APPLICATION OF THE WIENER-HOPF TECHNIQUE TO HALF PLANE DIFFRACTION OF CYLINDRICAL WAVES

by

DONALD EDWARD AMOS

A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1960

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Chairman of School Graduate Committee

Redacted for Privacy

Dean of Graduate School

Date thesis is presented May 6, 1960

Typed by Rose Amos

ACKNOWLEDGMENT

The author wishes to thank Professor F. Oberhettinger for his suggestions and direction during the development of the material in this thesis.

TABLE OF CONTENTS

APPLICATION OF THE WIENER-HOPF TECHNIQUE
TO HALF PLANE DIFFRACTION OF CYLINDRICAL WAVES

1. INTRODUCTION

In 1896 Sommerfeld (33) published his classical result on the diffraction of plane waves by a perfectly reflecting half plane. In this problem plane waves were incident on a perfectly reflecting half plane, and the field surrounding the plane was computed using multivalued solutions of the wave equation. Following this development, Carslaw (3) applied Sommerfeld's method to the half plane with cylindrical source excitation - in the electromagnetic case, a line of Hertz dipoles oriented parallel to the edge of the half plane; in the acoustic case, a line source parallel to the edge. In 1915 MacDonald (20) published a result for the corresponding wedge problem. Since that time many authors (2,18,22,23,26,28,29,39,40,41) have contributed results for the wedge. In this class of problems the half plane is a wedge with an angle of 2π, and the results clearly apply to the half plane problem.

The Wiener-Hopf technique for the solution of a class of integral equations was first published in 1931 (42, 31, p. 49-67). Magnus (21) expressed the solution of the half plane problem with incident plane waves in terms of an integral equation of the Wiener-Hopf type which he solved by computing coefficients in a series of Bessel functions. Later, Copson (5) solved the same integral equation by the

Wiener-Hopf technique. These developments lead the way to a variety of papers (1,12,13,14,15,16) which demonstrate the power of the method for complicated diffraction gratings and plane wave excitation. Karp (17) gives an expository paper in which separation of variables is combined with Wiener-Hopf technques to yield solutions of physically significant problems.

The Wiener-Hopf investigation for the half plane problem leaves the solution in the form of contour integrals in a complex plane. Gast (10) chose to manipulate the integrals along hyperbolic contours around branch cuts taken radially from the origin (in the manner of Copson). The resulting double integrals were subsequently reduced to the solutions of MacDonald and Sommerfeld, the plane wave (Sommerfeld) solution being obtained by letting the line source tend to infinity. In addition, geometric optics and diffraction terms emerged, and the results of Harrington (11) for the angular variation of the far field were expressed in terms of tabulated Fresnel integrals. (See also Oberhettinger for further asymptotic estimates.)

In the following investigation we chose to make branch cuts parallel to the imaginary axis and manipulate straight line and circular contours. This choice gives the results in the form of a single integral. Furthermore, aside from computational value, the wave solutions can easily be superimposed for the study of arbitrary pulses at the source. That is, in the secondary field the frequency occurs in only an exponential term which makes the Fourier analysis for arbitrary pulses particularly simple.

The entire analysis is based upon the Maxwell equations for free space,

$$\operatorname{curl} \vec{H} = \frac{\epsilon}{v}\frac{\partial \vec{E}}{\partial t} \qquad \operatorname{div} \vec{H} = 0$$

$$\operatorname{curl} \vec{E} = -\frac{\mu}{v}\frac{\partial \vec{H}}{\partial t} \qquad \operatorname{div} \vec{E} = 0$$

where the sources are line currents (Hertz dipoles) parallel to the z axis. This orientation restricts the field so that $E_x = E_y = H_z = 0$, and the Maxwell equations reduce to a two dimensional wave equation,

$$[1.0] \qquad \nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \quad , \quad c = \frac{v}{\sqrt{\epsilon\,\mu}}$$

for $\Phi = E_z(x,y)$. We seek a solution for Φ at a point $P(x,y)$ in the form of an inverse Laplace integral,

$$[1.1] \quad \Phi(P;Q,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{g}(\gamma)u(\gamma)e^{\gamma ct}d\gamma \quad , \ t>0 \ , \ c>0$$

where $\overline{g}(\gamma)$ is the Laplace transform of a source of strength $g(t)$ at a source point $Q(x',y')$. Under suitable conditions, this amounts to a Fourier synthesis with the time dependency e^{iwt} replaced by $e^{\gamma ct}$. With Φ expressed by [1.1], we have formally,

$$[1.2] \qquad \nabla^2 u - \gamma^2 u = \delta(x-x')\,\delta(y-y').$$

Thus, the entire problem with arbitrary pulse strengths at the source point $Q(x',y')$ is reduced to the solution of equation [1.2] in the space variables subject to the boundary condition $u = 0$ on the perfectly conducting surfaces.

The solution for u is assumed to be of the form

$$[1.3] \quad u = u_i + u_s \ , \ u_i = -\frac{1}{2\pi} K_o[\gamma \sqrt{(x-x')^2+(y-y')^2}]$$

where u_i is the incident field at a point $P(x,y)$ due to a unit source at $Q(x',y')$ and u_s, the secondary field, is regular throughout the entire region under consideration. Thus, under suitable conditions we find that the solution for Φ can be reduced to the construction of the secondary field, u_s. Chapters 2 and 3 are devoted primarily to this purpose for the infinite reflecting plane and the half plane respectively. Chapter 4 presents an asymptotic analysis of the current induced in the half plane by the incident radiation.

2. REFLECTION FROM AN INFINITE PLANE

The analysis for an infinite reflecting plane is inserted here to form a basis for some of the assumptions needed in the half plane case. In particular, we wish to study the induced current distribution in the conducting plane at large distances from the source.[1] That is, one expects currents to flow in the conducting plane in accordance with the laws of electromagnetic induction. Since the incident field has no H_z component, the currents flow parallel to the z axis. These currents act like sources of new radiation and re-radiate the energy incident upon the plane to create a secondary field u_s. Thus, in these problems for the conducting planes, the secondary field can be constructed in terms of the induced current distribution.

To be more precise, consider a cylindrical source parallel to the z axis at a point $Q(x',y')$ radiating on an infinite, perfectly conducting plane of zero thickness, $y = 0$, $-\infty < x < \infty$ in Figure 1. If currents are induced in the plane in the amount $I(\xi)$ per unit length in the x direction, then the total contribution to the field from

[1] The induced current is often identified with surface currents on the two sides of the plane. In either case, the invention of a current explains the discontinuous tangential component H_x at the surface of the perfect conductor.

all increments is

$$[2.0] \quad u_s = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} I(\xi)K_o[\gamma\sqrt{(x-\xi)^2+y^2}]d\xi$$

$$[2.0a] \quad = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{i\alpha x-|y|\sqrt{\alpha^2+\gamma^2}}}{\alpha^2+\gamma^2}\bar{I}(\alpha)d\alpha$$

where $\bar{I}(\alpha)$ is the Fourier transform of $I(x)$,

$$\bar{I}(\alpha) = \int_{-\infty}^{+\infty} I(x)e^{-i\alpha x}dx.$$

Here we have used a Fourier transform representation (7, vol. 1, p. 17) for $K_o[\gamma\sqrt{(x-\xi)^2+y^2}]$ and exchanged the order of the integrations to get equation [2.0a]. Naturally we assume that $\bar{I}(\alpha)$ exists and that the inversion of the integrations is valid. To proceed, we may express the total field, equation [1.3], by

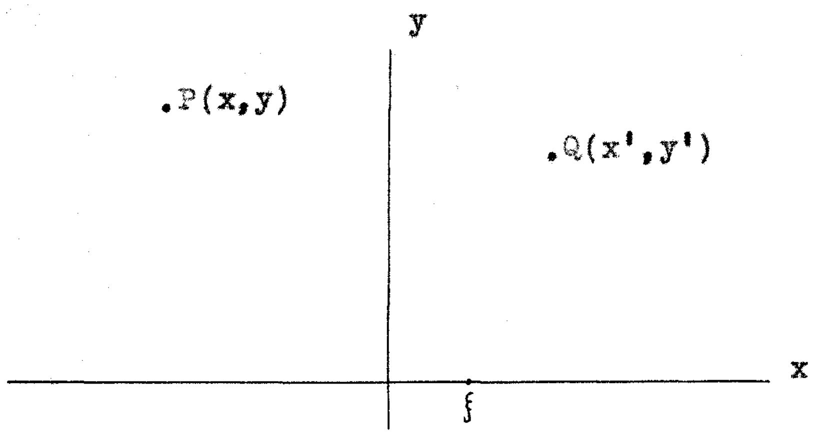

Figure 1

$$u = -\frac{1}{2\pi} K_o[\gamma \sqrt{(x-x')^2+(y-y')^2}] \tag{2.1}$$
$$- \frac{1}{2\pi} \int_{-\infty}^{+\infty} I(\xi)K_o[\gamma \sqrt{(x-\xi)^2+y^2}]d\xi$$

and determine $I(\xi)$ from the boundary condition $u(x,0) = 0$,

$$\int_{-\infty}^{+\infty} I(\xi)K_o[\gamma |x-\xi|]d\xi = -K_o[\gamma \sqrt{(x-x')^2+y'^2}], -\infty<x<\infty. \tag{2.2}$$

