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The Role of Mathematical Definitions in Mathematics and in Undergraduate Mathematics Courses

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Introduction

One of the earliest subjects of undergraduate mathematics education research was students' difficulties in writing formal mathematical proofs. Some research focused on the heuristics involved in proof-writing, but early attempts to show that the teaching of heuristics and strategies benefited students' proof-writing skills (Bittinger, 1968; Goldberg, 1973) failed to produce statistically significant results. Other difficulties have been identified, including students' weak understanding of logic and/or mathematical concepts and their definitions (cf. Hart, 1986; Moore, 1994). Several recent studies have looked further at students' proof-writing skills (cf. Dreyfus, 1999, Harel & Sowder, 1998; Selden & Selden, 2003); this topic is also addressed in this volume in chapters by Selden & Selden, Harel & Brown, and Zazkis. The purpose of this chapter is to look closely at one topic that arose from research on proof-writing, mathematical definitions, and most importantly, the role that these definitions play in the mathematical enterprise as well as in the teaching of undergraduate mathematics courses.

Mathematical definitions are of fundamental importance in the axiomatic structure that characterizes mathematics. The enculturation of college mathematics students into the field of mathematics includes their acceptance and understanding of the role of mathematical definitions, that the words of the formal definition embody the essence of and completely specify the concept being defined. But definitions also play a role in the students' experiences in mathematics courses themselves, in the sense that definitions are often used as a vehicle toward a more robust understanding of a given concept.

In this chapter we first discuss a framework for thinking about mathematical definitions derived from literature in the fields of mathematics, mathematics education, philosophy and lexicography. Next, we discuss research on student understanding and use of definition and on the role of definitions in the teaching of mathematics. Finally, we discuss the implications of this research and important pedagogical decisions that should govern the use of mathematical definitions in the teaching of mathematics.

Definitions

Definitions play a key role in mathematics, but their creation and use differs from those of "everyday language" definitions. This distinction is outlined by philosopher Richard Robinson (1962) and lexicographer Sidney Landau (2001), and from their work we derive the terms we use in our work. We distinguish between *extracted* definitions and *stipulated* definitions. According to Landau, extracted definitions are "definitions that are based on examples of actual

usage, definitions *extracted* from a body of evidence" (2001, p. 165). Robinson describes extracted definitions (which he refers to as *lexical* definitions) as "that sort of word-thing definition in which we are explaining the actual way in which some actual word has been used by some actual person" (1962, p. 135).

In contrast, *stipulated* definitions are an "explicit and self-conscious setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object)" (1962, p. 59). Their chief advantage is "the improvement of concepts or the creation of new concepts, which is the key to one of the two or three locks on the door of successful science" (1962, p. 68). Landau says such definitions "are imposed on the basis of expert advice" with the goal of "ease and accuracy of communication between those versed in the language of science" (2001, p. 165).

Thus, as we observed in Edwards and Ward (2004), *extracted* definitions report usage, while *stipulated* definitions create usage, indeed create concepts, by decree. Moreover, when a term is defined by stipulation, it is to be free from connotation, that is, free from all the associations the term may have acquired in its non-technical use. Finally, stipulated definitions have no truth value. Extracted definitions have a truth value. They either accurately report usage or they do not. Along with Robinson and Landau, we think of mathematical definitions as stipulated, whereas most, "everyday language" definitions are extracted.

It is important to note here that mathematical definitions frequently *do* have a history — that is, they can and do evolve. The definition we use for function, for instance, may not be the one that mathematicians favored two-hundred years ago; the concept of connectedness has two definitions, path-connected and set-theoretically connected. However, in formal mathematics we do not leave the meaning of a term to contextual interpretation; we *declare* our definition and expect there to be no variance in its interpretation in that particular work.

Mathematical definitions have many features, some critical to their nature and others, while not necessary to the categorization of mathematical definitions, preferred by the mathematics community. Van Dormolen and Zaslavsky (2003) outline the features of a good mathematical definition as follows.

Necessary Features

- Criterion of hierarchy: According to Aristotle, any new concept must be described as a special case of a more general concept — a square is a quadrilateral (general concept) with four congruent sides and one right angle (special case).
- Criterion of existence: Also required by Aristotle this criterion demands proof that at least one instance of the newly defined concept exists.
- Criterion of equivalence: If one gives more than one definition for the same concept, one must prove that they are equivalent.
- Criterion of acclimatization: A definition must fit into and be part of a deductive system.

