset \mathbb{P}_2(\mathbb{C})$ is denoted by $\widetilde{B}$ and $B(Y_2)$ is its image under the isomorphism $\mathcal{K} \simeq Y_2$. The curve $B(Y)$ is the A_5-orbit of $B(Y_2)$ as seen in Y and $B(X)$ is its image under the map that resolves the cusp singularity of the surface X.

Throughout this text we will refer to the involution τ. This is the automorphism in $\mathbb{H}^2$ defined as:

$$\begin{array}{cccc} \tau : & \mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ & (z_1, z_2) & \longmapsto & (z_2, z_1) \end{array} \tag{2.12}$$

This involution induces involutions on X and on X_2, see [11, 9]. We denote these induced involutions again by τ. The induced τ operate also on the smooth models Y and Y_2. The action on X is so that the points of order 2 and 3 are fixed and the points of order 5 are exchanged.

The curve F_1 in X is also fixed (pointwise) by τ. The action of τ in Y_2 is explained in [18].

3 UNIFORMIZING GROUPS FOR THE KLEIN INVARIANTS

3.1 The Main Result

Consider the modular curve F_1 of the Hilbert modular surface X. This is the image in X of the diagonal $\{z_1 = z_2\}$ of $\mathbb{H}^2$. The stabilizer in Γ of the diagonal is $\mathrm{PSL}_2(\mathbb{Z})$, an arithmetic group of signature $(2, 3, \infty)$.

Consider now the non-compact, non-arithmetic Fuchsian triangle groups that are contained in the Hilbert modular group Γ. These are of signature $(2, 5, \infty)$, $(3, 5, \infty)$, $(5, 5, \infty)$ and $(5, \infty, \infty)$. We want to relate the Klein invariants of $\mathbb{P}_2(\mathbb{C})$ to modular embeddings of the groups above.

T. Schmidt shows in [18] the correspondence between the Cohen-Wolfart embedding of a group of signature $(2, 5, \infty)$ and the image in $\mathcal{K}$ of the exceptional divisors E_i of the fundamental points e_i. The action of the involution τ in X and Y_2 gives also the proper transforms of the conics G_j.

Theorem 3.1.1. *Consider the Klein invariants A, B, C and D of $\mathbb{P}_2(\mathbb{C})$ and their proper transform in the Klein surface $\mathcal{K}$ obtained from $\mathbb{P}_2(\mathbb{C})$ by blowing-up 16 points as described in Section 2.5. Consider the Hilbert modular surface X, the level 2 surface X_2 and the surfaces resolving the corresponding cusp singularities, Y and Y_2 respectively. Under Hirzebruch's isomorphism $\mathcal{K} \simeq Y_2$ we have the following correspondence:*

1. *The proper transforms of A and C in $\mathcal{K}$ are the lifted images in Y_2 of curves uniformized by groups of signature $(3, 5, \infty)$ in X. The two corresponding curves are equivalent under the action of the involution τ.*

2. *The proper transform of B in $\mathcal{K}$ is the lifted image in Y_2 of a curve uniformized by a group of signature $(5, 5, \infty)$ in X.*

3. The proper transform of D in $\mathcal{K}$ is the union in Y_2 of the 15 lines resolving the cusp singularities of X_2.

Part (3) was proven by Hirzebruch in [10]. In the following sections we will prove the rest of the theorem.

3.2 Elliptic Points of X and the Points of the Icosahedron

We first give an interpretation of the A_5 orbits of $\mathbb{P}_2(\mathbb{C})$ of length less than 60 and the elliptic points of the surface X. Following the definitions of Section 2.6, such orbits are given by the following proposition:

Proposition 3.2.1. *The orbits of points under A_5 of size less than 60 are:*

1. *The 6 fundamental points e_i.*
2. *The 10 Brianchon points.*
3. *The 15 vertices of the five triangles of $D = 0$.*
4. *The 12 polars to the fundamental points.*
5. *The 20 polar points with respect to the Brianchon points.*
6. *The 30 points that are the intersection of A and the lines of D (there are 2 on each line).*
7. *Orbits of length 30 consisting of points on the 15 lines of D that are not on A and are not the fundamental points nor the vertices of the triangles.*

Proof. This has been discussed in [13] but for another proof, see for example [6].

Using the isomorphisms proved by Hirzebruch in [10], we look at the images of these orbits in the Hilbert modular surface X and its smooth model Y.

Definition 5. *Let γ be an elliptic element of order n of Γ, fixing the point $\sigma = (z_0', z_0'')$ of $\mathbb{H}^2$. We say that σ is of type $(n;p,q)$ with p and q relatively prime to n if the rotation factors of γ and $\gamma^{(2)}$ are equal to $e^{2i\frac{p\pi}{n}}$ and $e^{2i\frac{q\pi}{n}}$ respectively. That is, if γ and $\gamma^{(2)}$ are $PSL_2(\mathbb{R})$-conjugate to matrices of the form*

$$\begin{pmatrix} e^{i\frac{p\pi}{n}} & 0 \\ 0 & e^{-i\frac{p\pi}{n}} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} e^{i\frac{q\pi}{n}} & 0 \\ 0 & e^{-i\frac{q\pi}{n}} \end{pmatrix}$$

respectively.

The elliptic points of the surface X are two of type $(2;1,1)$, one of each type $(3;1,2)$, $(3;1,3)$, $(5;1,2)$ and $(5;1,3)$, as is shown in [7].

Proposition 3.2.2. *Under the isomorphism $\mathcal{K} \simeq Y_2$, the A_5-orbits of $\mathbb{P}_2(\mathbb{C})$ and the elliptic points of X are related as follows:*

1. *The 6 fundamental points e_i correspond to a curve Υ which is the Cohen-Wolfart embedding in X of a fuchsian group of signature $(2,5,\infty)$. The elliptic points of type $(2;1,1)$ and $(5;1,2)$ of this curve in X correspond in $\mathbb{P}_2(\mathbb{C})$ to specific tangent directions at the e_i.*

2. *The 10 Brianchon points correspond in X to the modular curve F_1. The elliptic points of type (2;1,1) and (3;1,1) of F_1 correspond in $\mathbb{P}_2(\mathbb{C})$ to specific tangent directions at the Brianchon points.*

3. *The 15 vertices of the five triangles of $D = 0$ correspond to the cusp ∞ of X. In Y, their image is the double point of the resolution of the cusp.*

4. *The 12 polars to the fundamental points become in X the elliptic point of type $(5;1,3)$ that is on $\tau(\Upsilon)$.*

5. *The 20 polar points with respect to the Brianchon points correspond in X to the elliptic point of type (3;1,2).*

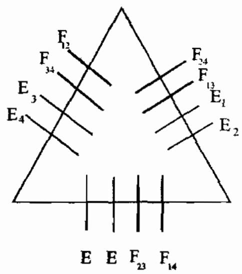

Figure 3.1: Triangular configuration on Y_2

6. *The 30 points orbits are sent on X to the cusp ∞. Their image in Y is a point on the curve of the resolution of the cusp, other than the double point.*

Proof. The surface Y_2 is obtained (up to isomorphism, see Section 2.5) by blowing-up the 6 points e_i and the 10 Brianchon points. The cusp ∞ of X lifts in Y_2 to the 5 triangles. Each divisor E_i intersects one of the lines of each triangle and each one of the 15 lines intersects two of the E_i. Each component of F_1 in Y_2 intersects one of the lines of 3 triangles and avoids the other two triangles. Each one of the 15 lines intersects two of the 10 components of F_1; see Figure 3.1. For all of this, see [10]. We will use these facts in the following argument.