This integral equation is readily solved by the Fourier transform since the integral on the left is a convolution integral. If the exchange of orders of integration is permitted, we have (7, vol. 1, p. 56)

$$\frac{\overline{I}(\alpha)\cdot\pi}{\sqrt{\alpha^2+\gamma^2}} = \frac{-\pi e^{-i\alpha x'-|y'|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha^2+\gamma^2}}$$

or

$$\overline{I}(\alpha) = -e^{-i\alpha x'-|y'|\sqrt{\alpha^2+\gamma^2}}. \tag{2.3}$$

Inversion of $\overline{I}(\alpha)$ gives (7, vol. 1, p. 56)

$$I(x) = \frac{-\gamma|y'|\, K_1[\gamma \sqrt{(x-x')^2+y'^2}]}{\pi\sqrt{(x-x')^2+y'^2}}. \tag{2.4}$$

From the asymptotics of $K_1(z)$ as $|z|\to\infty$, we obtain

$$I(x) \sim -|y'|\sqrt{\frac{\gamma}{2\pi}}\,\frac{e^{-\gamma|x|}}{|x|^{3/2}} \text{ for } |x|\to\infty. \tag{2.5}$$

The secondary field, using equations [2.3] and [2.0a] may be expressed by

$$u_s = \frac{1}{2\pi} K_o[\gamma \sqrt{(x-x')^2+(|y|+|y'|)^2}],$$

and we recognize that this is the field due to an image of opposite sign at $(x',-y')$. The total field becomes

$$[2.6] \quad u = \frac{1}{2\pi}\Big\{K_o[\gamma \sqrt{(x-x')^2+(|y|+|y'|)^2}] - K_o[\gamma \sqrt{(x-x')^2+(y-y')^2}]\Big\}.$$

Current-Field Relationship

The connection between the field and the current density in the plane is obtained by setting up the integral equation in another way. If we apply Green's formula,

$$[2.7] \quad \iint_R (\phi \nabla^2\psi - \psi \nabla^2\phi)d\xi d\eta = \int_S (\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n})ds$$

with

$$\phi(x,y) = u_s = u - u_1$$

$$\psi(x,y) = u_1 = -\frac{1}{2\pi} K_o[\gamma \sqrt{(x-x')^2+(y-y')^2}]$$

$$\nabla^2\psi - \gamma^2\psi = \delta(x-x')\,\delta(y-y')$$

$$\nabla^2\phi - \gamma^2\phi = 0 \;, \; u(x,0) = 0$$

to a semicircular region in the upper half plane and let the radius tend to infinity, we obtain

$$u_s = \int_{-\infty}^{+\infty} \left[- u_s \frac{\partial u_i}{\partial \eta} + u_i \frac{\partial u_s}{\partial \eta} \right]_{\eta=0} d\xi$$

or

$$u - u_i = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_o[\gamma \sqrt{(x-\xi)^2+y^2}] \left[\frac{\partial u(\xi,\eta)}{\partial \eta} \right]_{\eta=0} d\xi .$$

Thus, $I(x)$ can be identified with $(\partial u/\partial y)_{y=0+}$.

3. THE HALF PLANE FIELD

Wiener-Hopf Theory

The half plane problem in its formulation is very similar to that of the full plane. The difference lies in the integral equation for $I(\xi)$. In the full plane case, $I(\xi)$ was to be determined for all ξ knowing the field $u(x,0)$ for all x. In the half plane case, $I(\xi) = 0$ for $\xi < 0$ and $u(x,0) = 0$ for $x > 0$. The problem is to determine $I(\xi)$ for $\xi > 0$ and $u(x,0)$ for $x < 0$. These requirements are expressed in the following way.

Consider the half plane $y = 0$, $0 \leq x < \infty$ in Figure 2.

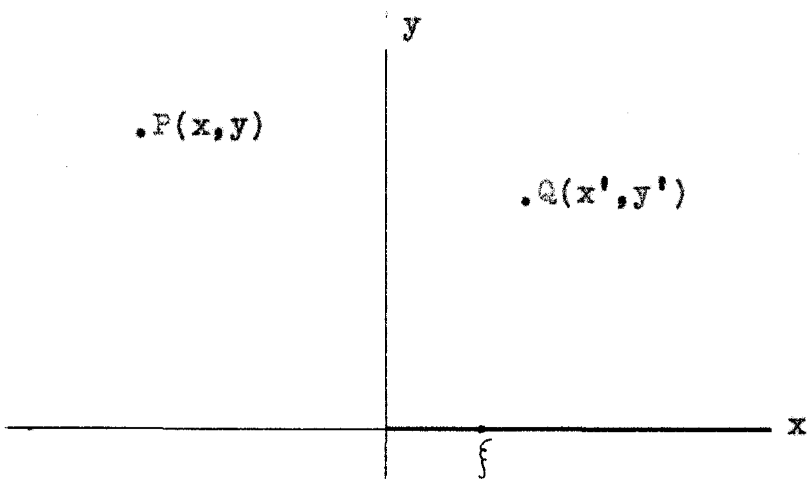

Figure 2

In the manner of Chapter 2 we obtain for the total field,

$$u = -\frac{1}{2\pi} K_o[\gamma \sqrt{(x-x')^2+(y-y')^2}]$$

[3.1]

$$-\frac{1}{2\pi}\int_0^{\infty} I(\xi)K_o[\gamma\sqrt{(x-\xi)^2+y^2}]d\xi \,.$$

When we insert the boundary condition $u(x,0) = 0$, $x > 0$, we have

[3.2]
$$\int_0^{\infty} I(\xi)K_o[\gamma\,|x-\xi|]d\xi \;+\; K_o[\gamma\sqrt{(x-x')^2+y'^2}]$$

$$= \begin{cases} 2\pi\; u(x,0) & x<0 \\ 0 & x>0. \end{cases}$$

That is, this is a dual pair of integral equations in which we have to find $I(\xi)$ for $\xi > 0$ and $u(x,0)$ for $x < 0$ knowing $I(\xi)$ for $\xi < 0$ and $u(x,0)$ for $x > 0$. This equation can be cast into the form

[3.3]
$$\int_{-\infty}^{+\infty} f(\xi)\,\ell(x-\xi)d\xi = g(x) + h(x) \qquad -\infty<x<\infty$$

by the relations

$$f(\xi) = \begin{cases} 0 & \xi < 0 \\ I(\xi) & \xi > 0 \end{cases}$$

$$\ell(x) = K_0[\gamma\ |x|] \qquad -\infty<x<\infty$$

$$g(x) = -\ K_0[\gamma \sqrt{(x-x')^2+y'^2}] \qquad -\infty<x<\infty$$

$$h(x) = \begin{cases} 2\pi\ u(x,0) & x < 0 \\ 0 & x > 0 \end{cases}$$

The technique of solution consists of examining the Fourier transform in the complex α plane and making deductions about the form of the transformed equation. To be more precise, consider the transform of equation [3.3],

$$[3.4] \qquad \bar{f}(\alpha)\,\bar{\ell}(\alpha) = \bar{g}(\alpha) + \bar{h}(\alpha)$$

where

$$\bar{f}(\alpha) = \int_0^{\infty} I(\xi)e^{-\alpha\xi}\, d\xi$$

$$(\alpha) = \int_{-\infty}^{+\infty} K_0[\gamma\ |x|]e^{-\alpha x}dx = \frac{\pi}{\sqrt{\alpha^2+\gamma^2}}$$

$$\bar{g}(\alpha) = -\int_{-\infty}^{+\infty} K_0[\gamma \sqrt{(x-x')^2+y'^2}]e^{-i\alpha x}dx = \frac{-\pi e^{-i\alpha x'-|y'|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha^2+\gamma^2}}$$

$$\bar{h}(\alpha) = 2\pi \int_{-\infty}^{0} u(x,0)e^{-i\alpha x}dx.$$

We see that certain assumptions about the forms of $I(\xi)$ and $u(x,0)$ are necessary before we can proceed. First, we require that $I(\xi)$ be absolutely integrable over any finite length. In addition, we want the behavior for large distances from the source to be like that of the full reflecting plane in Chapter 1,