Frequently Preferred Features

- Criterion of minimality: Only the minimal number of properties necessary to "reconstruct" the concept should be mentioned. Thus the definition of a square requires one right angle, not four.
- Criterion of elegance: When choosing between two equivalent definitions we want the one that uses fewer words and symbols, or the one that "looks" nicer.
- Criterion for degenerations: Sometimes the consequences of a definition allow degenerate cases that one may wish to exclude (or not).

The role and use of mathematical definitions is deeply imbedded in the culture of working mathematicians. However, to think about the use of definitions by novices in the field, a well-known framework that describes how mathematical knowledge in the form of conceptual ideas and their definitions is "stored" and used is helpful. This framework, known as *Concept Image/Concept Definition* (Tall, 1992; Vinner, 1991), describes the interplay of the individual's understanding of a particular mathematical concept and its formal definition. The concept image is the set of all the mental pictures associated in one's mind with the name of a particular concept as well as all the properties that characterize them. The concept image may be incomplete or mathematically incorrect, and can include naïve, non-mathematical associations with the concept name. For example, the notion of speed limit on a highway may be associated with the concept image of a mathematical limit.

On the other hand, the concept definition is the mathematical definition of the given concept. This may be known to the individual, he/she may be able to repeat a correctly memorized definition for a concept, or he/she may "remember" an incomplete or incorrect version of the definition. When faced with a task involving a given concept, rigorous mathematics demands that students base their solutions on the concept definition. Vinner (1991) postulated that students often rely instead upon the concept image. In the section that follows we discuss research on student understanding of the role of definitions, including some that have shown that even post-calculus mathematics majors may rely more on their concept images than on the concept definition when doing mathematical tasks.

Student Understanding of the Role of Definitions

It seems to be common knowledge in mathematics departments that many students do not "know" the definitions they need to know in order to perform mathematical tasks such as proving theorems. Often, in an attempt to solve this problem students are asked to memorize the pertinent definitions in the course and sometimes they are given credit in examinations for repeating those definitions. However, there has been research that shows that just knowing the definition may not be enough. Rasslan and Vinner (1998) studied 180 Israeli Arab high school students and their concept definitions and concept images of the increasing/decreasing function concept. One result of this study was that although 68 percent of the students could state the definition, only 36 percent of the students applied the definition successfully and well. Another 28 percent of the students applied the definition with varying levels of success.

The notion of "operable definition" is discussed by Bills and Tall (1998) in a report on their study of five students (three mathematics majors and two physics majors) in a twenty-week real analysis course that included work with the least upper bound property. The authors define operable in the following way: "A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument" (p. 104). Bills and Tall found that forming operable definitions is a task beyond that of just knowing the words of a definition and that "some students meet concepts at a stage when the cognitive demands are too great for them to succeed, others never have operable definitions, relying only on earlier experiences and inoperable concept images" (p.104).

It was necessary for the students in both of these studies to work with a concept definition that they could evoke from their own understanding. Their "success" (or lack thereof) in the given tasks thus depended at least in part upon their ability to remember and apply the appropriate definition. What if students could be relieved of that cognitive load and asked to work with definitions that were available to them throughout the tasks that they were asked to do? Edwards (1997a, 1997b) postulated that if students had mathematically correct definitions in front of them at all times during interviews and written tasks, it would be possible to see evidence of their understanding of *how mathematical definitions should be used,* unencumbered by worry about the actual wording of a particular definition.

Edwards' original study involved eight mathematics majors enrolled in an introductory real analysis course that had as one of its goals helping students learn to write proofs. Specifically, the course was described as an introduction to rigorous analytic proofs in the context of the properties of real numbers, continuity, differentiation, integration and infinite sequences and series. Although the nature of mathematical definitions was implicit in the delivery of the course in which the eight students were enrolled, the teacher did not explicitly draw attention to that issue. We will discuss the methodology and results of this study later, but briefly, Edwards found that even with the definitions in front of them, many of the undergraduate mathematics majors of her study had some difficulty using mathematical definitions in a mathematically appropriate way. The question arose, however, whether this result could have been influenced by the fact that many of the definitions in Edwards' study were of concepts that students had encountered in elementary calculus or were in some way related to those concepts (e.g., absolute continuity). Would this familiarity cause students to misread definitions and cause some uncertainty in the results? Although there was no evidence of this in the original study, these questions led to the second study conducted by Edwards and Ward (2004).

