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A NONLINEAR PSEUDOPARABOLIC DIFFUSION EQUATION*

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Abstract. Diffusion in a fissured medium with absorption or partial saturation effects leads to a pseudoparabolic equation nonlinear in both the enthalpy and the permeability. The corresponding initial-boundary value problem is shown to have a solution in various Sobolev–Besov spaces, and sufficient conditions are given for the problem to be well-posed.

Introduction. This is the second of two papers dealing with a certain pseudoparabolic diffusion equation. It was shown in [6] (see also [2]) that diffusion processes in fissured media lead to the following problem:

Let $G\subset\mathbb{R}^N$ be a bounded domain (the place where the diffusion process takes place), and denote by $S:=[0,T]$ a finite time interval. We are looking for functions $u=u(x,t)$ (concentration) and $v=v(x,t)$ (a flow potential), such that

$$
\begin{aligned}
&u'+\frac{1}{\varepsilon}(\alpha(u)-v)=f_1,\\
&-\operatorname{div}(k(u)\nabla v)+\frac{1}{\varepsilon}(v-\alpha(u))=f_2,\\
&u(x,0)=u_0(x),\qquad v_{\partial G}=0.
\end{aligned}
\tag{0.1}
$$

Here $u':=\partial u/\partial t$, "div" denotes the usual divergence operator, "∇" stands for the gradient with respect to $x=(x_1,\cdots,x_N)\in\mathbb{R}^N$, and $t\in S$. The functions $f_i=f_i(x,t)$, $i=1,2$, and u_0 are given and $k=k(u)$, $\alpha=\alpha(u)$ are specified by properties of the field or medium. For each $u\in L^1(G)$ define $A_u:=-\operatorname{div}(k(u)\nabla)$ and consider this elliptic operator subject to Dirichlet boundary conditions. By eliminating v in (0.1) we obtain the following equivalent ordinary differential equation involving only the single variable u

$$
\begin{aligned}
&u'(t)+\frac{1}{\varepsilon}\big(I-(I+\varepsilon A_u)^{-1}\big)\alpha(u)=f_1+(I+\varepsilon A_u)^{-1}f_2,\\
&u(0)=u_0.
\end{aligned}
\tag{0.2}
$$

Applying $I+A_{u(t)}$ to both sides of the equation in (0.2), one formally obtains the pseudo-parabolic problem

$$
\begin{aligned}
&u'+\varepsilon A_u(u')+A_u(\alpha(u))=(I+\varepsilon A_u)(f_1)+f_2,\\
&u(0)=u_0,\\
&\alpha(u)|_{\partial G}=0.
\end{aligned}
\tag{0.3}
$$

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We note that problems of the type (0.3) also arise in the quite different contexts of heat conduction modelled by two-temperature systems [8], certain weak formulations of two-phase Stefan problems [6], [15] and in the description of some non-Newtonian fluids [9], [17]. Further references can be found in [7].

In [6] we considered the case $k=k(x,t)$, $\alpha=\alpha(u)$ monotone, Lipschitz. There we showed existence and uniqueness of solutions under fairly weak assumptions on the data. Furthermore, for this case comparison and maximum principles were shown. Here we are concerned with the additional nonlinearity $k=k(u)$ which arises in the diffusion model [6] due to saturation or absorption effects on the permeability. Specific properties of k and α, as they are used later are listed below under (H1)–(H6).

We prove the existence of solutions to (0.2) or (0.3) in various spaces. Theorems 2.1, 2.3 and Corollary 2.2 contain existence results for solutions $u=u(t)$ taking their values in $BV(G)$, $W_0^{s,p}(G)$ and $H_0^1(G)$, respectively. As the formulation of Theorem 2.1 shows, there remains a "gap". If $p\in[1,N/2)$, $s\in(0,1]$, then there are solutions in $W_0^{s,p}(G)$ (provided, u_0, $f(t)\in W_0^{s,p}(G)$). If $p>N/2$, we only have some (sufficiently small) $s>0$, such that $u(t)\in W_0^{s,p}(G)$. Theorem 2.6 deals with $W^{2,p}$-existence of the solutions in the two-dimensional case. By interpolation methods we obtain results for $W_0^{1+\tau,p}$-existence for $\tau\in(0,1)$ (Corollary 2.4). As a consequence we get some sufficient conditions on the data which imply that $u(t)\in W_0^{1,p}(G)$ for $N=2$ and for certain $p>2$ (Corollary 2.5). These seem not to be optimal since the assumed regularity of the data is higher than that of the solutions obtained. We continue by proving a uniqueness- and continuous-dependence result, the assumptions of which can be met at least in the one- and two-dimensional cases. The final theorem states some useful pointwise estimates, which in particular imply a weak maximum principle for (0.2).

The paper is organized as follows. Section 1 contains notations and lists some function spaces which we use. Section 2 contains the precise formulation of the results. Section 3 is concerned with the proofs. We conclude with a short appendix which presents some facts on interpolation.

1. Notation and spaces. Let $G\subset\mathbb{R}^N$ be a smooth and bounded domain, $\Gamma:=\partial G$ the boundary,

$S:=[0,T]$—a finite (time) interval, $S_t:=[0,t]$ for $t\in S$,

$Q_t:=(0,t)\times G$, $Q:=(0,T)\times G$

$D^\alpha:=D^{\alpha_1}D^{\alpha_2}\cdots D^{\alpha_N}$ for a multiindex $\alpha=(\alpha_1,\cdots,\alpha_N)$, $D^{\alpha_i}:=\partial/\partial x_i$,

$x=(x_1,\cdots,x_N)\in G$,

$s\in[0,2]$, $p\in[1,\infty]$, $r\in[1,\infty]$, $\sigma:=Ns+p$ if $s\in(0,1)$,

$\lambda\in[0,1]$, $k\in\mathbb{N}$,

$$W^{s,p}(G):=\begin{cases} W^{s,p}(G)\text{—the usual Sobolev space, if } s \text{ is an integer,}^1\\ B^s_{p,p}(G)\text{—the usual Besov space, if } s \text{ is not an integer,}\end{cases}$$

the norm in $W^{s,p}(G)$ is denoted by $\|\cdot\|_{s,p}$,

$L^p(G):=$ the usual space of p-integrable real-valued functions, $L^p(G)$ normed by the usual L^p-norm $|\cdot|_p$. *Special case.* If $p=2$ and $s=1$, then $|v|:=|v|_2$, $\|v\|:=\|v\|_{1,2}$.

"$(\cdot,\cdot)$" denotes the usual scalar product in $L^2(G)$ and the dual pairing between $H^{-1}(G)$ and $H_0^1(G)$, "$((\cdot,\cdot))$" stands for the scalar product in $H_0^1(G)$.

By $W_0^{s,p}(G)$ we denote the Sobolev—Besov space of those functions in $W^{s,p}(G)$ having zero-trace on ∂G. $W_0^{1,p}(G)$ is assumed to be normed by $\|v\|_{1,p}:=\sum_{|\alpha|=1}|D^\alpha v|_p$.

[1] A complete reference for these spaces may be found in Adams [1].

This norm is on $W_0^{1,p}(G)$ equivalent to the usual $W^{1,p}(G)$-norm, so that we do not introduce extra notation.

Furthermore:

$C^k(\overline{G}) := \{v\colon \overline{G} \to \mathbb{R}, k$ times continuously differentiable$\}$, equipped with the usual max-norm $\|\cdot\|_{C^k(\overline{G})}$,

$C^{0,\mu}(\overline{G})$ denotes the space of all Hölder continuous functions with Hölder exponent $\mu \in (0,1]$, $\|\|_{C^{0,\mu}(\overline{G})}$-norm of this space.

$BV(G) := \{v \in L^1(G)\colon \|v\|_{BV(G)} < \infty\}$-the set of all L^1-functions with finite total variation,

$$\|v\|_{BV(G)} := |v|_{L^1(G)} + [v]_{BV(G)},$$

$$[v]_{BV(G)} := \sup\left\{ \int_G v \operatorname{div} \vec{w}\, dx, \vec{w} \in C_0^1(\overline{G})^N, |\vec{w}(x)| \leqq 1 \text{ for all } x \in \overline{G} \right\}.$$

$C_0^1(G) := \{v \in C^1(G)\colon \operatorname{supp} v \subset\subset G\}$.

Generally, we do not distinguish in our notation between the norms in the space $\{V, |\cdot|\}$ and the norm in $V \times V \times \cdots \times V$, e.g., "$|\nabla g|_p$" means $|\nabla g|_{L^p(G)^N}$. The same applies for scalar products.

If $\{V, |\cdot|\}$ is a normed space, then:

$L^r(S,V) := L^r(0,T;V)$—the usual space of V-valued, to the power r Bochner-integrable functions on S and equipped with the norm $\|\|_{L^r(S,V)}$ (sometimes we also write $||_{L^r(S,V)}$ for the same standard norm).

$W^{1,r}(S,V) := \{v \in L^r(S,V)\colon\ v' \in L^r(S,V)\}$.The norm in $W^{1,r}(S,V)$ is $\|v\|_{W^{1,r}(S,V)} := |v|_{L^r(S,V)} + |v'|_{L^r(S,V)}$.

$C(S,V) := C(0,T;V)$—the space of all continuous functions mapping S in V. $\|v\|_{C(S,V)} := \max\{|v(s)|\colon s \in S\}$,

$C^1(S,V) := \{v \in C(S,V)\colon v' \in C(S,V)\}$, $\|v\|_{C^1(S,V)} := \|v\|_{C(S,V)} + \|v'\|_{C(S,V)}$.