1. *The 6 fundamental points e_i.* These form an orbit of length 6, therefore each point is fixed by a dihedral group, in particular by some element of order 2 and some element of order 5.

The fact that the corresponding 6 exceptional divisors E_i of Y_2 descend to a curve Υ in X, which is the Cohen-Wolfart embedding of a Fuchsian group of signature $(2, 5, \infty)$, was proved by T. Schmidt [18].

The stabilizer in A_5 of each exceptional divisor E_i of Y_2, is isomorphic to D_5. In X the curve Υ passes through an elliptic point of order 2, the elliptic point of type $(5; 1, 2)$ and the cusp ∞ (see Lemma 4.1.2). The point of order 2 lifts to 30 points in X_2 as well as in Y_2 (since X_2 does not have elliptic singularities). By A_5-symmetry, each component E_i contains five of these 30 points, all distinct. None of these points lie on any of the 15 lines of the triangles of Y_2. An element of order 5 of the stabilizer of E_i permutes the 5 points on E_i. An element of order 2 fixes one of the points and permutes the others. When we blow down each E_i (as seen in $\mathcal{K}$) to $\mathbb{P}_2(\mathbb{C})$, the 5 points give five tangent directions at the corresponding point e_i, distinct from the lines of D. The element of order 5 of A_5 that fixes e_i permutes these tangents. The element of order 2 fixes one of them and permutes the others.

We do a similar argument for the point of order 5 of X. This point has a fiber of 12 points in X_2 and in Y_2. Each component E_i contains 2 of them, all distinct. Again, none of these 12 points lie on any of the 15 lines of the triangles. An element of order 5 of the stabilizer of E_i fixes the 2 points. An element of order 2 permutes them. When we blow down each E_i, the 2 points give two tangent directions at the corresponding point e_i, distinct from the lines of D. The element of order 2 that fixes e_i permutes these tangents. The element of order 5 fixes them.

Therefore the 7 ($= 5 + 2$) points that lie on each E_i and are in the fibers described above, give in $\mathbb{P}_2(\mathbb{C})$ seven tangent directions at e_i of any (irreducible) curve of X that intersects Υ at the points of order 2 and 5.

The curve $\tau(\Upsilon)$, where τ is induced by the involution in $\mathbb{H}^2$ given in Section 2.8, intersects Υ at the point of order 2 of X. Therefore the pull-back of $\tau(\Upsilon)$ to $\mathbb{P}_2(\mathbb{C})$ has 5 tangents directions at each of the fundamental points e_i. From [18] we know that $\tau(\Upsilon)$ corresponds in $\mathbb{P}_2(\mathbb{C})$ to the union of the G_i. Therefore the five tangent directions are the tangents to the 5 conics G_j, $j \neq i$ that go through e_i. This also implies that the intersection of Υ and $\tau(\Upsilon)$ in X occurs only at the point of order 2.

Remark 3.2.3. If a curve intersects Υ at some regular point of X, this intersection will give 10 tangent directions of the image of that curve at the points e_i of $\mathbb{P}_2(\mathbb{C})$.

Furthermore, any two curves in X intersecting Υ at one of its elliptic points are such that their pull-back to $\mathbb{P}_2(\mathbb{C})$ share their tangents at the fundamental points.

2. The 10 Brianchon points. The stabilizer of each point in A_5 is isomorphic to S_3, therefore each point is fixed by some element of order 2 and some element of order 3 of A_5. In Y_2 these points have been blown up and are the 10 components of the image of the diagonal F_1 of X. Here again, each component of F_1 is invariant under a subgroup S_3 of A_5 (see [10]).

In X, F_1 passes through an elliptic point of order 2 resulting from $(i, i) \in \mathbb{H}^2$ and different from the one of Υ, the elliptic point of type (3;1,1) and the cusp ∞, see [9].

The point of order 2 lifts to 30 points in Y_2. By A_5-symmetry, each component of F_1 contains three of them, all distinct and none on the intersection with the triangular configurations. Considering the action of the stabilizer of each component of F_1 and blowing down each component, we can see again that these three points give in $\mathbb{P}_2(\mathbb{C})$ three tangent directions at the Brianchon points.

There are 20 points in Y_2 above the point (3;1,1), two on each component of F_1. None of these points lie on the 15 lines of the triangles. These two points give in $\mathbb{P}_2(\mathbb{C})$ two tangent directions at the Brianchon points.

The five tangent directions above are all distinct from the lines of D.

As before, if two irreducible curves of X meet F_1 at a point of order 2 or 3, then their images in $\mathbb{P}_2(\mathbb{C})$ must share their corresponding tangents at the Brianchon points. Other tangent directions (in multiples of 6) may come from the intersections at regular points of X.

3. The 15 vertices of the five triangles of $D = 0$. The stabilizer of each vertex is of order 4, so these points are fixed by elements of order 2 of A_5. Since the vertices are distinct from the fundamental points and the Brianchon points, on Y_2 we have again 15 points, vertices of the five triangles of the cuspidal resolutions. Therefore the 15 vertices correspond to the cusp ∞ of X. In Y, their image is the double point of the resolution of the cusp (see Figure 2.1).

4. The 12 polars to the fundamental points. Their stabilizer is of order 5. Therefore these points are fixed by elements of order 5 of A_5. On Y_2 we have again 12 points, none of them belonging to the triangular configurations. Therefore these points become in X an elliptic point of order 5. Since the conics G_i in $\mathbb{P}_2(\mathbb{C})$ go through the 12 polars, the point in X is the one that is in $\tau(\Upsilon)$, this is the point of type $(5;1,3)$.

5. The 20 polar points with respect to the Brianchon points. The stabilizer of each point is of order 3. On Y_2 we have again 20 points, none of them belonging to the triangular configurations, therefore these points correspond in X to an elliptic point of order 3. Since the 20 polars are distinct from the Brianchon points, the point in X is the point that does not belong to F_1; therefore, it is of type (3;1,2).

6. Orbits of length 30 consisting of points on the 15 lines of D. By symmetry, there are two points on each line. On Y_2 they become 30 points, two on each line of the triangular configurations and distinct from the vertices. These points are sent to the cusp of X.

The stabilizer of a point of this orbit in Y_2 is of order two. The non-trivial element of A_5 that fixes it is the involution that fixes (pointwise) the corresponding line of the triangle and its opposite vertex. Taking the quotient of the resolution of a given cusp by its stabilizer on A_5 as it is done in Section 2.4.3; we can see that the image of the 30 points in Y consists of a single point on the curve of the resolution of the cusp, other than the double point (see Figure 3.2).