The Edwards and Ward study involved eight mathematics majors enrolled in an abstract algebra course taught by one of the researchers and observed by the other. In this study, the definitions that were chosen for the task-based interviews conducted by the observer were designed to minimize the chance of previous mathematical connections, in other words, students would be *forced* to work from the definitions that were provided for them because they would supposedly have no other mathematical "memories" to use in performing the tasks.

We discuss the methodologies and results of the two studies together since the findings from the second study corroborated those of the first. In our description and discussion of the two studies we will henceforth refer to the Edwards study as the analysis study and the Edwards and Ward study as the algebra study.

Design of the Studies

The purpose of both studies was to look beneath students' understandings of the *content* of mathematical definitions to discern their understandings of the *role* played by formal definitions in mathematics. This is somewhat tricky since it is possible that a student might apply a definition in a mathematically incorrect way for at least two reasons.

- A student could have an incomplete or faulty understanding of the content of a particular definition; or
- A student could have a mathematically incorrect understanding of the role or nature of mathematical definitions in general.

For example, a student may decide that $f(x) = 3$ is not a function because the symbolic form "has no x in it" (a faulty understanding of the function definition itself); or he may decide that the particular example is not a function even after reading the definition because the requirement of having an x is something we just know and it does not need to be mentioned in the definition (a faulty understanding of the role and character of mathematical definitions in general).

A further difficulty influencing the design of these studies arose from the possibility that, if asked directly, students might profess a seemingly adequate understanding of the role of formal definitions in mathematics without really understanding this role. It is not uncommon for students (or people in general) to repeat something they have heard without full understanding. For instance students may say, perhaps to please their teachers, that "mathematics is necessary in all walks of life," without being able to cite even one non-trivial example beyond the day-to-day interactions involved in commerce.

Both studies employed similar research methods, which we now briefly outline. Participants in both studies were volunteers from an upper division mathematics course, introductory real analysis or abstract algebra. Participants in both studies were in the last two years of an undergraduate major in mathematics. Each participated in task-based interviews spaced through the course. The intent of these interviews was to investigate each student's understanding and strategies in dealing with definitions including ones that the students had previously encountered, ones that students were currently encountering in their course, and ones that students had not encountered before.

The format of the interviews depended upon the student's familiarity with a given definition. For instance, if the definition had been encountered before but had not been discussed in the course, the students were asked first to explain in their own words their understanding of the associated concept and then to provide a definition for it if they could do so. The students were then given a copy of the stipulated definition and were asked to explain its meaning and to discuss how their previous explanation agreed or did not agree with their understanding of the given formal definition. Such was the approach used with the definitions of continuity and infinite decimal in the analysis study, for example. When students had no familiarity with a concept, the interview began with the student considering some stipulated formal definition. The definitions of "group" and "coset multiplication" from the algebra study fall into that category.

In all cases, following the introduction of the stipulated definition, students were given tasks to complete which required use of the definition. Two of the tasks were determining if a given function was continuous and determining if the set of cosets of a given subgroup formed a group under the operation of coset multiplication. In the latter case, definitions and tasks were selected because the researchers' experience and that of others suggested they would be difficult for the students (Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Brown, DeVries, Dubinsky, & Tomas, K, 1997). In all the tasks, the goal was to observe in what ways the students used, or did not use, the definitions to complete the task and to overcome their difficulties. Students had access to the written definitions at all times during the interview; thus, the researchers hoped that inaccurate memories of the mathematically correct definitions would not compromise the students' ability to do the given task in a way that was not consistent with the goals of the study.

In addition, in one of the interviews, all students were asked: "What is mathematics?" which was an indirect way for the interviewer to probe students' understandings of the role of mathematical definitions. The interviews were audio-taped and video-taped. Verbatim transcripts of the interviews were then made and analyzed by the researchers.[1]

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[1] For a description of the analysis process for the analysis study see (Edwards, 1997a or 1997b).

Results

The results of both studies are discussed in detail in Edwards & Ward (2004). Briefly, both studies showed that many undergraduate mathematics majors do not categorize mathematical definitions as stipulated, and that they may, under some circumstances, defer to their image of a concept rather than the definition if the two do not agree. Further, the algebra study showed that some students do not use definitions the way mathematicians do, even in the apparent absence of any other course of action.

