AN ABSTRACT OF THE DISSERTATION OF

Allison J. Dorko for the degree of Doctor of Philosophy in Mathematics Education presented on June 1, 2017.

Title: Students’ Generalisation of Function from Single- to Multivariable Settings.

Abstract approved:

_________________________________

Elise Lockwood

Generalisation is a key component of mathematical activity. Mathematicians often seek general formulae or wonder if a rule that holds in a particular dimension also holds in dimension \( n \). However, generalisation is not limited to professional mathematicians; kindergarteners engage in generalisation when they seek patterns and algebra is sometimes described as generalised arithmetic. Because generalisation is so critical to mathematical thought, research that investigates how people generalise is an important part of supporting student learning. In particular, students often struggle to form normatively correct generalisations (e.g., Dorko & Weber, 2014; Jones & Dorko, 2015; Kabael, 2011; Martínez-Planell & Gaisman, 2013, 2012; Martínez-Planell & Trigueros, 2012). Research can help us better understand what ways of thinking are productive for generalising and help us understand the nature of generalisation as a practise. To that end, this dissertation study focuses on how students generalise their notion of function from the single- to multivariable setting. I focus on function because functions are a fundamental mathematical concept and are relevant in everyday life.
The first manuscript describes results from a longitudinal study regarding how five calculus students generalised what it means for a relation to be a function from the single-to multivariable setting. In keeping with the use of the term ‘generalisation’ to refer to both a product (e.g., a theorem) and a process (c.f. Harel & Tall, 1991; Mitchelmore, 2002), I describe both the mathematical ideas students generalised and the nature of their generalising activity. Findings indicate that students generalised mathematical ideas such as function-as-equation, function-as-pattern, the vertical line test (VLT), function notation, inputs and outputs, and univalence. Focusing on equations seemed to prevent students from developing a normative understanding of what it means to be a function in \( \mathbb{R}^3 \). In contrast, some students considered generalisations of the VLT (e.g., applying the VLT to traces, lines parallel to the \( y \) axis on an \( \mathbb{R}^3 \) graph) that, while not normatively correct, led to correct generalisations. Similarly, thinking about function notation and inputs and outputs supported students in forming correct generalisations. Findings about students’ generalising activity indicate that students engaged in what Ellis (2007) terms relating, searching, and extending. For example, some students made sense of the notation \( f(x, y) \) by relating it to the notation \( f(x) \). One student engaged in searching as she sought to generalise the vertical line test. She searched for a way to draw a line on an \( \mathbb{R}^3 \) graph that would intersect the graph exactly once. Other students extended the range of applicability of prior ideas (e.g., input and output) and their definitions of function.

The second manuscript is more theoretical in nature. In this manuscript, I argue that Piaget’s constructs of assimilation and accommodation align with Harel and Tall’s (1991) framework for generalisation in advanced mathematics. Based on what they imagined to be the cognitive processes underlying generalisation, Harel and Tall proposed that generalisation might be expansive (occurring when a student expands the applicability range of an existing schema without reconstructing it), reconstructive (occurring when a student reconstructs a schema to widen its range of applicability), or disjunctive (occurring when a student constructs a new, disjoint schema to deal with a new context). I employ this framework to interpret data about how students generalise their notion of function from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) and how they first think about graphing in \( \mathbb{R}^3 \). I then interpret the same instances through the lens of assimilation and accommodation, arguing that they provide explanatory mechanisms for generalisation. I conclude by discussing how
assimilation and accommodation explain other empirical findings regarding students’ generalisation of function and graphing.

Taken together, the two manuscripts provide different, complementary insight into the generalisation phenomenon. The first manuscript employs a framework that describes generalisation at a small grain size, while the second focuses on describing generalisation from a much larger grain size. While the first manuscript describes generalisation empirically, the second seeks to contribute to a theoretical explanation for how generalisation occurs. Notably, these investigations occur among a population (undergraduate students) and content area (single and multivariable calculus) in which generalisation has not typically been studied. As a whole, the dissertation extends the existing body of literature by both empirically and theoretically examining the phenomenon of generalisation with undergraduate students in a relatively advanced mathematical setting.
Students’ Generalisation of Function from Single- to Multivariable Settings

by

Allison J. Dorko

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Major Professor, representing Mathematics Education

Dean of the College of Education

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Allison J. Dorko, Author
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AUTHOR’S NOTE

This dissertation is in fond memory of my friend and colleague John D. Ivanovitch. If you are a graduate student reading this, know that graduate school is hard but it should not feel unbearable. If you have thoughts of suicide, please reach out to someone who can help or call the national suicide hotline number (U.S.) at 1-800-273-8255.
DEDICATION

For my full-fridge-family, pick-you-up-in-Boston parents
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Chapter 1: Introduction

1. Background

Reports from the National Academy of Science indicate a need for increasing the number of science, technology, engineering, and mathematics (STEM) graduates in the US (National Academies, 2007). Calculus is often a barrier for would-be STEM majors, with even well-prepared students leaving the calculus sequence after only a single semester (Bressoud, Carlson, Mesa, & Rasmussen, 2013; Steen, 1988; Wake, 2011). Researchers have sought to address this problem in a number of ways. Some research explores students’ calculus experience, including students’ perceptions of pedagogy (Ellis, Kelton, & Rasmussen, 2014), the tension between amount of material to be covered and student success (Johnson, Ellis, & Rasmussen, 2014), and the relationship between the nature of homework and student success (Ellis, Hanson, Nuñez, & Rasmussen, 2015). A second line of research posits that university teaching may be a major factor in students leaving calculus, and that better examining teaching practices is key to improving student outcomes (Ellis et al., 2014; Speer, Smith, & Horvath, 2010). Still other researchers focus on how students understand specific topics in calculus (Orton, 1983a; Orton, 1983b; White & Mitchelmore, 1996; Zandieh, 2000), attacking the retention problem from the perspective that learning how students think about a concept will lead to better instruction and hence increased student learning, success, and retention. Such research has spurred changes in curriculum and instruction (e.g., Ferrini-Mundy & Graham, 1991; Hughes-Hallett, Lonzano, Gleason, Flath, & Frazer Lock, 2012). Collectively, this research contributes to improving the teaching and learning of calculus at the undergraduate level.

Most of the research about calculus learning has focused on differential calculus. Given that differential calculus turns so many students away from STEM careers, this emphasis makes sense. However, students who succeed in differential calculus must complete more calculus courses to remain on their STEM trajectories. For example,
engineers, chemists and computer scientists are required to take multivariable calculus.

Multivariable calculus is typically the first time that students encounter the multivariable functions that describe important concepts in physics and other sciences, engineering, statistics, and more advanced mathematics. Being able to reason about multivariable ideas is so important that the Mathematical Association of America recently recommended that the high school mathematics curriculum should include multivariable topics (Ganter & Haver, 2011; Shaughnessy, 2011). This recommendation came as the result of the Curriculum Renewal Across the First Two Years report (Ganter & Haver, 2011), which examined the mathematical needs of disciplines such as biology, chemistry, economics, engineering, and physics. If the future high school curriculum may include multivariable topics, research about how students understand such ideas becomes all the more important because it has the potential to affect the learning of a very large population. Further, the small body of research that does exist regarding student understanding of multivariable calculus documents that multivariable topics are difficult for students (e.g., Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012; McGee & Martínez-Planell, 2014; Yerushalmy, 1997). The importance of multivariable topics and the known difficulties students have with them point to a need for more research in this area.

Because many multivariable calculus topics are generalisations from single-variable contexts, research about students’ understanding of multivariable calculus connects to the body of research regarding how students generalise in mathematics. Beyond wanting to better understand how students understand multivariable calculus topics, my study provides an opportunity to learn more about generalisation as a mathematical practice. Generalisation is a hallmark of mathematical thinking (Dörfler, 1991; Ellis, 2007; Harel & Tall, 1991; Mitchelmore, 2002). For example, mathematicians often seek general formulae (e.g., the sum of the first \( n \) terms of a geometric series) or wonder if a rule that holds in a particular dimension also holds in dimension \( n \) (e.g., the Soddy-Gosset theorem generalises Descartes’ theorem for four mutually tangent circles to \( n \) dimensions). However, generalisation is not limited to professional mathematicians; kindergarteners engage in generalisation when they
seek patterns and algebra students learn how to write general rules for linear, exponential, and quadratic patterns. Because generalisation is so critical to mathematical thought, research that investigates how people generalise is an important part of supporting student learning. The abundance of studies about generalisation in contexts like angle (e.g., Mitchelmore, 2002), algebra (e.g., Ellis, 2007), combinatorics (e.g., Sriraman, 2003) and multivariable topics (e.g., Dorko & Weber, 2014; Kabael, 2011; Martínez-Planell & Gaisman, 2013, 2012; Martínez-Planell & Trigueros, 2012; McGee & Martínez-Planell, 2014) indicate the importance of generalisation to the mathematics education community.

To that end, this dissertation reports on a longitudinal study of how calculus students generalise their notion of function across differential, integral, and multivariable calculus. It addresses the following research questions:

1. In determining whether a multivariable relation represents a function, what mathematical ideas do students generalise from their understanding of what it means for a single-variable relation to represent a function?

2. What is the nature of students’ generalising activity as they generalise their understanding of what it means for a single-variable relation to represent a function to what it means for a multivariable relation to represent a function?

3. In what ways do the constructs of assimilation and accommodation relate to generalisation?

To address these research questions, this dissertation study will be presented in two manuscripts. The first, *Is it a Function? Generalising from the Single- to Multivariable Case*, addresses the first two research questions by reporting on data from how five students thought about what it means for a two-variable graph or a table to represent a function, and how they leveraged these understandings to make sense of what it would mean for a three-variable graph or table to represent a function. I have written this manuscript with the intent to send it to Research in Mathematics Education (RME). Supervised by the British Society for Research into Learning Mathematics, this journal publishes research papers that report on studies “involving empirical investigation and theoretical argumentation from which conclusions and implications can be drawn for future research and practise” (Taylor & Francis, n.d.).
My paper is an empirical study that helps expand the use of an actor-oriented theoretical perspective to advanced mathematics. Additionally, I have published a paper in RME about how students generalise domain and range in RME (Dorko & Weber, 2014). My first manuscript uses the same theoretical and analytic framework as the published paper, and both the manuscript and Dorko and Weber (2014) discuss how students generalise facets of the function concept. Other papers in RME about students’ understanding of function (e.g., Ayalon, Watson, & Lerman, 2017) and students’ understanding of the distinction between function and relation (e.g., Spyrou & Zagorianakos, 2010) indicate that this is a topic of interest to RME readers. Hence I feel RME is an appropriate journal for my first manuscript because the paper serves to continue an ongoing conversation within the journal.

The second manuscript, Exploring Expansive, Reconstructive, and Disjunctive Generalisation, addresses question 3. This manuscript links Harel and Tall’s (1991) generalisation framework to Piaget’s assimilation and accommodation constructs (von Glasersfeld, 1995). Specifically, I argue that assimilation and accommodation provide an explanatory mechanism for Harel and Tall’s (1991) categories of how students generalise in advanced mathematics. I have written this manuscript with the intent of sending it to For the Learning of Mathematics (FLM), a journal published by the Canadian Mathematics Education Study Group. There are several reasons for this. First, FLM published the Harel and Tall (1991) framework and submitting this paper to FLM would continue a conversation about the framework in the journal in which that conversation started. FLM seems to value this; for example, there has been a conversation within the journal spanning 1999 to 2017 about what in mathematics is arbitrary and what is necessary (c.f. Hewitt, 1999; 2001a; 2001b; Kontorovich & Zazkis, 2017). Secondly, part of the scope of the journal is to provide

space for articles which attempt to bring together ideas from several sources and show their relation to the theories or practises of mathematics education. It is a place where ideas may be tried out and presented for discussion (For the Learning of Mathematics, n.d.; emphasis added)
The aim of ‘trying out ideas and presenting them for discussion’ well characterises the thesis statement of my paper, which is that assimilation and accommodation provide theoretical underpinnings for generalisation.

In the next section, I describe the research frame that guided this work.

2. Research frame

Crotty’s (1998) framework for research informed the ways in which I developed a research frame for this dissertation. Specifically, Crotty’s (1998) framework includes an epistemological stance, a theoretical perspective, a methodological stance, a description of particular methods, and how the four are linked. I also include an ontological position as specifying this is important to understanding my epistemological stance.

My research was informed by a radical constructivist epistemological perspective. Radical constructivists contend that knowledge is based on an individual learner’s own experiences and that an individual cognitively constructs the reality they experience. The goal of knowledge acquisition is adaptive function, meaning that the purpose of cognition is to help people live in the world that they experience (von Glasersfeld, 1995). That is, individuals construct knowledge to help them live in and interpret the world.

Ontologically, radical constructivists contend that we cannot know if an objective reality exists. von Glasersfeld (1998) writes that the word “radical” distinguishes this view from

the traditional way of thinking according to which all human knowledge ought to or can approach a more or less “true” representation of an independent existing, or ontological reality (p. 21).

That is, radical constructivism posits that each person constructs the reality around them, and this reality may or may not be a reflection of an objective reality. Because each person constructs the reality he or she experiences, radical constructivists contend that it is impossible to know if there exists a singular, objective reality. Radical constructivists explain how we function as a society and act as if there exists objective truth via the idea of intersubjectivity. von Glasersfeld (1995) describes that a person constructs other people as existing and compares one’s own experience with
one’s interpretation of others’ experiences. When one interprets one’s own experience as similar to that of others, one’s experiential reality gains viability. Repeated similar comparisons reinforce multiple subjects’ experiences as seeming real, and this “additional viability can be interpreted as indicating intersubjectivity and constitutes the constructivist substitute for reality” (von Glasersfeld, 1995, p. 128). That is, people are able to interpret each other’s experiential realities and find points of what they interpret as agreement, thereby allowing them to act as if “concepts, theories, beliefs, and other abstract structures” (von Glasersfeld, 1995, p. 128) exist in the world.

Radical constructivists posit that people organise their experiences of the world by developing schemes. A scheme is a meaning that helps a person organise his or her experience (Thompson, 2016; von Glasersfeld, 1995). Piaget (1967) writes, “all knowledge is tied to action, and knowing an object or an event is to use it by assimilating it to an action scheme” (p. 14-15). ‘Action’ can be physical or mental activity (Thompson, 2016; von Glasersfeld, 1995). In short, under radical constructivism, to know something is to have a scheme for it. While in other theories ‘knowledge’ and ‘knowing’ may have different meanings (c.f. Sfard, 1998), for Piaget they are the same (Thompson, 2016).

This view of what knowledge is influenced my study’s theoretical perspective. I employed an actor-oriented theoretical perspective (Ellis, 2007; Lobato, 2003) to study generalisation. In this perspective, learners are seen as sense-makers who form connections across situations. This is in accordance with the ontological stance that learners construct the reality they experience. In an actor-oriented perspective, the observer treats what (s)he interprets as normatively correct\(^1\) and non-normatively correct connections as equally valid. This is important because my first two research questions focus on what students generalise and the nature of their generalising activity. I am interested in both normatively correct and non-normatively correct generalisations because capturing both provides a deeper description of the ways students think about multivariable functions. That is, I needed a perspective that could

\(^1\) By ‘normatively correct,’ I mean what the community of mathematicians takes to be true. Mathematicians arrive at an agreement on what is true via intersubjectivity.
capture all salient elements of learners’ schemes, not just elements that the mathematical community deem normatively correct. As such, a theoretical perspective that de-emphasises normative correctness is appropriate. In the first manuscript, I coded and reported on generalisations based on what the students saw as general, whether or not such generalisations were normatively correct. Similarly, in unpacking the relationship between generalisation, assimilation, and accommodation (research question 3), it was important to choose a perspective which focused on what students’ schemes were and how they assimilated new situations to those schemes (or accommodated those schemes to account for new situations) without being restricted by normative correctness.

Because I sought to understand participants’ thinking and reasoning based on their experiences, I conducted a qualitative study. Qualitative research provides rich descriptions of phenomena. As such, it is appropriate for investigating research questions that, taken together, focus on describing the phenomenon of what and how people generalise.

Specifically, in the first manuscript, I employed grounded theory methodology and Ellis’ (2007) grounded-theory based generalisation taxonomy. Grounded theory is an inductive approach in which the researcher describes categories that emerge from the data during analysis. In this methodology, “the subjective world of informants is analysed to produce conceptual understanding specific to the data collected” (Green, Camilli, & Elmore, 2006, p. 360). I used grounded theory in my first manuscript to answer research question 1 regarding what students generalised. Additionally, I used Ellis’ (2007) generalisation taxonomy to answer question 2. This taxonomy was developed through grounded theory, and as such the methodological approach is coherent for two distinct but related questions. The first manuscript provides a detailed description of Ellis’ (2007) framework and how I used it to make sense of my data.

In the second manuscript, which is primarily theoretical in nature, I employ Harel and Tall’s (1991) framework for describing “the different qualities of generalisation in advanced mathematics” (p. 1). I was specifically interested in investigating connections between Harel and Tall’s (ibid) framework and Piaget’s constructs of
assimilation and accommodation. The second manuscript provides a detailed
description of how I coded data using both Harel and Tall’s framework and von
Glasersfeld (1995) and Steffe’s (1991) definitions of assimilation and accommodation.

The use of two different generalisation frameworks was a purposeful choice. The
two frameworks are of different grain sizes, and hence provide complementary
insight into the generalisation phenomenon. Additionally, Ellis (2007) draws on Harel
and Tall’s (1991) definition of generalisation. Finally, the frameworks can be seen as
cohesive in that both can be used in an actor-oriented perspective. Ellis’ (2007)
framework is actor-oriented by design, and she writes of Harel and Tall’s (1991)
framework, “viewed through the actor-oriented transfer lens, one could study students’
scheme expansions and reconstructions without being restricted by normative notions
of correctness” (p. 226).

In the next sections, I describe the methods of data collection and analysis.

2.1 Data collection and analysis

In this section, I provide an overview of the participant selection, data corpus, and
analytic frameworks. The two manuscripts provide further detail.

2.1.1. Sample population

The sample population for this study consisted of the set of all students enrolled in
differential calculus at a large, public university in the pacific northwest in fall 2015.
After receiving permission for the study from the university’s Institutional Review
Board, I emailed all instructors teaching differential calculus during that term and
requested permission to visit their classes to recruit participants. Twelve students
agreed to take part in the study. Due to issues of attrition during the year-long data
collection, five of the twelve completed the entire study.

2.1.2. Data corpus

The main data source for the study are task-based interviews (Hunting, 1997) with
the five students who completed the entire study. Recall that the research questions
focus on what and how students generalise. I follow other researchers in finding it
tenable to assume that people can verbally describe and/or write what goes on in their
heads. That is, I take the epistemological stance that knowledge is represented in
schemes that reside in peoples’ brains but since it is impossible for a researcher to
experience someone else’s scheme, I rely on their verbal and written responses to
tasks.

Prior to data collection, mathematics education experts at two universities
reviewed the tasks regarding their ability to provide insight into how students
generalise. Additionally, I collected pilot data from five students. The piloting was
solely to test the tasks and as such was not collected under Institutional Review Board
approval. The pilot data is not reported on in this dissertation.

I collected data at four instances over the course of one academic year. This
longitudinal design allowed me to capture both students’ initial, pre-instruction
generalisations and the sense they had made after instruction. Studying generalisation
before students receive instruction on a topic provides information about students’
initial sense-making about a topic and how they may (or may not) leverage their prior
knowledge to understand new ideas (e.g., Dorko & Weber, 2014; Yerushalmy, 1997).
This provides instructors with information about what connections students have that
instruction can build upon. Additionally, studying how students generalise a topic
before instruction ensures that the data gathered represent students’ initial sense-
making, as opposed to their recall of what a particular professor said. On the other
hand, collecting data after instruction allows researchers to see what sense students
have made of normative ideas (e.g., Kabael, 2011). This is useful because it tells us if
students learned what was intended, and how they connected the new information to
previous understandings. For example, Kabael (2011) documented students’ post-
instruction generalisations of function and found that the use of the function machine
as a model helped students generalise in normatively correct ways. I felt that being
able to observe both students’ initial sense-making and their post-instruction
understandings would be useful, and hence designed my study such that I asked
students questions about multivariable functions while they were enrolled in
differential calculus (that is, before instruction about such functions) and near the end
of their multivariable calculus course (at which point they had studied multivariable
functions in class).

The five student participants each completed four task-based interviews. I
recorded these interviews using video, audio, and LiveScribe technology. The latter
produces a synched recording of audio along with the student’s written work (much like video, but in the form of a PDF). I transcribed the interviews and used the Grab application on my Mac laptop to take screenshots of students’ work. I embedded these clips within the text so that I could view both during analysis.

2.1.3. Data analysis

Each paper draws on different, complementary frameworks for analysing data. In the first manuscript, I employed a grounded-theory analysis and Ellis’ (2007) generalisation taxonomy. In the second manuscript, I analysed data via Harel and Tall’s (1991) framework for describing students’ generalisation in undergraduate mathematics and von Glasersfeld (1995) and Steffe’s (1991) definitions of assimilation and accommodation. Descriptions of the specific methods are given in the individual manuscripts.

In the next section, I discuss the significance of my study.