$$|I(\xi)| \le C\, e^{-\gamma\xi} \qquad \text{for } \xi \to \infty.$$

Then,

$$|\bar{f}(\alpha)| \le \int_0^A |I(\xi)|\, d\xi + C \int_A^\infty e^{-\mathrm{Re}(\gamma + i\alpha)\xi}\, d\xi$$

and we see that $\bar{f}(\alpha)$ represents an analytic function which is regular in a lower half plane, $\mathrm{Im}\,\alpha < \mathrm{Re}\,\gamma$. We also see that $\bar{f}(\alpha)$ is bounded in a proper half plane, $\mathrm{Im}\,\alpha - \epsilon < \mathrm{Re}\,\gamma$, $\mathrm{Re}\,\gamma > \epsilon > 0$. In a similar way we find that $\bar{h}(\alpha)$ is regular in an upper half plane $\mathrm{Im}\,\alpha > -\mathrm{Re}\,\gamma$ and bounded for $\mathrm{Im}\,\alpha + \epsilon \ge -\mathrm{Re}\,\gamma$ if we require the field to be absolutely integrable and decrease exponentially for $x \to -\infty$. The functions $\bar{\mathcal{L}}(\alpha)$ and $\bar{g}(\alpha)$ in the integral forms represent regular functions in the strip $-\mathrm{Re}\,\gamma < \mathrm{Im}\,\alpha < \mathrm{Re}\,\gamma$ since the Bessel function $K_0[\gamma|x|]$ behaves like $\sqrt{\pi}\, e^{-\gamma|x|}/\sqrt{2\gamma|x|}$ for $|x| \to \infty$; the closed forms however give the analytic continuations into the whole α plane when the plane is cut from $i\gamma$ to ∞ and $-i\gamma$ to ∞ along lines parallel to the imaginary axis

(Figure 3). The branches of the square roots are taken such that

$$-\frac{3\pi}{2} < \arg(\alpha - i\gamma) \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq \arg(\alpha + i\gamma) < \frac{3\pi}{2} .$$

When we put this information together, we find that the transformed equation [3.4] only applies in the strip $-\mathrm{Re}\,\gamma < \mathrm{Im}\,\alpha < \mathrm{Re}\,\gamma$ which is the overlapping region of regularity of $\overline{f}(\alpha)$ and $\overline{h}(\alpha)$.

The aim now is to separate equation [3.4] into two functions, $F_-(\alpha)$ and $F_+(\alpha)$, such that $F_-(\alpha)$ is regular in a lower half α plane and $F_+(\alpha)$ is regular in an upper half α plane, and $F_-(\alpha) = F_+(\alpha)$ in a strip about the real axis. If this can be accomplished, $F_-(\alpha)$ and $F_+(\alpha)$ will be analytic continuations of one another and define a function $P(\alpha)$ in the entire α plane such that

$$P(\alpha) = F_-(\alpha) \text{ in a lower half plane}$$

and

$$P(\alpha) = F_+(\alpha) \text{ in an upper half plane.}$$

From the behavior of $F_-(\alpha)$ and $F_+(\alpha)$ for $\alpha \to \infty$, we will be able to deduce that $P(\alpha) \equiv 0$ from which it follows that

$$F_-(\alpha) = 0 \quad \text{and } F_+(\alpha) = 0$$

in their respective half planes. Now, $F_-(\alpha)$ will contain $\bar{f}(\alpha)$ and $F_+(\alpha)$ will contain $\bar{h}(\alpha)$. These two results enable us to solve for $\bar{f}(\alpha)$ and $\bar{h}(\alpha)$ individually, and the problem, with the aid of equation [2.0a], will be solved.

If we multiply equation [3.4] by $\sqrt{\alpha+i\gamma}$, we obtain

$$[3.5] \quad \frac{\pi\,\bar{f}(\alpha)}{\sqrt{\alpha-i\gamma}} = \frac{-\pi\, e^{-i\alpha x'-|y'|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha-i\gamma}} + \sqrt{\alpha+i\gamma}\;\bar{h}(\alpha).$$

The first and last terms are regular in the two half planes $\operatorname{Im}\alpha < \operatorname{Re}\gamma$ and $\operatorname{Im}\alpha > -\operatorname{Re}\gamma$ respectively. The middle term however can be decomposed into terms $A_+(\alpha)$ and $A_-(\alpha)$ so that

$$\varphi(\alpha) \equiv \frac{\pi\, e^{-i\alpha x'-|y'|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha-i\gamma}} = A_+(\alpha) + A_-(\alpha),$$

where $A_+(\alpha)$ and $A_-(\alpha)$ are regular in the upper and lower, over-lapping half planes. Equation [3.5] separates to give

$$[3.6] \quad \frac{\pi\,\bar{f}_-(\alpha)}{\sqrt{\alpha-i\gamma}} + A_-(\alpha) = -A_+(\alpha) + \sqrt{\alpha+i\gamma}\;\bar{h}_+(\alpha).$$

Here $F_-(\alpha)$ is expressed by the left side and $F_+(\alpha)$ by the right side, $\bar{f}(\alpha)$ and $\bar{h}(\alpha)$ being tagged with - and + respectively to denote their half planes of regularity.

In order to proceed with the analytic behavior of $F(\alpha)$, defined implicitly by equation [3.6], we need to compute $A_+(\alpha)$ and $A_-(\alpha)$ and show that they are bounded in proper half planes Im $\alpha > -$ Re γ and Im $\alpha <$ Re γ respectively. If we apply Cauchy's theorem to $\varphi(\alpha)$ on a rectangular contour of Figure 3, and take the limit as $b \to \infty$, we obtain

$$\varphi(\alpha) = -\frac{1}{2\pi i}\int_{C_-}\frac{\varphi(z)}{z-\alpha}\,dz + \frac{1}{2\pi i}\int_{C_+}\frac{\varphi(z)}{z-\alpha}\,dz$$

where C_- and C_+ are straight line contours parallel to the real axis. We shall identify $A_+(\alpha)$ and $A_-(\alpha)$ with the integrals

$$[3.7]\quad A_+(\alpha) = \frac{1}{2\pi i}\int_{C_+}\frac{\varphi(z)}{z-\alpha}\,dz \text{ and } A_-(\alpha) = \frac{-1}{2\pi i}\int_{C_-}\frac{\varphi(z)}{z-\alpha}\,dz$$

and show that each is bounded in a proper half plane Im $\alpha \geq -a + \epsilon$ and Im $\alpha \geq a - \epsilon$, $a > \epsilon > 0$ respectively. We note again that these half planes overlap. The real formulations for $A_+(\alpha)$ and $A_-(\alpha)$ are given when $z = x \mp i\,a$ respectively. Then,

$$A_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(x-ia)}{x-ia-\alpha}dx \text{ and } A_-(\alpha) = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(x+ia)}{x+ia-\alpha}dx.$$

For $A_+(\alpha)$, the Cauchy-Schwarz inequality gives

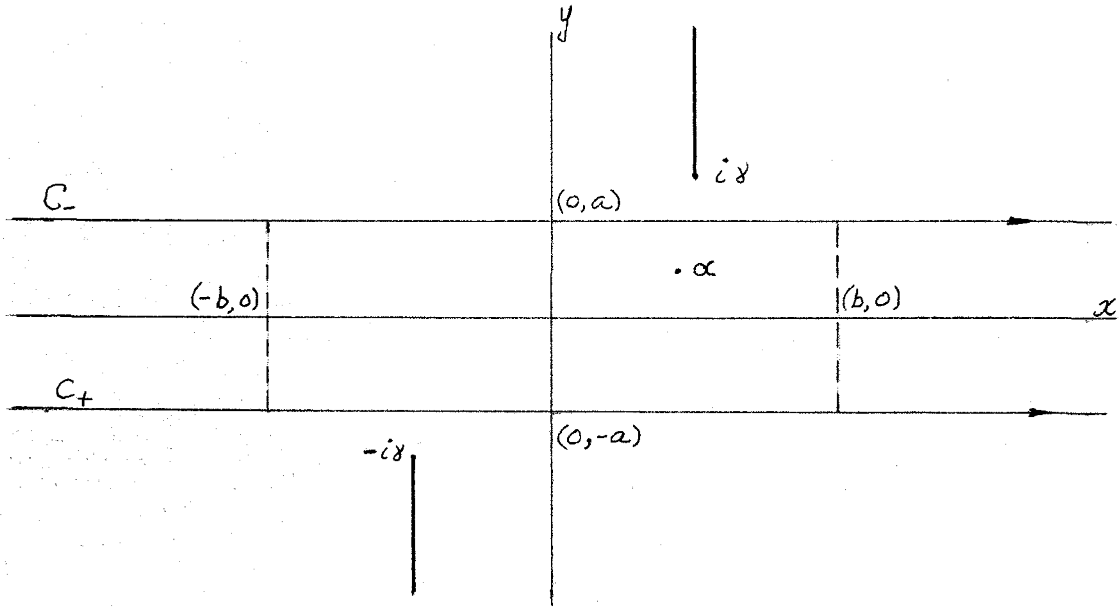

Figure 3

$$|A_+(\alpha)|^2 \leq \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} |\varphi(x-ia)|^2 dx \quad \int_{-\infty}^{+\infty} \frac{dx}{|x-ia-\alpha|^2}$$

$$\leq B^2 \int_{-\infty}^{+\infty} \frac{dx}{|x-ia-\alpha|^2} \, .$$

If $\alpha = u + iv$, we obtain

$$|A_+(\alpha)|^2 \leq \frac{\pi B^2}{|a+v|} \text{ where } B^2 = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} |\varphi(x-ia)|^2 dx.$$

