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Let E_2 be the set of ordered pairs of nonnegative real numbers and m^2 be the two-dimensional Lebesgue measure. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in E_2, then we write $x \preceq y$ if $x_i \leq y_i$ $(i = 1, 2)$. For $x \in E_2$ we let $L(x) = \{y \in E_2 : y \preceq x\}$. A bounded nonempty set $F \subseteq E_2$ is called fundamental if $x \in F$ implies $L(x) \subseteq F$. Let F be fundamental and $Cl(F)$ be the closure of F. Then F^* is the set of elements $x \in Cl(F)$ such that $y \notin Cl(F)$ if $x \preceq y$ and $x \neq y$.

Let A be a measurable subset of E_2. We define the density α of A by

$$\alpha = \text{glb} \frac{m^2(A \cap F)}{m^2(F)}$$

where F ranges over all fundamental sets with positive measure. If A and B are subsets of E_2 the sum set $C = A + B$ is

defined to be the set of all $a + b$ such that $a \in A$ and $b \in B$.

In this thesis we prove that if A and B are open subsets of E_2, $0 \in A \cap B$, A satisfies a restriction R described in the thesis, $F \sim C \neq \emptyset$, and $F^* \subseteq Cl(F \sim C)$, then $m^2(C \cap F) \geq \alpha m^2(F) + m^2(B \cap F)$. This is a continuous analogue of an extension to two dimensions of Mann's Second Theorem. Furthermore, a continuous analogue is obtained for Mann's $\alpha\beta$ Theorem where, in this case, the extension to two dimensions is unknown. We also obtain the Landau-Schnirelmann Inequality without the restriction R on A.

Furthermore, we obtain a one-dimensional continuous analogue of Mann's $\alpha\beta$ Theorem for measurable sets A and B where $0 \in A \cap B$.

Continuous Analogues of Density Theorems

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TABLE OF CONTENTS

CONTINUOUS ANALOGUES OF DENSITY THEOREMS

1. INTRODUCTION

In this chapter we will give a brief introduction to density theory, giving some of the major results concerned with subsets of nonnegative integers and extensions of these results to subsets of n-dimensional lattice points whose coordinates are nonnegative integers. We will then define the analogous concepts for subsets of nonnegative real numbers and extend these to certain subsets of ordered pairs of nonnegative real numbers. We will be mainly concerned with sets that are measurable and our proofs are strongly dependent on this assumption. The purpose of the thesis is to obtain results that concern real numbers that analogue the results obtained for integers. Thus, we obtain continuous results that analogue known discrete results.

1.1. Some Density Theorems for Subsets of Nonnegative Integers

Let A be a set of nonnegative integers. For $n > 0$ let $A(n)$ denote the number of positive integers in A that are less than or equal to n. In 1930 Schnirelmann [16] introduced the concept of density for A. He defined the density α of the set A to be $\alpha = \text{glb}\,\{\frac{A(n)}{n}: n \geq 1\}$. Erdös [4] defined another density α_1 of A

to be $\alpha_1 = \operatorname{glb}_{n \geq k} \frac{A(n)}{n+1}$ where k is the smallest positive integer missing from A.

For any two sets of nonnegative integers A and B, the sum set $C = A + B$ is defined by $C = A + B = \{a + b : a \in A, b \in B\}$. The density of B and C are denoted respectively by β and γ. We will now give a list of some of the theorems that relate the density γ of the sum set C to the densities of A and B if $0 \in A \cap B$:

(1.1) $\gamma \geq \alpha + \beta - \alpha\beta$ (Landau [10], Schnirelmann [16]),

(1.2) $\gamma \geq \frac{\beta}{1-\alpha}$ provided $\alpha + \beta < 1$ (Schur [17]),

(1.3) $\gamma \geq \min\{1, \alpha+\beta\}$ (Mann [12]).

Another theorem of Mann's [13] is

(1.4) $C(n) \geq \alpha_1(n+1) + B(n)$ for $n > 0$ and $n \notin C$.

1.2. Extensions to n-Dimensions of Density Theorems for Subsets of Nonnegative Integers

Let I_n be the set of all vectors $(x_1, \ldots, x_n)$ such that x_j is a nonnegative integer for $j = 1, \ldots, n$. A partial ordering $\precsim$ is defined on I_n by $x \precsim y$ if and only if $x_i \leq y_i$ for $i = 1, \ldots, n$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. If $x \precsim y$ and $x_k < y_k$ for some k, then we may write $x \prec y$. For $r \in I_n$ let $L(r) = \{x \in I_n : x \precsim r\}$.

A nonempty finite subset F of I_n that does not consist only of the origin is called fundamental if whenever $r \in F$, then $L(r) \subseteq F$. For A and T subsets of I_n with T finite, let $A(T)$ denote the number of non-zero vectors in $A \cap T$. Then the (Kvarda) density of A is

$$\alpha = \operatorname{glb} \frac{A(F)}{I_n(F)}$$

where F ranges over all fundamental sets of I_n. This is an extension of Schnirelmann density.

For n-dimensional space there is also a useful extension of the one-dimensional Erdős density. If A is any proper subset of I_n let

$$\alpha_1 = \operatorname{glb} \frac{A(F)}{I_n(F)+1}$$

where F ranges over all fundamental sets such that $A(F) < I_n(F)$.

Let A and B be subsets of I_n. Then the sum set $C = A + B$ is defined by $C = \{a + b\colon a \in A,\ b \in B\}$ where addition of vectors is done coordinatewise. Let β and γ be the respective densities of B and C. Then the following two extensions to n-dimensions have been made relating γ to α and β if $0 \in A \cap B$.

(2.1) $\gamma \geq \alpha + \beta - \alpha\beta$ (Kvarda [7]) and

(2.2) $$\gamma \geq \frac{\beta}{1-\alpha} \quad \text{if} \quad \alpha + \beta < 1 \quad \text{(Freedman [5])}.$$

A large open question remaining in additive number theory is whether Mann's $\alpha\beta$ Theorem (inequality 3, Section 1.1) can be extended to n-dimensions using Kvarda density.

Although Mann's $\alpha\beta$ Theorem has not been extended to n-dimensions, Kvarda [8] has extended Mann's Second Theorem (inequality 1.4) to n-dimensions. A theorem that satisfies the hypothesis of Kvarda's extension and has the same conclusion is as follows. If A and B are subsets of I_n, $0 \in A \cap B$, F is any fundamental set such that $C(F) < I_n(F)$, and the maximal points of F are contained in the complement of C, then

(2.3) $$C(F) \geq \alpha_1(I_n(F)+1) + B(F).$$

For $n = 1$ it can be shown that this result implies 1.4.

1.3. Continuous One-Dimensional Analogues

In 1961 A.M. Macbeath [11] proved a continuous analogue to Mann's $\alpha\beta$ Theorem. In the proof he assumed Dyson's [3] inequality for subsets of positive integers. To this author's knowledge Macbeath has been the only person to obtain a continuous $\alpha\beta$ theorem. His paper led us to ask if we could obtain continuous analogues of density theorems and, if possible, obtain them without assuming the results true in the integer setting. We succeeded in doing so in many cases and were able to extend our results with certain restrictions to two dimensions. We will now define the required terms so we can list in this section the main one-dimensional continuous analogues that we obtain in Chapter 3.

Let E_1 be the set of nonnegative real numbers and m be the one-dimensional Lebesgue measure. Let m_* be the inner measure.

<u>Definition 1.1.</u> Let K be any subset of E_1. Then $K(t) = m_*(K \cap [0,t])$.

We note $K(t)$ also equals $m_*(K \cap [0,t))$.

We notice that if K is measurable then $K(t) = m(K \cap [0,t])$.

<u>Definition 1.2.</u> Let K be any subset of E_1. Then the density of K, denoted by $d(K)$, is defined by $d(K) = \text{glb}\{\frac{K(t)}{t} : t > 0\}$.

We notice that $K(t) \geq t(d(K))$ for all $t \geq 0$.

<u>Definition 1.3.</u> Let A and B be subsets of E_1. Then the sum set $C = A + B$ is defined by $C = \{a + b : a \in A, b \in B\}$.

In Chapter 3, A and B will always be subsets of E_1 with

$0 \in A \cap B$. Hence $A \cup B \subseteq C$. We denote the densities of A, B, and C by α, β, and γ respectively.

In Chapter 3 we prove that if A and B are measurable subsets of E_1 and $0 \in A \cap B$ then

(3.1) $\gamma \geq \alpha + \beta - \alpha\beta$,

(3.2) $\gamma \geq \frac{\beta}{1-\alpha}$ if $\alpha + \beta < 1$,

(3.3) $\gamma \geq \min\{1, \alpha + \beta\}$,

(3.4) if $x \in E_1 \sim C$, $C(x) \geq \alpha x + B(x)$.

We note that if $x \in E_1 \sim C$, then $x \notin C$ and $x > 0$ since $0 \in C$.

We see that (3.1), (3.2), (3.3), and (3.4) are analogous to (1.1), (1.2), (1.3), and (1.4) of Section 1.1. We notice that for work with subsets of the reals we have not introduced a separate density analogous to the density of Erdös given in Section 1.1. Since the measure of a point is zero it turns out that the analogue of Erdös density that interests us is just Schnirelmann density in the continuous setting.

1.4. Continuous Two-Dimensional Analogues

Let E_2 be the set of ordered pairs of nonnegative real numbers. Let m^2 be the two-dimensional Lebesgue measure.

Definition 1.4. For x and y in E_2 we write $x \precsim y$ if and only if $x_i \leq y_i$ $(i = 1, 2)$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_k < y_k$ for either $k = 1$ or $k = 2$, we write $x \prec y$.

Definition 1.5. For $x \in E_2$ let $L(x) = \{y \in E_2 : y \precsim x\}$. We call $L(x)$ the lower set of x.

Definition 1.6. A nonempty bounded subset F of E_2 is called a fundamental set provided that $x \in F$ implies $L(x) \subseteq F$.

In Chapter 2 we show that any fundamental set is measurable. In Chapter 4 and Chapter 5 we will let $\mathfrak{F}$ be the family of all fundamental sets. We denote by $\mathfrak{F}^+$ the sub-family of $\mathfrak{F}$ that consists of the fundamental sets with positive measure; that is, $F \in \mathfrak{F}^+$ implies $F \in \mathfrak{F}$ and $m^2(F) > 0$.

Definition 1.7. Let K be a measurable subset of E_2. For any bounded measurable set S we let $K(S) = m^2(K \cap S)$.

Definition 1.8. Let A be a measurable subset of E_2. Then the density α of A is given by

$$\alpha = \text{glb}\{\frac{A(F)}{m^2(F)} : F \in \mathfrak{F}^+\}.$$

We note that if $F \in \mathfrak{F}^+$, then from Definition 1.8 we have $A(F) \geq \alpha m^2(F)$ and if $F \in \mathfrak{F}$ but $m^2(F) = 0$ we still have

$A(F) \geq \alpha m^2(F)$. Hence we have $A(F) \geq \alpha m^2(F)$ for all $F \in \mathcal{F}$.

Definition 1.9. Let F be a member of $\mathcal{F}$. Then $F^* = \{x \in Cl(F) : x \precsim y$ and $x \neq y$ implies $y \notin Cl(F)\}$ where $Cl(F)$ is the closure of F.

Definition 1.10. If A and B are subsets of E_2, then $C = A + B = \{a + b : a \in A, b \in B\}$ where addition is vector addition.

In Chapters 4 and 5 A and B will be subsets of E_2 with $0 \in A \cap B$. Therefore $A \cup B \subseteq C$.

When $0 \in A \cap B$, A satisfies a certain restriction defined in Chapter 4 and A and B are open, then we show

(4.1) $\gamma \geq \alpha + \beta - \alpha\beta$

(4.2) $\gamma \geq \dfrac{\beta}{1-\alpha}$ if $\alpha + \beta < 1$

(4.3) $\gamma \geq \min\{1, \alpha + \beta\}$

and

(4.4) if $F \setminus C \neq \emptyset$ and $F^* \subseteq Cl(F \setminus C)$, then $C(F) \geq \alpha m^2(F) + B(F)$.

We see (4.1), (4.2), and (4.4) are continuous analogues of results for lattice points, but (4.3) has no counterpart in the discrete case.

Müller [14] has done some work in this area in a more general setting. He considers any locally compact abelian group and uses the Haar measure. However, the only inequalities he obtains that are the same as the ones considered in this thesis are the Schur Inequality and the Landau-Schnirelmann Inequality. However, when he obtains these inequalities he is in the special case where his locally compact abelian group is the set of n-tuples of integers and his Harr measure is the counting measure. Thus, his results are different than the results of this thesis.

Other than Macbeath and Müller the author does not know of any other work that is closely related to the work in this thesis. In Chapter 6 we will suggest a few other ideas one can consider in this area.

In Chapter 2 we will state or prove some results that are needed from topology or measure theory. In Chapter 3 we will obtain our one-dimensional results and in Chapters 4 and 5 we will obtain our two-dimensional extensions.

1.5. Comparison of Theorems

In Section 1.1 we listed four density theorems. The four theorems are not all incomparable however. The $\alpha\beta$ Theorem implies both the Landau-Schnirelmann Inequality and the Schur Inequality, but on the other hand Lim [19] showed that the $\alpha\beta$ Theorem and Mann's Second Theorem are incomparable. Since the validity of the $\alpha\beta$ Theorem is not known in n-dimensions, the above comparisons can not be made at this time in n-dimensions.

In the continuous setting in both one dimension and two dimensions we obtain the corresponding four inequalities. However, since in the continuous setting the analogue of Erdös density we use is the continuous Schnirelmann density, we are able to show that Mann's Second Theorem implies the $\alpha\beta$ Theorem. As in the one-dimensional discrete case the $\alpha\beta$ Theorem implies both the Landau-Schnirelmann and the Schur Inequality.

The style and organization of this thesis is motivated by not only the results we establish, but also by the desire to emphasize the extent to which the theory for the discrete case may be carried over to analogous theory for the continuous case.

2. TOPOLOGICAL AND MEASURE THEORETIC PROPERTIES

In this chapter we state several results from real analysis that are topological or measure theoretic in nature and which are needed as tools in our development. The proof is included whenever the author has been unable to find the result in a standard textbook. Also some well known results are not stated here, but are used later in the thesis. These results either can be found in almost any standard text on real analysis; for example, Royden [15] or Asplund and Bungart [1], or if harder to locate will be referenced.

Some of the theorems will be stated or proved for n-dimensional space. However in the thesis we use these theorems only when $n = 1$ or $n = 2$. The proofs of these theorems are no more complicated for arbitrary n and this lets us avoid stating or proving these theorems separately for $n = 1$ and $n = 2$.

2.1. Topological Properties

Theorem 2.1. Every open set of real numbers is the union of a countable collection of disjoint open intervals.

Definition 2.1. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be points of E_2 such that $a_i < b_i$ $(i = 1, 2)$. Then the open rectangle $\{x = (x_1, x_2): a_i < x_i < b_i,\ i = 1, 2\}$ is denoted by $\square_a^b$.

<u>Definition 2.2.</u> Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be points of E_2 such that $a_i \leq b_i$ $(i = 1, 2)$ and either $a_1 < b_1$ or $a_2 < b_2$. Then the set $\{x = (x_1, x_2) : a_i \leq x_i \leq b_i,\ i = 1, 2\}$ is a closed rectangle of E_2.

We see that a closed line segment may be a closed rectangle of E_2.

In Definitions 2.1 and 2.2 we say the points (a_1, a_2) and (b_1, b_2) determine the defined rectangles.

<u>Definition 2.3.</u> The vertices of the rectangle determined by (a_1, a_2) and (b_1, b_2) are the points (a_1, a_2), (a_1, b_2), (b_1, a_2), and (a_2, b_2).

<u>Definition 2.4.</u> If T is an open rectangle of E_2 or if T is a closed rectangle of E_2 and K is any subset of E_2, then $T \cap K$ will be called a rectangle of K.

We notice that $T \cap K$ may not have vertices.

<u>Theorem 2.2.</u> Any open set of E_2 can be written as a countable union of open rectangles.

<u>Lemma 2.3.</u> If $\square_a^b$ is an open rectangle contained in A and $\square_c^d$ is an open rectangle contained in B, then $\square_a^b + \square_c^d = \square_{a+c}^{b+d}$ is an open rectangle contained in C.

Proof: Clearly $\square_a^b + \square_c^d \subseteq \square_{a+c}^{b+d}$ since $x \in \square_a^b$ implies $a \prec x \prec b$ and $y \in \square_c^d$ implies $c \prec y \prec d$. Hence $a + c \prec x + y \prec b + d$. Therefore $x + y \in \square_{a+c}^{b+d}$.

On the other hand, suppose $f = (f_1, f_2) \in \square_{a+c}^{b+d}$. Then $a + c \prec f \prec b + d$. Hence $a_1 + c_1 < f_1 < b_1 + d_1$ and $a_2 + c_2 < f_2 < b_2 + d_2$. We see $a_1 + c_1 < f_1 < b_1 + d_1$ implies

$$f_1 = (a_1+c_1) + [(b_1+d_1)-(a_1+c_1)]t$$

for some t such that $0 < t < 1$. Let

$$x_1 = a_1 + (b_1-a_1)t$$

and

$$y_1 = c_1 + (d_1-c_1)t.$$

Then

$$x_1 + y_1 = (a_1+c_1) + [(b_1+d_1)-(a_1+c_1)]t = f_1$$

and

$$a_1 < x_1 < b_1$$

and

$$c_1 < y_1 < d_1.$$

In the same way we can write $f_2 = x_2 + y_2$ where $a_2 < x_2 < b_2$ and $c_2 < y_2 < d_2$. Therefore $f = (f_1, f_2) = (x_1, x_2) + (y_1, y_2)$ where $(x_1, x_2) \in \square_a^b$ and $(y_1, y_2) \in \square_c^d$. Hence $\square_a^b + \square_c^d = \square_{a+c}^{b+d}$.

Since every element of $\square_{a+c}^{b+d}$ can be expressed as the sum of an element from A and an element from B we have $\square_{a+c}^{b+d} \subseteq C$.

Corollary 2.4. If A and B are open subsets of E_2, then $A + B = C$ is open.

Proof: Corollary 2.4 follows immediately from Theorem 2.2 and Lemma 2.3.

Lemma 2.5. If A and B are closed subsets of E_1, then $C = A + B$ is closed.

Proof: Let c be a limit point of C. Then there is an infinite sequance $<c_i>$ such that $c_i \to c$ and each $c_i \in C$. Thus there must be two infinite sequences $<a_i>$ and $<b_i>$ with $a_i \in A$ and $b_i \in B$ such that $a_i + b_i = c_i$. Since $<c_i>$ is a bounded sequence and $0 \leq a_i \leq c_i$ we see $<a_i>$ is a bounded sequence. Therefore $<a_i>$ has a convergent subsequence $<a^j>$. Suppose $a^j \to a$. Then $a \in A$ since A is closed. Let $<c^j>$ be the corresponding subsequence of $<c_i>$. Then $c^j \to c$. Thus $\lim b^j = \lim(c^j - a^j) = c - a$. Since B is closed we have $c - a \in B$. Hence $c = a + (c-a) \in C$. Therefore any limit point of C is in C. Thus C is closed.

2.2. Measure Theoretic Properties

Theorem 2.6. Let $\{A_j\}$ be a countable collection of measurable sets. Then $m^n(\bigcup_{j=1}^{\infty} A_j) = \lim_{N\to\infty} m^n(\bigcup_{j=1}^{N} A_j)$, where m^n is the n-dimensional Lebesgue measure.

Theorem 2.7. Let $\{A_i\}$ be a countable collection of measurable sets in E_1. If $m(A_1) < \infty$ and $A_{i+1} \subseteq A_i$, $i = 1, 2, \ldots$, then $m(\bigcap_{i=1}^{\infty} A_i) = \lim_{N\to\infty} m(A_N)$.

Theorem 2.8. If $K \subseteq E_1$ is measurable, then for any $\varepsilon > 0$ there is a closed set $H \subseteq K$ such that $m(K \setminus H) < \varepsilon$.

Theorem 2.9. Let H and K be disjoint sets of real numbers. Then

$$m_*(H) + m_*(K) \leq m_*(H \cup K) \leq m_*(H) + m^*(K)$$

where m^* is the Lebesgue outer measure.

Theorem 2.10. Suppose the function f on E_2 is Lebesgue integrable and $m^2 = m_1 \times m_2$ where $m_1 = m_2 = m$. Then the iterated integrals $\int_{E_1} (\int_{E_1} f dm_1) dm_2$ and $\int_{E_1} (\int_{E_1} f dm_2) dm_1$ exist and are equal to $\int_{E_2} f dm^2$.