3. Significance of the study

The significance of my study is best framed in terms of its two original motivations, which were to learn more about how students understand multivariable functions and to learn more about generalisation as a practise. Additionally, taken as a whole, the dissertation provides a comparison and contrast of two generalisation frameworks that the mathematics education community has found valuable (e.g., Dorko & Weber, 2014; Fisher, 2008; Greer & Harel, 1998; Jones & Dorko, 2015; Zazkis, & Liljedahl, 2002).

Per the first, this dissertation offers an empirical contribution regarding what students generalise about function from the single- to multivariable setting. Given that multivariable functions are so important to STEM fields and everyday life, knowing more about how students think about such functions can inform student learning about these functions, thereby improving students’ mathematical competence (Ganter & Haver, 2011; Shaughnessy, 2011). Studies about student thinking can be leveraged to improve instruction; for example, several groups of researchers in the multivariable calculus arena have used studies of student thinking to build instructional materials (e.g., Martinez-Planell & Trigueros, 2012).
Per the second, the dissertation makes a theoretical contribution regarding assimilation and accommodation as explanatory mechanisms for generalisation. Generalisation is a critical mathematical practise (e.g., Dörfler, 1991; Ellis, 2007) and there has been much work in the mathematics education community regarding what and how students generalise (e.g., Dorko & Weber, 2014; Ellis, 2007; Fisher, 2008; Harel & Tall, 1991; Jones & Dorko, 2015; Kabael, 2011). Understanding generalisation requires both empirical and theoretical work, and I contend linking generalisation to assimilation and accommodation situates it in language that is widely understood. This is of use to the field because it provides a possible explanation for why and how generalisation happens.

Finally, taken as a whole, this dissertation allows the reader to explore the affordances and limitations of two generalisation frameworks. In the first manuscript, I employ Ellis’ (2007) generalisation taxonomy to characterise how students generalised what it means for a table or graph to represent a function. This taxonomy allows a researcher to attend to the details of how students generalise, such as the objects students focus on (e.g., coordinate axes; Dorko & Weber, 2014), whether they implement or modify previously-learned rules, actions students might engage in while seeking patterns, and so on. The framework has 23 categories in total. In contrast, Harel and Tall’s (1991) framework, with three categories, has a much larger grain size. While Harel and Tall’s framework does not tell us the specific features of a situation a student attends to (as Ellis’ does), it allows us to see what ideas students see as straightforward extensions (expansive generalisations) of their current knowledge, which ideas cause them to adjust the ways in which they think about that idea (reconstructive generalisations), and which ideas they see as disconnected from prior knowledge (disjunctive generalisations/understandings; see Jones & Dorko, 2015). For example, both manuscripts describe how students made sense of \( f(x, y) \) notation based on their understanding of \( f(x) \) notation. Under Ellis’ (2007) framework, we learn how some students focused on notation as an object, attending to form and properties (e.g., the argument as representing an input). This is useful because it tells us the details of what students attend to (e.g., the argument as representing an input).
Under Harel and Tall’s (1991) framework, this sense-making is described as an expansive generalisation caused by an assimilation.
Chapter 2: Is it a Function? Generalising from Single- to Multivariable Settings

1. Abstract

A function is defined as a mapping from one nonempty set (the domain) to another nonempty set (the co-domain) such that each element of the domain maps to exactly one element in the co-domain. This paper reports on a longitudinal study of what five calculus students generalised as they reasoned about what it means for a relation to be a function from the single- to multivariable case, and what they attended to while generalising. Findings indicate that students generalised notions of function-as-equation, function-as-pattern, the vertical line test (VLT), function notation, inputs and outputs, and univalence. Focusing on equations seemed to prevent students from developing a normative understanding of what it means to be a function in $\mathbb{R}^3$. In contrast, some students considered generalisations of the VLT (e.g., applying the VLT to traces, lines parallel to the $y$ axis on an $\mathbb{R}^3$ graph) that, while not normatively correct, led to correct generalisations. Similarly, thinking about function notation and inputs and outputs supported students in forming correct generalisations. Findings about students’ generalising activity indicate that students engaged in what Ellis (2007) terms relating, searching, and extending. For example, some students made sense of the notation $f(x, y)$ by relating it to the notation $f(x)$. One student engaged in searching as she sought to generalise the vertical line test. She searched for a way to draw a line on an $\mathbb{R}^3$ graph that would intersect the graph exactly once. Other students extended the range of applicability of prior ideas (e.g., input and output) and their definitions of function. Overall, the findings indicate that most students generalised a conception of function consistent with a normative mathematical definition.

2. Introduction

Functions are fundamental to mathematics and are important in everyday life. Introduced in algebra, functions appear throughout calculus, real and complex analysis, transformational geometry, and many other areas of mathematics. For example, an electric bill is a function of kilowatt hours and the cost of a taxi is some
base rate plus a charge per mile. It follows that understanding function is critical for student success in mathematics and other STEM fields. While high school curricula have traditionally focused on functions of one variable, the Mathematical Association of America has proposed including multivariable topics in secondary mathematics curricula as a way to increase mathematical competence for all students (Ganter & Haver, 2011; Shaughnessy, 2011). Indeed, most real-world situations involve functions of more than one variable. The cost of a family movie outing, for instance, depends on the number of family members, ticket price, and how many bags of popcorn and cups of soda are purchased.

While we know that even young students can accurately model real-life situations of two variables (e.g., Yerushalmy, 1997, showed this among seventh graders), researchers are only beginning to explore students’ understanding of multivariable functions. Findings from the literature suggest that students struggle with facets of the multivariable function concept, including identifying domain and range (Dorko & Weber, 2014; Kabael, 2011; Martínez-Planell & Trigueros, 2012), drawing correct graphs (Dorko, 2016; Dorko & Lockwood, 2016; Martínez-Planell & Gaisman, 2012; Trigueros & Martínez-Planell, 2010), and working with free variables (Dorko, 2016; Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). These difficulties may prevent students from reasoning correctly about multivariable functions in applied settings such as physics, statistics, and engineering. Hence research regarding student thinking about multivariable functions can help inform instruction at the secondary and college levels.

Understanding how students think about multivariable functions necessarily includes how they determine what is and is not a function. Researchers have studied how students make such determinations for single-variable relations (e.g., Clement, 2001; Dubinsky & Wilson, 2013; Leinhardt, Zaslavsky, & Stein, 1990; Markovits, Eylon, & Bruckheimer, 1986; Montiel, Vidakovic, & Kabael, 2008; Williams, 1998), but we know less about how they determine whether a multivariable relation represents a function. Investigating this may also provide insight into students’ difficulties with domain and range and graphing. For example, if students do not have a mathematically correct understanding of what it means to be a multivariable
function, they may have trouble developing a correct understanding of domain and range or what a graph looks like.

In this paper, I use generalisation as a lens to explore how students make sense of what it means for a relation to be a function in a multivariable setting. This follows from evidence that when students think about multivariable topics, they generalise their understandings from single-variable settings (Dorko, 2016; Dorko & Weber, 2014; Fisher, 2008; Jones & Dorko, 2015; Kabael, 2011; Martínez-Planell & Gaisman, 2013; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). A focus on generalisation not only helps us understand how students make sense of what it means to be a function in $\mathbb{R}^3$, but also adds to the body of research regarding how undergraduate students generalise. Generalisation is a core component of mathematical activity (e.g., Dörfler, 1991; Ellis, 2007). However, most of what we know about how students generalise has come from studies of student generalisations about patterns and algebra. Because generalisation is a hallmark of mathematical thinking, it is important to extend our knowledge of how students generalise in mathematical contexts beyond patterns and algebra.

The term ‘generalisation’ is used to refer both to a product and a process (e.g., Mitchelmore, 2002). The mathematics community sometimes uses the word generalisation to mean a general rule or statement (e.g., a theorem) and at other times uses ‘generalisation’ to mean the activity of developing such a product. In this paper, I address both generalisation as a product and as a process (respectively) by focusing on the following research questions:

- In determining whether a multivariable relation represents a function, what mathematical ideas do students generalise from their understanding of what it means for a single-variable relation to represent a function?
- What is the nature of students’ generalising activity as they generalise their understanding of what it means for a single-variable relation to represent a function to what it means for a multivariable relation to represent a function?

To address these questions, I conducted a longitudinal study with five students across their differential, integral, and multivariable calculus courses. Each student completed four task-based, clinical interviews (Hunting, 1997) over the course
sequence. This paper focuses on students’ responses to tasks in which they were shown a relation and asked if it represented a function. The longitudinal design provided insight into how students thought about what it means to be a multivariable function before and after instruction about such functions. That is, students completed tasks about single- and multivariable relations both before and after instruction about multivariable functions. This afforded insight into students’ initial sense-making and the sense they made of normative ideas. Before providing more details about the study design, I review the definition of function and common characterisations of the function concept.

3. Mathematical discussion

3.1 Definitions and examples

In this section I provide the reader with definitions and examples of single- and multivariable functions. In doing so, I aim to help the reader consider how the function concept generalises to the multivariable case. This discussion should clarify how I am defining key mathematical terms and should help frame the subsequent presentation of student work.

A function\(^\text{2}\) is a mapping from one nonempty set (the domain) to another nonempty set (the co-domain) such that each element of the domain maps to exactly one element (called the image) in the co-domain. The condition that each element of the domain maps to exactly one element of the co-domain is termed \textit{univalence}\(^3\) (Dubinsky & Wilson, 2013; Even, 1993). The range of a function is the subset of the co-domain consisting of all images of elements of the domain under the mapping.

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\(^2\) Definitions in this section are adapted from Briggs and Cochran (2011), Dubinsky and Wilson (2013), Even (1993), and Rogawski (2008).

\(^3\) I follow Dubinsky and Wilson (2013) and Even (1993) in using the term ‘univalence’. The term ‘single-valued’ is synonymous, and is defined as having one and only one value of the range associated with each value of the domain (Merriam-Webster, 2017). The term ‘well-defined’ is synonymous with single-valued (Rotman, 1965). A function is well-defined if it gives the same output for different representations of an input value (Rotman, 1965). For instance, if a function is well-defined, \(f(0.25) = f(1/4)\).
For example, a common way to represent a single-variable function is as a mapping $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = y$. The use of $x$ and $y$ follows the standard use of a Cartesian coordinate system, in which $x$ is the independent variable and $y$ is the dependent variable. This notation specifies that the domain of $f$ is the set of real numbers and the co-domain of $f$ is the set of real numbers. Since $f$ is a function, each $x$ maps to exactly one value, denoted $f(x)$. The range of the function $f$ is the set of $y$ values for which $y = f(x)$ for some $x$ in the domain of $f$. The function $f$ is called a single-variable, real-valued function since its argument is specified by the value of one real number variable. As an example, $f(x) = x^2 - 2x$ is a single variable function with domain $x \in \mathbb{R}$ and range $y \geq -1$.

A common way to represent a multivariable function is as a mapping $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(x, y) = z$. Again, this follows the conventions of a standard Cartesian coordinate system. This notation specifies that the domain of $g$ is the set of points $(x, y)$ such that $x$ is a real number and $y$ is a real number. The co-domain of $g$ is the set of real numbers. Since $g$ is a function, each point $(x, y)$ maps to exactly one value, denoted $g(x, y)$. The range of the function $g$ is the set of $z$ values for which $z = g(x, y)$ for some $(x, y)$ in the domain of $g$. In this case, $g$ is called a multivariable real-valued function because each point in its domain is specified by more than one (real number) variable, and the image of each point is a real number value. As an example, $g(x, y) = x^2 + y^2$ is a multivariable function with domain $\{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$.

More generally, the term ‘multivariable real-valued function’ can refer to any function $h$ having a domain of points $(x_1, x_2, ..., x_n)$ where $n \geq 2$ and each $x_i$ is a real variable. The real number image of the point is denoted by $h(x_1, x_2, ..., x_n)$. However, in this paper when I say “multivariable function,” I will always mean a real-valued function of exactly two real variables (like $g$ above). Such functions can be represented graphically in $\mathbb{R}^3$ by plotting the collection of points $(x, y, z)$ for each $(x, y)$ in the domain. I chose to focus on multivariable function graphs in $\mathbb{R}^3$ for two reasons. First, much of multivariable calculus students’ work is in three-space and not more dimensions. Second, I want to better understand students’ generalisation and reasoning in three dimensions before moving to more than three dimensions.
3.2 Graphical representations and the univalence criterion

A function can also be represented by a graph. The graph of a single-variable function is the set of points \((x, y)\) in the \(xy\)-plane that satisfy the equation \(y = f(x)\). For example, the graph in Figure 1 shows the single-variable function \(f: \mathbb{R} \to \mathbb{R}\) such that \(f(x) = x^2\). All graphs in this paper follow the conventions of a Cartesian coordinate system.

![Figure 1](image1.png)

Figure 1. Graph of a single-variable function

The graph of a multivariable function is the set of points \((x, y, z)\) in \(\mathbb{R}^3\) that satisfy the equation \(z = g(x, y)\). For example, the graph in Figure 2 shows the multivariable function \(g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) such that \(g(x, y) = x^2 + y^2\).

![Figure 2](image2.png)

Figure 2. Graph of a multivariable function

Univalence of relations of a single variable is commonly assessed graphically by applying the vertical line test. To wit, a graph of a set of points \((x, y)\) represents a function if and only if every vertical line drawn in the plane intersects this graph at most once (Briggs & Cochran, 2011). Note that this tests for a function of the form \(f(x)\)
= y, graphed on a standard Cartesian coordinate plane. For example, the graph shown in Figure 1 passes the vertical line test and hence this graph represents a function.

The vertical line test generalises to graphs of points in $\mathbb{R}^3$ on a standard Cartesian coordinate system. Such a graph represents a function of the form $z = g(x, y)$ if and only if any vertical line drawn parallel to the $z$ axis intersects the graph at most once. For example, the graph shown in Figure 2 passes the vertical line test and hence this graph represents a function. In contrast, the graph shown in Figure 3 does not represent a single-variable function because the line $x = 0$ intersects the graph at both $y = -4$ and $y = 4$. Similarly, the graph shown in Figure 4 does not represent a function of two variables because the line parallel to the $z$ axis through $(0, 0)$ intersects the graph at $z = -1$ and $z = 1$.

Figure 3. Graph that does not represent a single-variable function

Figure 4. Graph that does not represent a function of two variables

3.3 Common models of function

Because this study focused on student thinking about functions, in this section I explain two models for function that students commonly employ (Dorko & Weber, 2014; Kabael, 2011; Yerushalmy, 1997). These are also frequently emphasised in instruction (e.g., Ayalon et al., 2017; Briggs & Cochran, 2011; Kabael, 2007; Rogawski, 2008) and follow conventions of a Cartesian coordinate system. The first is the input and output model. In $f(x) = y$, $x$ is the input and $y$ is the output. In $g(x, y) = z$, $(x, y)$ is the input and $z$ is the output. The second model is in terms of independent and dependent variables. In $f(x) = y$, $x$ is the independent variable and $y$ is the
dependent variable. In \( g(x, y) = z \), \( x \) and \( y \) are independent variables and \( z \) is the dependent variable. These characterisations emphasise the mapping idea.

3.4 Hypothesised cognitive shifts a student might undergo when learning about multivariable functions

The above definitions and characterisations of function imply cognitive shifts a student might have to make when learning about multivariable functions. I present these to help the reader think about what adjustments might occur in a student’s cognitive structures as he or she develops a normative understanding of multivariable functions. These cognitive shifts are inter-dependent, and might include:

- **Role of \( y \):** While \( y \) typically represents an output or dependent variable in a single-variable function, it is an input or dependent variable for the multivariable function \( g(x, y) = z \).
- **Input:** While the input of \( f(x) = y \) is a set of \( x \) values, the input of \( g(x, y) = z \) is a set of coordinate points (e.g., \( \{(x, y) \mid x \in \mathbb{R}, \ y \in \mathbb{R}\} \)).
- **Output:** While the output of \( f(x) = y \) is a \( y \) value, in \( g(x, y) \), \( y \) is part of the input.
- **Independent variable:** While \( f(x) \) has one independent variable, \( g(x, y) = z \) has two independent variables.
- **Dependent variable:** While \( y \) is the dependent variable of \( f(x) = y \), it is an independent variable in \( g(x, y) \).
- **Domain:** While the domain of \( f(x) = y \) is a set of numbers (e.g., \( x \in \mathbb{R} \)), the domain of \( g(x, y) = z \) is a set of coordinate points (e.g., \( \{(x, y) \mid x \in \mathbb{R}, \ y \in \mathbb{R}\} \)).
- **Range:** While the range of \( f(x) = y \) is a set of \( y \) values, the range of \( g(x, y) = z \) is a set of \( z \) values.
- **Univalence criterion:** While the univalence requirement for \( f(x) = y \) is such that each \( x \) in the domain maps to exactly one \( y \), the univalence requirement for \( g(x, y) = z \) is such that each \( (x, y) \) in the domain maps to exactly one \( z \).
- **Vertical line test:** While the graph of a single-variable relation represents a function if each line \( x = c \) for some constant \( c \) and \( x \) in the domain intersects the graph exactly once, the graph of a multivariable relation represents a
function if each vertical line drawn parallel to the z axis through each \((x, y)\) in the domain intersects the graph at most once.

In the next section, I review literature about how students think about some of these ideas and transitions.

4. Literature about student thinking regarding single- and multivariable functions

There is a large amount of research regarding how students understand single-variable functions. In this section, I review the subset of that literature relevant to this paper: to wit, how students think about single-variable function classification tasks, then describe what is known about student thinking about multivariable functions.

4.1 Student thinking about single-variable function classification tasks

Beginning algebra instruction typically includes ‘classification’ tasks in which students are given a relation (in the form of a graph, table, set of coordinate points, etc.) and asked if it represents a function (Leinhardt, Zaslavsky, & Stein, 1990). Students typically evaluate the representation to see if it violates the univalence criterion. Researchers have documented that students commonly solve graph classification tasks by applying the vertical line test (Clement, 2001; Montiel et al., 2008; Williams, 1998). Moreover, some students will convert a non-graphical representation to a graph, or a graph in polar coordinates to a graph in rectangular coordinates, in order to employ the vertical line test (Kabael, 2011; Montiel et al., 2008; Williams, 1998). Finally, students may solve classification tasks such as \(x = y^2\) by comparing graphs to prototypical examples (Clement, 2001; Montiel et al., 2008; Spyrou & Zagonianakos, 2010).

There are myriad difficulties students may experience when determining whether a relation represents a function. For example, students’ definition of function may not include the univalence criterion (Even, 1993; Kabael, 2011). Other students may exhibit \(x, y\) syndrome, in which students remember a procedure “in terms of the symbols used when it was first learned” (White & Mitchemore, 1996, p. 90) rather than a process connected to a particular concept. In the context of classification tasks, students look for \(x\)’s and \(y\)’s and classify representations such as \((x, 2x)\) for \(x \in R\) and \(y = c\) for \(c \in R\) as non-functions (Montiel et al., 2008; Williams, 1998). Students may
also conflate univalence with other properties, such as one-to-one-ness\(^4\) (Clement, 2001; Even, 1993; Markovits et al., 1986; Spyrou & Zagorianakos, 2010; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). Another noted difficulty is that students may state the univalence criterion backward, saying each element in the range is paired with exactly one element in the domain (Even, 1993) or apply a horizontal line test (Williams, 1998). Finally, students may generalise the vertical line test as applicable to any graph (e.g., a graph in polar coordinates, c.f. Montiel et al., 2008;).

Student success rates on classification tasks vary with curriculum and student population. In a study of an innovative algebra curriculum for low-SES students, Dubinsky and Wilson (2013) reported a 63% success rate on classification tasks that included arrow diagrams, tables, ordered pairs, and graphs. Instruction did not include the vertical line test, as the researchers felt it emphasised symbolic manipulation. Markovits et al. (1986) found most 9th-grade students were successful on classification tasks, while Clement (2001) reported a 40% success rate for precalculus students.

4.2 Student understanding of and generalisation about multivariable functions

Researchers have recently begun to investigate students’ understanding of multivariable functions and how students generalise their understanding of single-variable functions to the multivariable case. There exist findings about students’ understanding of the role of \(y\), domain and range, input/output and independent/dependent variable characterisations of multivariable functions, and multivariable graphing.

4.2.1 Role of \(y\)

Following the standard Cartesian coordinate system, in the single-variable function \(f(x) = y\), \(y\) is a dependent variable. However, in the multivariable function \(g(x, y) = z\), \(y\) is an independent variable. This difference may seem trivial, but research findings indicate that students struggle to see \(y\) as an independent variable or as part of the input to \(g(x, y)\). For example, Kabael (2011) found that some students offered

\[4\] A function is one-to-one (injective) if for all \(a\) and \(b\) in the domain, if \(f(a) = f(b)\), then \(a = b\). That is, each input maps to a unique output. A function is onto (surjective) if for each element in the codomain \(Y\), there is at least one element in the domain \(X\) such that \(f(x) = y\).
an \((x, y, z)\) tuple as an example of an element in the domain of a multivariable function, and some students offered an \((x, y, z)\) tuple as an example of an element in the range. This finding indicates that students struggle with the role of each variable. Similarly, other researchers have found that students may generalise that the domain and range of \(g(x, y)\) are a set of \(x\) values and a set of \(y\) values, respectively (Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012). Generalising that a set of \(y\) values is the range of \(g(x, y)\) is strong evidence that students may not realise that \(y\) is an independent variable in a multivariable function \(g(x, y)\).