$C^{0,\mu}(S,V) := \{v \in C(S,V)\colon \|v\|_{C^{0,\mu}(S,V)} < \infty\}$.

$$\|v\|_{C^{0,\mu}(S,V)} := \|v\|_{C(S,V)} + \sup\left\{ \frac{|v(s) - v(t)|}{|t-s|^{\mu}}, t \neq s, t, s \in S \right\}.$$

By "$\subset$" we denote (beside set theoretic inclusion) continuous imbeddings, and "$\subset\subset$" denotes compact imbeddings.

"c" always stands for a nonnegative constant. Sometimes, we indicate on what quantities c might depend.

"$\to$" denotes strong convergence, "$\rightharpoonup$"-weak convergence, "$\overset{*}{\rightharpoonup}$"-weak-star convergence.

2. Results. We will use the following hypotheses on the coefficients k and α, respectively.

(H1) $k\colon \mathbb{R} \to \mathbb{R}$-continuous, and there are constants k_0, k_1 such that $0 < k_0 \leqq k(u) \leqq k_1$ for $u \in \mathbb{R}$,

(H2) $\alpha\colon \mathbb{R} \to \mathbb{R}$-Lipschitz continuous with Lipschitz constant L_α,

(H3) α is monotone and $\alpha(0) = 0$,

(H4) $k \in W^{1,\infty}(\mathbb{R})$, $|k'|_\infty =: L_k$,

(H5) $\alpha \in W^{2,\infty}(\mathbb{R})$, $\|\alpha\|_{2,\infty} =: \alpha_1$,

(H6) $\gamma := k \cdot \alpha'$-Lipschitz continuous with Lipschitz constant L_γ.

Now we are going to formulate what we understand by a solution of (0.2) and (0.3), respectively. Assume k and α are sufficiently regular so that all the appearing terms make sense. For the sake of (purely technical) simplicity we set $f_1=f$ and $f_2=0$ hereafter.

Formulation of (0.2) *as an ordinary differential equation*. Let us assume for a moment, we are given a sufficiently regular solution u,v of (0.1). If $u(t)\in L^1(G)$, then $\hat{k}(t):=k(u(t))\in L^\infty(G)$ (k as in (H1)). Set $A(t)=-\operatorname{div}(\hat{k}(t)\nabla(\cdot))$. By the existence theory for elliptic operators which are subject to Dirichlet boundary conditions, we can define

$$B(t):=(I+\varepsilon A(t))^{-1} \quad (I=\text{identity})$$

and we have a continuous linear operator

$$B(t)\colon L^p(G)\to L^p(G) \quad \text{for all } p\in[1,\infty]$$

(see Lemma 3.1 below). Now, set $A_v:=-\operatorname{div}(k(v)\nabla(\cdot))$ for $v\in L^1(G)$. By the preceding remarks,

$$B_v:=(I+\varepsilon A_v)^{-1}\colon L^p(G)\to L^p(G) \quad \text{for all } p\in[1,\infty].$$

Set for abbreviation

$$A_v^\varepsilon:=\frac{1}{\varepsilon}(I-B_v).$$

Thus, (0.2) is equivalent to

$$u'(t)+A^\varepsilon_{u(t)}(\alpha(u(t)))=f(t) \quad \text{for } t\in S. \tag{2.1}$$

This leads to the formulation of (0.2) as an ordinary differential equation in $L^p(G)$: Let

$$r\in[1,\infty],\quad p\in[1,\infty],\quad f\in L^r(S,L^p(G)),\quad u_0\in L^p(G).$$

We call $u\in W^{1,r}(S,L^p(G))$ a solution, if (2.1) holds for *a.a.* $t\in S$ as an equation in $L^p(G)$ and

$$u(0)=u_0 \quad \text{in } L^p(G). \tag{2.2}$$

Formulation in variational form. Formally applying $I+\varepsilon A_{u(t)}$ on both sides of (2.1), we obtain (0.3). Given $r\in[1,\infty]$, $p\in[1,\infty]$, $1/p+1/p'=1$, $u_0\in W_0^{1,p}(G)$, $f\in L^r(S,W_0^{1,p}(G))$, we call $u\in W^{1,r}(S,W_0^{1,p}(G))$ a solution of (0.3) if

$$\begin{aligned}(2.3)\quad &(u'(t),v)+\varepsilon(k(u(t))\nabla u'(t),\nabla v)+(k(u(t))\nabla\alpha(u(t)),\nabla v)\\ &\quad=(f(t),v)+\varepsilon(k(u)\nabla f,\nabla v) \quad \text{for all } v\in W_0^{1,p'}(G),\quad \text{a.a. } t\in S,\\ (2.4)\quad &u(0)=u_0.\end{aligned}$$

Notice that under appropriate regularity assumptions on u, the fact that u satisfies (2.1), (2.2) implies that it also satisfies (2.3), (2.4) and vice versa.

Our results are as follows.

THEOREM 2.1. *Assume* (H1), (H2),

$$u_0\in W_0^{s,p}(G), \qquad f\in C(S,W_0^{s,p}(G))$$

and one of the following conditions is satisfied:

(a) $N\geq 2$, $p\in[1,\min\{2N/(N+2),N/2\}]$, $s\in(0,1]$,

(b) $2 \leqq N \leqq 6, p \in [2N/(N+2), N/2), s \in (0, (2-N)/2 + N/p)$,

(c) $p > \max\{1, N/2\}, s \in (0,1]$ *sufficiently small*,

(d) $N=1, p=1, s \in (0,1]$.

Then (2.1), (2.2) *has a solution* $u \in C^1(S, W_0^{s,p}(G))$.

Remark 2.1. Let k and α be as in Theorem 2.1. If $u_0 \in BV(G)$, $f \in C(S, BV(G))$, then (2.1), (2.2) is solvable by a $u \in C^1(S, BV(G))$. For merely integrable right-hand sides f we have

COROLLARY 2.2. *Let* k, α, p *and* s *be as in Theorem* 2.1, $r \in [1, \infty]$. *If* $f \in L^r(S, W_0^{s,p}(G))$, $u_0 \in W_0^{s,p}(G)$, *then there is a solution* $u \in W^{1,r}(S, W_0^{s,p}(G))$ *satisfying* (2.1), (2.2). *The corresponding remark holds for* $f \in L^r(S, BV(G))$, $u_0 \in BV(G)$.

THEOREM 2.3. *Let* $N \geqq 1$, $r \in [1, \infty]$.

(a) *If* $f \in L^r(S, H_0^1(G))$, $u_0 \in H_0^1(G)$, *then* (2.3), (2.4) *has a solution* $u \in W^{1,r}(S, H_0^1(G))$.

(b) *If* $f \in C(S, H_0^1(G))$, *then* $u \in C^1(S, H_0^1(G))$.

We remark that the proofs also yield several estimates for norms of u in terms of the data.

Looking at (2.1), (2.2), one should expect u to be exactly as regular as u_0 and f, since $B_{u(t)}$ is for many function spaces at least a regularity-preserving operator. But the nonlinearity of the problem causes some problems. With respect to a higher than square integrability of the first derivatives we get only a partial result which will be a consequence of Theorem 2.6 formulated below and the following corollary, the formulation of which seems to be rather technical.

COROLLARY 2.4. *Let* $N=2$, *and assume* (H1), (H2), (H4), (H5) *and* (H6). *Furthermore, let* θ, $\tau \in (0,1)$, $p^* \geqq 2$, a, $a' > 1$, $p > 2$ *be such that*

$$\frac{1}{a} + \frac{1}{a'} = 1, \quad \theta + \tau \leqq 1, \quad \theta = \frac{a'p-2}{2a'(p-1)}, \quad \frac{1}{p^*} = \frac{(1-\tau)}{2} + \frac{\tau}{p}, \quad \tau \geqq \frac{2}{p^*} - \frac{2}{ap}.$$

If $u_0 \in W_0^{1+\tau,p^*}(G)$, $f \in C(S, W_0^{1+\tau,p^*}(G))$, *then* (2.3) (2.4) *has a solution* $u \in C^1(S, W_0^{1+\tau,p^*}(G))$. *Furthermore*, $u \in C^1(S, W_0^{1,ap}(G))$.

To illustrate the assumptions under which this corollary is valid, we formulate

COROLLARY 2.5.

(a) *Let* $a > 1, p > 2$ *and set*

$$\tau := \frac{ap-2}{2a(p-1)}, \quad p^* := \frac{2ap(p-1)}{p(a+1)-2}, \quad \theta := \frac{a(p-2)+2}{2a(p-2)}.$$

These numbers satisfy the conditions imposed in Corollary 2.4.

(b) *Vice versa—let* $N=2, k$ *and* α *as in Corollary* 2.4, $q > 2$ *a given number. We have for a solution* $u \in C^1(S, W_0^{1,q}(G))$, *provided the data satisfy*

$$u_0 \in W_0^{1+\tau,p^*}(G), \qquad f \in C\big(S, W_0^{1+\tau,p^*}(G)\big),$$

where for a $p \in [2, q)$

$$p^* = \frac{2q(p-1)}{q+p-2}, \quad \theta = \frac{q(1-1/p)+2}{2q(1-2/p)}, \quad \tau = \frac{q-2}{2q(1-1/p)}.$$

Concerning $W^{2,p}$-regularity, we have

THEOREM 2.6. *Let* $N=2$, k *and* α *as in Corollary* 2.4, $\tau\in[1,\infty]$, *and* $p\in[2,\infty]$.