Lemma 2.11. Let $S \subseteq E_2$ be a measurable set such that $m^2(S) < \infty$. Let $T_{c_1} = \{(c_1, x_2) : x_2 \geq 0\}$ and $T_{c_2} = \{(x_1, c_2) : x_1 \geq 0\}$. Let $g(c_1) = m(T_{c_1} \cap S)$ and $h(c_2) = m(T_{c_2} \cap S)$. Then

$$m^2(S) = \int_{E_1} g(x)dm = \int_{E_1} h(x)dm.$$

Proof: Let χ_S be the characteristic function of S. Then

$$m^2(S) = \int_{E_1 \times E_1} \chi_S(x_1, x_2)dm^2$$

$$= \int_{E_1} (\int_{E_1} \chi_S(x_1, x_2)dm_2)dm_1$$

by Theorem 2.10. However $\int_{E_1} \chi_S(x_1, x_2)dm_2 = g(x_1)$ since $\int_{E_1} \chi_S(x_1, x_2)dm_2 = m(T_{x_1} \cap S)$. Hence $m^2(S) = \int_{E_1} g(x_1)dm$. An exactly similar argument shows $m^2(S) = \int_{E_1} h(x)dm$.

Theorem 2.12. (Royden [15]). Let f be a nonnegative integrable function on $(-\infty, \infty)$, and let m^2 be two-dimensional Lebesgue measure on the set of ordered pairs of real numbers. Then $m^2\{(x, y): 0 \leq y \leq f(x)\} = m^2\{(x, y): 0 \leq y < f(x)\} = \int f(x)dm$.

Lemma 2.13. If $F \in \mathfrak{F}$, then F is measurable and $m^2(F) = m^2(Cl(F))$.

Proof: For (x_1, x_2) in F let $f(x_1) = \sup\{y: (x_1, y) \in \partial F\}$ where ∂F is the boundary of F. Then $f(x)$ is a nonincreasing function. To see this suppose $\xi_1 < \xi_2$ and $f(\xi_2) > f(\xi_1)$. Then there exist open disks D_1 and D_2 about centers $(\xi_1, f(\xi_1))$ and $(\xi_2, f(\xi_2))$ respectively such that $(a_1, a_2) \in D_1$ and $(b_1, b_2) \in D_2$ implies $a_1 < b_1$ and $a_2 < b_2$. Since $(\xi_2, f(\xi_2))$ is a boundary point of F there exists a point $f = (f_1, f_2) \in D_2 \cap F$. Since $(\xi_1, f(\xi_1))$ is a boundary point of F there exists a point $g = (g_1, g_2) \in D_1 \setminus F$. But $g \in L(F)$ from the way we chose D_1 and D_2. Therefore $L(f) \not\subset F$. This contradicts that F is fundamental. Hence f is nonincreasing.

Since f is nonincreasing it has at most a countable number of discontinuities and therefore f is a nonnegative measurable function. Now $Cl(F) = \{(x, y) : 0 \leq y \leq f(x)\}$. Since $Cl(F)$ is bounded we see f is a nonnegative integrable function. Therefore from Theorem 2.12 we have $\int f dm = m^2(Cl(F))$.

Let $T_{x_1} = \{(x_1, x_2) : x_2 \geq 0\}$ and let $g(x_1) = m(T_{x_1} \cap F)$. Suppose x_1 is not a discontinuity of $f(x)$. Then $\{(x_1, y) : (x_1, y) \in \partial F\}$ is a singleton. To see this suppose $(x_1, y_1) \in \partial F$ and $(x_1, y_2) \in \partial F$ and $y_1 < y_2$. Then $f(x_1) \geq y_2$ and $\lim_{x \to x_1^+} f(x) \leq y_1 < y_2$. Hence $\lim_{x \to x_1^+} f(x) \neq f(x_1)$ and f would not be continuous at $x = x_1$. Hence if x_1 is a point of continuity of f, we have $g(x_1) = f(x_1)$. Thus $f = g$ almost everywhere. Therefore $\int f dm = \int g dm$. We have

$$\int g\,dm = m^2\{(x,y): 0 \leq y \leq g(x)\} = m^2\{(x,y): 0 \leq y < g(x)\}.$$

However

$$\{(x,y): 0 \leq y < g(x)\} \subseteq F \subseteq \{(x,y): 0 \leq y \leq g(x)\}.$$

Therefore $\int g\,dm = m^2(F)$. Thus $m^2(\mathrm{Cl}(F)) = m^2(F)$.

Theorem 2.14. Lebesgue measure is translation invariant.

Theorem 2.15. (Asplund [1]). Suppose φ is a linear transformation of R^n, the set of n-tuples of real numbers, with $\det \varphi \neq 0$ and m^n is the n-dimensional Lebesgue measure. If f is an integrable function on R^n, then $f \circ \varphi$ is also integrable and $\int f \circ \varphi\, dm^n = \frac{1}{|\det \varphi|} \int f\,dm^n$.

Definition 2.3. The set $a + K = \{a + k: k \in K\}$.

Theorem 2.16. If $T \subseteq R^n$ is measurable and $m^n(T) < \infty$ and $g \in R^n$, then $m^n(T) = m^n(\{g-t: t \in T\}$.

Proof: Let $H = \{g-t: t \in T\}$. Then $H = g + (-T)$ where $-T = \{-t: t \in T\}$. From Theorem 2.14 we see $m^n(H) = m^n(-T)$. Therefore to finish the proof it suffices to show $m^n(T) = m^n(-T)$.

Let χ_T and χ_{-T} be the characteristic functions of T and $-T$ respectively. Let φ be the linear transformation defined by $\varphi(t) = -t$. Then φ is given by $\begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$ and hence $|\det \varphi| = 1 \neq 0$.

We see that $\chi_T = \chi_{-T} \circ \varphi$ since

$$(\chi_{-T} \circ \varphi)(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases}$$

and

$$\chi_T = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases}.$$

Thus

$$m^n(T) = \int \chi_T dm^n = \int \chi_{-T} \circ \varphi dm^n.$$

However by Theorem 2. 15 we have

$$\int \chi_{-T} \circ \varphi dm^n = \frac{1}{|\det \varphi|} \int \chi_{-T} dm^n$$

$$= \frac{1}{|\det \varphi|} m^n(-T)$$

$$= m^n(-T).$$

Hence $m^n(T) = m^n(-T)$.

3. ONE-DIMENSIONAL CONTINUOUS ANALOGUES OF DENSITY THEOREMS

In this chapter we will develop continuous one-dimensional analogues of density theorems concerning sum sets of nonnegative integers. The set E_1 will be considered as the universal set.

3.1. The Landau-Schnirelmann Inequality

Before proving the Landau-Schnirelmann inequality in the case where A and B are measurable subsets of E_1, we will first prove a series of lemmas. We first obtain an inequality for open subsets A and B of E_1, then one for closed subsets of E_1 by taking intersections of certain open sets, and finally one for measurable sets using the fact that every measurable set contains a closed subset of nearly equal measure.

Lemma 3.1. If A and B are open subsets of E_1, $0 \in A \cap B$, and x is a positive real number, then $C(x) \geq (1-\beta)A(x) + \beta x$.

Proof: Suppose $x > 0$ is fixed. Since A is open, $A \cap [0,x]$ can be written as a disjoint union of open intervals in the relative topology. Since $m(A \cap [0,x]) < \infty$, then for any $\varepsilon > 0$ there exists a finite number of disjoint open intervals contained in

$A \cap [0,x]$, say $[0,a_1) = [a_0,a_1), (a_2,a_3), \ldots, (a_{2n},a_{2n+1})$, such that $m(A\cap[0,x]) - \varepsilon \leq m(\bigcup_{j=0}^{n} (a_{2j},a_{2j+1}))$ where $j < k$ implies $a_j \leq a_k$.

Let $A^\varepsilon = [\bigcup_{j=0}^{n} (a_{2j},a_{2j+1})] \cup [A\cap(x,\infty)]$ and let $C^\varepsilon = A^\varepsilon + B$. Then $A^\varepsilon \subsetneq A$ and $A(x) \geq A^\varepsilon(x) \geq A(x) - \varepsilon$. We see also that $C^\varepsilon \subsetneq C$. Let $h_1 = a_2 - a_1, \ldots, h_n = a_{2n} - a_{2n-1}$, and $h_{n+1} = x - a_{2n+1}$. Then $x - A^\varepsilon(x) = x - m(A^\varepsilon\cap[0,x]) = \sum_{j=1}^{n+1} h_j$.

For each i, $0 \leq i \leq n$, the numbers $a_{2i+1} + b$, where $b \in B$ and $a_{2i+1} \leq a_{2i+1} + b \leq a_{2i+1} + h_{i+1}$, are in C^ε but not in A^ε. That $a_{2i+1} + b \notin A^\varepsilon$ follows from the fact that (a_{2i},a_{2i+1}) and (a_{2i+2},a_{2i+3}) are two consecutive intervals of A^ε. That $a_{2i+1} + b \in C^\varepsilon$ follows from the fact that B is open. Thus $b \in B$ implies that there exists δ such that $0 < \delta < (a_{2i+1}-a_{2i})$ and $b' \in B$ if $b' \in (b-\delta,b+\delta)$. Then

$$a_{2i+1} + b = (a_{2i+1} - \frac{\delta}{2}) + (b+\frac{\delta}{2}) \in A^\varepsilon + B = C^\varepsilon.$$

The set of all such real numbers b is $B \cap [0,h_{i+1}]$.

Therefore,

$$C^{\varepsilon}(x) \geq A^{\varepsilon}(x) + \sum_{i=0}^{n} m(a_{2i+1} + B \cap [0, h_{i+1}])$$

$$= A^{\varepsilon}(x) + \sum_{i=0}^{n} m(B \cap [0, h_{i+1}])$$

$$\geq A^{\varepsilon}(x) + \beta \sum_{i=1}^{n+1} h_i$$

$$= A^{\varepsilon}(x) + \beta(x - A^{\varepsilon}(x))$$

$$\geq A(x) + \beta(x - A(x)) - \varepsilon.$$

However $C(x) \geq C^{\varepsilon}(x)$. Hence

$$C(x) \geq A(x) + \beta(x - A(x)) - \varepsilon .$$

Letting ε tend to zero we obtain

$$C(x) \geq A(x) + \beta(x - A(x))$$

$$= (1-\beta)A(x) + \beta x.$$

Now given any set $A \subseteq E_1$ we will construct an open set from A.

Definition 3.1. Let N be the set of positive integers. For a given set $A \subseteq E_1$ and for given h and k in N let
$A_{h/k} = \{x \in E_1 : |x-a| < \frac{h}{k}$ for some $a \in A\}$.

Lemma 3.2. If $0 \in A \cap B$, then $A_{1/k} + B_{1/k} \subseteq C_{2/k}$.

Proof: Let $x = a' + b'$ where $a' \in A_{1/k}$ and $b' \in B_{1/k}$. Then $|a-a'| < \frac{1}{k}$ for some $a \in A$ and $|b-b'| < \frac{1}{k}$ for some $b \in B$. Hence $|(a'+b') - (a+b)| \leq |a'-a| + |b'-b| < \frac{2}{k}$. Therefore $a' + b' \in C_{2/k}$. Hence $A_{1/k} + B_{1/k} \subseteq C_{2/k}$.

Lemma 3.3. If a set $K \subseteq E_1$ is closed, then $K \cap [0,x] = \bigcap_{k \in N} \{(K_{h/k} \cap [0,x]) : h \in N \text{ is fixed}\}$.

Proof: We have $K \cap [0,x] \subseteq K_{h/k}$ for all $k \in N$. Therefore $K \cap [0,x] \subseteq \bigcap_{k \in N} (K_{h/k} \cap [0,x])$. Assume that $\bigcap_{k \in N} (K_{h/k} \cap [0,x]) \not\subseteq K \cap [0,x]$. Choose $y \in \bigcap_{k \in N} (K_{h/k} \cap [0,x])$ such that $y \notin K \cap [0,x]$. Then we have two possibilities

(1) there exists $k \in N$ such that $\{z : |z-y| < \frac{h}{k}\} \cap (K \cap [0,x]) = \emptyset$

or otherwise

(2) for all $k \in N$ we have $\{z : |z-y| < \frac{h}{k}\} \cap (K \cap [0,x]) \neq \emptyset$.

If (1) is true, then $y \notin (K \cap [0,x])_{h/k} \subseteq K_{h/k} \cap [0,x]$, which is a contradiction. If (2) is true, then y is a boundary point of $K \cap [0,x]$, but since $K \cap [0,x]$ is closed $y \in K \cap [0,x]$. Hence we again have a contradiction. Therefore

$$\bigcap_{k \in N} (K_{h/k} \cap [0,x]) = K \cap [0,x].$$

Lemma 3.4. If $0 \in A \cap B$, A and B are closed subsets of

E_1, and x is a positive real number, then $C(x) \geq (1-\beta)A(x) + \beta x$.

Proof: Define $A_{1/k}$ and $B_{1/k}$ as in Definition 3.1. Then $A_{1/k} + B_{1/k} \subseteq C_{2/k}$. Therefore from Lemma 3.1 we have

$$\begin{aligned} C_{2/k}(x) &\geq (A_{1/k}+B_{1/k})(x) \\ &\geq (1-\beta_{1/k})A_{1/k}(x) + \beta_{1/k}x \\ &= A_{1/k}(x) + \beta_{1/k}(x-A_{1/k}(x)) \\ &\geq A_{1/k}(x) + \beta(x-A_{1/k}(x)) \\ &= (1-\beta)A_{1/k}(x) + \beta x \end{aligned}$$

where

$$\beta_{1/k} = \operatorname*{glb}_{y>0} \frac{B_{1/k}(y)}{y}$$

and hence $\beta_{1/k} \geq \beta$.

However $A \cap [0,x] = \bigcap_{k \in N} (A_{1/k} \cap [0,x])$ and $(A_{1/k+1} \cap [0,x]) \subseteq A_{1/k} \cap [0,x]$. Since $m(A_{1/1} \cap [0,x]) < \infty$ we have by Theorem 2.7

$$\begin{aligned} m(A \cap [0,x]) &= m(\bigcap_{k \in N} (A_{1/k} \cap [0,x])) \\ &= \lim_{k \to \infty} m(A_{1/k} \cap [0,x]). \end{aligned}$$

Likewise $m(C \cap [0,x]) = \lim_{k \to \infty} m(C_{2/k} \cap [0,x])$. Therefore, as $k \to \infty$ we have $C(x) \geq (1-\beta)A(x) + \beta x$.

Lemma 3.5. If p and q are real and $p > 1$ and $q < 1$ and $B \subseteq E_1$ is measurable, then there exists a closed set B' contained in B such that for all $t > 0$ we have $B'(t) \geq \frac{q}{p} \beta t$.

Proof: Let J_n denote the interval (p^n, p^{n+1}) for all integers n. By Theorem 2.8 we can choose a closed set $B' \subseteq B$ with $0 \in B'$ such that $m(B' \cap J_n) \geq qm(B \cap J_n)$ for all n. If t is such that $p^r \leq t < p^{r+1}$ for some integer r, we have
$B'(t) \geq B'(p^r) \geq qB(p^r) \geq q\beta p^r = q\beta \frac{p^{r+1}}{p} \geq q \frac{\beta t}{p}$. Hence for all $t > 0$ we have $B'(t) \geq \frac{q}{p} \beta t$.

Theorem 3.6. If A and B are measurable subsets of E_1 and $0 \in A \cap B$, then $\gamma \geq \alpha + \beta - \alpha\beta$.

Proof: Choose real $p > 1$ and real $q < 1$. Then by Lemma 3.5 there exists a closed set $B' \subseteq B$ such that $B'(t) \geq \frac{q}{p} \beta t$ for all $t > 0$. Then $\beta' \geq \frac{q}{p} \beta$ where β' is the density of B'. Let $x > 0$ be fixed and choose a closed set $A' \subseteq A$ such that $m(A' \cap [0,x]) \geq qm(A \cap [0,x])$. Then let $C' = A' + B'$. Then C' is closed. Now by Lemma 3.4 we have
$C'(x) \geq (1-\beta')A'(x) + \beta'(x) \geqslant (1-\beta)qA(x) + \frac{q}{p} \beta x$. However $C(x) \geq C'(x)$. Therefore as we let $\frac{q}{p} \to 1$ (and hence $q \to 1$) we obtain
$C(x) \geq (1-\beta)A(x) + \beta x$. Hence

$$\frac{C(x)}{x} \geq (1-\beta)\frac{A(x)}{x} + \beta$$

$$\geq (1-\beta)\alpha + \beta$$

$$= \alpha + \beta - \alpha\beta .$$

Therefore

$$\gamma = \operatorname*{glb}_{x>0} \frac{C(x)}{x} \geq \alpha + \beta - \alpha\beta .$$

3.2. The Schur Inequality

In this section we proceed as in Section 3.1 to prove a series of lemmas that enable us to establish the Schur Inequality for measurable sets.

Lemma 3.7. If A and B are open subsets of E_1 and $0 \in A \cap B$, then $(1-\alpha)C(y) \geq B(y)$ for $y \in E_1 \setminus C$.

Proof: Let $y \in E_1 \setminus C$ be fixed. Since $0 \in C$ we see $y > 0$. Since A is open than $A \cap [0,y]$ can be written as a countable union of disjoint open intervals. Since $m(A \cap [0,y]) < \infty$ then given any $\varepsilon > 0$ there exists a finite number of these disjoint open intervals of $A \cap [0,y]$, say $[0,a_1), \ldots, (a_{2n}, a_{2n+1})$ such that $m(A \cap [0,y]) \leq m([0,a_1) \cup \bigcup_{j=1}^{n} (a_{2j}, a_{2j+1})) + \varepsilon$ where $j < k$ implies $a_j \leq a_k$. Similarly $B \cap [0,y]$ is a countable union of disjoint open intervals and for $\varepsilon > 0$ there exists a finite number of them, say $[0, b_1), \ldots, (b_{2t}, b_{2t+1})$, such that

$$m(B \cap [0,y]) \leq m([0,b_1) \cup \bigcup_{k=1}^{t} (b_{2k}, b_{2k+1})) + \varepsilon .$$

Let $A^{\varepsilon} = [0, a_1) \cup [\bigcup_{j=1}^{n} (a_{2j}, a_{2j+1})] \cup (A \cap (y, \infty))$ and $B^{\varepsilon} = [0, b_1) \cup [\bigcup_{k=1}^{t} (b_{2k}, b_{2k+1})] \cup (B \cap (y, \infty))$. Let $C^{\varepsilon} = A^{\varepsilon} + B^{\varepsilon}$. Then $C^{\varepsilon} \subseteq C = A + B$. We note that $A(x) \geq A^{\varepsilon}(x) \geq A(x) - \varepsilon$ and $B(x) \geq B^{\varepsilon}(x) \geq B(x) - \varepsilon$ for all $x > 0$.

Let α_{ε} be the density of A^{ε}. Then $\lim_{\varepsilon \to 0} \alpha_{\varepsilon} = \alpha$. This is so since

$$\alpha = \text{glb}\{\frac{A(x)}{x} : x \geq a_1 > 0\} \geq \alpha_{\varepsilon} = \text{glb}\{\frac{A^{\varepsilon}(x)}{x} : x \geq a_1 > 0\}$$

$$\geq \text{glb}\{\frac{A(x) - \varepsilon}{x} : x \geq a_1 > 0\} \geq \alpha - \frac{\varepsilon}{a_1} .$$

Thus as $\varepsilon \to 0$ we have $\alpha_{\varepsilon} \to \alpha$.

Since A^{ε} and B^{ε} are open sets we see C^{ε} is open. Furthermore $C^{\varepsilon} \cap [0, y]$ is the union of a finite number of open intervals since an open interval of A^{ε} added to an open interval of B^{ε} is an open interval of C^{ε}. Let $C^{\varepsilon} \cap [0, y]$ where $0 = c_0$, equal $[c_0, c_1) \cup (c_2, c_3) \cup \ldots \cup (c_{2s}, c_{2s+1})$ where $i < j$ implies $c_i \leq c_j$.

If $a \in A^{\varepsilon}$ and $0 \leq a < c_{2i+1} - c_{2i}$, $0 \leq i \leq s$, then $c_{2i+1} - a \notin B^{\varepsilon}$ and $c_{2i} < c_{2i+1} - a \leq c_{2i+1}$. This is so, for if

$c_{2i+1} - a \in B^{\varepsilon}$ then $c_{2i+1} = a + (c_{2i+1} - a) \in A^{\varepsilon} + B^{\varepsilon} = C^{\varepsilon}$ contrary to $c_{2i+1} \notin C^{\varepsilon}$. Also $a \geq 0$ implies $c_{2i+1} - a \leq c_{2i+1}$ and $a < c_{2i+1} - c_{2i}$ implies $c_{2i} < c_{2i+1} - a$.