Participants in both studies were analyzed in light of what they said about the role played by definitions in mathematics as well as how each one used the definitions to complete the tasks involved in the interviews. Students were not necessarily consistent in their views and actions. For instance some students would seem to understand the role of mathematical definitions, but later on in the interview would use or not use a definition in a way that was mathematically inappropriate.

It is important to note that the students in both studies had all successfully completed at least one advanced mathematics course prior to their involvement in one of our studies and that all of the students comfortably passed the courses in which they were enrolled during the two studies. Thus our research indicates that some undergraduates with advanced mathematical training and decent, sometimes excellent, grades do not completely understand the nature and role of mathematical definitions.

We will illustrate the results of our studies with one example from each study, beginning with Jesse in the analysis study and his work with the definition for a point-wise continuous function. Before seeing the formal definition Jesse said he remembered a definition from high school for continuity, that a function was continuous at a point "if the limit at the point equals the actual value of the point." After some discussion Jesse and the interviewer formalized his (mathematically acceptable) definition and wrote it out in the following way.

A function f is continuous at a point $x = a$ if $f(a)$ exists and if $\lim_{x \to a} f$ exists and if $\lim_{x \to a} f = f(a)$.

He was then given the standard stipulated definition which was also the same definition that soon would be discussed in his introductory real analysis course.

Definition: Let f be a real-valued function whose domain is a subset of R. Then f is continuous at $x_0 \in$ dom(f) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x_0 \in$ dom (f) and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

When Jesse read the standard stipulated definition he remembered seeing it at one time and he was able to discuss the first two function tasks of the interview using both forms of the definition (although he preferred his version). The third task was to state whether or not the following function was continuous at $x = 0$.

$$f(x) = |x|$$

Jesse considered his definition and the "epsilon/delta" definition, and said,

> Jesse: Cusps, or there were a whole bunch of things that were not continuous. And, I think this is one of them. Although, it looks pretty continuous…. I'm pretty sure I remember that this is not continuous and my definition isn't cutting it, so I'm looking at the, at the real one. [Jesse pointed to the "epsilon/delta" definition. His definition wasn't "cutting it" because it was telling him that the given function was, in fact, continuous.]

For several minutes he went back and forth between "knowing" that this function was not continuous and seeing that both definitions before him indicated that the function was continuous. Finally, he said,

> Jesse: But that, but I know cusps, and sharp peaks are not, but from the definition, if we're saying that the limit of these two equals that, and f of a equals that, then that would be continuous. (Short pause.) But it's not.

Clearly, although Jesse seemed to understand the definition (especially his own version), was able to refer to it during the interview at any time, and had used it in previous tasks, he based his mathematical decision on (misleading) memories from his elementary calculus class.[2] It is interesting to note a comment that Jesse made about definitions toward the end of this interview — that they were a "lot of jargon." During the second interview Jesse had indicated, "After about the first day in calculus, we didn't care about this [formal definition] … if you had the concept right,

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[2] The following day he reported that he had remembered that the absolute value function *was* in fact continuous, it just was not differentiable at $x = 0$. The fact remained however that it was not the definition that convinced him.

not really the definition, that was all that really mattered." Although several current popular calculus texts and many calculus teachers (including the authors of this chapter) do not focus on formal definitions in elementary calculus, one must consider the impact of these pedagogical decisions and perhaps do something to mitigate against later misunderstandings by the student.[3]

Heidi, a student from the algebra study, represents another example of a student who seemed not to categorize mathematical definitions as stipulated. (See Edwards & Ward, 2004, for more details.) However, analyzing her understanding was more difficult. Sometimes she seemed to be exhibiting a mathematically correct view of definitions, as when she talked about a hierarchy of definitions and axioms being used to prove theorems. But a few lines later she said, "You have to make the definitions from what something actually is," which would seem to indicate a view of definitions as extracted. While doing a task involving cosets in the second interview she pointedly avoided using the definitions that were available to her. She tried instead to do the task by remembering how she had done similar tasks before.