(a) *If* $f\in L^r(S,W^{2,p}(G)\cap W_0^{1,p}(G))$, $u_0\in W^{2,p}(G)\cap W_0^{1,p}(G)$, *then we have for a solution of* (2.1), (2.2),

$$u\in W^{1,r}\big(S,W^{2,p}(G)\cap W_0^{1,p}(G)\big).$$

(b) *If* $f\in C(S,W^{2,p}(G)\cap W_0^{1,p}(G))$, *then* $u\in C^1(S,W^{2,p}(G)\cap W_0^{1,p}(G))$.

The next theorem reflects some sufficient conditions, which ensure unique solvability and continuous dependence of the solution on the data for problem (2.3), (2.4). In particular, these conditions are automatically fulfilled if $N=1$. For $N=2$, Corollary 2.5 provides information about such properties of u_0 and f giving at least one sufficiently regular solution which meets these conditions.

THEOREM 2.7. *Let* $N\geqq 1$, *and* k *satisfying* (H1), (H4), (H6). *Set* $p>2$ *if* $N=2$, *otherwise*, $p:=N$.

(a) *If* $u_0\in H_0^1(G)$, $f\in L^1(S,W_0^{1,p}(G))$ *and if there is at least one solution of* (2.3), (2.4) *with* $u\in W^{1,1}(S,W_0^{1,p}(G))$, *then* (2.1), (2.2) *is uniquely solvable*.

(b) *The map*

$$\{u_0,f\}\in H_0^1(G)\times L^r\big(S,W^{1,p}(G)\big)\mapsto u\in W^{1,r}\big(S,H_0^1(G)\big)$$

is locally Lipschitz for all $r\in[1,\infty]$.

Finally, we obtain some pointwise estimates on solutions and briefly indicate their usefulness.

THEOREM 2.8. *Let* k *and* α *satisfy* (H1)–(H3) *and let* $u\in W^{1,1}(S,L^1(G))$ *be a solution of* (2.1), (2.2). *Then, for a.a.* $t\in S$ *we have*

$$|u^+(t)|_\infty\leqq|u_0^+|_\infty+\int_0^t|f^+(s)|_\infty\,ds,$$

$$|u^-(t)|_\infty\leqq|u_0^-|_\infty+\int_0^t|f^-(s)|_\infty\,ds.$$

In particular, if $u_0(x)\geqq 0$ *a.e.* $f(s,x)\geqq 0$ *a.e., then* $u(t,x)\geqq 0$ *a.e. If, in addition, there is a number* $c_0>0$ *with* $\alpha(c_0)=0$ *and* $u_0(x)\geqq c_0$ *a.e. in* G, *then* $u(t,x)\geqq c_0$ *a.e. in* 0.

The preceding is particularly relevant in the diffusion model of [6] where there is some interval $[0,L]$ on which α is identically zero. (This occurs because of partial saturation or absorption in the model.) To illustrate the usefulness of Theorem 2.8, suppose in this situation we know only that $k(u)$ is defined and continuous at each $u>0$. With u_0 and f as given in Theorem 2.8, we choose k_0 to be the minimum and k_1 the maximum of $k(\cdot)$ on the interval $[c_0,\|u_0\|_{L^\infty}+\int_0^1\|f(s)\|_{l^\infty}\,ds]$. Then extend k outside this interval so as to satisfy (H1). By Theorem 2.8 it follows the solution is independent of the extension, so we may assume without loss of generality that the original k satisfies (H1). These remarks are useful in the diffusion model [6] where possibly $k(u)\to+\infty$ as $u\to 0^+$ or $k(u)\to 0$ as $u\to+\infty$.

3. Proofs. The formulation of (0.2) as an operator equation (2.1) involves the resolvent $B_v:=(I+\varepsilon A_v)^{-1}$. We have to justify that B_v exists. The point which has to be observed is that for fixed v the operator A_v has a coefficient $k=k(v)$, which is due to the lack of regularity of k not too smooth. The following lemma lists some properties of the resolvent B of a related operator A.

LEMMA 3.1. *Let $\hat{k}$: $\mathbb{R}^n \to \mathbb{R}$ be measureable, $0 < k_0 \leqq \hat{k}(x) \leqq k_1$ for a.a. $x \in \mathbb{R}^n$ (k_0, k_1 as in the definition of $k(\cdot)$). Set $A := -\operatorname{div}(\hat{k}(x)\nabla(\cdot))$, $B := (I+\varepsilon A)^{-1}$, and consider the elliptic operator A as subject to Dirichlet boundary conditions. In each of the following cases B is defined and we have*

(i) *If $1 < p \leqq 2N/(N+2)$, $1/p^* = 1/p - 1/N$, then B: $L^p(G) \to W_0^{1,p^*}(G)$.*
(ii) *If $p > N/2$, then there is a $\lambda \in (0,1)$ such that B: $L^p(G) \to H_0^1(G) \cap C^{0,\lambda}(\bar{G})$.*
(iii) *If $p = 1$, $q \in [1, N(N-1))$, then B: $L^1(G) \to W_0^{1,q}(G)$.*
(iv) *If $2N/(N+2) < p \leqq \min\{N/2, 2\}$, then B: $L^p(G) \to W_0^{1,2}(G)$.*
(v) *If $2 \leqq p < N$, $1/p^* = 1/p - 1/N$, then B: $L^p(G) \to W_0^{1,2}(G) \cap L^{p^*}(G)$.*

In each of these cases, B is a linear and continuous map and its norm depends at most on $G, p^*, p, N, \lambda, \varepsilon k_0, k_1$. λ depends at most on $G, p, N, \varepsilon k_0, k_1$.

Proof. (i), (iii) are part of [16, Thm. 4.5], for (ii) see [12, Chap. III], (iv) and (v) follows from (i). □

The next lemma list some known imbedding properties.

LEMMA 3.2. *Consider the situation in Lemma 3.1 and let p, p^*, λ, N be as in (i)–(v). Take in (i)–(iv) $s \in (0,1)$, in (v) $s \in (0(2-N)/2 + N/p)$. Then*

(i)[2] $W^{1,p^*}(G) \subset\subset W^{s,p^*}(G) \subset W^{s,p} \subset\subset L^p(G)$
(ii) $W^{s,p}(G) \subset\subset L^p(G)$, $C^{0,\lambda}(\bar{G}) \subset L^\infty(G)$
(iii) $W^{1,p^*}(G) \subset\subset W^{s,p^*}(G) \subset W^{s,1}(G) \subset\subset L^1(G)$
(iv) $W^{1,p^*}(G) \subset\subset W^{s,p}(G) \subset\subset L^p(G)$
(v) $W^{1,2}(G) \subset\subset W^{s,p}(G) \subset\subset L^p(G)$, *if* $N \leqq 6$.

Proof. See [1], [3]. The compactness results from $W^{1,p} \subset\subset L^p$ for any $p \geqq 1$, and the fact, that[3] $W^{s,p} = [L^p, W^{1,p}]_s$ and general interpolation theory. □

COROLLARY 3.3. *Let $v \in L^1(G)$, set, as before, $A_v := -\operatorname{div}(k(v(x))\nabla(\cdot))$, $B_v := (I+\varepsilon A_v)^{-1}$. B_v has exactly the same properties as B in Lemma 3.1. Furthermore, $A_v^\varepsilon := (1/\varepsilon)(I - B_v)$: $L^p(G) \to L^p(G)$ is Lipschitz. Thus, $A_v^\varepsilon \circ \alpha$: $L^p(G) \to L^p(G)$ is Lipschitz.*

It follows that (2.1) can be considered as an ordinary differential equation in $L^p(G)$. Moreover, we have the following.

LEMMA 3.4. *Let $v \in L^1(S, L^1(G))$ be given. Then*

(i) *For each $w \in L^p(G)$*

$$t \in S \to A_{v(t)}^\varepsilon(w) \in L^p(G) \tag{3.1}$$

is measureable.

(ii) *If $v \in C(S, L^1(G))$, then the map in (3.1) is continuous.*

Proof. (i) see [14].

(ii) Let $t_n \to t$ in S, set $g_n := (I+\varepsilon A_{v(t_n)})^{-1}(w)$. By definition g_n satisfies

$$\left(I+\varepsilon A_{v(t_n)}\right)(g_n) = w. \tag{3.2}$$

Consider case (i) in Lemma 3.1. We have $\|g_n\|_{1,p^*} \leqq c|w|_p$. Therefore, for a subsequence $g_{n_j} \to \tilde{g}$ in $L^{p^*}(G)$ and $g_{n_j} \rightharpoonup \tilde{g}$ in $W_0^{1,p^*}(G)$. By the continuity of $k, \tilde{g}$ satisfies

$$\left(I+\varepsilon A_{v(t)}\right)\tilde{g} = w. \tag{3.2'}$$

---

[2] "$\subset$" denotes algebraic and topological imbedding, "$\subset\subset$" the compact imbedding.

[3] "$[\cdot,\cdot]$" is the complex interpolation space generator (see appendix).