Therefore

$$\begin{aligned}(c_{2i+1} - c_{2i}) - m(B^{\varepsilon} \cap (c_{2i}, c_{2i+1})) &\geq m(\{c_{2i+1} - a : a \in A^{\varepsilon} \cap [0, c_{2i+1} - c_{2i})\}) \\ &= m(A^{\varepsilon} \cap [0, c_{2i+1} - c_{2i})) \\ &\geq \alpha_{\varepsilon}(c_{2i+1} - c_{2i}).\end{aligned}$$

Hence $(1 - \alpha_{\varepsilon})(c_{2i+1} - c_{2i}) \geq m(B^{\varepsilon} \cap (c_{2i}, c_{2i+1}))$. Therefore

$$(1 - \alpha_{\varepsilon}) \sum_{i=0}^{s} (c_{2i+1} - c_{2i}) \geq \sum_{i=0}^{s} m(B^{\varepsilon} \cap (c_{2i}, c_{2i+1})).$$

However

$$\sum_{i=0}^{s} (c_{2i+1} - c_{2i}) = m\left(\bigcup_{i=0}^{s} (c_{2i}, c_{2i+1})\right) = m(C^{\varepsilon} \cap [0, y]),$$

and since $B^{\varepsilon} \subseteq C^{\varepsilon}$ we also have $\sum_{i=0}^{s} m(B^{\varepsilon} \cap (c_{2i}, c_{2i+1})) = B^{\varepsilon}(y)$. Hence $(1 - \alpha_{\varepsilon}) C^{\varepsilon}(y) \geq B^{\varepsilon}(y)$. Therefore

$$\begin{aligned}(1 - \alpha_{\varepsilon}) C(y) &\geq (1 - \alpha_{\varepsilon}) C^{\varepsilon}(y) \\ &\geq B^{\varepsilon}(y) \\ &\geq B(y) - \varepsilon.\end{aligned}$$

Letting ε tend to zero we obtain $(1 - \alpha) C(y) \geq B(y)$.

Lemma 3.8. If A and B are closed subsets of E_1 and $0 \in A \cap B$, then $(1-\alpha)C(y) \geq B(y)$ for $y \in E_1 \setminus C$.

Proof: By Lemma 2.5 we know that C is closed. Therefore $E_1 \setminus C$ is open. Since $E_1 \setminus C$ is open and $y \in E_1 \setminus C$ there must exist $\delta > 0$ such that $(y-\delta, y+\delta) \subseteq E_1 \setminus C$. Choose $k \in N$ such that $0 < \frac{2}{k} < \delta$. Define $A_{1/k}$ and $B_{1/k}$ as before. Having chosen $\frac{2}{k} < \delta$ we see $y \notin A_{1/k} + B_{1/k}$. Since $0 \in A \cap B$ we have $0 \in A_{1/k} \cap B_{1/k}$. Therefore from Lemma 3.7 we have

$$\begin{aligned}(1-\alpha_{1/k})(A_{1/k}+B_{1/k})(y) &\geq B_{1/k}(y) \\ &\geq B(y)\end{aligned}$$

where $\alpha_{1/k}$ is the density of $A_{1/k}$. Since $\alpha_{1/k} \geq \alpha$ and $A_{1/k} + B_{1/k} \subseteq C_{2/k}$, then $(1-\alpha)C_{2/k}(y) \geq B(y)$. As before, letting $k \to \infty$ we have $(1-\alpha)C(y) \geq B(y)$.

Lemma 3.9. If A and B are measurable subsets of E_1 and $0 \in A \cap B$, then $(1-\alpha)C(y) \geq B(y)$ for $y \in E_1 \setminus C$.

Proof: Choose $p > 1$ and $q < 1$. Then as before there exists a closed set $A' \subseteq A$ with $0 \in A'$ such that $\alpha' \geq \frac{q}{p}\alpha$. Choose a closed set $B' \subseteq B$ such that $0 \in B'$ and $m(B' \cap [0,y]) \geq qm(B \cap [0,y])$. Then let $C' = A' + B'$. Now by Lemma 3.8 we have

$$(1-\alpha')C'(y) \geq B'(y).$$

Hence

$$(1-\frac{q}{p}\alpha)C(y) \geq (1-\alpha')C'(y) \geq B'(y) \geq qB(y).$$

Thus letting $\frac{q}{p} \to 1$ and hence letting $q \to 1$ we obtain $(1-\alpha)C(y) \geq B(y)$.

<u>Lemma 3.10.</u> If $0 \in A \cap B$ and C does not contain all the positive reals, then $\gamma = \text{glb}\{\frac{C(x)}{x}: x \in E_1 \setminus C\}$.

Proof: Clearly $\gamma \leq \text{glb}\{\frac{C(x)}{x}: x \in E_1 \setminus C\}$. On the other hand,

$$\gamma = \text{glb}\{\frac{C(z)}{z} : z > 0\}$$

$$= \text{glb}\{\frac{C(z)}{z} : z > 0 \text{ ther exists } x \text{ where } 0 < x \leq z \text{ and } x \notin C\}.$$

Let $z \in C$ and $x \notin C$ be such that $0 < x < z$. Let $s = \sup\{y: y \in E_1 \setminus C \text{ and } y < z\}$. Then $C(z) = C(s) + (z-s)$. This is so since $C(z) = m_*(C \cap [0,z])$ and

$$m_*(C \cap [0,s)) + m_*(C \cap [s,z]) \leq m_*(C \cap [0,z])$$
$$\leq m_*(C \cap [0,s)) + m^*(C \cap [s,z]).$$

However $m_*(C \cap [s,z]) = m^*(C \cap [s,z]) = z - s$. Therefore $C(z) = C(s) + (z-s)$.

Now $\frac{C(z)}{z} = \frac{C(s)+(z-s)}{s+(z-s)} \geq \frac{C(s)}{s}$. However $\frac{C(s)}{s} \geq \text{glb}\{\frac{C(y)}{y} : y \in E_1 \smallsetminus C\}$ since either $s \notin C$ or s is a limit point of elements in $E_1 \smallsetminus C$. If $s \notin C$ the inequality is immediate. If s is a limit point of $E_1 \smallsetminus C$, then there exists a sequence $\langle y_k \rangle$ of elements in $E_1 \smallsetminus C$ such that $\frac{C(y_k)}{y_k} \to \frac{C(s)}{s}$. Hence $\frac{C(z)}{z} \geq \text{glb}\{\frac{C(y)}{y} : y \in E_1 \smallsetminus C\}$. Therefore $\gamma \geq \text{glb}\{\frac{C(y)}{y} : y \in E_1 \smallsetminus C\}$. Finally $\gamma = \text{glb}\{\frac{C(y)}{y} : y \in E_1 \smallsetminus C\}$.

Theorem 3.11. If A and B are measurable subsets of E_1, $0 \in A \cap B$, and $\alpha + \beta < 1$, then $\gamma \geq \frac{\beta}{1-\alpha}$.

Proof: If $C(x) = x$ for all $x > 0$, then $\gamma = 1$ and we are done since $1 > \frac{\beta}{1-\alpha}$. Assume $C(x) < x$ for some x. Then there exists some y in $E_1 \smallsetminus C$. From Lemma 3.9 we have $(1-\alpha)C(y) \geq B(y)$ for all $y \in E_1 \smallsetminus C$. Therefore $(1-\alpha)C(y) \geq B(y) \geq \beta y$ for $y \in E_1 \smallsetminus C$. Hence $\frac{C(y)}{y} \geq \frac{\beta}{1-\alpha}$. Hence $\gamma = \underset{y \in E_1 \smallsetminus C}{\text{glb}} \frac{C(y)}{y} \geq \frac{\beta}{1-\alpha}$.

3.3. Mann's Second Theorem

In this section we will obtain a continuous analogue of Mann's Second Theorem for measurable sets.

Lemma 3.12. If A and B are open subsets of E_1, $0 \in A \cap B$, and $x \in E_1 \smallsetminus C$, then $C(x) \geq \alpha x + B(x)$.

Proof: Suppose $x \in E_1 \setminus C$ is fixed. Since A and B are open, then given any $\varepsilon > 0$ we can find a finite number of open intervals of $A \cap [0,x]$ and $B \cap [0,x]$, say

$[0,a_1),(a_2,a_3),\ldots,(a_{2n},a_{2n+1})$ and $[0,b_1),(b_2,b_3),\ldots,(b_{2r},b_{2r+1})$

respectively, such that

$$(1) \qquad m([0,a_1) \cup (\bigcup_{j=1}^{n} (a_{2j},a_{2j+1}))) \geq A(x) - \varepsilon$$

and

$$(2) \qquad m([0,b_1) \cup (\bigcup_{k=0}^{r} (b_{2k},b_{2k+1}))) \geq B(x) - \varepsilon.$$

Let $A^{\varepsilon} = [0,a_1) \cup (\bigcup_{j=1}^{n} (a_{2j},a_{2j+1})) \cup (A \cap (x,\infty))$ and $B^{\varepsilon} = [0,b_1) \cup (\bigcup_{k=1}^{r} (b_{2k},b_{2k+1})) \cup (B \cap (x,\infty))$. Let $C^{\varepsilon} = A^{\varepsilon} + B^{\varepsilon}$. Then $C^{\varepsilon} \subseteq C$, C^{ε} is open, and $C^{\varepsilon} \cap [0,x]$ is the union of a finite number of disjoint open intervals. Hence $C(x) \geq C^{\varepsilon}(x)$, $B^{\varepsilon}(x) \geq B(x) - \varepsilon$, and if α_{ε} is the density of A^{ε}, then $\lim_{\varepsilon \to 0} \alpha_{\varepsilon} = \alpha$. These follow as in Section 3.2.

If K is a set and I is an interval such that $K \subseteq E_1$ and $I \subseteq E_1 \setminus K$, then we call I an interval gap of K. We can write $[0,x] \setminus B^{\varepsilon}$ and $[0,x] \setminus C^{\varepsilon}$ as the union of a finite number of disjoint closed intervals (a point is considered a closed interval here).

Let $[c_1,c_2]$ be the first interval gap of C^{ε}. It must be contained in some interval gap $[b_{2m-1},b_{2m}]$ of B^{ε} since $B^{\varepsilon} \subseteq C^{\varepsilon}$. Let c be the largest element of $E_1 \setminus C^{\varepsilon}$ such that

$b_{2m-1} < c \leq b_{2m}$. If $c < x$, then let $S_1 = \{t \in E_1 : 0 \leq t \leq b_{2m}\}$. If $c \geq x$, then let $S_1 = \{t \in E_1 : 0 \leq t \leq x\}$. (Note: If $[c_1, c_2]$ is contained in an interval gap of B of the form $[b_{2m-1}, \infty)$, then $c \geq x$ and $S_1 = \{t \in E_1 : 0 \leq t \leq x\}$.)

Case I. $S_1 = \{t \in E_1 : 0 \leq t \leq x\}$

Let (b_{2k}, b_{2k+1}) be one of the finite number of open intervals contained in $B^\varepsilon \cap S_1$. Then $(x-b_{2k+1}, x-b_{2k})$ is an interval gap of $A^\varepsilon \cap S_1$. This is so since $x \geq x - b_{2k}$ and if there exists $a \in A^\varepsilon$ such that $x - b_{2k+1} < a < x - b_{2k}$, then there would exist $b \in (b_{2k}, b_{2k+1})$ such that $x = a + b$ Hence x would be in C^ε and hence in C, which would be a contradiction. If (b_{2j}, b_{2j+1}) and (b_{2k}, b_{2k+1}) are two of the finite open disjoint intervals whose union is $B^\varepsilon \cap S_1$, then $(x-b_{2j+1}, x-b_{2j})$ and $(x-b_{2k+1}, x-b_{2k})$ are disjoint intervals of S_1.

Let $G_{1,\delta} = \{d-b_{2m-1} + \delta : d \in S_1 \setminus C^\varepsilon\}$ where $0 < \delta < b_{2m-1} - b_{2m-2}$. We note that $b_{2m-1} = \sup\{b : b \in B^\varepsilon \cap S_1\}$. We see that $G_{1,\delta}$ is a union of interval gaps of A^ε in S_1. This is so since if $d - b_{2m-1} + \delta$ were in A^ε, then $d = (d-b_{2m-1} + \delta) + (b_{2m-1} - \delta) \in A^\varepsilon + B^\varepsilon = C^\varepsilon$. This contradicts $d \in S_1 \setminus C^\varepsilon$. Also $G_{1,\delta}$ is disjoint from $(x-b_{2k+1}, x-b_{2k})$ where $(b_{2k}, b_{2k+1}) \subsetneq B^\varepsilon \cap S_1$ and $b_{2k+1} \leq b_{2m-2}$, and $m(G_{1,\delta} \cap (x-b_{2m-1}, x-b_{2m-2})) \leq \delta$. This is so since the largest element of $G_{1,\delta}$ is $x - b_{2m-1} + \delta$ and

$x - b_{2m-1} + \delta < x - b_{2m-1} + (b_{2m-1} - b_{2m-2}) = x - b_{2m-2}$ and all the other intervals $(x-b_{2j+1}, x-b_{2j})$ are such that $x - b_{2m-2} \leq x - b_{2j+1}$. We also have $m(G_{1,\delta}) = m(S_1 \smallsetminus C^{\varepsilon})$.

Thus,

$$m(S_1 \smallsetminus A^{\varepsilon}) \geq m(\bigcup_{k=0}^{m-1} (x-b_{2k+1}, x-b_{2k})) + m(G_{1,\delta})$$

$$- m(G_{1,\delta} \cap (x-b_{2m-1}, x-b_{2m-2}))$$

$$\geq m(\bigcup_{k=0}^{m-1} (x-b_{2k+1}, x-b_{2k})) + m(G_{1,\delta}) - \delta$$

$$= m(B^{\varepsilon} \cap S_1) + m(S_1 \smallsetminus C^{\varepsilon}) - \delta.$$

Letting $\delta \to 0$ we obtain $m(S_1 \smallsetminus A^{\varepsilon}) \geq m(B^{\varepsilon} \cap S_1) + m(S_1 \smallsetminus C^{\varepsilon})$. Hence $m(S_1) - m(A^{\varepsilon} \cap S_1) \geq m(S_1) - m(C^{\varepsilon} \cap S_1) + m(B \cap S_1)$ and thus $m(C^{\varepsilon} \cap S_1) \geq m(A^{\varepsilon} \cap S_1) + m(B^{\varepsilon} \cap S_1)$. However $S_1 = [0, x]$ and so we have

$$\begin{aligned} C(x) &\geq C^{\varepsilon}(x) \\ &\geq A^{\varepsilon}(x) + B^{\varepsilon}(x) \\ &\geq \alpha_{\varepsilon} x + B^{\varepsilon}(x) \\ &\geq \alpha_{\varepsilon} x + B(x) - \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$C(x) \geq \alpha x + B(x).$$

Therefore if $S_1 = \{t: 0 \leq t \leq x\}$ we are done.

Case II. $S_1 = \{t: 0 \leq t \leq b_{2m}\}$

Recall that c is the largest element in $E_1 \setminus C^{\varepsilon}$ such that $b_{2m-1} < c \leq b_{2m}$. If we go through the same argument as we did in Case I considering the intervals $(c-b_{2k+1}, c-b_{2k})$ instead of the intervals $(x-b_{2k+1}, x-b_{2k})$ where $(b_{2k}, b_{2k+1}) \subsetneq B^{\varepsilon} \cap S_1$ we will arrive at the inequality

$$(1) \qquad C^{\varepsilon}(S_1) \geq a_{\varepsilon} m(S_1) + B^{\varepsilon}(S_1).$$

We now form a new set S_2. Let $[e_1, e_2]$ be the next interval gap of C^{ε} after c; that is, $e_1 = \inf\{z \in E_1 \setminus C^{\varepsilon} : z > c\}$. As before $[e_1, e_2] \subsetneq [b_{2p-1}, b_{2p}]$ where $[b_{2p-1}, b_{2p}]$ is an interval gap of B^{ε}. Let e be the largest element not in C^{ε} such that $b_{2p-1} < e \leq b_{2p}$. If $x \leq e$ we let $S_2 = \{t \in E_1 : b_{2m} \leq t \leq x\}$. If $x > e$, we let $S_2 = \{t \in E_1 : b_{2m} \leq t \leq b_{2p}\}$. We now let $S_2' = \{t - b_{2m} : t \in S_2\}$.

We now consider the subcase where $e \geq x$. Then $S_2 = \{t \in E_1 : b_{2m} \leq t \leq x\}$. The intervals $(x-b_{2j+1}, x-b_{2j})$ where $(b_{2j}, b_{2j+1}) \subsetneq B^{\varepsilon} \cap S_2$ are disjoint and are interval gaps of $A^{\varepsilon} \cap S_2'$. That they are disjoint and are interval gaps of A^{ε} follows as before. We see that they are in S_2' by observing that $b_{2j} \geq b_{2m}$.

Let $G_{2,\delta} = \{d - b_{2p-1} + \delta : d \in S_2 \setminus C^{\varepsilon}\}$ where

$0 < \delta < (b_{2p-1} - b_{2p-2})$. Recall that $(b_{2p-2}, b_{2p-1}) \subseteq B^{\varepsilon}$. By a similar argument to the one used above we have $G_{2,\delta}$ disjoint from $(x-b_{2j+1}, x-b_{2j})$ for $(b_{2j}, b_{2j+1}) \subseteq B^{\varepsilon} \cap S_2$ such that $b_{2j+1} \leq b_{2p-2}$, and $m(G_{2,\delta} \cap (x-b_{2p-1}, x-b_{2p-2})) \leq \delta$. Also as before $G_{2,\delta}$ is the union of interval gaps of A^{ε} in S_2' and $m(G_{2,\delta}) = m(S_2 \setminus C^{\varepsilon})$.

Thus we have

$$m(S_2' \setminus A^{\varepsilon}) \geq m\left(\bigcup_{j=m}^{p-1} (x-b_{2j+1}, x-b_{2j})\right) + m(G_{2,\delta})$$

$$- m(G_{2,\delta} \cap (x-b_{2p-1}, x-b_{2p-2}))$$

$$\geq m(B^{\varepsilon} \cap S_2) + m(G_{2,\delta}) - \delta$$

$$\geq m(B^{\varepsilon} \cap S_2) + m(S_2 \setminus C^{\varepsilon}) - \delta.$$

Letting $\delta \to 0$ we obtain $m(S_2' \setminus A^{\varepsilon}) \geq B^{\varepsilon}(S_2) + m(S_2 \setminus C^{\varepsilon})$.

Therefore we have

$$m(S_2') - m(A^{\varepsilon} \cap S_2') \geq m(B^{\varepsilon} \cap S_2) + m(S_2) - m(C^{\varepsilon} \cap S_2).$$

However $m(S_2') = m(S_2)$. Hence we have

$$m(C^{\varepsilon} \cap S_2) \geq m(A^{\varepsilon} \cap S_2') + m(B^{\varepsilon} \cap S_2)$$

$$\geq a_{\varepsilon} m(S_2') + m(B^{\varepsilon} \cap S_2)$$

$$= a_{\varepsilon} m(S_2) + B^{\varepsilon}(S_2).$$

Therefore since $S_1 \cup S_2 = [0,x]$ and $m(S_1 \cap S_2) = 0$ we have

$$\begin{aligned} C^\varepsilon(x) &= C^\varepsilon(S_1 \cup S_2) \\ &= C^\varepsilon(S_1) + C^\varepsilon(S_2) \\ &\geq a_\varepsilon m(S_1) + B(S_1) + C^\varepsilon(S_2) \\ &\geq a_\varepsilon m(S_1) + B(S_1) + a_\varepsilon m(S_2) + B^\varepsilon(S_2) \\ &= a_\varepsilon (m(S_1) + m(S_2)) + B^\varepsilon(S_1) + B^\varepsilon(S_2) \\ &= a_\varepsilon m(S_1 \cup S_2) + B^\varepsilon(S_1 \cup S_2) \\ &= a_\varepsilon x + B^\varepsilon(x). \end{aligned}$$

Hence

$$\begin{aligned} C(x) &\geq C^\varepsilon(x) \\ &\geq a_\varepsilon x + B^\varepsilon(x) \\ &\geq a_\varepsilon x + B(x) - \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$C(x) \geq a x + B(x).$$

Thus if $S_2 = \{t \in E_1 : b_{2m} \leq t \leq x\}$ we are done.