A possible explanation for students’ difficulties with \(y\) is what White and Mitchelmore (1996) term \(x, y\) syndrome. In this ‘syndrome,’ students recall procedures in terms of the specific variables involved rather than the underlying meaning of the procedure. In the case of students’ difficulties with the role of \(y\), it may be that they are so accustomed to the symbol \(y\) indicating a dependent variable that this causes issues in the multivariable case. In support of this, Dorko and Weber (2014) wrote that students who said the domain of \(g(x, y)\) was a set of \(x\) values and the range was a set of \(y\) values were generalising by relating objects (c.f. Ellis, 2007). That is, the students focused on the presence of the \(x\) and \(y\) symbols in the equations and related them to their experience with single-variable functions, in which domain usually was a set of \(x\) values and range usually was a set of \(y\) values. These findings suggest that students’ prior experience with functions of the form \(f(x) = y\) may make it difficult for students to understand that the role of \(y\) differs in a multivariable function.

4.2.2 Input, output, independence, and dependence

The input-output and independence-dependence characterisations of multivariable functions appear to be productive ways for students to think about multivariable functions (Dorko & Weber, 2014; Kabael, 2011; Yerushalmy, 1997).

Researchers have found that the input-output relationship is a key part of how students think about single-variable functions (e.g., Ayalon, Watson, & Lerman, 2017), and teaching multivariable functions using an input-output characterisation seems to allow students to generalise their knowledge of single-variable functions in normatively correct ways (Kabael, 2011). For example, Dorko and Weber (2014) observed that many students who thought about a single variable function’s domain
and range in terms of input and output generalised that a multivariable function would have multiple inputs. While not normatively correct (see Section 2), this generalisation did allow students to realise the domain of $f(x, y)$ is defined in terms of two variables.

Yerushalmy (1997) found that introducing a multivariable function in a modeling context naturally led to students thinking about independent and dependent variables, and ultimately determining that a multivariable function has multiple independent variables. Similarly, Dorko and Weber (2014) found that students who thought of domain as the possible values for the independent variable and range as the possible values for the dependent variable were able to generalise that $f(x, y)$ has two independent variables.

While Kabael (2011) purposefully taught functions using an input-output characterisation, Dorko and Weber (2014) and Yerushalmy’s (1997) study designs elicited students’ ways of thinking about function without researcher-influenced instruction. Collectively, findings from these studies indicate that the input-output and independence-dependence characterisations of function are powerful ways for students to generalise their thinking about function from the single- to multivariable case. Kabael (2011) argues this is because the input-output notion provides a cognitive root (c.f. Tall, McGowen, & DeMarios, 2000), meaning it is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence [that] allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction (p. 485).

Similarly, Dorko and Weber (2014) classified students’ generalisations using Ellis’ (2007) generalisation taxonomy, and found that students who generalised domain and range through the input-output and/or independence-dependence models tended to do so by extending their meanings for domain and range. Taken together, results from these studies suggest that input-output and independence-dependence are productive models for students’ generalisation about functions because students see them as cohesive across the single- and multivariable contexts.
4.2.3 Domain and range

Findings from research indicate that students struggle to identify multivariable functions’ domains and ranges (Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012; Kabael, 2011). For example, students may give an \((x, y, z)\)-tuple as an element of the domain or range. Similarly, students may think that the domain of a multivariable function is the set of all possible \(x\) values and the range is the set of all possible \(y\) values, as is commonly the case in \(\mathbb{R}^2\) (Dorko & Weber, 2014; Martínez-Planell, & Trigueros, 2012). This finding suggests several important points about how students generalise from the single-to multivariable case. First, the presence of particular symbols seems to provide a foothold for students when generalising. Dorko and Weber (2014) follow Ellis (2007) in calling this generalising by relating objects. Second, from a broader perspective, students who generalise that the domain of a multivariable function is the set of all possible \(x\) values and the range is the set of all possible \(y\) values appear to have generalised by applying an idea from \(\mathbb{R}^2\) directly to \(\mathbb{R}^3\).

Students may also generalise strategies for finding domain and range. When finding the domain and range of a single-variable function, students often project or ‘push’ the graph to each axis to find the domain and range, respectively (Cho & Moore-Russo, 2014). Some students seem to generalise this strategy. For instance, Kabael (2011) observed students projecting a multivariable function’s graph to the \(x\) axis and \(y\) axis to find the domain, rather than projecting the graph to the \(xy\) plane. Kabael (2011) suggested this might be related to difficulties conceptualising the nature of three-space. Given Cho and Moore-Russo’s (2014) finding, an alternative explanation is that students who project an \(\mathbb{R}^3\) graph to each axis are generalising a strategy from \(\mathbb{R}^2\). Regardless, students’ projection to each axis may indicate that students see \(x\) and \(y\) as two separate inputs rather than the normatively correct conception that \((x, y)\) is a single input. This is consistent with other findings that students often think of \(f(x, y)\) as having an \(x\) input and a \(y\) input (Dorko & Weber, 2014; Martinez-Planell, & Trigueros, 2012).
4.2.4 Graphing in R³

Findings from research indicate that students find R³ graphing challenging (Dorko, 2016; Dorko & Lockwood, 2016; Kabael, 2011; Martínez-Planell & Gaisman, 2013; Martínez-Planell & Trigueros, 2012). Specific difficulties include drawing graphs (particularly those with free variables), finding the intersection of a multivariable function and a fundamental plane, and determining whether a graph in R³ represents a function.

One reason students may struggle to draw multivariable graphs is that their correct thinking from R² may mislead them in R³. For example, students may draw \( f(x, y) = x^2 + y^2 \) as a cylinder or a sphere because they are accustomed to \( x^2 + y^2 \) representing a circle in R² (Martínez-Planell & Gaisman, 2013). Likewise, students may think the graph of \( y = 3 \) in R³ is a line, as it is in R² (Dorko, 2016; Dorko & Lockwood, 2016). Functions with free variables are particularly troublesome; for instance, students may draw \( f(x, y) = x^2 \) as a parabola (instead of a parabolic surface) in R³, or draw the parabola \( f(x) = x^2 \) in R². Students also have trouble determining the intersection of a surface with a fundamental plane (a plane of the form \( x = a, y = b, \) or \( z = c \) for \( a, b, c \in \mathbb{R} \); Trigueros & Martínez-Planell, 2010).

Martínez-Planell and Gaisman (2013) propose that students’ difficulties with multivariable graphs are tied to under-developed schemas for three-space, and other findings support this (Dorko, 2016; Dorko & Lockwood, 2016; Kabael, 2011). As an example, students sometimes struggle with conceptualising \( f(x, y) \) as an output or the height of the graph at a particular \((x, y)\)-tuple (Martinez-Planell & Trigueros, 2012). This may explain both their difficulties with intersecting a graph with a fundamental plane and Kabael’s (2011) finding that students have difficulty projecting a graph to the \( xy \) plane to determine its domain and range. Part of developing a useful schema for R³ seems to be explicitly attending to \( z \) as a quantity that varies (Dorko, 2016; Dorko & Lockwood, 2016). For example, students who do not attend to \( z \) as varying tend to draw \( y = 3 \) in R³ as a line, while students who draw \( y = 3 \) in R³ as a plane in some way attend to \( z \) as varying (Dorko, 2016; Dorko & Lockwood, 2016).

To my knowledge, only one study has documented how students solve graphical classification tasks in R³. Kabael (2011) found that following instruction about
multivariable functions, 21 of 23 students solved graphical classification tasks by converting the graphs to algebraic formulae. In a test at the end of the multivariable calculus course, 13 of the 23 solved graphical classification tasks with the vertical line test. Four students evaluated univalence by determining algebraic formulae for all classification tasks on both tests. Kabael’s (2011) findings indicate that some of the students had generalised univalence and the vertical line test. However, students also appeared to have made other generalisations as they sought to solve the classification problems. For example, students’ conversions of graphs to algebraic formulae is reminiscent of abundant evidence in the single-variable function literature about students’ preference for algebraic representations (e.g., Ferrini-Mundy & Graham, 1994).

These findings raise questions about what else students might generalise as they come to understand multivariable functions. My study seeks to fill that gap in the literature, building on Kabael’s (2011) work by focusing explicitly on what students generalise as they think about what it means to be a function in $\mathbb{R}^3$. In addition, I contribute to the literature about how students think about graphs in $\mathbb{R}^3$. Prior research about students’ multivariable graphing have focused on students constructing graphs from given equations (e.g., Dorko, 2016; Dorko & Lockwood, 2016; Martínez-Planell & Gaisman, 2013; Martínez-Planell & Trigueros, 2012). Given students’ difficulties with such constructions, it makes sense to learn more about how students think about graphs in $\mathbb{R}^3$. My study builds on prior work by investigating students’ reasoning about graphs that are given to them, rather than graphs they construct.

5. Theoretical Framework

I interpret students’ thoughts and generalisations from an actor-oriented perspective (Lobato, 2003; Ellis, 2007). The actor-oriented perspective focuses on what connections students identify across situations, even if their perceived similarities are not normatively correct. Given that research about undergraduate students’ thinking has revealed that students often make what seem to be unconventional connections through the eyes of the researcher (Dorko, 2016; Dorko
& Weber, 2014; Lockwood, 2011), it is important to adopt a theoretical perspective that captures the wide variety of ways students generalise.

The actor-oriented perspective has its roots in transfer research. Lobato (2003) proposed actor-oriented transfer as an alternative to traditional transfer research (e.g., Gick & Holyoke, 1980; Judd, 1908), in which researchers tend to focus on whether participants identify similarities that an expert has pre-determined. By attending to only what is normatively correct, traditional transfer perspectives may fail to capture the sense participants make of situations. By ‘normatively correct’ I mean the agreed-upon definition or understanding within a community of mathematicians. This language comes from Ellis (2007), whose actor-oriented taxonomy for characterising students’ generalisations de-emphasises normative correctness in order to capture a broader range of the sense students make of situations (which often includes ideas that are non-normatively correct).

5.1 Definition of generalisation

I define generalising as the activity of making connections between prior and novel situations. This draws from Ellis’ (2007) and Lobato’s (2003) definitions of generalisation and transfer which I describe below.

According to Ellis (2007) and Lobato (2003), actor-oriented transfer is defined as “the influence of a learner’s prior activities on his or her activity in novel situations” (Ellis, 2007, p. 225). One reason the actor-oriented perspective is appropriate for this study is that in the multivariable pre-interview (see Section 5.1), students looked at a novel context ($R^3$) that differed significantly from the original ($R^2$) context. Hence an actor-oriented perspective allowed me to capture what students saw as similar across these contexts. While the definition given is for the word transfer, there is a relationship between transfer and generalisation. Ellis (2007) writes that alternative approaches to traditional transfer research, including the actor-oriented perspective, characterise transfer as the generalisation of learning; thus, they capture a broader range of student actions as transfer, including students’ mathematical generalisations (p. 225).

Further down the same page, Ellis (2007) writes,
From this [actor-oriented perspective], transfer is the generalisation of learning, which can be seen as the influence of a learner’s prior activities on his or her activity in novel situations (p. 225).

Finally, Ellis (2007) lists three ways mathematics education researchers describe generalisation and states that these “could be captured under the umbrella of the transfer phenomenon” (p. 225). She writes that she developed a generalisation taxonomy to solve a limitation of the actor-oriented perspective, namely that

[the actor-oriented perspective] does not distinguish between types of generalisations, instead capturing a wide range of generalising acts as forms of transfer (Ellis, 2007, p. 226).

I argue that these three descriptions position generalisation as similar to transfer. Hence my definition of generalising is similar to Ellis’ (2007) definition of transfer. Specifically, I define generalising as the activity of making connections between prior and novel situations.

As an example, making connections might involve some of the specific generalizing actions that Ellis (2007) describes, including students “identifying commonality across cases, extending their reasoning beyond the range in which it originated, and deriving broad results from particular cases” (p. 227). The phrase ‘making connections’ in my definition of generalisation is broad because in coding data (discussed later), I kept Ellis’ three examples in mind but was also open to other, novel ways of making connections in which students might engage.

In the next section, I describe the taxonomy Ellis (2007) developed to distinguish between the different ways students might generalise and the different types of generalisations they might make.

5.2 Generalisation taxonomy

Ellis’s (2007) empirically-grounded taxonomy for characterising students’ generalising is shown in Figures 5 and 6. One reason this framework was useful to my work was that it distinguishes between generalisation as a product and as a process. This is empirically useful because it separates what students generalise from the process they underwent to arrive at that generalisation.
The taxonomy distinguishes between generalising actions (Figure 5) and reflection generalisations (Figure 6). Generalising actions “describe learners’ mental acts as inferred through the person’s activity and talk” (Ellis, 2007, p. 233), while reflection generalisations are students’ public statements of a generalisation. Students’ use of “the result of a generalisation, such as applying an idea from a prior situation to a new problem” also fall under the reflection generalisation header.

There are three major types of generalising actions. The first, relating, occurs “when a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects” (Ellis, 2007, p. 236). The second, searching, occurs when students “perform the same repeated action in an attempt to determine if an element of similarity will emerge” (Ellis, 2007, p. 239). Finally, when extending, students notice “a pattern or relationship of similarity” (Ellis, 2007, p. 238) and expand it into a more general structure. In extending, “a student expands his or her reasoning so that it reaches beyond the problem, situation, or case in which it originated” (Ellis, 2007, p. 241). Each of these three major categories has subcategories, as shown in Figure 5.

Reflection generalisations also come in three main forms. Identifications or statements occur when students “refer to a general pattern, property, rule, or strategy” or when students “explicitly identify a common element across different cases or problems” (Ellis, 2007, p. 245). Definitions capture “cases in which students ma[ke] statements conveying the fundamental character of a pattern, relation, class, or other phenomenon” (Ellis, 2007, p. 248). Finally, influence characterises “cases in which students implement previously developed generalisations in new problems or contexts” (Ellis, 2007, p. 250). The subcategories for each of these are shown in Figure 6.

As discussed in the next section, Ellis’ generalisation taxonomy served as an analytic framework because it provided language to describe both what students attended to and how they generalised.
GENERALISING ACTIONS

1. Relating situations: The formation of an association between two or more problems or situations. 
   - Connecting Back: The formation of a connection between a current situation and a previously-encountered situation.
   - Creating New: The invention of a new situation viewed as similar to an existing situation.

2. Relating objects: The formation of an association between two or more present objects.
   - Property: The association of objects by focusing on a property similar to both.
   - Form: The association of objects by focusing on their similar form.

Type I: Relating

Type II: Searching

1. Searching for the Same Relationship: The performance of a repeated action in order to detect a stable relationship between two or more objects.
2. Searching for the Same Procedure: The repeated performance of a procedure in order to test whether it remains valid for all cases.
3. Searching for the Same Pattern: The repeated action to check whether a detected pattern remains stable across all cases.
4. Searching for the Same Solution or Result: The performance of a repeated action in order to determine if the outcome of the action is identical every time.

Type III: Extending

1. Expanding the range of Applicability: The application of a phenomenon to a larger range of cases than that from which it originated.
2. Removing Particulars: The removal of some contextual details in order to develop a global case.
3. Operating: The act of operating upon an object in order to generate new cases.
4. Continuing: The act of repeating an existing pattern in order to generate new cases.

Figure 5. Generalising actions (Ellis, 2007, p. 235)
REFLECTION GENERALISATIONS

1. *Continuing Phenomenon:* The identification of a dynamic property extending beyond a specific instance.

2. *Sameness:* Statement of commonality or similarity.
   - **Common Property:** The identification of the property common to objects or situations.
   - **Objects or Representations:** The identification of objects as similar or identical.
   - **Situations:** The identification of situations as similar or identical.

   - **Rule:** The description of a general formula or fact.
   - **Pattern:** The identification of a general pattern.
   - **Strategy or Procedure:** The description of a method extending beyond a specific case.
   - **Global Rule:** The statement of the meaning of an object or idea.

4. *Class of Objects:* The definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon.

5. *Definition:* Type V:
   - **Prior Idea or Strategy:** The implementation of a previously-developed generalisation.
   - **Modified Idea or Strategy:** The adaptation of an existing generalisation to apply to a new problem or situation.

Figure 6. Reflection generalisations (Ellis, 2007, p. 245)

6. Methods

6.1 Data collection

The data analysed in this paper come from a longitudinal study of how five students generalised their notion of function from single- to multivariable calculus. Data collection for the larger study took place over one academic year at an institution on a quarter system, in which some students take differential calculus during the fall term, integral calculus during the winter term, and multivariable calculus during the spring term.

Each student completed four task-based clinical interviews (Hunting, 1997), the topics and timing of which are described in Table 1. Of particular importance is the timing of the multivariable tasks. I aimed to capture students’ initial sense-making about facets of the multivariable function concept before instruction, as well as how they understood those same facets after instruction. Hence students completed tasks...
about multivariable functions while they were enrolled in differential and integral calculus, as well as during their enrollment in multivariable calculus. The total time for all four interviews ranged from 4.25 to 5.67 hours per student. The total time spent on the tasks discussed in this paper ranged from 24 to 54 minutes per student.

I recorded each interview using a video camera, audio recorder, and LiveScribe pen. The LiveScribe pen creates a playable PDF recording that syncs audio with student writing. I transcribed each interview for use in analysis.

Table 1. Interview schedule

<table>
<thead>
<tr>
<th>Interview</th>
<th>Topics</th>
<th>Course at time of interview</th>
<th>Timing of interview in relation to course material</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Single-variable functions, some MV questions</td>
<td>Differential calculus</td>
<td>During instruction about limits</td>
</tr>
<tr>
<td>2</td>
<td>Limits and derivatives of single-variable functions; MV classification tasks if time permitted</td>
<td>Differential calculus</td>
<td>After instruction about limits and derivatives</td>
</tr>
<tr>
<td>3</td>
<td>Definite integrals and antiderivatives; MV classification tasks if not done in interview 2</td>
<td>Integral calculus</td>
<td>After instruction about definite integrals and antiderivatives</td>
</tr>
<tr>
<td>4</td>
<td>Multivariable functions</td>
<td>Multivariable calculus</td>
<td>After instruction about multivariable functions</td>
</tr>
</tbody>
</table>

Although the students were asked many tasks over the course of the interviews, this paper focuses on classification tasks. In particular, I examine the single-variable classification tasks from Interview 1 (hereafter “SV Interview”; Figure 7 and Figure 8), the multivariable classification tasks from Interview 2 or 3 (hereafter “MV Pre-Interview”, Figure 9), and the multivariable classification tasks from Interview 4 (hereafter “MV Post-Interview”; Figure 11).
Which of the following represent functions?

**SV Table 1**

\[
\begin{array}{ccccccc}
 x & -2 & 0 & 3 & 3 & 5 & 6 \\
 y & 9 & 2 & 4 & 5 & 7 & 7 \\
\end{array}
\]

**SV Table 2**

\[
\begin{array}{ccccccc}
 x & -2 & 0 & 3 & 5 & 6 & 7 \\
 y & -1 & 1 & 4 & 6 & 7 & 8 \\
\end{array}
\]

**SV Table 3**

\[
\begin{array}{ccccccc}
 x & -2 & 0 & 3 & 5 & 6 & 7 \\
 y & 0 & 0 & 8 & 15 & 31 & -5 \\
\end{array}
\]

*Figure 7. SV interview table classification tasks (used in interview 1)*

Which of the following graphs represent functions of \(x\)?

**SV Graph 1**

**SV Graph 2**

**SV Graph 3**

**SV Graph 4**

*Figure 8. SV Interview graph classification tasks (used in interview 1)*

In the first interview, I asked students whether single-variable relations represented functions (Figure 7 and Figure 8). The second interview, conducted while students were enrolled in integral calculus, contained \(\mathbb{R}^3\) graphs (Figure 9). Asking students about graphs in \(\mathbb{R}^3\) before instruction about multivariable functions allowed me to capture students’ initial sense-making about multivariable functions. Before starting the tasks I presented students with a picture of \(\mathbb{R}^3\) axes, pointed out the axis labels, and explained that the \(xy\) axis made a flat plane like a tabletop and the \(z\) axis was vertical and orthogonal/perpendicular to the \(xy\) plane. I demonstrated this using the corner of a table and a pen held orthogonal to the table top. Some of the students had plotted points on \(\mathbb{R}^3\) axes in high school algebra, but that was the extent of their experience with multivariable topics.
The phrasing “do the following represent functions?” in the Interview 2 and 3 tasks was intentionally ambiguous\(^5\). Since findings from research indicate that some students generalise domain and range by thinking about inputs and outputs and/or independence and dependence (Dorko & Weber, 2014; Kabael, 2011), I wanted to see if and how students would determine which variables were the input(s), output, independent, and dependent variables.

Which of the following graphs represent functions? (from Kabael, 2011)

Figure 9. MV pre-interview classification tasks (used in interview 2 and 3)

The fourth interview, conducted at the end of students’ multivariable calculus course, contained tables in \(\mathbb{R}^3\) and graphs in \(\mathbb{R}^2\) and \(\mathbb{R}^3\) (Figure 10, Figure 11). I explained how to read the tables and gave the example ‘in Table 1, when \(x\) is 0 and \(y\) is -1, \(z\) is 0.’ These post-instruction tasks aimed to capture the sense students made of normative ideas and how they connected those ideas to prior knowledge.