Thus, $A_+(\alpha)$ is bounded for $v = \text{Im}\ \alpha \geq -a + \epsilon$, $\epsilon > 0$ and

$$|A_+(\alpha)| = 0\left(\frac{1}{|a+\text{Im}\ \alpha|}\right)^{\frac{1}{2}} \text{ for } |\alpha| \to \infty .$$

In the same way, we see that $A_-(\alpha)$ is bounded for

$\text{Im}\ \alpha \leq a - \epsilon$ and

$$|A_-(\alpha)| = O\left(\frac{1}{|a-\operatorname{Im}\alpha|}\right)^{\frac{1}{2}} \text{ for } |\alpha|\to\infty.$$

Thus, we have all of the quantities $\bar{f}_-(\alpha)$, $\bar{h}_+(\alpha)$, $A_+(\alpha)$ and $A_-(\alpha)$ bounded in their respective half planes. It follows from equation [3.6] that

$$P(\alpha) = O(|\alpha|^{\frac{1}{2}}) \quad \text{for } |\alpha|\to\infty,\ \operatorname{Im}\alpha \geq -a_+ \quad ,$$

[3.8]

$$P(\alpha) = O(1) \quad \text{for } |\alpha|\to\infty,\ \operatorname{Im}\alpha \leq a_- \quad .$$

An extension of Liouville's theorem states that if $P(\alpha)$ is analytic for all finite values of α and if, as $|\alpha|\to\infty$, $P(\alpha) = O(|\alpha|^k)$, then $P(\alpha)$ is a polynomial of degree $\leq k$. From [3.8] we see that $P(\alpha)$ is of degree $\leq \frac{1}{2}$. It follows that $P(\alpha)$ is a constant. Furthermore, as $|\alpha|\to\infty$ along the negative imaginary axis we see from equation [3.6] that $P(\alpha)\to 0$. This means that $P(\alpha)\equiv 0$. Thus, we have in each of the half planes,

$$F_-(\alpha) = \frac{\pi f_-(\alpha)}{\sqrt{\alpha-i\gamma}} + A_-(\alpha) = 0 \ ,\ F_+(\alpha) = -A_+(\alpha) + \sqrt{\alpha+i\gamma}\ \bar{h}_+(\alpha) = 0,$$

and it follows that

$$[3.9]\quad \bar{f}_-(\alpha) = -\frac{1}{\pi}\sqrt{\alpha-i\gamma}\ A_-(\alpha)\ ,\ \bar{h}_+(\alpha) = \frac{A_+(\alpha)}{\sqrt{\alpha+i\gamma}}\ .$$

The Secondary Field

Since we shall be interested in the current distribution in addition to the secondary field, we need only to work with $\overline{f}_-(\alpha)$ and use relation [2.0a] for the secondary field,

$$[3.10]\quad u_s = \frac{-1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{e^{i\alpha x - |y|\sqrt{\alpha^2 + \gamma^2}}}{\sqrt{\alpha + i\gamma}} \left(\frac{1}{2\pi i} \int_{C_-} \frac{\varphi(z)}{z-\alpha}\, dz \right) d\alpha$$

and invert $\overline{f}_-(\alpha)$ for $I(x)$,

$$[3.11]\quad I(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\alpha x} \sqrt{\alpha - i\gamma} \left(\frac{1}{2\pi i} \int_{C_-} \frac{\varphi(z)}{z-\alpha}\, dz \right) d\alpha.$$

From this point onward, we will have occasions to invert the order of integrations. We recall that if $f(x,y)$ is a measurable function and if each of the Cauchy-Riemann integrals

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\,dx\,dy \quad , \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$$

exists, and one is absolutely convergent, then the two integrals are equal. This theorem can be used to justify changes in the order of integrations in all cases.

In this section we consider only the secondary field and defer the consideration of $I(x)$ for Chapter 4. Our aim here is to reduce the contour integral along C_- to a real integral by applying Cauchy's theorem to one of the contours of Figures 4 or 5.

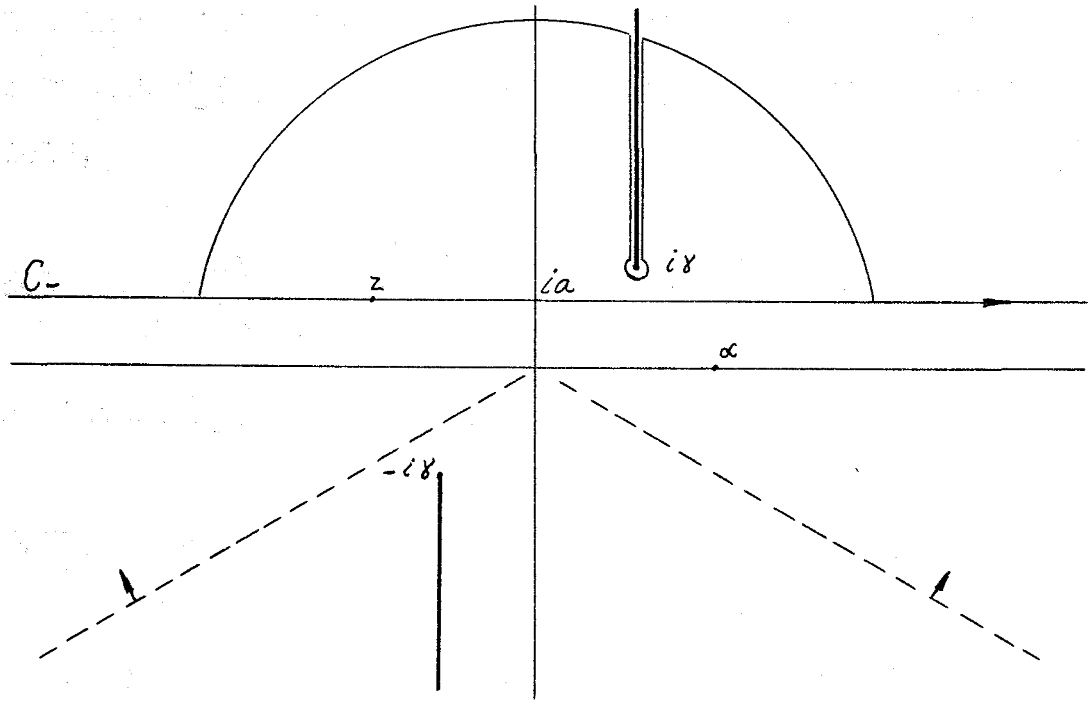

Figure 4

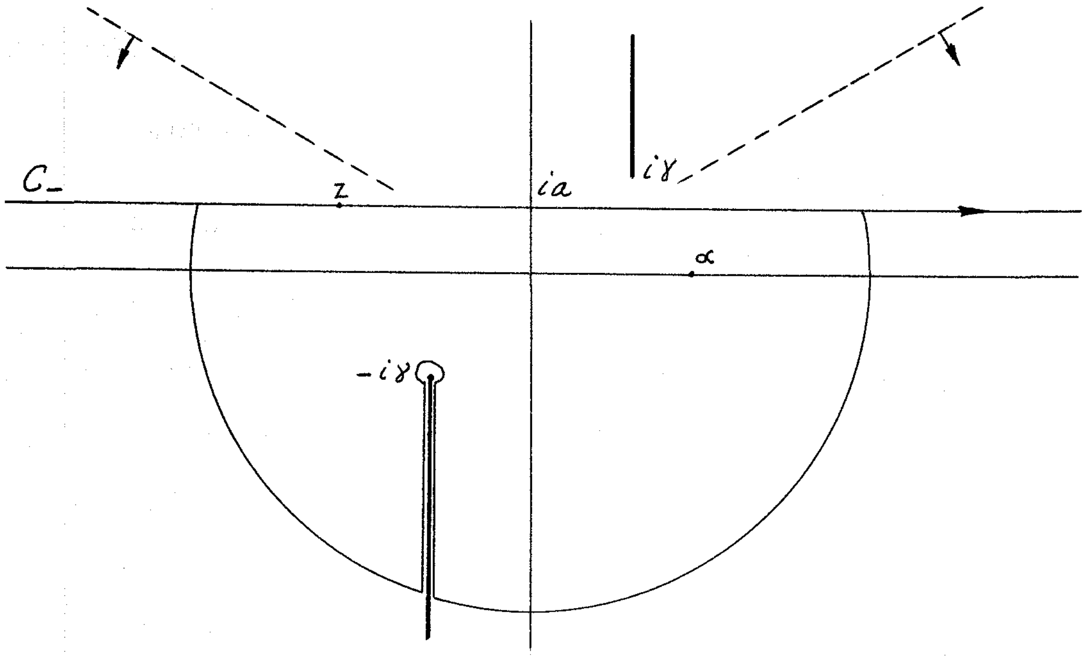

Figure 5

The asymptotic estimates

$$\varphi(z) \sim \begin{cases} \dfrac{e^{-ix'z-|y'|z}}{z} & -\frac{3\pi}{2} < \arg(z-i\gamma) \leq \frac{\pi}{2} \\ \dfrac{e^{-ix'z+|y'|z}}{z} & \frac{\pi}{2} \leq \arg(z+i\gamma) < \frac{3\pi}{2} \end{cases}$$