If $e < x$ then we are in the other subcase and $S_2 = \{t \in E_1 : b_{2m} \leq t \leq b_{2p}\}$. Recall that e is the largest element in $E_1 \sim C^\varepsilon$ such that $b_{2p-1} < e \leq b_{2p}$. Then going through exactly the same arguments as we did above, considering the intervals

$(e-b_{2j+1}, e-b_{2j})$ in place of the intervals $(x-b_{2j+1}, x-b_{2j})$ where $(b_{2j}, b_{2j+1}) \subsetneq B^{\varepsilon} \cap S_2$, we arrive at the following inequality,

$$(2) \qquad m(C^{\varepsilon} \cap S_2) \geq a_{\varepsilon} m(S_2) + m(B^{\varepsilon} \cap S_2).$$

We use the same process as before to construct a new set S_3 and arrive at the corresponding inequalities. If we continue in this manner, we obtain after a finite number r of steps a finite set of intervals $\{S_j : 1 \leq j \leq r\}$ where $[0, x] = \bigcup_{j=1}^{r} S_j$ and $m(S_j \cap S_k) = 0$ for $j \neq k$. That only a finite number of steps are required follows from the fact that there are a finite number of interval gaps of B^{ε} contained in $[0, x]$.

Since for each j we have

$$\begin{aligned} m(C^{\varepsilon} \cap S_j) &= C^{\varepsilon}(S_j) \\ &\geq a_{\varepsilon} m(S_j) + m(B^{\varepsilon} \cap S_j) \end{aligned}$$

it then follows that

$$\begin{aligned} C^{\varepsilon}(x) &= m(C^{\varepsilon} \cap [0, x]) \\ &= m(C^{\varepsilon} \cap \bigcup_{j=1}^{r} S_j) \\ &= m(\bigcup_{j=1}^{r} (C^{\varepsilon} \cap S_j)) \\ &= \sum_{j=1}^{r} m(C^{\varepsilon} \cap S_j) \geq \end{aligned}$$

$$\geq \sum_{j=1}^{r} (\alpha_\varepsilon m(S_j) + m(B^\varepsilon \cap S_j))$$

$$= \alpha_\varepsilon m(\bigcup_{j=1}^{r} S_j) + m(\bigcup_{j=1}^{r} (B^\varepsilon \cap S_j))$$

$$= \alpha_\varepsilon x + B^\varepsilon(x).$$

However

$$C(x) \geq C^\varepsilon(x)$$

$$\geq \alpha_\varepsilon x + B^\varepsilon(x)$$

$$\geq \alpha_\varepsilon x + B(x) - \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain

$$C(x) \geq \alpha x + B(x).$$

<u>Lemma 3.13.</u> If A and B are closed subsets of E_1, $0 \in A \cap B$, and if there exists a positive number x not in C, then $C(x) \geq \alpha x + B(x)$.

Proof: By Lemma 2.5 we know C is a closed set. Therefore $E_1 \sim C$ is open. Since $x \in E_1 \sim C$ and $E_1 \sim C$ is open, there must exist $\delta > 0$ such that $(x-\delta, x+\delta) \subsetneq E_1 \sim C$. Choose $k \in N$ such that $0 < \frac{2}{k} < \delta$. Define $A_{1/k}$ and $B_{1/k}$ as before. Having chosen $\frac{2}{k} < \delta$ we see $x \notin C_{2/k}$. Since $A_{1/k} + B_{1/k} \subsetneq C_{2/k}$ we see that $x \notin A_{1/k} + B_{1/k}$. Since $A \subsetneq A_{1/k}$ we have $A_{1/k}(t) \geq A(t) \geq \alpha t$ for $t \geq 0$. Hence $\alpha_{1/k} \geq \alpha$. Therefore from

Lemma 3.12 we have

$$C_{2/k}(x) \geq (A_{1/k}+B_{1/k})(x)$$

$$\geq a_{1/k}x + B_{1/k}(x)$$

$$\geq ax + B_{1/k}(x).$$

But as in Lemma 3.4 we have

$$m(A \cap [0,x]) = \lim_{k\to\infty} m(A_{1/k} \cap [0,x])$$

and

$$m(C \cap [0,x]) = \lim_{k\to\infty} m(C_{2/k} \cap [0,x]).$$

Therefore as we let $k \to \infty$ we obtain

$$C(x) \geq ax + B(x).$$

Theorem 3.14. If A and B are measurable subsets of E_1, $0 \in A \cap B$, and if there exists $x \in E_1 \setminus C$, then $C(x) \geq ax + B(x)$.

Proof: Choose two positive constants $p > 1$ and $q < 1$. Then as before there exists a closed set $A' \subseteq A$ such that $a' \geq \frac{q}{p}a$ and $0 \in A'$. Choose a closed set $B' \subseteq B$ such that $m(B' \cap [0,x]) \geq qm(B \cap [0,x])$ and $0 \in B'$. Then $C' \subseteq C$ and hence $x \notin C'$. Now we apply Lemma 3.13 to obtain

$C'(x) \geq \alpha' x + B'(x) \geq \frac{q}{p}\alpha x + qB(x)$. However $C(x) \geq C'(x)$. Hence as we let $\frac{p}{q} \to 1$ we have $C(x) \geq \alpha x + B(x)$.

3.4. The $\alpha\beta$ Theorem

In this section we will obtain a continuous analogue of Mann's $\alpha\beta$ Theorem. The $\alpha\beta$ Theorem follows from Theorem 3.14. Since in the discrete case the proof of Mann's Second Theorem is simpler than the proof of the $\alpha\beta$ Theorem, our method of proving the $\alpha\beta$ Theorem is inherently easier than the methods needed to prove the $\alpha\beta$ Theorem in the discrete case. It also does not assume anything as powerful as Dyson's Inequality which Macbeath used.

Theorem 3.15. If A and B are measurable subsets of E_1 and $0 \in A \cap B$, then $\gamma \geq \min\{1, \alpha+\beta\}$.

Proof: If $C = E_1$, then $\gamma = 1$. Suppose $C \neq E_1$. Let $x \in E_1 \smallsetminus C$. From Theorem 3.14 we have $C(x) \geq \alpha x + B(x)$. We have $B(x) \geq \beta x$. Hence $C(x) \geq \alpha x + \beta x = (\alpha+\beta)x$. Therefore, $\frac{C(x)}{x} \geq \alpha + \beta$. Thus $\gamma = \text{glb}\{\frac{C(x)}{x} : x \in E_1 \smallsetminus C\} \geq \alpha + \beta$.

3.5. Other Continuous Analogues

In this section we get continuous analogues of other interesting results known in the discrete case.

<u>Corollary 3.16.</u> If $\alpha + \beta \geq 1$, then $\gamma = 1$.

Proof: This corollary is an immediate consequence of Theorem 3.15.

For subsets of non-negative integers it has been shown that $\alpha + \beta \geq 1$ implies that C is the set of all nonnegative integers. We have shown $\alpha + \beta \geq 1$ implies $\gamma = 1$, but by example we will show $\alpha + \beta = 1$ does not imply $C = E_1$.

<u>Example 3.1.</u> Let $A = [0, \frac{1}{2}) \cup (1, \infty)$ and $B = [0, \frac{1}{2}) \cup (1, \infty)$. Then $\alpha = \beta = \frac{1}{2}$ and hence $\alpha + \beta = 1$. However $C = [0, 1) \cup (1, \infty)$ and hence $1 \notin C$.

<u>Theorem 3.17.</u> If A and B are measurable subsets of E_1, $0 \in A \cap B$, and $\alpha + \beta > 1$, then $C = E_1$.

Proof: Assume there exists $x \in E_1 \setminus C$. Let $H = A \cap [0, x]$ and $G = \{x - b : b \in B \cap [0, x]\}$. Then $H \cap G = \emptyset$. To see this suppose $x - b \in H \cap G$. Then $x = (x-b) + b \in C$. Hence $m(H) + m(G) = m(H \cup G) \leq x$. However $m(H) = A(x)$ and $m(G) = B(x)$. Therefore $A(x) + B(x) \leq x$ and $\alpha + \beta \leq \frac{A(x)+B(x)}{x} \leq 1$. This contradicts $\alpha + \beta > 1$. Hence $C = E_1$.

<u>Definition 3.2.</u> Let $n \geq 1$ and $A_1, \ldots, A_n$ be subsets of E_1

such that $0 \in \bigcap_{j=1}^{n} A_j$. Then $A_1 + \ldots + A_n = \{a_1+\ldots+a_n : a_j \in A_j, j = 1,\ldots,n\}$.

Lemma 3.18. Let $0 \in \bigcap_{j=1}^{n} A_j$ and let $d(A_1+\ldots+A_n)$ be the density of $A_1+\ldots+A_n$ and let α_i be the density of A_i, $i = 1,\ldots,n$. If $A_1,\ldots,A_n$ are open subsets of E_1 or if $A_1,\ldots,A_n$ are closed subsets of E_1, then
$1 - d(A_1+\ldots+A_n) \leq (1-\alpha_1)\ldots(1-\alpha_n)$.

Proof: If $n = 1$, then $1 - d(A_1) = 1 - \alpha_1 \leq 1 - \alpha_1$. Since the sum of any two open sets is open and the sum of any two closed sets is closed we have that any finite sum of open sets is open and any finite sum of closed sets is closed.

Assume that for some integer $k \geq 1$ we have

$$1 - d(A_1+\ldots+A_k) \leq (1-\alpha_1)\ldots(1-\alpha_k).$$

Then

$$1 - d(A_1+\ldots+A_k+A_{k+1}) = 1 - d([A_1+\ldots+A_k]+A_{k+1}).$$

However by Theorem 3.6

$$d([A_1+\ldots+A_k]+A_{k+1}) \geq d(A_1+\ldots+A_k) + \alpha_{k+1} - d(A_1+\ldots+A_k)\alpha_{k+1},$$

or

$$\begin{aligned}1 - d([A_1+\ldots+A_k]+A_{k+1}) &\leq 1 - d(A_1+\ldots+A_k) - \alpha_{k+1} + d(A_1+\ldots+A_k)\alpha_{k+1} \\ &= (1-d(A_1+\ldots+A_k))(1-\alpha_{k+1}) \\ &\leq (1-\alpha_1)\ldots(1-\alpha_k)(1-\alpha_{k+1}).\end{aligned}$$

Hence the lemma is true by induction.

Definition 3.3. Let k be a positive integer. We call A a basic set of E_1 of order k if $A_1 + \ldots + A_k = E_1$, where each $A_j = A$, $j = 1, \ldots, k$, and k is minimal. We write $A_1 + \ldots + A_k$ as kA.

Theorem 3.19. If $0 \in A$, A is open or A is closed, and $d(A) > 0$, then A is a basic set of E_1.

Proof: If $d(A) > 0$, then there exists a positive integer n such that $(1-d(A))^n < \frac{1}{2}$. Then by Lemma 3.18 we have $(1-d(nA)) \leq (1-\alpha)^n < \frac{1}{2}$. Hence $d(nA) > \frac{1}{2}$. Thus $d(nA) + d(nA) > 1$ and from Theorem 3.17 we have $nA + nA = 2nA = E_1$. Thus the order of A is less than or equal to $2n$.

The results of Lemma 3.18 and Theorem 3.19 follow exactly as the corresponding results for subsets of nonnegative integers. In the integer case there are subsets which have density zero, but are nevertheless basic. For instance, if $A = \{k^2 : k \text{ is an integer}\}$ then $d(A) = 0$, but A is basic of order 4.

We will now show by example that there are sets in E_1 with density zero, but basic of order k for any $k \geq 2$.

Example 3.2. Let $A = [0, \frac{1}{k}] \cup \{1, 2, \ldots, n, \ldots\}$ where k is an integer greater than or equal to 1. Then $d(A) = 0$, clearly $(k+1)A = E_1$, and also $h < k+1$ implies $hA \neq E_1$ since there exists $\delta > 0$ such that $2 - \delta \notin hA$. Any δ where $\delta < (1 - \frac{h-1}{k})$ will suffice.

The final result we will show in this chapter is that the $\alpha\beta$ Theorem is in a sense best possible.

Theorem 3.20. Given $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + \beta \leq 1$, then there exist sets A and B such that $d(A) = \alpha$, $d(B) = \beta$, and $d(A+B) = \alpha + \beta$.

Proof: Let $A = [0, \alpha] \cup [1, \infty)$ and $B = [0, \beta] \cup [1, \infty)$. Then $A + B = [0, \alpha + \beta] \cup [1, \infty)$ and we see that $d(A) = \alpha$, $d(B) = \beta$, and $d(A+B) = \alpha + \beta$.

4. TWO-DIMENSIONAL CONTINUOUS ANALOGUES OF THEOREMS OF MANN

In this chapter we prove two major theorems of the thesis. We first obtain a continuous analogue of the two-dimensional extension of Mann's Second Theorem using a method of Kvarda [9]. We then obtain an $\alpha\beta$ theorem for two dimensions. In this chapter A and B will always be open sets with $0 = (0,0) \in A \cap B$ and E_2 will be the universal set.

4.1. Preliminary Notation and Lemmas

Definition 4.1. Let $S = F' \setminus F''$ where $F', F'' \in \mathfrak{F}$ and $S \neq \emptyset$. Let "min"$S = \{s : s \in Cl(S),\ 0 \lessdot t \lessdot s$ implies $t \notin Cl(S)\}$.

We let "min"$S = \{\delta_j : j \in \Omega\}$ where Ω is the index set.

Definition 4.2. For each $j \in \Omega$ let $S_j = \{s : s \in Cl(S), \delta_j \lessdot s\}$.

Lemma 4.1. $\bigcup_{j \in \Omega} S_j = Cl(S)$.

Proof: We see that $\bigcup_{j\in\Omega} S_j \subseteq Cl(S)$ since by definition each $S_j \subseteq Cl(S)$. On the other hand, for each $s \in Cl(S)$ we have either $s \in$ "min"S or $s \notin$ "min"S. If $s \in$ "min"S, then $s = \delta_j$ for some $j \in \Omega$ and hence $s \in S_j$. Therefore $s \in \bigcup S_j$. If $s = (s_1, s_2) \notin$ "min"S, let $\bar{s}_1 = \min\{x : (x, s_2) \in Cl(S)\}$. Then since

Cl(S) is closed, $(\bar{s}_1, s_2) = s^{(1)} \in Cl(S)$ and $s^{(1)} \prec s$. Since F" is fundamental there does not exist $t = (t_1, t_2) \in Cl(S)$ such that $t_2 \leq s_2$ and $t_1 < \bar{s}_1$. If $s^{(1)} \in$ "min" S, then $s^{(1)} = \delta_j$ for some $j \in \Omega$, and $\delta_j = s^{(1)} \prec s$. Hence $s \in S_j \subseteq \cup S_j$.

If $s^{(1)} \notin$ "min" S let $\bar{s}_2 = \min\{x: (\bar{s}_1, x) \in Cl(S)\}$. Then $s^{(2)} = (\bar{s}_1, \bar{s}_2) \in Cl(S)$ and $s^{(2)} \prec s^{(1)} \prec s$. Furthermore $s^{(2)} \in$ "min" S since if $t = (t_1, t_2)$ is in Cl(S) and $t \prec s^{(2)}$ then $t_1 \leq \bar{s}_1$ and $t_2 \leq \bar{s}_2$ with strict inequality holding in at least one case, and we would contradict the way $s^{(1)}$ and $s^{(2)}$ were constructed. Thus $s^{(2)} = \delta_j$ for some $j \in \Omega$ and thus $\delta_j = s^{(2)} \prec s^{(1)} \prec s$. Hence $s \in S_j \subseteq \cup S_j$. Therefore $Cl(S) \subseteq \bigcup_{j \in \Omega} S_j$.

<u>Definition 4.3.</u> $S_j' = \{s - \delta_j : s \in S_j\}$.

<u>4.2. Fundamental Theorems for the Set S'</u>

<u>Lemma 4.2.</u> $S_j' \in \mathcal{F}$.

Proof: First we observe S_j' is bounded since $S_j' \subseteq Cl(F')$ and also that S_j' is nonempty by definition. Next suppose $x \in S_j'$ and $0 \preceq r \prec x$. Since $x \in S_j'$ we have $x = s - \delta_j$ for some $s \in S_j$. Thus $\delta_j \preceq r + \delta_j \prec x + \delta_j = s$. Hence $r + \delta_j \in S_j$. Therefore $r = (r+\delta_j) - \delta_j \in S_j'$, and we see that $L(x) \subseteq S_j'$. Thus $S_j' \in \mathcal{F}$.

Definition 4.4. $S' = \bigcup_{j \in \Omega} S'_j$.

Lemma 4.3. Let $D \subseteq E_2$ be a bounded set. Let $\mathcal{G}$ be a family of fundamental sets such that $G \in \mathcal{G}$ implies $G \subseteq D$. Then $\bigcup_{G \in \mathcal{G}} G \in \mathcal{F}$.

Proof: We see that $\bigcup_{G \in \mathcal{G}} G$ is bounded since it is a subset of D. Suppose $x \in \bigcup_{G \in \mathcal{G}} G$. Then there exists $H \in \mathcal{G}$ such that $x \in H$. Since H is fundamental we have $L(x) \subseteq H$. Hence $L(x) \subseteq \bigcup_{G \in \mathcal{G}} G$. Therefore $\bigcup_{G \in \mathcal{G}} G \in \mathcal{F}$.

Lemma 4.4. $S' \in \mathcal{F}$.

Proof: The lemma follows immediately from Definition 4.4 and Lemmas 4.2 and 4.3.

Lemma 4.5. $m^2(S') \leq m^2(S)$.

Proof: Let ω_i be the unit vector whose ith component is 1. Let $\delta_j = (\delta_{j_1}, \delta_{j_2})$ where $\delta_j \in$ "min" S. Suppose $1 \leq r \leq 2$. We now define

$$S_j \prod_{i=1}^{r} \lambda_i = \{s - \delta_{j_r} \omega_r : s \in S_j \prod_{i=1}^{r-1} \lambda_i\}$$

and

$$S_j \prod_{i=1}^{0} \lambda_i = S_j.$$

We define

$$S \prod_{i=1}^{r} \lambda_i = \bigcup_{j \in \Omega} (S_j \prod_{i=1}^{r} \lambda_i)$$

and

$$S \prod_{i=1}^{0} \lambda_i = \bigcup_{j \in \Omega} (S_j \prod_{i=1}^{0} \lambda_i)$$

Since $S_j' = S_j \prod_{i=1}^{2} \lambda_i$ we have

$$S' = \bigcup_{j \in \Omega} S_j' = \bigcup_{j \in \Omega} (S_j \prod_{i=1}^{2} \lambda_i) = S \prod_{i=1}^{2} \lambda_i .$$

Let $T^r_{(c_1, c_2)}$ be the ray normal to the axis $x_r = 0$ passing through the point (c_1, c_2). Thus if $r = 1$, then $T^r_{(c_1, c_2)} = \{(x_1, c_2) : x_1 \geq 0\}$ and if $r = 2$, then $T^r_{(c_1, c_2)} = \{(c_1, x_2) : x_2 \geq 0\}$. We now consider $E_2 = X_1 \times X_2$ where $X_1 = E_1$ and $X_2 = E_1$. Let the function $g : X_j \to E_1$ where $j \neq r$ be defined by

$$g(x_j) = m(T^r_{(x_1, x_2)} \cap S \prod_{i=1}^{r} \lambda_i).$$

Let the function $h : X_j \to E_1$ where $j \neq r$ be defined by

$$h(x_j) = m(T^r_{(x_1, x_2)} \cap S \prod_{i=1}^{r-1} \lambda_i).$$

Then we have

$$g(x_j) = m(T^r_{(x_1,x_2)} \cap S \prod_{i=1}^{r} \lambda_i)$$

$$= m(T^r_{(x_1,x_2)} \cap (\bigcup_{j \in \Omega} (S_j \prod_{i=1}^{r} \lambda_i)))$$

$$= m(\bigcup_{j \in \Omega} (T^r_{(x_1,x_2)} \cap S_j \prod_{i=1}^{r} \lambda_i))$$

$$= \sup_{j \in \Omega} m(T^r_{(x_1,x_2)} \cap S_j \prod_{i=1}^{r} \lambda_i)$$

$$= \sup_{j \in \Omega} m(T^r_{(x_1,x_2)} \cap S_j \prod_{i=1}^{r-1} \lambda_i)$$

$$\leq m(\bigcup_{j \in \Omega} (T^r_{(x_1,x_2)} \cap S_j \prod_{i=1}^{r-1} \lambda_i))$$

$$= m(T^r_{(x_1,x_2)} \cap (\bigcup_{j \in \Omega} (S_j \prod_{i=1}^{r-1} \lambda_i)))$$

$$= m(T^r_{(x_1,x_2)} \cap S \prod_{i=1}^{r-1} \lambda_i)$$

$$= h(x_j).$$

Therefore $g(x_j) \leq h(x_j)$ for $j = 1$ and $j = 2$.