□

Corollary 3.2.4. *With the notations above, any curve of X intersecting Υ at the point of order 2 is such that its pull-back in $\mathbb{P}_2(\mathbb{C})$ shares 5 tangents at each e_i with the 5 conics G_j, $j \neq i$.*

Proof. This is a consequence of Remark 3.2.3. □

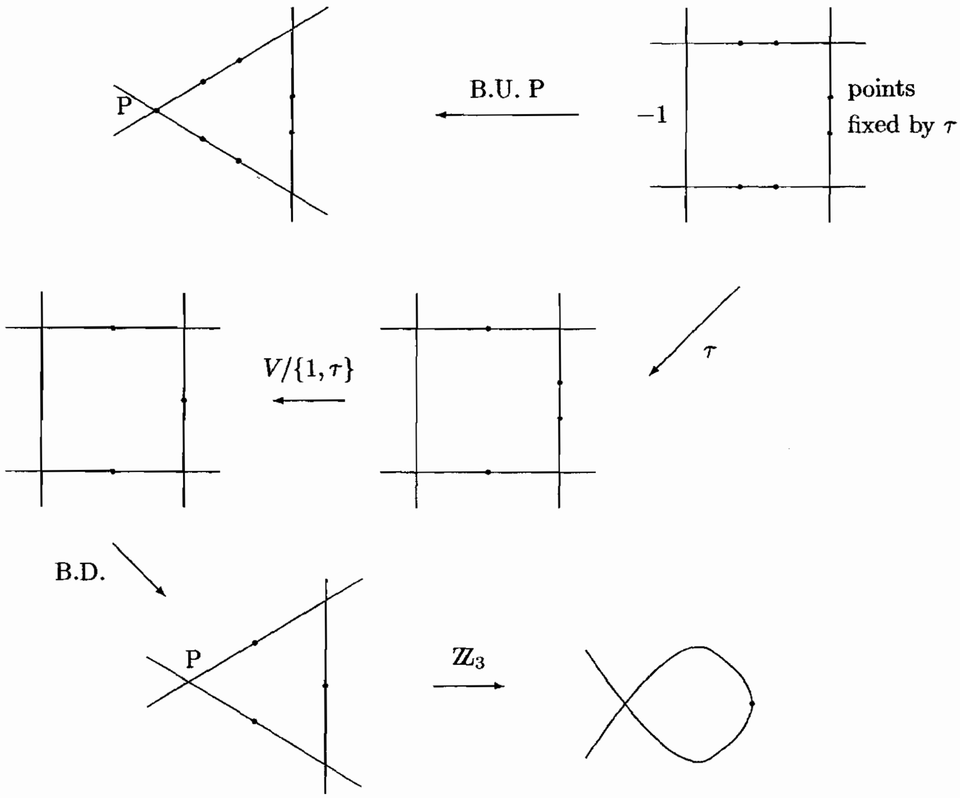

Figure 3.2: Intersections at the cusps

3.3 The Curve B and the Group $(5, 5, \infty)$

3.3.1 The Image of B in the Surface X

In $\mathbb{P}_2(\mathbb{C})$, the curve B passes with multiplicity 2 through each of the 6 fundamental points (its intersection with C). It has a node at each such point. In Y_2, these fundamental points have been blown up, hence $B(Y_2)$ intersects each of the exceptional divisors E_i at 2 different points. These points are not on any of the 15 lines of the triangular configuration because the tangents to B at the fundamental points are distinct from the lines of D. Indeed, consider one of the fundamental points, say e_1. There is an element σ of order 5 of A_5 that fixes e_1 (its stabilizer is of order 10). Any element of A_5, and in particular σ,

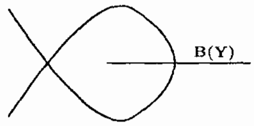

Figure 3.3: Intersection of B(Y) at the resolution of ∞

leaves B invariant. On the other hand, B has two tangents at e_1. But σ permutes the five lines of D that go through e_1 (the stabilizer of each line is of order 2). Hence these lines cannot be the tangents to B.
We have then 12 points of intersection $B(Y_2) \cap (\cup_i E_i)$ and $B(Y_2)$ is A_5-invariant. Therefore these 12 points form the orbit of a point in the fiber above a point of order 5. Furthermore, $B(X)$ intersects Υ at the point of order 5 and the intersection is transversal.

In $\mathbb{P}_2(\mathbb{C})$, B contains the 12 polar points to the e_i, and we know that B and G_i intersect transversally at those points (see [6]). Thus $B(X)$ intersects $\tau(\Upsilon)$ transversally at the corresponding elliptic point of order 5.

On the other hand, $B \subset \mathbb{P}_2(\mathbb{C})$ intersects D transversally at 30 points other than the fundamental points; two distinct points on each line of D. This can be checked by using Bézout's theorem for B and one of the lines $L_{i,j}$ of D. Indeed, the two curves should intersect in 6 points (counting multiplicities). They intersect at two fundamental points with multiplicity 2 at each one of them. Thus there are two other points of intersection. If these two remaining points of intersection between B and $L_{i,j}$ were not distinct, then the A_5-action on $B \cap L_{i,j}$ would give an orbit of length less than 30, but from Proposition 3.2.1 and from the geometry of B, we can see that this is impossible. So these other two points of intersection must be distinct.

Therefore $B \cap D$ form a A_5-orbit of length 30 and $B(X)$ goes through the cusp ∞. Its image $B(Y)$ intersects the curve of the resolution of the cusp transversally at a single point as shown in Figure 3.3.

The curve B does not go through the Brianchon points of $\mathbb{P}_2(\mathbb{C})$, thus $B(Y)$ does not intersect $F_1(Y)$ and $B(X)$ intersects F_1 only at the cusp ∞. In particular, $B(X)$ avoids the points of order 2 and 3 of F_1.

3.3.2 The Uniformizing Group for $B(X)$

We show now that $B(X)$ is the compactification of the quotient of $\mathbb{H}$ by a Fuchsian group of signature $(5, 5, \infty)$. We do this by proving that the elliptic points of order 5 are indeed quotient singularities in $B(X)$ and that $B(X)$ is a curve of genus zero.

The curve $B(Y)$ goes once through two points of order 5 and the resolution of the cusp ∞. It does not contain any other elliptic point. We need to check that the elliptic points are indeed quotient singularities in B(Y), in other words, that the A_5-covering $B(Y_2) \to B(Y)$ is ramified above those points. We know that $B(Y_2)$ is irreducible and invariant under the action of A_5.

We thus compute the ramification above the elliptic points of order 5 and above ∞. In Y_2, the fiber above each point of order 5 consists of $\frac{60}{5} = 12$ points. From the discussion in Section 3.3.1, we know that the curve $B(Y_2)$ passes through all of these points with multiplicity one. Therefore the ramification number equals $120 - 24 = 96$, 48 for each point.

There are 5 cusps in X_2 above the cusp ∞ of X. The curve $B(Y_2)$ intersects the resolution of each of them 6 times. Therefore the ramification number above ∞ equals $60 - 6 \cdot 5 = 30$.

We can now apply the Riemann-Hurwitz formula to $B(Y_2)$ and $B(Y)$ in order to compute the genus of $B(Y)$. The curve B has genus 4. Hence $B(Y_2)$ has genus 4 as well, and is smooth. The total ramification number is $96 + 30 = 126$. If g is the genus of $B(Y)$, we have

$$60 \cdot (2g - 2) = (8 - 2) - 126$$

and $g = 0$. Thence $B(X)$ is singular at the two points of order 5 and the cusp ∞ and has genus zero. Let $B(\Gamma\backslash\mathbb{H}^2)$ be the curve corresponding to $B(X)$ in the non-compact space $\Gamma\backslash\mathbb{H}^2$. The non-compact curve $B(\Gamma\backslash\mathbb{H}^2)$, seen as a Riemann surface, has a hyperbolic covering and the covering group is a subgroup of Γ with two elliptic elements of order 5, one parabolic element and no hyperbolic element. This is a non-arithmetic Fuchsian triangle group of signature $(5, 5, \infty)$.