for large (Z) show that the lines $(z = u + iv)$

$$\mathrm{Re}(ix'z+|y'|z) = 0 \quad ; \quad \mathrm{Re}(-ix'z+|y'|z) = 0$$

or

$$v = \frac{|y'|}{x'}u \quad , \quad u > 0 \quad ; \quad v = -\frac{|y'|u}{x'} \quad , \quad u<0$$

are boundaries between regions of exponential growth and exponential decay for $\varphi(z)$. These regions are separated by dashed lines in Figures 4 and 5 with arrows pointing toward the regions of exponential decay for $x' < 0$ and $x' > 0$ respectively. Thus, in order to obtain convergent integrals in the complex plane, there are two cases to consider.

Case I, $x' < 0$. If we apply Cauchy's theorem to the contour of Figure 4 and let $R \to \infty$, it follows that

$$\frac{1}{2\pi i}\int_{C_-} \frac{\varphi(z)}{z-\alpha}\,dz = -\,e^{\gamma x'} e^{\frac{\pi i}{4}} \int_0^\infty \frac{e^{x'r}\cos|y'|\sqrt{r^2+2\gamma r}}{\sqrt{r}\,(r+\gamma+i\alpha)}\,dr.$$

The substitution $r = -\gamma + \sqrt{p^2+\gamma^2}$ converts the right side to

$$-e^{\frac{\pi i}{4}} \int_0^\infty \frac{e^{x'\sqrt{p^2+\gamma^2}}\, p \cos|y'|p \, dp}{\sqrt{-\gamma+\sqrt{p^2+\gamma^2}}\,\sqrt{p^2+\gamma^2}\,[i\alpha+\sqrt{p^2+\gamma^2}]} .$$

This integral is a Fourier cosine transform and can be put into a more convenient form with the aid of the integral (7, vol. 1, p. 17).

$$\int_0^\infty \frac{e^{-v\sqrt{p^2+\gamma^2}}\, p \cos|y'|p \, dp}{\sqrt{-\gamma+\sqrt{p^2+\gamma^2}}\,\sqrt{p^2+\gamma^2}} = \sqrt{\frac{\pi}{2}}\, \frac{|y'|e^{-\gamma\sqrt{v^2+y'^2}}}{\sqrt{v^2+y'^2}\,\sqrt{-v+\sqrt{v^2+y'^2}}} .$$

Multiplying both sides by $e^{-i\alpha v}$ and integrating v over the interval $(-x',\infty)$, we have

$$u_s = \frac{|y'|e^{\frac{\pi i}{4}}}{4\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{i\alpha(x-x')}}{\sqrt{\alpha+i\gamma}} \int_{-x'}^{\infty} \frac{e^{-i\alpha v-\gamma\sqrt{v^2+y'^2}}}{\sqrt{v^2+y'^2}\,\sqrt{-v+\sqrt{v^2+y'^2}}}\, dv\, d\alpha .$$

Replacing v by $v-x'$ and inverting the order of integrations gives

$$u_s = \frac{|y'|e^{\frac{\pi i}{4}}}{4\pi \quad 2\pi} \int_0^\infty \frac{e^{-\gamma\sqrt{(v-x')^2+y'^2}}}{\sqrt{(v-x')^2+y'^2}\ \sqrt{-(v-x')+\sqrt{(v-x')^2+y'^2}}} \cdot$$

$$\int_{-\infty}^{+\infty} \frac{e^{i\alpha(x-v)-|y|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha+i\gamma}}\, d\alpha\, dv.$$

The integral

$$[3.12] \quad \int_{-\infty}^{+\infty} \frac{e^{i\alpha\beta-|y|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha \pm i\gamma}}\, d\alpha = \sqrt{2\pi}\, e^{\mp\frac{\pi i}{4}} \frac{|y|e^{-\gamma\sqrt{\beta^2+y^2}}}{\sqrt{\beta^2+y^2}\sqrt{\pm\beta+\sqrt{\beta^2+y^2}}}$$

is obtained from the contours of Figures 4 and 5 and the results of references (7, vol. 1, p. 17) and (7, vol. 1, p. 75). The final expression for u_s, $x' < 0$ is

[3.13]

$$\frac{|y||y'|}{4\pi} \int_0^\infty e^{-\gamma[\sqrt{(v-x')^2+y'^2}+\sqrt{(v-x)^2+y^2}]} \phi(v-x',y')\phi(v-x,y)\,dv$$

where

$$\phi(v,y) = \frac{1}{\sqrt{v^2+y^2}\ \sqrt{-v+\sqrt{v^2+y^2}}} \,.$$

<u>Case II, $x' \geq 0$</u>. The analytical details of this case are very similar to those above, except that we use the contour of Figure 5 to evaluate

$$\frac{1}{2\pi i}\int_{C_-}\frac{\varphi(z)}{z-\alpha}\,dz = -\varphi(\alpha)+e^{-\gamma x'}e^{\frac{\pi i}{4}}\int_0^{\infty}\frac{e^{-x'r}\sin|y'|\sqrt{r^2+2\gamma r}}{\sqrt{r+2\gamma}\;(r+\gamma-i\alpha)}\,dr.$$

Again, we use the substitution $r = -\gamma+\sqrt{p^2+\gamma^2}$ to reduce the second term to

$$e^{\frac{\pi i}{4}}\int_0^{\infty}\frac{e^{-x'\sqrt{p^2+\gamma^2}}\,p\,\sin|y'|p}{\sqrt{\gamma+\sqrt{p^2+\gamma^2}}\,\sqrt{p^2+\gamma^2}\,(-i\alpha+\sqrt{p^2+\gamma^2})}\,dp.$$

This integral is a Fourier sine transform and can be obtained by manipulating the result (7, vol. 1, p. 75)

$$\int_0^{\infty}\frac{e^{-v\sqrt{p^2+\gamma^2}}\,p\,\sin|y'|p}{\sqrt{\gamma+\sqrt{p^2+\gamma^2}}\sqrt{p^2+\gamma^2}}\,dp = \sqrt{\frac{\pi}{2}}\,\frac{|y'|e^{-\gamma\sqrt{v^2+y'^2}}}{\sqrt{v^2+y'^2}\sqrt{v+\sqrt{v^2+y'^2}}}\,.$$

Here we multiply both sides by $e^{i\alpha v}$, integrate v over (x',∞), replace v by $v+x'$ on the right and obtain

$$u_s = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{i\alpha(x-x')-(|y|+|y'|)\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha^2+\gamma^2}}\,d\alpha$$

$$-\frac{|y'|e^{\frac{\pi i}{4}}}{4\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{i\alpha x-|y|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha+i\gamma}}\int_0^{\infty}\frac{e^{i\alpha v-\gamma\sqrt{(v+x')^2+y'^2}}\,dv\,d\alpha}{\sqrt{(v+x')^2+y'^2}\sqrt{v+x'+\sqrt{(v+x')^2+y'^2}}}$$

If we evaluate the first integral (7, vol. 1, p. 17) and apply equation [3.12] to the inverted integrals in the second term, we have

[3.14]

$$-\frac{|y||y'|}{4\pi}\int_0^\infty e^{-\gamma[\sqrt{(v+x')^2+y'^2}+\sqrt{(v+x)^2+y^2}]}\psi(v+x',y')\psi(v+x,y)\,dv$$

where

$$\psi(v,y) = \frac{1}{\sqrt{v^2+y^2}\sqrt{v+\sqrt{v^2+y^2}}}$$

<u>The Total Field</u>

The Total Field expressions, using equations [1.3], [3.13], and [3.14], assume the forms