But, (3.2′) is uniquely solvable, so that $g_n \to \tilde{g}$ in $L^p(G)$, which proves the assertion. The other cases in Lemma 3.1 can be dealt with in a similar manner. □

Proof of Theorem 2.1. Fix $\bar{u} \in C(S, L^p(G))$. By Corollary 3.3 and Lemma 3.4, there is exactly one solution

$$u \in C^1(S, L^p(G)) \tag{3.3}$$

satisfying

$$u'(t) + A\frac{\varepsilon}{\bar{u}(t)}\alpha(u(t)) = f(t), \qquad u(0) = u_0. \tag{3.4}$$

We employ Schauder's theorem to show that the map

$$\mathscr{T}\colon C(S, L^p(G)) \to C(S, L^p(G))$$

defined by (3.3), (3.4) has a fixed point. Obviously, a fixed point of $\mathscr{T}$ solves (2.3), (2.4). The next lemma summarizes some properties of $\mathscr{T}$. In particular, the proof yields several estimates of the solutions of (2.1), (2.2).

LEMMA 3.5. (i) *$\mathscr{T}$ maps $C(S, L^1(G))$ into a bounded subset of $C^1(S, L^p(G))$.*

(ii) *Let s, N be as in Lemma* 3.2, *$t \in S$, $\bar{u} \in C(S, L^1(G))$. Then $(\mathscr{T}\bar{u})(t)$ is in a bounded subset of $W^{s,p}(G)$.*

(iii) *$\mathscr{T}\colon C(S, L^1(G)) \to C(S, L^p(G))$ is continuous.*

Proof. (i). Integrate (3.4) over $(0,t)$, and take the $L^p(G)$-norm on both sides. Then, by Lemmas 3.1, 3.2 and α's Lipschitz continuity

$$\begin{aligned}
|u(t)|_p &\leqq |u_0|_p + \int_0^t \left|A^{\varepsilon}_{\bar{u}(s)}\alpha(u(s))\right|_p ds + \int_0^t |f(s)|_p\, ds\\
&\leqq |u_0|_p + c\int_0^t |\alpha(u(s))|_p\, ds + \int_0^t |f(s)|_p\, ds\\
&\leqq |u_0|_p + c\int_0^t \left\{|\alpha(0)|\cdot|G| + L_\alpha |u(s)|_p\right\} ds + \int_0^t |f(s)|_p\, ds;
\end{aligned}$$

hence, by Gronwall's inequality

$$|u|_{C(0,t;L^p(G))} \leqq c\left\{|u_0|_p + 1 + |f|_{L^1(S,L^p(G))}\right\}, \tag{3.5}$$

where $c = c(\varepsilon k_0, k_1, |G|, N, p, L_\alpha, \lambda, s, T)$. By (3.4) and (3.5)

$$|u|_{C^1(0,t;L^p(G))} \leqq c\left\{|u_0|_p + 1 + |f|_{C(S,L^p(G))}\right\}. \tag{3.6}$$

To obtain further estimates, we notice that u as a solution of an ordinary differential equation is the $C(S, L^p(G))$-limit of the sequence $\{u_n\}$ defined by

$$u_1 := u_0, \tag{3.7}$$

$$u_{n+1}(t) = u_0 + \int_0^t f(s)\, ds - \int_0^t A^{\varepsilon}_{\bar{u}(s)}\alpha(u_n(s))\, ds,\ t \in S. \tag{3.8}$$

We have

LEMMA 3.6. *Let $u_n \in C(S, W^{s,p}(G))$, where s, p are taken as in Lemma* 3.1, 3.2, *$s \in (0, \lambda)$ if $p > N/2$ (λ arises in Lemma* 3.1, (ii)). *Then*

a) *$u_{n+1} \in C^1(S, W^{s,p}(G))$ and*

b)

$$\|u_n\|_{C(S,W^{s,p}(G))} \leqq c\left\{\|u_0\|_{s,p} + 1 + \|f\|_{L^1(S,W^{s,p}(G))}\right\}, \tag{3.9}$$

with c as in (3.5).

Proof. By assumption $u_n(t)\in W^{s,p}(G)$; hence $\alpha(u_n)\in W^{s,p}(G)\subset L^p(G)$. By Lemma 3.1, 3.2 $A^{\varepsilon}_{\bar{u}(t)}(\alpha(u_n(t)))\in W^{s,p}(G)$ if $p\leqq N/2$, and

$$\left\|A^{\varepsilon}_{\bar{u}(t)}(\alpha(u_n(t)))\right\|_{s,p}\leqq\frac{1}{\varepsilon}\|\alpha(u_n(t))\|_{s,p}+\frac{c}{\varepsilon}|\alpha(u_n(t))|_p \tag{3.10}$$

$$\leqq\frac{c}{\varepsilon}\|u_n(t)\|_{s,p}\qquad (c \text{ as in } (3.5)).$$

Therefore, by (3.8), $u_{n+1}(t)\in W^{s,p}(G)$. The t-continuity properties follow as in Lemma 3.4. Equations (3.8) and (3.10) yield (3.9) by an iteration argument (cf. [4]). To show (3.9) for $p>N/2$, take the $W^{s,p}$-norm on both sides of (3.8). Thus

$$\|u_{n+1}(t)\|_{s,p}\leqq\|u_0\|_{s,p}+\int_0^t\|f(s)\|_{s,p}ds+\frac{1}{\varepsilon}\int_0^t\|\alpha(u_n(s))\|_{s,p}ds \tag{3.11}$$

$$+\frac{1}{\varepsilon}\int_0^t\left\|\left(I+\varepsilon A_{\bar{u}(s)}\right)^{-1}\alpha(u_n(s))\right\|_{s,p}ds.$$

We have $\|\alpha(u_n)\|_{s,p}\leqq L_\alpha\|u_n\|_{s,p}$ and for

$$g(t,x):=\left(I+\varepsilon A_{\bar{u}(t)}\right)^{-1}(\alpha(u_n(t)))$$

by Lemma 3.1, 3.2.

(ii)

$$\|g(t)\|^p_{s,p}=\iint\limits_{GG}\frac{|g(t,x)-g(t,y)|^p}{|x-y|^\mu}\,dx\,dy,\qquad \mu=N+sp$$

$$\leqq\iint\limits_{GG}c\cdot|x-y|^{\lambda p-\mu}\,dx\,dy\leqq\text{const}.$$

Therefore (3.11) and Gronwall's inequality imply (3.9). □

To complete the proof of Lemma 3.5(ii), we note that (3.9) and (3.6) imply

$$\|u_n\|_{C^1(S,W^{s,p}(G))}\leqq c\left\{\|u_0\|_{s,p}+\|f\|_{C(S,W^{s,p}(G))}+1\right\}, \tag{3.12}$$

where c is as in (3.5). By weak-star compactness, $u\in W^{1,\infty}(S,W^{s,p}(G))$ and by (3.4), $u\in C^1(S,W^{s,p}(G))$, where (3.12) implies

$$\|u\|_{C^1(S,W^{s,p}(G))}\leqq c\left\{\|u_0\|_{s,p}+\|f\|_{C(S,W^{s,p}(G))}+1\right\}. \tag{3.13}$$

This proves (ii) of Lemma 3.5. To see the continuity of $\mathscr{T}$ let $\bar{u}_k\to\bar{u}$ in $C(S,L^1(G))$, set $g_k:=B_{\bar{u}_k}(\alpha(u_k)), u_k:=\mathscr{T}(\bar{u}_k)$, so that g_k and u_k, resp. satisfy

$$\varepsilon A_{\bar{u}_k}(g_k)=-g_k+\alpha(u_k), \tag{3.14}$$

$$u_k'(t)+\frac{1}{\varepsilon}\alpha(u_k(t))=\frac{1}{\varepsilon}g_k(t)+f(t),\qquad u_k(0)=u_0. \tag{3.15}$$

By estimates (3.6), (3.13) and α's Lipschitz continuity there is a subsequence

$$\begin{aligned} &u_{k_j}\overset{*}{\rightharpoonup}u\quad\text{in } W^{1,\infty}(S,W^{s,p}(G)),\\ &\alpha(u_{k_j})\overset{*}{\rightharpoonup}\alpha(u)\quad\text{in } L^\infty(S,W^{s,p}(G)).\end{aligned} \tag{3.16}$$

Using the estimates for u_k from (3.15) (cf. estimates (3.6), (3.13)) to estimate g_k in (3.14) ((3.14) is an elliptic problem for g_k, various norms of g_k can be estimated in terms of $\alpha(u_k)$ by Lemma 3.1, and $\alpha(u_k)$ can be estimated by (3.13), (3.6)), we arrive for $p \leqq (N/2)$ at

$$|g_k|_{L^\infty(S,W^{s,p}(G))} \leqq c|\alpha(u_k)|_{L^\infty(S,L^p(G))} \leqq c\left\{1+|u_k|_{L^\infty(S,L^p(G))}\right\}$$
$$\leqq c\left\{1+|u_0|_p+|f|_{L^1(S,L^p(G))}\right\},$$

and for $p > N/2$, at

$$|g_k|_{L^\infty(S,H_0^1(G))} \leqq c\left\{1+|u_0|_p+|f|_{L^1(S,L^p(G))}\right\}.$$

Therefore, we have for a subsequence

$$g_{k_j} \overset{*}{\rightharpoonup} g \text{ in } L^\infty\left(S, W_0^{1,p}(G)\right). \tag{3.17}$$

Equations (3.16), (3.17) and relation (3.14) imply that g satisfies

$$\varepsilon A_{\bar{u}}(g) = -g+\alpha(u). \tag{3.18}$$

(To pass to the limit $j \to \infty$ in (3.14), use the strong convergence properties of $k(\bar{u}_{k_j})$—remember, $\bar{u}_k \to \bar{u}$ in $C(S, L^1(G))$ and k is continuous.) Since (3.18) is uniquely solvable, we have $u_k \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(S, W^{s,p}(G))$, i.e., the whole sequence converges, in particular $u_k = \mathscr{T}(\bar{u}_k) \to u = \mathscr{T}(\bar{u})$ in $C(S, L^p(G))$, i.e., $\mathscr{T}$ is continuous, which finishes the proof of Lemma 3.5. □