3.4 The Curves A and C and the Group $(3, 5, \infty)$

3.4.1 The Image of A in the Surface X

In $\mathbb{P}_2(\mathbb{C})$, the curve A, a smooth conic, contains the 12 polar points to the e_i. Thus its image in Y_2 contains the 12 images of those points, and these are regular points for $A(Y_2)$. It follows that $A(X)$ passes once through the elliptic point of order 5 of $\tau(\Upsilon)$. Also we know that A and G_i share tangents at the 12 polar points (see [6]), hence $A(X)$ intersects $\tau(\Upsilon)$ at the elliptic point of order 5, sharing its tangent at that point.

In $\mathbb{P}_2(\mathbb{C})$, A passes through the 20 polar points to the Brianchon points (these correspond to the intersection of A and C). Therefore $A(X)$ goes through the point (3;1,2).

Furthermore, $A \subset \mathbb{P}_2(\mathbb{C})$ intersects D transversally in 30 points. Two points on each line of D, all distinct from the vertices. Therefore, $A(Y)$ intersects the curve of the resolution of the cusp transversally at a single point, distinct from the point of intersection of this curve with $B(Y)$.

The curve A does not go through the fundamental points nor through the Brianchon points of $\mathbb{P}_2(\mathbb{C})$. Therefore $A(X)$ intersects neither Υ nor F_1 except at ∞. In particular, $A(X)$ avoids the points of order 2 and 5 of Υ and the points of order 2 and 3 that belong to F_1.

3.4.2 The Uniformizing Group for $A(X)$

We show now that $A(X)$ is the compactification of the quotient of $\mathbb{H}$ by the Fuchsian group $(3, 5, \infty)$. The curve $A(X)$ goes once each through a point of order 3, a point of order 5 and the cusp ∞. It contains no other elliptic point. We also know that $A(Y_2)$ is irreducible and invariant under the action of A_5. We again check that the A_5-covering is actually ramified above the given points by computing the ramification indices.
The point of order 3 has $\frac{60}{3} = 20$ points above it. The curve $A(Y_2)$ passes through each of them with multiplicity one. Therefore the ramification number equals $60 - 20 = 40$. The point of order 5 has 12 points above it. The curve $A(Y_2)$ passes through all of them once. Therefore the ramification equals $60 - 12 = 48$.
There are 5 cusps in X_2 above the cusp ∞ of X. The curve $A(Y)$ intersects the resolution of ∞ once whereas $A(Y_2)$ intersects each one of the resolutions in Y_2 six times. Therefore the ramification equals $60 - 6 \cdot 5 = 30$.
Hence A has ramification above all these points. The curve $A(Y_2)$ has genus zero and is smooth. Since a curve of genus zero can only cover curves of genus zero, we conclude that $A(Y)$ has genus zero as well.

Thus $A(X)$ is a curve of genus zero with quotient singularities at the points of order 3, 5 and the cusp ∞. Therefore, $A(\Gamma \backslash \mathbb{H}^2)$ is uniformized by a non-arithmetic Fuchsian group of signature $(3, 5, \infty)$.

3.4.3 The Image of C in the Surface X

In $\mathbb{P}_2(\mathbb{C})$, the curve C has a double cusp at each of the 6 fundamental points. In Y_2, these fundamental points have been blown up, hence $C(Y_2)$ intersects each of the exceptional divisors E_i tangentially at 2 different points. These intersection points are the same as those of $\{B \cap E_i,\ i = 1, \ldots, 5\}$, because B and C share their tangents at the e_i. From Corollary 3.2.4, it follows that $C(X)$ intersects Υ at the point of order 5, going only once through that point but sharing its tangent with Υ.

Like A in $\mathbb{P}_2(\mathbb{C})$, C passes through the 20 polars to the Brianchon points. Therefore $C(X)$ also goes through the point (3;1,2).

Using the same argument as for B, we can see that $C \subset \mathbb{P}_2(\mathbb{C})$ intersects D transversally at 30 points other than the fundamental points; two distinct points on each line of D. Indeed, C and one of the lines $L_{i,j}$ of D must intersect in 10 points (counting multiplicities). They intersect at 2 fundamental points with multiplicity 4 (multiplicity of C) at each one of them. Thus there are 2 other points of intersection. But none of the lines L_{ij} is tangent to C, and C does not have other singularities than at the fundamental points. So these other two points of intersection must be distinct. They are also distinct from the 30 intersection points of B and D (since B and C intersect only at the fundamental points).

Consequently, $C(Y)$ intersects the curve of the resolution of the cusp transversally at a single point, distinct from the points of intersection with $A(Y)$ and with $B(Y)$. See Figure 3.4.

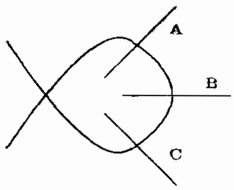

Figure 3.4: Intersections in Y

The curve C does not go through the Brianchon points of $\mathbb{P}_2(\mathbb{C})$, therefore $C(X)$ does not intersect F_1 but at the cusp ∞ and avoids in particular the points of order 2 and 3 of F_1.

3.4.4 The Uniformizing Group for $C(X)$

The final step is to prove that $C(X)$ is also the compactification of the quotient of $\mathbb{H}$ by the Fuchsian group of signature $(3, 5, \infty)$. This is the same signature as for A. Only the points of order 5 are different. This is consistent with fact that $\tau(A) = C$.

The curve $C(X)$ goes once through a point of order 3, a point of order 5 and the cusp ∞ and through no other elliptic point. We also know that $C(X_2)$ is irreducible and A_5-invariant. We need to check again that the A_5-covering $C(Y_2) \to C(Y)$ is actually ramified above those points.

The ramifications above the elliptic points of order 3 and 5 and above ∞ are as follows. Since $C(Y_2)$ passes through all of the 20 points above the point of order 3 with multiplicity one, the ramification at that point equals $60 - 20 = 40$. The curve $C(Y_2)$ passes through the 12 points above the point of order 5 with multiplicity one. Therefore the ramification equals $60 - 12 = 48$. The curve $C(Y_2)$ intersects each of the five resolutions of cusps above ∞ six times. Therefore the ramification equals $60 - 6 \cdot 5 = 30$.
Therefore $C(Y_2)$ has ramification above the given elliptic points of order 3 and 5 and the cusp. It is a curve of genus zero and smooth. It follows that $C(Y)$ has genus zero. Thence $C(X)$ is a genus zero curve with quotient singularities at a point of order 3 and a point of order 5. Therefore $C(\Gamma \backslash \mathbb{H}^2)$ is uniformized by a group of signature $(3, 5, \infty)$. The fact that the involution τ in $\mathbb{P}_2(\mathbb{C})$ induced by τ switches the curves A and C is well known. This completes the proof of Theorem 3.1.