$\underline{x' < 0}$

[3.15] $$u = -\frac{1}{2\pi}K_o[\gamma\sqrt{(x-x')^2+(y-y')^2}$$

$$\frac{|y||y'|}{4\pi}\int_0^\infty e^{-\gamma[\sqrt{(v-x')^2+y'^2}+\sqrt{(v-x)^2+y^2}]}\phi(v-x',y')\phi(v-x,y)\,dv$$

$\underline{x' \geq 0}$

[3.16] $$u = \frac{1}{2\pi}\Big\{K_o[\gamma\sqrt{(x-x')^2+(|y|+|y'|)^2}$$

$$-K_o[\gamma\sqrt{(x-x')^2+(y-y')^2}]\Big\}$$

$$-\frac{|y||y'|}{4\pi}\int_0^\infty e^{-\gamma[\sqrt{(v+x')^2+y'^2}+\sqrt{(v+x)^2+y^2}]}\psi(v+x',y')\psi(v+x,y)\,dv$$

with

$$\phi(v,y) = \frac{1}{\sqrt{v^2+y^2}\sqrt{-v+\sqrt{v^2+y^2}}}$$

and

$$\psi(v,y) = \frac{1}{\sqrt{v^2+y^2}\sqrt{v+\sqrt{v^2+y^2}}} .$$

It is appropriate now to observe that these formulae apply for Re $\gamma \geq 0$ and that this is just the requirement in order to apply equation [1.1]. One can show that the boundary condition $u(x,0) = 0$, $x > 0$ is indeed satisfied by examining the terms

$$[3.17] \quad \frac{|y|}{\sqrt{-(v-x)+\sqrt{(v-x)^2+y^2}}} , \quad \frac{|y|}{\sqrt{(v+x)+\sqrt{(v+x)^2+y^2}}} \quad \text{as } y \to 0.$$

For $x' \geq 0$ and $x \geq 0$ the first expression results in

$$\begin{array}{ll} 0 & 0 \leq v \leq x \\ \sqrt{2(v-x)} & x \leq v < \infty \end{array}$$

and we verify the identity

$$K_o[\gamma\sqrt{(x-x')^2+y'^2}]$$

$$= \frac{|y'|}{\sqrt{2}} \int_x^\infty \frac{e^{-\gamma[\sqrt{(v-x')^2+y'^2}+v-x]}}{\sqrt{v-x}\sqrt{(v-x')^2+y^2}\sqrt{-(v-x')+\sqrt{(v-x')^2+y'^2}}}\, dv$$

arising from equation [3.15] by the substitution

$$v' = \frac{v-x + \sqrt{(v-x')^2+y'^2}}{\sqrt{(x-x')^2 + y'^2}}$$

and the results of equation [29] of reference (7, vol. 1, p. 140). For $x' < 0$ and $x \geq 0$ the second limit in [3.17] is zero and the total field expressed by equation [3.16] is zero for $y = 0$. We note also that the expression for the full plane case of Chapter 1 appears as a leading term in equation [3.16], so that the first term is identically zero for $y \leq 0$ and $y' \geq 0$ or $y \geq 0$ and $y' \leq 0$.

4. ASYMPTOTICS FOR I(x)

The purpose of this chapter is to compute asymptotic forms for I(x) for both large and small x and to exhibit the current-field relationships for the half plane. We start with equation [3.11],

$$[4.0]\quad I(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha x}\sqrt{\alpha - i\gamma}\left(\frac{1}{2\pi i}\int_{C_-}\frac{\varphi(z)}{z-\alpha}\,dz\right)d\alpha,$$

and exchange the order of integrations,

$$[4.1]\quad I(x) = \frac{-1}{4\pi^2 i}\int_{C_-}\frac{e^{-ix'z-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}\int_{-\infty}^{+\infty}\frac{e^{i\alpha x}\sqrt{\alpha-i\gamma}}{\alpha-z}\,d\alpha\,dz.$$

The integrand in α is not absolutely integrable, but we circumvent this difficulty in the inversion process by deforming the path $(-\infty,\ \infty)$ to the contours of Figures 6 and 7. The second integral becomes

$$[4.2]\quad \int_{-\infty}^{+\infty}\frac{e^{i\alpha x}\sqrt{\alpha-i\gamma}}{\alpha-z}\,d\alpha = 2\pi i\, e^{izx}\sqrt{z-i\gamma}$$

$$- 2ie^{-\gamma x}\, e^{\frac{\pi i}{4}}\int_0^\infty \frac{e^{-xr}\sqrt{r}}{r+\gamma+iz}\,dr \qquad x>0$$

$$= 0 \qquad x<0,$$

and I(x) for $x > 0$ may be written in the form

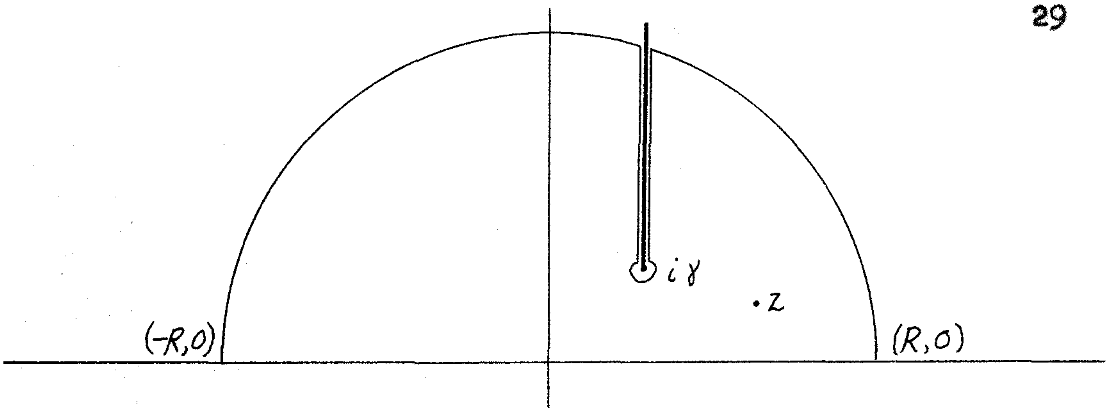

Figure 6

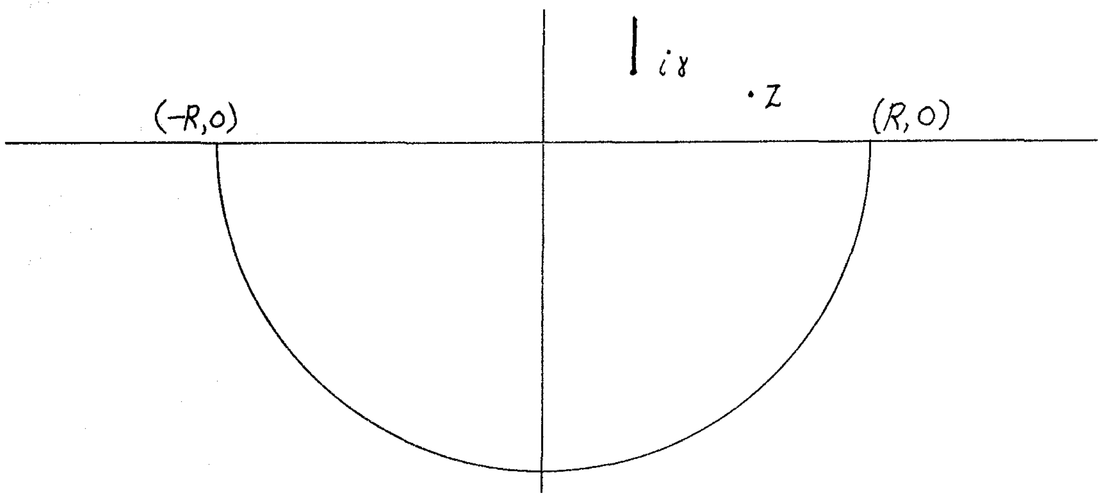

Figure 7

$$[4.3]\quad I(x) = -\frac{1}{2\pi}\int_{C_-} e^{iz(x-x')-|y'|\sqrt{x^2+\gamma^2}}\,dz$$

$$+\frac{e^{-\gamma x}e^{\frac{\pi i}{4}}}{2\pi^{\frac{3}{2}}}\quad\frac{1}{\sqrt{x}}\int_{C_-}\frac{e^{-ix'z-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}\,dz$$

$$-\frac{1}{2\pi}\int_{C_-} e^{+iz(x-x')-|y'|\sqrt{z^2+\gamma^2}}\ \mathrm{erfc}\,\sqrt{x(\gamma+iz)}\,dz \quad,\quad x > 0$$

when the Laplace transformation on the right in [4.2] is evaluated (7, vol. 1, p. 136). The contours along C_- can be deformed into the real axis by Cauchy's theorem, and we see immediately with the help of equation [3.12] that

$$I(x) \sim \frac{|y'|e^{-\gamma x}}{\pi\sqrt{2x}} \cdot \frac{e^{-\gamma\sqrt{x'^2+y'^2}}}{\sqrt{x'^2+y'^2}\sqrt{x'+\sqrt{x'^2+y'^2}}}$$

since all other integrals are bounded in x as $x \to 0_+$.