By applying Lemma 2.11 we have

$$\int_{X_j} g(x_j) dm = m^2(S \prod_{i=1}^{r} \lambda_i)$$

and

$$\int_{X_j} h(x_j)dm = m^2(S \prod_{i=1}^{r-1} \lambda_i)$$

for $j = 1$ and $j = 2$.

Since $g(x_j) \leq h(x_j)$, $j = 1, 2$, it follows that

$$m^2(S \prod_{i=1}^{r} \lambda_i) \leq m^2(S \prod_{i=1}^{r-1} \lambda_i).$$

Thus we finally obtain

$$m^2(S') = m^2(S \prod_{i=1}^{2} \lambda_i) \leq m^2(S \prod_{i=1}^{1} \lambda_i) \leq m^2(S \prod_{i=1}^{0} \lambda_i)$$

However

$$S \prod_{i=1}^{0} \lambda_i = \bigcup_{j \epsilon \Omega} (S_j \prod_{i=1}^{0} \lambda_i) = \bigcup_{j \epsilon \Omega} S_j = Cl(S)$$

and by Theorem 2.13 we have $m^2(S) = m^2(Cl(S))$. Hence $m^2(S') \leq m^2(S)$.

<u>Definition 4.5.</u> If T is a set, then $L(T) = \{L(t): t \epsilon T\}$.

<u>Lemma 4.6.</u> If $S = F' \smallsetminus F''$ where $F', F'' \epsilon \mathfrak{F}$, $P_1 = \{(x_1, x_2): x_1 = a_1\}$ is such that $P_1 \cap S \neq \emptyset$, $S_1 = \{(x_1, x_2): x_1 \leq a_1\} \cap S$ and $S_2 = \{(x_1, x_2): x_1 \geq a_1\} \cap S$, then $S = S_1 \cup S_2$ and both S_1 and S_2 are differences of two fundamental sets.

Proof: The equation $S = S_1 \cup S_2$ is immediate so we proceed to establish the second conclusion. We have

$$S_1 = \{(x_1, x_2): x_1 \leq a_1\} \cap (F' \smallsetminus F'')$$
$$= (\{(x_1, x_2): x_1 \leq a_1\} \cap F') \smallsetminus F''$$

and

$$S_2 = \{(x_1, x_2): x_1 \geq a_1\} \cap (F' \smallsetminus F'')$$
$$= F' \smallsetminus ((F' \cap \{(x_1, x_2): x_1 < a_1\}) \cup F'').$$

Therefore it suffices to show $F' \cap \{(x_1, x_2): x_1 \leq a_1\}$ is in $\mathcal{F}$ and $(F' \cap \{(x_1, x_2): x_1 < a_1\}) \cup F'' \in \mathcal{F}$. If $x \in F' \cap \{(x_1, x_2): x_1 \leq a_1\}$ and $y = (y_1, y_2) \prec x = (x_1, x_2)$, then $y_1 \leq x_1 \leq a_1$. Since F' is fundamental we see $y \in F'$. Thus $y \in F' \cap \{(x_1, x_2): x_1 \leq a_1\}$. Hence $F' \cap \{(x_1, x_2): x_1 \leq a_1\}$ is fundamental. If $F' \cap \{(x_1, x_2): x_1 < a_1\}$ is not empty, then the same argument shows that $F' \cap \{(x_1, x_2): x_1 < a_1\} \in \mathcal{F}$. Since the union of two fundamental sets is fundamental we have in both cases that $(F' \cap \{(x_1, x_2): x_1 < a_1\}) \cup F''$ is a fundamental set. Therefore the lemma is proved.

4.3. The Partition of a Fundamental Set

In this section we show how to partition a fundamental set F if $B \cap F$ and $C \cap F$ are finite unions of open rectangles such that these rectangles have a maximal vertex with respect to the partial ordering "$\preccurlyeq$". So we suppose F is a fundamental set satisfying the above conditions.

Since $B \cap F$ and $C \cap F$ are finite unions of open rectangles they determine a finite number of vertices. For each vertex (v_1, v_2) determined in this manner construct the two lines $x_1 = v_1$ and $x_2 = v_2$. Since there are only a finite number of vertices, we have constructed only a finite number of lines. Each line $x_1 = v_1$ intersects the boundary of F at one point, say (v_1, w_2). Construct the line $x_2 = w_2$. Each line $x_2 = v_2$ intersects the boundary of F at a point (w_1, v_2). Construct the line $x_1 = w_1$. Again we have constructed a finite number of lines. This network of lines determines a finite union of disjoint open rectangles. We will denote by $\mathcal{P}$ the collection of these open rectangles intersected with F and the network of lines intersected with F. The network intersected with F consists of a finite union of vertical and horizontal line segments. We see $\mathcal{P}$ is a partition of F.

<u>Lemma 4.7.</u> (1) If T is an open rectangle in $\mathcal{P}$, then either $T \subseteq B$ or $T \subseteq E_2 \setminus B$, and either $T \subseteq C$ or $T \subseteq E_2 \setminus C$.

(2) If U and V are any disjoint open rectangles in $\mathcal{P}$ and if u and v are the minimal vertices with respect to "$\preccurlyeq$" of

U and V respectively, then $u \prec v$ if and only if $L(U) \subsetneq L(V)$.

(3) If T is an open rectangle in this partition $\mathcal{P}$ such that $T \subsetneq E_2 \setminus C$, then the vertices of T belong to $E_2 \setminus C$.

Proof: The process we used in constructing $\mathcal{P}$ certainly gives us condition (1).

Let U be an open rectangle in $\mathcal{P}$ with minimal vertex $u = (u_1, u_2)$ and V be another open rectangle in $\mathcal{P}$ with minimal vertex $v = (v_1, v_2)$. If $u \prec v$, then $u_1 \leq v_1$ and $u_2 \leq v_2$ with either $u_1 < v_1$ or $u_2 < v_2$. Suppose $u_1 < v_1$. Let $(u_1, \bar{u}_2)$ be the other vertex of U with first coordinate u_1 and let $(v_1, \bar{v}_2)$ be the other vertex of V with first coordinate v_1. Then since $u_2 \leq v_2$ we have $\bar{u}_2 \leq \bar{v}_2$. Let $x = (x_1, x_2)$ be any point of U. Then $x_1 < v_1$ and $x_2 < \bar{u}_2 \leq \bar{v}_2$. Hence $x \in L(v_1, \bar{v}_2)$. Therefore since $(v_1, \bar{v}_2)$ is a vertex of V and $U \subsetneq L(v_1, \bar{v}_2)$ we have $L(U) \subsetneq L(V)$. An exactly similar argument applies if $u_2 < v_2$.

Conversely, if $L(U) \subsetneq L(V)$, then $u_1 \leq v_1$ and $u_2 \leq v_2$. For suppose $u_1 > v_1$. Then if $(\bar{v}_1, v_2)$ is the other vertex of V with second coordinate equal to v_2, we have $\bar{v}_1 \leq u_1$. Since every element of $L(V)$ has first coordinate less than or equal to $\bar{v}_1$ and every element of U has first coordinate greater than or equal to u_1, $U \not\subseteq L(V)$. Hence $u_1 > v_1$ is impossible. A similar argument shows $u_2 \leq v_2$. Therefore $u \preceq v$. However since $U \neq V$ we have

$u \not< v$. Thus condition (2) is proved.

To show condition (3) is true we notice that if (t_1, t_2) is a vertex of $T \subseteq E_2 \setminus C$, then (t_1, t_2) is a limit point of T and hence of $E_2 \setminus C$. Since $E_2 \setminus C$ is closed we have $(t_1, t_2) \in E_2 \setminus C$.

Whenever a fundamental set F has the property that $B \cap F$ and $C \cap F$ are finite unions of open rectangles with maximal vertices with respect to "$\preceq$" we will assume that F has been partitioned as above.

4.4. Sets of Type 2 and Theorem 4.8

Let F be a fundamental set and suppose that $B \cap F$ and $C \cap F$ are finite unions of open rectangles that have a maximal vertex with respect to "$\preceq$". Let $\mathcal{G}$ be the partition of F as above.

Definition 4.6. A set S is of type 2 with respect to B, C, and F provided

(1) $S = F \setminus F'$ where $F' \in \mathcal{F}$.

(2) $B \cap S \neq \emptyset$ and $S \setminus C \neq \emptyset$.

(3) If T is a rectangle in S, then T has a minimal vertex with respect to "$\preceq$".

(4) If $b \in B \cap S$ and $g \in S \setminus C$, then $b \in L(D)$ where D is the rectangle of $S \setminus C$ of which g is an element.

Theorem 4.8. If S is a set of type 2 with respect to B, C, and F, then $C(S) \geq am^2(S) + B(S)$.

Before we prove Theorem 4.8 we will give an informal discussion of the method of proof we shall use. We shall write S as the union of three sets in the manner described below and indicated by the following figure. Let g_1 be the minimum value taken on by any first coordinate of a vertex of a rectangle of $\wp$ in $S \setminus C$. Let D denote a rectangle of $S \setminus C$ such that its minimal vertex has as first coordinate g_1. Let h_1 be the first coordinate of the other vertex of D that is on the horizontal line that passes through the minimal vertex of D.

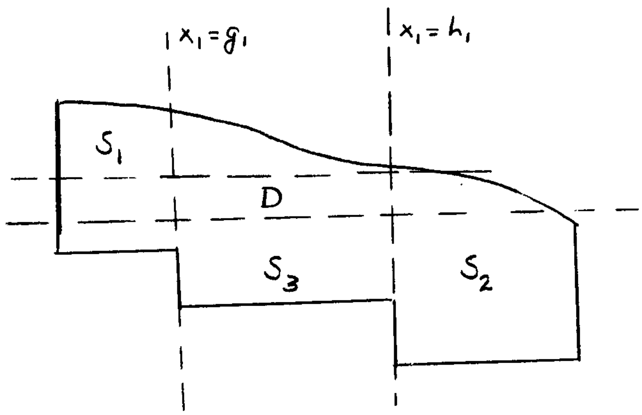

Figure 1.

We now let $S_1 = \{(x_1, x_2): x_1 \leq g_1\} \cap S$, $S_2 = \{(x_1, x_2): x_1 \geq h_1\} \cap S$, and $S_3 = \{(x_1, x_2): g_1 \leq x_1 \leq h_1\} \cap S$. Since S is of type 2 $B \cap S_2 = \emptyset$ for otherwise there would be a $b \in B \cap S$ such that $b \notin L(D)$. We see that $B \cap S_3$ may be empty or

$B \cap S_3$ may be nonempty. Our method of proof will be to show that if $B \cap S_3 = \phi$, then $C(S) \geq am^2(S) + B(S)$ and if $B \cap S_3 \neq \phi$, then $C(S_j) \geq am^2(S_j) + B(S_j)$ for $j = 1, 2, 3$. We will then again conclude that $C(S) \geq am^2(S) + B(S)$.

4.5. Proof of Theorem 4.8 When $B \cap S_3$ Is Empty

Since $B \cap S_3 = \phi$ we see that for all $b = (b_1, b_2)$ in $B \cap S$ we have $b_1 \leq g_1$. Let D be a rectangle of $S \setminus C$ with its minimal vertex with respect to "$\precsim$" having g_1 as its first coordinate. Let $\bar{g}$ be the vertex of D with first coordinate g_1 and second coordinate maximal. Then since S is a type 2 with respect to B, C, and F if $b \in B \cap S$, then $b \precsim \bar{g}$. If this were not true, then b would not be in $L(D)$. Hence S would not be of type 2 with respect to B, C, and F.

Let $G = \{\bar{g} - x : x \in B \cap S\}$. We proceed to show that

(5.1) $$G \subseteq E_2;$$

(5.2) $$G \cap A = \phi;$$

(5.3) $$G \subseteq S';$$

(5.4) $$m^2(G) = m^2(B \cap S).$$

To see that condition (5.1) is true we recall that we are supposing that $x \in B \cap S$ implies $x \precsim \bar{g}$. Hence $\bar{g} - x \in E_2$.

To see that condition (5.2) is true we suppose $G \cap A \neq \emptyset$. Pick $\bar{g} - x \in G \cap A$. Then $\bar{g} = (\bar{g}-x) + x \in A + B = C$. But this contradicts $\bar{g} \in E_2 \setminus C$. Hence condition (5.2) is true.

To see that (5.3) is true, pick $\bar{g} - x \in G$. Since $x \in B \cap S$ we have $x \in S$, and so $x \in S_j$ for some $j \in \Omega$. Again since $x \in B \cap S$, we have $\delta_j \precsim x \prec \bar{g}$. Hence $\bar{g} \in S_j$ since $\bar{g} \in S$. Thus $\bar{g} - \delta_j \in S_j'$. We also have $0 \prec \bar{g} - x \precsim \bar{g} - \delta_j$. Since S_j' is fundamental by Lemma 4.2, $\bar{g} - x \in S_j' \subseteq S'$. Therefore $G \subseteq S'$.

To see that condition (5.4) is true we note that $B \cap S$ is measurable and $m^2(B \cap S) < \infty$. Hence by Theorem 2.16 we have $m^2(G) = m^2(B \cap S)$.

Let ${}_wb_1$ be the minimum of the set of all first coordinates of vertices of rectangles in $B \cap S$. Let ${}_wb_2$ be the minimum of the set of all second coordinates of vertices of rectangles in $B \cap S$ with b_1^* as first coordinate. Let ${}_wb = ({}_wb_1, {}_wb_2)$. (The notation suggests that ${}_wb$ is a "lower west" vertex of a rectangle of $B \cap S$.) Since ${}_wb$ is a minimal vertex of a rectangle in $B \cap S$, we can choose an $\varepsilon > 0$ as small as we please such that $b_\varepsilon = ({}_wb_1+\varepsilon, {}_wb_2+\varepsilon)$ is in the rectangle of $B \cap S$ of which ${}_wb$ is the minimal vertex.

Let $H_\varepsilon = \{y - b_\varepsilon : y \in S \setminus C,\ b_\varepsilon \precsim y\}$. We will show that

(5.5) $$H_\varepsilon \subseteq E_2;$$

(5.6) $$H_\varepsilon \cap A = \emptyset;$$

(5.7) $$H_\varepsilon \subseteq S';$$

and for a positive constant K

(5.8) $$m^2(H_\varepsilon) \geq m^2(S \setminus C) - K\varepsilon;$$

(5.9) $$m^2(G \cap H_\varepsilon) \leq K\varepsilon .$$

To see that condition (5.5) is true we observe that for $y - b_\varepsilon$ to be in H_ε that we must have $b_\varepsilon \precsim y$. Hence $H_\varepsilon \subseteq E_2$.

To see that condition (5.6) is true, suppose $H_\varepsilon \cap A \neq \emptyset$. Pick $y - b_\varepsilon \in H_\varepsilon \cap A$. Then $g = (g-b_\varepsilon) + b_\varepsilon$ is in $A + B = C$. This contradicts $g \in S \setminus C$.

To see that condition (5.7) is true pick $y - b_\varepsilon \in H_\varepsilon$. Then $b_\varepsilon \in S$ and hence $b_\varepsilon \in S_j$ for some $j \in \Omega$. Thus $\delta_j \precsim b_\varepsilon \precsim y$. Hence $y \in S_j$ since $y \in S$. Therefore $y - \delta_j \in S_j'$. Also however, $0 \prec y - b_\varepsilon \precsim y - \delta_j$. Since S_j' is fundamental we have $y - b_\varepsilon \in S_j' \subseteq S'$.

To see that condition (5.8) is true we observe that $H_\varepsilon + b_\varepsilon \subseteq S \setminus C$ and $(S \setminus C) \setminus (H_\varepsilon + b_\varepsilon) \subseteq F \cap \{(x_1, x_2): {}_wb_2 \leq x_2 \leq {}_wb_2 + \varepsilon\}$ since S is a set of type 2 with respect to B, C, and F and ${}_wb$ is a minimal vertex of a rectangle of $B \cap S$. Hence $m^2((S \setminus C) \setminus (H_\varepsilon + b_\varepsilon)) \leq K\varepsilon$ where K is the least upper bound of the lengths of all line segments contained in F. Therefore

$$m^2(H_\varepsilon) = m^2(H_\varepsilon + b_\varepsilon) \geq m^2((S \setminus C)) - K\varepsilon.$$

To see that condition (5.9) is true, suppose $y = (y_1, y_2) \in G \cap H_\varepsilon$. Then $y_1 \leq \bar{g}_1 - {}_wb_1$ since $y \in G$ on the one hand, and $y_1 \geq \bar{g}_1 - ({}_wb_1 + \varepsilon)$ on the other hand, since $y \in H_\varepsilon$. Thus $\bar{g}_1 - {}_wb_1 - \varepsilon \leq y_1 \leq \bar{g}_1 - {}_wb_1$. Therefore we have

$$G \cap H_\varepsilon \subseteq \{(x_1, x_2) : \bar{g}_1 - {}_wb_1 - \varepsilon \leq x_1 \leq \bar{g}_1 - {}_wb_1\} \cap F.$$

Hence $m^2(G \cap H_\varepsilon) \leq K\varepsilon$ where K is as defined in the proof of condition (5.8).

Since the sets G and H_ε satisfy conditions (5.2) through (5.4) and conditions (5.6) through (5.9) we have

$$\begin{aligned} m^2(S' \setminus A) &\geq m^2(G) + m^2(H_\varepsilon) - m^2(G \cap H_\varepsilon) \\ &\geq m^2(G) + m^2(H_\varepsilon) - K\varepsilon \\ &\geq m^2(G) + m^2(S \setminus C) - K\varepsilon - K\varepsilon \\ &= m^2(B \cap S) + m^2(S \setminus C) - 2K\varepsilon. \end{aligned}$$

Noting that this inequality holds for all sufficiently small $\varepsilon > 0$, we let $\varepsilon \to 0$ to obtain

$$m^2(S' \setminus A) \geq m^2(B \cap S) + m^2(S \setminus C).$$

Hence we have

$$m^2(S') - m^2(A \cap S') \geq m^2(B \cap S) + m^2(S) - m^2(C \cap S)$$

Thus

$$m^2(C \cap S) \geq m^2(A \cap S') + m^2(S) - m^2(S') + m^2(B \cap S)$$

$$\geq \alpha m^2(S') + (m^2(S) - m^2(S')) + m^2(B \cap S).$$

Since $m^2(S) - m^2(S') \geq 0$ by Lemma 4.5, we have

$$m^2(C \cap S) \geq \alpha m^2(S') + \alpha(m^2(S) - m^2(S')) + m^2(B \cap S)$$

$$= \alpha m^2(S) + m^2(B \cap S).$$

Thus if $B \cap S_3 = \phi$, then Theorem 4.8 follows. We now consider the other case.

4.6. Proof of Theorem 4.8 When $B \cap S_3 \neq \phi$. Part 1

Since $B \cap S_3 \neq \phi$ there exists $b = (b_1, b_2) \in B \cap S$ such that $b_1 > g_1$ and since $B \cap S_2 = \phi$ we have $g_1 < b_1 < h_1$. By Lemma 4.6 we have that S_1, S_2, and S_3 are all differences of two fundamental sets. Furthermore $S = S_1 \cup S_2 \cup S_3$ and $m^2(S_i \cap S_j) = 0$, $i \neq j$. In this section we will show

$$C(S_1) \geq \alpha m^2(S_1) + B(S_1).$$

We notice that $S_1 \setminus C \neq \phi$. This is so since $\bar{g}$ is in $S_1 \setminus C$ where $\bar{g}$ is as defined in Section 4.5. Since S is of type 2 with respect to B, C, and F we have for all $b \in B \cap S_1$ that $b \prec \bar{g}$. We also notice that $m^2(S_1 \setminus C) = 0$ since $S_1 \setminus C \subseteq \{(x_1, x_2): x_1 = g_1\} \cap S$, and so

$$C(S_1) = m^2(C \cap S_1) = m^2(S_1) - m^2(C \cap S_1) = m^2(S_1).$$

If $B \cap S_1 = \emptyset$, then

$$C(S_1) = m^2(S_1) + B(S_1)$$

$$\geq a m^2(S_1) + B(S_1)$$

and we are through.