3.5 The Hilbert Modular Surface of Level $\sqrt{5}$

Consider the congruence subgroup Γ_5 of Γ associated to the ideal $\sqrt{5}\mathcal{O}$. The compactified quotient of $\mathbb{H}^2$ by Γ_5 gives the Hilbert modular surface of level $\sqrt{5}$, X_5 which has 6 cusps and no elliptic singularities. Let Y_5 be the surface of the resolution of the cusps. Here again Γ_5 is a normal subgroup of Γ and $\Gamma/\Gamma_5 \simeq A_5$. So X_5 is a A_5-cover of X. In [11], Hirzebruch shows that there is an isomorphism

$$\tau \backslash Y_5 \simeq \widetilde{\mathbb{P}}_2(\mathbb{C}), \tag{3.1}$$

where $\widetilde{\mathbb{P}}_2(\mathbb{C})$ is the surface obtained by blowing-up the 6 fundamental points in $\mathbb{P}_2(\mathbb{C})$.

It is also proven in [11] that under this isomorphism, the lifted image of F_5 to Y_5 projects under the τ-action to the Klein invariant D, and in a similar way the lifted image of F_1 corresponds to the Klein invariant C.

Using this isomorphism, we can show the following result, which gives a second interpretation of the elliptic points of X, this time in terms of the points of $\mathbb{P}_2(\mathbb{C})$.

Proposition 3.5.1. *Under the isomorphism $Y_5/\tau \simeq \widetilde{\mathbb{P}}_2(\mathbb{C})$, the elliptic points of X and the A_5-orbits of $\mathbb{P}_2(\mathbb{C})$ are related as follows:*

1. *The elliptic point of order 2 that belongs to F_1 corresponds in $\mathbb{P}_2(\mathbb{C})$ to the 30 points of intersection of C and D, distinct from the fundamental points;*

2. *The elliptic point of order 2 that F_1 avoids, corresponds in $\mathbb{P}_2(\mathbb{C})$ to the 15 vertices of the five triangles of D;*

3. *The elliptic point of type $(3;1,1)$ corresponds in $\mathbb{P}_2(\mathbb{C})$ to the 20 points of intersection of A and C, thus to the 20 polars to the Brianchon points;*

4. *The elliptic point of type $(3;1,2)$ corresponds in $\mathbb{P}_2(\mathbb{C})$ to the 10 Brianchon points;*

5. *The two elliptic points of order 5 correspond in $\mathbb{P}_2(\mathbb{C})$ to the 12 polars to the fundamental points;*

Proof. The proof is based on an argument similar to the proof of Proposition 3.2.2 and on the following facts.

The involution τ defined by (2.12) also induces an involution on X and on X_5, see [11]. The induced involutions that we denote again by τ, act in the following way. As we said in Section 2.8, τ fixes in X the elliptic points of order 2 and 3 and permutes the points of order 5. In fact, the set of fixed points of τ equals $F_1 \cup F_5$. On Y_5 the induced action of τ is such that the set of fixed points equals the pre-image of F_1. All of the above is given in [11].

On the other hand, the curves F_1 and F_5 intersect at one of the points of order 2. The curve F_5 passes through the other point of order 2 and through the point of type $(3;1,2)$. The point of type $(3;1,1)$ belongs to F_1. The points of order 5 belong to neither F_1 nor F_5. This is given in [9].

The remaining details of this proof can be easily checked.

□

4 FURTHER RESULTS

4.1 Embedding of $(2, 5, \infty)$

When studying the Cohen-Wolfart embedding of groups of signature $(2, 5, \infty)$, T. Schmidt considers in [18] the Hecke group G_5, as was suggested in [3], generated by S, T and U with:

$$S = \begin{pmatrix} 1 & \varepsilon_0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = ST. \tag{4.1}$$

One of the vertices of the fundamental domain of G_5 in $\mathbb{H}$ is the point $z = i$ fixed by T. Its image under F, see (2.10), is the point (i, i) that belongs to the diagonal of $\mathbb{H}^2$ and therefore its Γ-orbit in X is the point of order 2 of F_1. But Schmidt shows in [18] that the Cohen-Wolfart embedding for the signature $(2, 5, \infty)$ is unique and that the corresponding curve Υ in X does not intersect F_1.

Although not explicitly stated in [3], the geometric construction of the Cohen-Wolfart embedding depends upon the choice of the group —and its generators— for the given signature. In fact, this embedding cannot be applied directly to G_5 but to some $\mathrm{PSL}_2(\mathbb{R})$-conjugate of this group. We give here the correct choice of the group of signature $(2, 5, \infty)$ for the construction of the Cohen-Wolfart embedding.

The elliptic points of the surface X are given in [7]. The points of order 2 are the Γ-orbits of the points $v_{2,1} = (i, i)$ and $v_{2,2} = (-\varepsilon_0' i, \varepsilon_0 i)$, where $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$ and we denote by ε_0' the Galois conjugate $\varepsilon_0^{(2)}$. We know now that Υ should pass through the second point. The points of order 5 are the Γ-orbits of $v_{5,1} = (\zeta_{10}, \zeta_{10}^3)$ and $v_{5,2} = (\zeta_{10}^3, \zeta_{10})$, where ζ_n denotes the primitive n-th root of unity, $e^{i\frac{2\pi}{n}}$.

The point $v_{2,1}$ as we mentioned above is fixed by

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that in $\mathrm{PSL}_2(\mathcal{O})$ this is the same element T as the one given in (4.1). We make this different choice of sign here in order to preserve the normalization used in [7].

The point $v_{2,2}$ is fixed by the element:

$$T' = \begin{pmatrix} 0 & -\varepsilon_0' \\ -\varepsilon_0 & 0 \end{pmatrix}.$$

The matrices T and T' are conjugate by the element of $\mathrm{PSL}_2(\mathbb{R})$

$$M_2 = \begin{pmatrix} -\varepsilon_0' & 0 \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

that sends $v_{2,1}$ to $v_{2,2}$. Now we apply the same conjugation to all the generators (4.1) of G_5. This gives for S the element:

$$S' = M_2 S M_2^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

that fixes the cusp ∞.

As a matter of normalization, we again replace the element U given in (4.1) by one of its powers, this is $U = (S.T)^{-1}$, for T as in (4.1). In this way we preserve the relations (2.1) given in Section 2.1.2 as well. The conjugate of U,

$$U' = M_2 U M_2^{-1} = \begin{pmatrix} 0 & -\varepsilon_0' \\ -\varepsilon_0 & \varepsilon_0 \end{pmatrix},$$

fixes the point $v_5 = (-\varepsilon_0'\zeta_{10}, \varepsilon_0\zeta_{10}^2)$.

These three elements T', S' and U' are in $\mathrm{PSL}_2(\mathcal{O})$ and generate a group of signature $(2,5,\infty)$ that is $\mathrm{PSL}_2(\mathbb{R})$-conjugate to G_5. We have the relation $S' \cdot T' \cdot U' = -I_2$.

The point v_5 is not in the fundamental domain of Γ but must of course be Γ-conjugate to either $v_{5,1}$ or $v_{5,2}$. The rotation factor of U' (see Definition 5) equals $(e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}})$. Therefore v_5 is a point of type $(5;1,2)$ and hence it is conjugate under Γ to $\nu_{5,2}$, which is of same type.