Furthermore, the first integral in equation [4.3] is equal to (7, vol. 1, p. 56)

$$-\frac{\gamma|y'|}{\pi} \cdot \frac{K_1[\gamma\sqrt{(x-x')^2+y'^2}]}{\sqrt{(x-x')^2+y'^2}}$$

and is asymptotic to

$$-|y'|\sqrt{\frac{\gamma}{2\pi}}\,\frac{e^{-\gamma x}}{x^{3/2}} \qquad \text{for } x \to \infty .$$

The asymptotics for the last integral are obtained from the asymptotics for erfc z,

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{z^{2n+1}} \quad , \quad -\frac{3\pi}{4} < \arg z < \frac{3\pi}{4} .$$

Thus,

[4.4] $$\operatorname{erfc}\sqrt{x(\gamma+iz)}$$

$$\frac{e^{-\gamma x}e^{-izx}}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(n+\frac{1}{2})}{i^{n+\frac{1}{2}}\;x^{n+\frac{1}{2}}(z-i\gamma)^{n+\frac{1}{2}}}\;,\quad -2\pi < \arg(z-i\gamma) < \pi$$

and if we use equation [3.12] for n = 0, we have

$$I(x) \sim -\frac{\gamma\,|y'|K_1[\gamma\sqrt{(x-x')^2+y'^2}]}{\sqrt{(x-x')^2+y'^2}}$$

$$+\frac{e^{-\gamma x}}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,\Gamma(n+\frac{1}{2})}{i^{n+\frac{3}{2}}\;x^{n+\frac{1}{2}}}\int_{-\infty}^{+\infty}\frac{e^{-izx'-|y'|\sqrt{z^2+\gamma^2}}}{(z-i\gamma)^{n+\frac{1}{2}}}\,dz$$

upon inverting the sum and integral. This result is also an asymptotic expansion for the integral since the order relation in [4.4] to N terms holds uniformly in z (6, p. 16),

$$O\left(\frac{e^{-\gamma x}e^{-izx}}{\pi}\;\frac{(-1)^{N+1}\,\Gamma(N+\frac{3}{2})}{i^{N+\frac{3}{2}}\;x^{N+\frac{3}{2}}\;(z-i\gamma)^{N+\frac{3}{2}}}\right) = O\left(\frac{e^{-\gamma x}\,\Gamma(N+\frac{3}{2})}{x^{N+\frac{3}{2}}\;\gamma^{N+\frac{3}{2}}}\right).$$

It follows that

$$I(x) \sim \frac{C\,|y'|\,e^{-\gamma x}}{x^{3/2}}\qquad \text{for } x \to \infty.$$

if higher order terms are neglected.

Current-Field Relationship

If we apply Green's formula [2.7] to the region of Figure 2, we obtain a current-field relationship just as we did in Chapter 1. Thus, with the relationships

$$\phi(x,y) = u_s \quad , \quad \nabla^2 u_s - \gamma^2 u_s = 0$$

$$\psi(x,y) = u_1 \quad , \quad u_1 = -\frac{1}{2\pi} K_0[\gamma \sqrt{(x-x')^2+(y-y')^2}]$$

$$u(x,0) = 0, \; x>0; \quad \nabla^2 u_1 - \gamma^2 u_1 = + \delta(x-x')\,\delta(y-y').$$

Green's formula gives

$$u(x,y) = u_1 - \frac{1}{2\pi}\int_0^\infty K_0[\gamma \sqrt{(x-\xi)^2+y^2}] \left[\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0_+} + \left(\frac{\partial u}{\partial \eta}\right)_{\eta=0_-}\right] d\xi$$

and we identify I(x) with

$$I(x) = \left(\frac{\partial u}{\partial y}\right)_{y=0_+} + \left(\frac{\partial u}{\partial y}\right)_{y=0_-} = \left(\frac{\partial u_s}{\partial y}\right)_{y=0_+} + \left(\frac{\partial u_s}{\partial y}\right)_{y=0_-}.$$

The closed forms for I(x) can be obtained from the field expressions.

BIBLIOGRAPHY

1. Carlson, J. F. and A. E. Heins. The reflection of an electromagnetic plane wave by an infinite set of plates. I. Quarterly of Applied Mathematics 4: 313-329. 1946.

2. Carslaw, H. S. Diffraction of waves by a wedge of any angle. Proceedings of the London Mathematical Society 18: 291-306. 1920.

3. Carslaw, H. S. Some multiform solutions of the partial differential equations of physics and mathematics and their applications. Proceedings of the London Mathematical Society 30: 121-163. 1899.

4. Clemmow, P. C. A note on the diffraction of a cylindrical wave by a perfectly reflecting conducting half plane. Quarterly Journal of Mechanics and Applied Mathematics 3: 377-384. 1950.

5. Copson, E. T. On an integral equation arising in the theory of diffraction. Quarterly Journal of Mathematics, Oxford Series, 17: 19-34. 1946.

6. Erdelyi, A. Asymptotic expansions. New York, Dover, 1956. 108p.

7. Erdelyi, Arthur et al. Tables of integral transforms. New York, McGraw-Hill, 1954. 2 vols.

8. Friedlander, F. G. The diffraction of sound pulses. Proceedings of the Royal Society of London, ser. A, 186: 322-351. 1946.

9. Friedlander, F. G. On the half plane diffraction problem. Quarterly Journal of Mechanics and Applied Mathematics 4: 344-357. 1951.

10. Gast, R. C. Half plane diffraction with line source excitation. Pittsburg, Carnegie Institute of Technology Department of Mathematics. 1956. 49p. (Technical Report No. 24.)

11. Harrington, R. F. Current element near the edge of a conducting half plane. Journal of Applied Physics 24: 547-550. 1953.

12. Heins, A. E. and J. F. Carlson. The reflection of an electromagnetic plane wave by an infinite set of plates. II. Quarterly of Applied Mathematics 5: 82-89. 1947.

13. Heins, A. E. The Green's function for periodic structures in diffraction theory with an application to parallel plate media. I. Journal of Mathematics and Mechanics 6: 401-426. 1957.

14. Heins, A. E. The Green's function for periodic structures in diffraction theory with an application to parallel plate media. II. Journal of Mathematics and Mechanics 6: 629-639. 1957.

15. Heins, A. E. The radiation and transmission properties of a pair of semi-infinite parallel plates. I. Quarterly of Applied Mathematics 6: 157-166. 1948.

16. Heins, A. E. The radiation and transmission properties of a pair of semi-infinite parallel plates. II. Quarterly of Applied Mathematics 6: 215-220. 1948.

17. Karp, S. Wiener-Hopf techniques and mixed boundary value problems. Communications on Pure and Applied Mathematics 3: 411-426. 1950.

18. Kay, I. The diffraction of an arbitrary pulse by a wedge. Communications on Pure and Applied Mathematics 6: 419-434. 1953.

19. Keller, J. B. and A. Blank. Diffraction and reflection of pulses by wedges and corners. Communications on Pure and Applied Mathematics 4: 75-94. 1951.

20. MacDonald, H. M. A class of diffraction problems. Proceedings of the London Mathematical Society 14: 410-427. 1915.

21. Magnus, W. Uber die Beugung electromagnetischer Wellen an einer Halbebene. Zeitschrift fur Physik 117: 168-179. 1941.

22. Miles, J. W. On the diffraction of an acoustic pulse by a wedge. Proceedings of the Royal Society of London, ser. A, 212: 543-546. 1952.

23. Miles, J. W. On the diffraction of an electromagnetic pulse by a wedge. Proceedings of the Royal Society of London, ser. A, 212: 547-551. 1952.

24. Morse, P. M. and H. Feshbach. Methods of Theoretical Physics. Parts 1 and 2. New York, McGraw-Hill, 1953. 1978p.

25. Noble, B. Methods based on the Wiener-Hopf Technique for the solution of partial differential equations. New York, Pergamon Press, 1958. 246p.

26. Oberhettinger, F. Diffraction of waves by a wedge. Communications on Pure and Applied Mathematics 7: 551-563. 1954.

27. Oberhettinger, F. On a modification of Watson's lemma. Journal of Research of the National Bureau of Standards 63B: 15-17. 1959.

28. Oberhettinger, F. On asymptotic series for functions occurring in the diffraction of waves by wedges. Journal of Mathematics and Physics 34: 245-255. 1956.

29. Oberhettinger, F. On the diffraction and reflection of waves and pulses by wedges and corners. Journal of Research of the National Bureau of Standards 61: 343-365. 1958.