If $B \cap S_1 \neq \emptyset$ we recall that $b \in B \cap S_1$ implies $b \prec \bar{g}$. Let $G = \{\bar{g} - x : x \in B \cap S_1\}$. In exactly the same way as before we can show that

(6.1) $$G \subseteq E_2;$$

(6.2) $$G \cap A = \emptyset;$$

(6.3) $$G \subseteq S_1';$$

(6.4) $$m^2(G) = m^2(B \cap S_1).$$

Therefore,

$$m^2(S_1' \setminus A) \geq m^2(G \setminus A)$$

$$= m^2(G) - m^2(G \cap A)$$

$$= m^2(G) = m^2(B \cap S_1).$$

Hence $m^2(S_1') - m^2(A \cap S_1') \geq m^2(B \cap S_1)$. Adding $m^2(S_1)$ to both sides and transposing terms we obtain

$$\begin{aligned} m^2(S_1) &\geq m^2(A \cap S_1') + m^2(S_1) - m^2(S_1') + m^2(B \cap S_1) \\ &\geq am^2(S_1') + a(m^2(S_1) - m^2(S_1')) + m^2(B \cap S_1) \\ &= am^2(S_1) + m^2(B \cap S_1). \end{aligned}$$

However $C(S_1) = m^2(S_1)$, and hence

$$C(S_1) \geq am^2(S_1) + B(S_1).$$

4.7. Proof of Theorem 4.8 When $B \cap S_3 \neq \phi$. Part 2

In this section we show that $C(S_2) \geq am^2(S_2) + B(S_2)$.

If $m^2(S_2 \setminus C) = 0$, then $C(S_2) = m^2(S_2) \geq m^2(S_2) + B(S_2)$ since $m^2(B \cap S_2) = 0$. Hence $C(S_2) \geq am^2(S_2) + B(S_2)$. Hence we suppose $m^2(S_2 \setminus C) \neq 0$.

Since $B \cap S_3 \neq \phi$ there is a rectangle of $B \cap S_3$ with a vertex of the form (h_1, h_2) for some h_2. Let ${}_eb_2$ be the minimum of the set of all second coordinates of vertices of rectangles of $B \cap S_3$ with h_1 as first coordinate. Let ${}_eb = (h_1, {}_eb_2)$. The notation suggests ${}_eb$ is a "lower east" vertex of a rectangle of $B \cap S_3$. Since ${}_eb$ is a vertex of a rectangle of $B \cap S_3$, for any sufficiently small $\varepsilon > 0$ we see $(h_1 - \varepsilon, {}_eb_2 + \varepsilon)$ is an element of the rectangle of $B \cap S_3$ for which h_1 is the maximal first coordinate of one of its vertices and ${}_eb_2$ is the minimal second coordinate of one of its vertices.

Let $b_\varepsilon = (h_1 - \varepsilon, {}_eb_2 + \varepsilon)$ and $S_2(\varepsilon) = \{(x_1, x_2) : x_1 \geq h_1 - \varepsilon\} \cap S$. Let $H_\varepsilon = \{y - b_\varepsilon : y \in (S_2(\varepsilon) \setminus C), b_\varepsilon \preceq y\}$. In the same manner as before we have

(7.5) $$H_\varepsilon \subseteq E_2;$$

(7.6) $$H_\varepsilon \cap A = \phi;$$

(7.7) $$H_\varepsilon \subseteq S_2'(\varepsilon);$$

and for the positive constant K defined in the proof of condition (5.8) we have

(7.8) $$m^2(H_\varepsilon) \geq m^2(S_2(\varepsilon) \setminus C) - K\varepsilon.$$

We will also show that for the same constant K we have

(7.9) $$m^2(C \cap S_2) \geq m^2(C \cap S_2(\varepsilon)) - K\varepsilon.$$

That condition (7.9) is true follows from

$$C \cap S_2(\varepsilon) \subseteq (C \cap S_2) \cup (\{(x_1, x_2) : h_1 - \varepsilon \leq x_1 \leq h_1\} \cap S).$$

Thus

$$m^2(C \cap S_2(\varepsilon)) \leq m^2(C \cap S_2) + K\varepsilon.$$

Therefore from (7.6), (7.7), and (7.8) we have

$$m^2(S_2'(\varepsilon) \smallsetminus A) \geq m^2(H_\varepsilon)$$

$$\geq m^2(S_2(\varepsilon) \smallsetminus C) - K\varepsilon.$$

Hence

$$m^2(S_2'(\varepsilon)) - m^2(A \cap S_2'(\varepsilon)) \geq m^2(S_2(\varepsilon)) - m^2(C \cap S_2(\varepsilon)) - K\varepsilon,$$

which implies

$$m^2(C \cap S_2(\varepsilon)) \geq m^2(A \cap S_2'(\varepsilon)) + m^2(S_2(\varepsilon)) - m^2(S_2'(\varepsilon)) - K\varepsilon$$

$$\geq am^2(S_2'(\varepsilon)) + a(m^2(S_2(\varepsilon)) - m^2(S_2'(\varepsilon)) - K\varepsilon$$

$$= am^2(S_2(\varepsilon)) - K_1\varepsilon.$$

Hence from inequality (7.9) and the fact that $am^2(S_2(\varepsilon)) \geq am^2(S_2)$ we have

$$m^2(C \cap S_2) + K\varepsilon \geq am^2(S_2) - K\varepsilon$$

for all sufficiently small $\varepsilon > 0$. If we let $\varepsilon \to 0$ we obtain

$$C(S_2) = m^2(C \cap S_2)$$

$$\geq am^2(S_2)$$

$$= am^2(S_2) + B(S_2),$$

since $B \cap S_2 = \emptyset$.

4.8. Proof of Theorem 4.8 When $B \cap S_3 \neq \emptyset$. Part 3

In this section we show $C(S_3) \geq am^2(S_3) + B(S_3)$ and finally conclude that if $B \cap S_3 \neq \emptyset$ we have $C(S) \geq am^2(S) + B(S)$, thus proving Theorem 4.8.

Recall that any rectangle of $S_3 \setminus C$ has at least one vertex with first coordinate h_1. Let h_2 be the minimum of the set of all second coordinates of vertices of rectangles of $S_3 \setminus C$ such that the first coordinate is h_1. Let $\bar{h} = (h_1, h_2)$. Then since S is of type 2 with respect to B, C, and F we have for all $b \in B \cap S_3$ that $b \prec \bar{h} = (h_1, h_2)$.

Let $G = \{\bar{h} - x : x \in B \cap S_3\}$. As before we have

(8.1) $$G \subsetneq E_2;$$

(8.2) $$G \cap A' = \emptyset;$$

(8.3) $$G \subsetneq S_3';$$

(8.4) $$m^2(G) = m^2(B \cap S_3).$$

Let ${}_wb_2$ be the minimum of the set of all second coordinates of vertices of rectangles of $B \cap S_3$. Let ${}_wb = (g_1, {}_wb_2)$. Then ${}_wb$ is a minimal vertex with respect to " $\lesssim$ " of a rectangle of $B \cap S_3$. Therefore we can choose $\varepsilon > 0$ such that $b_\varepsilon = (g_1+\varepsilon, b_2+\varepsilon)$ is an element of the rectangle of which ${}_wb$ is the minimal vertex.

Let $H_\varepsilon = \{y - b_\varepsilon : y \in S_3 \setminus C, b_\varepsilon \lesssim y\}$. As before we have

(8.5) $$H_\varepsilon \subseteq E_2;$$

(8.6) $$H_\varepsilon \cap A = \emptyset;$$

(8.7) $$H_\varepsilon \subseteq S_3';$$

and for the constant K defined in the proof of condition 5.8 of Section 4.5 we have

(8.8) $$m^2(H_\varepsilon) \geq m^2(S_3 \setminus C) - K\varepsilon.$$

For the same constant K we also have

(8.9) $$m^2(G \cap H_\varepsilon) < K\varepsilon.$$

To see that (8.9) is true, suppose $y = (y_1, y_2) \in G \cap H_\varepsilon$. Then $y_2 \leq h_2 - {}_w b_2$ since $y \in G$ on the one hand, and $y_2 \geq h_2 - ({}_w b_2 + \varepsilon)$ on the other hand, since $y \in H_\varepsilon$. Thus $h_2 - {}_w b_2 - \varepsilon \leq y_2 \leq h_2 - {}_w b_2$. Therefore we have $G \cap H_\varepsilon \subseteq \{(x_1, x_2) : h_2 - {}_w b_2 - \varepsilon \leq x_2 \leq h_2 - {}_w b_2\} \cap F$. Hence $m^2(G \cap H_\varepsilon) \leq K\varepsilon$.

Since the sets G and H_ε satisfy the above conditions we have

$$
\begin{aligned}
m^2(S_3' \setminus A) &\geq m^2(G) + m^2(H_\varepsilon) - m^2(G \cap H_\varepsilon) \\
&\geq m^2(G) + m^2(S_3 \setminus C) - K\varepsilon - K\varepsilon \\
&= m^2(B \cap S_3) + m^2(S_3 \setminus C) - 2K\varepsilon.
\end{aligned}
$$

Since this inequality is true for all sufficiently small $\varepsilon > 0$ we obtain

$$
m^2(S_3' \setminus A) \geq m^2(B \cap S_3) + m^2(S_3 \setminus C).
$$

Hence

$$
m^2(S_3') - m^2(A \cap S_3') \geq m^2(B \cap S_3) + m^2(S_3) - m^2(C \cap S_3).
$$

Finally,

$$
\begin{aligned}
C(S_3) &\geq m^2(A \cap S_3') + m^2(S_3) - m^2(S_3') + m^2(B \cap S_3) \\
&\geq am^2(S_3') + a(m^2(S_3) - m^2(S_3')) + m^2(B \cap S_3) \\
&= am^2(S_3) + B(S_3).
\end{aligned}
$$

Therefore in Sections 4.6, 4.7, and 4.8 we have shown $C(S_j) \geq am^2(S_j) + B(S_j)$ for $j = 1, 2,$ and 3. Thus,

$$
\begin{aligned}
C(S) &= C(S_1 \cup S_2 \cup S_3) \\
&= C(S_1) + C(S_2) + C(S_3) \\
&\geq am^2(S_1) + B(S_1) + am^2(S_2) + B(S_2) + am^2(S_3) + B(S_3) \\
&= a(m^2(S_1) + m^2(S_2) + m^2(S_3)) + B(S_1 \dot\cup S_2 \cup S_3) \\
&= am^2(S) + B(S).
\end{aligned}
$$

This completes Theorem 4.8.

4.9. Sets of Type 1 and Theorems 4.9 and 4.10

Definition 4.7. A set $F \in \mathfrak{F}$ is of type 1 with respect to B and C provided

(1) $B \cap F$ and $C \cap F$ are finite unions of open rectangles that have a maximal vertex with respect to "$\precsim$".

(2) $F \smallsetminus C \neq \phi$.

(3) If $b \in B \cap F$ and $g \in F \smallsetminus C$, then $b \in L(D)$ where D is the rectangle of the partition $\mathcal{Q}$ of Section 4.3 of which g is an element.

Theorem 4.9. If F is a set of type 1 with respect to B and C, then $C(F) \geq \alpha m^2(F) + B(F)$.

Proof: Let $S = F \smallsetminus \{(0,0)\}$. Then $S = F \smallsetminus F'$ where $F' = \{(0,0)\} \in \mathfrak{F}$, $B \cap S \neq \phi$ since $0 \in B$ and B is open and hence 0 is not an isolated point, $S \smallsetminus C \neq \phi$ since $F \smallsetminus C \neq \phi$, and finally if $b \in B \cap S$ and $g \in S \smallsetminus C$, then $b \in L(D)$ where D is the rectangle of $F \smallsetminus C$, and hence of $S \smallsetminus C$, of which g is an element. Therefore since $B \cap F$ and $C \cap F$ are finite unions of open rectangles with maximal vertices we see S is of type 2 with respect to B, C, and F. Therefore by Theorem 4.8 we have

$$C(S) \geq \alpha m^2(S) + B(S).$$

However $C(S) = C(F)$ and $\alpha m^2(S) + B(S) = \alpha m^2(F) + B(F)$. Therefore,

$$C(F) \geq \alpha m^2(F) + B(F).$$

<u>Theorem 4.10.</u> If $F \in \mathcal{F}$, $A \cap F$ and $B \cap F$ are finite unions of open rectangles, $F \smallsetminus C \neq \phi$, and $F^* \subseteq Cl(F \smallsetminus C)$, then $C(F) \geq \alpha m^2(F) + B(F)$.

Proof: We first observe that since $A \cap F$ and $B \cap F$ are finite unions of open rectangles that $C \cap F$ is a finite union of open rectangles since the sum set of two open rectangles is again an open rectangle. Since $F^* \subseteq Cl(F \smallsetminus C)$ the open rectangles of $B \cap F$ and $C \cap F$ have a maximal vertex.

This lemma is proved by using induction on the number of rectangles in $F \smallsetminus C$. If $F \smallsetminus C$ consists only of one rectangle D, possibly a line segment, then since $F^* \subseteq Cl(F \smallsetminus C)$ we have $F = L(D)$. To see this is true we note that $L(D) \subseteq F$. Suppose $F \not\subseteq L(D)$. Then there exists $c \in C \cap F$ such that $c \notin L(D)$. Either $c \in F^*$ or $c \notin F^*$. If $c \in F^*$ we contradict that $F^* \subseteq Cl(F \smallsetminus C)$. If $c \notin F^*$ and $c \notin L(D)$, then for all $y \in F$ such that $c \prec y$ we have $y \in C$. However if this is so then there is an element of F^* that is not a limit point of $F \smallsetminus C$, which contradicts $F^* \subsetneq Cl(F \smallsetminus C)$.

Therefore if $b \in B \cap F$, then $b \in L(D)$. Hence F is of type 1 with respect to B and C. Therefore by Theorem 4.9 we have $C(F) \geq \alpha m^2(F) + B(F)$. Thus we suppose that $F \smallsetminus C$ consists of $n \geq 2$ rectangles and the conclusion of the lemma holds for any fundamental set F' satisfying the hypotheses of the lemma with $F' \smallsetminus C$ consisting of less than n rectangles.

Let $V = \{D_1, \ldots, D_n\}$ where each D_j $(j = 1, 2, \ldots, n)$ is a rectangle of $F \smallsetminus C$. Since $F^* \subsetneq Cl(F \smallsetminus C)$ we have as before $F = \bigcup_{j=1}^{n} L(D_j)$.

Suppose $B \cap F \subsetneq \bigcap_{j=1}^{n} L(D_j)$. Then F is of type 1 with respect to B and C. Hence the conclusion follows by Lemma 4.9.

Therefore the case remains where there exists an element $b \in B \cap F$ and a rectangle $D_k \in V$ such that $b \notin L(D_k)$. Thus $B \cap (F \smallsetminus L(D_k)) \neq \emptyset$. But since $F = \bigcup_{j=1}^{n} L(D_j)$ we have $B \cap (F \smallsetminus \bigcup_{j=1}^{n} L(D_j)) = \emptyset$. From the preceding two conditions we see that there must exist a subset P of V such that

(1) $B \cap (F \smallsetminus \bigcup_{D \in P} L(D)) \neq \emptyset$;

(2) $B \cap (F \smallsetminus \bigcup_{D \in P \cup \{K\}} L(D)) = \emptyset$ if $K \in V \smallsetminus P$;

and

(3) $\{D_k\} \subsetneq P \subsetneq V$.

We now partition F into two parts W and $F \smallsetminus W$. Let

$W = \bigcup_{D \in P} L(D)$. Then W satisfies the following properties:

(4) $W \in \mathfrak{F}$;

(5) $W^* \subseteq Cl(W \setminus C)$;

(6) $W \setminus C \neq \emptyset$;

(7) Both $A \cap W$ and $B \cap W$ are finite unions of open rectangles;

(8) The set $W \setminus C$ consists of fewer than n rectangles of the partition $\mathcal{P}$.

That property (4) holds follows from Lemma 4.3.

That property (5) holds follows from the fact that

$W^* \subseteq \bigcup_{D \in P} D \subseteq E_2 \setminus C$.

That property (6) holds follows from condition (3) above.

That property (7) holds follows from the relation $W \subseteq F$ and the hypothesis that $A \cap F$ and $B \cap F$ are finite unions of open rectangles.

That property (8) holds follows from (3) above. The number of rectangles of $W \setminus C$ is equal to the cardinality of P which is in turn less than the cardinality n of V.

We have therefore demonstrated that W is a fundamental set satisfying the hypothesis of the lemma and that $W \setminus C$ consists of less than n rectangles. Hence the inequality

$$(9)\quad C(W) \geq am^2(W) + B(W)$$

holds.

We now show that $F \smallsetminus W$ is a set of type 2 with respect to B, C, and F. First $B \cap F$ and $C \cap F$ are finite unions of open rectangles that have a maximal vertex with respect to "$\lesssim$" which we showed above. Next $F \in \mathscr{F}$ by hypothesis and $W \in \mathscr{F}$ from (4) above. Also $B \cap (F \smallsetminus W) \neq \emptyset$ and $W \smallsetminus C \neq \emptyset$ from conditions (1) and (6). Furthermore if T is a rectangle of $F \smallsetminus W$, then T has a minimal vertex since W is the union of lower sets of rectangles of F by the definition of W. Finally suppose $b \in B \cap (F \smallsetminus W)$ and $g \in (F \smallsetminus W) \smallsetminus C$. Let K be the rectangle in $V \smallsetminus P$ to which g belongs. Then

$$B \cap (F \smallsetminus (W \cup L(K))) = B \cap (F \smallsetminus \bigcup_{D \in P \cup \{K\}} L(D)) = \emptyset$$

from the definition of W and property (2). Therfore $b \in L(K)$. Thus $F \smallsetminus W$ is a set of type 2 with respect to B, C, and F.

We now have from Lemma 4.8 that

$$(10)\quad C(F \smallsetminus W) \geq am^2(F \smallsetminus W) + B(F \smallsetminus W).$$

Hence by (9) and (10) we have

$$\begin{aligned} C(F) &= C(W) + C(F \smallsetminus W) \\ &\geq am^2(W) + B(W) + am^2(F \smallsetminus W) + B(F \smallsetminus W) \\ &= am^2(F) + B(F). \end{aligned}$$

4.10. Conditions P and Q

Definition 4.8. An open set A is said to satisfy condition P if there exists $\xi_1 > 0$ and $\xi_2 > 0$ such that both $A \cap \{(x_1, x_2): 0 \leq x_1 \leq \xi_1\}$ and $A \cap \{(x_1, x_2): 0 \leq x_2 \leq \xi_2\}$ are the union of a finite number of open rectangles.

Lemma 4.11. Let A satisfy condition P and F be a given fundamental set. For $\varepsilon > 0$ let $A_1, A_2, \ldots, A_k$ be a finite number of open rectangles of $A \cap F$, including all those rectangles within ξ_j units of the x_j-axis $(j = 1, 2)$, such that $m^2((A \cap F) \setminus \bigcup_{j=1}^{k} A_j) < \varepsilon$. Let

$$A^\varepsilon = (\bigcup_{j=1}^{k} A_j) \cup ((E_2 \setminus F) \cap A).$$

Let

$$\alpha_\varepsilon = \text{glb}\{\frac{A^\varepsilon(G)}{m^2(G)} : G \in \mathfrak{F}^+\}.$$

Then $\lim_{\varepsilon \to 0} \alpha_\varepsilon = \alpha$.

Proof: For all $G \in \mathfrak{F}^+$ such that

$$G \subseteq \{(x_1, x_2): 0 \leq x_1 \leq \xi_1\} \cup \{(x_1, x_2): 0 \leq x_2 \leq \xi_2\}$$

we have $A^\varepsilon(G) = A(G)$. Therefore $\frac{A^\varepsilon(G)}{m^2(G)} = \frac{A(G)}{m^2(G)} \geq \alpha$ for all of these G.

If G is not in this union, then $m^2(G) \geq \xi_1 \xi_2$. Hence

$$A(G) - A^{\varepsilon}(G) \leq A(F) - A^{\varepsilon}(F) < \varepsilon$$

and so

$$\frac{A^{\varepsilon}(G)}{m^2(G)} \geq \frac{A(G)-\varepsilon}{m^2(G)} = \frac{A(G)}{m^2(G)} - \frac{\varepsilon}{m^2(G)} \geq \alpha - \frac{\varepsilon}{\xi_1 \xi_2} .$$

Therefore,

$$\alpha \geq \alpha_{\varepsilon} \geq \alpha - \frac{\varepsilon}{\xi_1 \xi_2}$$

and as $\varepsilon \to 0$ we have $\alpha_{\varepsilon} \to \alpha$.

Definition 4.9. Set A is said to satisfy condition Q if A is open and for any fundamental set F the following condition is satisfied. Given any $\varepsilon > 0$ there exists a finite number of open rectangles, say $A_1, \ldots, A_k$, of $A \cap F$ such that $m(\pi_1((A \cap F) \sim \bigcup_{j=1}^{k} A_j)) < \varepsilon$ and $m(\pi_2((A \cap F) \sim \bigcup_{j=1}^{k} A_j)) < \varepsilon$ where for any $S \subseteq E_2$ we define $\pi_1(S)$ and $\pi_2(S)$ by $\pi_1(S) = \{(x_1, 0): (x_1, x_2) \in S\}$ and $\pi_2(S) = \{(0, x_2): (x_1, x_2) \in S\}$.