COROLLARY 3.7. *Lemma* 3.5 *implies that $\mathscr{T}$ has a fixed point u. u solves problem* (2.3), (2.4) *and satisfies the estimates given by* (3.6), (3.9). *Moreover, u is—according to* (3.7), (3.8)—*the limit of the sequence $\{u_n\}$ defined by*

$$\begin{aligned} &u_1 := u_0, \\ &u_{n+1}(t) = u_0 + \int_0^t f(s)\,ds - \int_0^t A^\varepsilon_{u_n(s)}\left(\alpha(u_n(s))\right)ds, \end{aligned} \tag{3.19}$$

and

$$u_n \to u \quad \text{in } C\left(S, L^p(G)\right), \qquad u_n \overset{*}{\rightharpoonup} u \quad \textit{in } W^{1,\infty}\left(S, L^p(G)\right). \tag{3.20}$$

This finishes the proof of Theorem 2.1. □

Proof of Remark 2.1. Construct as in (3.3), (3.4) an operator

$$\mathscr{T}\colon C\left(S, L^1(G)\right) \to C^1\left(S, L^1(G)\right).$$

One has already estimate (3.6), so that Lemma 3.5(i) follows. The third statement of this lemma has already been proved and the second has to be changed to

LEMMA 3.5′. (ii′) *If $\bar{u} \in C(S, L^1(G))$, then $(\mathscr{T}u)(t)$ is in a bounded subset of $BV(G)$, which does not depend either on t or on $\bar{u}$.*

To see this, look at the iteration procedure (3.7), (3.8) which yields u as the $C(S,L^1(G))$—limit of $\{u_n\}$. By Lemma 3.1(iii) we have $(I+\varepsilon A_{\bar{u}(s)})^{-1}\alpha(u_n(s))\in W^{1,p^*}(G)\subset BV(G)$ so that by the same lemma

$$\left\|A^{\varepsilon}_{\bar{u}(s)}\alpha(u_n(s))\right\|_{BV(G)}\leqq c\cdot\frac{1}{\varepsilon}\left\{\|\alpha(u_n)\|_{BV(G)}+\|\alpha(u_n)\|_{L^1(G)}\right\}$$

$$\leqq c\cdot\frac{1}{\varepsilon}\left\{1+\|u_n\|_{BV(G)}\right\}.$$

Taking the $BV(G)$-norm on both sides of (3.8) yields

$$\|u_n\|_{C^1(S,BV(G))}\leqq c\left\{\|u_0\|_{BV(G)}+\|f\|_{C(S,BV(G))}+1\right\}.$$

Since $BV(G)\subset\subset L^{p^*}(G)\subset L^1(G)$ for $p^*\in(1,N/(N-1))$, Arzela–Ascoli's theorem yields for a subsequence

$$u_{n_j}\to u \quad \text{in } C(S,L^{p^*}(G)).$$

Because of the reflexivity of $BV(G)$ we have by weak-star compactness

$$u_{n_j}\overset{*}{\rightharpoonup}u \quad \text{in } W^{1,\infty}(S,BV(g)) \quad \text{and}$$

$$\|u\|_{W^{1,\infty}(S,BV(G))}\leqq c\left\{\|u_0\|_{BV(G)}+1+\|f\|_{C(S,BV(G))}\right\}.$$

Since u satisfies (3.4), we obtain after a short calculation

$$\begin{aligned}&u'\in C(S,BV(G)) \quad \text{and}\\ &\|u\|_{C^1(S,BV(G))}\leqq c\left\{\|u_0\|_{BV(G)}+1+\|f\|_{C(S,Bv(G))}\right\}.\end{aligned}\tag{3.21}$$

Therefore $\mathscr{T}$ has a fixed point $u\in C^1(S,BV(G))$ which solves (2.3), (2.4). □

Proof of Corollary 2.2. We modify estimates (3.6), (3.13). One has

$$|u'(t)|_p\leqq\left|A^{\varepsilon}_{u(t)}(\alpha(u(t)))\right|_p+|f(t)|_p$$

$$\leqq c\left\{1+|u(t)|_p\right\}+|f(t)|_p$$

from arguments which led to estimate (3.5). Taking (3.5) into account, one gets for $r\in[1,\infty]$

$$|u'|_{L^r(S,L^p(G))}\leqq c\left\{1+|u_0|_p+|f|_{L^2(S,L^p(G))}\right\}.\tag{3.22}$$

Similarly, (3.11) implies

$$\|u\|_{C(S,W^{s,p}(G))}\leqq c\left\{1+\|u_0\|_{s,p}+|f|_{L^1(S,W^{s,p}(G))}\right\},$$

and (2.3) yields

$$\|u'(t)\|_{s,p}\leqq\left\|A^{\varepsilon}_{u(t)}(\alpha(u(t)))\right\|_{s,p}+\|f(t)\|_{s,p}.$$

Therefore,

$$|u'|_{L^r(S,W^{s,p}(G))}\leqq c\left\{1+\|u_0\|_{s,p}+|f|_{L^r(S,W^{s,p}(G))}\right\}.\tag{3.23}$$

□

Proof of Theorem 2.3. We employ the Galerkin method. Let $\{w_i\} \subset H_0^1(G) \cap H^2(G)$ be an orthonormal eigenvalue basis of the Laplacian subject to Dirichlet boundary conditions, i.e,.

$$-\Delta w_i = \lambda_i w_i,\ (w_i, w_j) = \delta_{ij},\ w_i \perp w_j \quad \text{in } H_0^1(G). \tag{3.24}$$

Set $V_n := \operatorname{span}\{w_1, \cdots, w_n\}$, denote by P_n the orthogonal projection in $H_0^1(G)$ onto V_n. We are looking for an absolutely continuous

$$u_n(t) := \sum_{j=1}^{n} h_{nj}(t) w_j, \tag{3.25'}$$

which satisfies

$$\begin{aligned} &\big(u_n'(t), v\big) + \varepsilon\big(k(u_n)\nabla u_n', \nabla v\big) + \big(k(u_n)\nabla\alpha(u_n), \nabla v\big) \\ &\qquad = \varepsilon\big(k(u_n)\nabla f, \nabla v\big) + (f, v) \quad \forall v \in V_n, \text{ a.a. } t \in S, \end{aligned} \tag{3.25}$$

$$u_n(0) = u_{0n} := P_n u_0. \tag{3.26}$$

By using a fixed point argument and applying the results of [6] or a reduction of (3.25), (3.26) to an ordinary differential equation (cf. [5, Lemma 1]) to obtain the standard form for an application of Caratheodory's theorem, one shows, that (3.25')–(3.26) is (at least) locally solvable. The following lemma implies that these solutions are globally defined. One has

LEMMA 3.8. *There is a constant $c = c(\varepsilon k_0, k_1, L_\alpha, T, |G|, \varepsilon)$ such that for $r \in [1, \infty]$*

$$\|u_n\|_{W^{1,r}(S, H_0^1(G))} \leqq c\left\{\|u_0\| + \|f\|_{L^1(S, H_0^1(G))}\right\}. \tag{3.27}$$

Proof of Lemma 3.8. Choose in (3.25) $v := u_n'(t)$, use the boundedness properties of $k(\cdot)$ and $\alpha(\cdot)$ and apply Hölder's and Young's inequalities

$$\begin{aligned} &|u_n'(t)|^2 + \varepsilon k_0 |\nabla u_n'(t)|^2 \\ &\quad k_1 L_\alpha |\nabla u_n| |\nabla u_n'| + \varepsilon k_1 |\nabla f| |\nabla u_n'| + |f| \cdot |u_n'| \\ &\quad \leqq \frac{\varepsilon k_0}{2} |\nabla u_n'|^2 + c\left\{|\nabla u_n|^2 + |\nabla f|^2 + |f|\right\} + \frac{1}{2}|u_n'|^2, \end{aligned} \tag{3.28}$$

$$|\nabla u_n'(t)|^{\hat r} \leqq c\left\{|f|^{\hat r} + |\nabla f|^{\hat r} + |\nabla u_{0n}|^{\hat r} + \int_0^t |\nabla u_n'(s)|^{\hat r}\, ds\right\}. \tag{3.29}$$

If $r < \infty$, then $\hat r := r$; if $r = \infty$, then $\hat r := 1$.

Gronwall's inequality in connection with (3.26) yields

$$\|u_n'\|_{L^r(S, H_0^1(G))} \leqq c\left\{\|f\|_{L^r(S, H_0^1(G))} + \|u_0\|\right\}$$

and therefore (3.27). The usual compactness argument completes the proof and shows that (2.1), (2.2) has a solution, which satisfies

$$\|u\|_{W^{1,r}(S, H_0^1(G))} \leqq c\left\{\|u_0\| + \|f\|_{L^r(S, H_0^1(G))}\right\}. \tag{3.30}$$ □

Before proving Corollaries 2.4, 2.5, we prove Theorem 2.6.