Remark 4.1.1. In [7], Gundlach gives the rotation factor corresponding to $v_{5,1}$ as being $(e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}})$ and the rotation factor corresponding to $v_{5,2}$ as $(e^{i\frac{2\pi}{5}}, e^{i\frac{6\pi}{5}})$. After verifying the formulas given in [7], we found that these should be switched. We should have indeed, $v_{5,1}$ of type $(5;1,3)$ and $v_{5,2}$ of type $(5,1,2)$.

Using the notation $z = (z', z'') \in \mathbb{H}^2$, the hyperbolic triangle $\mathcal{F}$ given by the vertices $v'_{2,2}, v'_5$ and ∞ has angles $(\frac{\pi}{2}, \frac{\pi}{5}, 0)$. The hyperbolic triangle $\mathcal{F}^{(2)}$ given by $v''_{2,2}, v''_5$ and ∞ has angles $(\frac{\pi}{2}, \frac{2\pi}{5}, 0)$. Therefore the calculations of the volume given in [18] are correct, as well as the main result concerning the embedding of the group of signature $(2,5,\infty)$. Moreover, we can now state:

Lemma 4.1.2. *The image Υ of the Cohen-Wolfart embedding in X of the group of signature $(2,5,\infty)$ conjugate to the Hecke group G_5 by the matrix* (4.2)*, passes through the elliptic point of type $(2;1,1)$ represented in $\mathbb{H}^2$ by $(-\varepsilon'_0 i, \varepsilon_0 i)$ and the elliptic point of type $(5;1,2)$ given by $(\zeta_{10}^3, \zeta_{10})$.*

4.2 The Curve $(5,\infty,\infty)\backslash\mathbb{H}$

We have described the geometry of all the triangle non-compact and non-arithmetic modular embeddings in $\mathbb{P}_2(\mathbb{C})$, except for groups of signature $(5,\infty,\infty)$. We are investigating this embedding and we discuss in this chapter the results we have found so far.

Consider the non-arithmetic triangle group Δ_0 of signature $(5, \infty, \infty)$. We wish to find the pre-image in $\mathbb{P}_2(\mathbb{C})$ of the Cohen-Wolfart embedding

$$\overline{(5,\infty,\infty)\backslash\mathbb{H}} \hookrightarrow \overline{\Gamma\backslash\mathbb{H}^2}$$

in terms of the Klein invariants.

Remark 4.2.1. This group is not maximal. It is contained in the Hecke group of index 10, of signature $(10, 2, \infty)$.

4.2.1 Generators of Δ_0

The following set of generators for a group of signature $(5, \infty, \infty)$ is given in [3]. The parabolic matrices are

$$\gamma_Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_T = \begin{pmatrix} 1 & 0 \\ \frac{-3+\sqrt{5}}{2} & 1 \end{pmatrix}.$$

The element of order 5 of the group is

$$\gamma_P = \pm(\gamma_Q\gamma_T)^{-1}.$$

The fixed points for the action of these elements in $\mathbb{H}$, are ∞ , 0 and $\frac{1}{2}\,(1 + i\,\sqrt{(5+2\sqrt{5})})$. Unfortunately, these do not form a hyperbolic triangle with angles $0, 0$ and $\frac{\pi}{5}$, but rather $0, 0$ and $\frac{4\pi}{5}$. We need to find therefore a group Δ_0 with the appropriate set of generators. In order to respect the construction of the Cohen-Wolfart embedding of this group, this set must in particular be such that the elliptic point fixed by the generator of order 5 corresponds to one of the elliptic points of order 5 of the surface X.

Lemma 4.2.2. *Consider a group $\Delta_0 \subset SL_2(\mathcal{O})$ of signature $(5, \infty, \infty)$. A set of generators for the image of Δ_0 in $PSL_2(\mathcal{O})$ consists of the two parabolic elements:*

$$U = \begin{pmatrix} 1 & 2+\varepsilon_0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

fixing the cusps $z = \infty$ *and* $z = [-1 : 1]$, *respectively; and the elliptic element of order five*

$$L = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon_0 \end{pmatrix}$$

fixing the point $\nu_{5,1} = (\zeta_{10}, \zeta_{10}^3)$.

Proof. We first consider the standard set of generators given in [17] for the signature $(5, \infty, \infty)$. These are a translation U that fixes ∞, the elliptic element E of order 5 that fixes in $\mathbb{H}$ the point $z = \lambda i$, with in our case $\lambda = \frac{\sin(\frac{\pi}{5})}{2+2\cos(\pi/5)}$:

$$E = \begin{pmatrix} \cos(\frac{\pi}{5}) & \lambda \sin(\frac{\pi}{5} \\ -\frac{1}{\lambda}\sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{5}) & \frac{1}{2} - \frac{1}{2}\cos(\frac{\pi}{5}) \\ -2 - 2\cos(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) \end{pmatrix}$$

and the parabolic element P that fixes the point $z = [-\frac{1}{2} : 1]$:

$$P = \begin{pmatrix} -\cos(\frac{\pi}{5}) + \frac{1}{\lambda}\sin(\frac{\pi}{5}) & \lambda\sin(\frac{\pi}{5}) + \cos(\frac{\pi}{5}) \\ -\frac{1}{\lambda}\sin(\frac{\pi}{5}) & -\cos(\frac{\pi}{5}) \end{pmatrix} = \begin{pmatrix} 2 + \cos(\frac{\pi}{5}) & \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi}{5})) \\ -2 - 2\cos(\frac{\pi}{5}) & -\cos(\frac{\pi}{5}) \end{pmatrix}$$

with the relation $U \cdot P \cdot E = I_2$.

One of the elliptic points of order 5 of the surface X is given by the point $\nu_{5,1} = (\zeta_{10}, \zeta_{10}^3) \in \mathbb{H}^2$, (see [7]). Since the Cohen-Wolfart embedding is given by

$$\begin{aligned} f : \quad \mathbb{H} & \to \quad \mathbb{H}^2 \\ z & \mapsto \quad (z, f(z)), \end{aligned}$$

we can require the fixed point of order 5 of Δ_0 to be $z = \zeta_{10}$. Therefore, we need to conjugate the generators above by the element of $\mathrm{GL}_2(\mathbb{R})$ that sends $z = \lambda i$ to $z = \zeta_{10}$.

This element is the matrix

$$M_\lambda = \begin{pmatrix} 2+2\cos(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) \\ 0 & 1 \end{pmatrix}.$$

The elliptic element E becomes then

$$L = M_\lambda E M_\lambda^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon_0 \end{pmatrix}.$$

One can check that this is the same element of order 5 of $\mathrm{PSL}_2(\mathcal{O})$ given by Gundlach in [7]. The parabolic element P becomes:

$$T = M_\lambda P M_\lambda^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

and T fixes the cusp $z = [-1:1] \in \mathbb{P}^1(K)$. Finally, the parabolic element U (we recycle the notation used above for its conjugate) that fixes $z = \infty$ is

$$U = \pm(T.L)^{-1} = \pm \begin{pmatrix} 1 & 2+\varepsilon_0 \\ 0 & 1 \end{pmatrix}.$$

In $\mathrm{PSL}_2(\mathcal{O})$, this becomes simply

$$U = \begin{pmatrix} 1 & 2+\varepsilon_0 \\ 0 & 1 \end{pmatrix}.$$

□

4.2.2 The Curve $\Delta_0\backslash\mathbb{H}$ Embedded in X

To construct the Cohen-Wolfart modular embedding in X,

$$\Psi : \overline{\Delta_0\backslash\mathbb{H}} \to X$$

we consider the image Δ_0' of Δ_0 under the non-trivial Galois automorphism of the field K (see [3]). The corresponding hyperbolic triangles $\mathcal{F}$ and $\mathcal{F}'$ have angles $(\frac{\pi}{5}, 0, 0)$ and $(\frac{3\pi}{5}, 0, 0)$. The fundamental domain $\mathcal{R}$ in $\mathbb{H}$ for the action of Δ_0 is $\mathcal{F} \cup \widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}$ is the reflection of $\mathcal{F}$ about the line joining two of its vertices. The image of $\mathcal{R}$ under F as defined in (2.10), Section 2.7, is $\mathcal{R}' = \mathcal{F}' \cup \widetilde{\mathcal{F}'}$ and Δ_0 acts on $\mathcal{R}$.