Heidi seemed to teeter on the cusp of understanding, however, and we see her case as evidence that a student's understanding of the categorization of mathematical definitions is not necessarily clear cut. Students do not fit nicely into one group (those who understand) or the other (those who do not understand). This notion is consistent with what Burger and Shaughnessy (1986) describe as transitional stages of students' understanding between van Hiele levels in geometry. According to Burger and Shaughnessy, it is possible that students may exhibit different levels of understanding on different tasks and some may even oscillate between levels of understanding on the same task.

For us, Heidi's case is evidence that the notion of mathematical definitions may be a "teachable" concept. According to Vygotsky (1978) each individual's understanding of a given concept resides in a zone that reaches somewhat beyond his or her understanding. With the help of a teacher, that individual's understanding can grow within that zone. We interpret Heidi's understanding to be in the zone of a mathematically correct understanding of the role of definitions in mathematics, one that would categorize mathematical definitions as stipulated. In the next section we will discuss the role of definitions in mathematics courses as a pedagogical question.

The Role of Definitions in Undergraduate Mathematics Courses

So far we have focused on how students perceive the way mathematical definitions are used in advanced mathematics. However, there is also the issue of the role mathematical definitions play in the teaching of mathematics. This is a separate issue, and it is possible that in some ways these two roles may conflict. There is mathematics education literature that focuses on the role of mathematical definitions in the teaching of mathematics. Some researchers have addressed the issue in the context of K–12 students, often in the teaching of geometry (de Villiers, 1998; Mariotti, & Fishbein, 1997; Van Dormolen & Zaslavsky, 2003). Others have focused on college classrooms, especially in courses populated by prospective teachers (Hershkowitz, Bruckheimer, & Vinner, 1987; Rasmussen, & Zandieh, 2000; Harel & Brown, this volume; Zazkis, this volume). One prominent notion in this literature is that students should have experiences creating their own definitions. De Villiers (1998) writes about Felix Klein's notion of the *bio-genetic* principle as a way to employ definitions to enhance students' understanding of mathematical concepts. In Klein's view, mathematics topics should not be presented to students as completed axiomatic-deductive systems. Rather students should retrace (to some extent) the path of the original thinking about, or discovery/invention of the concept. In this paper, de Villiers (1998) describes a study that focused on developing students' abilities to construct formal definitions for geometric concepts. This study involved tenth grade students in 19 schools and showed that students who were given defining activities in the course of learning geometry had much more success when asked to complete tasks involving writing correct, economical definitions of geometric concepts.

Like Klein, Freudenthal (1973) strongly criticized the traditional pedagogical practice of providing for students extant definitions of geometric concepts. He believed that since definitions were not preconceived by the students themselves, but were the final touch of a mathematical activity, mathematics instructors were denying students the chance to participate in the entire activity by merely giving them the final product. In his view students should be allowed to participate in the mathematical enterprise from the beginning, including creating definitions.

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[3] In the authors' view, this would *not* include returning to a "formal definitions" approach in the teaching of elementary calculus, but it could include discussions and activities focusing on the role of definitions in mathematics. Some of these are described later in this chapter.

Definition activities in mathematics courses can have several pedagogical objectives, some of which could be

- promoting deeper conceptual understanding of the mathematics involved,
- promoting an understanding of the nature or the characteristics of mathematical definitions, and/or
- promoting an understanding of the role of definitions in mathematics.

Definitions are frequently and most obviously used to promote the first objective, deeper conceptual understanding of mathematics. Indeed the traditional method of communicating mathematics between professional mathematicians begins with a statement of the pertinent definition or definitions. Activities that involve studying a mathematical definition carefully and deciding from a collection of items which are and which are not examples of the defined concept are plentiful. Some of the tasks of the interviews for both the algebra study and the analysis study are examples of such activities. These activities also indirectly address the role of mathematical definitions. For a related discussion on this topic see Wilson (1990).

Activities that address the second objective are common also, although we feel that these alone might also not be sufficient for encouraging proper use of definitions in formal mathematics. An activity used by both of the authors is to ask students working in groups to define a given concept, for example, prime number. Each group agrees upon a definition for prime number and all definitions are then displayed for the whole class to compare and then choose the "best" definition. Of course, what it means to be "best" is also discussed. Students often mention criteria from Van Dormolen and Zaslavsky (2003) and when necessary we guide the discussion toward considering the entire list. We also encourage discussions about the consequences of various conditions included in the definition for prime. For instance, do we want 1 to be prime? Further discussion of the second objective and some additional activities for undergraduate students can be found in Hershkowitz, Bruckheimer & Vinner (1987); Winicki-Landman, & Leikin (2000); and Leikin, & Winicki-Landman (2000).