---

\(^5\) The question is ambiguous in that it does not specify the domain. For example, MV Graph 1 in Figure 9 does not represent a function of two variables, but it could represent a function of three variables.
Do the following represent functions?

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<tr>
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Figure 10. MV Post-interview table classification tasks (used in interview 4)
Do the following represent functions?

Figure 11. MV Post-interview graph classification tasks (used in interview 4)

6.2 Rationale for tasks

The tasks shown in Figure 7, Figure 8, Figure 9, Figure 10, and Figure 11 were designed to gain insight into how students determined whether a particular tabular or graphical relation represented a function. These classification tasks are typical when relations and functions are first introduced (Leinhardt, Zaslavsky, & Stein, 1990), and researchers use them to offer insight into students’ understanding of various components of functions. Moreover, the tasks were designed to elicit some of the documented difficulties students often experience with the function concept. For example, students often believe functions show a pattern or can always be expressed with an equation (e.g., Carlson, 1998; Clement, 2001; Even, 1993; Ferrini-Mundy & Graham, 1994; Sierpinska, 1992). Hence the \((x, y)\) pairs given in SV Table 2 could be generated from the equation \(y = x + 1\), and the \((x, y, z)\) tuples given in MV Table I could be generated from the equation \(z = xy\). Similarly, students may think that all equations represent functions. Hence I included the graph of a circle (SV Graph 4, Figure 8) and sphere (MV Graph 1, Figure 9, Figure 11) on the hypothesis that students might be familiar with their equations.
Another common error I targeted was some students’ tendency to conflate univalence with one-to-one-ness\(^6\) (Clement, 2001; Even, 1993; Markovits et al., 1986; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). For example, I included a table that represented a one-to-one function (SV Table 2, Figure 7) and a table that represented a function\(^7\) that is not one-to-one (SV Table 3, Figure 7). Since other researchers (Williams, 1998) have found that students may use a horizontal line test (rather than a vertical line test) to determine whether a particular situation represents a function, I included an R\(^2\) graph that would pass a horizontal line test (SV Graph 1, Figure 8).

### 6.3 Data analysis

The first step in data analysis was to identify instances in which it seemed students were generalising. In doing so, per my definition of generalising, I looked for places where students were making connections between prior and novel situations.

In a first read of all the transcripts, I noticed that students seemed to use similar words or engage in similar behaviours across interviews. For example, students who talked about functions in terms of inputs and outputs on the classification tasks in SV Interview 1 also tended to use the words ‘input’ and ‘output’ in the classification tasks in the other interviews. I took students’ use of similar words and behaviours as evidence of generalising because this repetition across interviews (which happened weeks or months apart) seemed to constitute the influence of prior activity on new activity. I read all the transcripts and identified repeated words and behaviours. The list of words included input, output, vertical line test, pattern, repeated [\(x\) value], and equation. The behaviours included drawing lines parallel to an axis and determining an algebraic formula.

The second step involved taking each instance of generalising and identifying what was being generalised. For example, if students talked about inputs and outputs in both the R\(^2\) and R\(^3\) context, I took that as evidence of generalising the notion of

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\(^6\) Per Briggs and Cochran (2011), “a function \(f\) is one-to-one on a domain \(D\) if each value of \(f(x)\) corresponds to exactly one value of \(x\) in \(D\). More precisely, \(f\) is one-to-one on \(D\) if \(f(x_1) = f(x_2)\) whenever \(x_1 = x_2\) for \(x_1\) and \(x_2\) in \(D\). The horizontal line test says that every horizontal line intersects the graph of a one-to-one function at most once” (Briggs & Cochran, 2011, p. 25).

\(^7\) Recall that a function is univalent by definition, but not all functions are one-to-one.
input and output to the multivariable case. If students drew a line parallel to one of
the $\mathbb{R}^3$ axes or verbally described such a line, I took that as evidence of generalising
the vertical line test. I used a constant comparative analysis (Strauss & Corbin, 1998)
to form categories of similar generalisations. I found that students generalised
function-as-equation, the notion of input and output, vertical line test, and univalence.
These were frequently interrelated. For example, one student, Kyle$^8$, leveraged
function notation (function-as-equation) to generalise that $z$ would be the output of a
multivariable function (notion of input and output) and then generalised that since $z$
was the output, the vertical line test in $\mathbb{R}^3$ would involve a line parallel to the $z$ axis
(vertical line test). Hence the three main categories (function-as-equation, input and
output, vertical line test, and univalence) have some overlap, but in general provide a
way to organise what students generalised about what it means to be a function.

In the third step, Ellis’ (2007) generalisation taxonomy (Figure 5, Figure 6)
provided an analytic framework for characterising what students attended to while
generalising, and how they generalised. I coded each instance of generalisation
identified in step two according to Ellis’ generalisation taxonomy. The subcategories
are not mutually exclusive, and hence some instances of generalisation had several
codes.

As I coded data, I wrote theoretical memos in which I included data excerpts that
I felt were evidence of generalisation; my coding as to what was being generalised;
justification for what I contended was being generalised; my coding per Ellis’ (2007)
framework; and my justification for that coding. Another mathematics education
researcher who was currently working with Ellis’ (2007) framework and who had
extensive experience in researching generalisation reviewed these memos. This is one
of the ways I ensured reliability in coding. The other way is in the manuscript itself.
That is, I have presented abundant data excerpts to support my claims, and the reader
can compare my codes to how Ellis defines particular categories (see Figures 5 and 6).

The following section provides two examples of coding.

$^8$ Names are gender-preserving pseudonyms.
6.3.1 Example of coding

The following excerpts provide an example of coding. Excerpt 1 provides evidence that the vertical line test was part of Wendy’s conception of single-variable function. She generalised the vertical line test to $\mathbb{R}^3$ by *searching for the same procedure* (Excerpt 2). In these excerpts and in the remainder of the paper, bold text indicates phrases that served as evidence for particular categories or codes and italics indicate codes from Ellis’ (2007) framework.

Excerpt 1. SV Interview, SV Tables 1, 2, 3

Wendy: So these are tables of $x$ and $y$ variables… I always think of functions in graphs… Basically what we want is that for every $x$ variable there is one $y$ value so that it doesn’t *double cross on the graph*… So on the first one I see that 3 is 4 and 5 simultaneously, and for every one $x$ value you need one $y$ value, so I don’t think that A is a function because it doesn’t pass the *vertical line test*.

Excerpt 2. MV Pre-interview, MV graph 1 (sphere), MV graph 3 (plane $z = 4$)

Wendy: You see that [MV graph C] is what I’m initially led to believe is more of a function than that [MV graph A] … because it can, you can have *a line here* [draws line parallel to $z$ axis on MV graph C] and it only hits on *like the plane there*. So that’s leading me to believe that that’s more of a function.

In the first phase of coding, I identified Excerpt 2 as an example of generalising because Wendy talked about and drew lines, which she had also described in the SV interview (Excerpt 1). Hence she exhibited similar language and behaviour across interviews, which I took as evidence of the influence of prior activity on activity in a novel situation. Later, I grouped this with other instances of students drawing lines; this category became the generalisation (product) of the vertical line test.

In the second phase of coding, I identified what Wendy was generalising. Since she had drawn lines and talked about the vertical line test in both the SV and MV pre-interviews, it seemed that she was generalising the vertical line test.
In the third step, I categorized Wendy’s reasoning according to Ellis’ (2007) generalisations taxonomy. I coded her reasoning about the vertical line test as *searching for the same procedure*, or “the repeated performance of a procedure in order to test whether it remains valid for all cases” (Ellis, 2007, p. 235). Wendy performed the vertical line test on two different graphs (performing a procedure repeatedly), and she seemed to attend to the result (how many times the line intersected the graph) as a way to think about whether the vertical line test would determine whether or not an $\mathbb{R}^3$ graph represented a function. As such, she seemed to be evaluating if the vertical line test were valid for the new $\mathbb{R}^3$ cases.

7. Results

Findings from the constant comparative analysis (Strauss & Corbin, 1998) indicate that when determining whether a multivariable relation represents a function, students generalised the following four ideas: (a) a conception of function as an equation or pattern; (b) the notion of input and output; (c) the vertical line test; and (d) univalence. There exists overlap among these categories and some are intertwined from a mathematical perspective (e.g., the vertical line test is used to evaluate if a relation is univalent). However, based on the coding scheme, these four categories captured students’ main generalisations as they sought to determine whether multivariable relations represented functions. Hence the results section is organized in terms of these four categories.

Within each of these main categories, I discuss the codes from Ellis’ (2007) framework, which provide language to describe the mathematical features students attended to. For ease of identification, codes from Ellis’ (2007) framework are written in italics.

7.1 Generalising a conception of function as an equation

All the students leveraged either algebraic representations of functions or function notation to complete the classification tasks. In function-as-equation, students generalised the belief that if an equation exists, then that equation represents a function. In leveraging function notation, students attended to similarities in the
symbols $f(x)$ and $f(x, y)$ to support thinking about generalising the vertical line test and univalence.

7.1.1 Function-as-equation

Three students, Stan, Wendy, and Ike, seemed to believe that if an equation existed for a graph, then the graph represented a function. They appear to have developed this idea in $\mathbb{R}^2$, and generalised it to $\mathbb{R}^3$ by seeking equations for the multivariable graphs. Given that many researchers have noted that students often believe a function must be defined by an algebraic formula (e.g., Carlson, 1998; Clement, 2001; Even, 1994; Ferrini-Mundy & Graham, 1994; Sierpinska, 1992), it is unsurprising that Stan, Wendy, and Ike thought so much about equations as they solved the classification tasks. This study extends prior research by documenting that students who hold a functions-are-equations belief may generalise that belief to the multivariable case.

Students’ generalisations about functions-as-equations were both productive and unproductive. Stan and Wendy’s thinking provide examples of an unproductive generalisation. Function-as-equation was unproductive for Stan (Excerpts 3 and 4) because it prevented him from thinking about univalence. It was unproductive for Wendy because it conflicted with her knowledge that a function should be univalent (Excerpt 5). In contrast, Ike used equations productively, comparing the graph of a sphere to the equation of a circle and reasoning that since a circle did not represent a function in $\mathbb{R}^2$, the sphere did not represent a function in $\mathbb{R}^3$. Ike’s reasoning provides an instance of a student reasoning by comparing a situation to a prototypical example, a strategy students often use on single-variable classification tasks (Clement, 2001). In the remainder of this section, I provide excerpts from the data that illustrate students’ generalisations of function-as-equation and I explain how these excerpts are coded according to Ellis’ (2007) framework.

Equation and pattern were two key, inter-related components of Stan’s notion of single-variable function, and he generalised this to multivariable situations by extending: expanding the range of applicability. In the single-variable interview, Stan described that he looked for patterns because if he could find a pattern, he could write a formula:
Excerpt 3. SV Interview, SV Tables 1 and 2

Stan: I’m trying to see how you get from the $x$ value to the $y$ value, and if there’s a pattern of how the $x$ gets to the $y$ then you can determine a formula off of that. But for [SV Table 1] I can’t see anything that, I don’t see a necessary pattern… so there’s not a set function that can describe what’s going to happen… the [SV Table 2] table, to get from the $x$ value to the $y$ value it’s +1 [writes $f(x) = x + 1$].

Stan classified SV Table 1 as a non-function because it did not have a pattern and hence there was not a “set function that can describe what’s going to happen.” By “set function,” I infer that he meant an equation. Stan’s $f(x) = x + 1$ equation for SV Table 2 supports this inference. Additionally, Stan’s phrasing in the MV pre-interview seemed to indicate that the words ‘function’ and ‘equation’ were synonymous to him:

Excerpt 4. MV Pre-interview, MV Graph 1 (sphere)

Stan: This is like a sphere, there’s equations for spheres. These look like half spheres, so it would just be like a different kind of rearrangement of that equation… a function is an, an equation that, depending on what kind of input along a particular axis you want, will give you an output on another axis…

Int.: Okay, can you draw a circle… do you know what the equation is for that?

Stan: It’s $x^2 + y^2$ equals your radius squared. [writes $x^2 + y^2 = r^2$]

Int.: Is that a function?

Stan: Yup.

Stan’s definition, “a function… is an equation” is evidence that he thought the two words were somewhat synonymous. Moreover, based on his acceptance of $x^2 + y^2 = r^2$ as a function, Stan seemed to believe that if one could write an equation for a graph, then the graph represented a function.

Ellis (2007) writes that when extending, “a student expands his or her reasoning so that it reaches beyond the problem, situation, or case in which it originated. Through this action the student generates something new, such as a new domain of validity, new members of a class, a new relationship, a new structure, or a new
description of a general phenomenon” (p. 241). In describing the sphere (and other multivariable graphs) as functions because equations existed for them, Stan extended his notion of function-as-equation. Stan expanded the range of applicability because he applied the function-as-equation phenomenon to a larger range of cases (R^3) than that from which it had originated (R^2).

Like Stan, Wendy believed that if an equation existed for a graph, then the graph represented a function. In the MV pre-interview, Wendy was conflicted regarding whether MV graph A represented a function. In seeking to generalise what it means to be a function in R^3, Wendy compared MV graph A with MV graph C:

Excerpt 5. MV Pre-interview, MV Graphs A and C

Wendy: You see that [MV graph C] is what I’m initially led to believe is more of a function than that [MV graph A] … because it can, you can have a line here [draws line parallel to z axis on MV graph C] and it only hits on like the plane there. So that’s leading me to believe that that’s more of a function. But I know that you can write these [points to MV graph A] down in f(x, y).

Int.: The sphere you can’t, actually. So if you wanted an equation for the sphere it would be \(x^2 + y^2 + z^2\) and then you would have to give it some radius value, so I don’t know, 9. So that one doesn’t actually have an \(f(x, y)\) equation.

Wendy: Okay, so that one has an \(x, y, z\) equation.

Wendy thought the plane was more function-like than the sphere because she could draw a vertical line that intersected the plane exactly once, while a line drawn on the graph of the sphere intersected the graph twice (Section 6.2.1). However, Wendy thought an \(f(x, y)\) equation existed for the sphere. I infer that Wendy thought the sphere might be a function because she thought she could write an equation for it.

Wendy did not talk about equations in the single-variable interview, but her search for how to write an equation seems to be evidence that she was connecting back to prior experiences in which ‘equation’ and ‘function’ were synonymous. Additionally, in comparing MV graphs A and C, Wendy showed evidence of generalising by discerning differences.
Ike thought of functions as ‘one variable in terms of another,’ which seemed to be part of his definition of function. Ike explained that a circle was not a function because $x$ and $y$ were not “directly related” to each other. He then reasoned that a sphere would not be a function because its equation would not have one variable in terms of another, while the plane $z = 4$ would represent a function because it fit his definition (Excerpt 6).

Excerpt 6. MV Pre-interview, MV Graph 1 (sphere), MV Graph 3 (plane $z = 4$)

Ike: I remember a circle isn’t a function because it’s $x^2 + y^2$ equals a number, which isn’t a function because they aren’t directly related to each other, like one doesn’t equal something times or plus the other one. So a sphere I imagine would be the same concept, but I don’t know what the equation is.

Ike: It looks like [MV Graph 3] $z = 4$ in that case… I guess one way to define a function is you have one variable by itself on one side and then stuff on the other side.

In bringing his definition of function as ‘one variable in terms of another’ to bear in the $\mathbb{R}^3$ tasks, Ike expanded the range of applicability and implemented a prior idea. He implemented his prior idea that a function had to have variables ‘directly related to one another’, by which he seemed to mean that each side of the equation could only have one variable. He expanded the range of applicability because he implemented this idea in questions about graphs in $\mathbb{R}^3$, which was a larger range of cases than that in which his idea of variables being ‘directly related’ had developed.

In summary, three students generalised a notion of function-as-equation. Specifically, Stan and Wendy seemed to believe that all equations represent functions, and they generalised this to the $\mathbb{R}^3$ case. Ike generalised a notion of function-as-equation by comparing the graph of the sphere to an equation he knew did not represent a function. As such, students used their ideas about equations productively and unproductively. In the next section, I provide examples of how students productively leveraged function notation in their generalising.

7.1.2 Leveraging function notation to generalise the vertical line test and univalence

Attending to function notation proved to be a productive way for two students, Kenny and Kyle, to support generalising the vertical line test and univalence. Relating
the notation $z = f(x, y)$ to $y = f(x)$ allowed the students to think about $f(x, y)$ in terms of input and output, which in turn allowed them to generalise the vertical line test and univalence. Other researchers have documented students’ use of notation and the symbols in equations in generalising (e.g., Dorko & Weber, 2014; Martínez-Planell & Gaisman, 2013), so it is not surprising that students in this study also focused on notation. However, the students in other studies largely used notation to form non-normatively correct generalisations (see Section 3.2.3). One reason students in this study may have been successful in making generalisations based on notation is that they engaged in comparing notations from $\mathbb{R}^2$ and $\mathbb{R}^3$, while students in the other studies looked only at multivariable equations and notation. That is, while none of the tasks directed the students to compare notations, students often spontaneously talked about the notations $f(x)$ and $f(x, y)$.

In the remainder of this section, I provide data excerpts that illustrate how Kenny and Kyle leveraged function notation to support generalising the vertical line test and univalence.

Kenny leveraged function notation to support generalising the vertical line test. In the MV post interview, Kenny attended to the notations $y = f(x)$ and $z = f(x, y)$ to generalise the orientation of the vertical line test in $\mathbb{R}^3$. Specifically, relating objects (form and property) supported his generalisation that the line would be “parallel to whatever axes is being output.”

Kenny had applied the vertical line test in some of the single variable graphs, but had not mentioned it in the multivariable graph tasks. The researcher wondered if he thought the vertical line test would generalise to functions of more than one variable:
Excerpt 7. MV Pre-interview

Int.: You had talked about the vertical line test in, in the first question. Does the vertical line test apply to these functions of more than one variable?

Kenny: I think so and it’s, it makes sense to me now because when, last time [MV pre-interview] when we were talking about this I was getting a little hung up on the multiple variables, but because this [underlines the x, y in f(x, y); Figure 12] returns a single variable or output... Same as this [underlines f(x); Figure 1] only has one input but it returns one output, so you only need one line to test the vertical line test... the vertical line must be parallel to whatever axes is being output.

Figure 12. Kenny relates objects based on form and property

Relating the notations f(x, y) and f(x) supported Kenny’s generalising. Kenny related the notations based on form and property. His underlining is evidence of relating objects’ forms. In particular, Kenny seemed to focus on the form f([stuff]) = [single output]. He also related the property that both functions returned a single output. This allowed Kenny to conclude that he only needed one line for the vertical line test, and that the line would be parallel to whichever axis represented the output.

Kyle attended to function notation to help him generalise univalence and the orientation of the vertical line test in R^3. In the single-variable interview, Kyle described the graphs as functions if each x mapped to exactly one y. I took this as evidence that he understood the univalence criterion. In the multivariable pre-interview, thinking about function notation helped Kyle generalise univalence to R^3. Specifically, Kyle generalised that in R^3, univalence meant that each x and y mapped to exactly one z.

In thinking about how the notion of function would generalise to R^3, Kyle first thought about the three variables involved and produced the notation f(x, y, z). The
interviewer provided the correct notation for a function in R³. Kyle unpacked this $f(x, y)$ notation, thinking about plotting a particular point that would be ‘over’ some $(x, y)$ and ‘up’ some $z$ value. He leveraged this to conclude that $z$ was the output, then compared MV Graphs A and B to generalise univalence in R³ as each $(x, y)$ mapping to a unique $z$:

Excerpt 8. MV Pre-interview, MV Graph C

Int.: When you write equations for functions in three space they’re $f$ of $x$ comma $y$ so it’s like, can I get you to write that? So $f$ open parenthesis $x$ comma $y$ close parenthesis. [Kyle writes $f(x, y)$] And then you’d have equals.

Kyle: Right, that equals something, and that equals the $z$ value [Figure 13]… So the function, so there’s a variable $x$ and a variable $y$, and if you plot that you would go over $x$ and then across $y$ and then that would be where the $z$ part is somewhere. So if $x$ was like 2, this would be 2, and $y$ was like 1, this is 1, somewhere like here, 3D, this would be the value of $z$. [draws coordinate axes and correctly indicates finding the point (2, 1, z)]

Int.: And how’d you come up with that? Can you say a little more about that?

Kyle: I know if this is the $x$ axis and $x$ is always the first one, variable, number in the, in the parenthesis, and $y$ is the second number in the parenthesis [points to the $x$ and $y$ in $f(x, y)$], then $z$ would be what we’re, the output of it… So which, is it a function, maybe. I’m not sure… Cuz I think if, if [MV Graph] A is, if A is not a function, for sure [MV Graph] B is not a function also because it’s basically the same princ-, same idea… That for every value, for every $x$ and $y$ value there’s a different $z$ value.
Kyle generalised by *relating objects*. Unlike Kenny, who related back to his experience with \( f(x) \), Kyle related two new objects: the \( f(x, y) \) notation and the \( \mathbb{R}^3 \) coordinate axes. As evidence, Kyle wrote \( f(x, y) = z \), then described that he would go “over \( x \) and across \( y \)” because “\( x \) is always the first one… in the parenthesis, and \( y \) is the second number.” Directly following his linking of representations, Kyle concluded that \( z \) was the output. Hence it seems that connecting the representations afforded his realization that \( z \) was the output. Kyle then generalised univalence by *relating objects* again, this time MV Graphs A and B. Specifically, he related them by focusing on a *property* (which he called the “same idea”). Comparing graphs seemed to support Kyle’s generalisation of univalence. He seems to have noticed that MV Graphs A and B were similar in that they were not univalent, which led to Kyle’s correct statement of univalence in \( \mathbb{R}^3 \) as each \((x, y)\) mapping to exactly one \( z \).