Lemma 4.12. Let A satisfy condition Q and let F be any given fundamental set. For $\varepsilon > 0$ let $A_1, A_2, \ldots, A_k$ be open rectangles of $A \cap F$ such that $m(\pi_1((A \cap F) \sim \bigcup_{j=1}^{k} A_j)) < \varepsilon$ and $m(\pi_2((A \cap F) \sim \bigcup_{j=1}^{k} A_j)) < \varepsilon$. Let $A^{\varepsilon} = (\bigcup_{j=1}^{k} A_j) \cup ((E_2 \sim F) \cap A)$. Then $\lim_{\varepsilon \to 0} \alpha_{\varepsilon} = \alpha$.

Proof: Since $\alpha_{\varepsilon} \leq \alpha$, then $\lim_{\varepsilon \to 0} \alpha_{\varepsilon} \leq \alpha$. Since $0 \in A$ and A is open we can assume that there exists a square, say A_1, with side ξ and minimal vertex at the origin contained in A and in

A^{ε} provided we choose $A_1, \ldots, A_k$ appropriately. For any fundamental set $G \in \mathfrak{F}^+$ which is contained in this square A_1 we have $A(G) = A^{\varepsilon}(G)$ or $\frac{A^{\varepsilon}(G)}{m^2(G)} \geq a$.

Let K be the least upper bound of the lengths of all line segments contained in F. Suppose the square A_1 is contained in the fundamental set G where $G \in \mathfrak{F}^+$, then $m^2((A^{\varepsilon} \cap G) \geq m^2(A \cap G) - 2K\varepsilon$. This is so since

$$
\begin{aligned}
m^2((A \cap G) \smallsetminus (A^{\varepsilon} \cap G)) &\leq (m(\pi_1((A \cap F) \smallsetminus \bigcup_{j=1}^{k} A_j))K \\
&\quad + (m(\pi_2(A \cap F) \smallsetminus \bigcup_{j=1}^{k} A_j))K \\
&\leq \varepsilon K + \varepsilon K \\
&= 2K\varepsilon.
\end{aligned}
$$

Hence

$$\frac{A^{\varepsilon}(G)}{m^2(G)} \geq \frac{A(G) - 2K\varepsilon}{m^2(G)} \geq a - \frac{2K\varepsilon}{\xi^2}.$$

The only remaining possibility for a fundamental set $G \in \mathfrak{F}^+$ is $A_1 \not\subset G$ and $G \not\subset A_1$. Then the boundaries of A_1 and G intersect in two points (ℓ_1, ξ) and (ξ, ℓ_2) as in Figure 2 below or in one point (ℓ_1, ξ) or (ξ, ℓ_2) as in Figure 3 below.

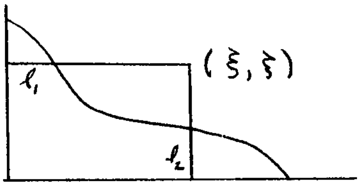

Figure 2.

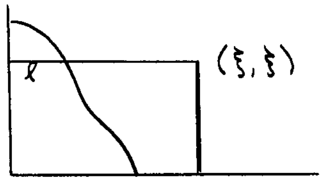

Figure 3.

If (ℓ_1, ξ) and (ξ, ℓ_2) are intersection points of the boundaries A_1 and G, then assume $\ell_1 \geq \ell_2$.

Then

$$\frac{A^{\varepsilon}(G)}{m^2(G)} \geq \frac{A(G)-(\ell_1+\ell_2)\varepsilon}{m^2(G)}$$

$$\geq a - \frac{(\ell_1+\ell_2)\varepsilon}{\ell_1 \xi}$$

$$\geq a - \frac{\left(1+\frac{\ell_2}{\ell_1}\right)\varepsilon}{\xi}$$

$$\geq a - \frac{2\varepsilon}{\xi} .$$

If $\ell_1 \leq \ell_2$, then a symmetric argument shows that again we have

$$\frac{A^{\varepsilon}(G)}{m^2(G)} \geq a - \frac{2\varepsilon}{\xi} .$$

If the boundaries of A_1 and G intersect in only one point, say (ℓ_1, ξ), (it could be a point of the form (ξ, ℓ_2)) we have

$$\frac{A^{\varepsilon}(G)}{m^2(G)} \geq \frac{A(G)-\ell_1\varepsilon}{m^2(G)}$$

$$\geq \alpha - \frac{\ell_1\varepsilon}{\ell_1\xi}$$

$$= \alpha - \frac{\varepsilon}{\xi} .$$

If the point of intersection had been (ξ, ℓ_2), then a symmetric argument shows that again we have

$$\frac{A^{\varepsilon}(G)}{m^2(G)} \geq \alpha - \frac{\varepsilon}{\xi} .$$

Thus

$$\lim_{\varepsilon \to 0} \alpha_{\varepsilon} = \lim_{\varepsilon \to 0} \text{glb}\{\frac{A^{\varepsilon}(G)}{m^2(G)} : G \in \mathfrak{F}^{+}\} \geq \alpha .$$

Finally $\lim_{\varepsilon \to 0} \alpha_{\varepsilon} = \alpha .$

4.11. Mann's Second Theorem

In this section we finally obtain a two-dimensional continuous analogue for Mann's Second Theorem. This is the main theorem of the thesis.

Theorem 4.13. If $0 \in A \cap B$, A and B are open, A satisfies either condition P or condition Q, $F \in \mathfrak{F}$, $F \setminus C \neq \emptyset$, and $F^* \subseteq Cl(F \setminus C)$, then $C(F) \geq \alpha m^2(F) + B(F)$.

Proof: Since B is open and $m^2(B \cap F) < \infty$ we can find a finite number of open rectangles of $B \cap F$, say $B_1, \ldots, B_n$, such that $m^2((B \cap F) \setminus \bigcup_{j=1}^{n} B_j) < \varepsilon$. Let $B^\varepsilon = (\bigcup_{j=1}^{n} B_j) \cup ((E_2 \setminus F) \cap B)$. Then

$$B^\varepsilon(F) \geq B(F) - \varepsilon.$$

If A satisfies condition P, let A^ε be defined as in Lemma 4.11. If A satisfies condition Q, let A^ε be defined as in Lemma 4.12. In either case $A^\varepsilon \cap F$ is a finite union of open rectangles. Let $C^\varepsilon = A^\varepsilon + B^\varepsilon$. Then $C^\varepsilon \subseteq C$ and so both $F \setminus C^\varepsilon \neq \emptyset$ and $F^* \subseteq Cl(F \setminus C^\varepsilon)$. Thus by Theorem 4.10 we have

$$\begin{aligned} C^\varepsilon(F) &\geq a_\varepsilon m^2(F) + B^\varepsilon(F) \\ &\geq a_\varepsilon m^2(F) + B(F) - \varepsilon. \end{aligned}$$

However $C(F) \geq C^\varepsilon(F)$. Therefore

$$C(F) \geq a_\varepsilon m^2(F) + B(F) - \varepsilon.$$

By Lemma 4.11 and 4.12 we have $\lim_{\varepsilon \to 0} a_\varepsilon = a$. Hence letting $\varepsilon \to 0$ in the above inequality we obtain

$$C(F) \geq a m^2(F) + B(F).$$

4.12. A Continuous Two-Dimensional $\alpha\beta$ Theorem

In this section we obtain a continuous $\alpha\beta$ Theorem in two dimensions. A two-dimensional $\alpha\beta$ Theorem is not known in the discrete case.

Lemma 4.14. If C is measurable and $E_2 \setminus C \neq \emptyset$, then

$$\gamma = \text{glb} \frac{C(F)}{m^2(F)}$$

where $F \in \mathfrak{F}^+$ and $F^* \subseteq \text{Cl}(F \setminus C)$.

Proof: Let γ' denote this glb. Clearly $\gamma \leq \gamma'$. Let G be any fundamental set in $\mathfrak{F}^+$. If $C(G) \neq m^2(G)$, then $G \setminus C \neq \emptyset$. In this case let $F = \bigcup_{g \in G \setminus C} L(g)$. Then F is fundamental by Lemma 4.3. We also know $F \in \mathfrak{F}^+$ since $G \setminus C$ must contain a point $g = (g_1, g_2)$ such that $g_1 g_2 \neq 0$ when $C(G) < m^2(G)$. Furthermore $F^* \subseteq \text{Cl}(F \setminus C)$ since if $x \in F^*$ then $x \in F \setminus C$ or x is a limit point of points of $F \setminus C$.

According to the construction of F we also have $C(G \setminus F) = m^2(G \setminus F)$. Therefore

$$\frac{C(G)}{m^2(G)} = \frac{C(F)+C(G \setminus F)}{m^2(G)}$$

$$= \frac{C(F)+m^2(G \setminus F)}{m^2(F)+m^2(G \setminus F)}$$

$$\geq \frac{C(F)}{m^2(F)}$$

$$\geq \gamma'.$$

Hence $\gamma \geq \gamma'$. Therefore $\gamma = \gamma'$.

<u>Theorem 4.15.</u> If $0 \in A \cap B$, A and B are open, and A satisfies either condition P or condition Q, then $\gamma \geq \min(1, \alpha+\beta)$.

Proof: If $m^2(E_2 \setminus C) = 0$, then $\gamma = 1$. Assume $m^2(E_2 \setminus C) > 0$. Let F be any set in $\mathfrak{F}^+$ where $F \setminus C \neq \emptyset$ and $F^* \subseteq Cl(F \setminus C)$. Then by Theorem 4.13 we have

$$C(F) \geq \alpha m^2(F) + B(F)$$

$$\geq \alpha m^2(F) + \beta m^2(F)$$

$$= (\alpha+\beta) m^2(F),$$

and so

$$\frac{C(F)}{m^2(F)} \geq \alpha + \beta .$$

Hence by Lemma 4.14 we have

$$\gamma = \text{glb}\{\frac{C(F)}{m^2(F)} : F \in \mathfrak{F}^+,\ F^* \subseteq Cl(F \setminus C)\} \geq \alpha + \beta .$$

Thus in both cases we have $\gamma \geq \min(1, \alpha+\beta)$.

5. TWO-DIMENSIONAL CONTINUOUS ANALOGUES OF THE LANDAU-SCHNIRELMANN INEQUALITY AND THE SCHUR INEQUALITY

In this chapter we obtain continuous analogues of the Landau-Schnirelmann and the Schur Inequalities in two dimensions.

5.1. The Landau-Schnirelmann Inequality

In this section we will use a method of Kvarda [8] to obtain the Landau-Schnirelmann Inequality.

Theorem 5.1. If A and B are open subsets of E_2 and $0 \in A \cap B$, then $\gamma \geq \alpha + \beta - \alpha\beta$.

Proof: It suffices to show that

$$(1) \quad \frac{C(F)}{m^2(F)} \geq \alpha + \beta - \alpha\beta$$

for all $F \in \mathcal{F}^+$. If $C(F) = m^2(F)$ for all $F \in \mathcal{F}^+$, then (1) holds since $(1-\alpha)(1-\beta) \geq 0$ implies $1 \geq \alpha + \beta - \alpha\beta$. Therefore we assume $C(F) < m^2(F)$. This implies $A(F) < m^2(F)$ and hence $F \setminus A \neq \emptyset$.

Since $A \cap F$ is the union of at most a countable number of open rectangles and $m^2(A \cap F) < \infty$, then given $\varepsilon > 0$ there exists a finite number of these rectangles, say $A_1, \ldots, A_k$, such that

$m^2((A \cap F) \sim \bigcup_{j=1}^{k} A_j) < \varepsilon$. We choose A_1 such that $0 = (0,0)$ is the minimal vertex of A_1. Let $A^\varepsilon = \bigcup_{j=1}^{k} A_j \cup (A \cap (E_2 \sim F))$. Then

(2) $A^\varepsilon(F) > A(F) - \varepsilon$.

Let $C^\varepsilon = A^\varepsilon + B$. Then $C^\varepsilon \subsetneq C$.

Through each vertex (a_1, a_2) of a rectangle of $A^\varepsilon \cap F$ construct the lines $x_1 = a_1$ and $x_2 = a_2$. These finite number of lines "partition" F into a finite number of rectangles. Let us call this partition $\mathcal{P}_1$. The interior of each of these rectangles lies either entirely in $A^\varepsilon \cap F$ or entirely in $F \sim A^\varepsilon$. The rectangles of $F \sim A^\varepsilon$ are closed rectangles where a closed line segment in the universe F is considered a rectangle. The rectangles in $F \sim A^\varepsilon$ are measure disjoint.

Let $H = F \sim A^\varepsilon$. Note that $H \neq \emptyset$. Let us denote the rectangles that are contained in H after the partition $\mathcal{P}_1$ by $H_1, \ldots, H_p$. For each $H_j \subsetneq H$ we proceed to determine a unique element $a^{(j)} \in F$.

For $H_j \subsetneq H$ let $(h^*_{j_1}, h^*_{j_2})$ be the minimal vertex of H_j with respect to "$\precsim$", the partial ordering defined in Definition 1.4. Let U_j be the set of vertices $(\bar{a}_1, \bar{a}_2)$ of rectangles of $A^\varepsilon \cap F$ for the partition $\mathcal{P}_1$ where $\bar{a}_k \leq h^*_{j_k}$ for $k = 1, 2$. Let the elements of F be ordered so that $(x_1, x_2) > (x'_1, x'_2)$ if

$x_1 > x_1'$ or if $x_1 = x_1'$ and $x_2 > x_2'$. This is a linear ordering instead of a partial ordering. Since U_j is a finite set it contains a largest vertex in this lexicographic ordering. Let $a^{(j)} = (a_1^{(j)}, a_2^{(j)})$ be this vector. Let $a^{(1)}, \ldots, a^{(s)}$ be all the distinct vectors determined in this manner. Let $L_i = \bigcup_{k \in \Delta_i} H_k$ where $a^{(i)}$ is the vertex associated with H_k if and only if $k \in \Delta_i$.

We now show that

(1) $L_i \subseteq H$ for $i = 1, \ldots, s$;

(2) the sets $L_i' = \{x - a^{(i)} : x \in L_i\}$ are fundamental for $i = 1, \ldots, s$;

(3) $m^2(L_i \cap L_j) = 0$ if $i \neq j$, $1 \leq i \leq s$, $1 \leq j \leq s$;

(4) $H = \bigcup_{i=1}^{s} L_i$.

That (1) and (4) are true follows from the definition of the L_i. To see that (3) is true, notice that for $i \neq j$ we have that $L_i \cap L_j$ consists of a line segment or is empty.

Lastly we must show $L_i' = \{x - a^{(i)} : x \in L_i\}$ is a fundamental set. To prove this consider a vector $y = (y_1, y_2)$ such that $x_j \geq y_j \geq a_j^{(i)}$ $(j = 1, 2)$ where $x = (x_1, x_2)$ is in L_i. Suppose $y \in L_k$. Since $x_j \geq y_j \geq a_j^{(i)}$ $(j = 1, 2)$, then $x_j \geq a_j^{(k)}$ and so $a^{(i)} \geq a^{(k)}$. Since $y_j \geq a_j^{(i)}$ $(j = 1, 2)$, then $a^{(k)} \geq a^{(i)}$. Therefore $a^{(k)} = a^{(i)}$ and so $k = i$. The only other possibility is for y to be in A^{ε}. However $a^{(i)} \notin A^{\varepsilon}$ and the rectangle with $a^{(i)}$ as minimal vertex with respect to "$\lesssim$" is a rectangle of H. Therefore $y \notin A^\varepsilon$ for if $y \in A^\varepsilon$ then the minimal vertex a^* of the rectangle of $A^\varepsilon \cap F$ of the partition $\mathcal{P}_1$ to which y belonged would be such that $a^* > a^{(i)}$ and the minimal vertex h^* of the rectangle of H to which x belongs would be such that $a^* \lesssim h^*$. However then $x \notin L_i$, a contradiction. Hence $y \in L_i$. Thus $y - a^{(i)} \in L_i'$. Therefore $x - a^{(i)} \in L_i'$ and $0 \lesssim y - a^{(i)} \lesssim x - a^{(i)}$ imply $y - a^{(i)} \in L_i'$. Therefore L_i' is fundamental since it is a bounded nonempty subset of E_2.

Let $b \in B \cap L_i'$. Then $a^{(i)} + b \in C^\varepsilon \cap L_i$. This is so since $a^{(i)}$ is a vertex of a rectangle of $A^\varepsilon \cap F$ and hence for any $\xi > 0$ there is a point $a = (a_1, a_2) \in A^\varepsilon$ such that the distance from $a^{(i)}$ to a is less than ξ. Since B is open there exists a real number $\xi > 0$ such that $b + \delta \in B$ whenever the distance from b to $b + \delta$ is less than ξ. Let $a + \delta = a^{(i)}$ where the distance from the origin to δ is less than ξ and $a \in A^\varepsilon$. Then $a^{(i)} + b = (a^{(i)} - \delta) + (b+\delta) \in A^\varepsilon + B = C^\varepsilon$. Also $a^{(i)} + b \in L_i$ since $b \in L_i'$. Hence $a^{(i)} + b \in C^\varepsilon \cap L_i$.

Therefore, we have

$$C^{\varepsilon}(F) \geq A^{\varepsilon}(F) + \sum_{i=1}^{s} C^{\varepsilon}(L_i)$$

$$\geq A^{\varepsilon}(F) + \sum_{i=1}^{s} m^2(a^{(i)}+(B \cap L_i'))$$

$$= A^{\varepsilon}(F) + \sum_{i=1}^{s} m^2(B \cap L_i')$$

$$\geq A^{\varepsilon}(F) + \beta \sum_{i=1}^{s} m^2(L_i')$$

$$= A^{\varepsilon}(F) + \beta \sum_{i=1}^{s} m^2(L_i)$$

$$= A^{\varepsilon}(F) + \beta m^2(H)$$

$$= A^{\varepsilon}(F) + \beta(m^2(F) - A^{\varepsilon}(F))$$

$$= (1-\beta)A^{\varepsilon}(F) + \beta m^2(F).$$

From (2) and the above inequality we obtain

$$C^{\varepsilon}(F) \geq (1-\beta)(A(F)-\varepsilon) + \beta m^2(F).$$

However $C(F) \geq C^{\varepsilon}(F)$. Hence

$$C(F) \geq (1-\beta)(A(F)-\varepsilon) + \beta m^2(F).$$

Since ε is an arbitrary positive real number, we have

$$C(F) \geq (1-\beta)A(F) + \beta m^2(F)$$
$$\geq (1-\beta)\alpha m^2(F) + \beta m^2(F),$$

and hence

$$\frac{C(F)}{m^2(F)} \geq \alpha + \beta - \alpha\beta .$$

5.2. The Schur Inequality

In this section we use a method of Freedman [5] to obtain the Schur Inequality. We recall that in Section 4.10 we defined conditions P and Q. In this section we employ these conditions again.

Theorem 5.2. Let A and B be open subsets of E_2 such that A satisfies either condition P or condition Q, and suppose $0 \in A \cap B$. If $F \in \mathfrak{F}$, $F \sim C \neq \phi$, and $F^* \subseteq Cl(F \sim C)$, then $C(F) \geq \alpha C(F) + B(F)$.

Proof: Since B is open and $m^2(B \cap F) < \infty$, then given any $\varepsilon > 0$ we can find a finite number of open rectangles of $B \cap F$, say $B_1, \ldots, B_m$, such that $m^2((B \cap F) \sim \bigcup_{j=1}^{m} B_j) < \varepsilon$. Let $B^\varepsilon = (\bigcup_{j=1}^{m} B_j) \cup (B \cap (E_2 \sim F))$. Then

(1) $B^\varepsilon(F) > B(F) - \varepsilon$.

If A satisfies condition P let A^ε be defined as in Lemma 4.11. If A satisfies condition Q let A^ε be defined as in Lemma 4.12. In either case $A^\varepsilon \cap F$ is a finite union of open rectangles. Let $C^\varepsilon = A^\varepsilon + B^\varepsilon$. Then since $C^\varepsilon \subseteq C$ we have $F^* \subseteq Cl(F \sim C^\varepsilon)$. Since $A^\varepsilon \cap F$ and $B^\varepsilon \cap F$ consist of a finite number of open rectangles and since the sum of two open rectangles is an open rectangle we have that $C^\varepsilon \cap F$ consists of a finite number of open rectangles. Since $F^* \subseteq Cl(F \setminus C^\varepsilon)$ then every rectangle of $C^\varepsilon \cap F$ has a maximal vertex with respect to the partial ordering "$\precsim$".

Through every vertex (c_1, c_2) of a rectangle in $C^\varepsilon \cap F$ construct the lines $x_1 = c_1$ and $x_2 = c_2$. Since there are only a finite number of such vertices then there are only a finite number of such lines constructed. These lines partition $F \setminus T$ into a finite number of disjoint open rectangles relative to F where T is the finite union of the line we constructed. We call this partition $\wp_2$. Each of these open rectangles of $\wp_2$ is entirely in C^ε or entirely in $F \setminus C^\varepsilon$.