It seems that the key issue for many of the students from the analysis and algebra studies, however, was understanding that mathematical definitions are stipulated and thus different from everyday definitions. To develop this understanding requires treating mathematical definition as a concept in its own right by promoting an understanding of the role of definitions in mathematics, our third objective.

In Edwards and Ward (2004), some activities are given that are designed to promote this third pedagogical objective in undergraduate mathematics courses. These activities include directly addressing the topic of mathematical definitions including discussing the notion of Concept Image/Concept Definition with students, or exploring the dictionary definitions of such words as *radical*, which has both stipulated and extracted definitions.

Probably most important, however, is providing activities for students that involve them in the process of creating their own definitions in authentic ways as practicing mathematicians might do. For instance, how can *triangle* on the sphere or a hyperbolic surface be defined so that the congruence theorems in Euclidean geometry will hold on these non-Euclidean surfaces? Or, how can *continuous function* be defined so that it can only describe a function that can be drawn "without lifting one's pencil"? It seems that authentic experiences in defining might help students understand that although they have a certain amount of freedom in the creation of the definitions, the purpose for having a definition in the end is that there will be no misunderstanding about the exact nature of the concept being defined.

It is, however, in activities involving the creation of definitions where we see the potential for conflicting messages if the goal of the activity is that students understand a given concept more deeply. It is our view that implementing the ideas of Klein and Freudenthal, using definitions to increase students' understanding of mathematical concepts must be done with caution and potentially can be problematic. If students are asked to create definitions in such a way that their task is actually one of discovering the *"correct"* definition for a concept, it may give them the impression that definitions can be right or wrong and that they are extracted rather than stipulated. In the analysis study, one student, Stephanie, decided that a given definition for infinite decimal was wrong in the case of the decimal .999…. (Edwards, 1997b). Stephanie's work with definitions indicated a subconscious belief that one's concept image should rule in the case of a conflict between concept image and concept definition. Some defining activities could inadvertently reinforce this notion.

A central focus in the definition work of Klein and Freudenthal was that students will understand mathematics better if they are involved in its development from start to finish. This is indeed an excellent way to learn mathematics especially because the student can have a better notion of what mathematics is (cf. Maher & Martino, 1996). Too

often, however, defining activities are like games that are won if the student can guess what the teacher is thinking. We believe that if students create their own definitions for concepts with existing definitions, they should first be allowed to carry the activity through to its logical conclusion, including realizing unintended consequences. Later students can compare their definitions to the definitions that were created before them, probably with a much greater understanding of the defining process. This is possibly a more difficult and longer process, but it is more authentic.

Conclusion

In this paper we have outlined a framework that looks at everyday definitions as extracted and mathematical definitions as stipulated. We have also discussed the notion of Concept Image/Concept Definition as a way of describing how students work with definitions in mathematical tasks and we have described research on students' understanding of the role of definitions in mathematics as well as research on the role of definitions in mathematics education. We believe there are still questions regarding the interplay between students' understanding of the nature and role of mathematical definitions and their experiences with definitions while learning mathematics. Research is needed to determine appropriate pedagogical approaches that encourage the development of the concept of definition throughout a student's experience in mathematics classes. Further, research is needed to assess the effectiveness of activities designed to help students create more robust understandings of the concept of definition.

Definitions are essential to the mathematical enterprise, thus it seems that focus on the role of definition should be central to the education of mathematics majors. However, according to Vinner (1991),

> The role of definition in mathematical thinking is somehow neglected in official contexts…. We are not sure whether this is because it is taken for granted or because it is overlooked. It is obligatory to remember that there are some contexts in which referring to the formal definition is critical for a correct performance on a given task (p. 80).

Although many students will eventually "figure out" how to use formal definitions in a mathematically correct way, it seems that it is important not to leave this to chance. This is especially true for students who will become K–12 mathematics teachers for it is these students who will have the greatest impact on our future students. Our research has shown that there exist mathematics majors who have been successful in advanced courses in mathematics, at least from the standpoint of the grades they earn, but who really do not understand the role of mathematical definitions in a mathematically acceptable way. This concept, like any other mathematical concept, can and should be addressed in undergraduate mathematics classrooms.

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