Proof of Theorem 2.6(a). We use the Galerkin method and continue at (3.25), (3.26). We have already estimate (3.27) and will show that

$$\|u_n\|_{W^{1,r}(S, W^{2,p}(G) \cap W_0^{1,p}(G))} \leqq \text{const}. \tag{3.31}$$

To see this, note that due to (3.24)

$$v := -\Delta u_n'(T) \in V_n$$

is an admissible function in (3.25). Integration by parts and reordering yield

$$\begin{aligned} &|\nabla u_n'(t)|^2 + \varepsilon(k(u_n)\Delta u_n', \Delta u_n') \\ &\quad = \varepsilon(k'(u_n)\nabla u_n', \nabla u_n v) + (f, v) - \varepsilon(k'(u_n)\nabla u_n, \nabla f v) \\ &\qquad - \varepsilon(k(u_n)\Delta, f, v) + (k'(u_n)\Delta u_n \cdot \nabla u_n, \alpha'(u_n)v) \\ &\qquad + \left(k(u_n)\alpha''(u_n)|\nabla u_n|^2, v\right) + (k(u_n)\alpha'(u_n)\Delta u_n, v). \end{aligned}$$

By (H1), (H2), (H4), (H5), Hölder's, Young's and one of Sobolev's inequalities we obtain

$$\begin{aligned} &|\nabla u_n'(t)|^2 + \varepsilon k_0 |\Delta u_n'(t)|^2 \\ &\quad \leqq \varepsilon L_k |\nabla u_n'|_4 |\nabla u_n|_4 |v|_2 + |f|_2 |v|_2 + \varepsilon L_k |\nabla u_n|_4 |\nabla f|_4 |v|_2 \\ &\qquad + \varepsilon k_1 |\Delta f|_2 |v|_2 + L_k \alpha_1 |\nabla u_n|_4^2 |v|_2 + k_1 \alpha_1 |\nabla u_n|_4^2 |v|_2 + k_1 \alpha_1 |\Delta u_n|_2 |v|_2 \\ &\quad \leqq \frac{\varepsilon k_0}{4} |v|_2^2 + c\Big\{ |\nabla u_n'|_2 |\Delta u_n'|_2 |\nabla u_n|_2 + |f|_2^2 \\ &\qquad + |\nabla u_n||\Delta u_n||\nabla f|_4^2 + |\Delta f|_2^2 + |\nabla u_n|^2 |\Delta u_n|^2 + |\Delta u_n|^2 \Big\}. \end{aligned}$$

(This is the only place where the restriction $N=2$ is essential. The case $N=3$ allows only estimates like $|\nabla u_n'|_4 \leqq c|\nabla u_n'|_2^{1/3} |\Delta u_n'|_2^{2/3}$, which finally would require some restrictions on $|\nabla f_4$ and $\|u_0\|_{2,2}$, to obtain (3.31).)

From now on we use again our convention concerning the notation of L^2-norms. Using again Young's inequality, we arrive at

$$\begin{aligned} |\Delta u_n'(t)|^2 \leqq c\Big\{ &|\nabla u_n'|^2 |\nabla u_n|^2 |\Delta u_n|^2 + |f|^2 + |\Delta f|^2 \\ &+ |\nabla u_n||\Delta_n||\nabla f|_4^2 + |\nabla u_n|^2 |\Delta u_n|^2 + |\Delta u_n|^2 \Big\} \end{aligned}$$

$$\begin{aligned} |\Delta u_n'(t)| \leqq c\Big\{ &|\nabla u_n'||\nabla u_n||\Delta u_n| + |f| + |\nabla u_n| + |\Delta u_n||\nabla f|_4^2 \\ &+ |\Delta f| + |\nabla u_n||\Delta u_n| + |\nabla u_n||\nabla f|_4^2 + |\Delta u_n| \Big\}. \end{aligned}$$

By (3.27),

$$|\nabla u_n|_{C(S, L^2(G)^N)} \leqq c\left\{ \|u_0\| + \|f\|_{L^1(S, H_0^1(G))} \right\} =: \bar{c},$$

so that

$$(3.32)\qquad |\Delta u_n'(t)| \leqq c\left\{ \bar{c}|\nabla u_n'|\left(|\Delta u_{0n}| + \int_0^t |\Delta u_n'(s)|\,ds\right) + \|f\|_{2,2} + \bar{c} + \left(|\Delta u_{0n}| + \int_0^t |\Delta u_n'(s)|\,ds\right)|\nabla f|_4^2 + \cdots \right\}.$$

If $r=\infty$, then estimate (3.30) and Gronwall's inequality yield $|\Delta u_n'|_{C(0,t;\,L^2(G))} \leqq$ const., where the const. depends on

$$(3.33)\qquad \varepsilon k_0,\ k_1,\ L_k,\ T, N=2,\ |G|,\ |f|_{L^r(S,W^{1,4}(G))},\ |\Delta u_0|_2.$$

If $r \in [1,\infty)$, take on both sides of (3.32) the rth power and integrate over $(0,t) \subset [0,T]$, which yields by Gronwall's inequality

$$|\Delta u_n'|_{L^r(0,t;\,L^2(G))} \leqq \text{const.}$$

Because of $u_n/\partial G = \Delta u_n'/\partial G = 0$, this implies

$$(3.34')\qquad \|u_n\|_{W^{1,r}(S,W^{2,2}(G)\cap H_0^1(G))} \leqq \text{const} =: \hat{c},$$

$\hat{c}$ as in (3.33). The usual compactness argument finishes the proof and yields an estimate

$$(3.34)\qquad \|u\|_{W^{1,r}(S,W^{2,2}(G)\cap H_0^1(G))} \leqq \hat{c}$$

for the solution u of (2.1), (2.2).

Proof of part (b). (Sketch, a similar argument is used in the proof of Corollary 2.4.) Under the conditions of part (b) we set that (2.1), (2.2) holds in $L^1(G)$ a.e. Therefore, we can multiply equation (2.2) on both sides by $v := -|\Delta u'(t)|^{p-2}\Delta u'(t)$ and arrive, using similar arguments as in (a) and estimate (3.34), at

$$(3.35)\qquad \|u\|_{W^{1,r}(S,W^{2,p}(G))} \leqq c\left\{1 + |\Delta f|_{L^r(S,L^p(G))} + |\Delta u_0|_p\right\}$$

which is sufficient to complete the proof. c depends via (3.34) on the data. □

Proof of Corollary 2.4. We approximate f and u_0 respectively by regular f_δ, $u_{0\delta}$, and obtain by Theorem 2.6 regular solutions u_δ for which we show the estimate (3.39). The basic tool to obtain (3.39) are the estimates (3.42), (3.43) for a related linear problem (3.40). Let

$$(3.36)\qquad f_\delta \in C\left(S, W^{2,\infty}(G) \cap W_0^{1,\infty}(G)\right),\ u_{0\delta} \in W^{2,\infty}(G) \cap W_0^{1,\infty}(G)$$

such that

$$(3.37)\qquad f_\delta \to f \ \text{ in } C\left(S, W_0^{1+\tau,p^*}(G)\right), \qquad u_{0\delta} \to u_0 \ \text{ n } W_0^{1+\tau,p^*}(G).$$

By Theorem 2.6, there are solutions

$$(3.38)\qquad u_\delta \in C^1\left(S, W^{2,\infty}(G) \cap W_0^{1,\infty}(G)\right),$$

satisfying (2.1), (2.2) with $u_{0\delta}$, f_δ as data. Set $\hat{k}(t,x) := k(u_\delta(t,x))$, $\hat{\gamma}(t,x) := \hat{k}(t,x)\cdot \alpha'(u_\delta(t,x))$. By our hypotheses, $0 < k_0 \leqq \hat{k}(t,x) \leqq k_1$ $\forall t \in S$, $x \in G$ and $|\hat{\gamma}|_\infty < \infty$. It is our goal to show the following estimate. There is a constant c merely depending on the bounds of the coefficients k, α and their derivatives as appearing in (H1)–(H6) and on

$$\|u_0\|_{1+\tau,p^*},\ \|f\|_{C(S,W^{1+\tau,p^*}(G))},$$

such that

$$\|u_\delta\|_{C^1(S,W_0^{1+\tau,p^*}(G))} \leqq c. \tag{3.39}$$

To this end consider for given g, v_0 the linear problem

$$\begin{aligned} &v' - \varepsilon\,\mathrm{div}(\hat{k}\nabla v') - \mathrm{div}(\hat{\gamma}\nabla v) = g - \varepsilon\,\mathrm{div}(\hat{k}\nabla\ ),\\ &v(0) = v_0. \end{aligned} \tag{3.40}$$

We note that choosing $g := f_\delta$, $v_0 := u_{0\delta}$, $v := u_\delta$ satisfies this equation.