Let V be the set of all g^* such that g^* is a vertex of a rectangle of $\wp_2$ in $C^\varepsilon \cap F$. Since this set is finite it contains finitely many elements with respect to the partial ordering "$\precsim$". Let $g_1^*, \ldots, g_p^*$ be the maximal elements of V with respect to "$\precsim$". Then

$$C^{\varepsilon} \cap F \subseteq \bigcup_{i=1}^{p} L(g_i^*)$$

and letting

$$F' = \bigcup_{i=1}^{p} L(g_i^*)$$

we have

$$C^{\varepsilon}(F) = C^{\varepsilon}(F')$$

and F' is a fundamental set. If g_i^* is a maximal element of V, then g_i^* is also a vertex of a rectangle $F \smallsetminus C^{\varepsilon}$. Hence g_i^* is a limit point of $E_2 \smallsetminus C^{\varepsilon}$. Since $E_2 \smallsetminus C^{\varepsilon}$ is closed we must have $g_i^* \in E_2 \smallsetminus C^{\varepsilon}$, and hence $g_i^* \notin C^{\varepsilon}$.

From the construction of F' we see that every rectangle of $\mathcal{P}_2$ of $F' \smallsetminus C^{\varepsilon}$ must have a maximal vertex. There are only a finite number of rectangles (possibly a line segment on boundary of F') in $F' \smallsetminus C^{\varepsilon}$. Let us denote them by $D_1, \ldots, D_r$. Let H be the set of vertices of the rectangles D_j, $1 \leq j \leq r$. Let $J = H \cup \{g_1^*, \ldots, g_p^*\}$. Then J is finite.

Let us write $J = \{\ell_1, \ldots, \ell_s\}$ where the indexing is determined as follows. Let ℓ_1 be any one of the minimal vertices in J with respect to the partial ordering "$\precsim$". Let ℓ_2 be in the set of minimal elements of $J \smallsetminus \{\ell_1\}$ with respect to "$\precsim$", let ℓ_3 be in the set of minimal elements of $J \smallsetminus \{\ell_1, \ell_2\}$, and so on. Since J is finite the set of ℓ_i obtained by this process exhausts J. We now show that the resulting indexing satisfies the condition that $\ell_i \prec \ell_j$ implies $i < j$. Let $\ell_i \prec \ell_j$. Since $\ell_i \neq \ell_j$ we have either $i < j$ or $j < i$. If $j < i$, then ℓ_i is in $J \setminus \{\ell_1, \ldots, \ell_{j-1}\}$. However ℓ_j is in the set of minimal elements of $J \setminus \{\ell_1, \ldots, \ell_{j-1}\}$. Thus $\ell_i \prec \ell_j$ is impossible. Hence we have $i < j$.

Define

$$H_1 = L(\ell_1)$$

and

$$H_{i+1} = L(\ell_{i+1}) \setminus \bigcup_{j=1}^{i} H_j$$

for $1 \leq i < s$. Then

(2) the sets H_i, $1 \leq i \leq s$, are pairwise disjoint;

(3) $\bigcup_{i=1}^{s} H_i = F'$;

(4) if ℓ_k is a maximal vertex of a rectangle D_j, $1 \leq j \leq r$, of $F' \setminus C^\varepsilon$, then $H_k = D_j$ almost everywhere.

Now property (2) follows from the definition of the sets H_i and property (3) follows since $\bigcup_{i=1}^{s} H_i = \bigcup_{i=1}^{s} L(\ell_i) = F'$. To prove property (4) notice that ℓ_k is the maximal vertex of D_j, then D_j equals almost everywhere $L(\ell_k) \setminus \cup L(\ell_g)$ where $\ell_g \prec \ell_k$. However $\ell_g \prec \ell_k$ implies $g < k$. Suppose $j < k$ but $\ell_j \not\prec \ell_k$. Then we still know $\ell_k \not\prec \ell_j$. Hence $L(\ell_k) \setminus \cup L(\ell_g)$ where $\ell_g \prec \ell_k$ equals $L(\ell_k) \setminus \bigcup_{j<k} L(\ell_j) = H_k$.

For each i, $1 \leq i \leq s$, let

$$tH_i = \{\ell_i - x : x \in H_i\}.$$

Then

(5) tH_i is a fundamental set,

and

(6) $m^2(tH_i) = m^2(H_i)$.

To show property (5) is true, let z be an arbitrary element in tH_i and let $y \in L(z)$. We must show $y \in tH_i$. We have $0 \lesssim \ell_i - z \lesssim \ell_i - y \lesssim \ell_i$. Thus $\ell_i - y \in L(\ell_i)$. Since $z \in tH_i$ we have $z = \ell_i - h$ for some $h \in H_i$ and so $\ell_i - z = h \in H_i$. We must have $\ell_i - y \in H_i$, for if $\ell_i - y \notin H_i$, then $\ell_i - y \in \bigcup_{j=1}^{i-1} H_j$. However then $\ell_i - z \in \bigcup_{j=1}^{i-1} H_j$ since $0 \lesssim \ell_i - z \lesssim \ell_i - y$, and hence $\ell_i - z \notin L(\ell_i) \smallsetminus \bigcup_{j=1}^{i-1} H_j = H_i$, a contradiction. Thus $y = \ell_i - (\ell_i - y) \in tH_i$. Therefore tH_i is a fundamental set since it is a nonempty bounded subset of E_2.

Equation (6) follows from Theorem 2.16.

Now for each $a \in A^\varepsilon \cap tH_i$ we have $a = \ell_i - x$ where $x \in H_i$. Furthermore $x \notin B^\varepsilon$ for otherwise $\ell_i = a + x \in A^\varepsilon + B^\varepsilon = C^\varepsilon$ which contradicts the fact that $\ell_i \notin C^\varepsilon$. Therefore $A^\varepsilon \cap tH_i \subseteq \{\ell_i - x : x \in H_i \smallsetminus B^\varepsilon\}$. Hence

$$m^2(A^\varepsilon \cap tH_i) \leq m^2(\{\ell_i - x : x \in H_i \smallsetminus B^\varepsilon\}).$$

However by Theorem 2.16 we have

$$m^2(\{\ell_i - x : x \in H_i \setminus B^\varepsilon\}) = m^2(H_i \setminus B^\varepsilon).$$

Thus $m^2(A^\varepsilon \cap tH_i) \leq m^2(H_i \setminus B^\varepsilon)$. Hence we have

$$\begin{aligned} m^2(H_i \setminus B^\varepsilon) &\geq m^2(A^\varepsilon \cap tH_i) \\ &\geq \alpha_\varepsilon m^2(tH_i) \\ &\geq \alpha_\varepsilon m^2(H_i) \end{aligned}$$

from (5) and (6). Therefore

$$m^2(H_i) - m^2(B^\varepsilon \cap H_i) \geq \alpha_\varepsilon m^2(H_i)$$

which implies

(7) $$(1-\alpha_\varepsilon) m^2(H_i) \geq m^2(B^\varepsilon \cap H_i).$$

Let W be the set of all ℓ_k such that ℓ_k is the maximal vertex of one of the rectangles D_j, $1 \leq j \leq r$. Then if $\ell_k \in W$ we have $H_k = D_j$ a.e. for some j such that $1 \leq j \leq s$ by (4). Hence $B^\varepsilon(H_k) = 0$ since $B^\varepsilon \subseteq C^\varepsilon$. Therefore from (2) and (7) we have

$$(1-\alpha_\varepsilon) \sum_{\ell_i \in J \setminus W} m^2(H_i) \geq \sum_{\ell_i \in J \setminus W} m^2(B^\varepsilon \cap H_i).$$

Since $\ell_k \in W$ implies $B(H_k) = 0$ we have

$$\sum_{\ell_i \in J \setminus W} m^2(B^\varepsilon \cap H_i) = \sum_{\ell_i \in J} m^2(B^\varepsilon \cap H_i)$$

$$= B^\varepsilon(F')$$

Hence

$$(1-a_\varepsilon) \sum_{\ell_i \in J \setminus W} m^2(H_i) \geq B^\varepsilon(F').$$

However,

$$m^2\Big(\bigcup_{\ell_i \in J \setminus W} H_i\Big) = m^2\Big(F' \setminus \bigcup_{\ell_i \in W} H_i\Big)$$

$$= m^2\Big(F' \setminus \bigcup_{j=1}^{r} D_j\Big)$$

by (4). But we have $m^2(F' \setminus \bigcup_{j=1}^{r} D_j) = C^\varepsilon(F') = C^\varepsilon(F)$. Therefore

$$(1-a_\varepsilon)C^\varepsilon(F) \geq (1-a_\varepsilon)m^2\Big(\bigcup_{\ell_i \in J \setminus W} H_i\Big)$$

$$= (1-a_\varepsilon) \sum_{\ell_i \in J \setminus W} m^2(H_i)$$

$$\geq \sum_{\ell_i \in J \setminus W} m^2(H_i)$$

$$= B^\varepsilon(F')$$

$$= B^\varepsilon(F).$$

However $C^\varepsilon \subseteq C$. Hence

$$(1-a_\varepsilon)C(F) \geq B^\varepsilon(F)$$

and by (1) we have

$$(1-a_\varepsilon)C(F) \geq B(F) - \varepsilon.$$

Since A satisfies either condition P or condition Q, then when we let ε tend to zero we obtain

$$(1-a)C(F) \geq B(F),$$

or equivalently,

$$C(F) \geq aC(F) + B(F).$$

<u>Theorem 5.3.</u> If A and B are open subsets of E_2 such that A satisfies condition P or condition Q, $0 \in A \cap B$, and $a + \beta < 1$, then $\gamma \geq \frac{\beta}{1-a}$.

Proof: Since $\beta < 1 - a$, then $\frac{\beta}{1-a} < 1$. Hence if $\gamma = 1$ we are done. If $\gamma \neq 1$, then $E_2 \smallsetminus C \neq \emptyset$. Now for a set $F \in \mathfrak{F}^+$ such that $F \smallsetminus C \neq \emptyset$ and $F^* \subseteq Cl(F \smallsetminus C)$ we have from Theorem 5.2 that

$$C(F) \geq aC(F) + B(F).$$

Hence

$$\frac{C(F)}{m^2(F)} \geq a\frac{C(F)}{m^2(F)} + \frac{B(F)}{m^2(F)}$$

$$\geq a\gamma + \beta.$$

By Lemma 4.14 we have $\gamma \geq \alpha\gamma + \beta$. Hence $\gamma \geq \frac{\beta}{1-\alpha}$ since $1 - \alpha > \beta \geq 0$.

5.3. Other Continuous Analogues

In this section we will define some terms and prove some results that correspond to the material in Section 3.5 up to Theorem 3.19.

Let $E_2^+ = E_2 \setminus (\{(0, x_2): x_2 \geq 0\} \cup \{(x_1, 0): x_1 \geq 0\})$. Then E_2^+ is all of E_2 except the x_1-axis and the x_2-axis.

Theorem 5.4. If A and B are measurable subsets of E_2, $0 \in A \cap B$, and $\alpha + \beta > 1$, then $E_2^+ \subseteq C$.

Proof: Assume there exists $r = (r_1, r_2) \in E_2^+ \setminus C$. Let $F = L(r)$. Then $F \in \mathfrak{F}^+$ since $r_1 r_2 > 0$. Let $H = A \cap F$ and $G = \{r - b : b \in B \cap F\}$. Then $H \cap G = \emptyset$. This is so since suppose $r - b \in G \cap H$. Then $r = (r-b) + b \in A + B = C$, a contradiction. Hence $m^2(H) + m^2(G) = m^2(H \cup G) \leq m^2(F)$. However $m^2(H) = m^2(A \cap F)$ and $m^2(G) = m^2(B \cap F)$ by Theorem 2.16. Therefore

$$m^2(A \cap F) + m^2(B \cap F) \leq m^2(F).$$

This implies

$$\alpha + \beta \leq \frac{m^2(A \cap F)}{m^2(F)} + \frac{m^2(B \cap F)}{m^2(F)} \leq 1$$

But this contradicts $\alpha + \beta > 1$. Hence if $r \in E_2^+$, then $r \in C$ and it follows that $E_2^+ \subseteq C$.

We can not prove that $\alpha + \beta > 1$ implies $E_2 = C$ as can be seen by the following example.

Example 5.1. Let $A = B = \{(0,0)\} \cup E_2^+$. Then $\alpha + \beta = 2 > 1$. We see that any element on either axis with one positive coordinate is not in $A + B$.

Definition 5.1. Let $n \geq 1$ and $A_1, \ldots, A_n$ be subsets of E_2 such that $0 = (0,0) \in \bigcap_{j=1}^{n} A_j$. Then

$$A_1 + \ldots + A_n = \{a_1 + \ldots + a_n : a_j \in A_j, \ j = 1, \ldots, n\}.$$

Lemma 5.5. If $A_1, \ldots, A_n$ are open subsets of E_2 such that $0 \in \bigcap_{j=1}^{n} A_j$, if $d(A_1 + \ldots + A_n)$ is the density of $A_1 + \ldots + A_n$, and if α_i is the density of A_i $(i = 1, \ldots, n)$, then
$1 - d(A_1 + \ldots + A_n) \leq (1-\alpha_1) \ldots (1-\alpha_n)$.

Proof: The proof follows in the same way as that of Lemma 3.18.

Definition 5.2. Let k be a positive integer. We call A a basic set of E_2^+ of order k if $E_2^+ \subseteq A_1 + \ldots + A_k$ where each $A_j = A$, $j = 1, \ldots, k$, and k is minimal.

<u>Theorem 5.6.</u> If $0 \in A$, A is open, and $d(A) > 0$, then A is a basic set of E_2^+.

Proof: The proof follows in the same way as that of Theorem 3.19.

6. FURTHER QUESTIONS

In this chapter we present further questions of interest that are closely related to the topics in this thesis. We also discuss possible improvements of the results we have obtained for two dimensions. Finally we will indicate problems we encountered when attempting to extend our results to n-dimensions.

6.1. One-Dimensional Problems

In this thesis we have used a method of Kvarda [7] to obtain a continuous analogue of Mann's Second Theorem. In the continuous setting we have used this theorem to obtain a continuous analogue of Mann's $\alpha\beta$ Theorem. However there are three other methods for obtaining Mann's Second Theorem, namely one due to Mann [13], one due to Besicovitch [2], and another by Kvarda [7] which is found in the same reference as her first listed method. We have not attempted to see whether or not these methods can be used to obtain the continuous analogue of Mann's Second Theorem.

There are three known methods to obtain Mann's $\alpha\beta$ Theorem. These are due to Mann [12], Dyson [3], and Kvarda [6]. One could attempt to use these methods to obtain a continuous $\alpha\beta$ Theorem directly rather than as a corollary to Mann's Second Theorem in the continuous setting. The author attempted to use Dyson's method to do this, but failed.

The reason the author used Kvarda's first method for proving the continuous analogue of Mann's Second Theorem is that it is the only known method to have been extended to n-dimensions. In Chapter 4 the author used Kvarda's extension to n-dimensions and Stalley's [18] refinement of it to obtain a continuous analogue of Mann's Second Theorem in two dimensions.

Another one-dimensional problem that has not been settled is as follows. If A and B are measurable sets of E_1, $0 \in A \cap B$, and $C = A + B$, then does it follow that C is measurable? If we could answer this question in the affirmative we could extend Theorem 3.19 to measurable sets. Proving Theorem 3.19 for measurable sets without knowing the sum of two measurable sets is measurable is yet another problem.

6.2. Two-Dimensional Problems

In Lemma 4.11 and Lemma 4.12 we showed that if A is an open set that satisfies an additional restriction, then given any fundamental set F and any $\varepsilon > 0$ we can construct a set $A^{\varepsilon} \subseteq A$ such that the following three conditions hold:

(1) $m^2((A \cap F) \setminus (A^{\varepsilon} \cap F)) < K\varepsilon$ for a positive constant K;

(2) $A^{\varepsilon} \cap F$ is a finite union of open rectangles;

(3) $\lim_{\varepsilon \to 0} \alpha_\varepsilon = \alpha$.

The reason for the additional restriction on A is to ensure that condition (3) holds. There do exist open sets A for which conditions (1) and (2) hold, but condition (3) fails. Below we give such an example.

<u>Example 6.1.</u> Let $A = E_2 \smallsetminus (\bigcup_{j=0}^{\infty} V_j \cup H_1 \cup H_2)$ where $V_0 = \{(x,y): 0 \leq y \leq 1, x = 1\}$, $V_j = \{(x,y): 1 \leq y \leq 2,\ x = \frac{1}{j}\}$ where $j = 1,2,\ldots$, $H_j = \{(x,y): 0 \leq x \leq 1, y = j,\ j = 1,2\}$. Thus A consists of the open unit square with minimal vertex at the origin and a countable number of open rectangles contained in the unit square with minimal vertex of $(0,1)$; each rectangle in this square being of the form $\{(x,y) : \frac{1}{j+1} < x < \frac{1}{j},\ 1 < y < 2\}$ where j is a positive integer, and everything in E_2 that is in the exterior of the closed rectangle with vertices $(0,0)$, $(1,0)$, $(0,2)$ and $(1,2)$. Then $m^2(A \cap F) = m^2(F)$ for all $F \in \mathcal{F}$ since $E_2 \smallsetminus A$ has measure zero. Hence $\alpha = 1$.

Let F be the fundamental set which is a square with sides of length 2. Then $A \cap F$ is a countable union of disjoint open rectangles. For any $\varepsilon > 0$ we can choose a finite number of these rectangles, say $A_1, A_2, \ldots, A_k$, such $m^2((A \cap F) \smallsetminus \bigcup_{j=1}^{k} A_j) < \varepsilon$. Let $A^\varepsilon = (\bigcup_{j=1}^{k} A_j \cup (E_2 \smallsetminus F)) = (\bigcup_{j=1}^{k} A_j \cup (A \cap (E_2 \smallsetminus F)))$. Then since A^ε contains only a finite number of these rectangles, there must exist $\delta > 0$ such that $A^\varepsilon \cap \{(x,y): 0 < x < \delta, 1 < y < 2\}$ is empty. Let G be the fundamental set $\{(x,y): 0 \leq x < \delta, 0 \leq y \leq 2\}$. Then $\frac{A(G)}{m^2(G)} = \frac{\delta}{2\delta} = \frac{1}{2}$. Hence $\alpha_\varepsilon \leq \frac{1}{2}$. Thus, since $\alpha_\varepsilon \leq \frac{1}{2}$ for all $\varepsilon > 0$, then $\lim_{\varepsilon \to 0} \alpha_\varepsilon \leq \frac{1}{2}$. Therefore $\lim_{\varepsilon \to 0} \alpha_\varepsilon \neq \alpha$.

A closer study of the above example shows that $\alpha_\varepsilon = \frac{1}{2}$ whenever $0 < \varepsilon \leq 1$.

Further research might give conditions that are weaker than either condition P or condition Q that still ensure that $\lim_{\varepsilon \to 0} \alpha_\varepsilon = \alpha$. Such conditions would improve the results of Chapter 4.

Another interesting problem in two dimensions is the extension of the Landau-Schnirelmann Inequality to closed sets or even measureable sets. A related problem in two dimensions is the extension of Theorem 5.6 to closed or even to measurable sets.

6.3. n-Dimensional Problems

The author is not able to extend the results of Chapter 4 to $n \geq 3$ dimensions. The main difficulty is that we are unable to construct a partition of rectangles as fine as we did in Chapter 4 for two dimensions. This problem arises as follows. In two dimensions if (u_1, u_2) is a vertex of a certain rectangle contained in a fundamental set F we construct the line $x_1 = u_1$. If (u_1, w_2) is the point where the line $x_1 = u_1$ intersects the boundary of F we construct the line $x_2 = w_2$. The analogous construction can not be carried out in three dimensions. Suppose (u_1, u_2, u_3) is a vertex of a corresponding three-dimensional "fundamental" set F. Proceeding in an analogous manner to that for two dimensions we would construct the plane $x_1 = u_1$. However in three dimensions the plane $x_1 = u_1$ will usually intersect the boundary of F in an arc. There is nothing we can now construct that will correspond to the line $x_2 = w_2$ that we constructed in the two-dimensional case.

Another problem that the author can not solve is the extension of Condition Q to three dimensions.

A further problem is that of showing that a fundamental set in n dimensions is a measurable set for n-dimensional Lebesgue measure. If we solve this problem it appears that we can extend the Landau-Schnirelmann Inequality to n-dimensions for open sets and, if set A satisfies condition P, the Schur Inequality to n dimensions for open sets.

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