### 7.1.3 Discussion

Function notation served as a useful tool for students in generalising. Kenny and Kyle made sense of \( f(x, y) \) as having a single output. This allowed Kenny to conclude he could perform a vertical line test with a line parallel to the output axis, and led to Kyle correctly stating the univalence criterion for a function of two variables. This finding implies that unpacking function notation during instruction may be one way to support students in building a normative understanding of univalence and the vertical line test.

### 7.2 Generalising a notion of function as having an input(s) and an output

Thinking about inputs and outputs provided a powerful way for students to generalise their notion of function from the single- to multivariable case. Wendy and Ike stated that in the single-variable case, a function had one output for each input. In
the multivariable post-interview, both stated that they believed this was also the case for a multivariable function, but perhaps there were two inputs. Ike’s thinking differed from Wendy in two ways. First, he thought of the input and output for a multivariable function in the context of a trace (a two-dimensional cross-section) of the graph. Secondly, Ike’s notion of input-and-output was tied to an idea of function as equation or function as pattern.

7.2.1 Generalising one input, one output to one (or two) inputs, one output

Both Wendy and Ike generalised a notion of single-variable functions having one input and one output to conceiving of multivariable functions as having two inputs and one output. Moreover, in generalising, both followed a similar path in which they stated and modified a definition. Their definitions were as follows:

Excerpt 9. MV Pre-interview, MV Graphs A and B

Wendy: [explaining why she drew lines on the graph of the sphere; Figure 14] I knew that a function in 2D is not a function unless it only generates one $y$ output for every $x$ input.

Ike: So I guess for a function generally there’s only one output for one input.

Figure 14. Wendy’s sphere, MV Pre-interview
Wendy applied her definition to the first task in the multivariable post-interview:

Excerpt 10. MV Post-interview, MV Table 1

Wendy: You have your input of $x$ and $y$, and then your output is $z$. And there’s only one number in each of the cells that are answers… I’m pretty sure that one of the definitions of a function is that it only outputs one value. That’s my reasoning.

In evaluating MV Table I based on her definition of function, Wendy implemented a prior idea. In the next task (MV Table II), she modified her idea. Ike, in considering MV Graph 3, made a similar modification. The modification was subtle, and consisted of conceiving of two inputs rather than one:

Excerpt 11. MV Post-interview, MV Table 2

Wendy: It has all the same components of the last [table], like really all you need is an input and an output… it still has one input, or two inputs and one output.

Excerpt 12. MV Post-interview, Graph B

Ike: It makes me question what a function is when you get to 3D because it’s a lot different… it’s like it’s never really assured by just one variable… I guess with the thing about functions is it’s, you have an input variable and an output variable… But then when it’s three variables I guess it might be that there’s, there’s two inputs and one output … Because I guess if you define two points then you probably know the third point, because like if you said $y$ and $z$ then it would only go through the $x$ spot in one point.

Wendy and Ike both modified their definitions from one input mapping to one output to “two inputs and one output.” It is unclear what generalising action supported Wendy’s modified idea, but for Ike it seemed to be discerning differences (Ellis, 2007, p. 225). In particular, Ike identified 3D as “a lot different”. He compared single-variable functions, in which a single independent variable determines the value of a dependent variable, and multivariable functions, in which the values of two
independent variables determine the value of a dependent variable. Viewing a multivariable function as requiring one to fix two variables’ values seems to have resulted in Ike thinking of a multivariable function as having two inputs.

7.2.2 Input, output, and traces

For two tasks, Ike thought about input and output in the context of a two-dimensional trace of the graphs. A trace of a surface “is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes” (Briggs & Cochran, 2011, p. 778). For example, the $xy$ trace of $f(x, y) = x^2 + y^2$ is a circle. Though Ike did not use the word trace, I will use this term because in my judgment it accurately captures his reasoning. Ike thought about two tasks by considering traces. First, he considered the $yz$ trace of the graph of $f(x, y) = y^2$. Ike noticed that $x$ was a free variable and in reasoning about whether the graph represented a function, he attended only to $y$ and $z$ (Excerpt 13). He thought of this as ‘collapsing’ (Excerpt 14). Ike used the ‘collapsed’ graph (trace) to reason that the graph represented a function because each input corresponded to exactly one output.

**Excerpt 13.** MV Post-interview, MV Graph 7 (parabolic surface)

Ike: It’s a, like I think it’s this [MV Graph b2, a parabola] put in 3D... it looks roughly like $z = y^2$, which there is an input of $y$ and an output of $z$. $x$ is irrelevant so you can extend it outward however far you want. **I would say it is a function because there is an input that gives you a certain output.** And given an input of $y$ it only corresponds to one $z$ value.

Ike then considered traces of MV Graph 8, attending to inputs and outputs as he sought to determine whether it represented a function.
Excerpt 14. MV Post-interview, MV Graph 8 (cylinder)

Ike: I feel like this doesn’t work like the previous [MV Graph 7] because in the previous the, when you collapsed it to a two dimensional shape it was a function, but this circle isn’t a function… $xz$ would be easier to see. It would just be two lines, which wouldn’t make sense as a function because, for example if you had those two lines and you set $z$ to a certain value, there’s two corresponding values. But if you had those two lines and you set $x$ to certain values, there’s only two values where $x$ can even happen and then there’s infinite $z$’s. So I feel like there’s, there’s no input-output relationship for the entire shape.

In thinking about inputs and outputs, Ike generalised by implementing a prior idea. Input and output had been one of the ways he thought about whether a single-variable relation represented a function (Excerpt 15), and he had also considered inputs and outputs in the multivariable pre-interview (Excerpt 16).

Excerpt 15. SV Interview, SV Tables

Ike: I guess if you can get two possible outputs from one input, that doesn’t make sense… because if you do an operation to a number, say the input, then it doesn't, you shouldn’t be able to, if you simply do an operation on an input then it can’t equal three different numbers at once.

Excerpt 16. MV Pre-interview, MV Graph 2

Ike: I guess for a function generally there’s only one output for one input.

These excerpts provide evidence that Ike generalised that a multivariable function would have exactly one output for each input. It is notable that part of Ike’s reasoning was tied to a notion of an equation. That is, Ike’s justification that a function should
have exactly one output for each input was “if you simply do an operation on an input then it can’t equal three different numbers at once.” The phrase ‘do an operation’ suggests he was imagining computing an output via an algebraic rule.

7.2.3 Discussion

Other researchers have noted that thinking of functions in terms of inputs and outputs provides a useful way for students to generalise (Dorko & Weber, 2014; Kabael, 2011) and findings from this study provide further evidence that this is the case. Students in this study thought about the univalence requirement in terms of inputs and outputs. For example, Wendy and Ike defined univalence in the single-variable case as each input having exactly one output. They generalised that a multivariable function would have two inputs, but still one output. Students’ use of this input-output model is a good example of their generalising by stating and modifying definitions and implementing prior ideas.

7.3 Generalising the vertical line test

One of the behaviours students repeated across interviews was drawing lines parallel to an axis (or axes) of the graphs. In doing so, the students sought to generalise the vertical line test. The instances of generalising captured in this category include students either drawing a line or lines parallel to an axis or axes of a graph or verbally describing such a line.

Recall that a graph of \( z = f(x, y) \) represents a function if a line parallel to the \( z \) axis for each \((x, y)\) in the domain intersects the graph exactly once. This is a normatively correct generalisation of the vertical line test for \( \mathbb{R}^3 \). Students often arrived at this normatively correct understanding, but frequently also considered non-normatively correct generalisations of the vertical line test. Specifically, students considered (i) applying the vertical line test to a trace of the graph and (ii) non-normative orientations, such as lines parallel to the \( x \)-axis and \( y \)-axis.

Students’ lines parallel to the \( y \) axis suggest that conceptualising \( y \) as an independent variable can be an obstacle in coming to understand multivariable functions. Indeed, a key moment in students’ determining the correct orientation was their realisation that \( y \) was independent (e.g., Excerpt 23).
7.3.1 Applying the vertical line test to a trace of a graph

Kenny generalised the vertical line test in the multivariable pre-interview by applying the test to traces of the \( \mathbb{R}^3 \) graphs. His use of the vertical line test on traces of the graphs appeared to be a way of transforming the new situation (\( \mathbb{R}^3 \) graphs) into a familiar situation (\( \mathbb{R}^2 \) graph). For example, he described,

Excerpt 17. MV Pre-interview

Kenny: The ones that are planes, basically [are functions]. [Draws a plane] Because like a plane would just be a line extended into the \( z \), I think, so all the other ones have, would fail if it was just two dimensional, it would fail the line test for sure [draws a circle with two vertical lines through it]. So if any of them are functions then it would be...the planes, I think... I haven’t really done a whole lot with like 3D graphs, but as a default if you have a line that’s like, you can’t distinguish whether or not it’s a line in the, or if it’s like a completely a plane if you’re only looking at it in two dimensions. So I’m just kind of assuming that it’s all like \( z \) coordinates as well, like if this line is all these \( x \) and \( y \) coordinates, then it’s also the, all the different \( z \) coordinates as you extend it like these different ways.

Kenny: [MV Graph 6] if it was in two dimensions it would just look like that, I think, something like that [Figure 15], and it’s extended that way, so that fails that [draws a vertical line] so that’s definitely not a function, I think. And same with the ones that are like spheres, or like different circular shapes. They would fail that.

Figure 15. Kenny’s \( xz \) traces of MV Graph 6, MV Pre-interview

Kenny made several generalisations as he worked with the graphs in the multivariable pre-interview. His first generalisation was that if an \( \mathbb{R}^3 \) graph does not pass the
vertical line test, it is not a function. Kenny’s use of the vertical line test on all the tasks and his statements that some of the graphs ‘failed’ are evidence of this generalisation. His generalisation that the line test could be applied to $\mathbb{R}^3$ graphs is an example of extending: expanding the range of applicability because Kenny applied the line test to a larger range of cases ($\mathbb{R}^3$) than that from which the line test procedure had originated ($\mathbb{R}^2$). In applying the line test to traces, Kenny related situations: creating new. He viewed these traces as similar to the $\mathbb{R}^3$ graphs, as evidenced by his explanation “is all these $x$ and $y$ coordinates, then it’s also the, all the different $z$ coordinates as you extend it…” . That is, he seemed to view the $\mathbb{R}^2$ traces as only being able to see one slice of something that exists in $\mathbb{R}^3$. In taking traces, he transformed the $\mathbb{R}^3$ cases into something familiar, which afforded applying a known procedure. When the interviewer asked Kenny what he would do if he had to consider the $\mathbb{R}^3$ graph in $\mathbb{R}^3$ (rather than taking a trace), Kenny considered non-normative orientations of the line. This is discussed in the next section.

7.3.2 Non-normative orientations of lines

In generalising the vertical line test to $\mathbb{R}^3$, four students considered non-normative line orientations. Recall that the vertical line test for a multivariable function $f(x, y) = z$ is such that a graph in $\mathbb{R}^3$ represents a function if and only if every line drawn parallel to the $z$ axis through each $(x, y)$ in the domain intersects the graph exactly once. The students’ non-normative orientations included lines parallel to the $x$ axis (Wendy), the $y$ axis (Wendy, Ike, and Kyle), and what one student (Kenny) called “horizontal” as he drew a line within a plane. All four of these students generalised by expanding the range of applicability.

Wendy’s first attempt at applying the vertical line test in $\mathbb{R}^3$ included drawing lines parallel to each axis on the graph of the sphere (Figure 16). The interviewer asked why she had drawn the lines. Wendy said:
Excerpt 18. MV Pre-interview

Wendy:  [Because] I knew that a function in 2D is not a function unless it only generates one $y$ output for every $x$ input.

In looking at MV Graph 3 (the plane $z = 4$), Wendy returned to the graph of the sphere and compared the two graphs (Excerpt 1). Here, she used a vertical line test. Later, Wendy stated that she thought the line parallel to the $z$ axis was the correct orientation for the vertical line test in $\mathbb{R}^3$. When asked why, Wendy explained

Excerpt 19. MV Pre-interview

Wendy: Honestly I just did it **because it’s vertical**.

In drawing lines parallel to each axis and then deciding that the correct orientation was parallel to the $z$ axis, Wendy *searched for the same solution or result* (see Section 5.3.1) and *expanded the range of applicability* of the vertical line test. The lines she drew (Figure 16) in conjunction with her reference to a function in $\mathbb{R}^2$ provide evidence that Wendy was applying a phenomenon (the vertical line test as a procedure to determine whether a relation is a function) to a larger range of cases from that in which it had originated.

Ike talked about applying a line test with the line parallel to the $y$ axis for MV Graph 2 in the MV pre-interview. He chose this orientation because he was accustomed to $y$ as an output variable (Excerpt 20).

Excerpt 20. MV Pre-interview, MV Graph 2

Ike: **I wonder if there’s a sort of parallel to the vertical line test**… I’m trying to think if you could do it like $y$ equals something $x$, something $z$… like if you defined the $x$ and the $z$… **like it could be different $y$’s for the same $x$ and $z$**…
and then it would, kind of the same terms of the vertical line test, it couldn’t have, you couldn’t have two different values for two inputs and be a function.

Int.: Can you tell me why you’re thinking about picking an $x$ and a $z$?

Ike: I guess I’m used to $y$ being the output variable.

Like Wendy, Ike expanded the range of applicability when he wondered if there were a “parallel” version of the vertical line test. His use of the vertical line test for $f(x, z) = y$ is an example of relating situations by connecting back. In particular, Ike connected back to $\mathbb{R}^2$ situations in which $y$ is typically the output variable.

Like Wendy and Ike, Kyle considered lines that were parallel to the $y$-axis in the MV Pre-interview. Kyle said,

Excerpt 21. MV Pre-interview, MV Graph 1 (Sphere)

Kyle: So for something to be a function, for every $x$ value there can only be one $y$ value… and you can test that with the vertical line test because what it’s doing is it’s, it’s keeping $x$ the same but showing, showing different values of $y$… and if there’s more than one value of $y$, then it, then it doesn’t pass… but if it’s $x$, $y$, and $z$, it could be, it could be either the vertical line test for $z$ [gestures to indicate lines parallel to the $z$ axis], or the I guess the same principle applies if it’s going that way… if this was the graph and these were parallel to the $y$ [Figure 17].

Figure 17. Kyle's lines parallel to the $y$ axis

Kyle then thought about multivariable function notation (Excerpt 8) and talked about graphs in $\mathbb{R}^3$ as every $(x, y)$ having a $z$ value. This supported his generalising the
vertical line test as parallel to the \( z \) axis, as evident in his responses to MV Graph 3 and MV Graph 6:

Excerpt 22. MV Pre-interview, MV Graph 3 (plane \( z = 4 \)) and MV Graph 6

Kyle: That one looks like it could be a function because the \( z \) values… because, cuz for every \( x \), \( x \) and \( y \), okay, so in, if it’s linear like this, \( f(x) = y \). For every \( x \) value there’s only one \( y \) value. **So my thought now is that every \( x \) and \( y \) value there’s only one \( z \) value**… that would be a function… because for every \( x \) and \( y \) value, so every point in like the \( x \) and \( y \) plane I guess if you go up \( z \), there’s only one other point like that. If, if it was, if there was things on top if it, then it wouldn’t work because then there’d be multiple points… and then [MV Graph 6] **I’d say is not a function because there’s something on top of that plane** [draws vertical line, Figure 18].

Figure 18. Kyle's vertical line on MV Graph 6

Like Wendy and Ike, Kyle explicitly referred back to the \( R^2 \) case and **expanded the range of applicability** of the vertical line test. As evidence, Kyle said he thought “the same principle applies.”

Similar to Wendy, Ike, and Kyle, Kenny **expanded the range of applicability** of the vertical line test and drew a non-normative line orientation. However, while Wendy, Ike, and Kyle expanded the vertical line test as a procedure that could be used in \( R^3 \), Kenny expanded the idea of uniqueness of a function ‘not repeating.’ This had been an important characteristic for him in the single-variable classification tasks:

Excerpt 23. SV Interview, SV Tables 2 and 3

Kenny: [SV Table 2] does [represent a function] I think, yeah, it does because **none of the \( x \) values have a repeating, repeat**. And then [SV Table 3] does as well for the same reasons.
Recall that Kenny had originally applied the vertical line test to traces of the graph (Excerpt 17, Figure 15). When asked what he would do if he had to consider the graphs in $\mathbb{R}^3$, Kenny said

**Excerpt 24. MV Pre-interview, MV Graph 3 (plane $z = 4$)**

Kenny: I don’t know, because I don’t know if it’s a rule that you can’t have the same, … I don’t think it would be because even for this [MV Graph 3] it fails like, not a vertical line test but like a horizontal line test for like a plane, because there’s more than one $z$… so if that’s how that works, which is probably how it works, then I don’t think any of them are functions.

Int.: Can you give me like, how would you define function?

Kenny: A function is any manipulation of variables that do not like, I want to use the word ‘repeat’, but it’s where they don’t have the, they don’t double back and they don’t have like any $x$, $y$’s, or $z$’s that once you, they don’t repeat at all, so they’re all like unique and individual.

Kenny was concerned with his horizontal line intersecting the plane at what he perceived to be multiple $z$ values. He explained that he thought of a function as not ‘doubling back’ and that each coordinate should be individual. In stating this, he also engaged in generalising by modifying an idea. Specifically, he modified his conception of function in $\mathbb{R}^2$ as not having repeating $x$ values (Excerpt 23) to not having repeating $x$, $y$, or $z$ values.

7.3.3 Discussion

Though four of the five students sought to generalise the vertical line test, they were unsure about how it would generalise, considering non-normative orientations of the line and applying it to traces of the graph. Students’ lines parallel to the $y$ axis suggest that conceptualising $y$ as an independent variable can be an obstacle in coming to understand multivariable functions. Indeed, a key moment in students’ determining the correct orientation was their realisation that $y$ was independent (e.g., Excerpt 23).
8. Overall discussion and conclusions

Collectively, the findings from this study reinforce others’ findings that generalising aspects of the function concept is non-trivial. However, findings also indicate four of the five students generalised that a multivariable function must be univalent and that in the multivariable function $g(x, y)$, $(x, y)$ is an input and $z$ is an output. These generalisations are consistent with the mathematical community’s models for function.

While other studies have focused on domain and range and graphing, this study builds on that body of literature by documenting that generalising what it means to be a function from the single- to multivariable case can be difficult for students. In particular, the findings reinforce Dorko and Weber’s (2014) finding that the shift from conceptualising $y$ as a dependent variable in $y = f(x)$ to an independent variable in the $f(x, y)$ case is an important transition for students. Evidence in support of this claim includes students’ non-normative orientations of lines for the vertical line test, particularly the use of lines parallel to the $y$ axis (e.g., Figure 17; Excerpt 20) and one student’s statement that he was accustomed to $y$ as an output variable (Excerpt 20). Findings also indicate, however, that attending to function notation can support students in rethinking the role of $y$ (e.g., Excerpts 7 and 8).

It follows that two suggestions for instruction are (1) recognising that students may experience confusion as to the role of $y$ in $f(x, y)$, resulting in difficulties such as identifying whether or not a multivariable situation represents a function and finding domain and range and (2) unpacking function notations to support students in conceptualising $y$ as part of the input and/or as an independent variable.

Additionally, students’ trouble with $y$ might be alleviated if we did not use it so much in algebra, or if we occasionally made it an independent variable. For example, we might ask questions like those shown in Figure 19.

<table>
<thead>
<tr>
<th>$m$</th>
<th>-4</th>
<th>9</th>
<th>0</th>
<th>0</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>4</td>
<td>-6</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

Could $m$ be a function of $n$? Could $x$ be a function of $y$?
Could $n$ be a function of $m$? Could $y$ be a function of $x$? $x = y^2$
As another point of discussion, findings from this study and others indicate the power of function notation and equations in students’ thinking (Dorko, 2016; Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Martínez-Planell & Gaisman, 2013). In particular, students seemed to have associated meaning to the $f(\_\_\_)$ symbol template. Moreover, given other studies’ findings that algebraic expressions such as $x^2 + y^2$ influence how students think about graphs in $\mathbb{R}^3$, I contend this suggests the existence of symbolic forms (Sherin, 2001) for function notation and canonical expressions such as $x^2$ and $x^2 + y^2$.

In the following sections, I answer the research questions, then offer ideas for future research.

8.1 Answers to research questions

One question this study sought to answer was *In determining whether a multivariable relation represents a function, what mathematical ideas do students generalise?* Findings indicate that students generalise the vertical line test, the notion of input and output, and univalence. Students may also generalise common conceptions about single-variable functions, such as conflating ‘function’ with ‘equation.’ It is important to note that in generalising the vertical line test, students may consider non-normative orientations, particularly generalising that the vertical line test in $\mathbb{R}^3$ is performed with a line parallel to the $y$ axis. A conception of function-as-equation is at odds with how the mathematical community defines function. However, only one student retained this as a definition for function throughout multivariable calculus. The other four students, despite initial non-normative conceptions, arrived at what the mathematical community would deem a normative understanding of function.