Denote by $S_t := [0,t]$, $t \in S$, a subinterval of S. As in Theorems 2.3 and 2.5 one shows the existence of solution operators $P_{i,t}$, $i = 0, 1$,

$$\begin{aligned} &P_{0,t}\colon H_0^1(G) \times C(S_t, H_0^1(G)) \to C(S_t, H_0^1(G)),\\ &P_{1,t}\colon (W^{2,p}(G) \cap W_0^{1,p}(G)) \times C(S_t, W^{2,p}(G)) \to C(S_t, W^{2,p}(G) \cap W_0^{1,p}(G)),\\ &P_{i,t}\colon \{v_0, g\} \mapsto v',\ v_0, g, v, v' \quad \text{as in (3.40)}. \end{aligned}$$

(The numbers p and p^* are related by the hypotheses of this corollary.) These operators are linear and bounded in the given spaces and we denote by $M_{i,t}$ their respective norms. By interpolation it follows that $P_{1,t}$ can be restricted to $W_0^{1+\tau,p^*}(G) \times C(S_t, W^{1+\tau,p^*}(G))$ (see Lemma A5, appendix) and, denoting the restriction by $P_{\tau,t}$,

$$P_{\tau,t}\colon W_0^{1+\tau,p^*}(G) \times (G) \times C(S_t, W_0^{1+\tau,p^*}(G)) \to C(S_t, W_0^{1+\tau,p^*}(G)), \qquad \tau \in (0,1).$$

$P_{\tau,t}$ is linear and bounded and its norm $M_{\varepsilon,t}$ can be estimated by

$$M_{\tau,t} \leqq c M_{0,t}^{1-\tau} M_{1,t}^{\tau} \tag{3.41}$$

with some numerical constant depending on τ, but independent of t. We prove the following

LEMMA 3.9. *There exists a constant c, depending at most on τ, G, T, p^* the several bounds of the coefficients as appearing in the hypotheses of this corollary, such that*

$$\text{(a)} \quad M_{0,t} \leqq c, \tag{3.42}$$

$$\text{(b)} \quad M_{1,t} \leqq c\left\{1 + \|u_\delta\|_{C(S_t,W^{1+\tau,p^*}(G))}^{1/(1-\theta)}\right\}. \tag{3.43}$$

Proof of Lemma 3.9. First we notice that due to the rather technical assumptions of this corollary we have

$$W_0^{1+\tau,p^*}(G) \subset W_0^{1,ap}(G), \tag{3.44}$$

$$[W_0^{1,2}(G), W^{2,p}(G) \cap W_0^{1,p}(G)]_\tau = W_0^{1+\tau,p^*}(G) \tag{3.45}$$

(see the appendix, Lemmas A4, A6). The proof of (a) is at the beginning almost identical to that of estimate (3.27) in Lemma 3.8. To show (3.43) we begin with the case $p = 2$. Computing the div-terms in (3.40), multiplying by $w := -\Delta v'(t)$ and integrating over G by parts, one obtains for $s \in S_t$

$$|\nabla v'(s)|^2 + (\varepsilon\hat{k}\Delta v', \Delta v')$$
$$= (\varepsilon\hat{k}\Delta g + \hat{\gamma}\Delta v, \Delta v') + (g, -\Delta v') + (\varepsilon\nabla\hat{k}\cdot\nabla v' + \nabla\hat{\gamma}\cdot\nabla v, -\Delta v') + (\varepsilon\nabla\hat{k}\cdot\nabla g, \Delta v').$$

To obtain (3.43), it is sufficient to assume that

$$\|v_0\|_{2,p} + \|g\|_{C(S_t, W^{2,p}(G))} \leqq 1. \tag{3.46}$$

Using the boundedness properties of $\hat{k}$ and $\hat{\alpha}$ and Hölder's inequality, we obtain with $c = c(\varepsilon k_0, k_1, L_\alpha)$

$$|\nabla v'(s)|^2 + \varepsilon k_0 |\Delta v'(s)|^2$$
$$\leqq c\{|\Delta g| + |\Delta v| + |g| + |\nabla\hat{k}|_4 |\nabla v'|_4 + |\nabla\hat{\gamma}|_4 |\nabla v|_4 + |\nabla\hat{k}|_4 |\nabla g|_4\} |\Delta v'|.$$

By $v'/\partial G = 0$, there is a constant $c = c(G)$,

$$|\Delta v'(s)|_p + |v'(s)|_p \geqq c\|v'(s)\|_{2,p}; \tag{3.47}$$

by interpolation, $|\nabla v'|_4^2 \leqq c(G)|\Delta v'|_2 \|v'\|_{2,2}$. This and (3.46) imply by using Young's inequality and (3.42),

$$\|v'(s)\|_{2,2} \leqq c\Big\{\|g\|_{2,2} + \|v_0\|_{2,2} + \int_0^s \|v'(r)\|_{2,2}\,dr$$
$$+ |\nabla\hat{k}|_4^2 |\nabla v'|_2 + |\nabla\hat{\gamma}|_4^2 |\nabla v|_2 + \|v\|_{2,2} + |\nabla\hat{\gamma}|_4 \|g\|_{2,2}\Big\}$$
$$\leqq c\Big\{1 + |\nabla\hat{k}|_4^2 + |\nabla\hat{\gamma}|_4^2 + \int_0^s |v'(r)|_{2,2}\,dr\Big\}.$$

Because of

$$|\nabla\hat{k}|_4 + |\nabla\hat{\gamma}|_4 \leqq c\|u_\delta\|_{1,4} \qquad (\text{cf. def. of } \hat{k}, \hat{\gamma})$$
$$\leqq \|u_\delta\|_{3/2,2} \qquad (\text{cf. } (3.44)),$$

this and Gronwall's inequality imply (3.43) for $p = 2$, $\tau = \frac{1}{2}$. To deal with the general case $p \geqq 2$, $\tau \in (0,1)$ as in the assumptions of this corollary, we compute in (3.40) the div-terms, multiply by $w := -|\Delta v'(s)|^{p-2}\Delta v'(s)$, integrate over G parts, use the boundedness of $\hat{k}$ and $\hat{\gamma}$ as implied by (H1), (H2) and apply Hölder's inequality to arrive at

$$|\Delta v'(s)|_p \leqq c\Big\{|v'(s)|_p + |g|_p + |\nabla\hat{k}\cdot\nabla g|_p + |\hat{k}_{\Delta g}|_p$$
$$+ |\hat{\gamma}\Delta v|_p + |\nabla\hat{\gamma}\cdot\nabla v|_p + |\nabla\hat{k}\cdot\nabla v'|_p\Big\} |\Delta v'(s)|_p. \tag{3.48'}$$

By (3.42),

$$|v'(s)|_p \leqq c(G)|v'|_{C(S_t, H_0^1(G))} \leqq c\{\|v_0\| + \|g\|_{C(S_t, H_0^1(G))}\},$$

so that by (3.46) and the embedding theorems

$$|v'(s)|_p \leqq c.$$

Furthermore, $|\nabla g|_{a'p} \leqq c\|g\|_{2,p}$ (we have $N=2$, $p \geqq 2!$), so that by (3.46) $|\nabla g|_{a'p} \leqq c$ (indep. of g). By Sobolev's inequalities $|\nabla v'|_{a'p} \leqq c(G,a',p,p)|\nabla v'|_2^{1-\theta}\|v'\|_{2,p}^{\theta}$ (θ defined in the assumptions) $|v|_{a'p} \leqq c(G,a',p,p)|\nabla v'|_2^{1-\theta}\|v'\|_{2,p}^{\theta}$,

$$\begin{aligned}|\nabla v'(s)|_2 + |\nabla v(s)|_2 &\leqq c\left\{\|v_0\| + \|g\|_{C(S_t,H_0^1(G))}\right\} \quad \text{by (3.42)}\\ &\leqq c \quad \text{by (3.46)}.\end{aligned}$$

Finally, by (3.44), $|\nabla \hat{k}\|_{ap} + |\nabla \hat{\gamma}|_{ap} \leqq c\|u_\delta\|_{1,ap} \leqq c\|u_\delta\|_{1+\tau,p^*}$. Therefore, (3.47), (3.48′) and Young's inequality imply

$$\begin{aligned}\|v'(s)\|_{2,p} &\leqq c\left\{1 + \|u_\delta\|_{1+\tau,p^*}^{1/(1-\theta)} + \|v(s)\|_{2,p}\right\}\\ &\leqq c\left\{1 + \|u_\delta\|_{1+\tau,p^*}^{1/(1-\theta)} + \|v_0\|_{2,p} + \int_0^s |v'(r)|_{2,p}\,dr\right\}.\end{aligned}$$

Set for abbreviation $V := W_0^{1+\tau,\tau^*}(G)$. Gronwall's inequality and (3.46) show that

$$\begin{aligned}&|v'|_{C(S_t,W^{2,p}(G))} \leqq c\left\{1 + \|u_\delta\|_{C(S_t,V)}^{1/(1-\theta)}\right\} \cdot \exp(cT),\\ &\text{where } c = c\left(\varepsilon k_0, k_1, L_\alpha, |G|, T, p, \theta, \tau, p^*, |k'|_\infty, |\alpha''|_\infty\right),\end{aligned} \tag{3.48}$$

which implies (3.43). Lemma 3.9 is proved. □

To show (3.9), we first notice, that by (3.37), (3.41)–(3.43) and by $\theta + \tau \leqq 1$,[4]

$$\begin{aligned}\|u_\delta'\|_{C(S_t,V)} &\leqq c\left\{1 + \|u_\delta\|_{C(S_t,V)}^{1/(1-\theta)}\right\}^{\tau}\left\{\|u_{0\delta}\|_V + \|f_\delta\|_{C(S_t,V)}\right\}\\ &\leqq c\left\{1 + \|u_\delta\|_{C(S_t,V)}\right\}\left\{\|u_0\|_V + \|f\|_{C(S,V)}\right\}\\ &\leqq c\left\{1 + \|u_\delta\|_{C(S_t,V)}\right\}\\ &\leqq c\left\{1 + \|u_{0\delta}\|_V + \int_0^t \|u_\delta'(s)\|_V\,ds\right\}.\end{aligned}$$

Gronwall's inequality and (3.37) yield (3.39). Therefore, problem (3.40) with $u_{0\delta}$, f_δ as data has u_δ as a solution, which by (3.39) satisfies

$$\|u_\delta\|_{W^{1,\infty}(S,V)} \leqq \text{const}.$$

The usual compactness argument yields a (sub-) sequence

$$u_\delta \stackrel{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(S,V).$$

By means of the approximate equation (3.40) one easily shows that u satisfies (2.1), (2.2). This completes the proof of Corollary 2.4. □

Proof of Corollary 2.5. (a) follows directly from Corollary 2.4.