30. Oberhettinger, F. On the propagation of pulses. Zeitshrift fur Physik 146: 423-435. 1956.

31. Paley, R. E. A. C. and N. Wiener. Fourier transforms in the complex domain. New York, American Mathematical Society, 1934. 184p. (American Mathematical Society Colloquium Publications, Vol. 19.)

32. Senior, T. B. A. The diffraction of a dipole field by a perfectly conducting half plane. Quarterly Journal of Mechanics and Applied Mathematics 158: 101-114. 1953.

33. Sommerfeld, A. Mathematische Theorie der Diffraction. Mathematische Annalen 47: 317-341. 1896.

34. Stakgold, I. Boundary value problems for $\nabla^2 u + k^2 u = 0$. Stanford, Stanford University Applied Mathematics and Statistics Laboratory, 1954. 18p. (Technical Report No. 17.)

35. Titchmarsh, E. C. Introduction to the theory of Fourier integrals. Oxford, Oxford University Press, 1937. 390p.

36. Turner, R. D. The diffraction of a cylindrical pulse by a half plane. Quarterly of Applied Mathematics 14: 63-73. 1956.

37. Vandakurov, Y. V. Diffraction by a perfectly conducting half plane of electromagnetic waves emitted by an arbitrarily oriented electric and magnetic dipole. Zhurnal eksperimental noi teoreticheskoi fiziki 26: 3-18. 1954. (U. S. Air Force, Cambridge Research Center. Translation No. 1.)

38. Wait, J. Diffraction of a spherical wave pulse by a half screen. Canadian Journal of Physics 35: 693-696. 1957.

39. Wait, J. On the theory of an antenna with infinite corner reflector. Canadian Journal of Physics 32: 365-371. 1954.

40. Whipple, F. J. Diffraction by a wedge and kindred problems. Proceedings of the London Mathematical Society 16: 94-111. 1917.

41. Wiegrefe, A. Uber einige mehrwertige Losungen der Wellengleichungen in der Beugungstheorie. Annalen der Physik, ser. 4, 39: 449-484. 1912.

42. Wiener, N. and E. Hopf. Uber eine Klasse singular Integralgleichungen. Sitzung-berichte der Preussischen Akademie Wissenschaften, Mathematische-Physikalische Klasse 30: 696-706. 1931.

APPENDIX

A DIRECT APPROACH TO $\bar{f}(\alpha)$

Equation [3.5] relating $\bar{f}(\alpha)$ and $\bar{h}(\alpha)$ can be separated by the method of Harrington (11). In this reference, the Laplace transformation was used, but it is interesting to see the corresponding Fourier analysis and the relationship to the Wiener-Hopf separation. We re-write equation [3.5],

$$[I.0] \qquad \frac{\pi e^{-i\alpha x' - |y'|\sqrt{\alpha^2+\gamma^2}}}{\sqrt{\alpha - i\gamma}} = \frac{\pi\bar{f}(\alpha)}{\sqrt{\alpha - i\gamma}} + \sqrt{\alpha + i\gamma}\;\;\bar{h}(\alpha).$$

If we apply the inverse transform at this stage, we obtain

$$[I.1] \quad \frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{i\alpha(x-x')-|y'|\sqrt{\alpha^2+\gamma^2}}\,d\alpha}{\sqrt{\alpha-i\gamma}} = \begin{cases} \pi\displaystyle\int_{-\infty}^{+\infty}\frac{e^{i\alpha x}\bar{f}(\alpha)\,d\alpha}{\sqrt{\alpha-i\gamma}} & x>0 \\ \displaystyle\int_{-\infty}^{+\infty} e^{i\alpha x}\sqrt{\alpha+i\gamma}\;\bar{h}(\alpha)\,d\alpha & x<0 \end{cases}$$

Just how the right side of [I.1] comes about can be seen by taking a semi-circular contour of radius R about the lower half α plane for $x < 0$ and its reflection in the real axis for $x > 0$. In the limit as $R \to \infty$ we obtain

$$\int_{-\infty}^{+\infty}\frac{e^{i\alpha x}\bar{f}(\alpha)}{\sqrt{\alpha-i\gamma}} = 0\;,\; x<0 \text{ and } \int_{-\infty}^{+\infty} e^{i\alpha x}\sqrt{\alpha+i\gamma}\;\bar{h}(\alpha)\,d\alpha = 0\;,\; x>0.$$

Here we have used the assumptions that $\bar{f}(\alpha)$ and $\bar{h}(\alpha)$ are regular and bounded in their respective half planes. Thus, we have the integral involving $\overline{f}(\alpha)$ for all x,

$$[I.2] \quad \int_{-\infty}^{+\infty} \frac{e^{i\alpha x}\overline{f}(\alpha)}{\sqrt{\alpha-i\gamma}}d\alpha = \begin{cases} 0 & , x<0 \\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{iz(x-x')-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}dz, & x>0. \end{cases}$$

The Fourier transform of the function on the left gives

$$[I.3] \quad \frac{\overline{f}(\alpha)}{\sqrt{\alpha-i\gamma}} = \frac{1}{2\pi}\int_0^{\infty} e^{-i\alpha x}\int_{C_-} \frac{e^{iz(x-x')-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}\,dz\,dx$$

since by Cauchy's theorem on the rectangular contour of Figure 8 with $b \rightarrow \infty$,

$$\int_{-\infty}^{+\infty} \frac{e^{iz(x-x')-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}dz = -\int_{C_-} \frac{e^{iz(x-x')-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}\,dz.$$

We justify the exchange of integrals in [I.3] by absolute convergence if $\text{Im}\ \alpha < \text{Im}\ z$,

$$[I.4] \quad \overline{f}(\alpha) = -\frac{\sqrt{\alpha-i\gamma}}{2\pi}\int_{C_-} \frac{e^{-x'z-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}}\,dz\int_0^{\infty} e^{ix(z-\alpha)}dx$$

and

$$[I.5] \quad \overline{f}(\alpha) = \frac{\sqrt{\alpha-i\gamma}}{2\pi i}\int_{C_-} \frac{e^{-ix'z-|y'|\sqrt{z^2+\gamma^2}}}{\sqrt{z-i\gamma}\ (z-\alpha)}\,dz.$$

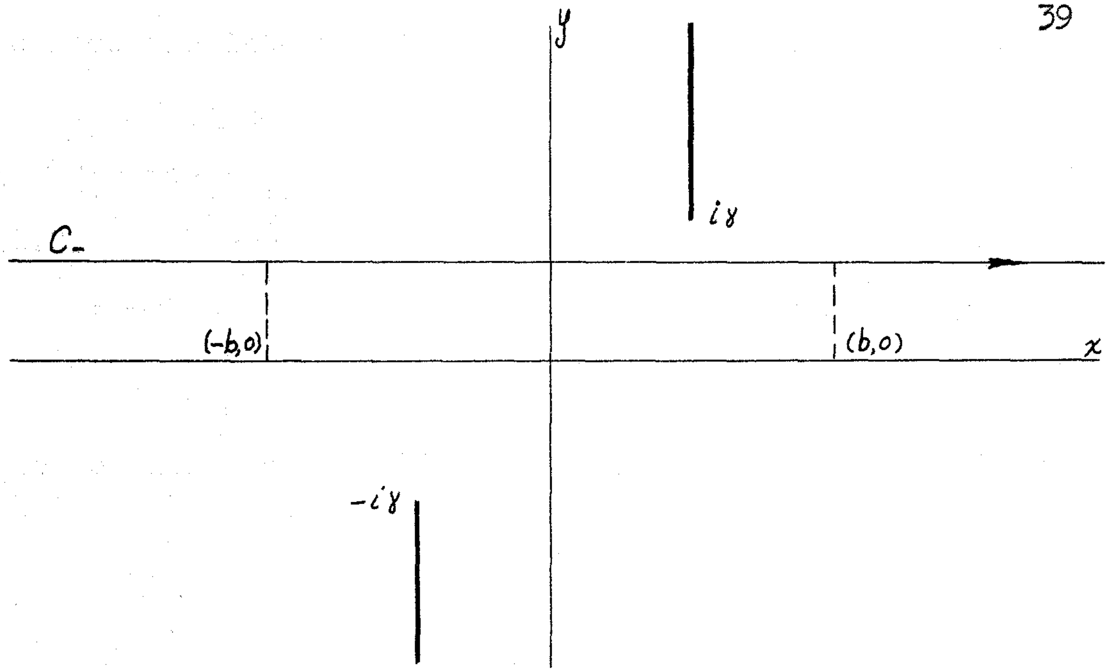

Figure 8

The condition Im α < Im z is no longer a restriction since the function $1/(z-\alpha)$ gives the analytic continuation of the second integral in α in [I.4] for all $\alpha \neq z$. Inversion of [I.5] gives equation [3.11].