The existence of non-normative generalisations provides evidence of the value of the actor-oriented perspective. For example, the actor-oriented lens allows researchers to identify that the transition of $y$ from an independent to dependent variable is difficult for students, and why students have difficulty with this transition. That is, in this example the actor-oriented perspective allows researchers to identify that students focus on objects (e.g., thinking of the vertical line test as a test in which the line is
parallel to the \( y \) axis). This explains students’ difficulty with \( y \) as an independent variable in \( f(x, y) \).

The second question this study sought to answer was *What is the nature of students’ generalising activity as they generalise their understanding of what it means for a single variable relation to represent a function to what it means for a multivariable relation to represent a function?* Findings indicate that students relate back to prior situations (particularly single-variable situations), implement previous ideas, sometimes with modifications (e.g., univalence, the vertical line test, input-output), search for the same result (e.g., to see if a procedure remains valid for the multivariable case), apply definitions, and expand the range of applicability of previous ideas.

Researchers have noted that two ways undergraduates generalise from two-space to three-space are treating \( \mathbb{R}^3 \) exactly as they would \( \mathbb{R}^2 \) and adapting \( \mathbb{R}^2 \) ideas for the \( \mathbb{R}^3 \) context (Dorko, 2016; Dorko & Lockwood, 2016; Dorko & Weber, 2014; Jones & Dorko, 2015; Martínez-Planell & Trigueros, 2012; Martínez-Planell & Gaisman, 2013). Students’ use of the vertical line test on traces of graphs (Section 6.3.1) and thinking about “collapsing” graphs (Section 6.2.1) provide evidence for a third way of generalising: modifying the \( \mathbb{R}^3 \) situation. The use of the word ‘collapsing’ indicates that students were thinking about transforming the \( \mathbb{R}^3 \) situation, which distinguishes this way of generalising from the aforementioned two ways. In creating traces, students created an \( \mathbb{R}^2 \) situation that they could reason with, and they used that \( \mathbb{R}^2 \) situation to inform their thinking about the \( \mathbb{R}^3 \) situation.

8.2 Suggestions for future research

One area for future research is exploring how instructors teach multivariable functions. That is, excepting Kabael’s (2011) evaluation of teaching using the input-output model, and Gaisman and Martínez-Planell’s (2011) analysis of how textbooks present multivariable graphing, studies about student learning of multivariable functions have focused on students’ perspectives. Now that we know more about how students think about multivariable functions, looking at the ways instruction supports or does not support such thinking is a logical next step for improving students’ learning.
Relatedly, studies about how instructors think about generalising and how they try to support students in generalising can complement studies like this one about students’ own attempts to generalise. What in-class opportunities do students have to generalise, and what activities are most productive for their generalising? We can build on research of the ways students tend to think about ideas to develop, pilot, and refine instructional activities.
Chapter 3: Expansive Generalisation, Reconstructive Generalisation, Assimilation, and Accommodation

1. Abstract

The purpose of this paper is to argue that Piaget’s constructs of assimilation and accommodation align with Harel and Tall’s (1991) framework for generalisation in advanced mathematics. Based on what they imagined to be the cognitive processes underlying generalisation, Harel and Tall proposed that generalisation might be expansive (occurring when a student expands the applicability range of an existing schema without reconstructing it), reconstructive (occurring when a student reconstructs a schema to widen its range of applicability), or disjunctive (occurring when a student constructs a new, disjoint schema to deal with a new context). I contend that expansive and reconstructive generalisation align with assimilation and accommodation, respectively. I provide ‘proof of concept’ using data from a study of students’ generalisation of function and graphing from $\mathbb{R}^2$ to $\mathbb{R}^3$. Further, I show how linking Piagetian constructs to Harel and Tall’s work provides a theoretical explanation for other empirical findings about generalisation.

2. Introduction

In 1991, Harel and Tall suggested a framework for describing the “different qualities of generalisation in advanced mathematics” (p. 31). Based on what they imagined to be the cognitive processes underlying generalisation, they proposed that generalisation might be expansive (occurring “when a student expands the applicability range of an existing schema without reconstructing it”), reconstructive (occurring “when a student reconstructs a schema to widen its range of applicability”), or disjunctive (occurring when a student “constructs a new, disjoint schema to deal with a new context”) (Harel & Tall, 1991, p. 1). Empirical use of this framework has provided examples of these types of generalisation in student thinking about algebraic patterns, limits, and integrals (Fisher, 2008; Jones & Dorko, 2015; Zazkis & Liljedahl, 2002). These studies provide evidence
that researchers (myself included) have found Harel and Tall’s (1991) framework useful for describing how students generalise.

Generalisation is a key component of mathematics. For example, mathematicians often seek general formulae (e.g., the sum of the first \( n \) terms of a geometric series) or wonder if a rule that holds in a particular dimension also holds in dimension \( n \). However, generalisation is not limited to professional mathematicians. For example, kindergarteners engage in generalisation when they seek the next shape in the pattern

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Multivariable calculus students must generalise their notion of function from the single-to multivariable case. Because generalisation is so critical to mathematical thought, research that investigates how people generalise is an important part of supporting student learning. In particular, students often struggle to form correct generalisations (e.g., Dorko, in preparation; Dorko & Weber, 2014; Jones & Dorko, 2015; Kabael, 2011; Martínez-Planell & Gaisman, 2013, 2012; Martínez-Planell & Trigueros, 2012). One cost of this is that students who struggle with mathematics may switch out of STEM studies. Hence efforts to better understand how students generalise and how instructors can support their generalisations could help solve the problem of retaining STEM majors (c.f. Bressoud, Carlson, Mesa, & Rasmussen, 2013; Rasmussen & Ellis, 2013; Uysal, Ellis, & Rasmussen, 2013).

Descriptions of how people generalise often come in the form of frameworks. Such frameworks are useful because they provide language to describe and account for qualitative differences in students’ thinking and activity, and knowing what students attend to when generalising can inform instruction and the development of mathematical activities to support productive generalisation. For instance, Harel and Tall’s stated aim was to “rationali[s]e the use of [the] terms [generalisation and abstraction] in a cognitive context… using this analysis we will be able to suggest pedagogical principles designed to assist students’ comprehension of advanced mathematical concepts” (1991, p.1). The pedagogical principles they suggest are that instructors should provide experiences for
students to understand a current situation and expansively generalise to a more general\textsuperscript{9} case. However, Harel and Tall note that some mathematical ideas require reconstructive generalisation, and instructors should “provide the learner with the conditions in which this reconstruction is more likely to take place” (p. 3). Hence some of the value of this framework seems to be identifying what mathematical ideas students tend to generalise expansively, what ideas they must reconstruct, and the specific facets that must be reconstructed (c.f. Jones & Dorko, 2015).

Other frameworks (e.g., Ellis, 2007) allow researchers to distinguish between generalisation as a product\textsuperscript{10} and generalisation as a process. Separating these is empirically useful because it helps to identify what students have generalised (product or “reflection generalisation”) and what they attended to while generalising (process or their “generalising actions”). For example, Dorko and Weber (2014) described a student who generalised that the range of a multivariable function \( f(x, y) \) would be a set of \( z \) values because her experience in the single-variable case had been that range applied to the values on the vertical axis. Hence she reasoned that the range of \( f(x, y) \) would be a set of \( z \) values because \( z \) is the vertical axis in standard \( \mathbb{R}^3 \) axes. Dorko (in preparation) noted a similar example of a student correctly generalising the vertical line test to \( \mathbb{R}^3 \) by focusing on the word ‘vertical’. In both examples, the students’ reflection generalisation was correct, but their generalising activity attended to surface features. This knowledge is useful to instructors because it illustrates that students may form correct generalisations that are not based on ideal mathematical reasoning. Frameworks can also help reveal ways students arrive at the same generalisation via different means.

In my work with Harel and Tall’s (1991) framework, I was struck by what seemed to be parallels between expansive and reconstructive generalisation and Piaget’s constructs of assimilation and accommodation (Piaget, 1980). The similarity occurred to me as I attempted to decide whether instances in my data regarding students’ generalisation of function and graphing from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) were expansive or reconstructive generalisations.

\textsuperscript{9} I use ‘general’ here in its mathematical sense. For example, students might work with several examples of Pythagorean triples and generalise the general rule (case) \( a^2 + b^2 = c^2 \).

\textsuperscript{10} That is, “a relationship that holds between all members of some set of objects” (Mitchelmore, 2002, p. 160).
There were several instances that I felt could fall in either category. I sought empirical studies that employed Harel and Tall’s framework to use as examples of how others had classified data. However, I found only three studies (Fisher, 2008; Jones & Dorko, 2015; Zazkis & Liljedahl, 2002). Though many researchers cite Harel and Tall’s definition of generalisation (e.g., Ellis, 2007; Mitchelmore, 2002) or offer hypothetical examples of the three types of generalisation (e.g., Greer & Harel, 1998; Mitchelmore, 2002), I was surprised by the lack of papers that have applied the framework to empirical data. This made me wonder if others had also experienced difficulty distinguishing between expansive and reconstructive generalisation and hence chosen different frameworks with which to conduct their work. Difficulty distinguishing between categories is problematic for researchers when coding data, and a clearer delineation might allow more researchers to take up what I contend is a powerful framework for thinking about how students generalise.

Because there seemed tenable links between expansive generalisation and assimilation and reconstructive generalisation and accommodation, I wondered if a student experiencing disequilibrium or perturbation, the distinguishing feature between assimilation and accommodation, could serve as a criterion to distinguish between expansive and reconstructive generalisation (respectively). Tall (1991) hints that assimilation and accommodation might underlie what Skemp (1979) called expansion and reconstruction of cognitive structures, and other researchers have described assimilation and accommodation as possible mechanisms for transfer (e.g., Čadež & Kolar, 2015; Hohensee, 2014; Wagner, 2010). Hence the purpose of this paper is to answer the following research question: *In what ways do the constructs of assimilation and accommodation relate to generalisation?* As such, the purpose of this paper is to explain the parallels I saw between expansive generalisation and assimilation and reconstructive generalisation and accommodation. I argue that assimilation and accommodation align with expansive and reconstructive generalisation, respectively. Specifically, I contend that students generalise expansively by assimilating a new situation to an existing scheme, and that when they form reconstructive generalisations, they do so by accommodating existing schemes.
The paper is structured as follows. First, I review the use of Harel and Tall’s framework in the literature and provide a description of the data set I worked with to tease out possible relationships between assimilation, accommodation, expansive generalisation, and reconstructive generalisation. Then, I provide an example of an expansive generalisation that can be explained in terms of assimilation, followed by an example of a reconstructive generalisation that can be explained in terms of perturbation and accommodation. I conclude by discussing how connecting expansive and reconstructive generalisation to assimilation and accommodation has explanatory power for empirical findings about students’ generalisations in advanced mathematics. The examples from my own data and new interpretation of published empirical findings provide proof of concept that assimilation and accommodation align with Harel and Tall’s (1991) framework. This benefits the field by linking and building theory. Further, I contend these links make Harel and Tall’s framework easier to use by situating it in widely-understood language, which benefits the field by improving an analytic tool. Finally, I argue a value of this work is that it provides new insight into why certain ways of thinking (e.g., function machine) are likely to yield productive generalisations.

3. Harel and Tall’s framework

Harel and Tall (1991) distinguish between three types of generalisation and provide examples in the context of systems of equations and vector addition. Table 2 contains definitions of the three types and two examples of what each type to help the reader understand what each type might look like in practise.
<table>
<thead>
<tr>
<th>Type</th>
<th>Harel and Tall’s example in the context of systems of equations</th>
<th>Harel and Tall’s example in the context of vector addition (see also Jones &amp; Dorko, 2015)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Expansive generalisation</strong></td>
<td>A student understands that when solving linear equations in one variable, adding an expression to each side or multiplying by a non-zero constant does not change the solution set. When learning how to solve a 2 x 2 system of equations, she uses these understandings to eliminate x. This yields an equation she can solve for y, the value of which she substitutes to solve for x.</td>
<td>A student understands vector addition (&lt;a,b&gt; + &lt;c,d&gt;) as performing addition twice. The student generalises her understanding of addition in (\mathbb{R}) by “repeating it across more terms” (Jones &amp; Dorko, 2015, p. 156).</td>
</tr>
<tr>
<td><strong>Reconstructive generalisation</strong></td>
<td>A student understands that when solving linear equations in one variable, adding an expression to each side or multiplying by a non-zero constant does not change the solution set. He learns how to solve a 2 x 2 system of equations as a list of procedures: eliminate (x), solve for (y), substitute the (y) value to find (x). When presented with a 3 x 3 system, he “begins to see the underlying meaning of the solution process” (p. 2) and reinterprets his list of procedures as adding expressions or multiplying by non-zero constants to eliminate variables, then using the value of isolated variables to solve for other variables.</td>
<td>A student understands vector addition (&lt;a,b&gt; + &lt;c,d&gt;) as performing addition twice. The student generalises her understanding of addition in (\mathbb{R}) by repeating it across more terms. The student learns the geometric interpretation of vector addition as placing vectors head to tail and finding the resultant vector. The idea of vector addition “may not exist for the student in basic addition in (\mathbb{R}), and consequently the underlying idea of “addition” itself is reconstructed for the new (\mathbb{R}^2) context” (Jones &amp; Dorko, 2015, p. 156).</td>
</tr>
</tbody>
</table>

Table 2. Harel and Tall’s (1991) framework
**Disjunctive generalisation**
“occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available” (Harel & Tall, 1991, p.1).

A student understands that when solving linear equations in one variable, adding an expression to each side or multiplying by a non-zero constant does not change the solution set. He learns how to solve a $2 \times 2$ system of equations as a list of procedures: eliminate $x$, solve for $y$, substitute the $y$ value to find $x$. When presented with a $3 \times 3$ system, he learns it as yet another list of procedures: eliminate $x$, eliminate $y$, solve for $z$, use the $z$ value to solve for $y$, use the $y$ and $z$ values to solve for $x$. In both the $2 \times 2$ and $3 \times 3$ cases, the student does not connect the idea of eliminating variables to the original understanding that adding an expression to each side or multiplying by a non-zero constant does not change the solution set.

One critique of this framework is that disjunctive generalisation is not actually a form of generalisation. Jones and Dorko (2015) argue that because the new conception is separate from the original conception, the student is not generalising. They suggest renaming this category as *disjunctive understanding*. Because I did not observe instances of disjunctive understandings in the data set reported on in this paper, I was unable to tease out any relationships between this category and assimilation and accommodation; whether such relationships exist is a potential avenue for future work.

Because my purpose is to argue that assimilation and accommodation provide a theoretical underpinning for students’ expansive and reconstructive generalisations, I now turn to definitions of assimilation and accommodation.
4. Assimilation and accommodation in the literature

Piaget theorised that assimilation and accommodation are the mechanisms by which people learn. Piaget believed that a person constructs the reality around him or her, organising their experiences by developing schemes. A scheme is an “organis[ation of mental and affective activity whose contents could be highly nuanced and could contain several layers of structure” (Thompson, 2016, p. 436).

Because Ernst von Glasersfeld and Les Steffe are widely recognised as leading scholars of Piagetian constructs in mathematics education, I employ their definitions of assimilation and accommodation. *Assimilation* is “the integration of new objects or new situations and events into previous schemes” (Piaget, 1980, p. 164 as cited in Steffe, 1991, p. 192). von Glasersfeld (1995) writes, assimilation “comes about when a cognising organism fits an experience into a conceptual structure it already has” (p. 62).

In contrast, *accommodation* is a modification of a scheme that occurs due to perturbation (Glasersfeld, 1995). Steffe and Thompson (2000) clarify that an accommodation is a *permanent* modification, meaning an observer would see the modification if the scheme were used in a situation independent from the one in which it was modified. Accommodation occurs when a person’s attempt to assimilate a situation to a scheme has an unexpected result, causing a perturbation. The person reviews the scheme as a collection of sensory elements… [which] may reveal characteristics that were disregarded by assimilation… one or more of the newly noticed characteristics may effect a change of the recognition pattern and thus the conditions that will trigger the activity in the future. Alternatively… a new recognition pattern may be formed to include the new characteristic, and this will constitute a new scheme. In both cases there would be an act of learning and we would speak of an accommodation (von Glasersfeld, 1995, p. 65).

In the next section, I explain the context of the data I used to investigate connections between Harel and Tall’s framework and assimilation and accommodation. The purpose of these data is to show an ‘existence proof’ that assimilation and accommodation align with expansive and reconstructive generalisation (respectively).
5. Context

The data excerpts in this paper come from a longitudinal study of calculus students’ generalisation of the function concept from the single- to multivariable setting. I conducted four task-based clinical interviews (Hunting, 1997) with each of five students over the span of their differential, integral, and multivariable calculus courses. The total time for all four interviews ranged from 4.25 to 5.67 hours per student. Students answered questions about single- and multivariable topics (e.g., classifying relations as function or non-function; domain and range; graphing) while they were enrolled in differential calculus and while they were enrolled in multivariable calculus. This design provided insight into both students’ initial sense-making of how ideas from $\mathbb{R}^2$ might generalise to $\mathbb{R}^3$, and the sense students made of those topics after instruction. Space constraints prohibit listing all the tasks students answered, but the tasks discussed in this paper are as follows. The italicised writing indicates the course in which students were enrolled when they answered that task. In all graphing questions, students were provided with a set of $\mathbb{R}^3$ coordinate axes.

1. What does $f(x)$ mean to you?
   *Differential calculus*

2. What do you think $f(x, y)$ means?
   *Differential calculus*

3. Graph $y = x$ in $\mathbb{R}^3$.
   *Multivariable calculus*

4. Graph $y = 2x + 1$ in $\mathbb{R}^3$.
   *Multivariable calculus*

5. Graph $z = 4$ in $\mathbb{R}^3$.
   *Multivariable calculus*

Because I wished to tease out possible connections between expansive generalisation, reconstructive generalisation, assimilation, and accommodation, the first step in my data analysis was to review the data and identify instances of generalisation. I followed Harel and Tall’s (1991) definition of generalisation as “the process of applying a given argument in a broader context” (p. 1). I then coded these instances as expansive generalisation, reconstructive generalisation, or disjunctive understanding based on the
definitions from Harel and Tall’s (1991) framework (Table 1). Finally, I took each instance and sought to code it as assimilation or accommodation based on the definitions provided above.

I gave the data presented in this paper to another researcher, who at the time was studying generalisation in real analysis from a Piagetian perspective and hence was knowledgeable about and experienced with identifying assimilation and accommodation in practise. This person coded the data separately, and this persons’s codes were the same as my own. The reader may wonder why I did not have the second coder review all of the data. There are two reasons for this. The first is that the goal of independent coders was not to measure inter-rater reliability, but rather to check that I understood assimilation and accommodation as others were thinking about these constructs. The second was that I was trying to explore proof of concept of a couple of data excerpts to support what I contend is a theoretical argument, not an empirical one. That is, I wished to provide a rich example or two of assimilation as connected to expansive generalisation and accommodation as connected to reconstructive generalisation. I believe these, in conjunction with the literature findings I re-interpret (See section 8), form an adequate argument from data. An empirical paper in which two coders independently code all the data would be a useful follow-up, and one I intend to write; however, it is not the focus of this paper. I argue that because this paper contains large data excerpts, the reader can verify the veracity of coding for his or herself. Finally, other data pertaining to the tasks listed above can be found in Dorko (2016) and Dorko and Lockwood (2016).

6. Expansive generalisation and assimilation: An example

I observed examples of expansive generalisation when I asked students what they thought \( f(x, y) \) would mean. When students answered this question, they were enrolled in differential calculus. That is, they had not received instruction about multivariable functions, and hence their answers represent their initial sense-making of the notation \( f(x, y) \). Wendy’s\(^{11}\) thinking is representative of the way the students tended to answer this question.

\(^{11}\) Gender-preserving pseudonym
Excerpt 25. Assimilating $f(x, y)$ to $f(x)$

Wendy: So I know what $f(x)$ means. That means you’re, you’re using $x$ to solve for $y$, so it would be like $2x + 4$ and then whatever you get when you plug in $x$ is your $y$ coordinate. But it looks like in this one you would have something like $f(x, y) = x^2 + y^2$… Because I know that when $x$ is in the parentheses here it’s what you’re putting in for the equation. So if you’re putting $x$ into the equation when there’s just this, if there’s $y$ too, then you would put $y$ into the equation.

The second coder and I coded this as expansive generalisation because Wendy seemed to apply the notion of the function’s argument as specifying the variables in the expression to the unfamiliar $f(x, y)$ case. I propose that Wendy’s comments about the variables “in the parentheses” indicate that her function notation schema entailed a sort of $f(\ )$ template in which the specified what symbols would appear on the right side of the equation. Wendy’s generalisation that $f(x, y)$ specified an equation with $x$’s and $y$’s was expansive because she interpreted $f(x, y)$ within her already-existing meaning for $f(\ )$. As such, she expanded the applicability range of an existing schema without reconstructing it.