(b) set $a := q/p$ and apply (a). Then, $u \in C^1(S, W_0^{1+\tau,p^*}(G))$ and by (3.44), $u \in C^1(S, W_0^{1,q}(G))$. □

Proof of Theorem 2.7. Let u_i be a solution of (2.1), (2.2) with respect to the initial value u_{0i} and the right-hand side f_i, $i=1,2$. Set for abbreviation $f := f_1 - f_2$, $w := u_1 - u_2$,

[4] $V := W_0^{1+\tau,p^*}(G)$, $\|\cdot\|_V := \|\cdot\|_{1+\tau,p^*}$.

$w_0 := u_{01} - u_{02}$ and assume

$$u_2 \in W^{1,1}(S, W_0^{1,p}(G)), \quad u_1 \in W^{1,1}(S, H_0^1(G)),$$
$$f_2 \in L^1(S, W^{1,p}(G)), \qquad f_1 \in L^1(S, H_0^1(G)).$$

Subtract (2.2) for u_2 from that for u_1, choose w' as a test-function in the variational formulation, integrate over G and make some rearrangements. Thus, for $t \in S$

$$\begin{aligned}
&|w'(t)|^2 + \varepsilon k_0 |\nabla w'(t)|^2 \\
&\quad \leqq |w'(t)|^2 + \varepsilon(k(u_1)\nabla w', \nabla w') \\
&\quad = (f, w') + \varepsilon(k(u_1)\nabla f, \nabla w') + \varepsilon((k(u_1) - k(u_2))\nabla f_2, \nabla w') \\
&\qquad + \varepsilon((k(u_1) - k(u_2))\nabla u_2', \nabla w') + (\gamma(u_1)\nabla w, \nabla w') \\
&\qquad + ((\gamma(u_1) - \gamma(u_2))\nabla u_2, \nabla w').
\end{aligned}$$

By using the Lipschitz continuity of k and γ and applying Hölder's inequality, we obtain

$$\begin{aligned}
&|w'(t)|^2 + \varepsilon k_0 |\nabla w'(t)|^2 \\
&\leqq |f||w'| + \left\{ \varepsilon k_1 \|f\| + \varepsilon L_k |w|_{p'} \|f_2\|_{1,p} + \varepsilon L_k |w|_{p'} \|u_2'\|_{1,p} + |\gamma|_\infty \|w\| + L_\gamma \|u_2\|_{1,p} |w|_{p'} \right\} \|w'(t)\|.
\end{aligned}$$

By the imbedding theorems, $|w|_{p'} \leqq c(G)\|w\|$. Also, $\|w(t)\|^r \leqq \|w_0\|^r + c\int_0^t \|w(r)\|^r dr$, if $r \in [1, \infty)$, with $c = c(r, T)$. This and Young's inequality imply for $\bar{r} := r$ if $r < \infty$, $\bar{r} = 1$ if $r = \infty$,

$$\begin{aligned}
&\psi := \|f_2\|_{1,p}^{\bar{r}} + \|u_r'\|_{1,p}^{\bar{r}} + \|u_2\|_{1,p}^{\bar{r}}, \\
&\|w'(t)\|^r \leqq \varepsilon k_1 \|f(t)\|^r = c\psi(t)\left\{ \|w_0\|^r + \int_0^t \|w'(s)\|^r ds \right\}.
\end{aligned}$$

Integration and Gronwall's inequality yield

$$\|w'\|_{L^r(S, H_0^1(G))} \leqq c\left\{ \|f\|_{L^r(S, H_0^1(G)} + \|w_0\| \cdot |\psi|_{L^1(S)} \right\} \cdot \exp\left\{ c|\psi|_{L^1(S)} \right\}.$$

For $r := \infty$ we obtain

$$\|w'\|_{L^\infty(S, H_0^1(G))} \leqq c\left\{ \|f\|_{L^\infty(S, H_0^1(G))} + \|w_0\| \cdot |\psi|_{L^\infty(S)} \right\} \cdot \exp\left\{ c|\psi|_{L^\infty(S)} \right\}.$$

This proves (b) of the theorem, and (a) follows by setting $u_{01} = u_{02}$, $f_1 = f_2$. □

Proof of Theorem 2.8. First of all, notice that $\hat{k}(t) := (u(t, \cdot)) \in L^\infty(G)$, $\hat{k}(t, x) \geqq k_0 > 0$ for a.a. $t \in S$, $x \in G$. Thus, $A(t) := -\operatorname{div}(\hat{k}(t)\nabla(\cdot))$ is an m-accretive operator on $L^1(G)$. Now, [6, Thm. 1] yields the desired estimates. If $u_0 \geqq 0$, $f(T) \geqq 0$, then obviously, $|u(t)|_{L^\infty(G)} = 0$, i.e., $u(t, s) \geqq 0$ a.e. □

Appendix. We list some facts on complex interpolation of B-spaces and, in particular, Sobolev spaces. Basic references are Bergy–Löfstrom [3], Lions–Magenes [13], and Triebel [18].

Let $\tau\in[0,1]$ be a parameter, $\bar{A}:=\{A_0,A_1\}$, $\bar{B}:=\{B_0,B_1\}$—two couples of compatible B-spaces "$[\cdot,\cdot]_\tau$" denotes the interpolation functor for the complex interpolation method (cf. [3], [13], [18]), $\bar{A}_\tau:=[A_0,A_1]_\tau$. We have

LEMMA A1. $[A_0\times B_0,A_1\times B_1]_\tau=[A_0,A_1]_\tau\times[B_0,B_1]_\tau$ *(algebraically and topologically).*

LEMMA A2. *Let* $\mathscr{P}\in\mathscr{L}(A_i,B_i)$ *with norm* M_i, $i=0,1$. *Then* $\mathscr{P}\in\mathscr{L}(\bar{A}_\tau,\bar{B}_\tau)$ *and* $|\mathscr{P}|_{\mathscr{L}(\bar{A}_\tau,\bar{B}_\tau)}\leqq M_0^{1-\tau}M_1^\tau$.

LEMMA A3. $[C(S,A_0),C(S,A_1)]_\tau=C(S,\bar{A}_\tau)$ *(algebraically and topologically).*

LEMMA A4. (a) *Let* $s_i\in\mathbb{R}$, $p_i\in(1,\infty)$, $\tau\in(0,1)$, $s^*:=(1-\tau)s_0+\tau s_2$,

$$\frac{1}{p^*}=\frac{1-\tau}{p_0}+\frac{\tau}{p_1},$$

$s_0\neq s_1$, $G\subset\mathbb{R}^N$ *a (bounded) domain. Then* $[W^{s_0,p_0}(G),W^{s_1,p_1}(G)]_\tau=W^{s^*,p^*}(G)$.

(b) *Let*

$$p\in(1,\infty),\qquad \tau\in(0,1),\quad \frac{1}{\bar{p}}=\frac{1-\tau}{2}+\frac{\tau}{p}.$$

Then $[W_0^{1,2}(G),W^{2,p}(G)\cap W_0^{1,p}(G)]_\tau=W_0^{1+\tau,\bar{p}}(G)$.

LEMMA A5. *Let* τ,p *and* $\bar{p}$ *be as in Lemma* A4(b), $S:=[0,T]$ *an interval of any finite length,*

$\mathscr{P}_1$: $(W^{2,p}(G)\cap W_0^{1,p}(G))\times C(S,W^{2,p}(G))\to C(S,W^{2,p}(G)\cap W_0^{1,p}(G))$
a linear and bounded operator with norm M_1,

$\mathscr{P}_0$ *a linear and bounded extension of* $\mathscr{P}_1$, *such that*
$\mathscr{P}_0$: $W_0^{1,2}(G)\times C(S,W^{1,2}(G))\to C(S,W_0^{1,2}(G))$.

Then

$\mathscr{P}_\tau:=$ *restriction of* $\mathscr{P}_0$ *to* $W_0^{1+\tau,\bar{p}}(G)\times C(S,W^{1+\tau,\bar{p}}(G))$ *maps continuously into* $C(S,W_0^{1+\tau,\bar{p}}(G))$.

Let M_0 *denote the norm of* $\mathscr{P}_0$, M_τ *that of* $\mathscr{P}_\tau$. *Then*

$$M_\tau\leqq M_0^{1-\tau}M_1^\tau.$$

Note that this estimate does not depend on S.

References/proofs. Lemma A1 follows from [3, Thm. 4.1.2]. Lemma A2 is a special case of a theorem in [11]. Lemma A3 can be found in [13], if A_1, A_0 are Hilbert spaces, otherwise cf. [10]. Lemma A4(a)—cf. [3, Thm. 6.4.5], (b)—cf. [18, Thm. 4.3.3]. Lemma A5 follows from Lemmas A1–A4.

Lemma A6. [3, Thm. 6.5.1]. Let $s_i\in\mathbb{R}$, $p_i>1$, $i=1,2$. If $s_1-N/p_1\geqq s_2-N/p_2$, then $W^{s_1,p_1}(G)\subset W^{s_2,p_2}(G)$.

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