We can therefore evaluate the volume of the embedded curve $\mathcal{T} = \Psi\ (\overline{\Delta_0 \backslash \mathbb{H}})$. This is $[\omega](\mathcal{T})$, where $\omega = \omega_1 + \omega_2$ is as in (2.3), Section 2.2 and ω_i applied to $\mathcal{T}$ give the normalized area of $\mathcal{R}$ and $\mathcal{R}'$. Using the Gauss-Bonnet formula for the area of a hyperbolic triangle of angles α, β and γ (see for example [12], pp. 13):

$$\text{area} = \pi - \alpha - \beta - \gamma,$$

we find

$$\text{area}(\mathcal{R}) = 2\pi(1 - \frac{1}{5} - 0 - 0) = 2\pi \cdot \frac{4}{5};$$

$$\text{area}(\mathcal{R}') = 2\pi(1 - \frac{3}{5}) - 0 - 0) = 2\pi \cdot \frac{2}{5}.$$

And the volume equals:

$$[\omega_1 + \omega_2](\mathcal{T}) = -(\frac{4}{5}) - (\frac{2}{5}) = -\frac{6}{5}.$$

We define now the congruence subgroup of level 2 of Δ_0. This is the image in $\mathrm{PSL}_2(\mathcal{O})$ of

$$\Delta_2 := \{\gamma \in \Delta_0;\ \gamma \equiv I_2 \bmod(2\mathcal{O})\}\ .$$

Since $\Delta_0 \subset G$, it is easy to see that

$$\Delta_2 = \Delta_0 \cap \Gamma_2$$

and Δ_2 is a normal subgroup of Δ_0. The quotient Δ_0/Δ_2 in $G/\Gamma \simeq A_5$ is isomorphic to D_5, the dihedral group of order 10 . Indeed, this quotient is the subgroup generated by the cosets of U, T and L that we denote again by U, T and L in Δ_0/Δ_2. It is therefore the group of coset representatives:

$$H := \Delta_0/\Delta_2 = \{I_2, U, T, L, L^2, L^3, L^4, LT, UL, UL^2\} \simeq D_5 .$$

A component of the pre-image of $\mathcal{T}$ in X_2 corresponds to a copy of the embedding of Δ_2 in Γ_2. We denote it by $\mathcal{T}_2$. There are 6 such components in X_2 because the subgroup of G/Γ_2 that stabilizes one of them is Δ_0/Δ_2. The volume of $\mathcal{T}_2$ equals 10 times the volume of the curve in X, this is -12.

The curve $\mathcal{T}_2$ does not have elliptic singularities because Δ_2 is contained in Γ_2 which acts freely on $\mathbb{H}$. The number of cusps is at least equal to the number of cusps of Γ_2, this is 5. We can explicitly compute this number by considering the H-orbit of each of the cusps of $\mathcal{T}$. For the cusp $\infty := [1 : 0] \in \mathbb{P}^1(K)$, we have the following images:

$I_2(\infty) = U(\infty) = [0 : 1] = \infty;$

$L(\infty) = [0 : 1];$

$L^2(\infty) = [-\varepsilon_0' : 1];$

$L^3(\infty) = [1 : 1];$

$L^4(\infty) = [\varepsilon : 1],$

that give us 5 cusps inequivalent under Γ_2 and therefore certainly inequivalent under Δ_2. The remaining points of the orbit:

$T(\infty) = [-2 : 1];$

$LT(\infty) = [1 : 2 + \varepsilon_0];$

$UL(\infty) = [-2 - \varepsilon_0 : 1];$

$UL^2(\infty) = [2 + \sqrt{} : 1],$

are actually Δ_2 equivalent to the first five, and therefore represent the same cusps. Indeed,

$$LUT^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \equiv I_2 \bmod (2\mathcal{O})$$

is an element of Δ_2 and sends $T(\infty)$ to $L(\infty)$. In the same way,

$$LTUL^2 = \begin{pmatrix} -1-4\varepsilon_0 & 2+2\varepsilon_0 \\ -6-14\varepsilon_0 & 7+8\varepsilon_0 \end{pmatrix} \in \Delta_2 \quad \text{sends } L^2(\infty) \text{ to } LT(\infty);$$

$$-(UL^2)^2 \in \Delta_2 \quad \text{sends } L^3(\infty) \text{ to } UL^2(\infty);$$

$$-(UL)^2 \in \Delta_2 \quad \text{sends } L^4(\infty) \text{ to } UL(\infty).$$

It follows that the curve $\overline{\Delta_2\backslash\mathbb{H}}$ has 5 distinct cusps above ∞. Consider now the H-orbit of the cusp $\tau := [-1:0]$:

$I_2(\tau) = T(\tau) = \tau = [-1:1] \equiv [1:1] \bmod \Gamma_2$;

$U(\tau) = L^4(\tau) = [\varepsilon_0^2 : 1] \equiv [\varepsilon_0^2 : 1] \bmod \Gamma_2$;

$L(\tau) = LT(\tau) = [\varepsilon_0'^2 : 1] \equiv [\varepsilon_0 : 1] \bmod \Gamma_2$;

$L^2(\tau) = [\varepsilon_0 : 2] \equiv \infty \bmod \Gamma_2$;

$L^3(\tau) = [2\varepsilon_0' : 1] \equiv [0:1] \bmod \Gamma_2$;

$UL(\tau) = [4:1]$;

$UL^2(\tau) = [-3\varepsilon_0' + 7 : 2]$.

The first five images give us 5 distinct cusps for the action of Δ_2. The points $UL(\tau)$ and $UL^2(\tau)$ are Δ_2 equivalent to $L^3(\tau)$ and $L^2(\tau)$ respectively. One can check again that:

$$-ULTL^2 \in \Delta_2 \quad \text{sends } L^3(\tau) \text{ to } UL(\tau);$$

$$-UL^2TL^3 \in \Delta_2 \quad \text{sends } L^2(\tau) \text{ to } UL^2(\tau).$$

We conclude that the curve $\overline{\Delta_2\backslash\mathbb{H}}$ has 5 distinct cusps above τ and 5 above ∞. Therefore, the curve $\mathcal{T}_2$, seen inside of Y_2, goes through the five Γ_2 cusps, intersecting each one twice.