I argue that Wendy was able to generalise expansively because she assimilated $f(x, y)$ to her scheme for $f(x)$. I interpret Wendy’s comment “so I know what $f(x, y)$ means” as evidence that seeing $f(x, y)$ activated an existing schema Wendy had for function notation. Wendy’s schema entailed $f(x)$ as relating to an expression with $x$’s (such as $2x + 4$) because she saw the $x$ “in the parentheses” as “what you’re putting in for the equation.” She generalised that $f(x, y)$ would indicate an equation with $x$’s and $y$’s, reasoning that “if there’s $y$ [in the parentheses] too, then you would put $y$ in the equation.” As such, Wendy fit $f(x, y)$ into a conceptual structure she already had: she assimilated it.

In the next section, I provide an example of a reconstructive generalisation and how accommodation afforded the student’s reconstruction of a scheme.

7. Reconstructive generalisation and accommodation: An example

I observed reconstructive generalisation when I asked students to graph the following three equations in $\mathbb{R}^3$: $y = x$, $y = 2x + 1$, and $z = 4$. Wendy’s thinking provides an
example. Wendy is one of four students who initially drew $y = x$, $y = 2x + l$ as lines and $z = 4$ as a plane. These students then reconsidered their responses and drew all three equations as planes. The fifth student drew all three equations as lines and resisted my attempts to perturb this reasoning.

I argue that Wendy engaged in reconstructive generalisation as a result of (1) assimilating $y = x$, $y = 2x + l$ to one scheme and $z = 4$ to another, (2) comparing the [different] results, which caused a perturbation, and (3) sought to re-equilibrate by examining how she had treated $z$ in each case. This led to her reconstructing her interpretation of a free variable as meaning that variable is set to zero to an interpretation in which that variable could take on any value.

Wendy appeared to have two schemes for linear equations in $\mathbb{R}^2$: a scheme for $y = mx + b$ ($m \neq 0$) and a scheme for $y = b$. She assimilated $y = x$ and $y = 2x + l$ in $\mathbb{R}^3$ to the former and $z = 4$ to the latter. The result of the first assimilation was that Wendy drew $y = x$ and $y = 2x + l$ in $\mathbb{R}^3$ as lines on the $xy$ plane. In contrast, she drew $z = 4$ as a plane. Unprompted by me, Wendy compared her graphs, and was puzzled by the fact that two graphs were lines and one was a plane. This comparison served as a perturbation that caused Wendy to reconstruct her scheme.

In the Excerpts 26, and 27, I provide what I take as evidence of Wendy’s assimilating $y = x$ and $y = 2x + l$ in $\mathbb{R}^3$ to a scheme for $y = mx + b$ ($m \neq 0$) in $\mathbb{R}^2$. “Int.” is short for “interviewer.”

**Excerpt 26.** Assimilating $y = x$ in $\mathbb{R}^3$ to a scheme for $y = x$ in $\mathbb{R}^2$

Wendy: So if you just plug in values for $x$ and then pull out values for $y$, you’re gonna get like 0, 0, 1, 1, 2, 2 [plots these on the $xy$ plane as she says them] and then it’s just going to continue being a straight line like this… you could choose any $x$ value, really. I chose like 1. So if $x$ is 1, then $y$ is equal to $x$, so that’s also 1.

Int.: Can you label some of the coordinates that you plotted?

Wendy: Okay, so this is going to be like 1, 1, 0 and then 2, 2, 0.

Int.: Why do we get a line here?

Wendy: The way I think of it is it’s just like having a 2D graph and plotting $y = x$ and that’ll give you a line, you’re just taking it and adding and then ignoring the $z$ component… if $y = x$, you can just always assume that $z$ is 0.
Excerpt 27. Assimilating \( y = 2x + 1 \) in \( R^3 \) to a scheme for \( y = 2x + 1 \) in \( R^2 \)

Wendy: I’m thinking that it will be like the same kind of concept where we’re just ignoring \( z \) so you can say like +0z here and that will give you the same equation [writes \( y = 2x + 1 + 0z \)]. So if you went \( 2x + 1 \) that would be 0, 1 and then 1, 3… basically you would just take the same line that you would have with your \( x \) and \( y \).

Int.: And do we get a line there?

Wendy: Yeah, that’s a line… like I said we’re ignoring the \( z \) component, but you can think of it as there, you’re just, have it, 0 set to it.

I argue that Wendy assimilated these equations to her \( R^2 \) scheme. Wendy talked about the coordinate points as \( (x, y) \) tuples (e.g., “0, 0, 1, 1, 2, 2”) as she was plotting the points (Excerpt 26). Though she described the points as \( (x, y, z) \) tuples when I asked her to identify points, I posit that her thinking of the points as \( (x, y) \) tuples during the act of graphing indicates that she had assimilated the question about creating a graph in \( R^3 \) to a schema for graphing in \( R^2 \). My inference is supported by Wendy’s explicit statement that she saw \( y = x \) in \( R^3 \) as “just like having a 2D graph and plotting \( y = x \)”.

I argue Wendy’s decision about \( z \) facilitated her assimilation. We know that Wendy considered \( z \) because she said in both graphs that she was “ignoring \( z \)” (Excerpt 26 and 27) or setting it to 0 (Excerpt 3). Further, when I asked her what points she had plotted on her \( y = x \) graph, Wendy gave \( (x, y, z) \) tuples. I draw the reader’s attention to this because had Wendy not stated anything about \( z \), the possibility would exist that she did not notice \( z \) was involved. If this were the case, then Wendy’s work would be a less powerful example of assimilation. However, Wendy’s statements about \( z \) provide evidence that she (a) explicitly considered \( z \) and (b) treated it in a way that allowed her to assimilate the \( y = x \) and \( y = 2x + 1 \) in \( R^3 \) tasks to her scheme from \( R^2 \). This is in accordance with von Glasersfeld’s (1995) note that “assimilation always reduces new experiences to already

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12 It is important to note that for Wendy, writing +0z meant that she was setting \( z \) to 0 (Excerpt 26). This contrasts a normative interpretation of \( y = 2x + 1 + 0z \), in which one sees \( z \) as varying.
existing sensorimotor or conceptual structures” (p. 63). The result of Wendy’s assimilation was that she drew these graphs as lines.

I argue that Wendy assimilated $z = 4$ to a different scheme. Specifically, Wendy appeared to have a scheme for $y = c$ in $\mathbb{R}^2$. She drew a plane for $z = 4$, as described in Excerpt 28.

Excerpt 28. Assimilating $z = 4$ in $\mathbb{R}^3$ to a scheme for $y = c$ in $\mathbb{R}^2$

Wendy: I’m thinking that whenever, no matter what $x$ and $y$ equal, $z$ is always going to equal 4. So you get a plane here at 4. That’s a really bad drawing of it, but, no matter what these [gestures to $x$ axis] equal, you’re always just going to get 4.

Int.: Can you tell me a little bit more about the ‘no matter what these equal’?

Wendy: So if you’re graphing, so $z = 4$, it’s like saying $y = 4$ on a normal graph you get a line at $y$, or 4. You just get that [sketches $y = 4$ in $\mathbb{R}^2$]. Because it doesn’t matter what $x$ equals. So here I’m kind of thinking that it’s the same concept, that no matter what $y$ or $x$ equals, $z$ is always going to equal 4.

Int.: Do you, as you graphed that $z = 4$, so you pretty immediately said oh, this is a plane. Did you think about this $y$ and $x$ graph? [points to Wendy’s graph of $y = 4$ in $\mathbb{R}^2$].

Wendy: I basically, I took the concept of it and applied it.

Int.: And what’s the concept of it?

Wendy: Yeah, the concept of it is like I said even though there’s no $x$ in this equation, like we always know that $y$ is going to be equal to 4 so it really doesn’t matter what $x$ is, so that’s why there’s no $x$ in the equation.

Int.: How come $z = 4$ isn’t just a line?

Wendy: Because you’re in 3D, so if say like $x$ was 1 and $y$ was 2, you’re always, $z$ is going to equal 4.

I take Wendy’s comment “it’s the same concept” as evidence that she assimilated $z = 4$ to an already-existing scheme. What Wendy appeared to see as the “same concept” was that $y = 4$ in $\mathbb{R}^2$ “[no] matter what $x$ equals”, so $z$ would equal 4 in $\mathbb{R}^3$ “no matter what $y$ or $x$ equals.” Wendy argued that $z = 4$ was a plane using the example of $(1, 2, 4)$ as being on the graph.
I posit that Wendy assimilated $z = 4$ to a different scheme than the scheme to which she assimilated $y = x$ and $y = 2x + 1$. That is, Wendy appeared to have a scheme for constant functions in $\mathbb{R}^2$, and an element of that scheme was that $x$ was free. She expanded this to the $\mathbb{R}^3$ case by viewing $x$ and $y$ as free. In contrast, she appeared to have a scheme for non-constant linear functions, an element of which was that such functions’ graphs are lines. Wendy expanded this scheme to the $\mathbb{R}^3$ case by choosing to “ignore $z$” or, equivalent in her mind, ‘set it to 0’.

The result of Wendy’s assimilations to two different schemes resulted in two different graphs, triggering a perturbation that subsequently caused Wendy to reconstruct her scheme for non-constant linear equations in $\mathbb{R}^3$. In Excerpt 29, I began to point out to Wendy that she had drawn lines for two graphs and a plane for a third, but Wendy had already noticed this and was perturbed by it.

Excerpt 29. Perturbation

Int.: So it’s interesting to me that –
Wendy: It’s interesting to me too.
Int.: What’s interesting to you too?
Wendy: That I think of that [\(z = 4\)] like that, and then the other ones [\(y = x\) and \(y = 2x + 1\)] I don’t think of like that. So if I, if I applied what I did in \([z = 4]\) to \([y = x\) and \(y = 2x + 1]\) I would get planes again, which would look like this… because \(y\) is going to equal \(x\). I feel like I’m confusing myself.

Wendy then compared her work on the three graphs, which led her to reconstruct her notion of a free variable (Excerpt 30).
Excerpt 30. Accommodation

Int.: Okay, so do you want to look at these again? [puts $y = x$ and $y = 2x + 1$ graphs in front of Wendy]

Wendy: So if I think about it like this [points to $z = 4$ graph], so if I thought of this [$z = 4$] like I think of this [points to $y = 2x + 1$], then this [$z = 4$] would just be a point.

Int.: Can you, can you say that sentence, maybe talking about, using equations rather than ‘this’, the word ‘this’ gets hard when I do the audio, when I transcribe it.

Wendy: Okay so on the previous ones I was thinking of, I was thinking of this [$y = x$] as – this – the $y = x$ as just like $y = x$ and then I was thinking of it as $+0z$. And so out of that you get a line. But instead of thinking of this $+0y + 0x$, I thought of it as more of the $y = 4$. That no matter what the, no matter what the $y$ and $x$ values are here, the $z$ is always going to equal 4… so if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this… because $y$ is going to equal $x$. I feel like I’m confusing myself.

Int.: So, so do you think $y = x$ in $\mathbb{R}^3$ is a plane or a line?

Wendy: My initial thought was that it was a line, but now I’m unsure… my initial thought process of it’s a line is because I was thinking that you didn’t change this $x$ and $y$ coordinate, you just laid it flat, and that is the only thing you did to make it 3D here. And so you could just graph it in 2D and then just lay it flat and put a $z$ axis in it and that wouldn’t change the $y = x$. But that was if I was thinking $+0z$ which there isn’t a $+0z$. So I think that no matter what $z$ is, $y$ is always going to equal $x$. So whatever $x$ and $y$ are, you’re going to have that plane.

I interpret the change in Wendy’s graph from a line to a plane as occurring as a result of the following cognitive acts. Wendy’s statement that she found it “interesting” that she had drawn a line for two of the graphs and a plane for the third suggests that she expected the graphs to look similar. The unexpected results (the graphs did not look similar) caused a perturbation. Wendy sought to re-equilibrate (remove the perturbation) by
comparing how she approached the $y = \ldots$ equations and the $z = 4$ equation. In doing so, she noticed that in the $y = \ldots$ equations she had assumed $z = 0$, while in the $z = 4$ equation she had assumed $x$ and $y$ could take on any value. Wendy accommodated her scheme for $y = x$ from $y = x + 0z$ to $y = x, z \in R$.

In summary, I offer the above analysis to establish that assimilation and accommodation can explain students’ cognitive activity while generalising. In the next section, I offer additional examples of students’ generalising from single- to multivariable functions and how assimilation and accommodation have explanatory power for other researchers’ findings about generalisation in undergraduate mathematics contexts.

8. The explanatory power of assimilation and accommodation for generalisation

I argue that assimilation and accommodation explain a variety of empirical findings about what students generalise from $R^2$ to $R^3$. For example, consider students’ expansive generalisation of Abstract Space in Jones and Dorko (2015). We might interpret this as students assimilating two or more integrals to a scheme of integrals as measuring space. In this section, I explore empirical findings about students’ generalisation of function machine and graphing in terms of expansive generalisation, reconstructive generalisation, assimilation, and accommodation. I offer this as further evidence of the usefulness of thinking about generalisation in terms of assimilation and accommodation. Specifically, I argue that connecting assimilation, accommodation, and generalisation reveals why function machine is such a powerful model for students and why they struggle so much to graph in $R^3$. Because assimilation and accommodation are widely recognised terms in learning theory, I argue that this lens provides an easily-understood way to think about generalisation. Moreover, it provides a theoretical explanation for empirical findings.

Specifically, I will argue that the function machine is a powerful model for students’ learning of multivariable function because it provides a structure to which they can assimilate. I will also provide more evidence about students’ difficulties graphing in $R^3$ as caused by assimilating particular graphs to students’ schemes for $R^2$, similar to the example of Wendy’s thinking above.
8.1 Function machine

Thinking about functions in terms of inputs and outputs seems to help students generalise the function notion from the single- to multivariable context (Dorko & Weber, 2014; Kabael, 2011). Tall, McGowen, and DeMarois (2000) suggest that the function machine model is powerful because it provides a *cognitive root*, or “a concept that is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence [that] allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction [and] contains the possibility of long-term meaning in later developments” (p.3). The phrase “cognitive expansion rather than… reconstruction” suggests that Tall, McGowen, and DeMarois (2000) consider the function machine an idea that allows students to engage in expansive generalisation. I theorise that viewed through the lens of assimilation and accommodation, the function machine provides a scheme to which students can assimilate. Empirical findings bear this out. For example, Dorko and Weber (2014) describe how some students generalise domain and range in terms of inputs and outputs. They provide an example of a student answering *What are the domain and range of* \( f(x, y) = x^2 + y^2 \)? by saying

“Domain is your input values… the range is your output… there would be two different domains. You have your \( x \) input and your \( y \) input. Your \( x \) domain and your \( y \) domain give you a range of a different variable. It’s the range of \( z \) or \( f(x, y) \)” (Dorko & Weber, 2014, p. 8).

Dorko and Weber (2014) described this generalisation as *extending* (c.f. Ellis, 2007) a meaning that domain corresponded to inputs and range corresponded to outputs. I argue that this student could extend the meaning because he assimilated the multivariable function to his already existing scheme of function as having inputs and outputs. Kabael (2011) studied students’ generalisation of function when taught multivariable functions using the input-output model, and gives examples of students engaging in similar extension (e.g., p. 492). I theorise that Kabael’s (2011) students could do so because the function machine allowed them to assimilate multivariable functions to their input-output scheme of single-variable functions.
8.2 Graphing

Researchers have observed student difficulties with graphing in \( \mathbb{R}^3 \), particularly equations with free variables (Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). For example, students may draw \( f(x,y) = x^2 + y^2 \) as a cylinder or a sphere because they are accustomed to \( x^2 + y^2 \) representing a circle in \( \mathbb{R}^2 \) (Martínez-Planell & Gaisman, 2013). Likewise, students may think the graph of \( y = 3 \) in \( \mathbb{R}^3 \) is a line on the \( xy \) plane (Dorko, 2016; Dorko & Lockwood, 2016). As another example, students may draw \( f(x,y) = x^2 \) as a parabola (instead of a parabolic surface) in \( \mathbb{R}^3 \) (Martínez-Planell & Gaisman, 2013). I posit students assimilate the \( f(x,y) = x^2 + y^2 \), \( f(x,y) = x^2 \), and \( y = 3 \) equations to their \( \mathbb{R}^2 \) schemes for the expressions \( x^2 + y^2 \), \( x^2 \), and \( y = 3 \), respectively. In support of this, Moore, Liss, Silverman, Paoletti, LaForest, and Musgrave (2013) have documented that students often create graphs based on shape thinking, or “conceiving of graphs as pictoral objects” (p. 441). That is, a possible explanation for students’ graphs of the aforementioned equations is that they assimilate \( f(x, y) = x^2 \) (etc.) to their schemes for the shapes of graphs in \( \mathbb{R}^2 \), which allows them to expansively generalise by creating similar shapes on \( \mathbb{R}^3 \) axes.

9. Conclusion

I have argued that assimilation and accommodation align with Harel and Tall’s (1991) expansive and reconstructive generalisation categories, respectively. I contend there is value in identifying connections between frameworks because they help us better understand phenomena of interest. There are many other generalisation frameworks and hence it is likely many more such connections can be made. For example, the Generalization Across Multiple Mathematical Areas project is currently examining Piaget’s various forms of abstraction as underpinning categories of Ellis’ (2007) generalisation taxonomy (Ellis, Tillema, Lockwood, & Moore, 2017). One might examine other generalisation frameworks in light of assimilation and accommodation.

Researchers also might investigate places in which researchers use different words to describe the same construct. For example, Mitchelmore (2002) documented three categories of how people tend to define generalisation and within those, places where different researchers used different terms synonymously. Making such connections
explicit serves the field by connecting bodies of work, which in turn can lend insight both to theory and practice.

Finally, Harel and Tall’s (1991) framework was developed with the aim of “suggest[ing] pedagogical principles designed to assist students’ comprehension of advanced mathematical concepts” (p. 1). They write, “we believe that the most desirable approach to generalisation is to provide experiences which lead to a meaningful understanding of the current situation, to allow the move to the more general case to occur by expansive generalisation, but that there are times when the situation demands a re-construction [sic] and, in such cases, it is necessary to provide the learner with the conditions in which this reconstruction is more likely to take place” (Harel & Tall, 1991, p. 3). I propose that identifying assimilation and accommodation as aligning with Harel and Tall’s (1991) framework make this instructional suggestion more user-friendly. That is, we can think about designing activities to support generalisation by identifying schemes students could assimilate to. Kabael’s (2011) study provides a good model. In the cases where it seems students often must reconstruct their schemes, we can use empirical studies to identify the sorts of accommodations they make.
Chapter 4: Conclusion

1. Introduction

The ability to generalise is a hallmark of mathematical thinking. However, generalising can be difficult and students often generalise in non-normatively correct ways. This dissertation sought to add to the body of literature regarding how undergraduate students generalise. Across the dissertation, two manuscripts explored undergraduates’ generalisation of function from two different, complementary frameworks. The first manuscript described what mathematical features students generalised about their notion of function from the single- to multivariable setting and how they made those generalisations. The ‘how’ focused on a small grain-size, such as students’ attention to particular notations or properties of mathematical objects (e.g., a coordinate axis as having a vertical position). The second manuscript considered generalisation from a much larger grain size, linking Harel and Tall’s (1991) expansive, reconstructive, and disjunctive generalisation framework to Piaget’s assimilation and accommodation constructs. More specifically, the two manuscripts addressed the following research questions:

1. In determining whether a multivariable relation represents a function, what mathematical ideas do students generalise from their understanding of what it means for a single-variable relation to represent a function?

2. What is the nature of students’ generalising activity as they generalise their understanding of what it means for a single-variable relation to represent a function to what it means for a multivariable relation to represent a function?

3. In what ways do the constructs of assimilation and accommodation relate to generalisation?

Taken as a whole, this dissertation addresses students’ generalisation about function as both a product (e.g., what students generalise about what it means to be a function) and as
a process, both from what details of representations students attend to as they generalise and, more broadly, how they organise that knowledge by expanding or reconstructing schemas.

In this concluding chapter, I summarise each manuscript’s findings and coordinate the findings across manuscripts. I then discuss the generalisability of my findings and the limitations of my study. I conclude by describing the implications for research and practise.

2. Summary of findings

2.1 Manuscript 1

In the first manuscript, I described an empirical analysis of what students generalise about what it means to be a function from the single- to multivariable case. I found that students generalised the vertical line test, the notion of input and output, univalence, and a notion of function-as-equation. An important takeaway is that four of the five students, despite some initial non-normative conceptions, arrived at what the mathematical community would deem a normative understanding of function.

I also described the nature of students’ generalising activity as they generalised their notion of function. Using Ellis’ (2007) framework, I found that students generalised by relating situations and objects, searching for the same procedure, searching for the same solution or result, expanding the range of applicability, and implementing and modifying definitions and prior strategies. This is unsurprising given the nature of the tasks. Taken as a whole, these tell us that students attend to mathematical and perceptual features and that often, those work together to help students generalise. When students work with definitions and implement prior ideas such as input and output, they attend to mathematical features of a situation. When they attend to the vertical-ness of a line, they attend to a perceptual feature. However, students are able to use the idea of ‘vertical’ to reason that the output of a multivariable function is $z$. As such, the language of Ellis’ (2007) framework allows us to see how the different features students attend to work in tandem to help them generalise.