We want to find the genus g of $\mathcal{T}_2$. For this, we apply the Riemann-Hurwitz formula to the curves $\mathcal{T}_2$ and $\mathcal{T}$.

$$2 - 2g = 10 \cdot 2 - R, \tag{4.3}$$

where $R = r_5 + r_\tau + r_\infty$ is the sum of the ramification indices at the fixed points of Δ_0. Above the elliptic point of order 5 of $\mathcal{T}$ there are $\frac{60}{5} = 12$ points, 2 on each component of the cover (because of symmetry), hence 2 in $\mathcal{T}_2$. Therefore, the ramification index at the point of order 5 equals $r_5 = 10 - 2 = 8$. We saw that above the cusp at ∞ of Δ_0 there are 5 cusps in $\mathcal{T}_2$. Therefore $r_\infty = 10 - 5 = 5$ and for the cusp τ we have also $r_\tau = 5$. From 4.3, we find:

$$2 - 2g = 20 - 8 - 5 - 5;$$

$$g = 0.$$

Therefore, the curve $\mathcal{T}_2$ in Y_2 is a genus zero curve with no elliptic points and which intersects twice each of the 5 cusps resolutions of Y_2.

Using the isomorphism between Y_2 and the blow-up of the cubic surface given in [10], we have various ways to express the canonical divisor K_2 of Y_2 (see [18]). One of them is

$$K_2 = -[w_1 + w_2] - \sum \pi^* L_{ij} ,$$

where $\pi : \ Y_2 \ \rightarrow \ X_2$ is the map resolving the cusp singularities of X_2 and L_{ij} are the curves of the resolution. We have then for any irreducible curve C on Y_2,

$$K_2 \cdot \mathcal{C} = -\mathrm{vol}(\mathcal{C}) - \sum (\pi^* L_{ij}) \cdot \mathcal{C},$$

and using the adjunction formula,

$$K_2 \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{C} + 2 - 2g(\mathcal{C})$$

we get

$$\mathcal{C} \cdot \mathcal{C} = 2g(\mathcal{C}) - 2 + \mathrm{vol}(\mathcal{C}) + \sum (\pi^* L_{ij}) \cdot \mathcal{C}. \tag{4.4}$$

Since $\mathcal{T}_2$ intersects each cusp resolution twice, we find using (4.4) that its self-intersection number equals $\mathcal{T}_2 \cdot \mathcal{T}_2 = 0 - 2 - 12 + 10 = -4$.

Next we can apply the formulas from [18], pp. 537 to obtain the following system:

$$\begin{cases} 3a - \sum b_i - \sum c_i = -2 \\ 15a - 5\sum b_i - 3\sum c_i = 10 \end{cases} \tag{4.5}$$

We have then

$$\sum c_i = 10\,.$$

Thus the curve $\mathcal{T}_2$ intersects the 10 components of the diagonal $F_1(Y_2)$ in 10 points. The curve $F_1(X)$ goes through a point of order 2 and a point of order 3 in X. The curve $\mathcal{T}$ does not have singularities at any of these points. The curve $F_1(X)$ avoids the points of order 5. Therefore we believe that all the possible intersections of $F_1(X)$ and $\mathcal{T}$ are the following. We give these results without proof since we are still verifying them.

Case 1: The intersection is a simple point of both curves away from the resolution of ∞. In this case $\mathcal{T}_2$ intersects five of the components of $F_1(Y_2)$ with multiplicity 2, this is $c_i = 2$ for five values of i, and each component of the diagonal meets twice three of the components of the A_5 orbit of Υ_2 at six different points.

Case 2: The two curves intersect in X at the point of order 2 of $F_1(X)$. In this case $c_i = 1$, $\forall i$. The curve $\mathcal{T}_2$ intersects the 10 components of $F_1(Y_2)$, two copies of $\mathcal{T}_2$ meet each component of F_1 at a single point and each component of $F_1(Y_2)$ meets the six components of $\mathcal{T}_2(Y_2)$.

Case 3: The intersection in X occur at the point of order 3 of $F_1(X)$ which is a regular point for $\mathcal{T}$. Here $c_i = 2$, $\forall i$, but three copies of $\mathcal{T}_2$ meet each component of F_1 at a single point.

Case 4: The curves $F_1(Y)$ and $\mathcal{T}(Y)$ intersect at the resolution of the cusp ∞. Then $c_i = 1$, $\forall i$ and 2 copies of $\mathcal{T}_2$ intersect D_{34} at one single point and D_{34} intersects the six copies of $\mathcal{T}_2$.

4.3 Elliptic Points and Uniformizing Groups

One of the questions that arose during this research is whether the knowledge of the elliptic points that belong to a curve $\mathcal{C}$ on X determines unequivocally the signature of the uniformizing group of $\mathcal{C}$. The answer is clearly no.

Consider indeed the modular curve $F_5 \subset X$ defined as the Γ-orbit of the curve $\{\lambda z_2 - \lambda^{(2)} z_1 = 0\}$ in $\mathbb{H}^2$. This curve passes through the two elliptic points of order 2 and through the point of type $(3;1,2)$, see [9]. However, the uniformizing group of F_5 (see [9]) is a degree 2 extension of the congruence subgroup $\Gamma_0[5]$ of $\mathrm{SL}_2(\mathbb{Z})$, which has signature $(2,2,\infty,\infty)$. The extension is given by the matrix

$$\begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} \\ \sqrt{5} & 0 \end{pmatrix}.$$

From [19] we have therefore that the uniformizing group of F_5 has signature $(2,2,2,\infty)$. This means that the point of order 3 that F_5 passes through is not a singularity of the curve itself. We can check this by computing the ramification index above that point.

For this we consider the Hilbert modular surface of level $\sqrt{5}$, X_5 given by the congruence subgroup Γ_5 of Γ associated to the ideal $\sqrt{5}\mathcal{O}$. Let Y_5 be its smooth model. Here again Γ_5 is a normal subgroup of Γ and $\Gamma/\Gamma_5 \simeq A_5$. So X_5 is a A_5-cover of X. In [11], Hirzebruch shows that the quotient $\tau\backslash Y_5$ is isomorphic to the surface $\widetilde{\mathbb{P}}_2(\mathbb{C})$ obtained by blowing-up the 6 fundamental points in $\mathbb{P}_2(\mathbb{C})$. It is also proven in [11] that under this isomorphism, the lifted image of F_5 in Y_5 corresponds to the Klein invariant D.
Indeed, the curve F_5 has 15 components on Y_5, the stabilizer of each one in Γ/Γ_5 is of order 4 and each component is a non-singular rational curve. Each component corresponds in $\widetilde{\mathbb{P}}_2(\mathbb{C})$ to one of the 15 lines of D. Fix one of these components and denote it by $F_{5,1}$. We want to show that $F_{5,1}$ is not ramified above the elliptic point $(3;1,2)$ of X.
From Proposition 3.5.1, we know that the fiber in $\tau\backslash Y_5$ above the elliptic point $(3;1,2)$ corresponds in $\widetilde{\mathbb{P}}_2(\mathbb{C})$ to the 10 Brianchon points. Each line of D contains two such points and each point belongs to 3 distinct lines. Under the action of τ, these 10 points give 20 points in Y_5, the curve $F_{5,1}$ passes therefore through 4 of them. These are 4 of the 20 points on the fiber of the point $(3;1,2)$ under the A_5-cover: $X_5 \to X$. The stabilizer of $F_{5,1}$ being of order 4, this proves that $F_{5,1}$ is not ramified above $(3;1,2)$.

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