Students generalised univalence via considering generalisations of the vertical line test and the input-output model of function. The vertical line test is a procedure to
determine if a graph in standard Cartesian coordinates represents a function. One draws a line parallel to the vertical axis and if it intersects the graph exactly once, then the graph represents a function. Some teacher-researchers recommend not teaching students the vertical line test because they believe students remember the procedure without remembering why it works (Dubinsky & Wilson, 2013). The students in this study, however, seemed to be aware of the vertical line test both as a procedure and as connected to univalence. For example, Wendy discussed the inputs and outputs as part of her justification for the vertical line test (Excerpt 18), as did Ike (Excerpt 20). However, these two students, and one more, all considered nonstandard orientations of lines. I argue that this is not a result of students’ remembering a procedure (i.e., ‘draw a line parallel to the $y$ axis’) but rather an important part of their generalisations. In considering non-normative orientations of lines, students had to think about what the input and output of a multivariable function would be. Hence I contend the vertical line test served as a tool that afforded their generalisation.

Students also generalised the input and output model of function. This generalisation was particularly useful because it supported students’ generalisation of univalence. Students who thought about input and output also tended to think about the vertical line test (which evaluates univalence). These ideas are connected mathematically, and students using them together suggests that they saw these important mathematical connections.

Lastly, students generalised a notion of function-as-equation to varying degrees. One student, Stan, seemed to believe that all functions can be written as equations, and this was his primary criterion for evaluating the classification tasks. For example, Stan only classified as functions the single-variable tables for which he could generate a symbolic rule. He did the same for the tables in the multivariable post-interview. That is, a multivariable calculus course (presumably with instruction about multivariable functions) seemed to have no effect on Stan’s function conception. Stan is not the only one who connected functions and equations. Other students believed that if an equation existed, a graph represented a function. For example, Wendy believed a sphere might be a function because one can write an equation for its graph (Excerpt 5). Students’ insistence about
equations being functions emphasises the need for students to develop a robust function conception in the early grades.

2.2 Manuscript 2

In the second manuscript, I provided a theoretical look at generalisation. Specifically, I proposed that Piaget’s constructs of assimilation and accommodation might provide an explanatory mechanism for Harel and Tall’s (1991) expansive and reconstructive generalisation categories. This extends the current literature by linking Harel and Tall’s (ibid) framework to a major learning theory.

I gave an example of each expansive and reconstructive generalisation from my own data and argued that assimilation and accommodation provided the cognitive mechanism for those generalisations (respectively). Specifically, I explained how a student used an expansive generalisation to make sense of the notation $f(x, y)$ and argued that she was able to do so because she assimilated $f(x, y)$ to her scheme for $f(x)$. I also explained how the same student made a reconstructive generalisation when I asked her to graph $y = x$, $y = 2x + 1$, and $z = 4$ in $\mathbb{R}^3$. I argue that the student first drew $y = x$ and $y = 2x + 1$ as lines because she assimilated the equations to a schema for graphing linear equations in $\mathbb{R}^2$. However, she drew $z = 4$ as a plane and, unprompted by me, returned to the first two graphs and reasoned that they should be planes. I argue that comparing the plane and the two lines caused a perturbation and subsequent accommodation to her scheme for graphing linear equations.

A second important finding detailed in the second manuscript is an ‘existence proof’ that assimilation and accommodation may explain other researchers’ empirical findings about how students generalise ideas from $\mathbb{R}^2$ to $\mathbb{R}^3$. I give two examples. In the first, I explain how the function machine model may be so productive for students in learning about multivariable functions because it provides a scheme to which they can assimilate the multivariable function concept, and hence understand multivariable functions via expansive generalisation. In the second, I explain how some findings about students’ difficulties graphing functions in $\mathbb{R}^3$ may arise because students assimilate particular expressions to their graphical shapes in $\mathbb{R}^2$ (e.g., thinking the graph of $f(x, y) = x^2 + y^2$ is a circle).
2.3 Coordination of findings across manuscripts

My manuscripts have two common themes: function and generalisation. In this section, I describe the insights I gained about each of these (interrelated) themes across the two manuscripts. One broad theme is students’ success in generalising multivariable ideas in ways that are consistent with how the mathematical community thinks about them. For example, in the first manuscript, four of five students developed a notion of multivariable function that included the univalence criterion and an input-output relationship. In the second manuscript, one student successfully reasoned about the shapes of graphs in $\mathbb{R}^3$. Both manuscripts provide evidence of students generalising a normative understanding of multivariable function notation. Because the data presented in the manuscripts primarily focus on students’ initial sense-making, the overall correct nature of student thinking supports that students’ initial ideas can be successfully leveraged to help them make sense of multivariable functions and graphing planes in $\mathbb{R}^3$.

2.3.1 Insights about function and relation

An important insight from both manuscripts is that on the whole, students’ generalisation of what it means to be a multivariable function is in accordance with a normative understanding of this topic. That is, of the five subjects, four generalised that a multivariable function must be univalent. Given that univalence is a key part of the definition of function, this is a positive finding. Though students generalised univalence in multiple ways, unpacking notation was a critical part of students’ generalising. In the first manuscript, I describe this as relating objects. In the second manuscript, I offer a more theoretical explanation, namely that notation is part of a scheme to which a student can assimilate.

In manuscript 2, I described how Wendy thought about graphing $y = x$, $y = 2x + 1$, and $z = 4$ in $\mathbb{R}^3$. Though only the latter is a function of the form $z = g(x, y)$, the $y = x$ and $y = 2x + 1$ tasks provide insight into students’ understanding of function because they reveal student thinking about free variables. Trigueros and Martínez-Planell (2010) found that students struggled to graph multivariable functions with free variables. I have written previously that student difficulties with graphing planes may be a result of not attending to $z$ as varying (Dorko, 2016; Dorko & Lockwood, 2016). Wendy’s thinking provides another example of this. In Excerpt 26, for instance, she describes plotting $(x, y)$-tuples
instead of \((x, y, z)\)-tuples. That is, she did not attend to \(z\) at all. Then, Wendy thought about the equation as \(y = 2x + 1 + 0z\) and said this meant one could “ignore \(z\),” which she used synonymously with “setting it equal to zero” (Excerpt 27). After graphing \(z = 4\) as a plane, Wendy looked at the line she had drawn for the graph of \(y = x\) and drew a plane, describing “no matter what \(z\) is, \(y\) is always going to equal \(x\).” In this case, she attended to \(z\) as a variable that \textit{varies} and drew the correct graph. By ‘varies’, I mean that she considered that \(z\) can take on multiple values. This contrasts the perspective in which she saw \(0z\) as meaning that \(z\) always equals zero, rather than as 0 times a non-zero quantity.

An implication is that instructors may need to help students think about free variables as varying. One way to do this could be asking students for examples of \((x, y, z)\)-tuples or writing 0 times the free variable (as Wendy did) to draw students’ attention to the variable. In the second manuscript I argue that students have trouble graphing in \(\mathbb{R}^3\) because they assimilate equations to their schemes for those equations’ shapes in \(\mathbb{R}^2\), and asking students to plot \((x, y, z)\)-tuples for a function with a free variable, or writing (e.g.) \(f(x, y) = x^2\) as \(f(x, y) = x^2 + 0y\) could cause perturbations that would lead to students reconstructing their graphing schemas. An implication is that notation could be particularly helpful because symbols seem to be a major focal point for students when generalising (discussed in the next section).

Finally, the manuscripts reveal the impact of students’ knowledge of single-variable function on the sense they made of multivariable functions. For example, four students understood single-variable functions in terms of inputs and outputs, and they generalised this to the multivariable case. In contrast, Stan, who thought of single-variable functions only in terms of symbolic rules, generalised that a multivariable relation represented a function only if he could find an equation for it. In the second manuscript, we see the impact of Wendy’s understanding of both function notation and the shapes of graph in \(\mathbb{R}^2\) as heavily impacting the sense she made of multivariable function notation and graphing in \(\mathbb{R}^3\). These findings indicate that the notions of function and graphing students form in algebra are long-lasting. An implication is that algebra instructors should take care to ensure that students develop a robust function conception, as this is what students will bring to bear on future mathematics. Additionally, Stan’s case indicates that multivariable calculus instructors may not want to assume that students arrive with robust function
conceptions, and might provide students with an opportunity to review their knowledge about single-variable functions before introducing multivariable functions.

2.3.2 Insights about generalisation and generalisation frameworks

My findings reveal the power of mathematical objects as a focal point for students when generalising. Relating objects is one of the categories in Ellis’ (2007) generalisations taxonomy, and is defined as “the formation of an association of similarity between two or more present objects” (p. 235). She lists equations, graphs, tables, and “other representations” (Ellis, 2007, p. 238) as examples of objects. In the first manuscript, I describe how I observed students relate the function notations $f(x)$ and $f(x, y)$. I also observed one student (Wendy) generalise that the vertical line test in $\mathbb{R}^3$ would require a line parallel to the $z$ axis. Wendy made this generalisation by relating the $z$ axis in $\mathbb{R}^3$ to the $y$ axis in $\mathbb{R}^2$ because both are vertical. Other researchers have also found that objects are a focal point for students while generalising. For example, Dorko and Weber (2014) describe students generalising domain and range based on the presence of particular symbols in equations and the positions of coordinate axes.

This dissertation employed two generalisation frameworks and as such, provide some insight into the different ways researchers have operationalised generalisation. One major difference between the two frameworks is the grain size at which they characterise students’ generalisation. For example, Ellis’ (2007) framework provides detail such as students generalising by modifying a rule or repeating an action (e.g., some sort of arithmetic) to see if the outcome is the same each time. This provides insight into the mechanics of generalising, and findings like this are useful to inform instruction. For instance, if we know that computing a quantity (e.g., slope) helps students generalise, then we should provide opportunities for students to do such computations. In my opinion, Harel and Tall’s (1991) framework, which characterises generalisation at a much larger grain size, does not inform instruction as directly. Studying how students generalise a particular topic might tell us that, on the whole, that topic requires most students to reconstructively generalise. However, language at a smaller grain-size (e.g., Ellis’) to help us learn what particular features of their ideas students must reconstruct.

In contrast, a limitation of Ellis’ framework is that it does not offer a theoretical mechanism for how students generalise. Work to unpack the underlying theory is
ongoing (Ellis et al., 2017). Harel and Tall’s (1991) framework, on the other hand, offers a description in terms of scheme construction and, if I have convinced the reader, assimilation and accommodation. I argue that mathematics educators are interested in theory and theory-building, and hence it is important to connect frameworks to fundamental learning theories.

2.3.3 Extending the literature base

The first manuscript extends multiple bodies of literature. First, to my knowledge, it is the first longitudinal study of students from single- to multivariable calculus, and as such, it provides insight as to how students’ ideas develop over time. One insight gained from the longitudinal design was that students’ conceptions of single-variable functions play a key role in how they come to think about multivariable functions. Students’ conceptions of single-variable functions seem to persist through multivariable calculus. For example, four students understood that single-variable functions are univalent and they generalised this to the multivariable case. In contrast, one student retained a notion of function-as-equation, even in multivariable calculus. Many studies about multivariable topics focus on students who are currently enrolled in multivariable calculus (e.g., Dorko & Weber, 2014; Kabael, 2011). As such, students’ ideas may have already evolved. In contrast, my research design captures students’ thinking about single-variable topics while they are still grappling with those ideas. This supports validity of my claim that student’s conceptions of single-variable functions seem to persist through multivariable calculus.

One notable extension relates to the literature about how undergraduate students generalise their notion of function. My findings indicate that students’ initial conceptions of what it means to be a function in $\mathbb{R}^3$ often involve non-normative ideas, such as positioning $y$ as the dependent variable. This may explain why some students believe the domain and range of a multivariable function $g(x, y) = z$ are $x$ and $y$, respectively (Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012; Kabael, 2011). That is, while one explanation is White and Mitchelmore’s (1996) $x, y$ syndrome, another possible explanation is that students see $y$ as the dependent variable in $g(x, y) = z$ and as such, describe the range in terms of $y$. My research also extends Kabael’s (2011) work. Kabael (2011) studied students’ generalisation of function in terms of APOS levels, and found
that a student’s APOS level for single-variable functions influenced their level for multivariable functions. My findings support Kabael’s (ibid) and extend them by providing a finer-grained perspective of what students generalise, and how they generalise it.

The second manuscript extends the literature base in several ways. First, other researchers have linked generalisation and transfer to assimilation and accommodation (e.g., Čadež & Kolar, 2015; Hohensee, 2014; Wagner, 2010). My research builds on their work by establishing further links between theoretical work about generalisation and transfer. Secondly, my work builds on empirical findings about how students generalise multivariable ideas by offering an explanatory mechanism for the cognition behind students’ generalising activity. Specifically, I offered a theory-based explanation for some findings about students’ difficulty graphing in $\mathbb{R}^3$ (Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010) and why function machine is such a powerful model for students (Dorko & Weber, 2014; Kabael, 2011; Tall, McGowen, & DeMarois, 2000). In the latter, I build on Tall, McGown, and DeMarois’ (ibid) argument that the function machine is powerful because it is a ‘cognitive root’ by suggesting that it is a cognitive root because it offers students a model to which they can assimilate.

Both papers build on Yerushalmy’s (1997) finding that seventh graders can successfully reason about and graph multivariable functions. This initial study focused on the students developing a graphing system and reasoning about two independent variables. My work provides insight into how undergraduates think about these ideas. Given Shaughnessy (2011) and Ganter and Haver’s (2011) suggestion that including multivariable topics in the high school mathematics curriculum could increase students’ mathematical competency, it is useful to know how students of different ages and mathematical backgrounds initially approach multivariable ideas.

3. Generalisability

It is reasonable to question whether a study of five students is representative of the entire population of students, and hence whether it is generalisable. One argument for a study with a small sample size being generalisable is that in studying student thinking,
the architecture of the knowledge structures documented is general, as is the nature of the structural change (Schoenfeld, Smith, & Arcavi, 1993). This is similar to what Maxwell (1996) calls *face generalisability*, or that there is “no obvious reason *not* to believe that the results apply more generally” (p. 97, emphasis original). That is, even a case study of a few students is likely to generate results that apply more generally. I can think of no reason my results would not apply more generally. Moreover, my findings are similar to what has already been documented in the literature (see section 2.3). This supports the position that my results apply more generally.

Generalisability in qualitative studies may also come from developing a theory that extends to other cases (Maxwell, 1996). I contend my second manuscript is a good example of qualitative work developing theory. In particular, when I began trying to unpack the relationship between assimilation, accommodation, expansive generalisation, and reconstructive generalisation with my own data, I realised that assimilation might explain why the idea of a function machine is so powerful for students and why students struggle to graph in $\mathbb{R}^3$. Hence I argue I have developed theory that explains not only my own data, but also others’.

4. Limitations

4.1 Impact of instruction

There are many factors that influence how students might think about a particular topic. While I collected data that would allow me to provide detailed descriptions of student thinking, I did not collect data about what might have influenced that thinking. For example, in the multivariable pre-interview (Excerpt 9) and post-interview (Excerpt 10), Wendy described a multivariable function in terms of input and output (Excerpt 10). Because she thought about multivariable functions in terms of inputs and outputs before instruction about multivariable functions, it is possible that Wendy’s use of this model in the post-interview is a generalisation she made on her own. On the other hand, it is equally possible that her instructor taught function using input-output and hence provided an opportunity for Wendy to generalise using this model.

Similarly, Stan’s conception of function was largely based on the idea of equation and pattern. This was evident in the pre-interview (e.g., Excerpts 3, 4) and the post-interview
That is, whatever the instructor presented seemed to have had little effect on Stan’s conception of function. It would be useful to know what instruction Stan received and why it did not appear to affect how he thought about function.

4.2 Disjunctive understanding (or disjunctive generalisation)

While I proposed that assimilation and accommodation could serve as explanatory mechanisms for expansive and reconstructive generalisation (respectively), I did not have data that allowed me to investigate possible relationships between assimilation and accommodation and disjunctive generalisation, Harel and Tall’s (1991) third category of generalisation. That is, I did not observe any instances of students engaging in what Harel and Tall (1991) term disjunctive generalisation and Jones and Dorko (2015) termed disjunctive understanding. Harel and Tall (1991) defined disjunctive generalisation as occurring when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available (p. 1).

As an example, Harel and Tall describe that a student who memorises a list of procedures for solving a 2 x 2 system of equations and another list of procedures for a 3 x 3 system, and sees these as disconnected, separate lists has generalised disjunctively. Jones and Dorko (2015) argued that since the new schema (e.g., the list of steps to solve a 3 x 3 system) is not related to a previous schema (e.g., the list of steps for a 2 x 2 system), this is not a generalisation. They hence suggested that the category be termed ‘disjunctive understanding.’ This makes sense given Harel and Tall’s (1991) definition of generalisation as “the process of applying a given argument in a broader context” (p. 1). That is, students who are disjunctively ‘generalising’ are not applying an argument in a broader context because, by Harel and Tall’s (1991) definition of disjunctive generalisation, they see the new context as new rather than a broader version of a familiar context. In short, I contend that Harel and Tall’s definition of disjunctive generalisation is incoherent with how they define generalisation.

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13 In contrast, a student who sees the procedures as similar in that they isolate one variable and use its value to find the value of the other variable(s) has generalised expansively or reconstructively; see Table 2.
A limitation of my study is that I lacked empirical examples of disjunctive understandings. I unpacked the relationship between expansive generalisation and assimilation and reconstructive generalisation and accommodation by looking at data. In retrospect, the tasks I designed afforded many opportunities for students to generalise expansively or reconstructively but limited (if any) opportunities for students to construct a disjunctive understanding. For example, Harel and Tall’s (1991) example (which is hypothetical, not empirical) is about procedures, but the only procedural tasks I had students complete were evaluating functions (e.g., given \( f(x, y) = 7x + y^3 \), find \( f(1, 2) \)). I designed tasks that I saw as parallel (e.g., the single- and multivariable classification tasks discussed in the first manuscript) and since students also seemed to see these as similar, I did not observe any disjunctive understandings. Hence a limitation of this study is that it does not propose any theoretical underpinnings of disjunctive understandings. This is an area for future research.

4.3 Limitations of the radical constructivist epistemological perspective

A limitation of the task-based interview methodology is that it is almost like a laboratory setting, and as such, does not reflect the social environments in which students learn. These limitations are best understood by considering how a student might work on one of the interview tasks in a typical college setting. A student working on the classification tasks, for example, might be working with peers in a recitation session. In discussing their ideas, each would interpret what others are saying based on their individually-constructed realities. My design did not allow me to investigate this important aspect of student learning.

Radical constructivists do not deny that learning has a social component; in fact, von Glasersfeld (1989) contends that social interaction is the most common cause of perturbation. However, my design did not allow me to investigate the social component of learning. Moreover, experts argue that “radical constructivism does not proffer an adequate explanation of how the socio-cultural and the personal components of learning interact” (Hardy & Taylor, 1997, p. 141). Given that relatively little is known about student understanding of multivariable functions, it makes sense to start with a study of individual student cognition before trying to study how social aspects of learning might
interact with this learning. However, it is important to acknowledge that such a position focused my attention on only some aspects of the learning phenomenon.

4.4 Equity

Equity has been a recent area of focus for the research in undergraduate mathematics education community (RUME). The Conference for Research in Undergraduate Mathematics Education held working groups focused on equity in mathematics education in 2016 and 2017. The RUME community is also in the process of developing a position statement on equity and mentoring. Work to make mathematics education more equitable is of the utmost importance, and a limitation of my study is that it does not contribute to current efforts to increase equity.

One way I will attend to equity in my future work is to report participants’ gender and ethnicity. As Adiredja and Andrews-Larson (2017) write,

> accounting for students’ demographic information has the potential to uncover hidden narratives in our studies. Whose voice do we privilege in our presentation of our data? To what extent do our findings perpetuate or challenge existing narratives about who are capable doers of mathematics? (p. 18)

Aside from the use of gender-preserving pseudonyms, my study contains no demographic data. Collecting such information requires approval from the Institutional Review Board (IRB), and I developed my study and secured IRB approval before learning that reporting demographics could help address issues of equity. Reporting such information is a practise I will take up in future work.

5. Future directions

As discussed above, one area for future theory building is identifying the cognitive mechanism for disjunctive understandings. In this section, I describe two additional future directions.

This work studied how students generalise in a somewhat isolated and artificial context. That is, students worked on mathematical tasks without accessing resources such as textbooks or computer algebra systems. I hypothesise that if a student were completing a graphing task such as those described in the second manuscript, he or she might sketch a graph by hand and then generate a graph using computer software and compare the two. Alternatively, the student might read the textbook section about graphing in $\mathbb{R}^3$. 
A common motivation for studies of student thinking, this one included, is that they can inform instruction. For example, my work adds support to Kabael’s (2011) finding that function machine is a useful model for students when generalising their function notion to the multivariable case. Future research could investigate what opportunities students experience in class to generalise. That is, in what ways (if any) do instructors support students in generalising?

Finally, an important motivation for this study was Shaughnessy (2011) and Ganter and Haver’s (2011) suggestion that including multivariable topics in the high school mathematics curriculum could increase students’ mathematical competency. My work has added to the body of literature regarding how students think about multivariable functions and graphing in $\mathbb{R}^3$. I believe that using research to inform practise is important, and hence one future direction is using what we know from my work and others’ to inform K-12 instruction. In the past, I have led workshops for teachers based on research findings (e.g., Dorko & Speer, 2015a) and written for a practitioner journal (Dorko & Speer, 2015b). I hope to do the same with this work.


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